Applied Signal Processing & Computer Science

Chapter 3: Signals

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Chapter 3: Signals

3.1 Signal Classes

- energy(-limited) vs. power(-limited)

Energy(-limited) Signals:

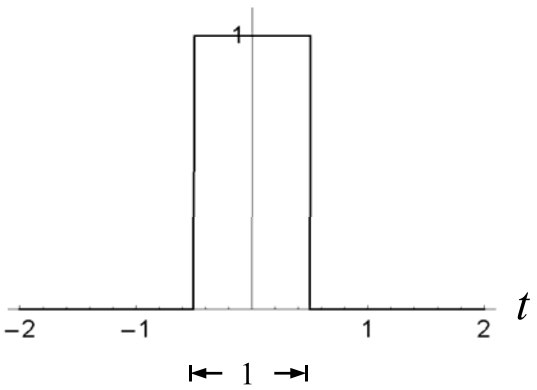
$$E = \int_{-\infty}^{+\infty} |u(t)|^2 \, dt < \infty$$

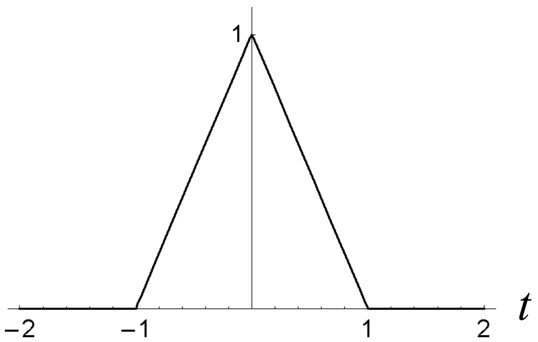
Examples of important energy-limited signals:

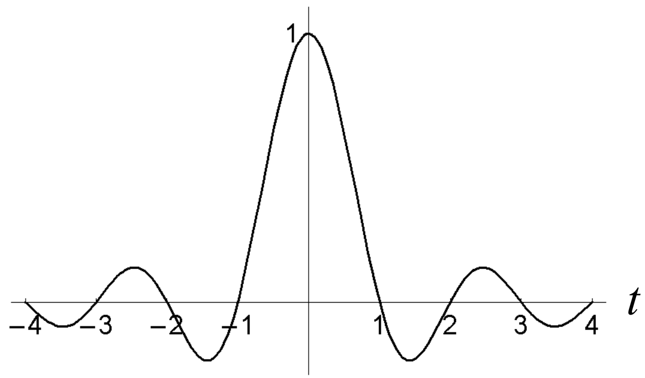
$$\mathrm{rect}(t) = \begin{cases} 1 & |t| < 1/2 \\ 1/2 & |t| = 1/2 \\ 0 & |t| > 1/2 \end{cases}$$

$$\mathrm{tri}(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & |t| \geq 1 \end{cases}$$

$$\mathrm{sinc}(t) = \mathrm{si}(\pi t) = \frac{\sin(\pi t)}{\pi t}$$







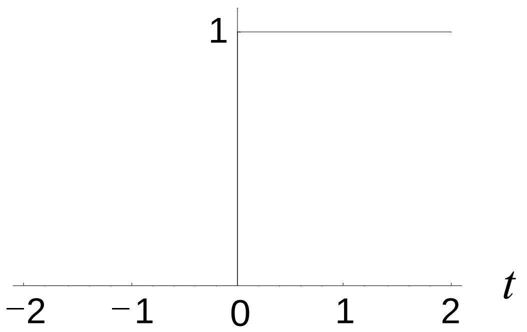
Power(-limited) Signals:

$$\overline{P} = \left\langle \left| u(t) \right|^2 \right\rangle = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{+T} \left| u(t) \right|^2 dt \quad < \infty$$

Examples:

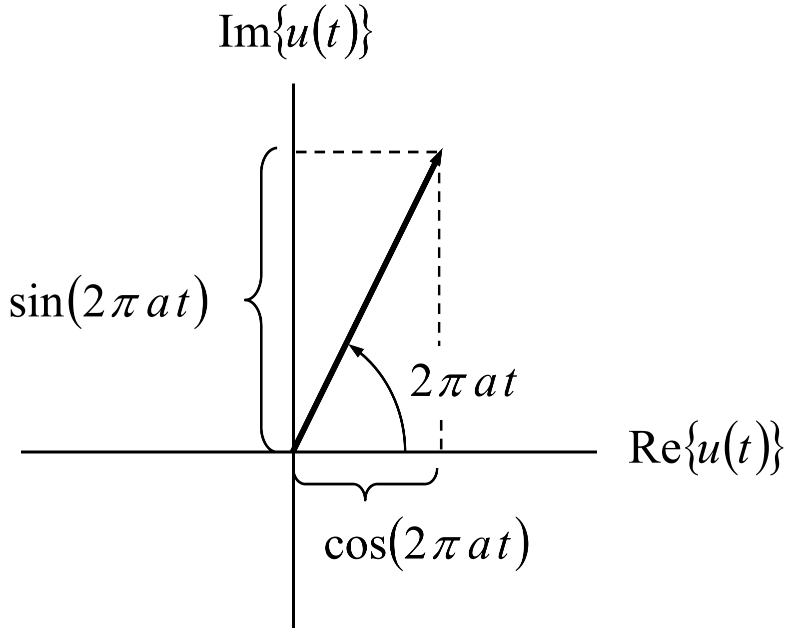
$$\gamma(t) = \begin{cases} 0 & t < 0 \\ 1/2 & t = 0 \\ 1 & t > 0 \end{cases}$$

step function



$$u(t) = \exp\left(j\, 2\pi\, a\, t \right)$$

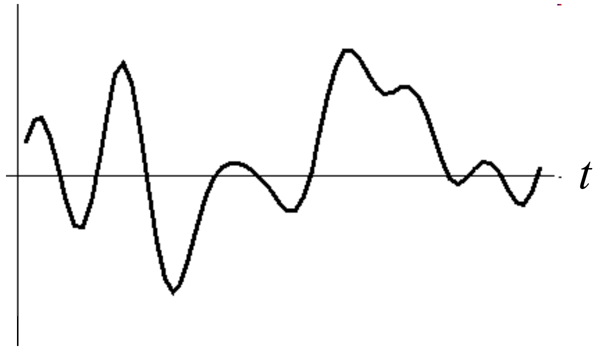
harmonic oscillation ($a \in \Re$)



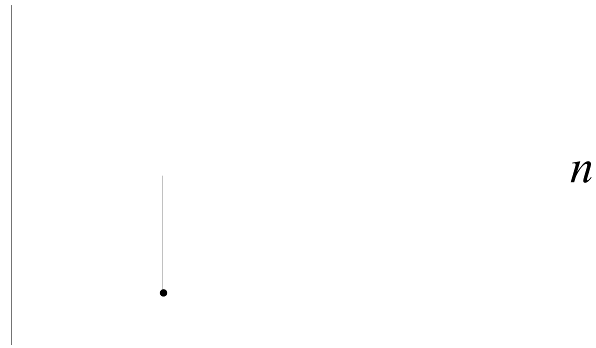
Signal Classes

- energy(-limited) vs. power(-limited)
- continuous-time vs. discrete-time

Continuous-time Signals: $u(t)$ mit $t \in \Re$



Discrete-time Signals: $u[n]$ mit $n \in Z$



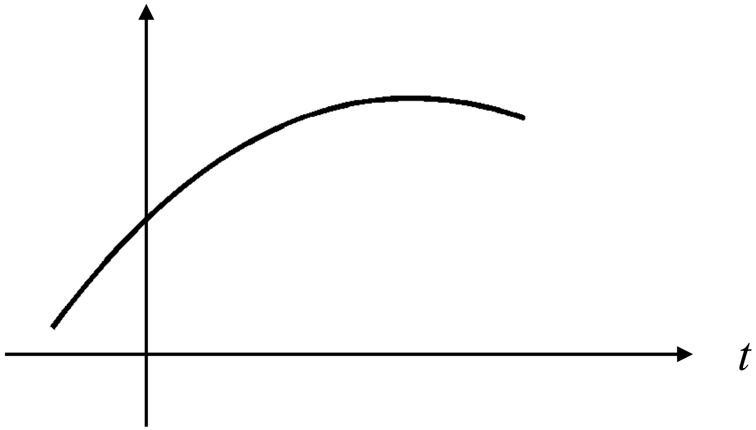
sequence representation: $u[n] \mathrel{\hat{=}} \{\ldots, u[-1], u[0], u[1], u[2], \ldots\}$

vector representation: $u[n] \mathrel{\hat{=}} (\ldots, u[-1], u[0], u[1], u[2], \ldots)^T = \begin{pmatrix} \ldots \\ u[-1] \\ u[0] \\ u[1] \\ u[2] \\ \ldots \end{pmatrix}$

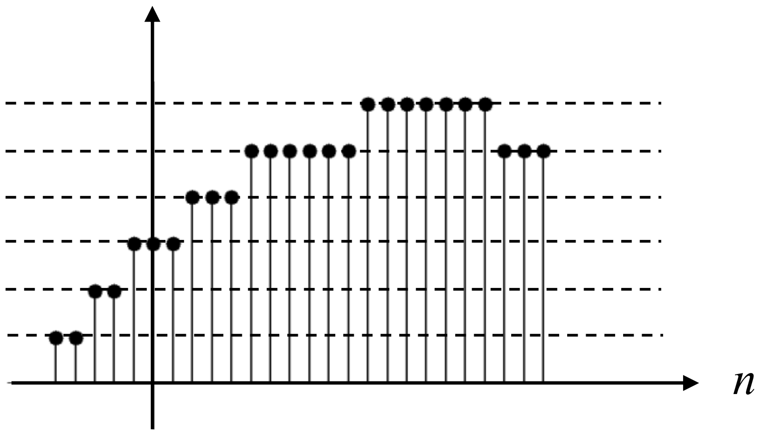
Signal Classes

- energy(-limited) vs. power(-limited)
- continuous-time vs. discrete-time
- analog and digital (discretizing amplitude of the signal)

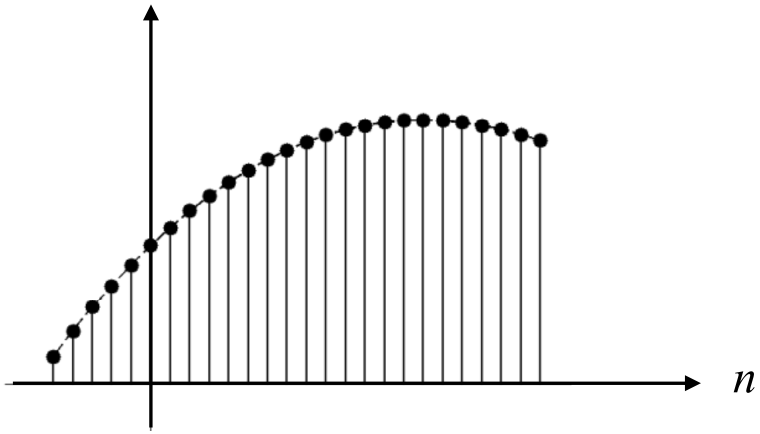
CT/DT vs. A/D



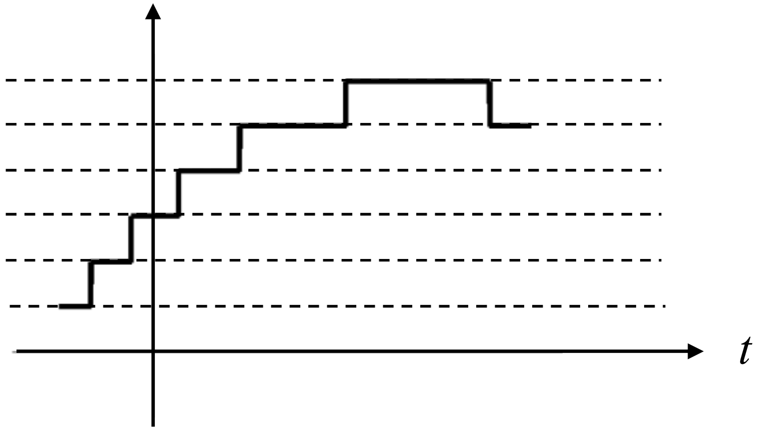
Analog & continuous-time
(analog signal)



Digital & discrete-time
(digital signal)



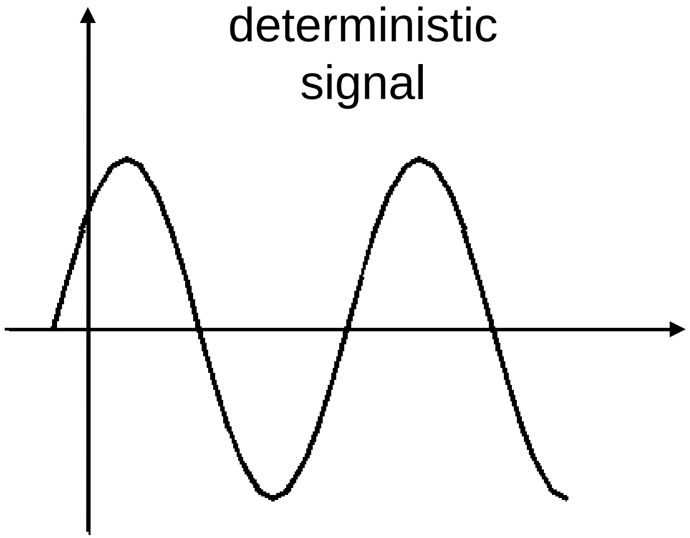
Analog & discrete-time
(sampled-data signal)

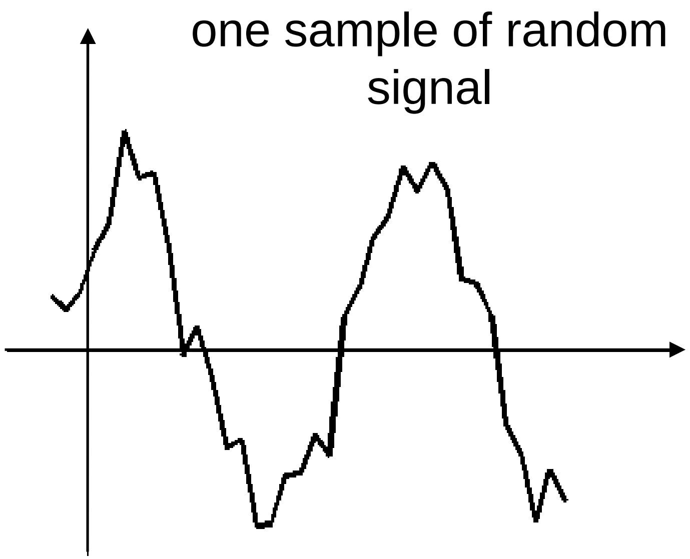


Digital & continuous-time
(quantized boxcar signal)

Signal Classes

- energy(-limited) vs. power(-limited)
- continuous-time vs. discrete-time
- analog and digital
- deterministic vs. non-deterministic: random signals vs. irregular signals





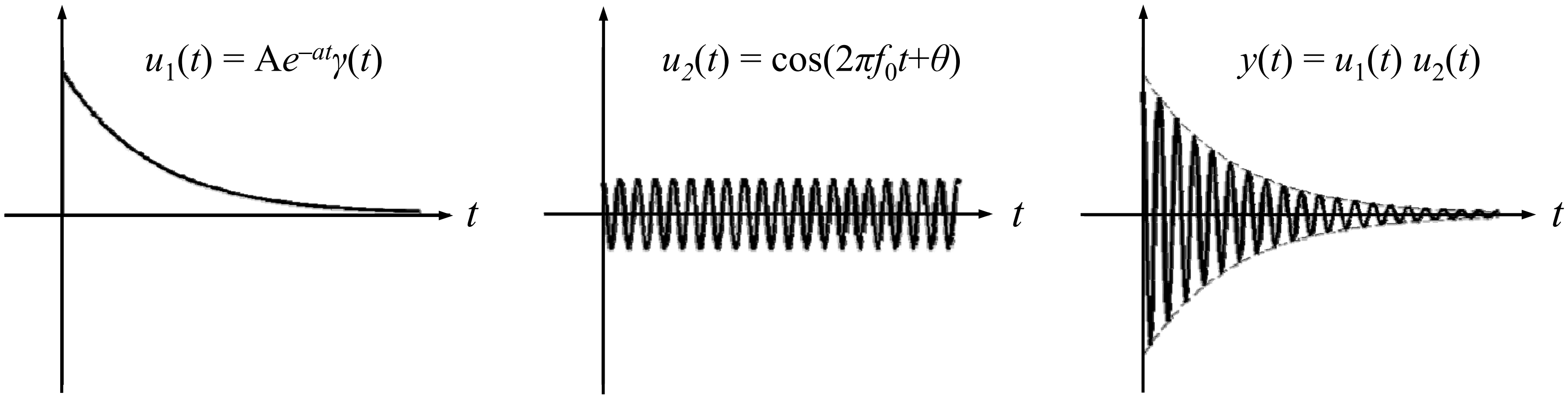
Signal Classes

- energy(-limited) vs. power(-limited)
- continuous-time vs. discrete-time
- analog and digital
- deterministic vs. non-deterministic
- <u>periodic</u> vs. non-periodic: $u(t) = u(t + mT),\ m \in Z\ ,\quad T > 0$
- <u>odd</u> vs. even: $u(t) = -u(-t)$

3.2 Some Useful Signal Operations

- Scalar multiplication: $y(t) = c\,u(t);\ \ y[n] = c\,u[n];\ \ c = \text{const}$
- Summation: $y(t) = u_1(t) + u_2(t);\ \ y[n] = u_1[n] + u_2[n]$
- Multiplication: $y(t) = u_1(t)\,u_2(t);\ \ y[n] = u_1[n]\,u_2[n]$

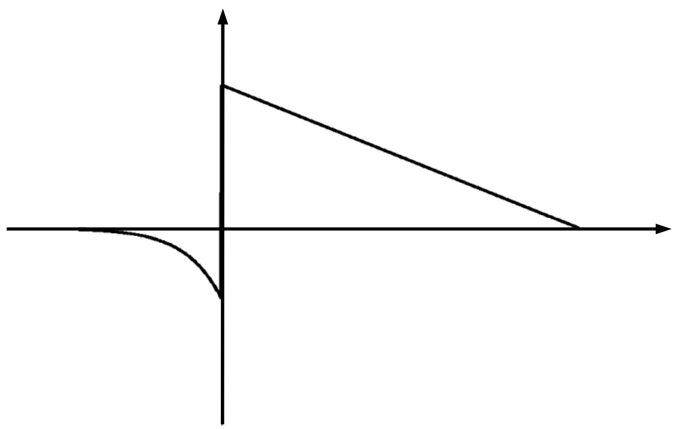
<u>Example:</u> amplitude-modulated sinusoid



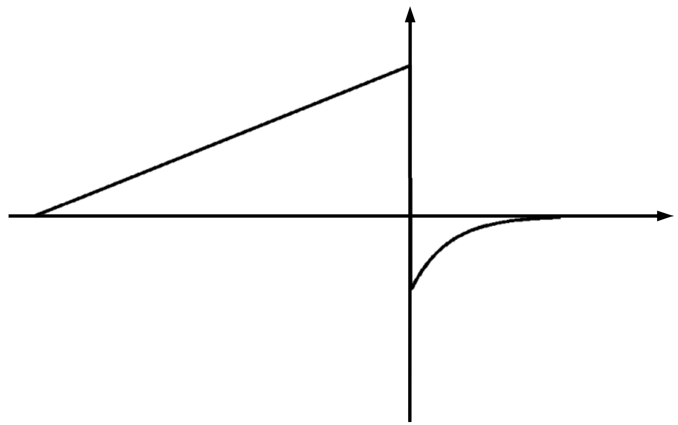
Some Useful Signal Operations

- **Reflection**: $u(t) \xrightarrow{reflection} y(t) = u(-t)$

$u(t)$



$y(t)$



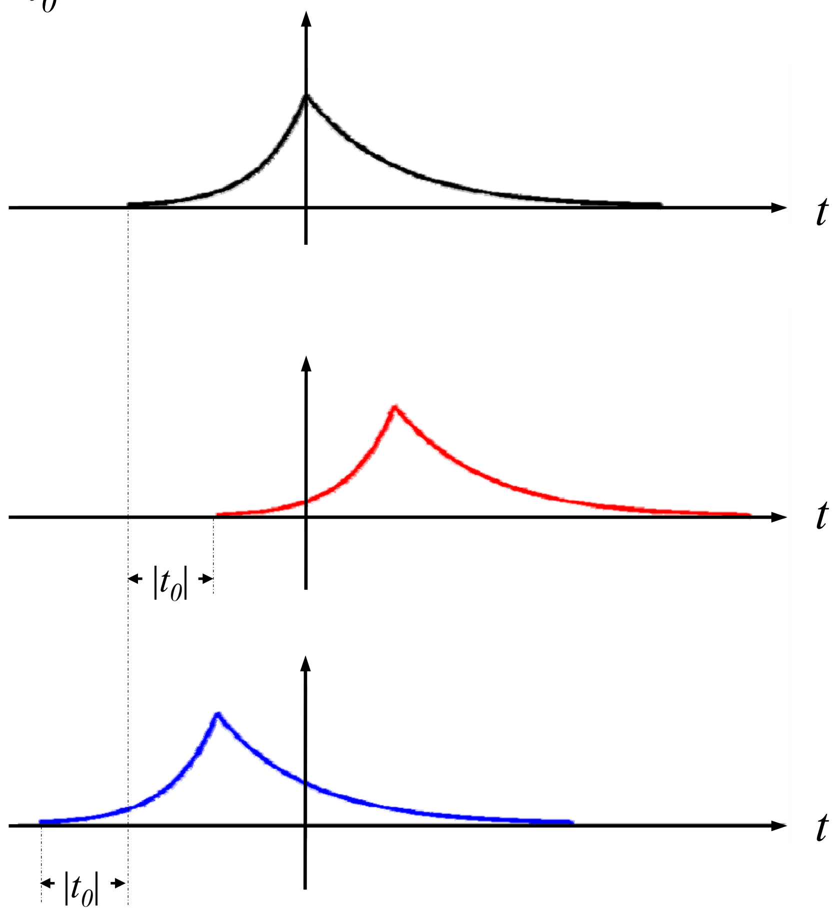
I.e. $y(t)$ is the mirror image of $u(t)$ about the vertical axis

Some Useful Signal Operations

- **Shifting**: $u(t) \xrightarrow{shifting} y(t) = u(t + t_0)$

  I.e. to shift a signal by t_0 = replace t by $t + t_0$

1. If $t_0 = 0$, $y(t) = u(t)$ =>
2. If $t_0 < 0$, shifted to the right by $|t_0|$ =>
3. If $t_0 > 0$, shifted to the left by $|t_0|$ =>

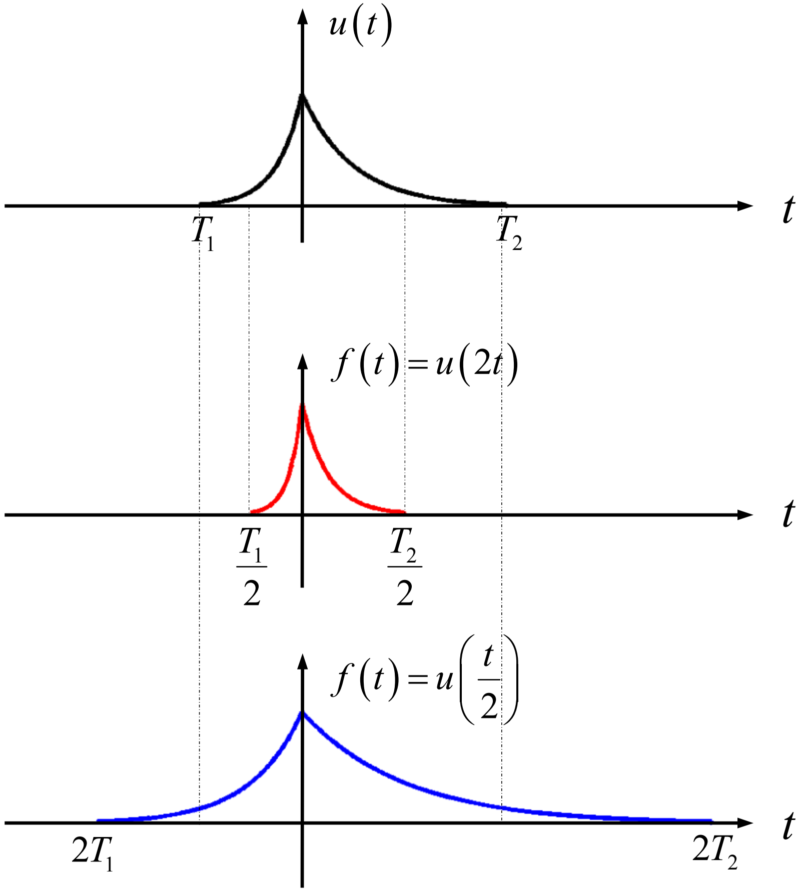


Some Useful Signal Operations

- **Scaling**: $u(t) \xrightarrow{scaling} y(t) = u(k\,t)$

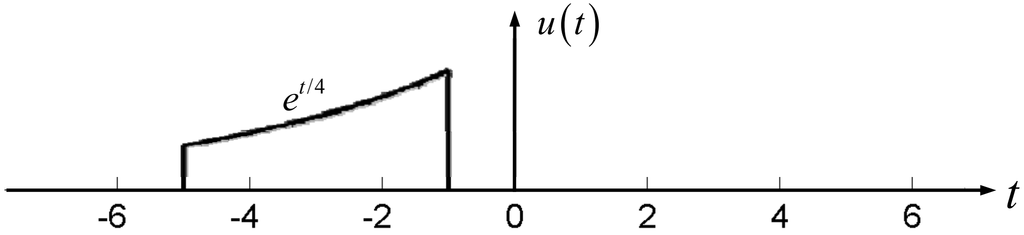
  I.e. compression and expansion of a signal in time

1. If $k = 1$, $y(t) = u(t)$ =>
2. If $k > 1$, compressed in t by a factor of k =>
3. If $k < 1$, expanded in t by a factor of k =>



Some Useful Signal Operations

- Exercise: Given $u(t)$, please sketch $u(-2t-3)$!

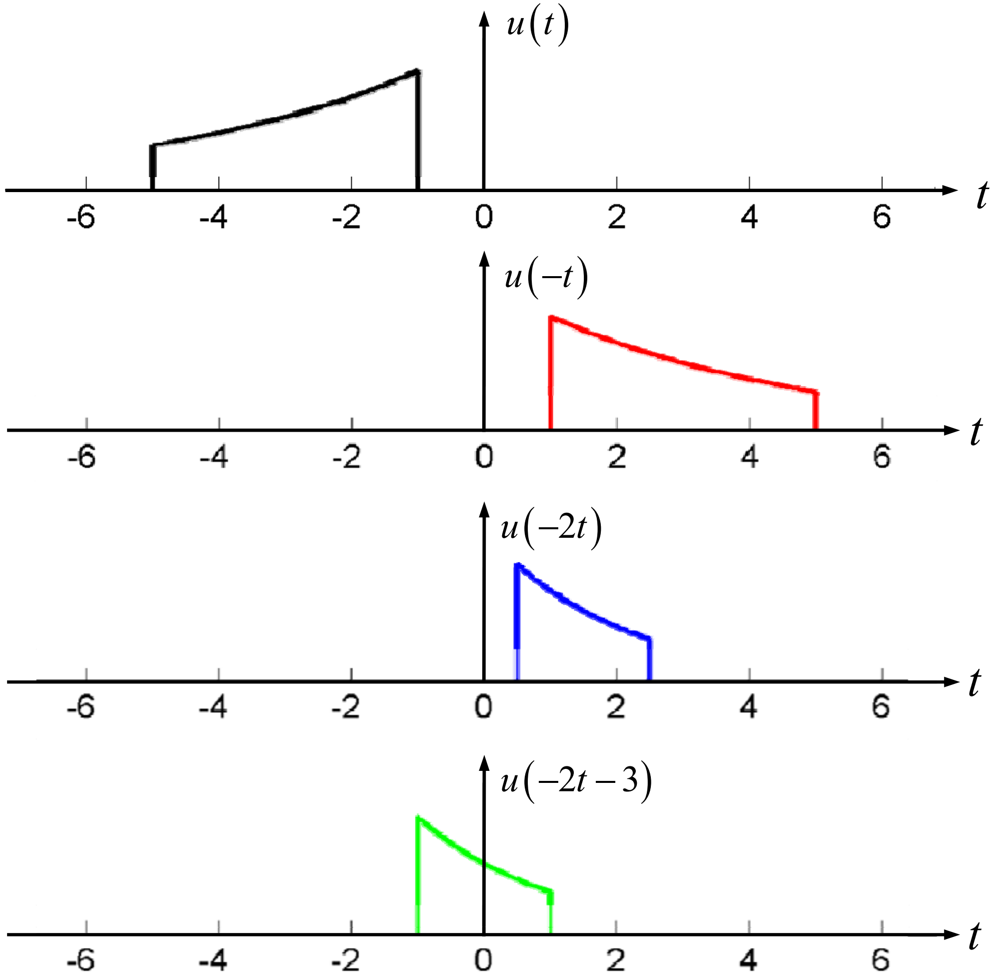


Some Useful Signal Operations

- <u>Exercise</u>: Given $u(t)$, please sketch $u(-2t-3)$!

- <u>Solution</u>: $u(-2t-3) = u\left(-2\left(t+\frac{3}{2}\right)\right)$

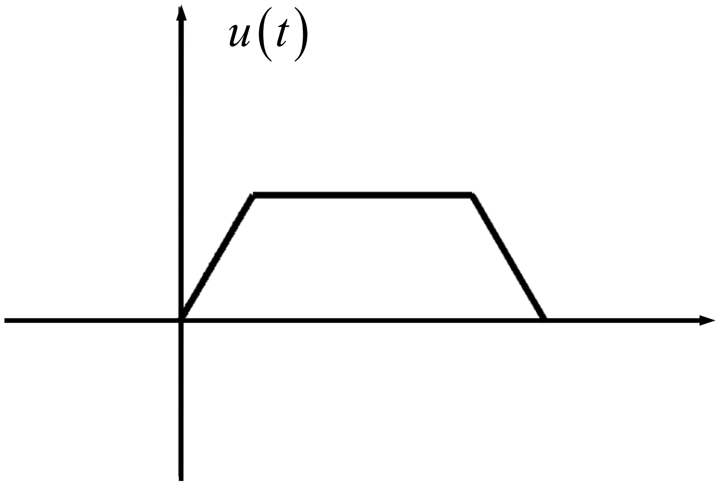
I.e.
  - Step 1: reflection
  - Step 2: compress by a factor of 2
  - Step 3: shift in time by 3/2

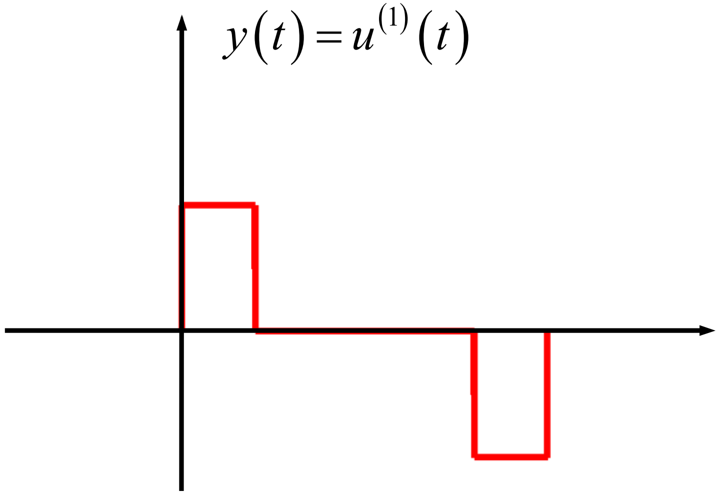


Some Useful Signal Operations

- **Differential:** $u(t) \xrightarrow{differential} y(t) = u^{(1)}(t) = \frac{d}{dt}u(t)$

  All about finding **rates of change** of one quantity compared to another.



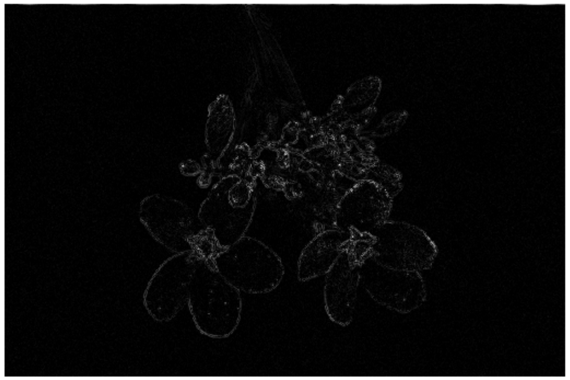


it emphasizes the change of a signal.

Some Useful Signal Operations

<u>Example</u>: Image sharpening by differentiation

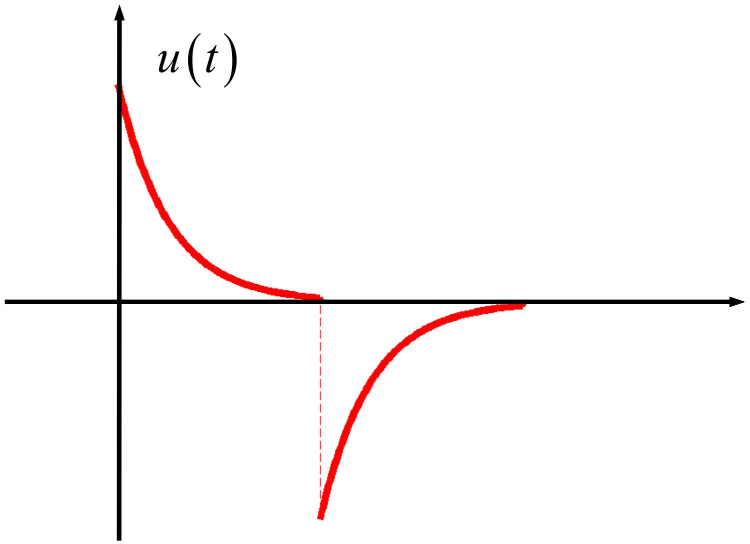
RGB image

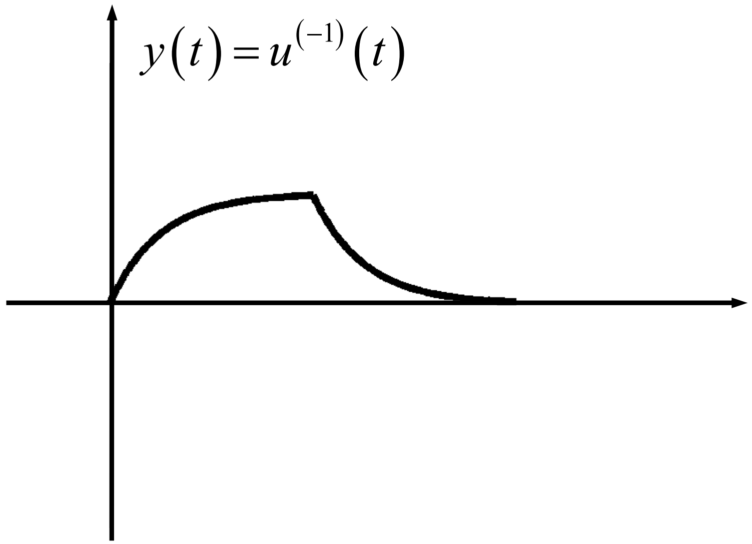


Some Useful Signal Operations

- **Integral:** $u(t) \xrightarrow{integral} y(t) = u^{(-1)}(t) = \int_{-\infty}^{t} u(\tau)\,d\tau$

  By differentiation we find the derivative of the given function, whereas by integration we **find the function whose derivative is known**.





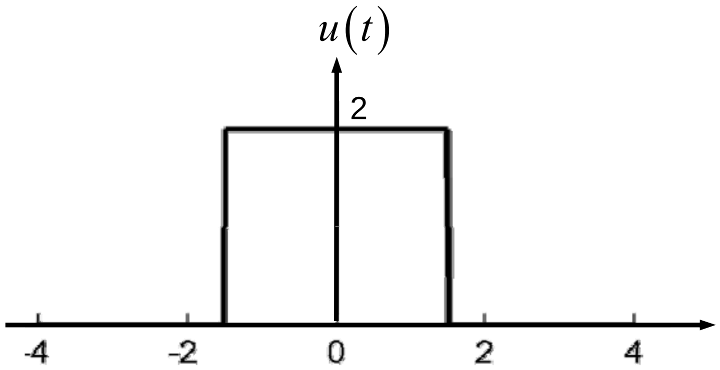
Integral table: http://www.integral-table.com/

Some Useful Signal Operations

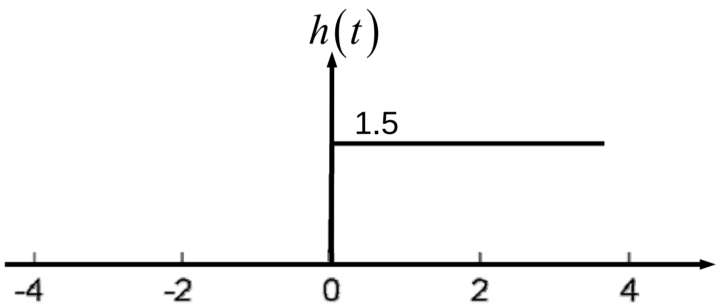
- <u>Exercise</u>: Combined Operations

  Given $u(t)$, $h(t)$ and $\tau = -1$, solve $\int_{-\infty}^{\infty} u(t)h(\tau - t)\,dt$ graphically.

  - $u(t) = 2\ \mathrm{rect}(\frac{t}{3})$



  - $h(t) = 1.5\,\gamma(t)$

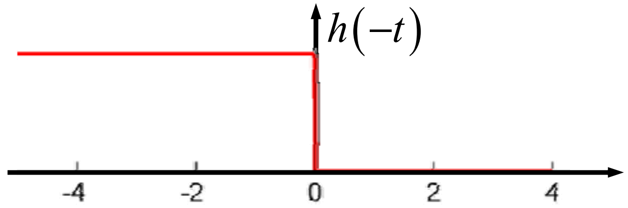


Some Useful Signal Operations

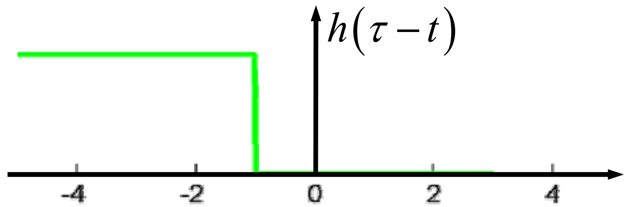
- Given $u(t)$, $h(t)$ and $\tau = -1$, solve $\int_{-\infty}^{\infty} u(t)h(\tau - t)\,dt$ graphically.

- Solution:

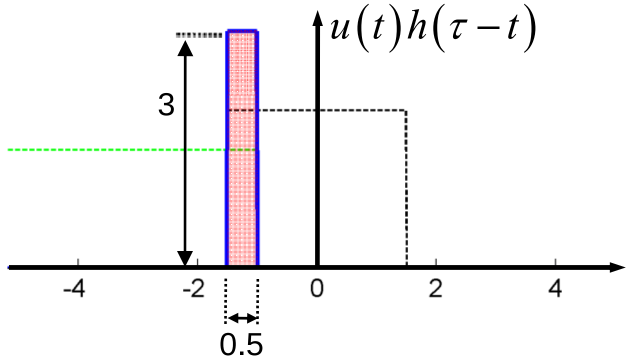
  - Reflection of $h(t)$



  - Shifting to the left by 1



  - Multiplication with $u(t)$



  - Integral

$$\boxed{\int_{-\infty}^{\infty} u(t)h(\tau - t)\,dt = 0.5 \times 3 = 1.5}$$

Convolution

= linear time-invariant operation

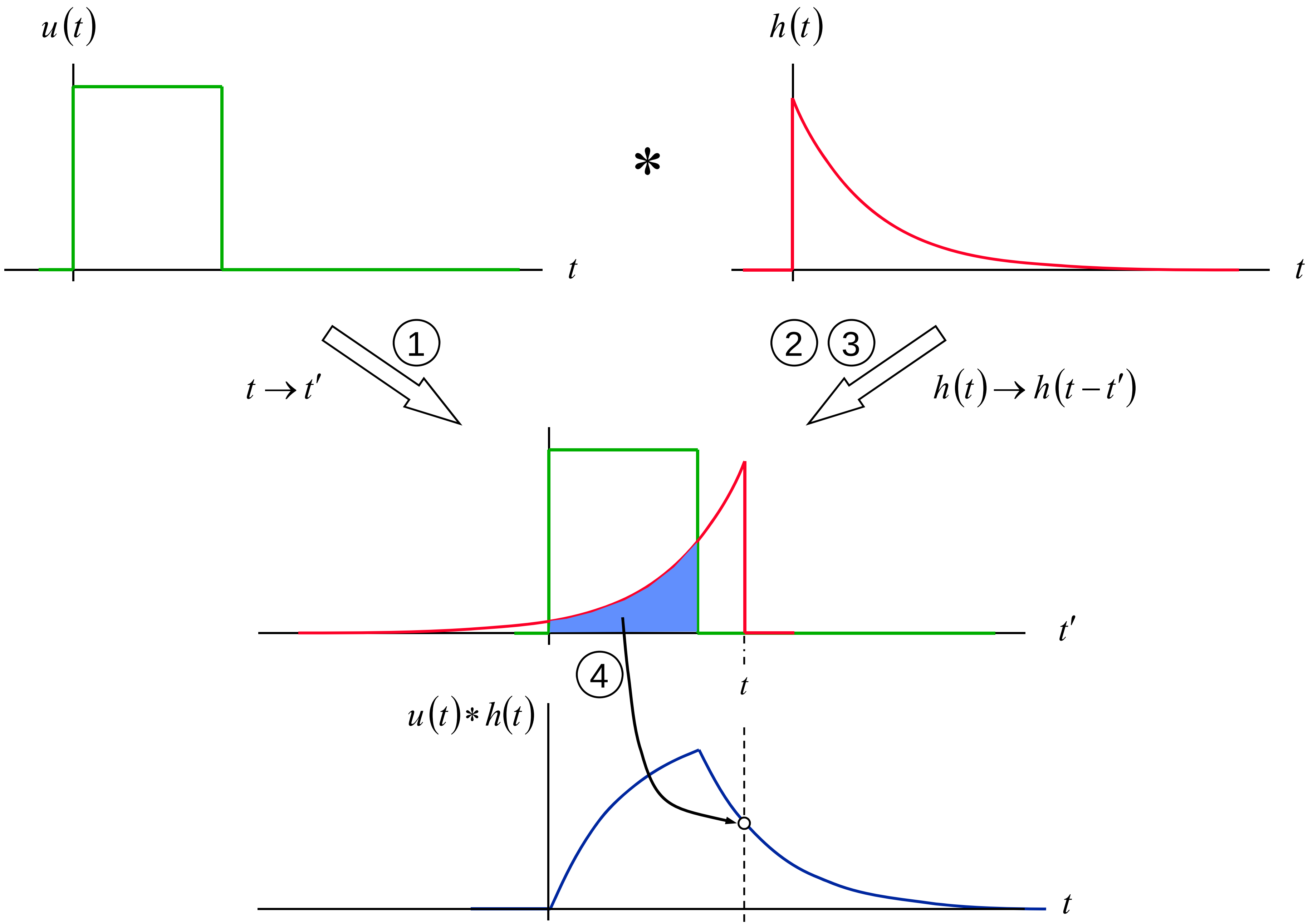
Definition:

$$u(t) * h(t) = \int_{-\infty}^{+\infty} u(t')\, h(t-t')\, dt'$$

$$= \int_{-\infty}^{+\infty} u(t-t')\, h(t')\, dt' = h(t) * u(t)$$

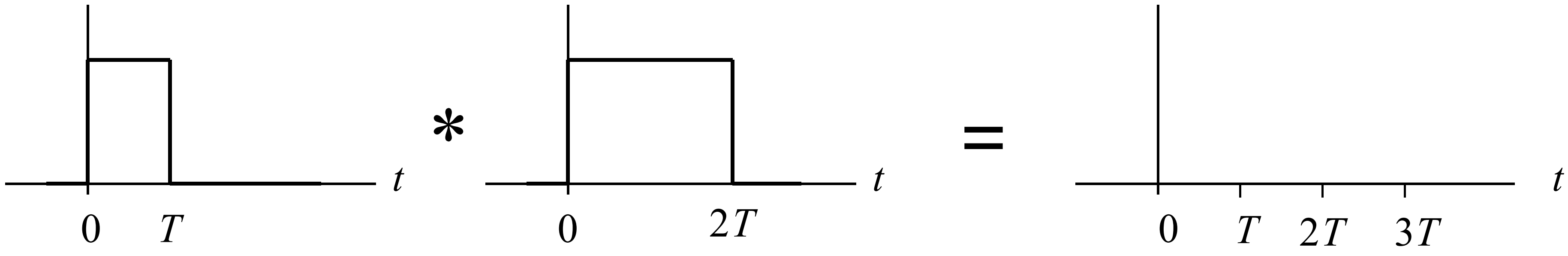
Interpretation 1: $h(t)$ as Integration Kernel

1. $u(t) \rightarrow u(t')$
2. $h(t) \rightarrow h(-t')$ Reflection at ordinate
3. $h(-t') \rightarrow h(t-t')$ Shift to position t
4. $\int_{-\infty}^{+\infty} u(t')h(t-t')dt'$ Multiplication and integration of the product



$u(t)$
t
$*$
$h(t)$
t
1
$t \to t'$
2
3
$h(t) \to h(t-t')$
t'
4
t
$u(t) * h(t)$
t

Tutorial: Convolution



graphical Convolution on-line ("Joy of Convolution") and many other toys:

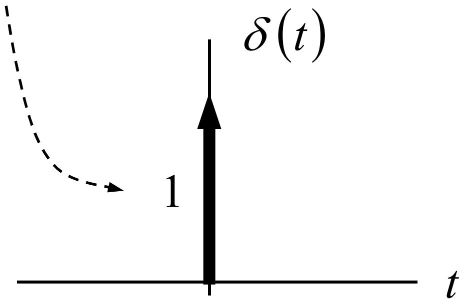
www.jhu.edu/~signals/index.html

3.3 Delta “Functions”: unit impulse function

Dirac‘s Delta Impulse $\delta(t)$:

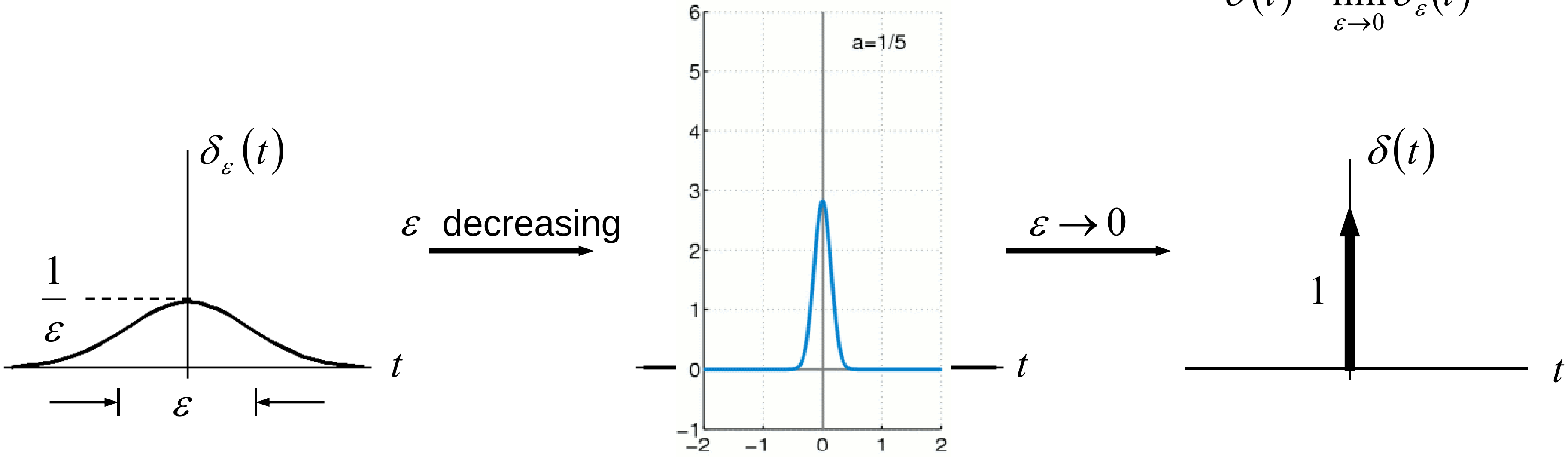
$$\int_{-\infty}^{+\infty} \delta(t-t_0)\, u(t)\, dt = u(t_0) \qquad u(t) \text{ continuous at } t = t_0$$

$$\Rightarrow \quad \int_{-\infty}^{+\infty} \delta(t)\, dt = 1 \qquad \text{and} \qquad \delta(t) = 0 \quad \forall \quad t \neq 0$$

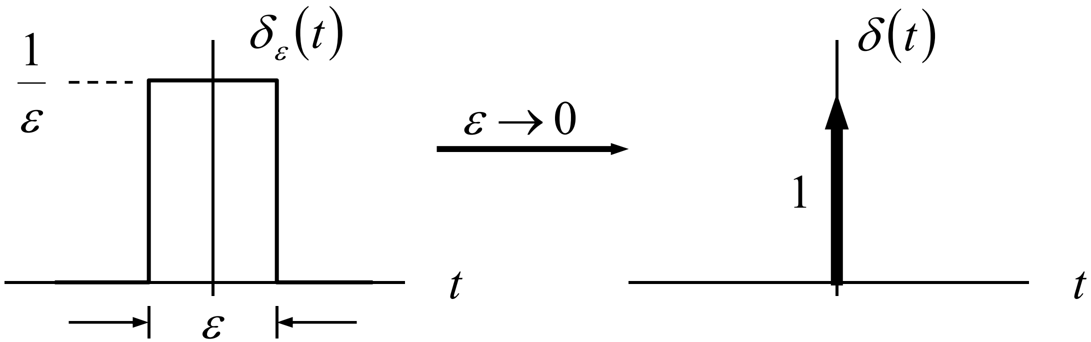


$\delta(t)$ as a limit of a sequence of functions $\delta_\varepsilon(t)$:

$$\delta(t) = \lim_{\varepsilon \to 0} \delta_\varepsilon(t)$$



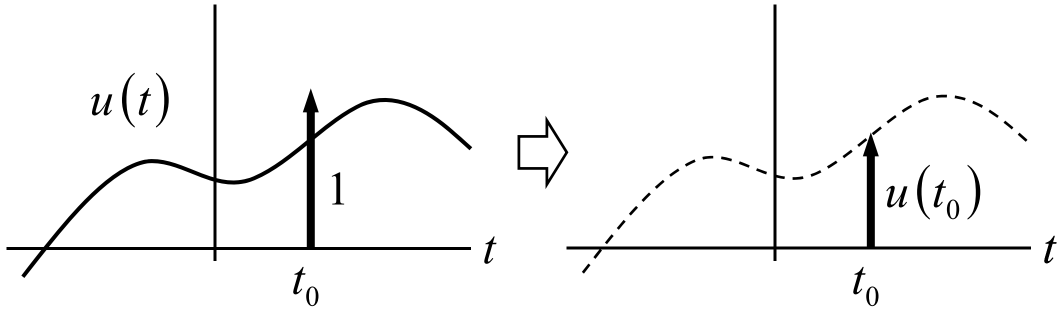
Simplified graphical representation: $\delta_\varepsilon(t) = \frac{1}{\varepsilon} rect\left(\frac{t}{\varepsilon}\right)$



Properties of Delta Impulses (1)

- Sifting Property:

$$\delta(t-t_0)\,u(t)=\delta(t-t_0)\,u(t_0)$$

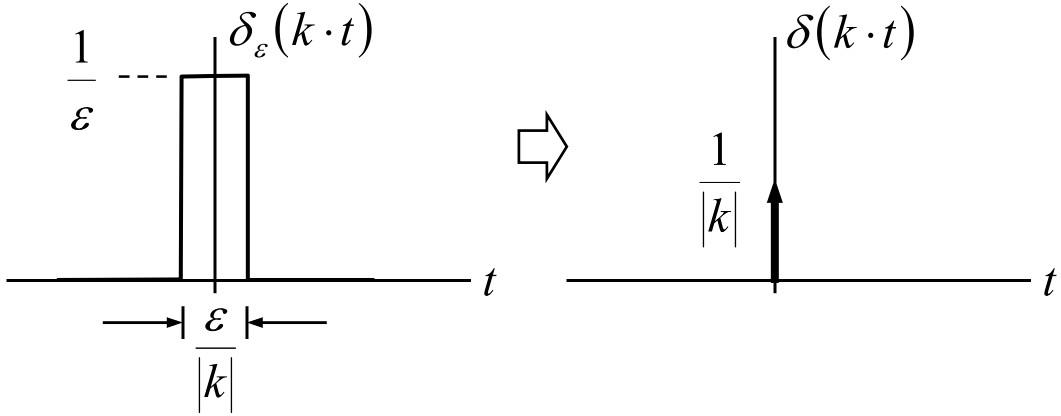


- Symmetry:

$$\delta(-t)=\delta(t)$$

- Coordinate Transformation:

$$\delta(k\cdot t)=\frac{1}{|k|}\,\delta(t) \qquad k\in\Re$$



Proof: $$\int_{-\infty}^{+\infty}\delta(k\cdot t)\,dt=\frac{1}{|k|}\int_{-\infty}^{+\infty}\delta(k\cdot t)\,d(k\cdot t)=\frac{1}{|k|}$$

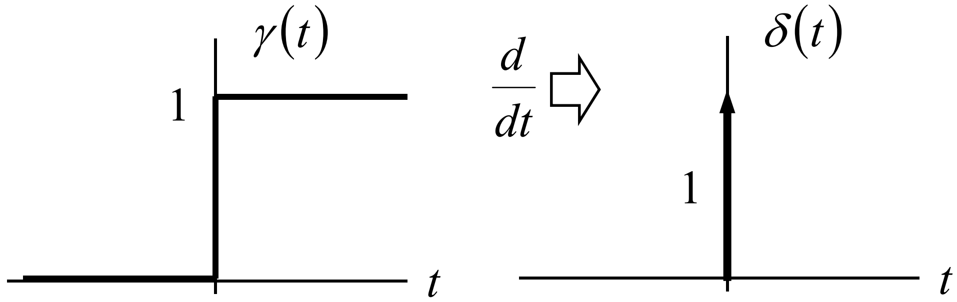
Properties of Delta Impulses (2)

- Orthogonality:

$$\int_{-\infty}^{+\infty} \delta(t-t_1)\,\delta(t-t_2)\,dt = \delta(t_1-t_2)$$

- Relationship to the Step Function:

$$\int_{-\infty}^{t} \delta(\tau)\,d\tau = \gamma(t) \quad \Rightarrow \quad \frac{d}{dt}\gamma(t) = \delta(t)$$



Derivatives of Delta Impulses

- 1^{st} Derivative:

$$\int_{-\infty}^{+\infty} \delta'(t-t_0)\, u(t)\, dt = -u'(t_0)$$

with $u(t)$ 2 times differentiable at $t = t_0$

- ν^{th} Derivative:

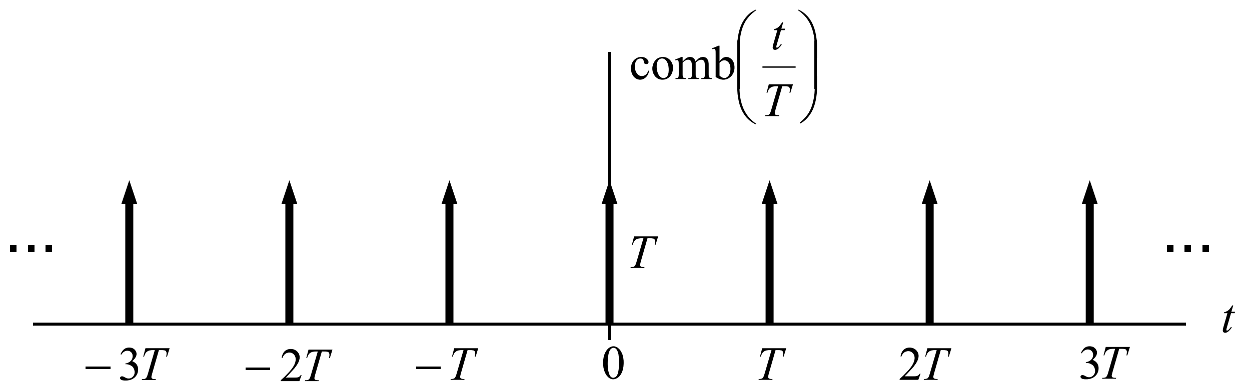
$$\int_{-\infty}^{+\infty} \delta^{(\nu)}(t-t_0)\, u(t)\, dt = (-1)^{\nu}\, u^{(\nu)}(t_0)$$

with $u(t)$ $(\nu+1)$ times differentiable at $t = t_0$

Periodic Delta Pulse (Comb Function)

$$\mathrm{comb}(t) = \sum_{n=-\infty}^{+\infty} \delta(t-n)$$

$$\mathrm{comb}\left(\frac{t}{T}\right) = T \sum_{n=-\infty}^{+\infty} \delta(t-nT) \qquad\qquad T \geq 0$$



3.4 Why Digital Signal Processing?

- Flexibility in system reconfiguration
- Better control of accuracy
- Perfectly reproducible
- No performance drift with temperature or age
- Signal processors are becoming more powerful and cheaper
- **Educational purpose: Easy to start!**

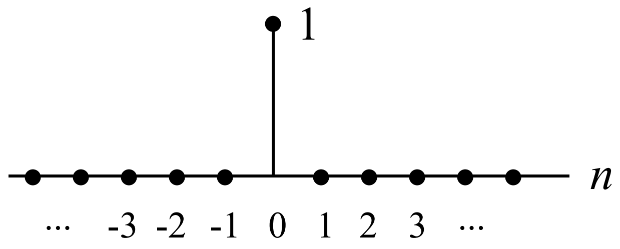
3.5 Some Discrete-time Signals

- Unit Impulse Sequence
- Unit Step Sequence
- Sinusoidal and Exponential Sequences
- Fundamental and Harmonic Component
- Example: Beat Rate

Unit Impulse Sequence

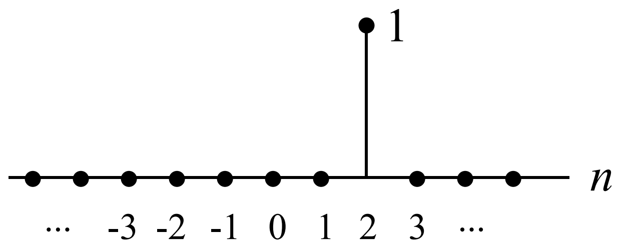
- Unit impulse sequence $\delta[n]$ (also called discrete-time impulse or the unit impulse)

$$\delta[n] = \begin{cases} 1, & n = 0, \\ 0, & n \neq 0. \end{cases}$$



- The unit sample sequence shifted by k samples:

$$\delta[n-k] = \begin{cases} 1, & n = k, \\ 0, & n \neq k. \end{cases}$$

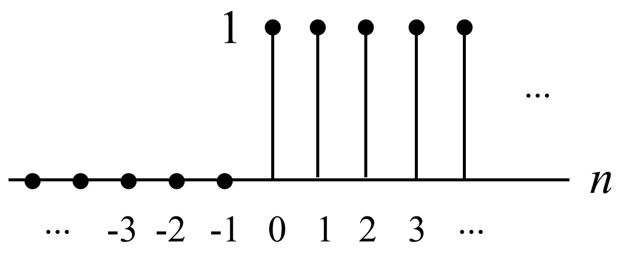


example with $k=2$

Unit Step Sequence

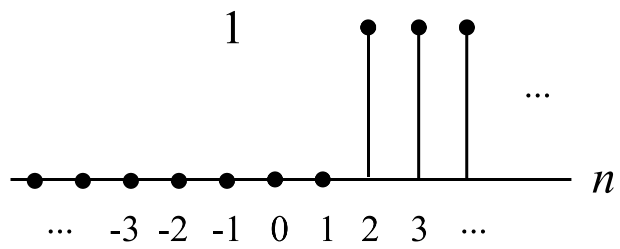
- Unit step sequence $\gamma[n]$:

$$\gamma[n]=\begin{cases}1, & n\geq 0,\\ 0, & n<0.\end{cases}$$



- Unit step sequence shifted by k samples:

$$\gamma[n-k]=\begin{cases}1, & n\geq k,\\ 0, & n<k.\end{cases}$$



- The unite step sequences are related as follows:

$$\gamma[n]=\sum_{m=0}^{\infty}\delta[n-m]=\sum_{k=-\infty}^{n}\delta[k],$$

$$\delta[n]=\gamma[n]-\gamma[n-1].$$

Sinusoidal and Exponential Sequences

- Real sinusoidal sequence

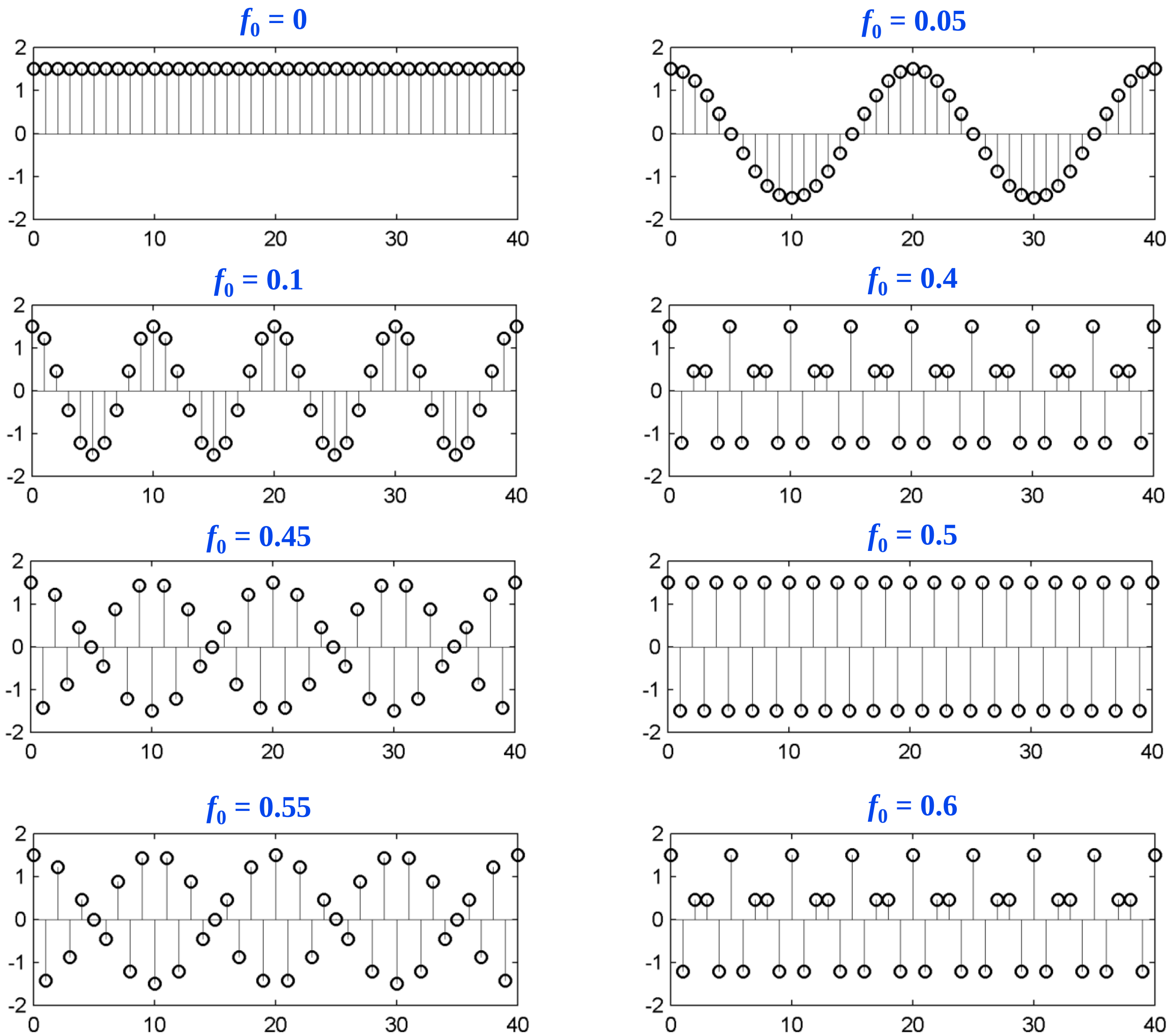
$$u[n] = \underset{\substack{\uparrow \\ \text{amplitude}}}{A} \cos\big(2\pi \underset{\substack{\uparrow \\ \text{frequency}}}{f_0} n + \underset{\substack{\uparrow \\ \text{phase}}}{\phi}\big), \quad -\infty < n < \infty$$

- In-phase and quadrature components

$$u[n] = u_i[n] + u_q[n]$$

$$\underbrace{u_i[n] = A\cos\phi\cos(2\pi f_0 n)}_{\text{in-phase component}}, \quad \underbrace{u_q[n] = -A\sin\phi\sin(2\pi f_0 n)}_{\text{quadrature component}}$$

- A family of sinusoidal sequences given by $x[n]=1.5\cos(2\pi f_0 n)$



- Exponential sequence

$$u[n] = A\ \alpha^n, \quad -\infty < n < \infty \qquad A,$$ are real or complex numbers

$$\alpha = e^{(\sigma_0 + j2\pi f_0)}, \quad A = |A| e^{j\phi}$$

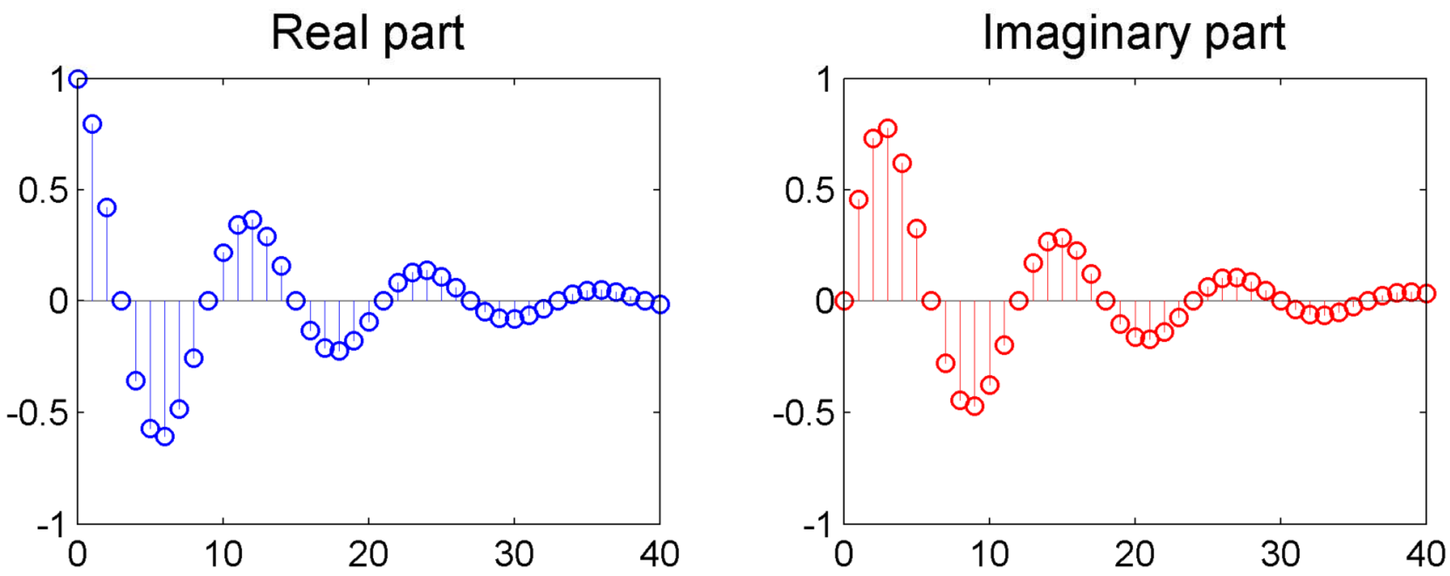
- Real part and imaginary part

real part: $$u_{\mathrm{re}}[n] = |A| e^{\sigma_0 n} \cos(2\pi f_0 n + \phi)$$

imaginary part: $$u_{\mathrm{im}}[n] = |A| e^{\sigma_0 n} \sin(2\pi f_0 n + \phi)$$

- Real part and imaginary part are real sinusoidal sequences with constant ($\sigma_0 = 0$), growing ($\sigma_0 > 0$), or decaying ($\sigma_0 < 0$) amplitudes for $n > 0$

- A complex exponential sequence $x[n] = e^{(-1/12+j\pi/6)n}$



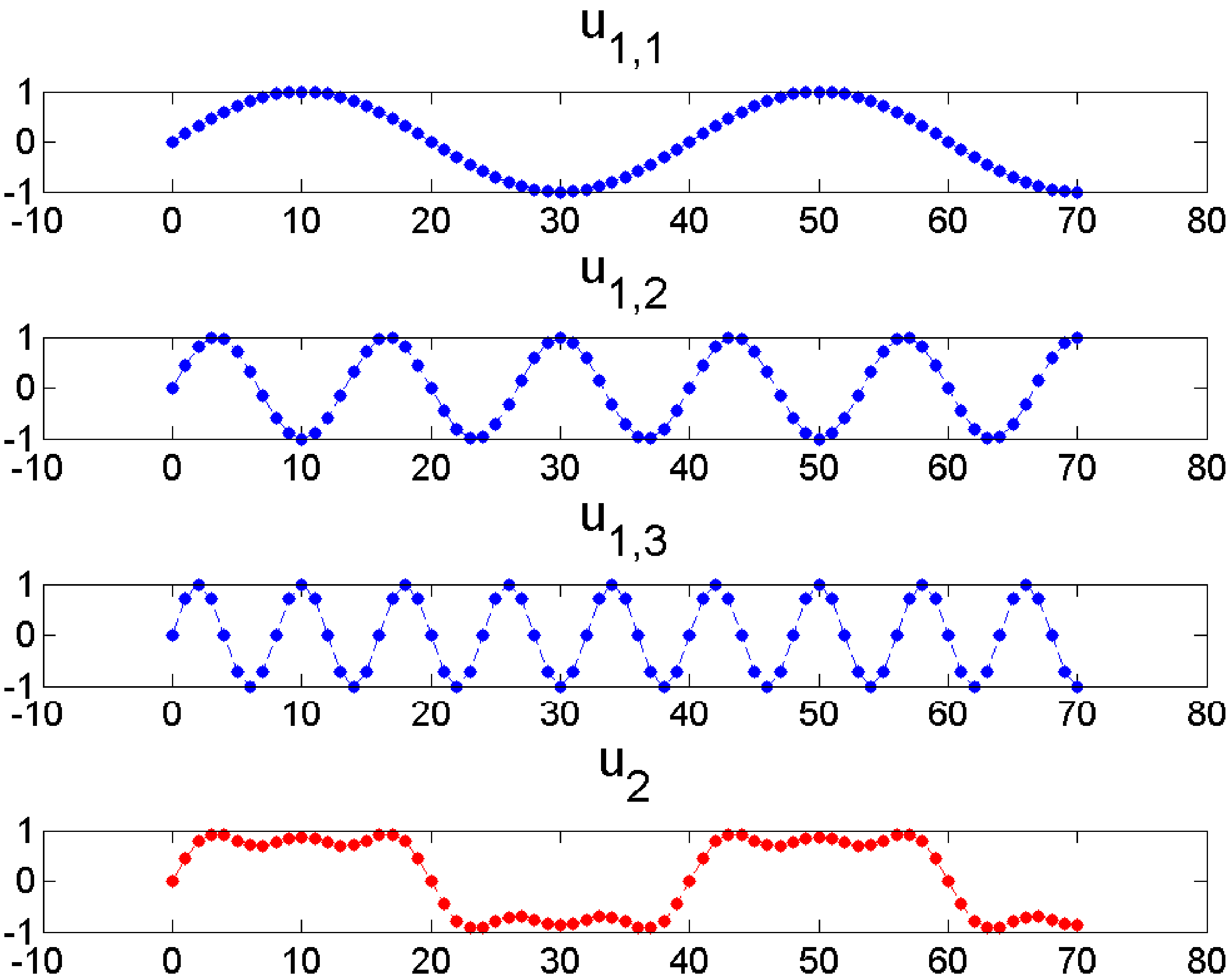
Programming : Generation of a square wave sequence

$$u_{1,1}[n] = \sin(2\pi f_1 n)\,\gamma[n], \quad f_1 = 0.025$$

$$u_{1,2}[n] = \sin(2\pi f_2 n)\,\gamma[n], \quad f_2 = 3 f_1$$

$$u_{1,3}[n] = \sin(2\pi f_3 n)\,\gamma[n], \quad f_3 = 5 f_1$$

$$u_2[n] = u_{1,1}[n] + \frac{1}{3} u_{1,2}[n] + \frac{1}{5} u_{1,3}[n]$$



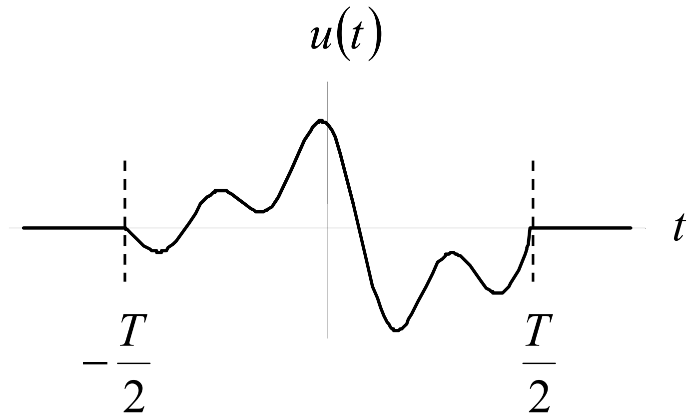
Program Download: www.lmf.bv.tum.de => teaching => links

Fundamental and Harmonic Component

- ***Fundamental component***: the sinusoidal sequence with the lowest frequency, with its frequency being called ***fundamental frequency***
  - e.g. $u_{1,1}[n]$ in the pre-exercise is the fundamental component
  - and f_1 is the fundamental frequency
- ***Harmonics***: the sinusoidal sequences whose frequency are interger multiplies of the lowest frequency
  - e.g. $u_{1,2}[n]$ and $u_{1,3}[n]$ are the 3-rd and the 5-th harmonic, respectively.
- ***Fourier series expansion***: expression of a periodic signal in the form of linearly weighted combination of a fundamental and a series of harmonic components.
- ***Fourier series coefficients***: the weights associated with each component in the expansion.

Fourier Series

- Expansion of energy- and time-limited signals in series of orthogonal harmonic basis functions



- w.l.o.g.: $-\frac{T}{2} < t < +\frac{T}{2}$

- Fourier series: $u(t) = \sum_{n=-\infty}^{+\infty} c_n \cdot \Psi_n(t) \quad \forall \quad |t| < \frac{T}{2}$ with $\Psi_n(t) = \exp\left(j 2\pi \frac{n}{T} t \right)$

Fourier coefficient of order *n*

basis function (harmonic oscillation)

frequency $n \cdot f_0$ with $f_0 = 1/T$

Fourier Series:

$$u(t) = \sum_{n=-\infty}^{+\infty} c_n \exp\left(j 2\pi \frac{n}{T} t \right) \qquad \forall \quad |t| < \frac{T}{2}$$

$$= a_0 + \underbrace{\sum_{n=1}^{+\infty} a_n \cos\left(2\pi \frac{n}{T} t \right)}_{\textit{even} \text{ part of the signal}} + \underbrace{\sum_{n=1}^{+\infty} b_n \sin\left(2\pi \frac{n}{T} t \right)}_{\textit{odd} \text{ part of the signal}}$$

- Complete signal description on the interval $\pm T/2$ for signals satisfying Dirichlet's conditions (e.g. finite number of discontinuities),
- Converges at points of discontinuity to the average of left-hand and right-hand limits
- Replicates the signal periodically outside of $\pm T/2$, hence poor convergence at discontinuities near the boundaries, i.e. for $u(-T/2) \neq u(T/2)$.

Basis functions of Fourier series :

$$\Psi_n(t) = \exp\left(j\,2\pi \frac{n}{T}\, t \right) = \cos\left(2\pi \frac{n}{T}\, t \right) + j \sin\left(2\pi \frac{n}{T}\, t \right)$$

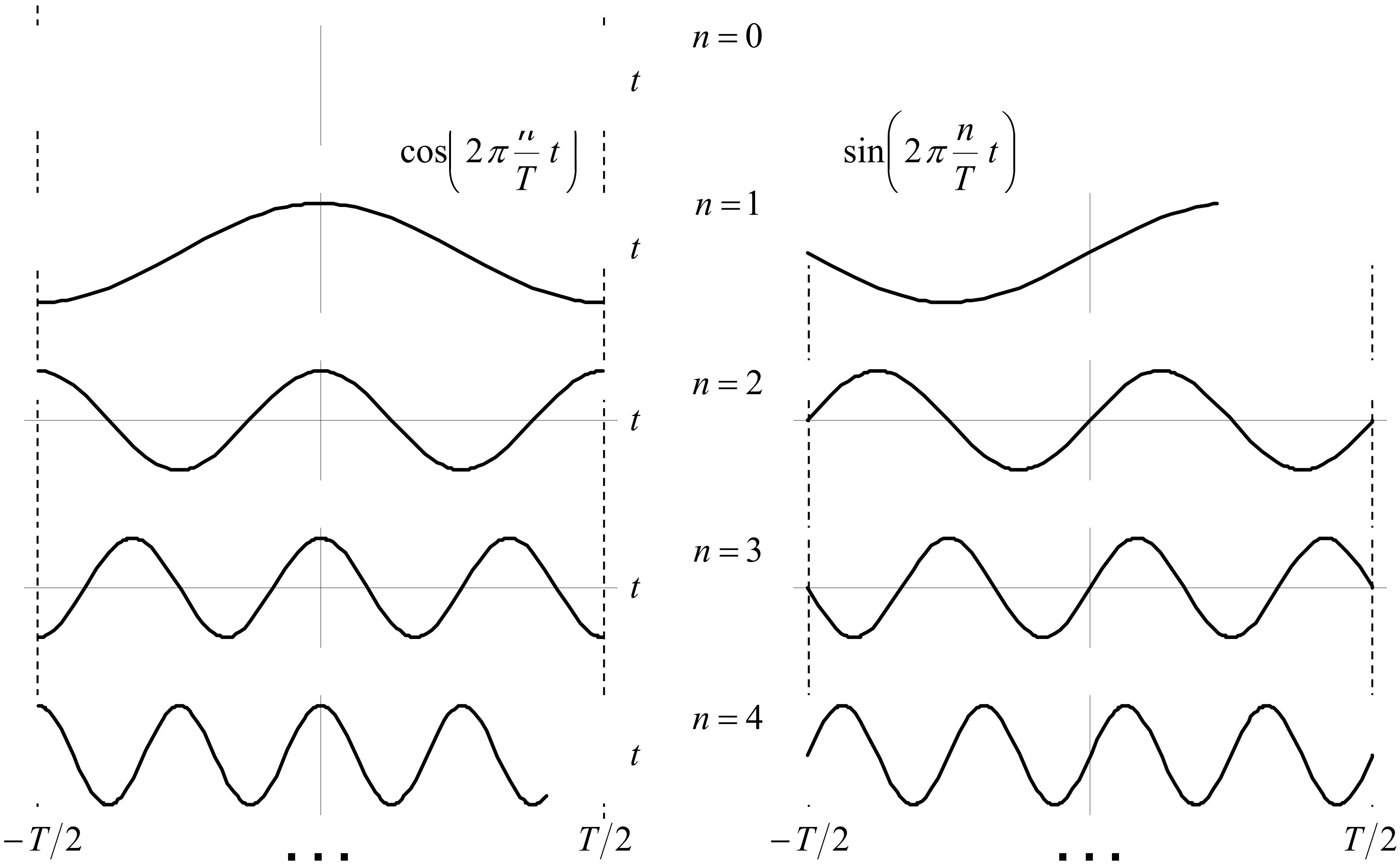
Orthogonality:

$$\int_{-T/2}^{T/2} \Psi_n(t)\Psi_m^*(t)\, dt \;=\; \begin{cases} 0 & n \neq m \\ T & n = m \end{cases}$$

Energy conservation (Parseval‘s Equation):

$$\int_{-T/2}^{T/2} \left| u(t) \right|^2 dt \;=\; T \sum_{n=-\infty}^{\infty} \left| c_n \right|^2$$

Basis functions of Fourier Series



Computation of Fourier coefficients :

complex:

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} u(t)\,\underbrace{\exp\left(-j\,2\pi\frac{n}{T}\,t\right)}_{\Psi_n^*(t)}\,dt \qquad c_n = \frac{1}{2}\left(a_n - j\,b_n\right)$$

cos-coefficients:

$$a_0 = c_0 = \frac{1}{T}\int_{-T/2}^{T/2} u(t)\,dt$$ (constant component, average of signal)

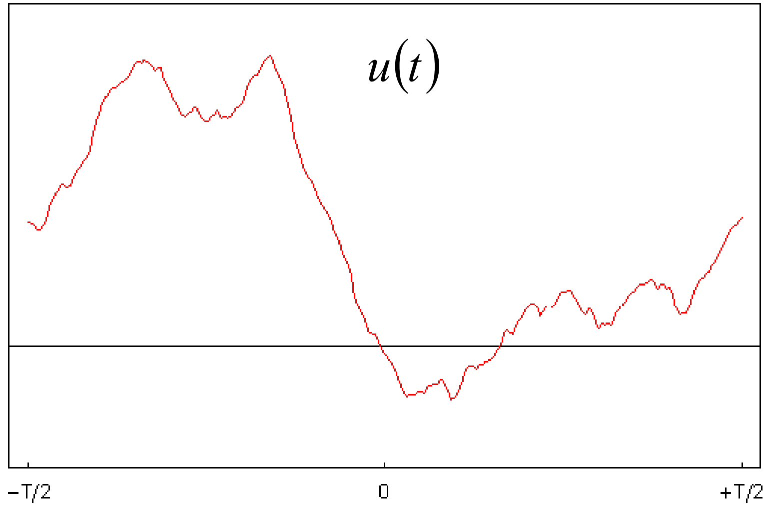
$$a_{n\geq 1} = \frac{2}{T}\int_{-T/2}^{T/2} u(t)\cos\left(2\pi\frac{n}{T}\,t\right)dt$$

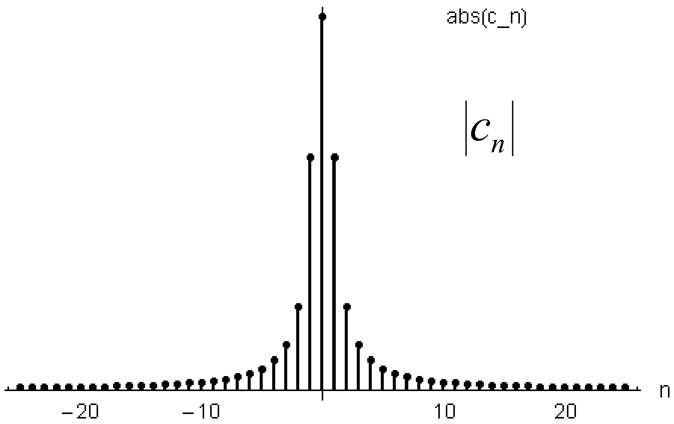
sin-coefficients:

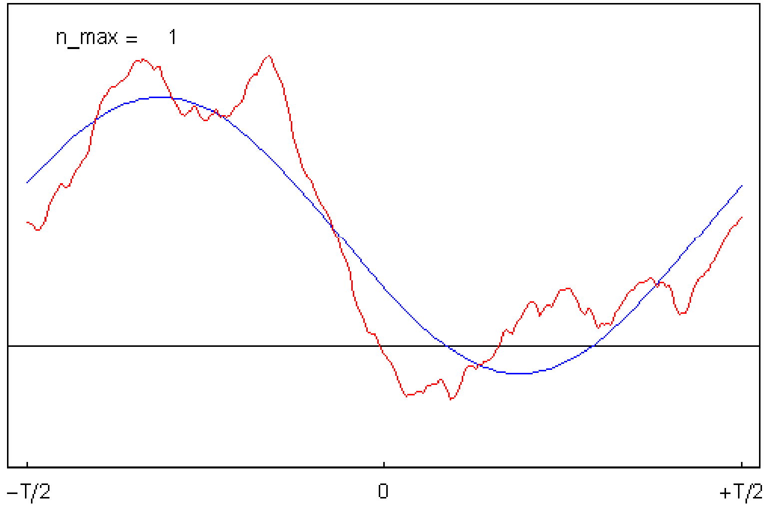
$$b_{n\geq 1} = \frac{2}{T}\int_{-T/2}^{T/2} u(t)\sin\left(2\pi\frac{n}{T}\,t\right)dt$$

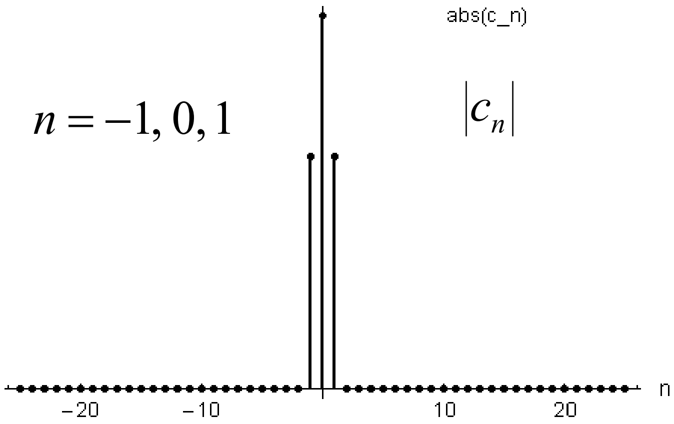
for real-valued signals: $c_{-n} = c_n^*$ and: $a_0, a_n, b_n \in \Re$

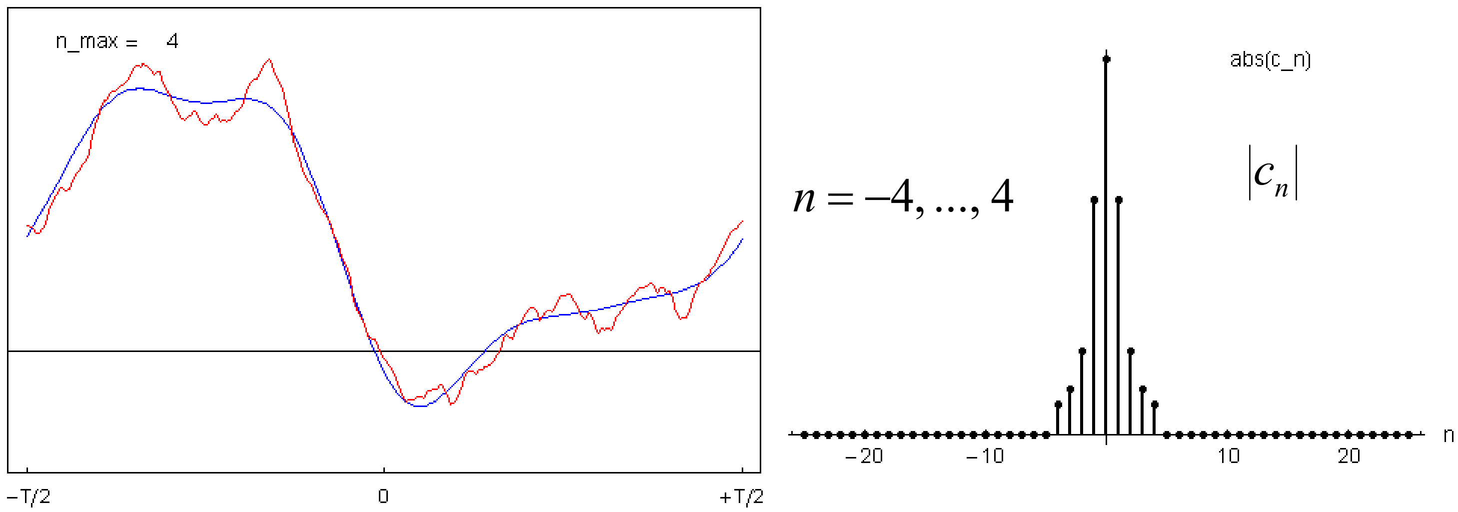
Approximation of a time-limited (but periodic continuous) Signal by finite Fourier Series



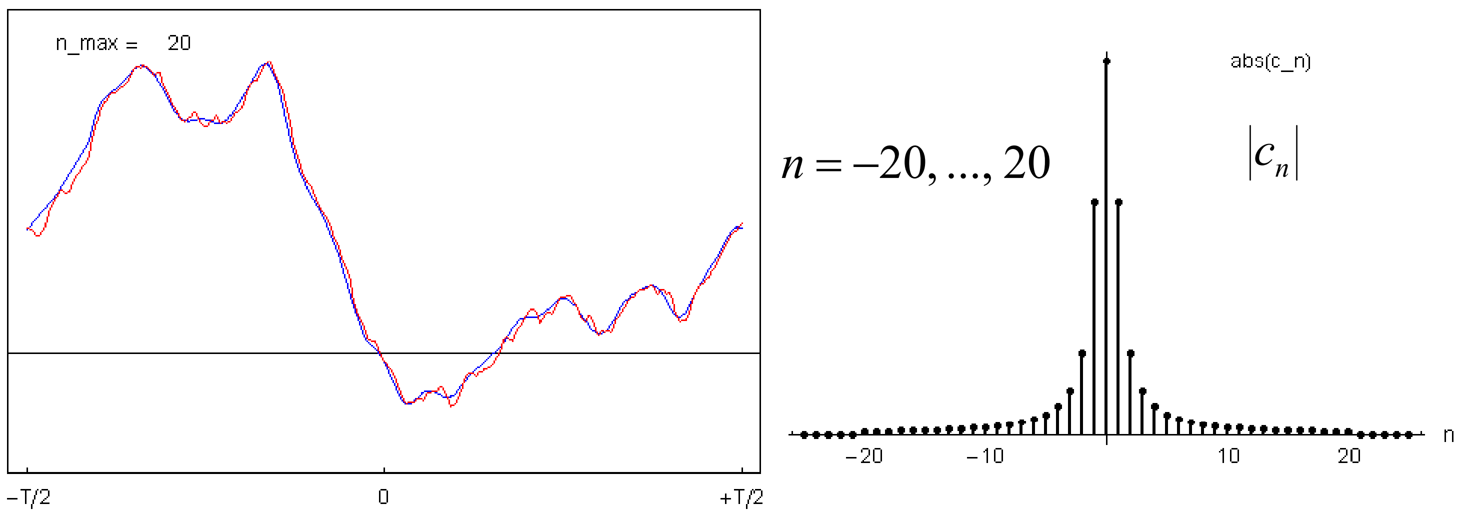








n_max = 4
-T/2
0
+T/2
$n = -4, \ldots, 4$
abs(c_n)
$|c_n|$
-20
-10
10
20
n



n_max = 20
-T/2
0
+T/2
$n = -20, \ldots, 20$
abs(c_n)
$|c_n|$
-20
-10
10
20
n

Example: Beat Note

- <u>Exercise</u>: Illustration of the modulation operation

$$u_{1,1}[n] = \cos(2\pi f_1 n),$$
$$u_{1,2}[n] = \cos(2\pi f_2 n),\ f_1 = 0.05,\ \ f_2 = 0.005$$

$$u_2[n] = u_{1,1}[n] \cdot u_{1,2}[n] = \frac{1}{2}\cos(2\pi(f_1 + f_2)n) + \frac{1}{2}\cos(2\pi(f_1 - f_2)n).$$

beat note

- ***Beat note***: the high frequency signal generated by the multiplication

- <u>Practice</u>: Play with the Audacity.

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Continuous-time Fourier Transform

$$U(f) = \int_{-\infty}^{+\infty} u(t) \exp(-j2\pi f t)\, dt$$ Fourier Transform

$$u(t) = \int_{-\infty}^{+\infty} U(f) \exp(j2\pi f t)\, df$$ inverse Fourier Transform

with $u(t)$ absolute integrable: $\int_{-\infty}^{+\infty} |u(t)|\, dt < \infty$ (sufficient condition)

Some power-limited signals can also be Fourier transformed :

$$U(f) = \lim_{\varepsilon \to 0} \int_{-\infty}^{+\infty} u(t) \exp(-\varepsilon|t|) \exp(-j2\pi f t)\, dt$$

Common symbols :

$$u(t) \quad \circ\!\!-\!\!\bullet \quad U(f)$$
$$u(t) \quad \rightarrow \quad U(f)$$
$$u(t) \quad \leftrightarrow \quad U(f)$$

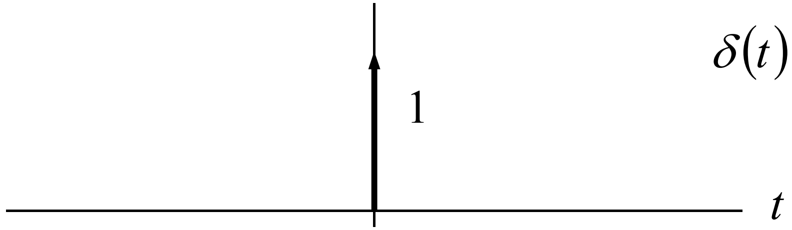
Alternative Expression for Fourier Transform (not used in this lecture):

- ω-notation (very common): $\omega = 2\pi f$

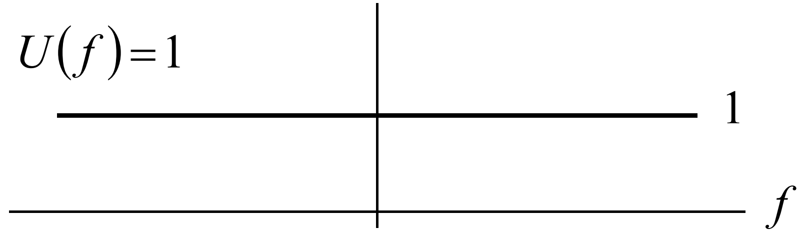
$$U(\omega) = \int_{-\infty}^{+\infty} u(t)\exp(-j\omega t)\,dt$$

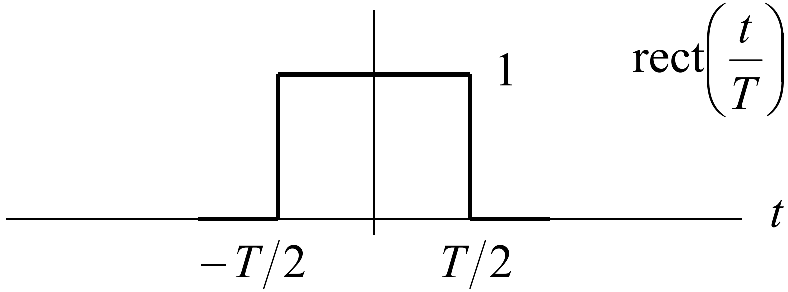
$$u(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} U(\omega)\exp(j\omega t)\,d\omega$$

Important 1-D Fourier Transform Pairs (1)

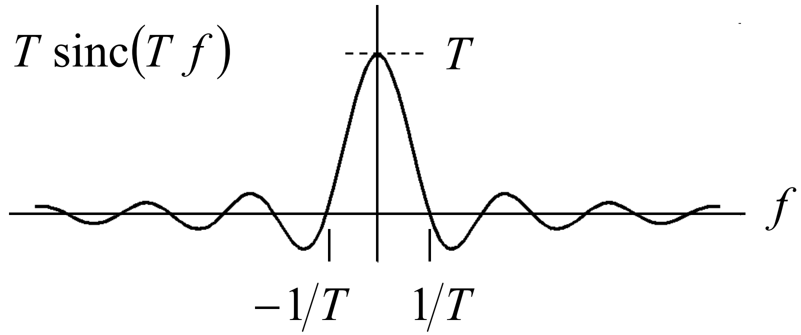


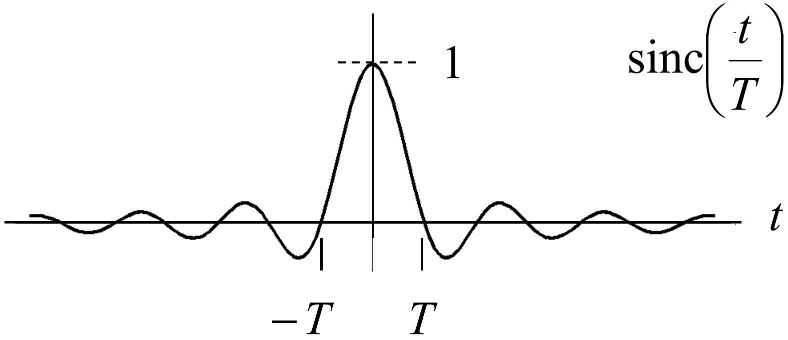
$$\delta(t) \longleftrightarrow U(f)=1$$



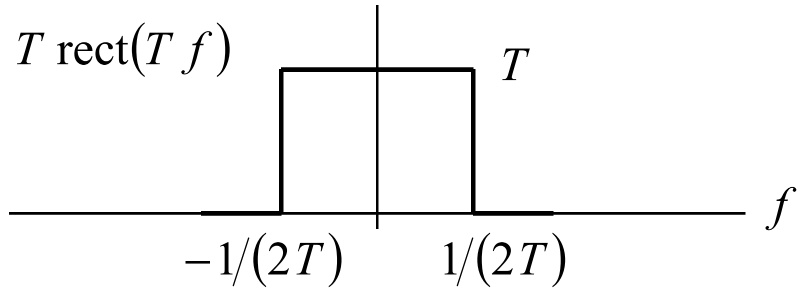


$$\mathrm{rect}\left(\frac{t}{T}\right) \longleftrightarrow T\,\mathrm{sinc}(T\,f)$$



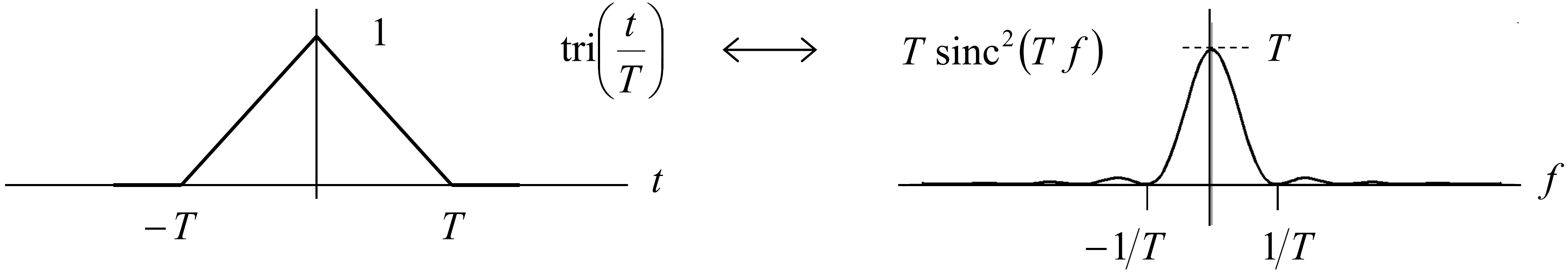


$$\mathrm{sinc}\left(\frac{t}{T}\right) \longleftrightarrow T\,\mathrm{rect}(T\,f)$$



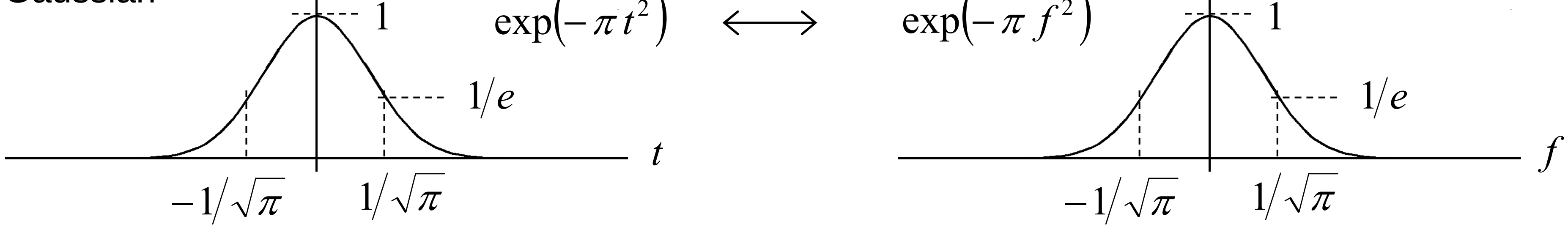
Important 1-D Fourier Transform Pairs (2)

$$\mathrm{tri}\left(\frac{t}{T}\right) \longleftrightarrow T\,\mathrm{sinc}^2(T\,f)$$



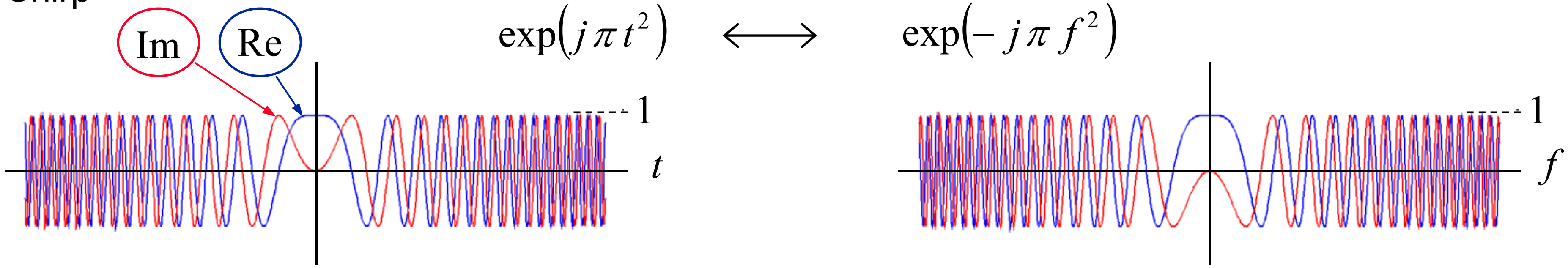
Gaussian

$$\exp\left(-\pi t^2\right) \longleftrightarrow \exp\left(-\pi f^2\right)$$



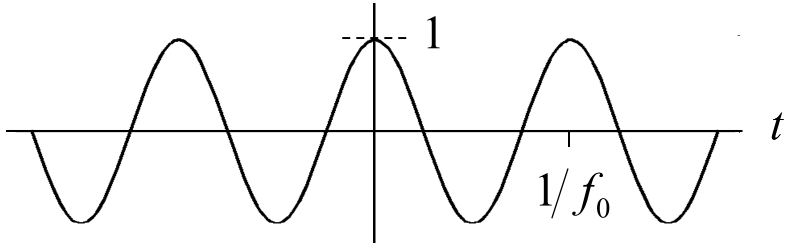
Chirp

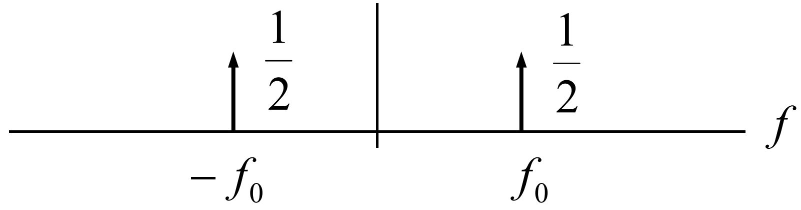
$$\exp\left(j\pi t^2\right) \longleftrightarrow \exp\left(-j\pi f^2\right)$$



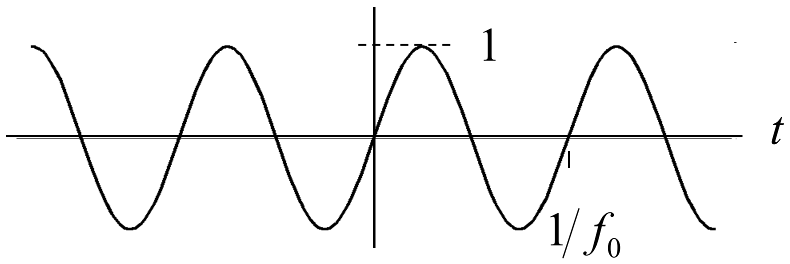
Important 1-D Fourier Transform Pairs (3)

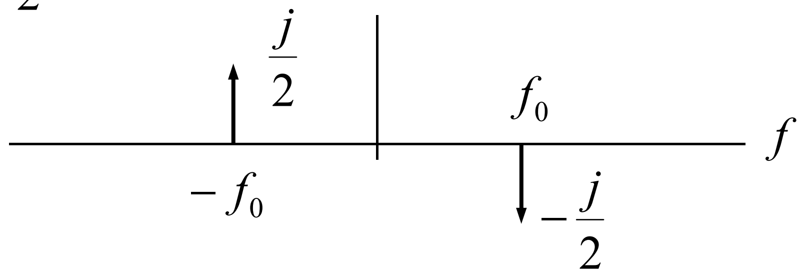
$$\cos(2\pi f_0 t) \longleftrightarrow \frac{1}{2}\left(\delta(f+f_0)+\delta(f-f_0)\right)$$





$$\sin(2\pi f_0 t) \longleftrightarrow \frac{j}{2}\left(\delta(f+f_0)-\delta(f-f_0)\right)$$





$$\exp(j2\pi f_0 t) \longleftrightarrow \delta(f-f_0)$$