Applied Signal Processing and Computer Science

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Solution 3: Fourier-Series

1. Fourier Series

1.1

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \Psi_n(t)\Psi_m^*(t)dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j2\pi\frac{n}{T}t} \cdot e^{-j2\pi\frac{m}{T}t} dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j2\pi\frac{(n-m)}{T}t} dt$$

$if \quad n = m$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j2\pi\frac{(n-m)}{T}t} dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} dt = T$$

$if \; n \neq m$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} e^{j2\pi\frac{(n-m)}{T}t} dt = \frac{1}{j2\pi\frac{(n-m)}{T}} e^{j2\pi\frac{(n-m)}{T}t} \Bigg|_{-\frac{T}{2}}^{\frac{T}{2}} = T\sin((n-m)\pi) = 0$$

=> $(\Psi_n(t), \Psi_m(t)) = T \cdot \delta_{n,m}$

=> Fouries-Series base functions $\Psi_n(t)$ are orthogonal!

1.2

$$u(t) = \sum_{n=-\infty}^{\infty} c_n \Psi_n(t)$$

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} |u(t)|^2 dt = \int_{-\frac{T}{2}}^{\frac{T}{2}} \left| \sum_{n=-\infty}^{\infty} c_n \Psi_n(t) \right|^2 dt = \sum_{n=-\infty}^{\infty} |c_n|^2 \int_{-\frac{T}{2}}^{\frac{T}{2}} |\Psi_n(t)|^2 dt$$

where, $$\int_{-\frac{T}{2}}^{\frac{T}{2}} |\Psi_n(t)|^2 dt = T$$

=> $$\int_{-\frac{T}{2}}^{\frac{T}{2}} |u(t)|^2 dt = T \sum_{n=-\infty}^{\infty} |c_n|^2$$

1.3

$$u(t) = \sum_{m=-\infty}^{\infty} c_m \Psi_m(t)$$

$$\frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)\cdot\Psi_n^*(t)dt = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} (\sum_{m=-\infty}^{\infty} c_m \Psi_m(t))\cdot\Psi_n^*(t)dt = \frac{1}{T}\sum_{m=-\infty}^{\infty} c_m \int_{-\frac{T}{2}}^{\frac{T}{2}} \Psi_m(t)\cdot\Psi_n^*(t)dt$$

where, $\int_{-\frac{T}{2}}^{\frac{T}{2}} \Psi_m(t)\cdot\Psi_n^*(t)dt = T\cdot\delta_{n,m}$

$$=> \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)\cdot\Psi_n^*(t)dt = \frac{1}{T}\sum_{m=-\infty}^{\infty} c_m\cdot T\delta_{n,m} = \sum_{m=-\infty}^{\infty} c_m\delta_{n,m} = c_n$$

1.4

$$u(t) = rect(t/\Delta t) = \begin{cases} 1 & |t| < \Delta t/2 \\ \frac{1}{2} & |t| = \Delta t/2 \qquad \Delta t < T \\ 0 & |t| > \Delta t/2 \end{cases}$$

$$u(t) = \sum_{n=-\infty}^{\infty} c_n \exp(j2\pi\frac{n}{T}t) = a_0 + \sum_{n=1}^{\infty} a_n\cos(2\pi\frac{n}{T}t) + \sum_{n=1}^{\infty} b_n\sin(2\pi\frac{n}{T}t)$$

Compute c_n :

$$c_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)\cdot\Psi_n^*(t)dt = \frac{1}{T}\int_{-\frac{\Delta t}{2}}^{\frac{\Delta t}{2}} 1\cdot e^{-j2\pi\frac{n}{T}t}dt = \begin{cases} \frac{\Delta t}{T} & n = 0 \\ \frac{1}{n\pi}\sin(\frac{n\pi}{T}\Delta t) & n \neq 0 \end{cases}$$

Or alternatively, compute a_0, a_n and b_n :

$$a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)dt = \frac{1}{T}\int_{-\frac{\Delta t}{2}}^{\frac{\Delta t}{2}} 1\cdot dt = \frac{\Delta t}{T}$$

$$a_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)\cos(2\pi\frac{n}{T}t)dt = \frac{2}{T}\int_{-\frac{\Delta t}{2}}^{\frac{\Delta t}{2}} 1\cdot\cos(2\pi\frac{n}{T}t)dt = \frac{2}{n\pi}\sin(\frac{n\pi}{T}\Delta t)$$

$$b_n = \frac{2}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(t)\sin(2\pi\frac{n}{T}t)dt = 0$$

1.5

Insert $\Delta t = T/2$ into the result of 2.4:

=>

$$a_0 = \frac{\Delta t}{T} = \frac{1}{2} \;;\; a_n = \frac{2}{n\pi}\sin(\frac{n\pi}{T}\Delta t) = \frac{2}{n\pi}\sin(\frac{n\pi}{2}) \;;\; b_n = 0$$

=>

$$u(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{n\pi}\sin(\frac{n\pi}{2})\cos(2\pi\frac{n}{T}t)$$

$f = 100Hz$ => $T = 0.01s$

Primary oscillation $(n=1)$: $\frac{2}{\pi}\cos(\frac{2\pi}{T}t) = \frac{2}{\pi}\cos(200\pi t)$

second oscillation $(n=2)$: 0