# Applied Signal Processing & Computer Science

## Chapter 1: Mathematical Basics

Xiaoxiang Zhu

xiao.zhu@dlr.de

Remote Sensing Technology
TU München
&
Remote Sensing Technology Institute
German Aerospace Center (DLR)
Oberpfaffenhofen

# Real Number

Natural Number
1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, … , $+\infty$

Integer Number
$-\infty$, …, -3, -2, -1, 0, 1, 2, 3, … , $+\infty$

Real Number
for example: 0,1, -2.33, $\pi$, -5, 99.99…

# Imaginary Numbers

$$\sqrt{9} = \pm 3 \qquad \sqrt{-9} = ?$$

$$\sqrt{-9} = \sqrt{9} * \sqrt{-1} = \pm 3\sqrt{-1} = \pm 3\, j$$

$$\sqrt{-1} = j$$

# Complex Numbers

$$2 + 3j = ?$$

$$2 + 3j = 2 + 3j$$

# Complex Numbers

Motivation: Polynomials of order *n* should have *n* roots.

$$z^2 - 1 = 0 \quad \Rightarrow \quad (z-1)(z+1) = 0 \quad \Rightarrow \quad z_1 = 1; \quad z_2 = -1$$

$$z^2 = 0 \quad \Rightarrow \quad z_1 = 0; \quad z_2 = 0$$

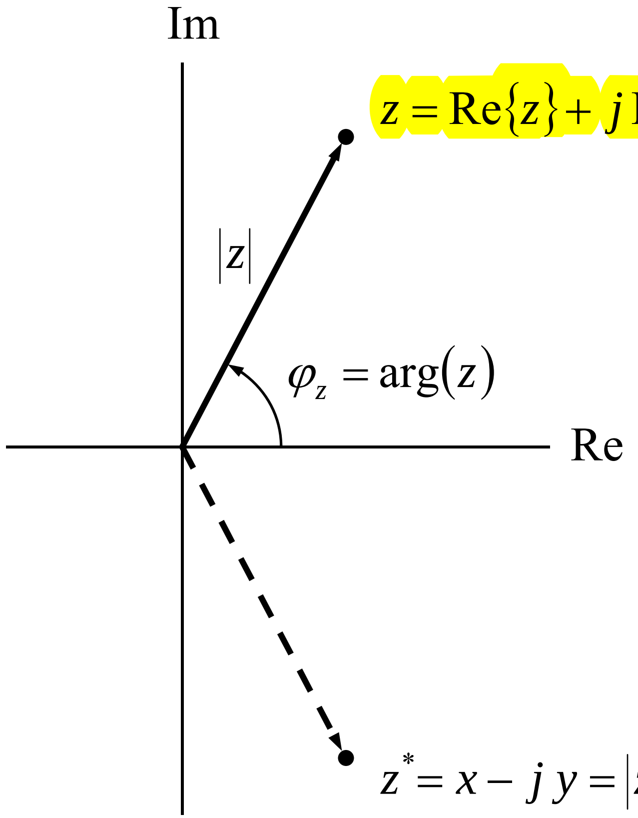
$$z^2 + 1 = 0 \quad \Rightarrow \quad z_1 = \sqrt{-1}; \quad z_2 = -\sqrt{-1}$$

Def.: $j = \sqrt{-1}$ imaginary unit (in mathematics mostly: „ $i$ “)

$z = x + j\,y \qquad x, y \in \Re$ complex number

$x = \mathrm{Re}\{z\}$ real part

$y = \mathrm{Im}\{z\}$ imaginary part




$$z = \mathrm{Re}\{z\} + j\,\mathrm{Im}\{z\} = |z| \exp(j\varphi_z)$$

$$z^* = x - j\,y = |z| \exp(-j\varphi_z)$$

complex conjugate

$$|z| = \sqrt{\mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2}$$

$$\varphi_z = \arg(z) = \arctan\left(\frac{\mathrm{Im}\{z\}}{\mathrm{Re}\{z\}}\right)$$

Check for quadrants!

$$\mathrm{Re}\{z\} = |z| \cos(\varphi_z)$$

$$\mathrm{Im}\{z\} = |z| \sin(\varphi_z)$$

## Euler's Formula

$$\exp(j\varphi) = \cos(\varphi) + j\sin(\varphi) \qquad \Rightarrow \qquad \begin{cases} \cos(\varphi) = \dfrac{1}{2}\left(\exp(j\varphi) + \exp(-j\varphi)\right) \\ \sin(\varphi) = \dfrac{1}{2j}\left(\exp(j\varphi) - \exp(-j\varphi)\right) \end{cases}$$

$$\left|\exp(j\varphi)\right| = 1 \quad \forall \quad \varphi \in \Re$$

Proof of Euler's formula using Taylor series:

$$\exp(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots$$

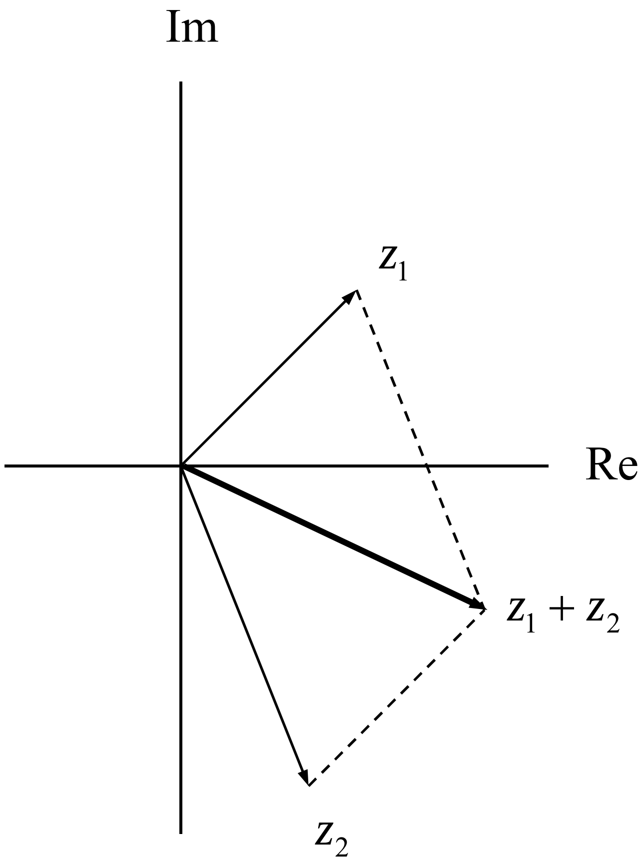
$$\begin{aligned} \exp(j\varphi) &= 1 + \frac{j\varphi}{1!} + \frac{(j\varphi)^2}{2!} + \frac{(j\varphi)^3}{3!} + \frac{(j\varphi)^4}{4!} + \frac{(j\varphi)^5}{5!} + \ldots \\ &= 1 + j\frac{\varphi}{1!} - \frac{\varphi^2}{2!} - j\frac{\varphi^3}{3!} + \frac{\varphi^4}{4!} + j\frac{\varphi^5}{5!} + \ldots \\ &= \underbrace{\left(1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \ldots\right)}_{\cos(\varphi)} + j\underbrace{\left(\frac{\varphi}{1!} - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!}\ldots\right)}_{\sin(\varphi)} \end{aligned}$$

$$j^2 = -1$$
$$j^3 = -j$$
$$j^4 = 1$$

# Sum of Complex Numbers

$$z_1 + z_2 = \mathrm{Re}\{z_1\} + \mathrm{Re}\{z_2\} + j\left(\mathrm{Im}\{z_1\} + \mathrm{Im}\{z_2\}\right)$$

corresponds to vector sum:



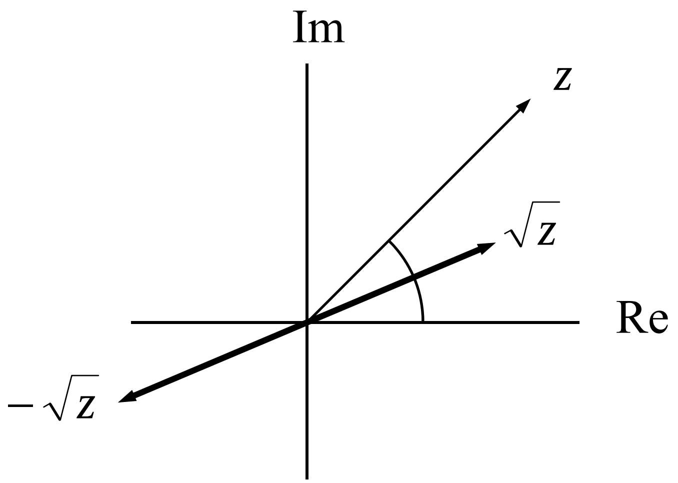
## Product of Complex Numbers

$$z_1 \cdot z_2 = \mathrm{Re}\{z_1\}\mathrm{Re}\{z_2\} - \mathrm{Im}\{z_1\}\mathrm{Im}\{z_2\} + j\left(\mathrm{Re}\{z_1\}\mathrm{Im}\{z_2\} + \mathrm{Re}\{z_2\}\mathrm{Im}\{z_1\}\right)$$

$$= |z_1|\,|z_2| \exp\left(j\left(\varphi_{z_1} + \varphi_{z_2}\right)\right)$$

$$z \cdot z^* = |z|^2 = \mathrm{Re}\{z\}^2 + \mathrm{Im}\{z\}^2$$

## Square Root

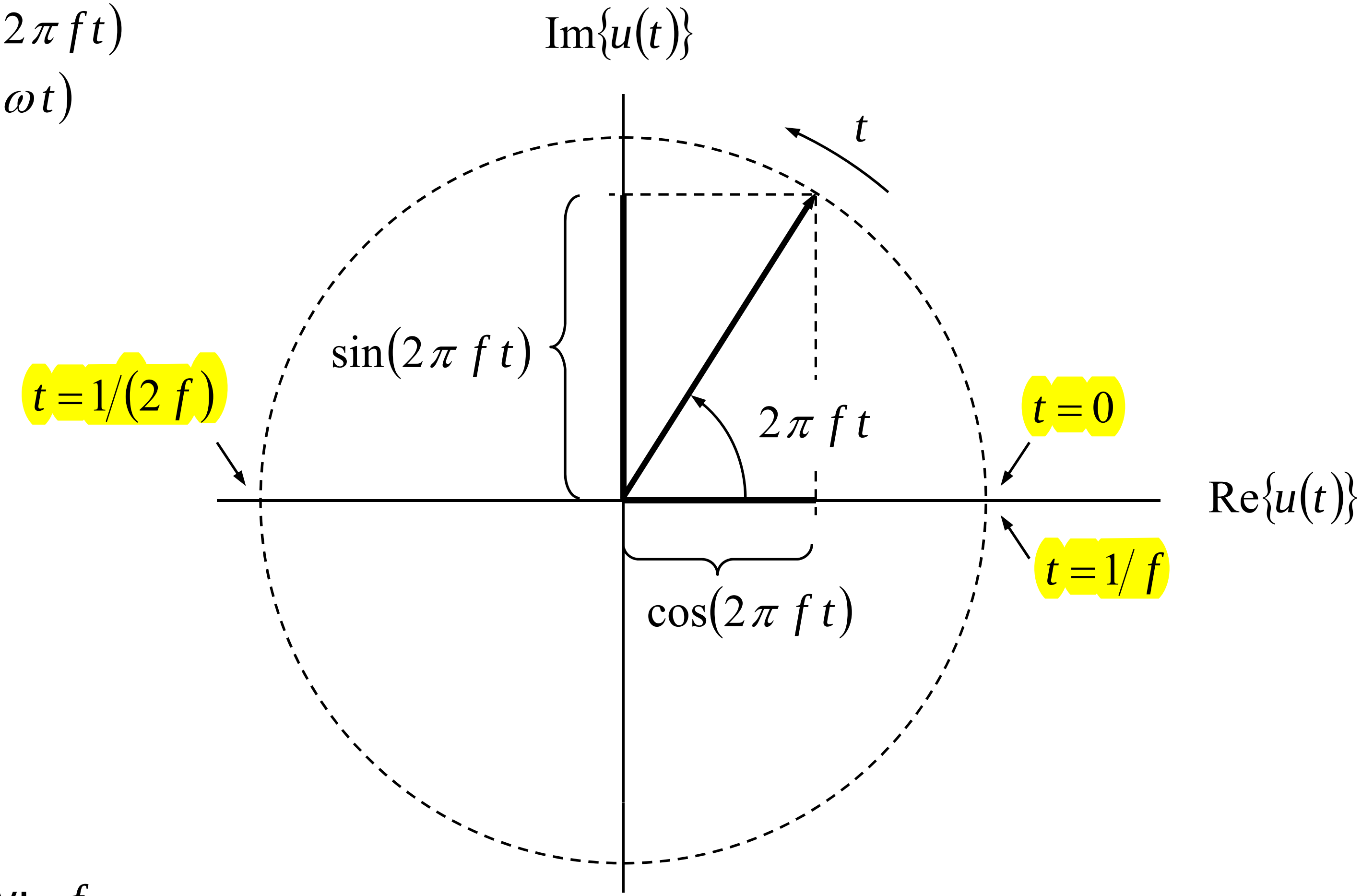



1st solution: $\sqrt{z} = \sqrt{|z|}\exp\left(j\dfrac{\varphi_z}{2}\right)$

2nd solution: $\sqrt{z} = \sqrt{|z|}\exp\left(\dfrac{j\left(\varphi_z + 2\pi\right)}{2}\right) = \sqrt{|z|}\exp\left(j\dfrac{\varphi_z}{2}\right)\underbrace{\exp\left(j\pi\right)}_{-1}$

# Complex Harmonic Oscillation (I)

$$u(t) = \exp(j\,2\pi f t) = \exp(j\,\omega t)$$




frequency: $f$

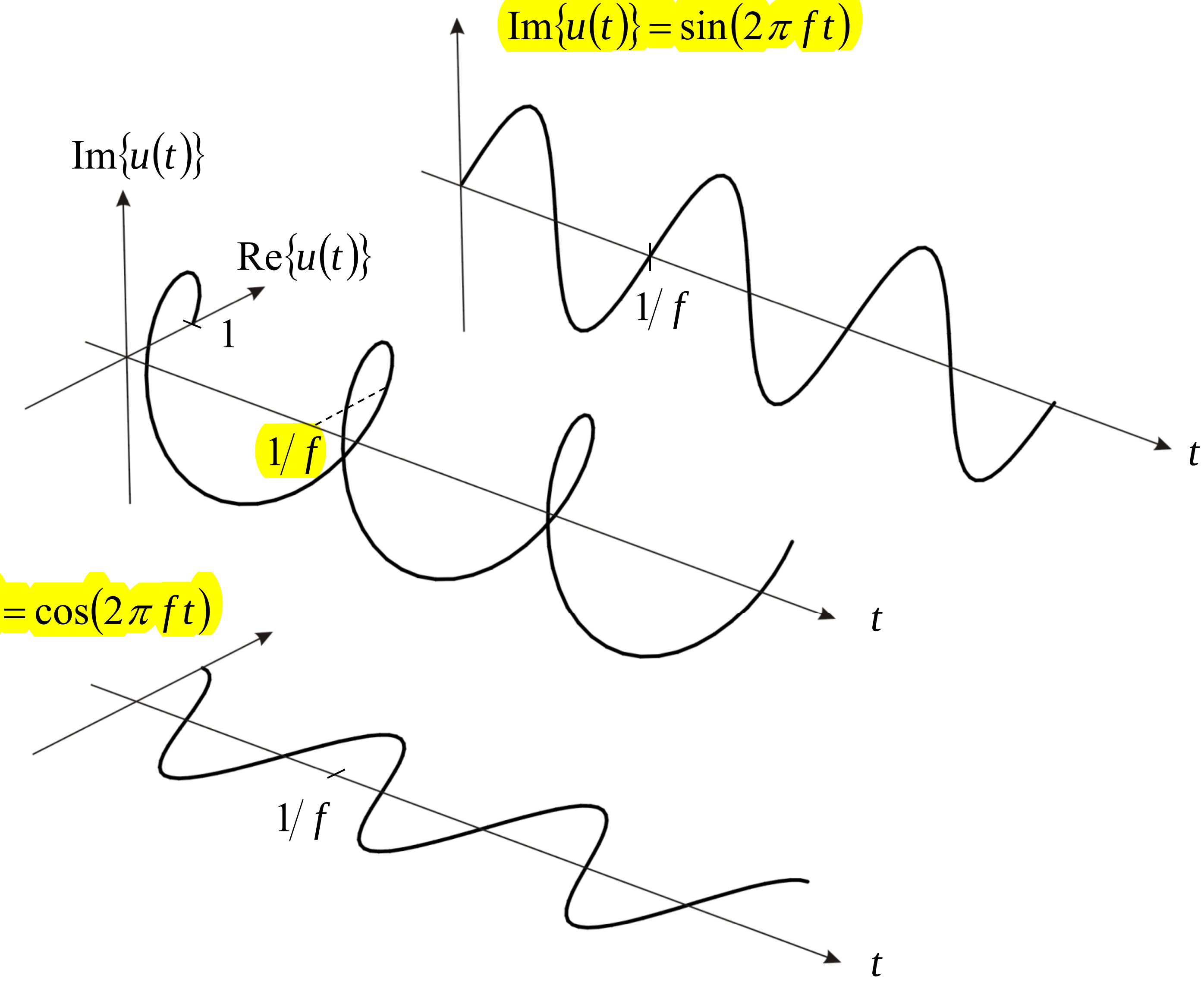
angular frequency: $\omega = 2\pi f$

# Complex Harmonic Oscillation (II)

(shown for $t \geq 0$)

$$u(t) = \exp(j2\pi f t)$$

$$\mathrm{Im}\{u(t)\} = \sin(2\pi f t)$$




$$\mathrm{Re}\{u(t)\} = \cos(2\pi f t)$$

**phase shift = Multiplication with a complex constant:**

$$\exp\left(j\left(2\pi\ f\ t+\varphi_0\right)\right)=\exp\left(j\,\varphi_0\right)\exp\left(j\,2\pi\ f\ t\right)$$

**derivative with respect to time = multiplication with** $j\,2\pi f$ **:**

$$\dot{u}(t)=\frac{d}{dt}\exp\left(j\,2\pi\,f\ t\right)=j\,2\pi\,f\ \exp\left(j2\pi\,ft\right)=j\,2\pi\,f\ u(t)$$

**differential equation of a harmonic oscillation:**

$$k^2\,\ddot{u}(t)+u(t)=0 \quad \xrightarrow{u(t)=\exp\left(j\,2\,\pi\,f\,t\right)} \quad \left(k^2\left(j\,2\pi\ f\right)^2+1\right)u(t)=0$$

$$-k^2\left(2\pi\ f\right)^2+1=0$$

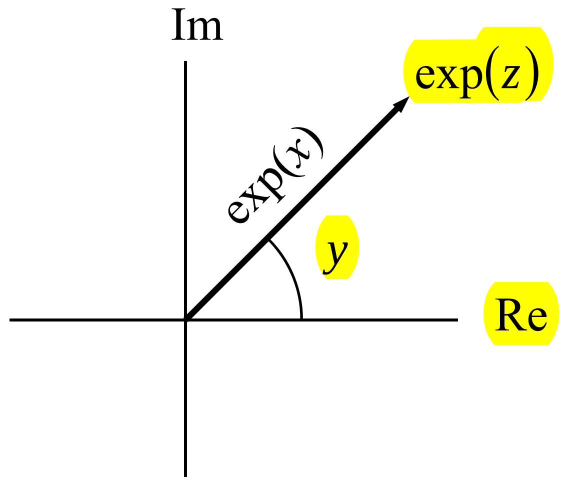
$$f=\pm\frac{1}{2\pi\,k}$$

## Complex Exponential

$$\exp(z) = \exp(x + jy) = \exp(x) \cdot \exp(jy) \qquad x, y \in \Re$$

$$|\exp(z)| = \exp(x)$$

$$\arg(\exp(z)) = y$$




## Complex Logarithm

$$\ln(z) = \ln(|z|) + j \arg(z)$$

Proof:

$$z = \exp(\ln(z)) = \exp(\ln(|z|) + j \arg(z)) = \exp(\ln(|z|)) \cdot \exp(j \arg(z)) = |z| \cdot \exp(j \arg(z)) = z$$

# dB or not dB

- In order to cover large dynamic ranges, logarithmic scales are often used

- For dimensionless quantities or ratios of quantities, i.e. ratio between input and output of a system in terms of
  - signal amplitudes (voltages, acoustic pressure, …): $|u_2| / |u_1|$
  - powers: $|u_2|^2 / |u_1|^2$

- Logarithmic dB scale:

$$10\log_{10}\frac{|u_2|^2}{|u_1|^2}\,dB = 20\log_{10}\frac{|u_2|}{|u_1|}\,dB$$

| power ratio | dB value |
|---:|---:|
| 100 | 20 dB |
| 10 | 10 dB |
| 2 | ≈ 3 dB |
| 1 | 0 dB |
| 1/2 | ≈ - 3 dB |
| 1/10 | - 10 dB |
| 1/100 | - 20 dB |

- *B* = *Bel* (in honor of Alexander Graham Bell)

- *dB* = *Dezibel* (“Dezi” = 1/10)