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Math and Geometry

Introduction to Math and Geometry

This chapter tackles several problems relating to important math and geometry concepts in programming, which regularly appear in technical interviews.

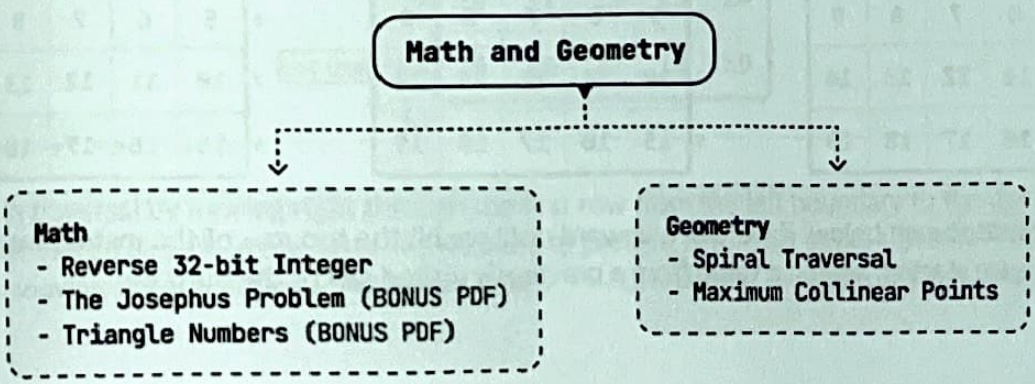
We explore subjects such as:

- Greatest common divisor (GCD).
- Modular arithmetic.
- Handling floating-point precision.
- Handling integer overflow and underflow.
- Recognizing patterns.

Real-world Example

Computer graphics: In 3D rendering, geometry is used to represent objects as shapes like polygons, and mathematical transformations such as rotation, scaling, and translation, are applied to these objects to animate them, or change their perspective. Algorithms that use trigonometry, vector math, and matrix operations, are critical for determining how objects move, interact with light, or cast shadows in virtual environments.

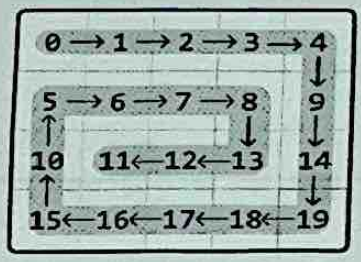
Chapter Outline



Spiral Traversal

Return the elements of a matrix in clockwise spiral order.

Example:



Output: `[0, 1, 2, 3, 4, 9, 14, 19, 18, 17, 16, 15, 10, 5, 6, 7, 8, 13, 12, 11]`

Intuition

To create the expected output for this problem, let's try simulating exactly what the problem describes and traverse the matrix in spiral order, adding each value to the output as we go. How can we do this?

Spiral traversal involves moving through the matrix in one direction until we can't go any further, then changing direction and continuing. Specifically, the sequence of directions is right, down, left, and up, repeated until all elements are traversed. To achieve this, we need to determine the exact conditions for switching directions.

Initially, our approach may seem simple: we start by moving right until reaching the right-most column of the matrix, at which point we switch directions. We can move and switch directions like this three times without running into any problems:

`move right until we reach the rightmost column:`

	0	1	2	3	4
0	0 →	1 →	2 →	3 →	4
1	5	6	7	8	9
2	10	11	12	13	14
3	15	16	17	18	19

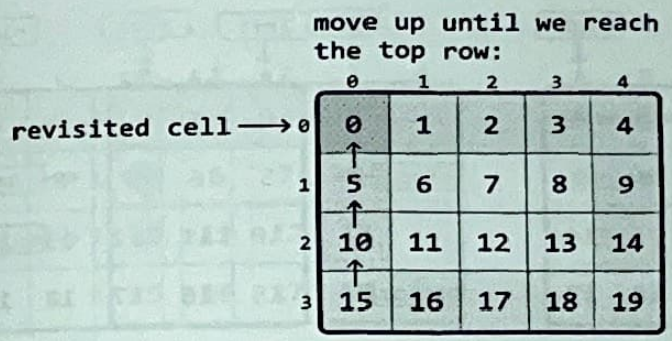
`move down until we reach the bottow row:`

	0	1	2	3	4
0	0	1	2	3	4 ↓
1	5	6	7	8	9 ↓
2	10	11	12	13	14 ↓
3	15	16	17	18	19

`move left until we reach the leftmost column:`

	0	1	2	3	4
0	0	1	2	3	4
1	5	6	7	8	9
2	10	11	12	13	14
3	15 ←	16 ←	17 ←	18 ←	19

However, as shown below, if we move upward until we hit the top row of the matrix, we'll return to where we started, adding a value from a previously visited cell to the output:



A potential solution to this is to keep track of all cells visited by using a hash set. This allows us to stop moving in a direction when we encounter a visited cell. While this approach is effective, it requires $O(m \cdot n)$ space, where m and n are the dimensions of the matrix. This is because we need to store every cell of the matrix in the hash set. Is there a way to avoid revisiting cells without using an additional data structure?

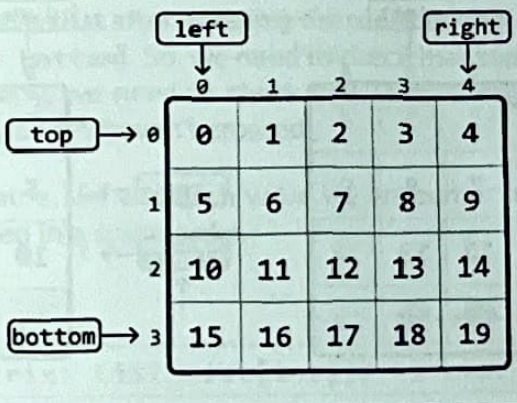
Notice in the above diagrams that when we move in a certain direction, we continue until we reach one of the boundary rows or columns (i.e., the top or bottom row, or the leftmost or rightmost column).

What if we adjust these boundaries as we traverse the matrix, to avoid revisiting previous cells?

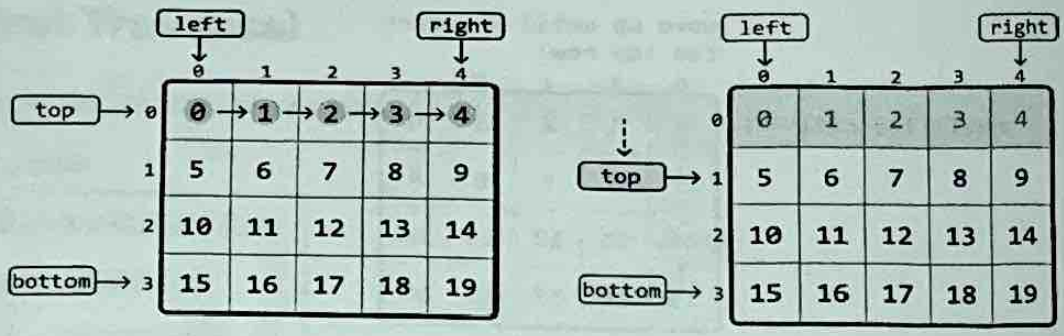
Adjusting boundaries

Let's initialize the four boundaries (`top`, `bottom`, `left`, `right`) with their initial positions:

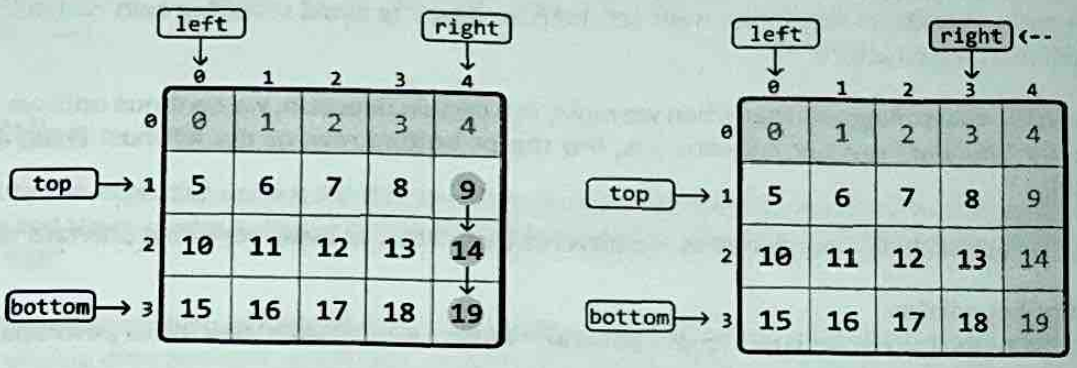
- `top = 0`
- `bottom = m - 1`
- `left = 0`
- `right = n - 1`



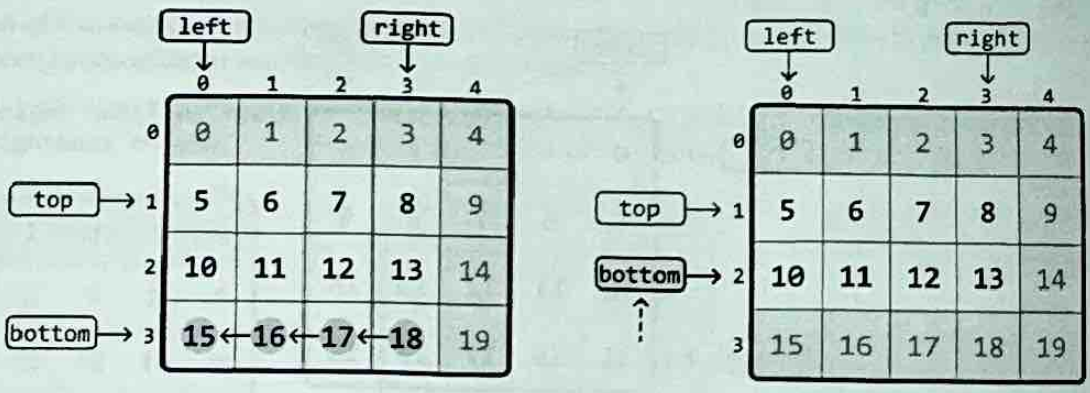
We begin traversal by moving **right** through the first row from the left boundary to the right. Since we've just visited all cells in the first row, we need to prevent future access to this row. This can be done by moving the top boundary down by 1 (`top += 1`), ensuring the top row can't be accessed:



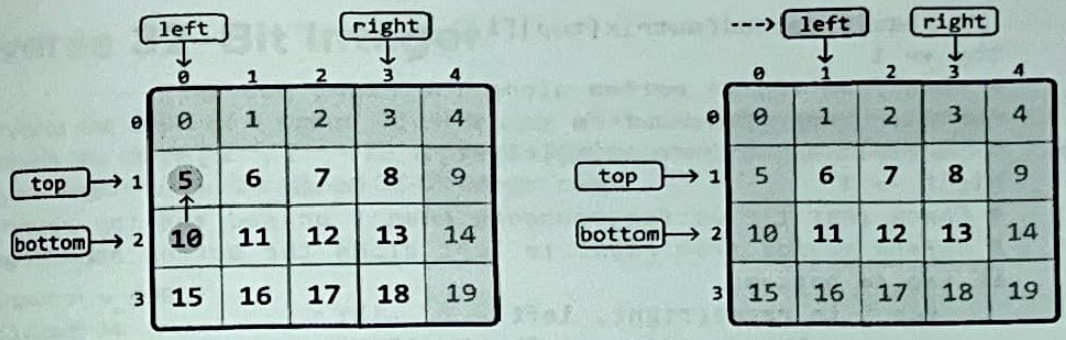
Next, we move **down** from the top boundary to the bottom boundary. To ensure this column is not revisited, update the right boundary (`right -= 1`):



Next, we move **left** from the right boundary to the left. To ensure this row doesn't get revisited, update the bottom boundary (`bottom -= 1`):



Next, we move **up** from the bottom boundary to the top boundary. To ensure this column isn't revisited, update the left boundary (`left += 1`):



We've just discussed how to traverse in each of the four directions and update the corresponding boundaries. These traversals are repeated until either the top boundary surpasses the bottom boundary, or the left boundary surpasses the right boundary. Either of these indicate there are no more cells left to traverse.

In summary, we traverse the matrix in spiral order by repeating the following sequences of traversals:

1. Move from left to right along the top boundary, then update the top boundary (`top += 1`).
2. Move from top to bottom along the right boundary, then update the right boundary (`right -= 1`).
3. Move from right to left along the bottom boundary, then update the bottom boundary (`bottom -= 1`).
4. Move from bottom to top along the left boundary, then update the left boundary (`left += 1`).

This continues while top $\leq$ bottom and `left` $\leq$ `right`.

A crucial thing to keep in mind is that after updating the top boundary, the top boundary might pass the bottom boundary (`top > bottom`). So, we need to check that `top` $\leq$ `bottom` before traversing the bottom boundary. Similarly, we need to check that `left` $\leq$ `right` before traversing the left boundary to ensure the boundaries haven't crossed.

As we move through the matrix, we add each value we encounter to the output array. This way, the matrix values are recorded in a spiral order.

Implementation

```
def spiral_matrix(matrix: List[List[int]]) -> List[int]:
    if not matrix:
        return []
    result = []
    # Initialize the matrix boundaries.
    top, bottom = 0, len(matrix) - 1
    left, right = 0, len(matrix[0]) - 1
    # Traverse the matrix in spiral order.
    while top <= bottom and left <= right:
        # Move from left to right along the top boundary.
        for i in range(left, right + 1):
```

```
            result.append(matrix[top][i])
        top += 1
        # Move from top to bottom along the right boundary.
        for i in range(top, bottom + 1):
            result.append(matrix[i][right])
        right -= 1
        # Check that the bottom boundary hasn't passed the top boundary
        # before moving from right to left along the bottom boundary.
        if top <= bottom:
            for i in range(right, left - 1, -1):
                result.append(matrix[bottom][i])
            bottom -= 1
        # Check that the left boundary hasn't passed the right boundary
        # before moving from bottom to top along the left boundary.
        if left <= right:
            for i in range(bottom, top - 1, -1):
                result.append(matrix[i][left])
            left += 1
    return result
```

Complexity Analysis

Time complexity: The time complexity of `spiral_matrix` is $O(m \cdot n)$ because we traverse each cell of the matrix once.

Space complexity: The space complexity is $O(1)$. The `res` array is not included in the space complexity.

Reverse 32-Bit Integer

Reverse the digits of a signed 32-bit integer. If the reversed integer overflows (i.e., is outside the range $[-2^{31}, 2^{31} - 1]$), return 0. Assume the environment only allows you to store integers within the signed 32-bit integer range.

Example 1:

```
Input: n = 420
Output: 24
```

Example 2:

```
Input: n = -15
Output: -51
```

Intuition

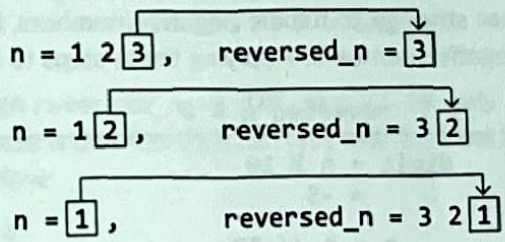
The primary challenge with this problem is in handling its edge cases. Before tackling these edge cases, let's first try handling the more basic cases and later see how we would need to modify our strategy.

Reversing positive numbers

Consider `n = 123`. Let's try building the reversed integer one digit at a time. The first thing to figure out is how to iterate through the digits of n to build our reversed number (initially set to 0):

```
n = 1 2 3 ,    reversed_n = 0
```

One way to do this is by starting at the last digit of n and appending each digit to `reversed_n`:



Let's explore how we can do this. To extract the last digit, we can use the modulus operator: `n % 10`. This operation effectively finds what the remainder of n would be if divided by 10:

```
n = 123
digit = n % 10
      = 3
```

After extracting the last digit, we can remove it by dividing n by 10, which shifts the second-to-last digit to the last position, preparing it for the next iteration:

```
n = n // 10
  = 12
```

Once that's done, let's add the last digit extracted to our reversed number:

```
reversed_n += digit
           = 3
```

Below, we see the current states of n and reversed_n:

```
n = 1 2 , reversed_n = 3
```

To process the next digit, let's extract it from n using the modulus operation, then remove it by dividing n by 10:

```
digit = n % 10
      = 2

    n = n // 10
      = 1
```

To append this digit to reversed_n, we can multiply reversed_n by 10 to shift its digits to the left, making space for the new digit. Then, we just add the new digit as before:

```
reversed_n = 10 * reversed_n + digit
           = 30 + 2
           = 32
```

We can repeat the above process until all digits of n are appended to reversed_n (i.e., until n equals 0). Here's a breakdown of this process:

1. Extract the last digit: `digit = n % 10`.
2. Remove the last digit: `n = n // 10`.
3. Append the digit: `reversed_n = reversed_n * 10 + digit`.

Reversing negative numbers

Before considering a separate strategy to handle negative numbers, let's first check if the set of steps above also work for negative numbers. Applying these steps to `n = -15` gives:

```
n = -15 ,  reversed_n = 0
```

```
digit = n % 10
      = -5

    n = n // 10
      = -1

reversed_n = 10 * reversed_n + digit
           = -5
```

```
n = -1 ,  reversed_n = -5
```

```
digit = n % 10
      = -1

    n = n // 10
      = 0

reversed_n = 10 * reversed_n + digit
           = -51
```

As we can see, it works for negative numbers. Now, let's tackle situations in which reversing a number could result in integer overflow or underflow.

Detecting integer overflow

If the reverse of a positive number is larger than $2^{31} - 1$, it will overflow, and we should return 0. Let's call this maximum value `INT_MAX`.

$$\texttt{INT\_MAX} = 2^{31} - 1 = 2147483647$$

Initially, it might seem sufficient to reverse the number completely, check if it exceeds $2^{31} - 1$, and return 0 if it does. However, in an environment where integers larger than $2^{31} - 1$ cannot be stored, attempting to reverse such an integer would cause an overflow:

$$1999999999 \xrightarrow{\text{reverse}} 9999999991 > \texttt{INT\_MAX} \longrightarrow \texttt{overflow}$$

So, let's think of another way to detect overflow.

We're constructing the number `reversed_n` one digit at a time, which means we need to ensure not to cause the number to overflow with each new digit we add. Let's think about when adding a new digit might cause `reversed_n` to become too large. Here's how we can analyze this:

If `reversed_n` is equal to 214748364 (i.e., `INT_MAX // 10`), then the final digit we can add to it must be less than or equal to 7 to avoid an overflow (since 2147483647 == `INT_MAX`):

```
                reversed_n = 21478364,  digit = 7
append digit:   reversed_n = reversed_n * 10 + digit
                           = 214783647 == INT_MAX ⟶ no overflow
```

```
                reversed_n = 21478364,  digit = 8
append digit:   reversed_n = reversed_n * 10 + digit
                           = 214783648 > INT_MAX ⟶ overflow
```

Now, keep in mind that when `reversed_n == INT_MAX // 10`, only one more digit can be added to it. The key observation here is that this digit can only ever be 1 because a larger final digit would be impossible, as shown below:

```
                INT_MAX // 10
               ┌──────────┐
reversed_n = 2147483641  ⟶ implies ⟶ n = 1463847412
               └──────────┘
                INT_MAX // 10
               ┌──────────┐
reversed_n = 2147483642  ⟶ implies ⟶ n = 2463847412 > INT_MAX ⟶ impossible input
               └──────────┘
```

This means that when `reversed_n == INT_MAX // 10`, the last digit added to it won't cause an overflow, meaning we don't need to check the last digit in this case.

If `reversed_n` is already larger than `INT_MAX // 10`, adding any digit will cause it to overflow. We can handle this case with the following condition:

```
if reversed_n > INT_MAX // 10:
    return 0
```

Detecting integer underflow
We can apply similar logic to the above for handling integer underflow. Here, we just need to check that `reversed_n` never falls below `INT_MIN // 10`:

```
if reversed_n < INT_MIN // 10:
    return 0
```

Implementation

In Python, using the modulus operator (%) with a negative number gives a positive result. To avoid this, we can instead use **`math.fmod`** and cast its result to an integer to attain a negative modulus value.

For division, Python's `//` operator performs floor division, which can result in an undesired value when dealing with negative numbers (e.g., `-15 // 10` results in -2, instead of the desired -1). To achieve the desired behavior, use `/` for division and cast its result to an integer.

```
def reverse_32_bit_integer(n: int) -> int:
    INT_MAX = 2**31 - 1
    INT_MIN = -2**31
    reversed_n = 0
    # Keep looping until we've added all digits of 'n' to 'reversed_n'
    # in reverse order.
    while n != 0:
        # digit = n % 10
        digit = int(math.fmod(n, 10))
        # n = n // 10
        n = int(n / 10)
        # Check for integer overflow or underflow.
        if (reversed_n > int(INT_MAX / 10)
            or reversed_n < int(INT_MIN / 10)):
            return 0
        # Add the current digit to 'reversed_n'.
        reversed_n = reversed_n * 10 + digit
    return reversed_n
```

Complexity Analysis

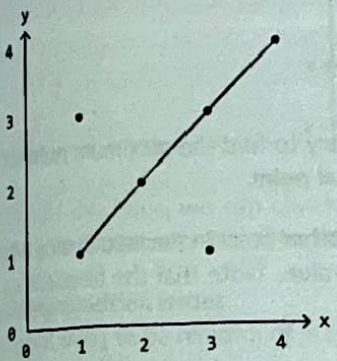
Time complexity: The time complexity of `reverse_32_bit_integer` is $O(\log(n))$ because we loop through each digit of n, of which there are roughly $\log(n)$ digits. As this environment only supports 32-bit integers, the time complexity can also be considered $O(1)$.

Space complexity: The space complexity is $O(1)$.

Maximum Collinear Points

Given a set of points in a two-dimensional plane, determine the maximum number of points that lie along the same straight line.

Example:



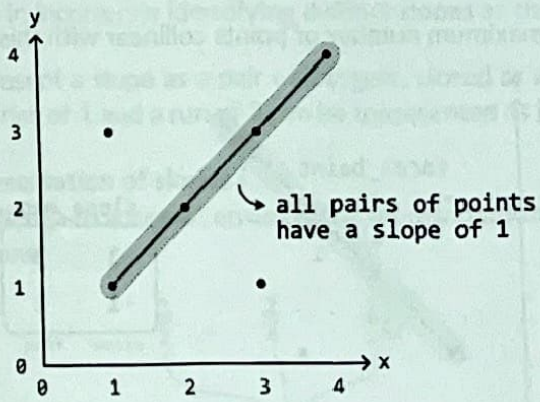
Input: `points = [[1, 1], [1, 3], [2, 2], [3, 1], [3, 3], [4, 4]]`
Output: 4

Constraints:

- The input won't contain duplicate points.

Intuition

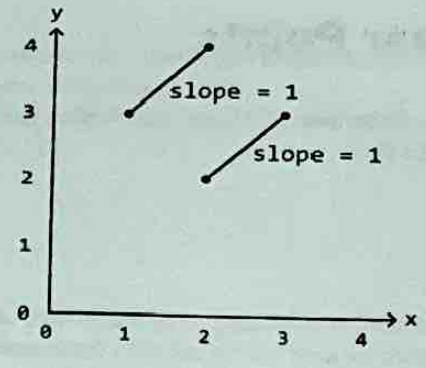
Two or more points are collinear if they lie on the same straight line. In other words, the **slope** between any pair of points among them will be equal:



Let's remind ourselves how the slope of a line is calculated given two points (x_a, y_a) and (x_b, y_b):

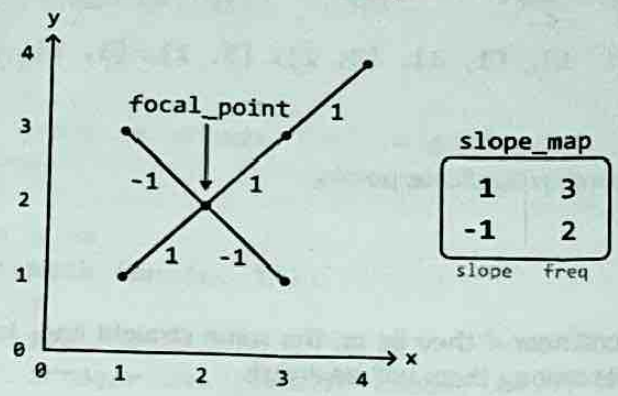
$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{y_b - y_a}{x_b - x_a}$$

Using this formula, we can calculate the slope between all pairs of points from the input, and determine the largest number of pairs that share the same slope. However, this approach is flawed because two pairs of points with the same slope value may not be collinear:

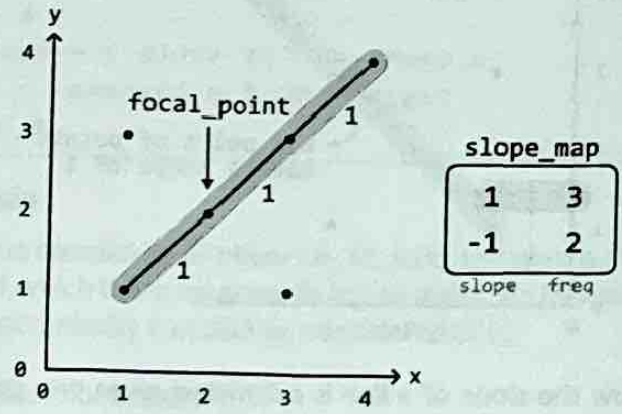


Let's think of a different way to find the answer. What if we try to find the maximum number of points collinear with a specific point? Let's call this point a **focal point**.

We can calculate the slope between the focal point and every other point in the input, using a hash map to count how many points correspond with each slope value. Note that the frequencies of points stored in the hash map do not include the focal point.



This allows us to find the maximum number of points collinear with this focal point:



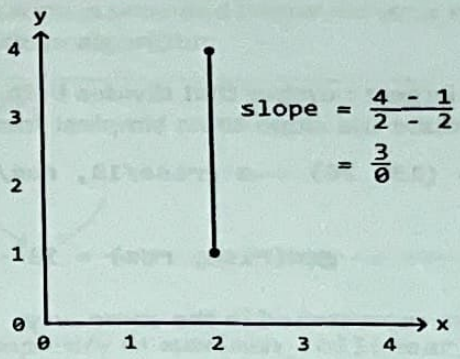
Here, the slope with the highest frequency is 3, which means the number of points on the line defined by that slope is 3 + 1 = 4, where +1 is used to account for the focal point itself.

By repeating the process for every point, we can determine the maximum number of points that are collinear with each focal point. Our final answer is equal to the largest of these maximums.

Edge case: two points on the same x-axis

When two collinear points share the same x-axis, the denominator of the slope equation will equal

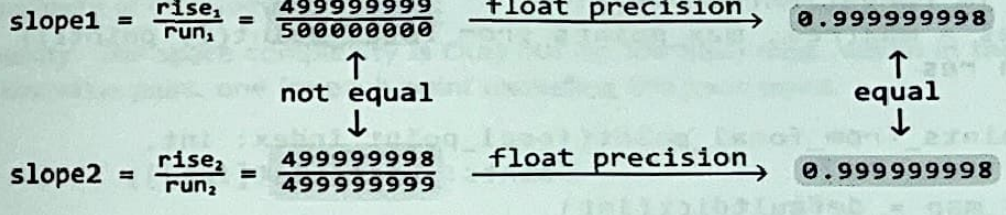
0 (i.e., run = $x_b - x_a = 0$). This is problematic because dividing by 0 is undefined:



To handle this issue, we can check if our run value is equal to 0. If so, we can just use infinity (`float('inf')`) to represent the value of this slope.

Avoiding precision issues

A crucial thing to be mindful of is the precision of slopes when storing them as floats or doubles. Consider the following slopes:

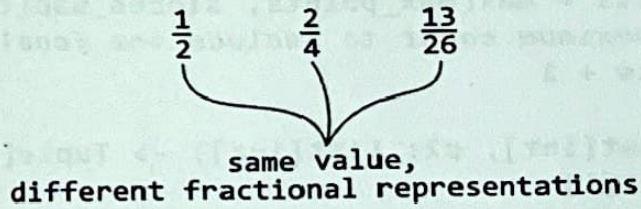


As we can see, despite the fractions themselves representing different slopes, presenting them as a float or double doesn't provide enough decimal-point precision to distinguish between them accurately. This can result in incorrectly identifying distinct slopes as the same.

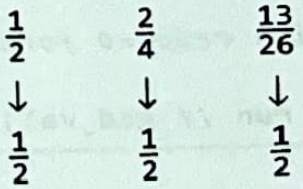
To avoid this, we can represent a slope as a pair of integers, stored as a tuple: `(rise, run)`. For example, the slope with a rise of 1 and a run of 2 can be represented as (1, 2) instead of 1 / 2 = 0.5.

Ensuring consistent representation of slopes

There's just one more challenge to address: ensuring the representation of a slope is consistent for all equivalent slope fractions:

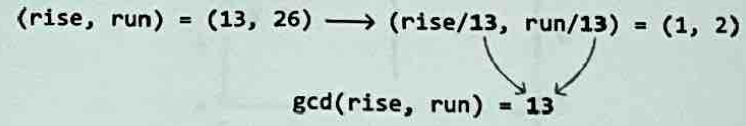


If we reduce fractions to their simplest forms, we can consistently represent equal fractions that have different initial representations.



But how can we do this? We can reduce fractions by dividing both the rise and run by their greatest common divisor (GCD).

Greatest common divisor
The GCD of two numbers is the largest number that divides both of them exactly. By dividing the rise and run by their GCD, we reduce the slope to its simplest form:



This ensures all equal fractions are represented in the same way.

Implementation

```
def maximum_collinear_points(points: List[List[int]]) -> int:
    res = 0
    # Treat each point as a focal point, and determine the maximum
    # number of points that are collinear with each focal point. The
    # largest of these maximums is the answer.
    for i in range(len(points)):
        res = max(res, max_points_from_focal_point(i, points))
    return res

def max_points_from_focal_point(focal_point_index: int,
                                points: List[List[int]]) -> int:
    slopes_map = defaultdict(int)
    max_points = 0
    # For the current focal point, calculate the slope between it and
    # every other point. This allows us to group points that share the
    # same slope.
    for j in range(len(points)):
        if j != focal_point_index:
            curr_slope = get_slope(points[focal_point_index], points[j])
            slopes_map[curr_slope] += 1
            # Update the maximum count of collinear points for the
            # current focal point.
            max_points = max(max_points, slopes_map[curr_slope])
    # Add 1 to the maximum count to include the focal point itself.
    return max_points + 1

def get_slope(p1: List[int], p2: List[int]) -> Tuple[int, int]:
    rise = p2[1] - p1[1]
    run = p2[0] - p1[0]
    # Handle vertical lines separately to avoid dividing by 0.
    if run == 0:
        return (1, 0)
    # Simplify the slope to its reduced form.
    gcd_val = gcd(rise, run)
    return (rise // gcd_val, run // gcd_val)
```

While some programming languages, Python included, have their own internal implementation of the GCD function, its implementation is provided below for your information. This implementation is commonly known as the Euclidean algorithm:

```
# The Euclidean algorithm.
def gcd(a, b):
    while b != 0:
        a, b = b, a % b
    return a
```

Complexity Analysis

Time complexity: The time complexity of `maximum_collinear_points` is $O(n^2 \log(m))$, where n denotes the number of points, and m denotes the largest value among the coordinates. Here's why:

- The time complexity of the `gcd(rise, run)` function is $O(\log(min(\texttt{rise}, \texttt{run}))$, which is approximately equal to $O(\log(m))$ in the worst case.
- The helper function `max_points_from_focal_point`, calls the gcd function a total of $n - 1$ times: one for each point excluding the focal point, giving a time complexity of $O(n \log(m))$.
- The `max_points_from_focal_point` function is called a total of n times, resulting in an overall time complexity of $O(n^2 \log(m))$.

Space complexity: The space complexity is $O(n)$ due to the hash map, which in the worst case, stores $n - 1$ key-value pairs: one for each point excluding the focal point.