Applied Signal Processing & Computer Science

Chapter 5: Frequency Domain Analysis

Xiaoxiang Zhu

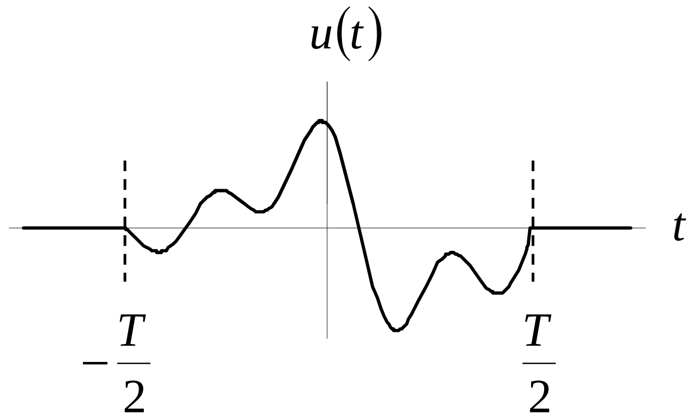
Xiao.zhu@dlr.de

Remote Sensing Technology
TU München
&
Remote Sensing Technology Institute
German Aerospace Center (DLR)
Oberpfaffenhofen

Chapter 5: Frequency Domain Analysis

Fourier Series

- Expansion of energy- and time-limited signals in series of orthogonal harmonic basis functions



- Without loss of generality: $-\frac{T}{2} < t < +\frac{T}{2}$

- Fourier series: $u(t) = \sum_{n=-\infty}^{+\infty} c_n \cdot \Psi_n(t) \quad \forall \quad |t| < \frac{T}{2}$ with $\Psi_n(t) = \exp\left(j 2\pi \frac{n}{T} t\right)$

Fourier coefficient of order *n*

basis function (harmonic oscillation)

frequency $n \cdot f_0$ with $f_0 = 1/T$

Continuous-time Fourier Transform

$$U(f) = \int_{-\infty}^{+\infty} u(t) \exp(-j 2\pi f t)\, dt$$ Fourier Transform

$$u(t) = \int_{-\infty}^{+\infty} U(f) \exp(j 2\pi f t)\, df$$ inverse Fourier Transform

with $u(t)$ absolute integrable: $\int_{-\infty}^{+\infty} |u(t)|\, dt < \infty$ (sufficient condition)

Some power-limited signals can also be Fourier transformed :

$$U(f) = \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} u(t) \exp(-\varepsilon |t|) \exp(-j 2\pi f t)\, dt$$

Common symbols :

$$u(t) \circ\!\!-\!\!\bullet\ U(f)$$
$$u(t) \rightarrow U(f)$$
$$u(t) \leftrightarrow U(f)$$

Discrete-Time and Discrete-Frequency Signals

$u[n]$ time-limited, e.g. for $n = 0,1,2,\ldots,N-1$

$\Rightarrow$ spectrum $U_d(f)$ can be sampled
sampling interval $\leq 1/(NT)$

$\Rightarrow$ $u[n]$ continues periodically with period $\geq N$

Caution: A time-limited signal can *never* be *exactly* frequency band-limited. Therefore, $u[n]$ is only an *approximation* for a physical time-limited signal. In general the approximation gets better for large N.

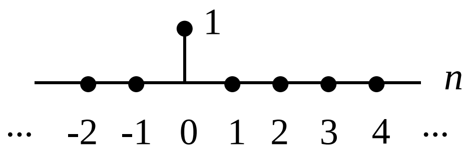
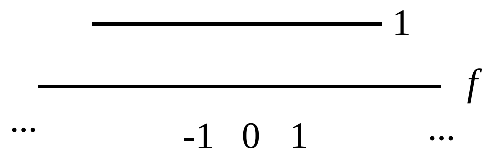
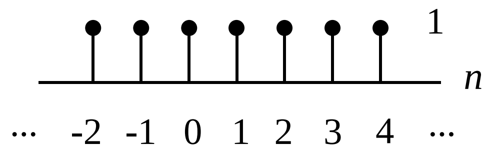
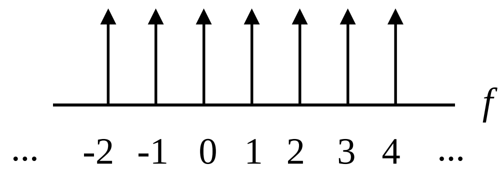
Discrete-Time Fourier Transform (DTFT)

$$U_d(f) = \sum_{n=-\infty}^{+\infty} u[n] \exp(-j 2\pi f nT)$$

$$u[n] = T \int_{-1/(2T)}^{+1/(2T)} U_d(f) \exp(j 2\pi f nT) df$$

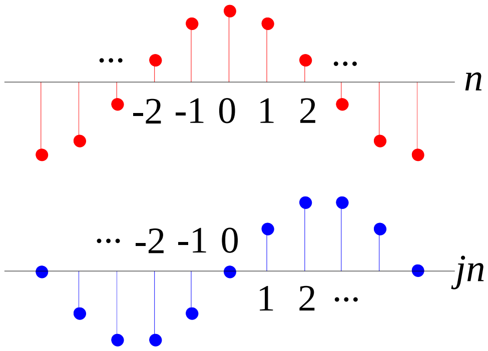
often used: normalized angular frequency: $\Omega = 2\pi fT \quad \Rightarrow \quad -\pi < \Omega \leq \pi$

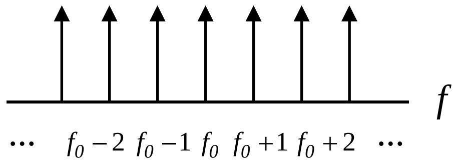
Commonly Used DTFT Pairs

Sequence	Discrete-Time Fourier Transform
$\delta[n]$	1
	
$1, \left(-\infty < n < \infty\right)$	$\sum_{k=-\infty}^{+\infty} \delta\left(f + k\right)$
	

Commonly Used DTFT Pairs

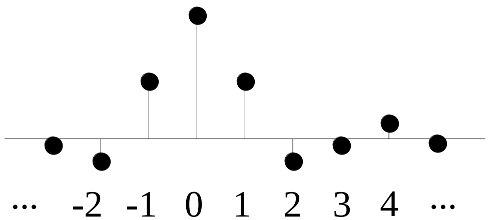
Sequence	Discrete-Time Fourier Transform
$e^{j2\pi f_0 n}$	$\sum_{k=-\infty}^{\infty} \delta(f - f_0 + k)$

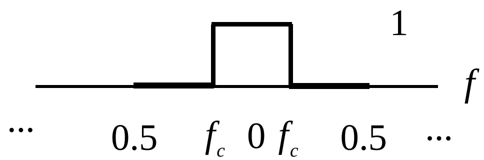




$$h_{LP}[n] = \frac{\sin 2\pi f_c n}{\pi n}, (-\infty < n < \infty)$$

$$H_{LP}\left(e^{j2\pi f}\right) = \begin{cases} 1, & 0 \le |f| \le f_c \\ 0, & f_c < |f| < \frac{1}{2} \end{cases}$$





Discrete-time Fourier Transform Theorems

$$u[n] \quad \leftrightarrow \quad U_d(f)$$

$$h[n] \quad \leftrightarrow \quad H_d(f)$$

Linearity ($\alpha, \beta \in \mathbb{C}$): $$\alpha\, u[n] + \beta\, h[n] \quad \leftrightarrow \quad \alpha\, U_d(f) + \beta\, H_d(f)$$

Time-reversal: $$u[-n] \quad \leftrightarrow \quad U_d^*(f)$$

Shift: $$u[n-n_0] \quad \leftrightarrow \quad U_d(f)\exp(-j\,2\pi\, n_0 T f)$$

$$u[n]\exp(j\,2\pi\, f_0\, nT) \quad \leftrightarrow \quad U_d(f-f_0)$$

Convolution: $u[n]*h[n] \quad \leftrightarrow \quad U_d(f)\,H_d(f)$

Modulation: $u[n]\,h[n] \quad \leftrightarrow \quad \frac{1}{T}\int\limits_{-1/(2T)}^{1/(2T)} U_d(f')*H_d(f-f')\,df'$

Correlation: $u[n]\otimes h[n] \quad \leftrightarrow \quad U_d(f)\,H_d^*(f)$

Derivatives: $-j\,2\pi\,nT\,u[n] \quad \leftrightarrow \quad \frac{d}{df}U_d(f)$

Parseval‘s theorem: $\sum\limits_{n=-\infty}^{+\infty} u[n]h^*[n] = \frac{1}{T}\int\limits_{-1/(2T)}^{1/(2T)} U_d(f)H_d^*(f)\,df$

energy preservation: $\sum\limits_{n=-\infty}^{+\infty} \left|u[n]\right|^2 = \frac{1}{T}\int\limits_{-1/(2T)}^{1/(2T)} \left|U_d(f)\right|^2 df$

symmetry:

index: e: even part
o: odd part

$$\mathrm{Re}\{u_e[n]\} \leftrightarrow \mathrm{Re}\{U_{d,e}(f)\}$$
$$\mathrm{Re}\{u_o[n]\} \leftrightarrow j\,\mathrm{Im}\{U_{d,o}(f)\}$$
$$j\,\mathrm{Im}\{u_e[n]\} \leftrightarrow j\,\mathrm{Im}\{U_{d,e}(f)\}$$
$$j\,\mathrm{Im}\{u_o[n]\} \leftrightarrow \mathrm{Re}\{U_{d,o}(f)\}$$

real-valued signals:

$$u[n] \in \Re \quad \leftrightarrow \quad U_d(f) = \mathrm{Re}\{U_d(f)\} + j\,\mathrm{Im}\{U_d(f)\} = |U_d(f)| \exp\left(j\phi_{U_d}(f)\right)$$

$$U_d(-f) = U_d^*(f)$$

$\mathrm{Re}\{U_d(f)\}$: even $\qquad |U_d(f)|$: even

$\mathrm{Im}\{U_d(f)\}$: odd $\qquad \phi_{U_d}(f)$: odd

Discrete Fourier Transform (DFT)

$$U[k] = \sum_{n=0}^{N-1} u[n] \exp\left(-j\frac{2\pi}{N} n k\right)$$

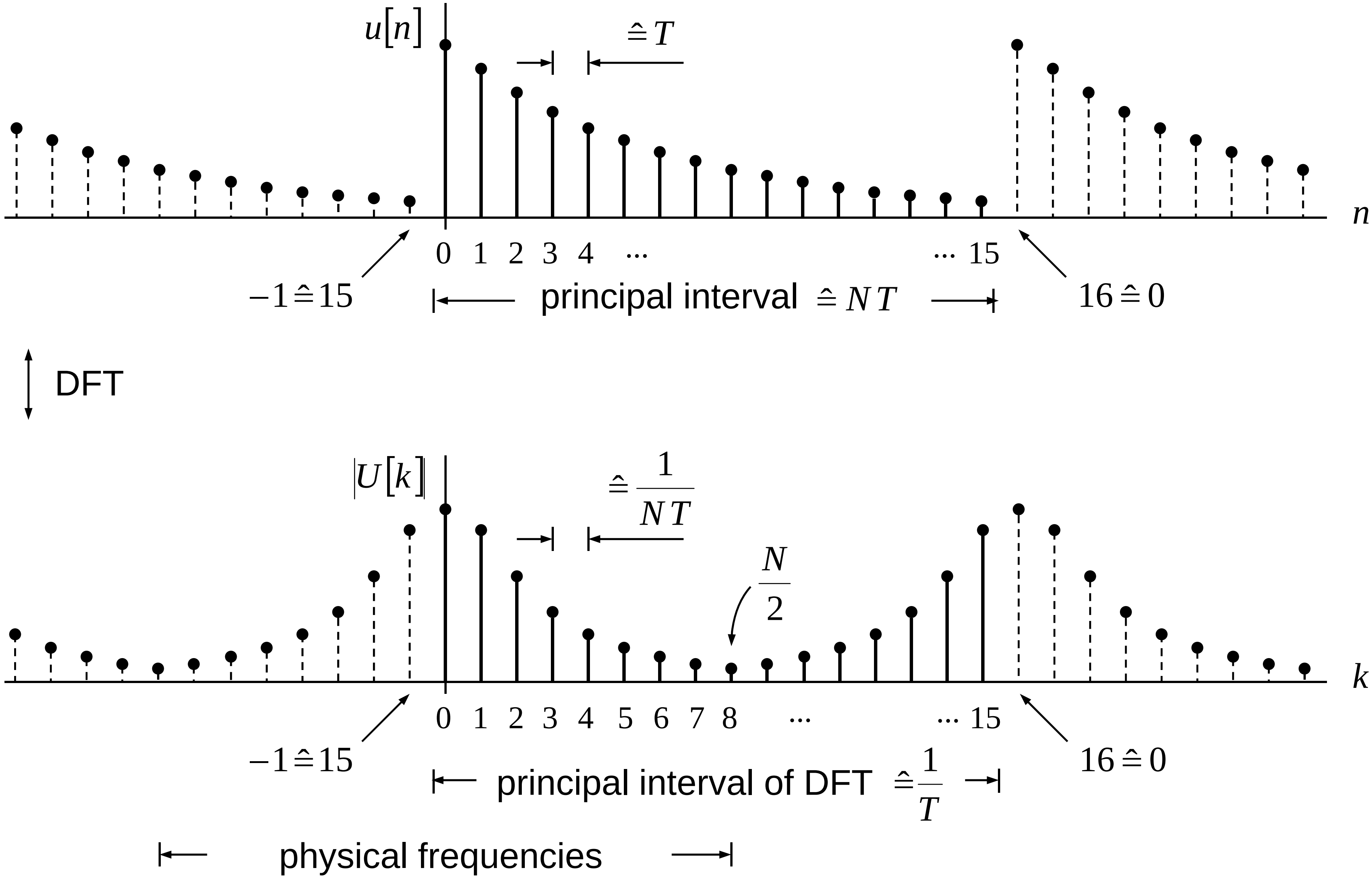
$$u[n] = \frac{1}{N} \sum_{k=0}^{N-1} U[k] \exp\left(j\frac{2\pi}{N} n k\right)$$

$\Rightarrow$ Both signal and spectrum consist of a finite number of samples ("Finite Signals")

Parseval's Theorem:

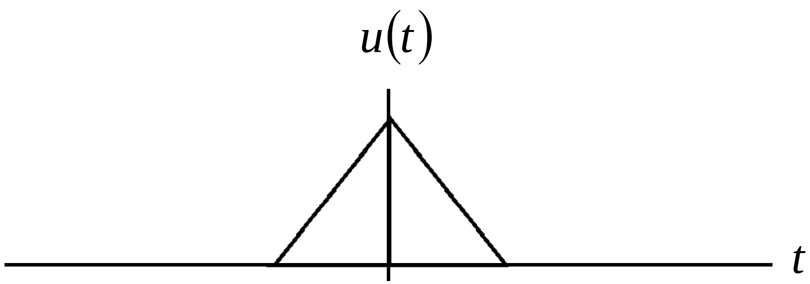
$$\sum_{n=0}^{N-1} \left|u[n]\right|^2 = \frac{1}{N} \sum_{k=0}^{N-1} \left|U[k]\right|^2$$

Example DFT with N = 16

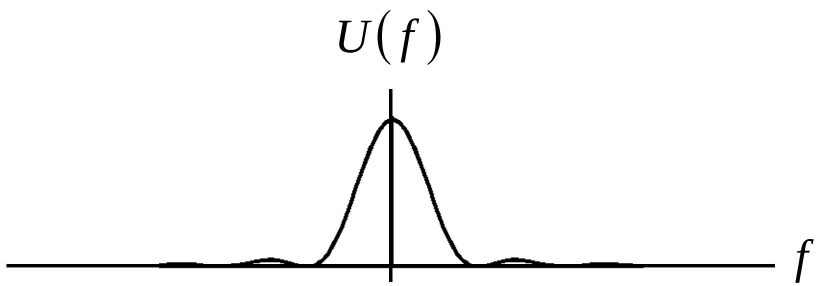


Summary:
discrete-time, discrete-frequency and finite signals

continuous signal (energy-limited):

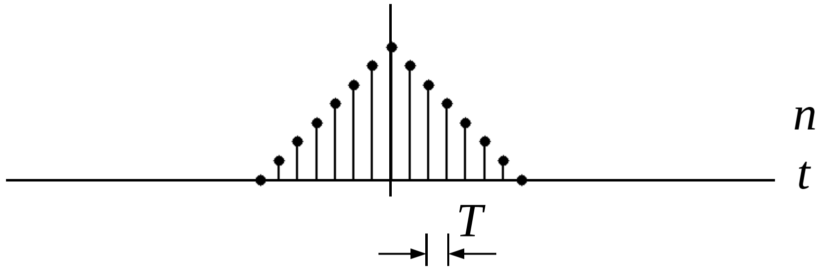


FT
$\leftrightarrow$



discrete-time signal

$u[n] \hat{=} u_d(t)$

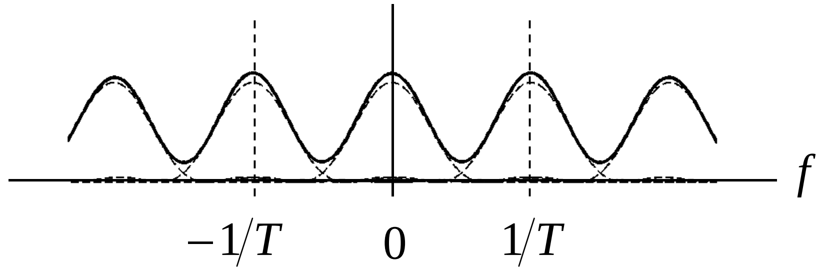


DTFT

$\leftrightarrow$

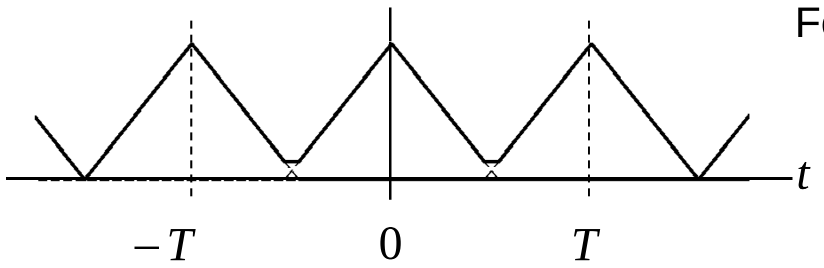
periodic spectrum

$U_d(f)$



periodic signal

$u_p(t)$

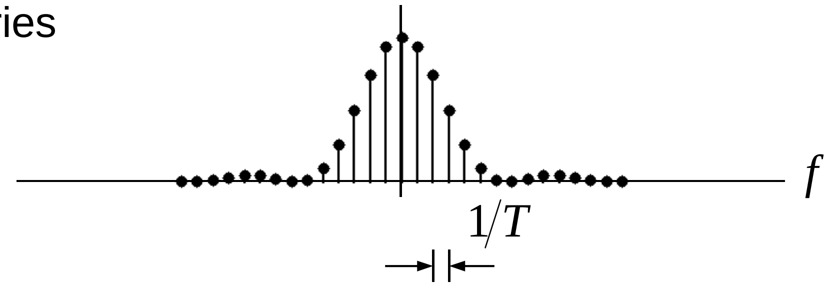


Fourier Series

$\leftrightarrow$

discrete-frequency spectrum

$U_p(f)$



finite signal

finite spectrum