Fourier transform, in 1D and in 2D

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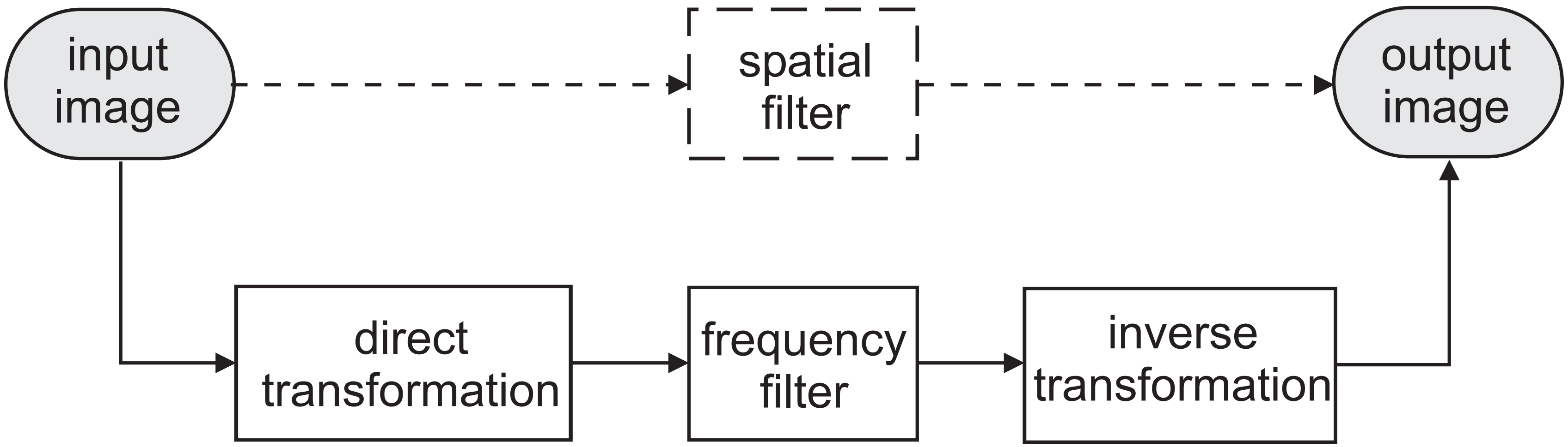
Outline of the talk:

- Fourier tx in 1D, computational complexity, FFT.
- Fourier tx in 2D, centering of the spectrum.
- Examples in 2D.

Initial idea, filtering in frequency domain

Image processing $\equiv$ filtration of 2D signals.



Filtration in the spatial domain. We would say in time domain for 1D signals. It is a linear combination of the input image with coefficients of (often local) filter. The basic operation is called convolution.

Filtration in the frequency domain. Conversion to the 'frequency domain', filtration there, and the conversion back.

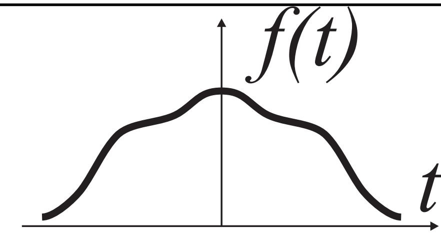
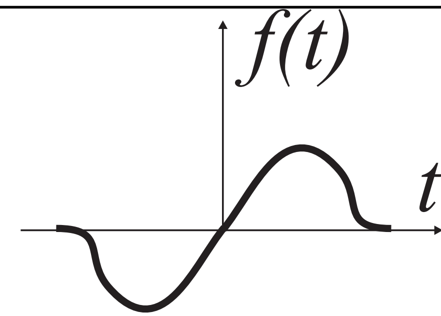
We consider *Fourier transform*, but there are other linear integral transforms serving a similar purpose, e.g., cosine, wavelets.

1D Fourier transform, introduction

- Fourier transforme is one of the most commonly used techniques in (linear) signal processing and control theory.
- It provides one-to-one transform of signals from/to a time-domain representation $f(t)$ to/from a frequency domain representation $F(\xi)$.
- It allows a frequency content (spectral) analysis of a signal.
- FT is suitable for periodic signals.
- If the signal is not periodic then the Windowed FT or the linear integral transformation with time (spatially in 2D) localized basis function, e.g., wavelets, Gabor filters can be used.

Joseph Fourier
1768-1830

Odd, even and complex conjugate functions

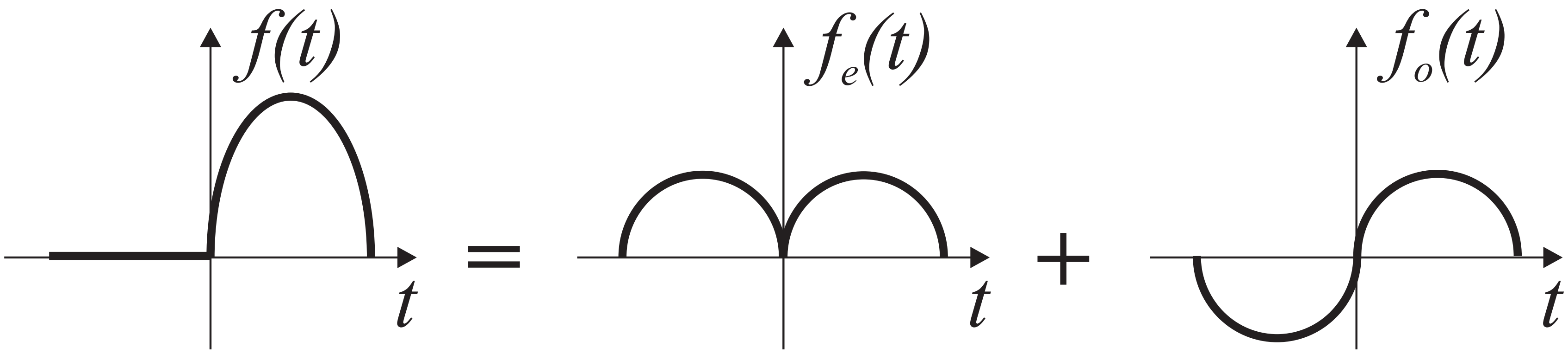
Even	$f(t) = f(-t)$	
Odd	$f(t) = -f(-t)$	
Conjugage symmetric	$f(\xi) = f^*(-\xi)$	$f(\ \ 5) = 2+3i$ $f(-5) = 2-3i$

- f^* denotes a complex conjugate function.
- i is a complex unit.

Any function can be decomposed as a sum of the even and odd part

$$f(t) = f_e(t) + f_o(t)$$

$$f_e(t) = \frac{f(t) + f(-t)}{2} \qquad f_o(t) = \frac{f(t) - f(-t)}{2}$$



Fourier Tx definition: continuous cased

$\mathcal{F}\{f(t)\} = F(\xi)$, where ξ [Hz=s^{-1}] is a frequency and$2\pi\xi$ [s^{-1}] is the angular frequency.

Fourier Tx	Inverse Fourier Tx
$F(\xi) = \int\limits_{-\infty}^{\infty} f(t)\, e^{-2\pi i \xi t}\, \mathrm{d}t$	$f(t) = \int\limits_{-\infty}^{\infty} F(\xi)\, e^{2\pi i \xi t}\, \mathrm{d}\xi$

What is the meaning of the inverse Fourier Tx? Express it as a Riemann sum:

$$f(t) \doteq \left(\ldots + F(\xi_0)\, e^{2\pi i \xi_0 t} + F(\xi_1)\, e^{2\pi i \xi_1 t} + \ldots \right) \Delta\xi \,,$$

kde $\Delta\xi = \xi_{k+1} - \xi_k$ pro $\forall\ k$.

$\Rightarrow$ Any 1D function can be expressed as a the weighted sum (integral) of many different complex exponentials (because of Euler's formula $e^{j\xi} = \cos\xi + j \sin\xi$, also of cosinusoids and sinusoids).

Existence conditions of Fourier Tx

1. $\int\limits_{-\infty}^{\infty} \mid f(t) \mid \mathrm{d}t < \infty$, i.e. $f(t)$ has to grow slower than an exponential curve.
2. $f(t)$ can have only a finite number of discontinuities and maxima, minima in any finite rectangle.
3. $f(t)$ need not have discontinuities with the infinite amplitude.

Fourier transformation exists always for digital images as they are limited and have finite number of discontinuities.

Fourier Tx, symmetries

- Symmetry with regards to the complex conjugate part, i.e., $F(-i\xi) = F^*(i\xi)$.
- $|F(i\xi)|$ is always even.
- The phase of $F(i\xi)$ is always odd.
- $Re\{F(i\xi)\}$ is always even.
- $Im\{F(i\xi)\}$ is always odd.
- The even part of $f(t)$ transforms to the real part of $F(i\xi)$.
- The odd part of $f(t)$ transforms to the imaginary part of $F(i\xi)$.

Convolution, definition, continuous case

- Convolution (in functional analysis) is an operation on two functions f and h which produces a third function $(f * h)$, often used to create a modification of one of the input functions.
- Convolution is an integral 'mixing' values of two functions, i.e., of the function $h(t)$, which is shifted and overlayed with the function $f(t)$ or vice-versa.
- Consider first the continuous case with general infinite limits

$$(f * h)(t) = (h * f)(t) \equiv \int_{-\infty}^{\infty} f(\tau)\, h(t-\tau)\, \mathrm{d}\tau = \int_{-\infty}^{\infty} f(t-\tau)\, h(\tau)\, \mathrm{d}\tau\,.$$

- The limits can be constraint to the interval $[0, t]$, because we assume zero values of functions for the negative argument

$$(f * h)(t) = (h * f)(t) \equiv \int_{0}^{t} f(\tau)\, h(t-\tau)\, \mathrm{d}\tau = \int_{0}^{t} f(t-\tau)\, h(\tau)\, \mathrm{d}\tau\,.$$

Convolution, discrete approximation

$$(f * h)(i) = (h * f)(i) \equiv \sum_{m \in \mathcal{O}} h(i-m)\, f(m) = \sum_{m \in \mathcal{O}} h(i)\, f(i-m)\,,$$

where $\mathcal{O}$ is a local neighborhood of a 'current position' i and h is the convolution kernel (also convolution mask).

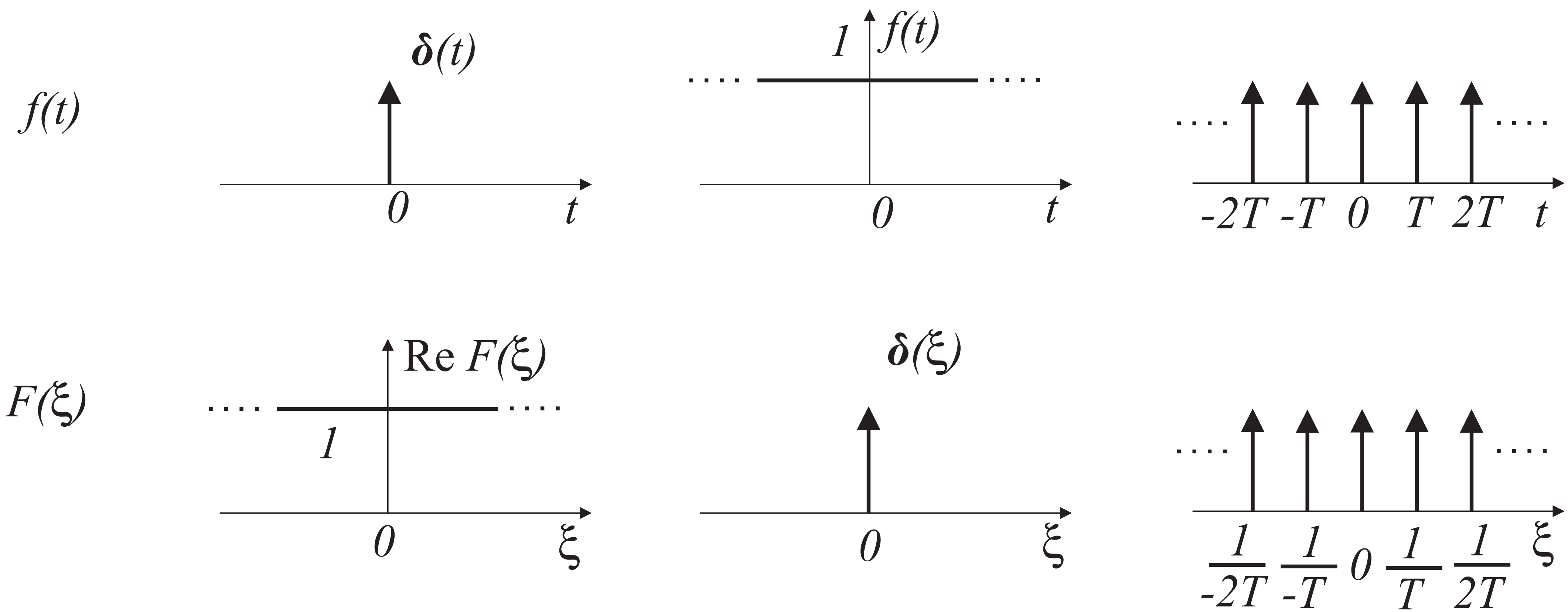
Fourier Tx, properties (1)

Property	$f(t)$	$F(\xi)$
Linearity	$a f_1(t) + b f_2(t)$	$a\,F_1(\xi) + b\,F_2(\xi)$
Duality	$F(t)$	$f(-\xi)$
Convolution	$(f * g)(t)$	$F(\xi)\,G(\xi)$
Product	$f(t)\,g(t)$	$(F * G)(\xi)$
Time shift	$f(t - t_0)$	$e^{-2\pi i \xi t_0}\,F(\xi)$
Frequency shift	$e^{2\pi i \xi_0 t} f(t)$	$F(\xi - \xi_0)$
Differentiation	$\frac{\mathrm{d} f(t)}{\mathrm{d} t}$	$2\pi i \xi F(\xi)$
Multiplication by t	$t\,f(t)$	$\frac{i}{2\pi}\frac{\mathrm{d} F(\xi)}{\mathrm{d}\xi}$
Time scaling	$f(a\,t)$	$\frac{1}{\lvert a\rvert} F\,(\xi/a)$

Fourier Tx, properties (2)

Area in time	$F(0) = \int\limits_{-\infty}^{\infty} f(t)dt$	Area function $f(t)$.
Area in freq.	$f(0) = \int\limits_{-\infty}^{\infty} F(j\xi)d\xi$	Area under $F(j\xi)$
Parseval's th.	$\int\limits_{-\infty}^{\infty} \lvert f(t)\rvert^2 dt = \int\limits_{-\infty}^{\infty} \lvert F(j\xi)\rvert^2 d\xi$	f energy $=$ F energy

Basic Fourier Tx pairs (1)

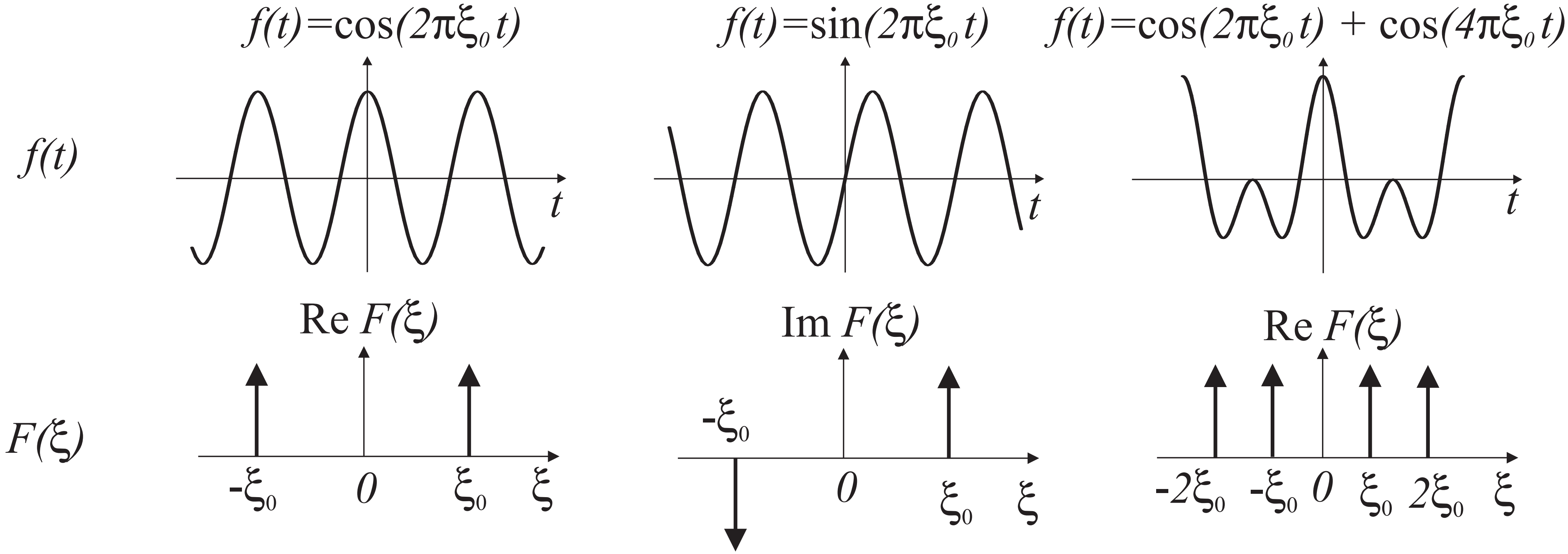


Dirac

constant

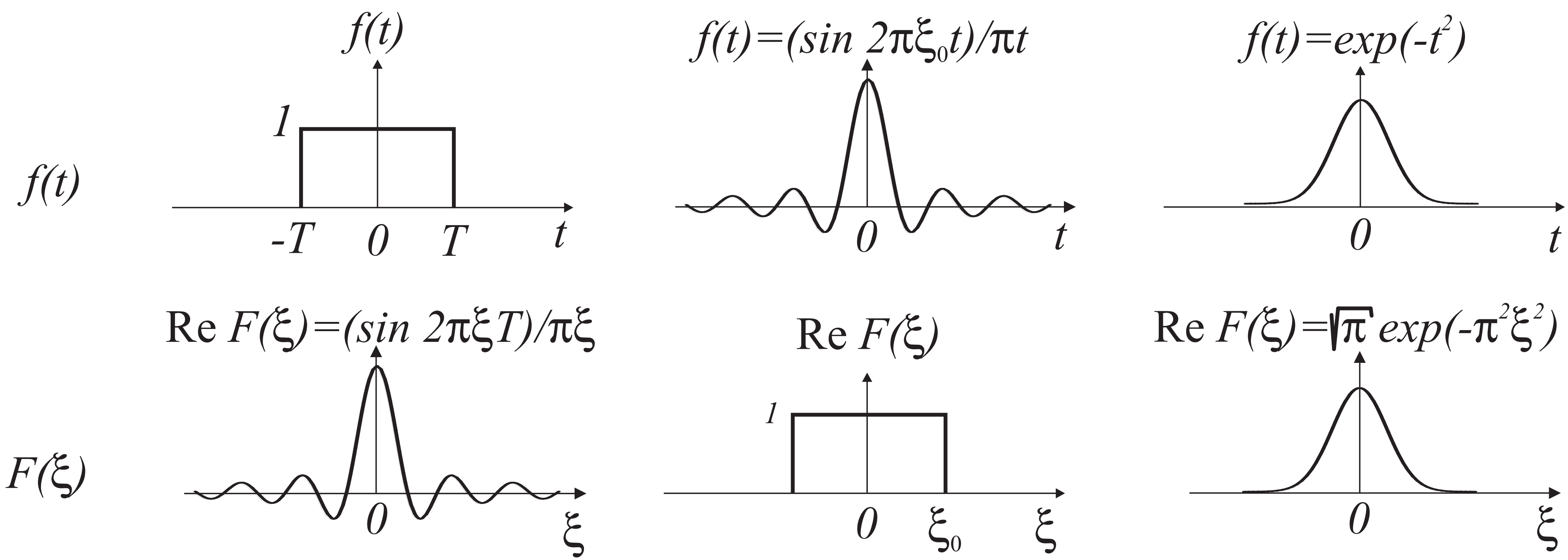
∞ sequence of Diracs

Basic Fourier Tx pairs (2)



sine cosine two cosines mixture

Basic Fourier Tx pairs (3)



rectangle in t | rectangle in ξ | Gaussian

Uncertainty principle

- All Fourier Tx pairs are constrained by the uncertainty principle.
- The signal of short duration must have wide Fourier spectrum and vice versa.
- $\boxed{\text{(signal duration) (frequency bandwith)} \geq \frac{1}{\pi}}$
- Observation: Gaussian e^{-t^2} modulated by a sinusoid (Gabor function) has the smallest duration-bandwidth product.
- The principle is an instance of the general uncertainty principle introduced by Werner Heisenberg in quantum mechanics.

Non-periodic signals

Fourier transform assumes a periodic signal. What if a non-periodic signal has to be processed? There are two common approaches.

1. To process the signal in small chunks (windows) and assume that the signal is periodic outside the windows.
   - The approach was introduced by Dennis Gabor in 1946 and it is named Short time Fourier transform.

     Dennis Gabor, 1900-1979, inventor of holography, Nobel price for physics in 1971, studied in Budapest, PhD in Berlin in 1927, fled Nazi persecution to Britain in 1933.
   - Mere cutting of the signal to rectangular windows is not good because discontinuities at windows limits cause unwanted high frequencies.
   - This is the reason why the signal is convolved by a dumping weight function, often Gaussian or Hamming function ensuring the zero signal value at the limits of the window and beyond it.
2. Use of more complex basis function, e.g., wavelets in the wavelet transform.

Discrete Fourier transform

- Let $f(n)$ be an input signal (a sequence), $n = 0, \ldots, N-1$.
- Let $F(k)$ be a Frequency spectrum (the result of the discrete Fourier transformation) of a signal $f(n)$.
- Discrete Fourier transformation

$$F(k) \equiv \sum_{k=0}^{N-1} f(n)\, e^{\frac{-2\pi i k n}{N}}$$

- Inverse discrete Fourier transformation

$$f(n) \equiv \frac{1}{N} \sum_{k=0}^{N-1} F(k)\, e^{\frac{2\pi i k n}{N}}$$

Computational complexity, a reminder of a notation

- While considering complexity, it is abstracted from a specific computer and only asymptotic behavior of algorithms is concerned.
- An asymptotic upper bound for the magnitude of a function (i.e., its growth) in terms of another, usually simpler, function is sought.
- 'Big $\mathcal{O}$' notation; for example, $\mathcal{O}(n^2)$ means that the number of algorithm steps will be roughly proportional to the square of the number of samples in the worst case.
- Additional terms and multiplicative constants are not taken into account because a qualitative comparison is sought.
- The quadratic complexity $\mathcal{O}(n^2)$ is worse than say $\mathcal{O}(n)$ (linear) or $\mathcal{O}(1)$ (constant, independent of the length n), but is better than $\mathcal{O}(n^3)$ (cubic). If the complexity is exponential, e.g., $\mathcal{O}(2^n)$, then it often means that the algorithm cannot be applied to larger problems (in practical terms).

Computational complexity of the discrete Fourier transform

- Let W be a complex number, $W \equiv e^{\frac{-2\pi i}{N}}$.

$$F(k) \equiv \sum_{n=0}^{N-1} f(n)\, e^{\frac{-2\pi i k n}{N}} = \sum_{n=0}^{N-1} W^{nk}\, f(n)$$

- The vector $f(n)$ is multiplied by the matrix whose element (n, k) is the complex constant W to the power $N \cdot k$.
- This has the computational complexity $\mathcal{O}(N^2)$.

Fast Fourier transform

- A fast Fourier transform (FFT) is an efficient algorithm to compute the discrete Fourier transform and its inverse.
- Statement: FFT has the complexity $\mathcal{O}(N \log_2 N)$.
- Example (according to Numerical recepies in C):
  - A sequence of $N = 10^6$, 1 μsecond computer.
  - FFT 30 seconds of CPU time.
  - DFT 2 weeks of CPU time, i.e., 1,209,600 seconds, which is about 40.000 $\times$ more.
- A FFT idea (Danielson, Lanczos, 1942): The DFT of length N can be expressed as sum of two DFTs of length $N/2$, the first one consisting of odd and the second of even samples. Note: FFT exists also for a general length N.

FFT, the proof

$$
\begin{aligned}
F(k) &= \sum_{n=0}^{N-1} e^{\frac{-2\pi i k n}{N}} f(n) \\
&= \sum_{n=0}^{(N/2)-1} e^{\frac{-2\pi i k (2n)}{N}} f(2n) + \sum_{n=0}^{(N/2)-1} e^{\frac{-2\pi i k (2n+1)}{N}} f(2n+1) \\
&= \sum_{n=0}^{(N/2)-1} e^{\frac{-2\pi i k n}{N/2}} f(2n) + W^n \sum_{n=0}^{(N/2)-1} e^{\frac{-2\pi i k n}{N/2}} f(2n+1) \\
&= F^e(k) + W^n F^o(k)\,, \quad k = 1, \ldots, N
\end{aligned}
$$

- The key idea: recursiveness and N is power of 2.
- Only $\log_2 N$ iterations needed.

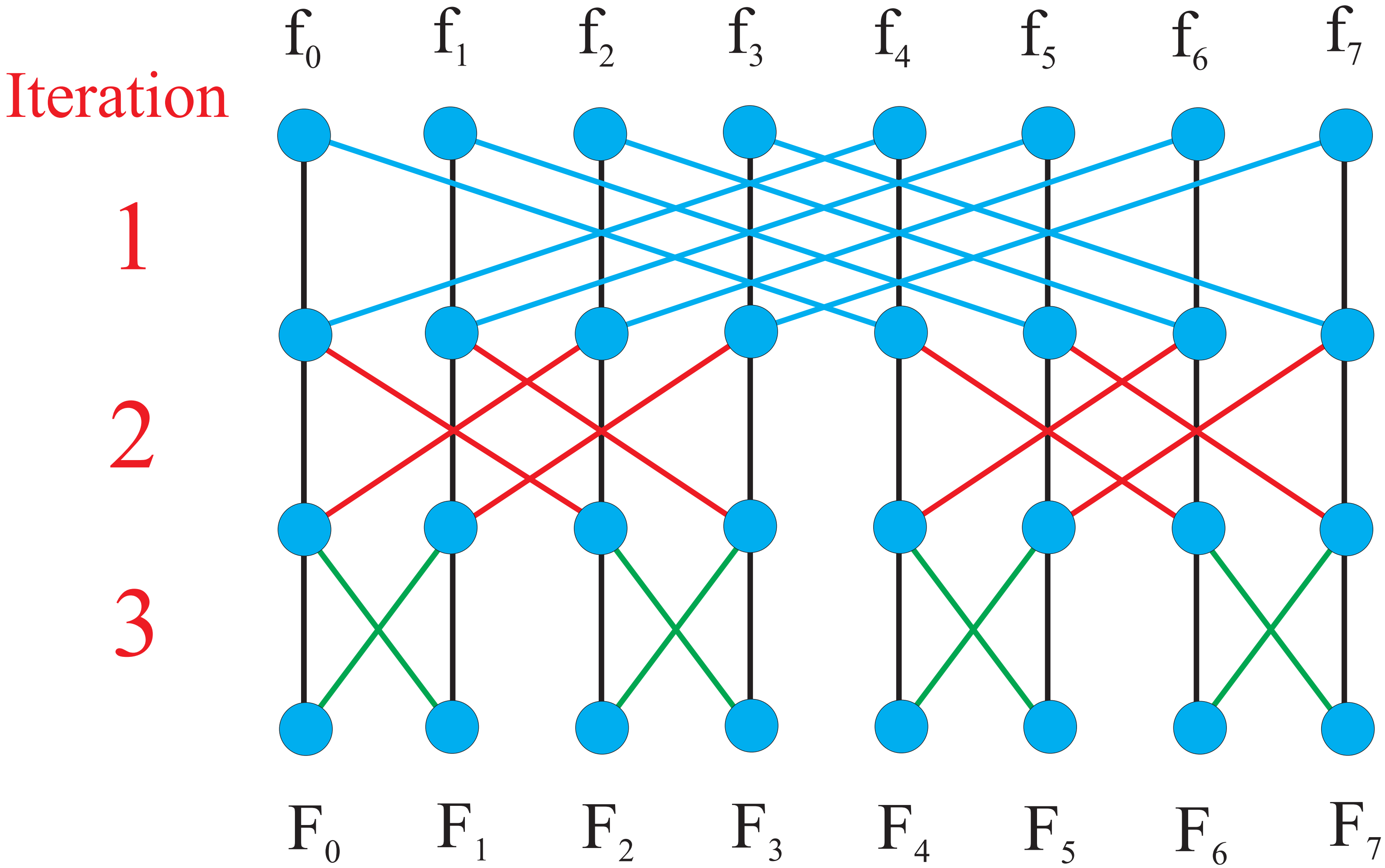
FFT, THE PROOF (2)

- Spectra $F^e(k)$ and $F^o(k)$ are periodic in k with length $N/2$.
- What is Fourier transform o length 1? It is just identity.
- For every pattern of $\log_2 N$ e's and o's, there is a one-point transform that is just one of input numbers $f(n)$,

$$F^{eoeeoeo\ldots oee}(k) = f(n) \quad \text{for some } n \ .$$

- The next trick is to utilize partial results $\Longrightarrow$ butterfly scheme of computations.

FFT butterfly scheme



2D Fourier transform

The idea. The image function $f(x, y)$ is decomposed to a linear combination of harmonic (sines and cosines, more generally orthogonal) functions.

Definition of the direct transform. u, v are spatial frequencies.

$$F(u, v) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, e^{-2\pi i (xu + yv)}\, \mathrm{d}x\, \mathrm{d}y$$

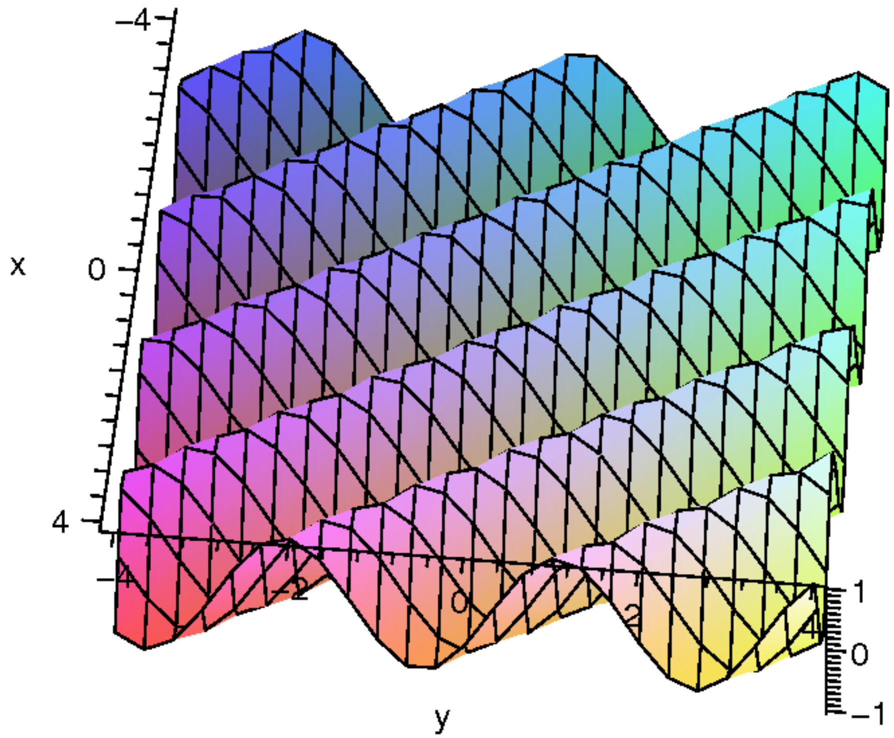
Inverse Fourier tranform

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)\, e^{2\pi i(xu+yv)}\, \mathrm{d}u\, \mathrm{d}v$$

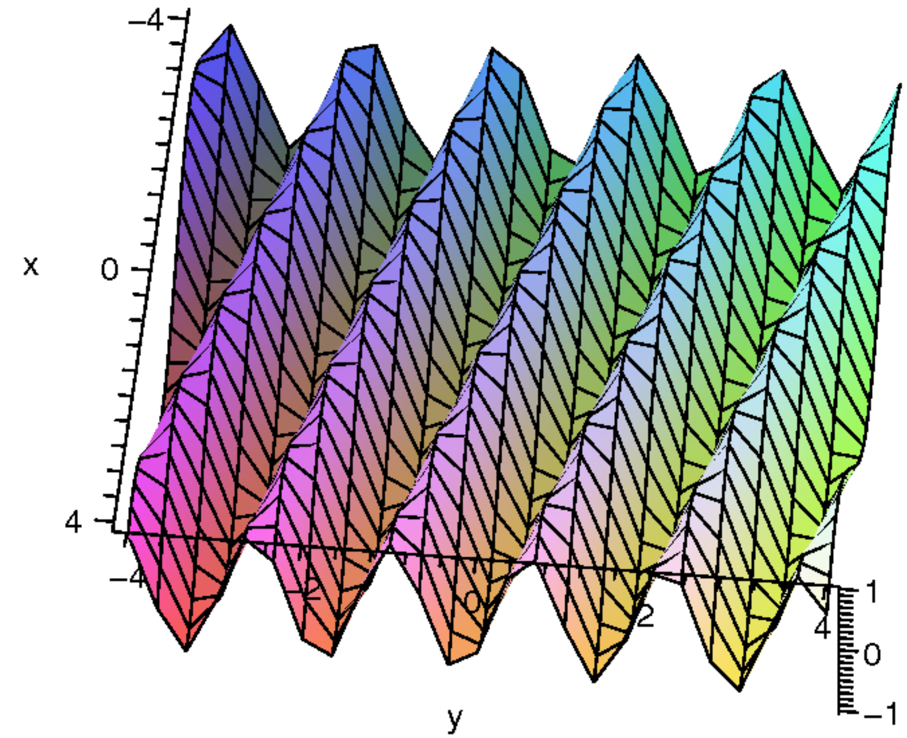
- $f(x,y)$ is a linear combination of simple harmonic functions (components) $e^{2\pi i(xu+uv)}$.
- Thanks to Euler formula ($e^{iz} = \cos z + i \sin z$), $\cos$ corresponds to the real part and $\sin$ corresponds to the imaginary part.
- Function $F(u,v)$ (complex spectrum) gives weights of harmonic components in the linear combination.

Illustration of 2D FT bases vectors

Analogy – corrugated iron.

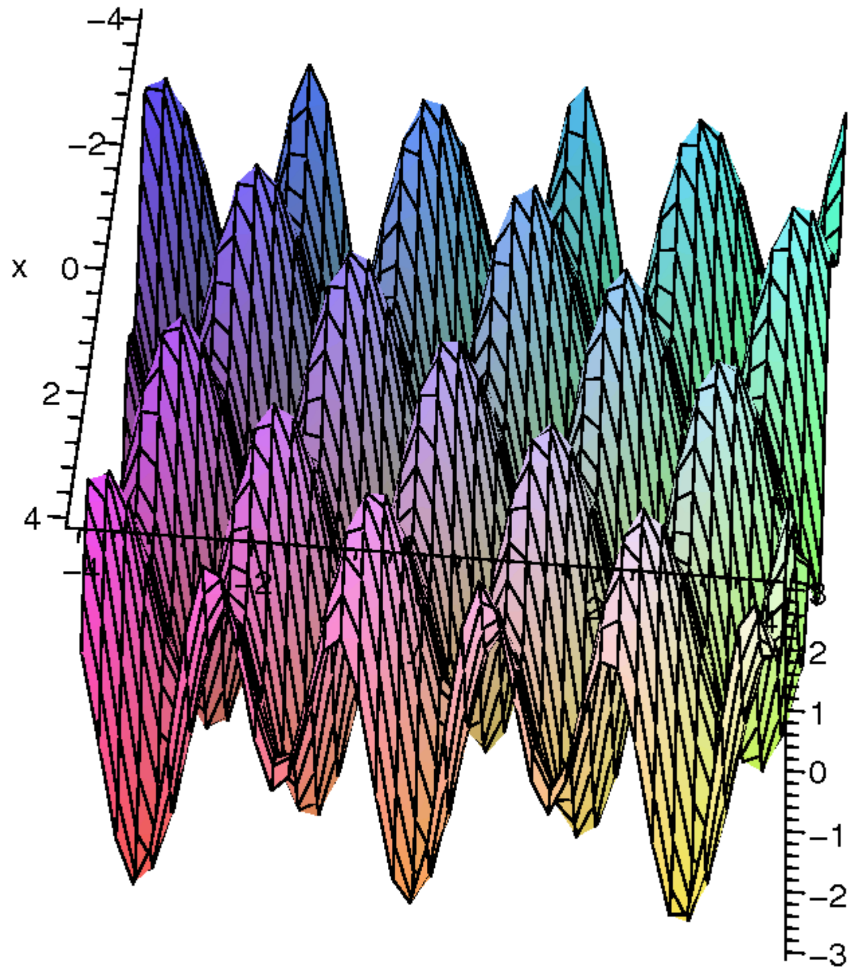


$\sin(3x + 2y)$

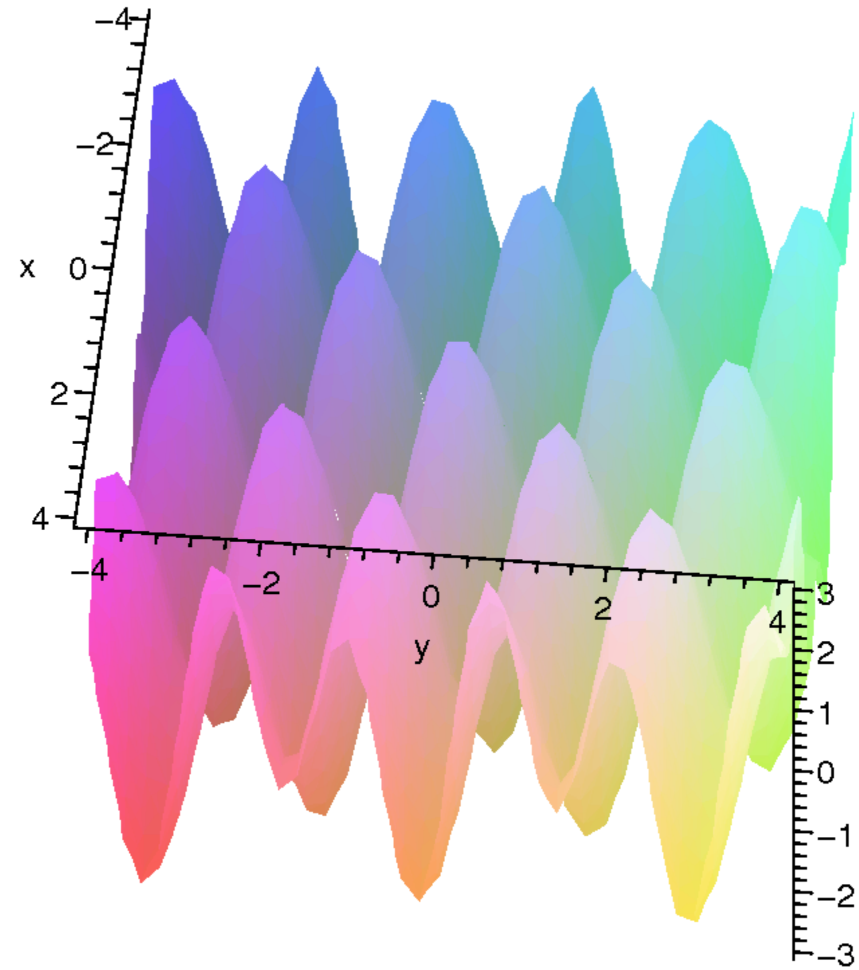


$\cos(x + 4y)$

Linear combination of base vectors



$\sin(3x + 2y) + \cos(x + 4y)$



different display only

2D discrete Fourier transform

Direct transform

$$F(u,v) = \frac{1}{MN} \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n) \, \exp\left[-2\pi i \left(\frac{mu}{M} + \frac{nv}{N}\right)\right] \, ,$$

$$u = 0, 1, \ldots, M-1 \, , \qquad v = 0, 1, \ldots, N-1 \, ,$$

Inverse transform

$$f(m,n) = \sum_{u=0}^{M-1} \sum_{v=0}^{N-1} F(u,v) \, \exp\left[2\pi i \left(\frac{mu}{M} + \frac{nv}{N}\right)\right] \, ,$$

$$m = 0, 1, \ldots, M-1 \, , \qquad n = 0, 1, \ldots, N-1 \, .$$

2D Fourier Tx as twice 1D Fourier Tx

2D direct FT can be modified to

$$F(u,v) = \frac{1}{M} \sum_{m=0}^{M-1} \left[\frac{1}{N} \sum_{n=0}^{N-1} \exp\left(\frac{-2\pi i n v}{N} \right) f(m,n) \right] \exp\left(\frac{-2\pi i m u}{M} \right) ,$$

$$u = 0, 1, \ldots, M-1 \, , \qquad v = 0, 1, \ldots, N-1 \, .$$

- The term in square brackets corresponds to the one-dimensional Fourier transform of the m^{th} line and can be computed using the standard fast Fourier transform (FFT).
- Each line is substituted with its Fourier transform, and the one-dimensional discrete Fourier transform of each column is computed.

Spatial frequencies spectrum

The outcome of the Fourier transform $F(u, v)$ is a function of complex variables.

(Complex) spectrum $F(u, v) = R(u, v) + i\, I(u, v)$

Amplitude spectrum $|F(u, v)| = \sqrt{R^2(u, v) + I^2(u, v)}$

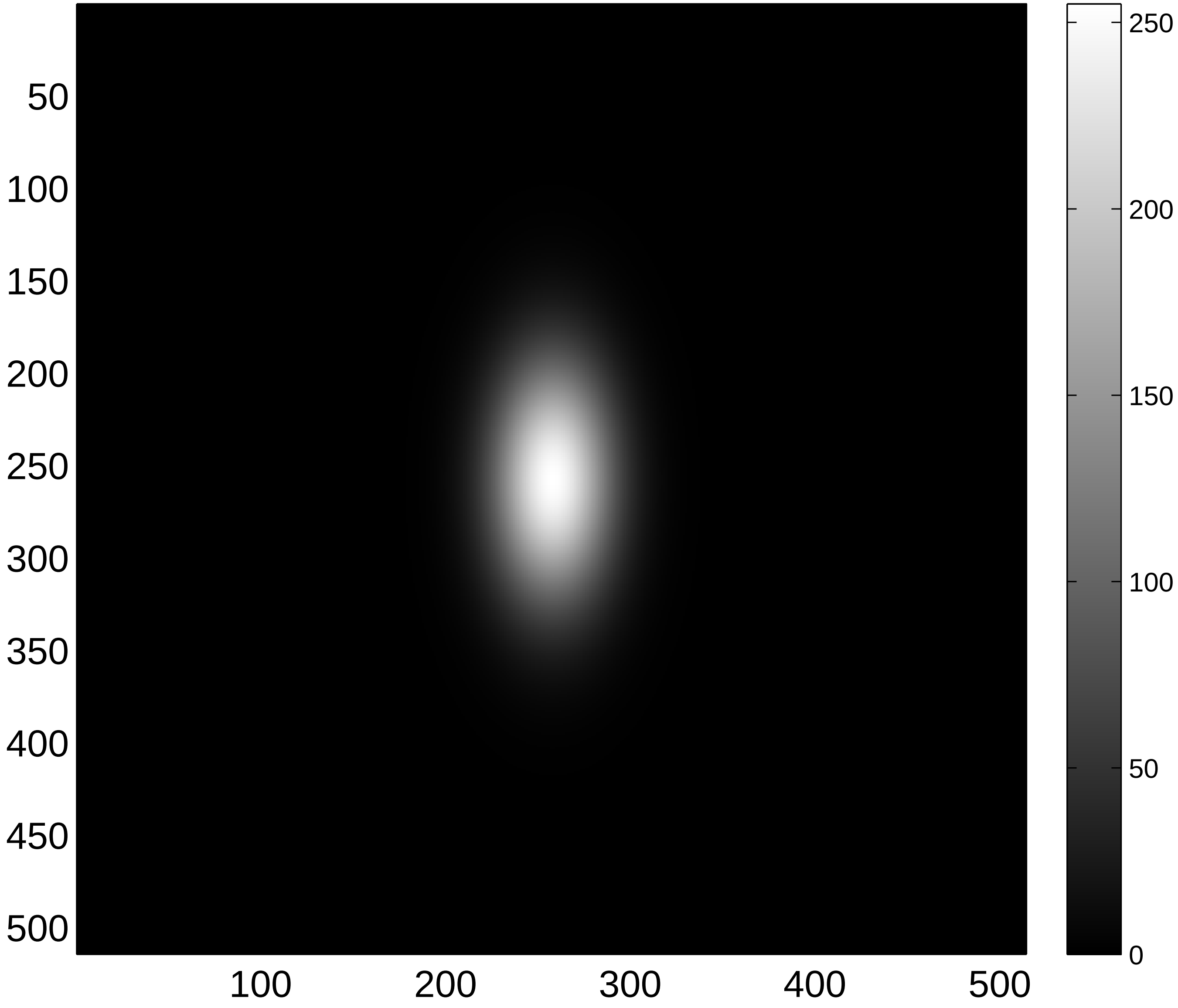
Phase spectrum $\phi(u, v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]$

Power spectrum $P(u, v) = |F(u, v)|^2 = R^2(u, v) + I^2(u, v)$

Displaying spectra, 2D Gaussian example

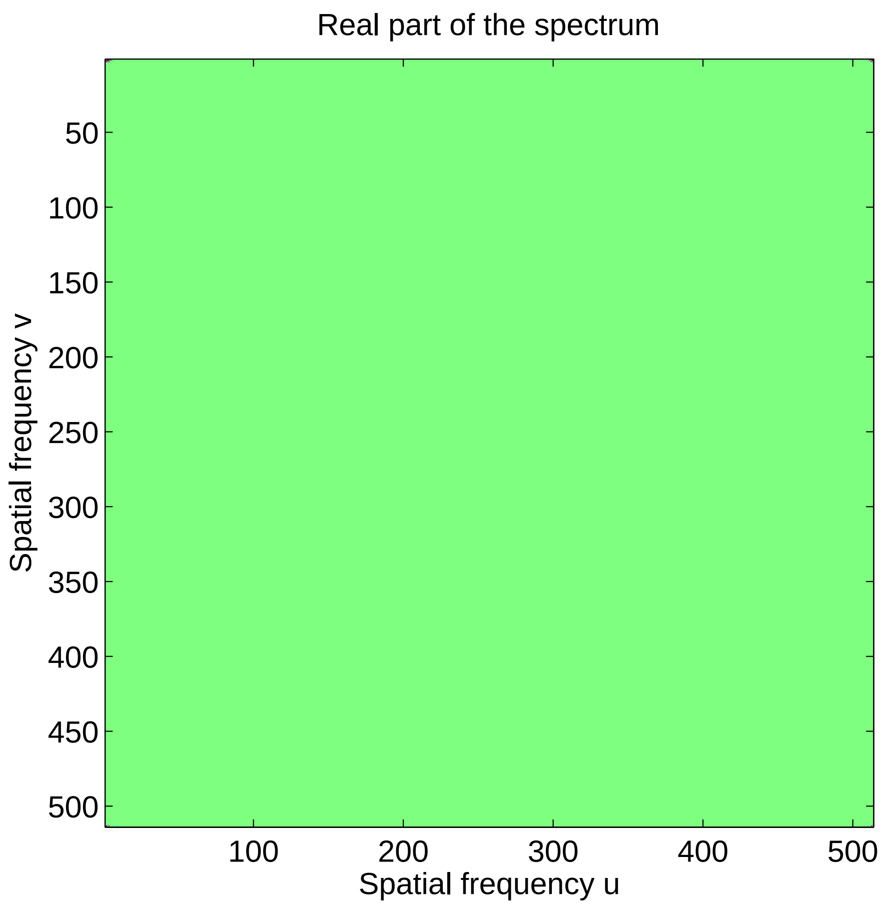
Gaussian is selected for illustration because it has a smooth spectrum, cf. uncertainty principle.

Input intensity image, coordinate system

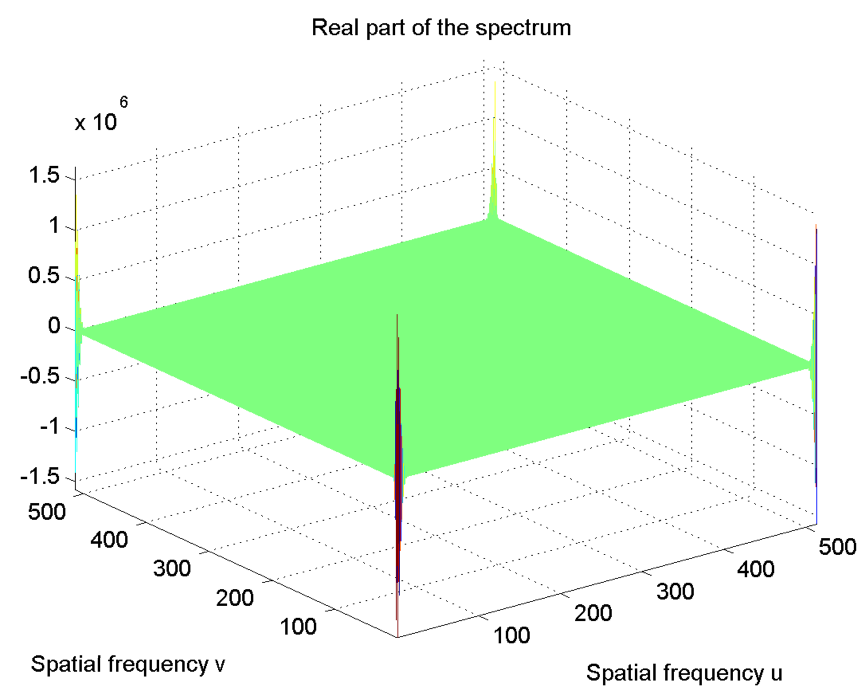


Real part of the spectrum, image and mesh

Problem with the image related coordinate system related to the image: interesting information is in corners, moreover divided into quarters. Due to spectrum periodicity it can be arbitrarily shifted.

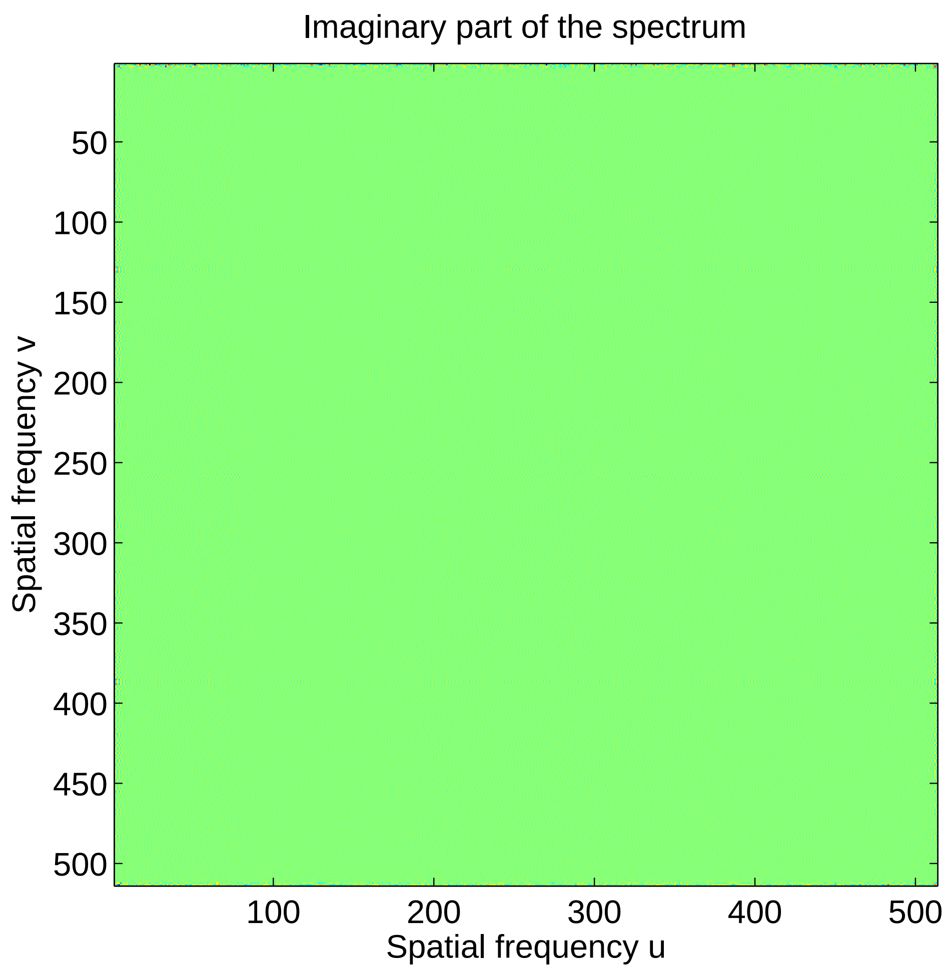


real part, image

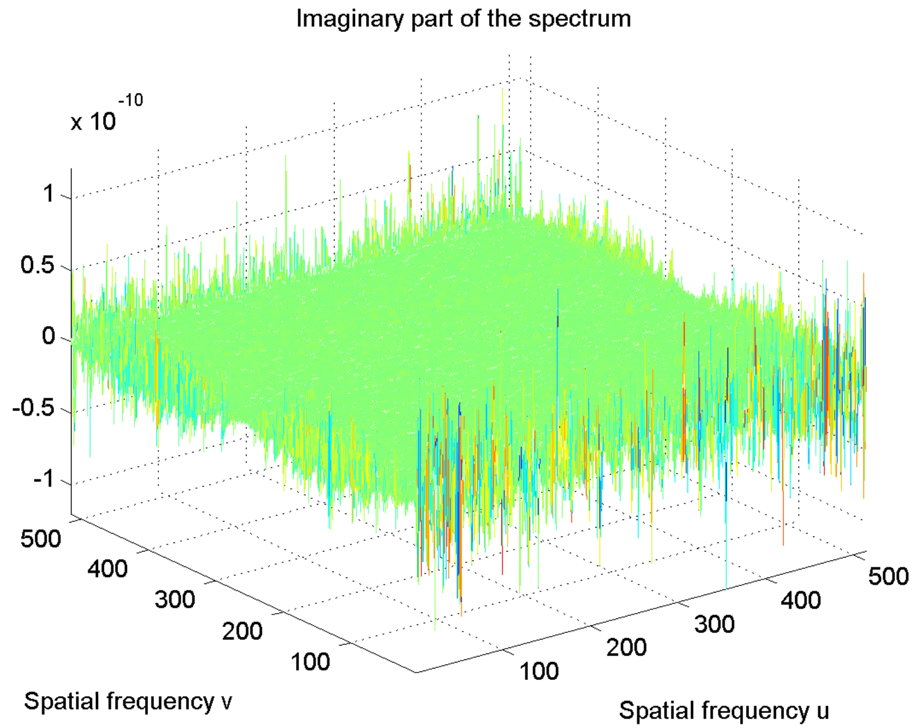


real part, mesh

Imaginary part of the spectrum, image and mesh

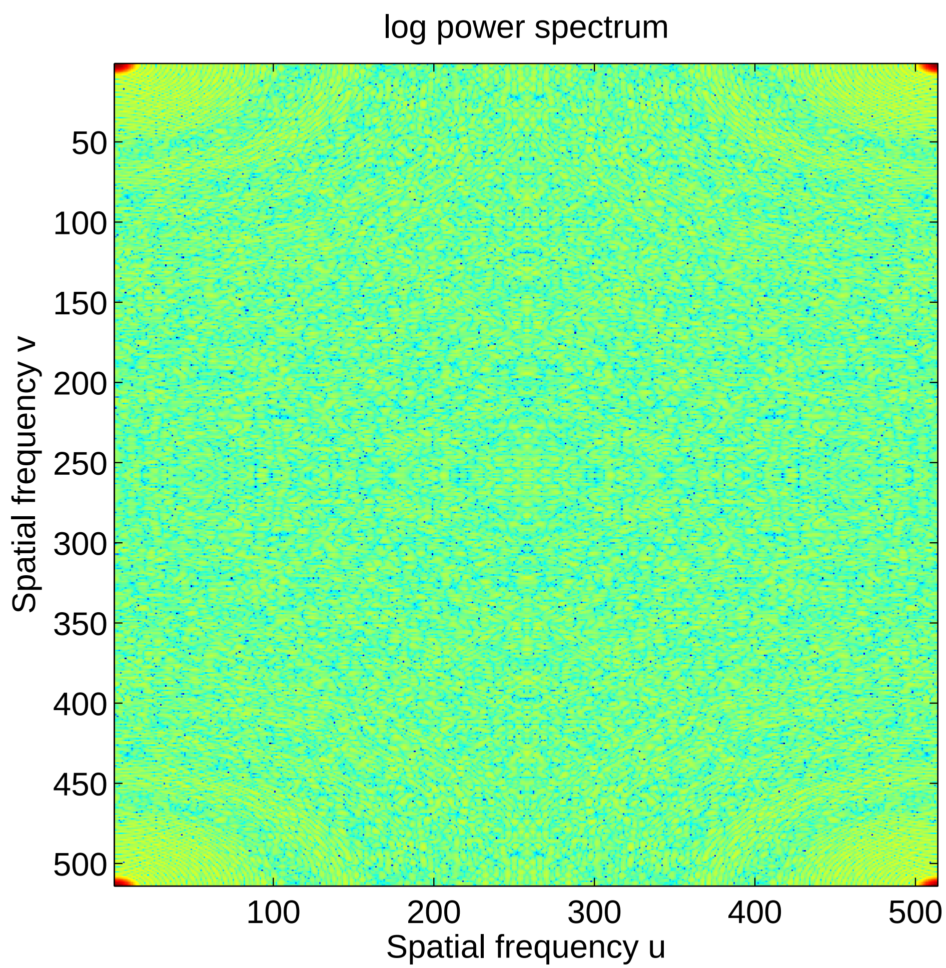


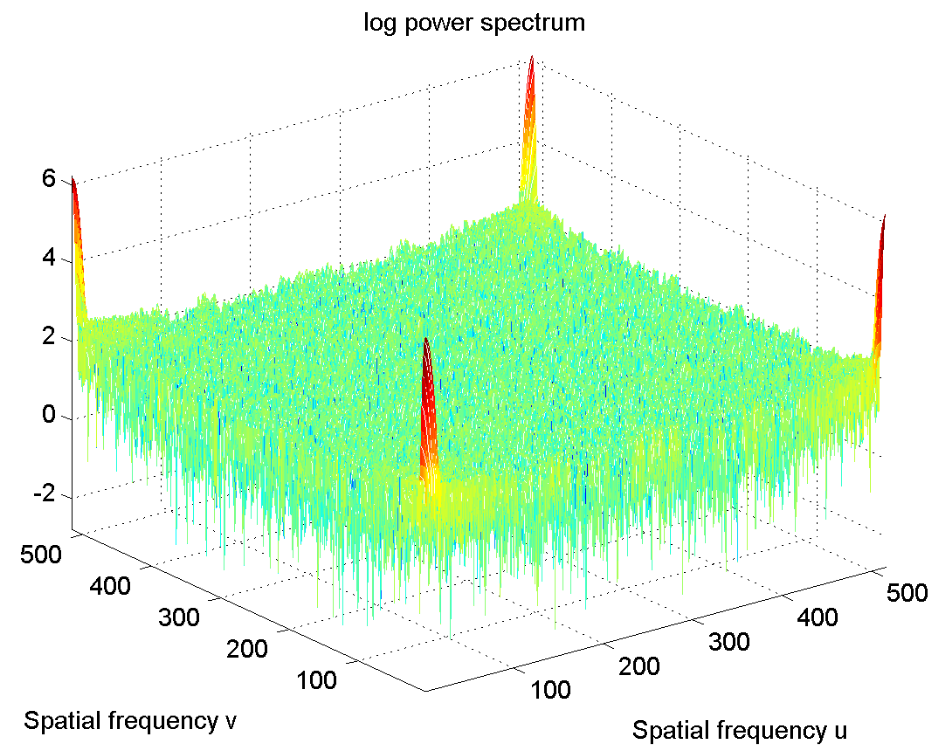
imaginary part, image



imaginary part, mesh

Log power of the spectrum, image and mesh



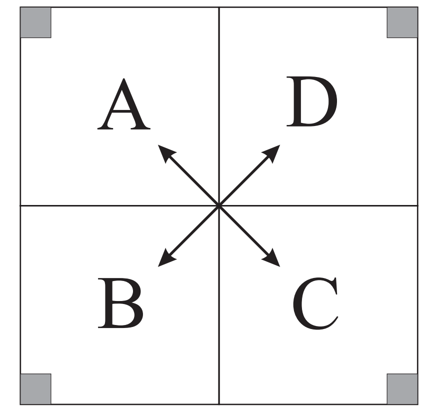


image

mesh

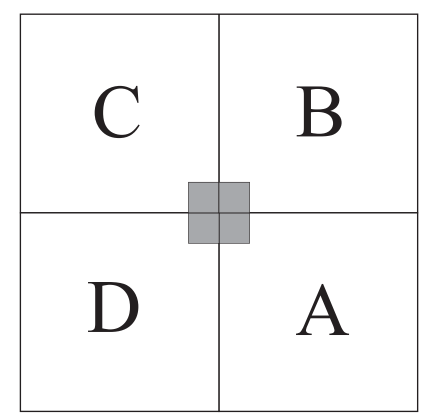
Centered spectra

- It is useful to visualize a centered spectrum with the origin of the coordinate system $(0, 0)$ in the middle of the spectrum.
- Assume the original spectrum is divided into four quadrants. The small gray-filled squares in the corners represent positions of low frequencies.
- Due to the symmetries of the spectrum the quadrant positions can be swapped diagonally and the low frequencies locations appear in the middle of the image.
- MATLABu provides function `fftshift` which converts nencentered $\longleftrightarrow$ centered spectra by switching quadrants diagonally.



original spectrum

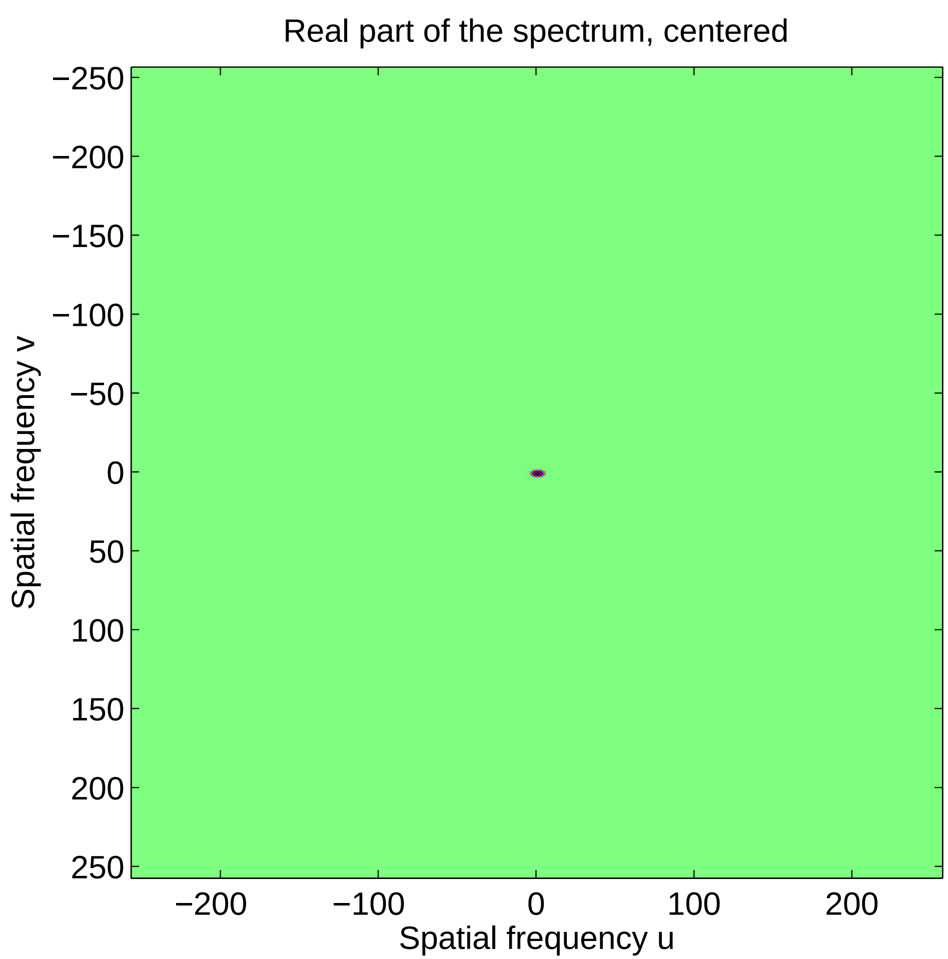
low frequencies in corners



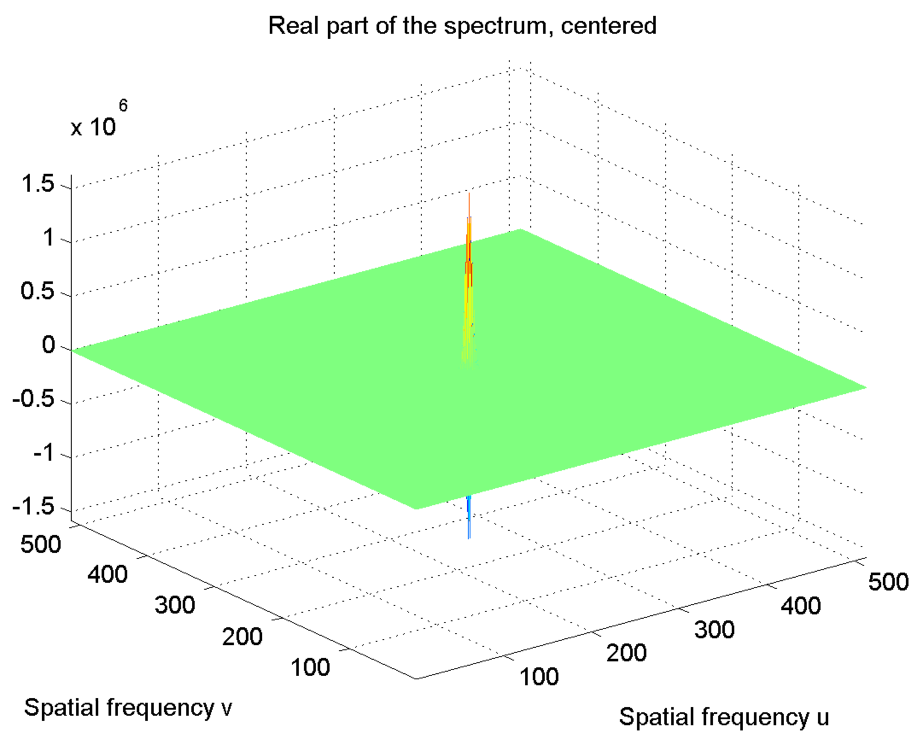
shifted spectrum

with the origin at $(0, 0)$

Real part of the centered spectrum, image and mesh

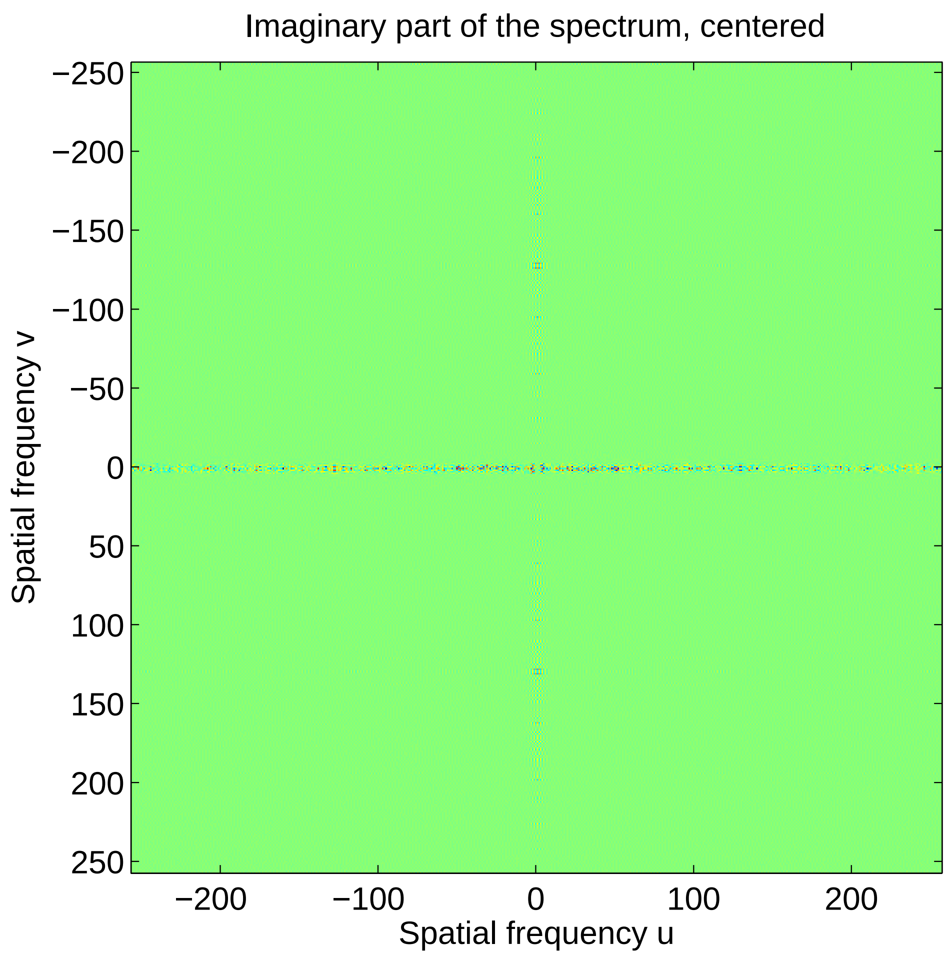


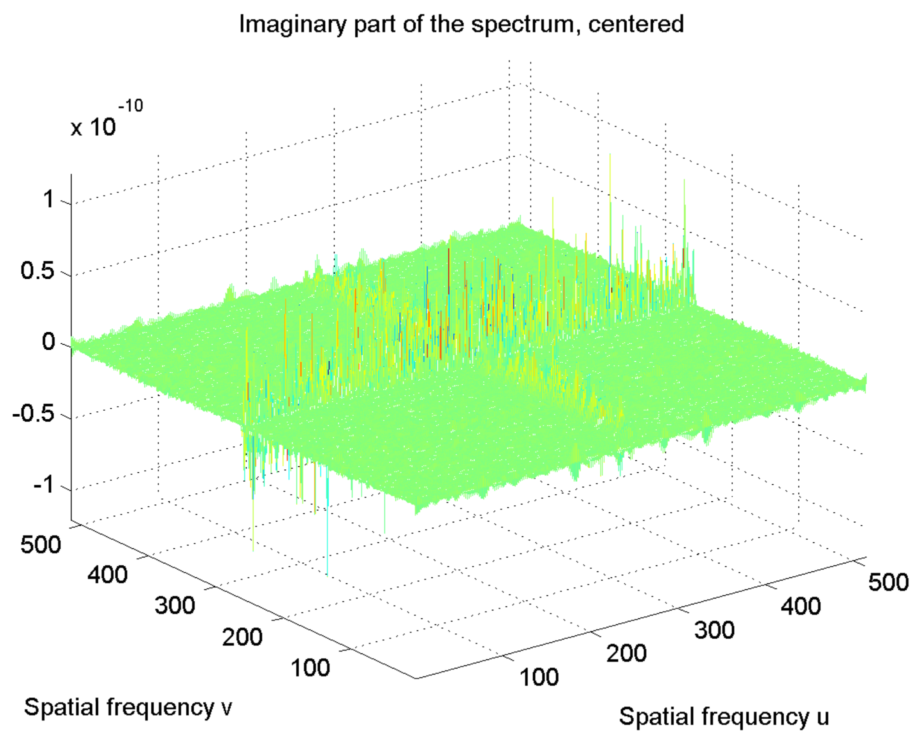
real part, image



real part, mesh

Imaginary part of the centered spectrum image and mesh

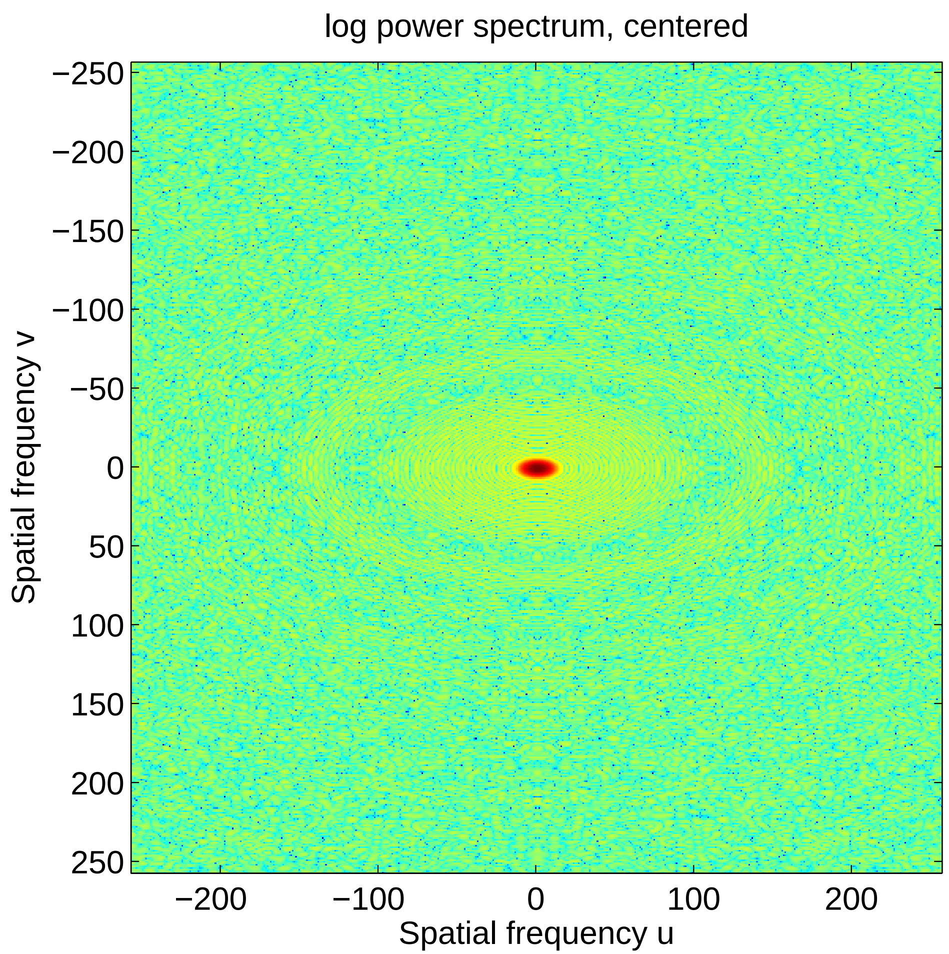


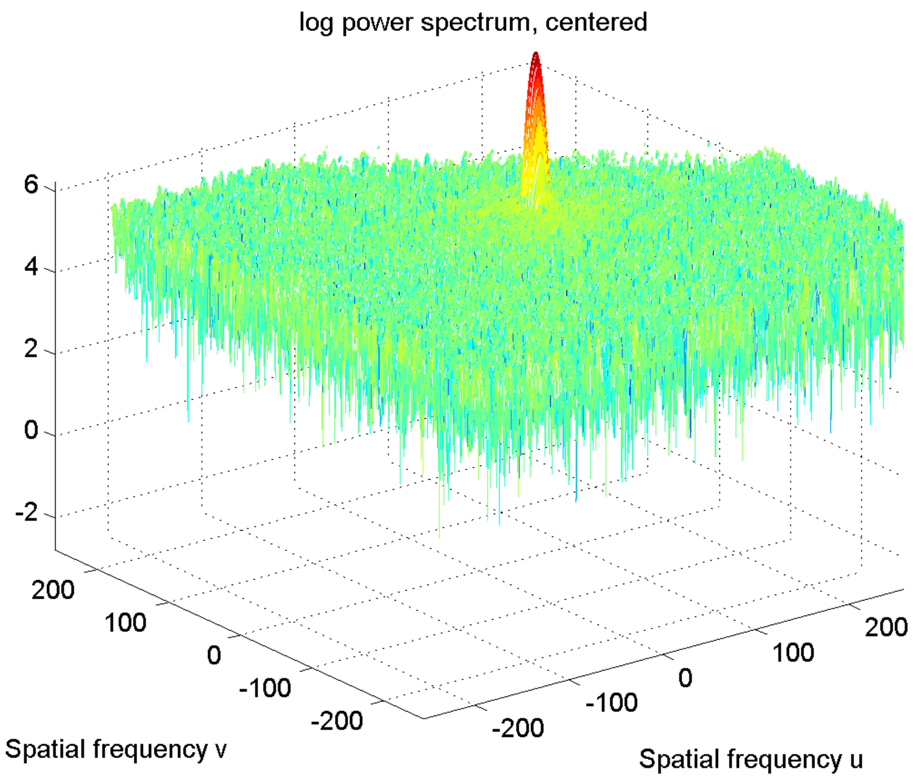


imaginary part, image

imaginary part, mesh

Log power of the centered spectrum image and mesh

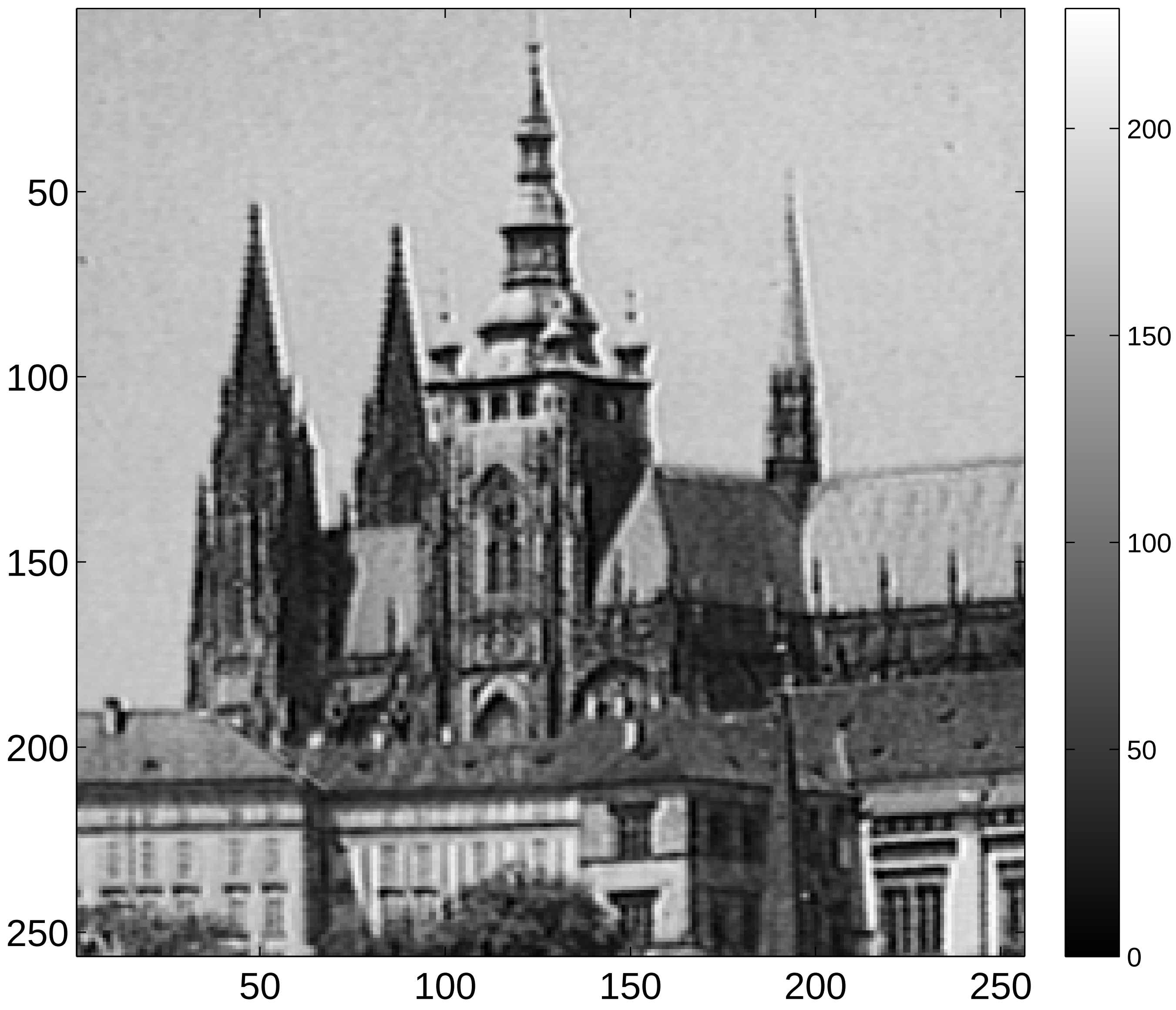




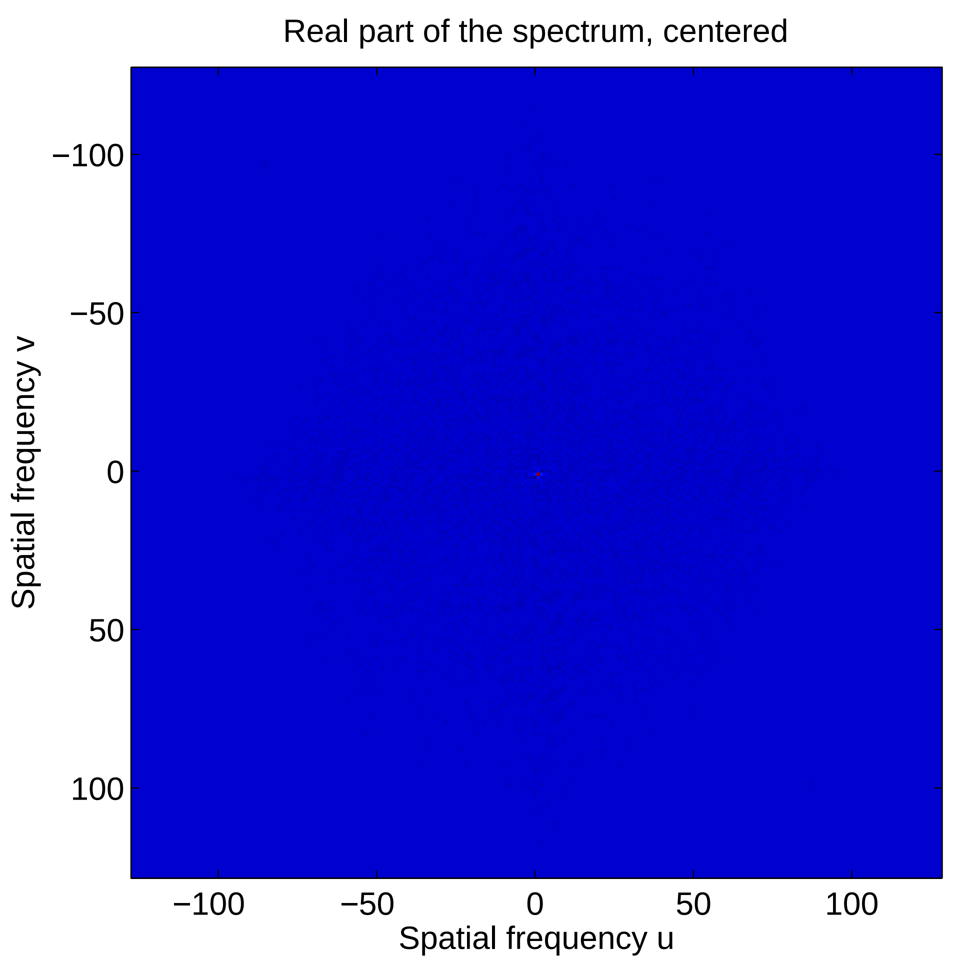
image

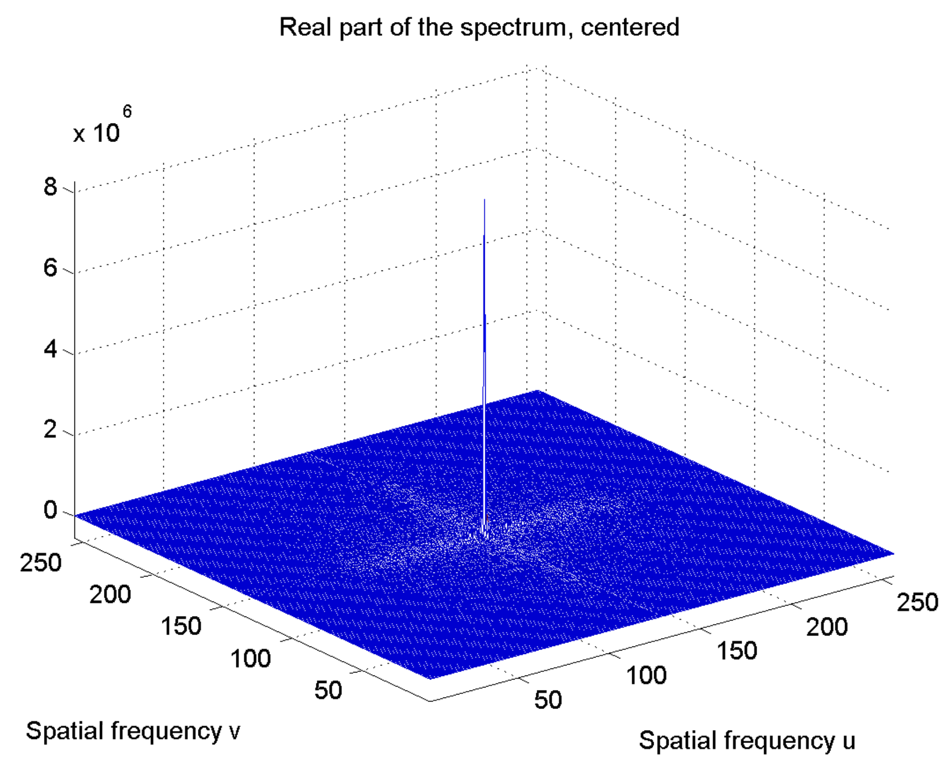
mesh

Prague Castle example, input image 265×256



Real part of the centered spectrum, image and mesh

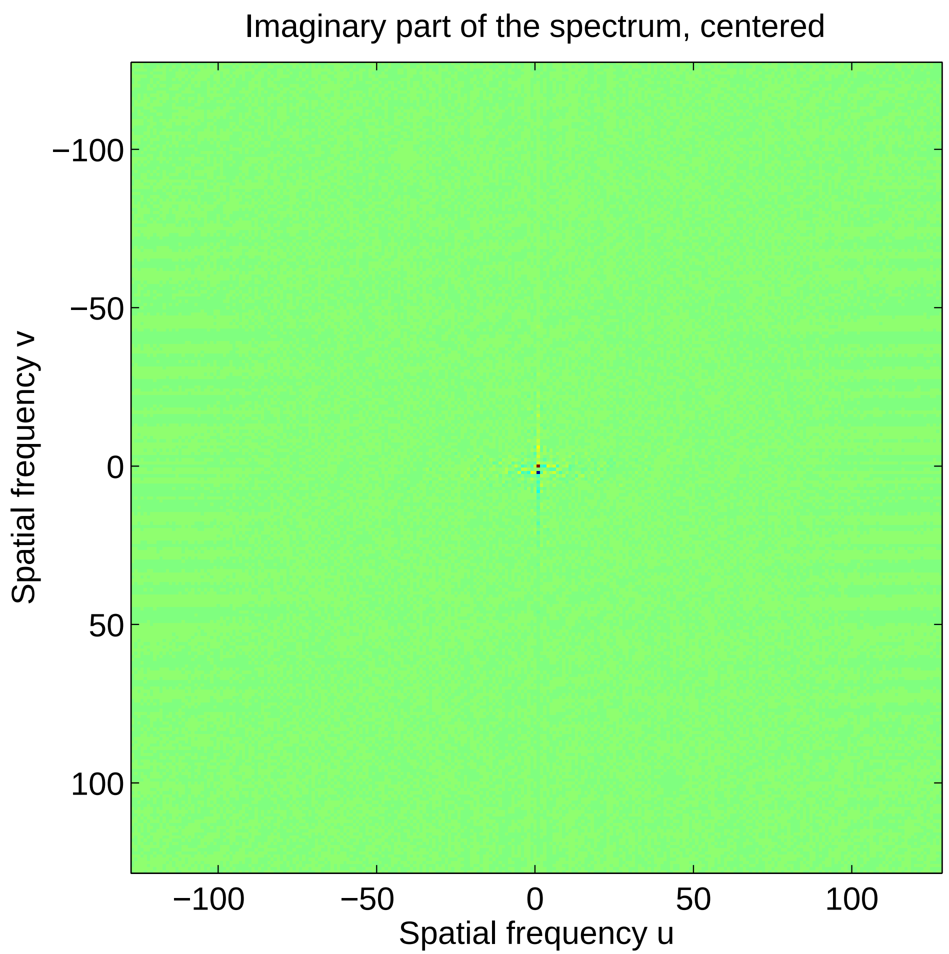


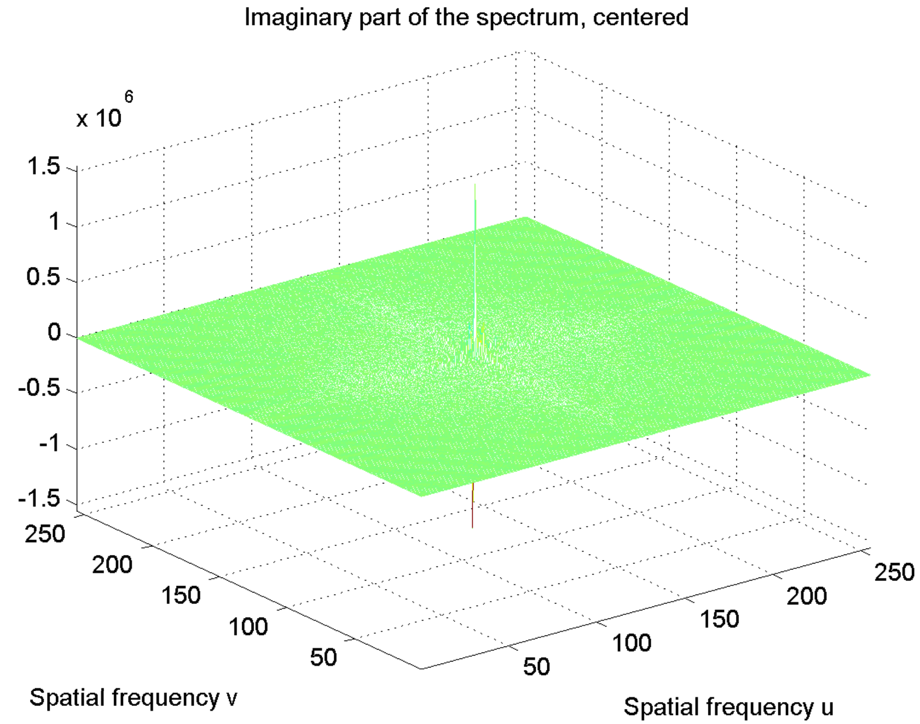


real part, image

real part, mesh

Imaginary part of the centered spectrum image and mesh

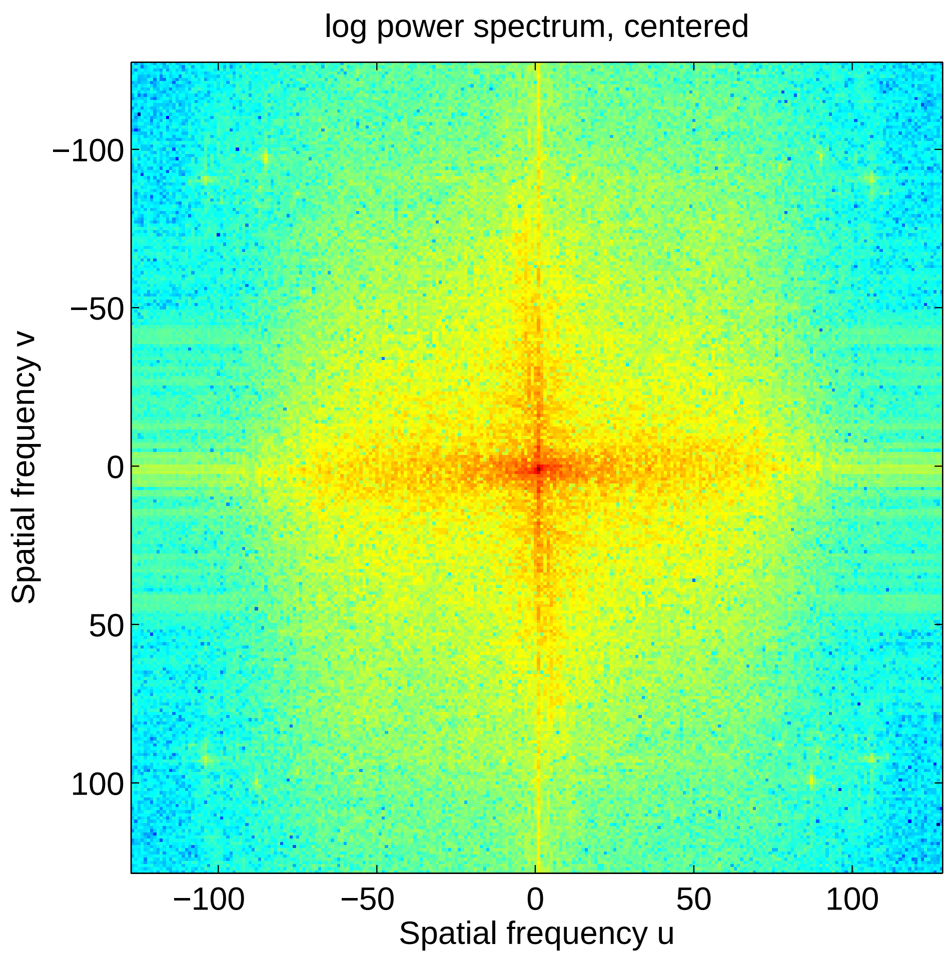




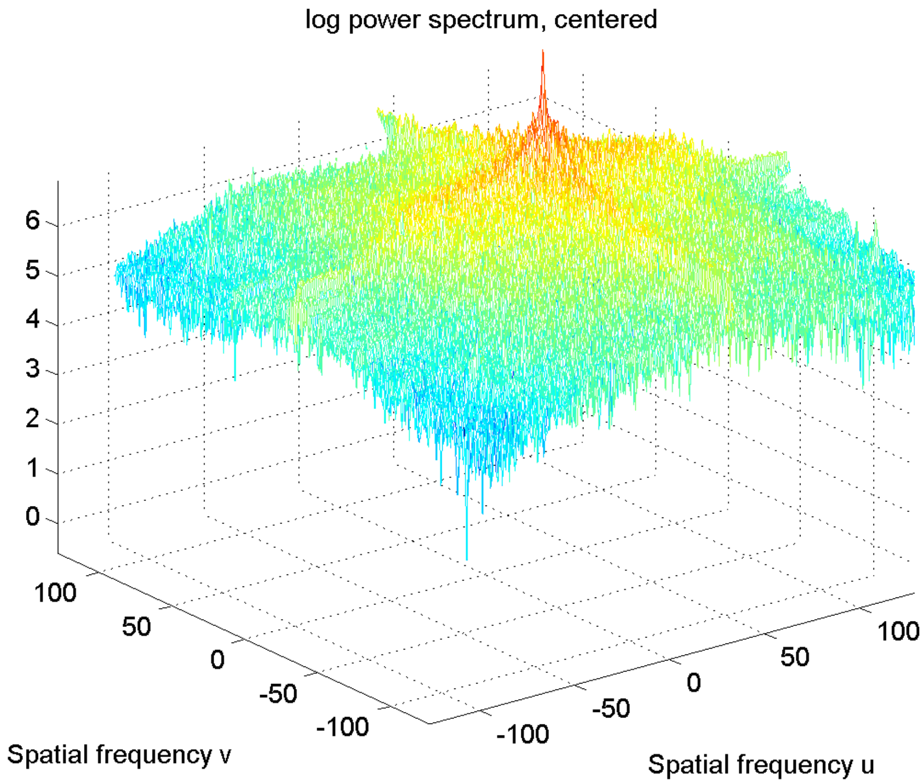
imaginary part, image

imaginary part, mesh

Log power of the centered spectrum image and mesh

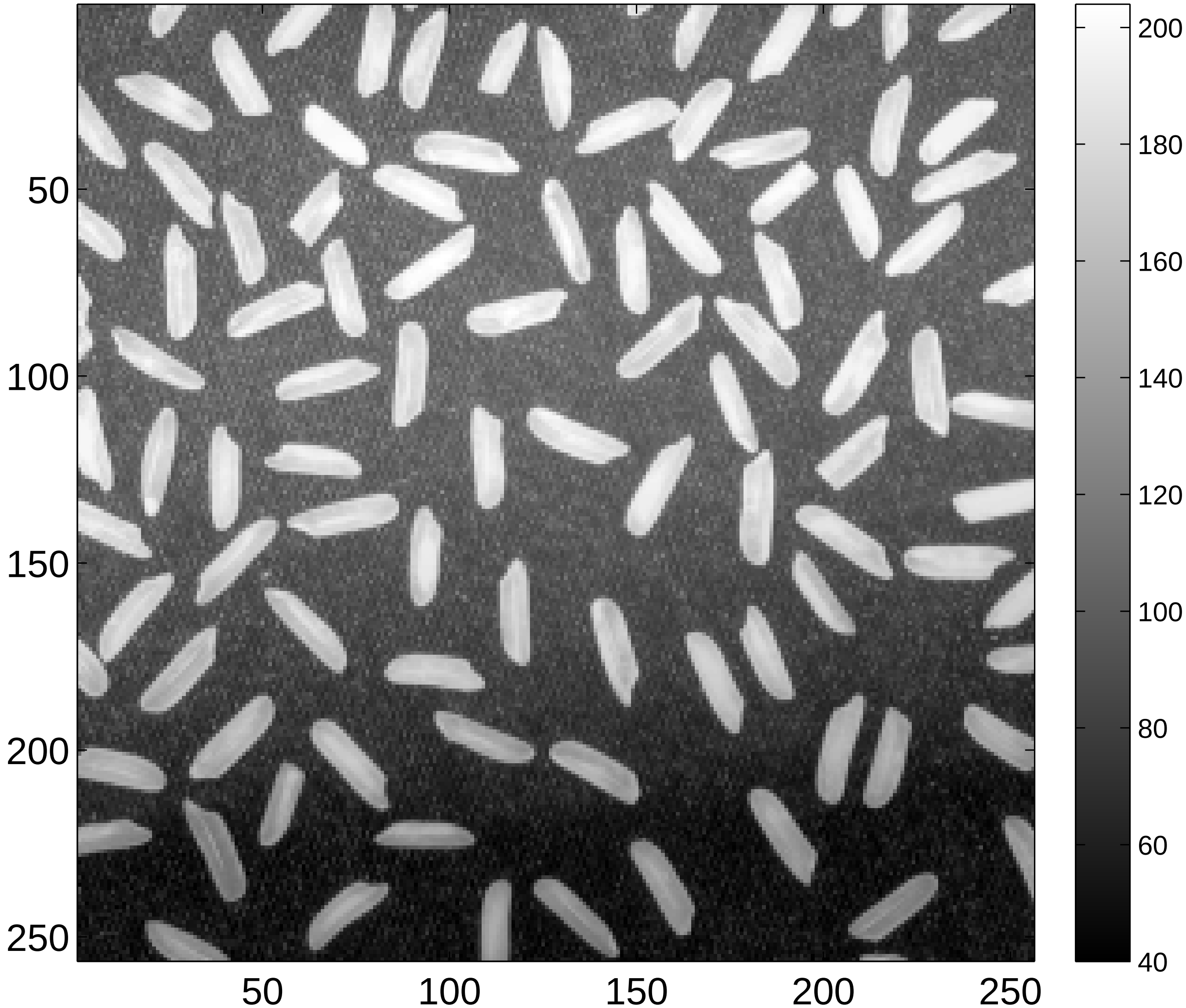


image

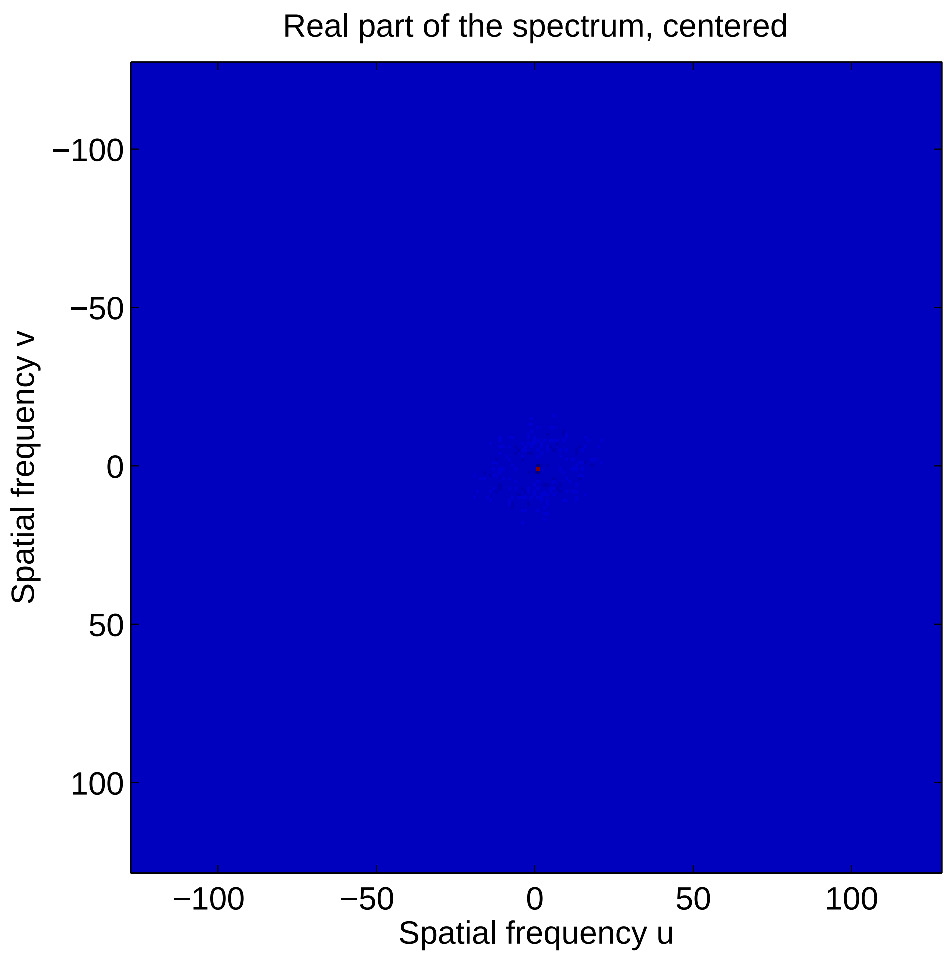


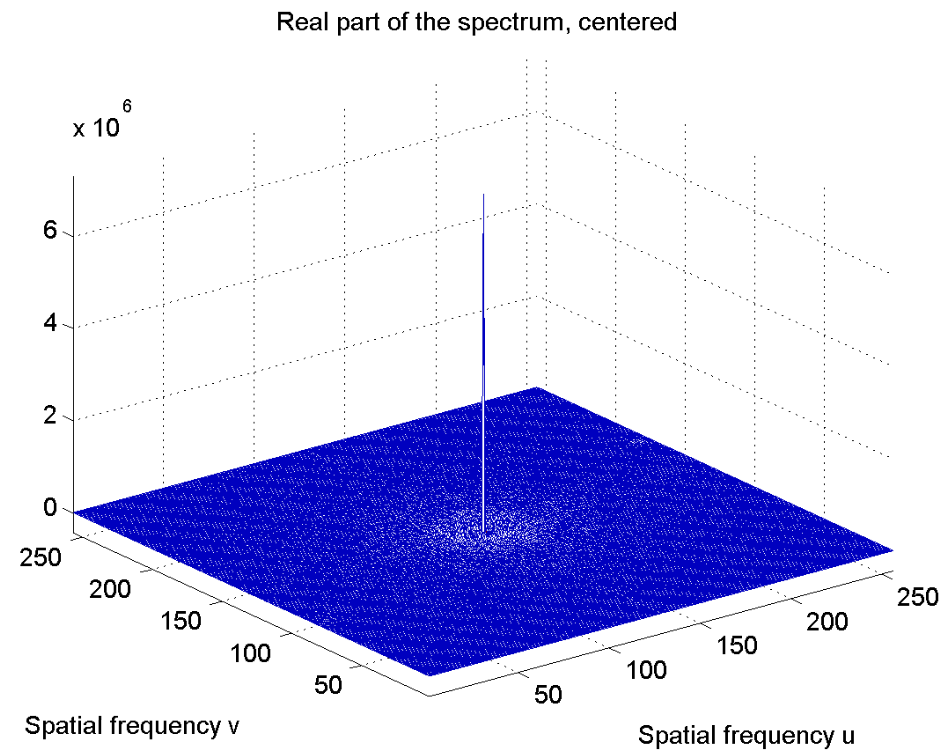
mesh

Rice example, input image 265×256



Real part of the centered spectrum, image and mesh

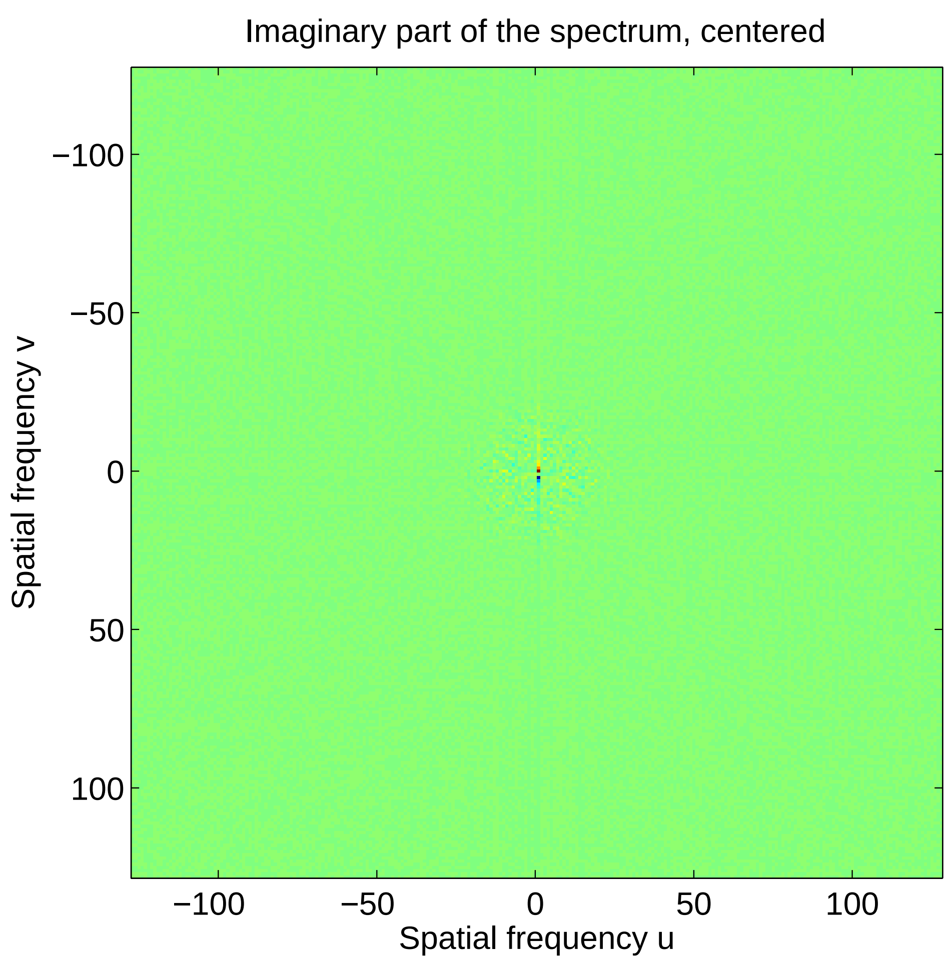




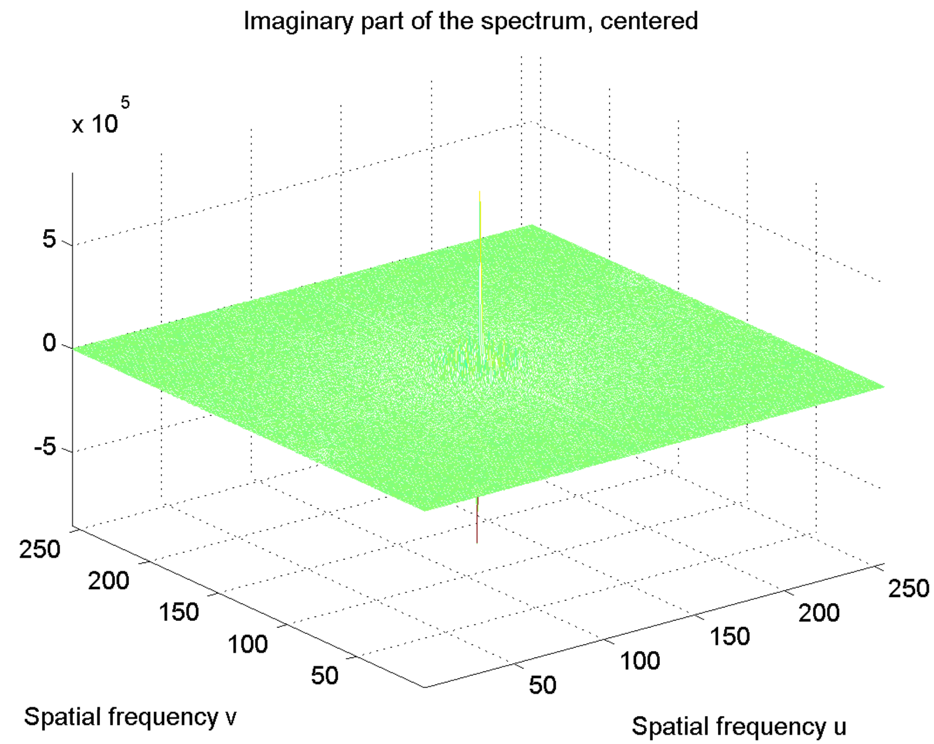
real part, image

real part, mesh

Imaginary part of the centered spectrum image and mesh

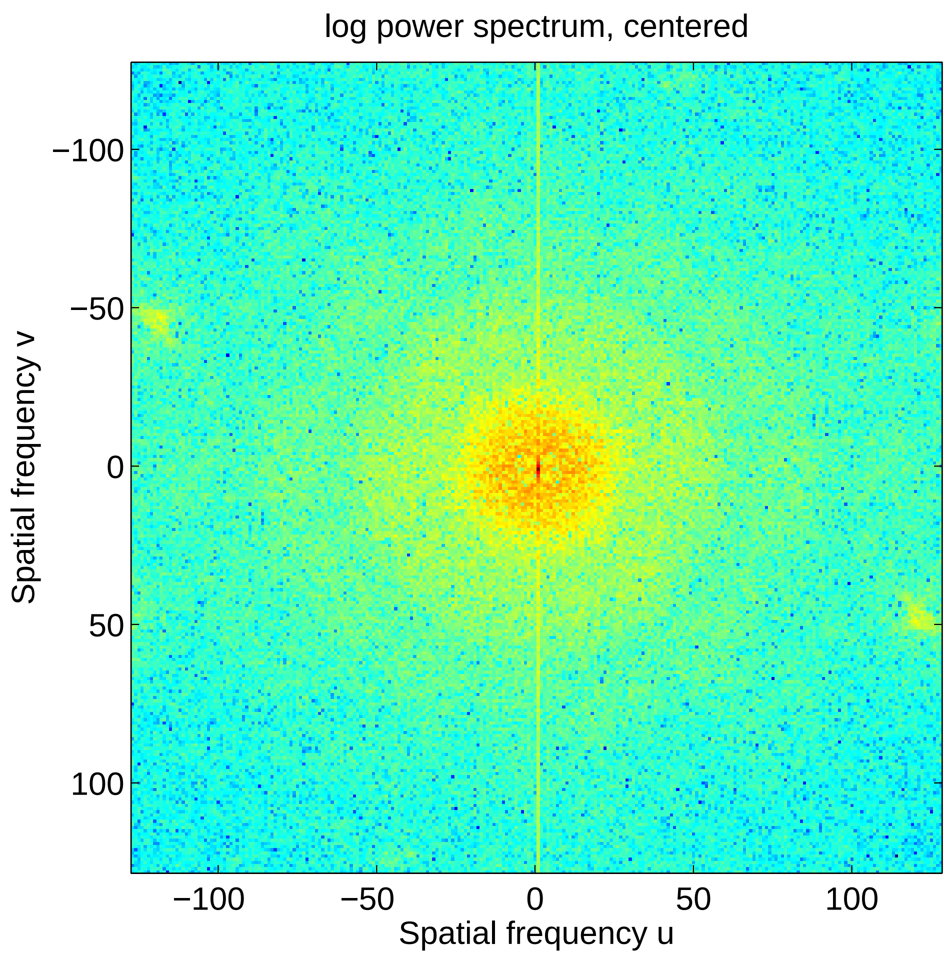


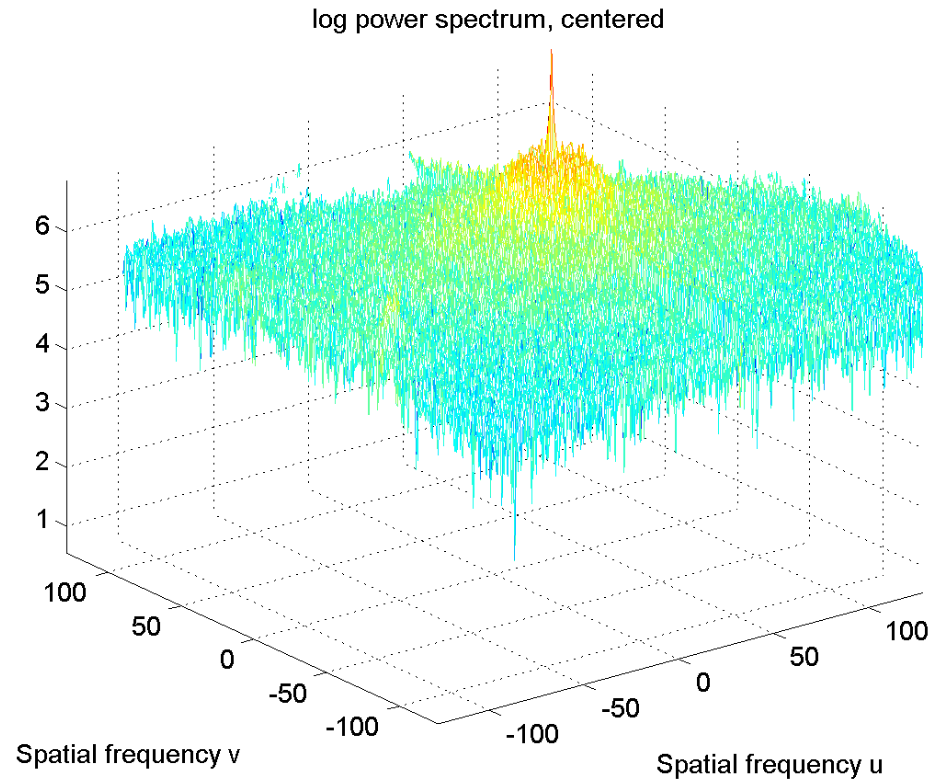
imaginary part, image



imaginary part, mesh

Log power of the centered spectrum image and mesh



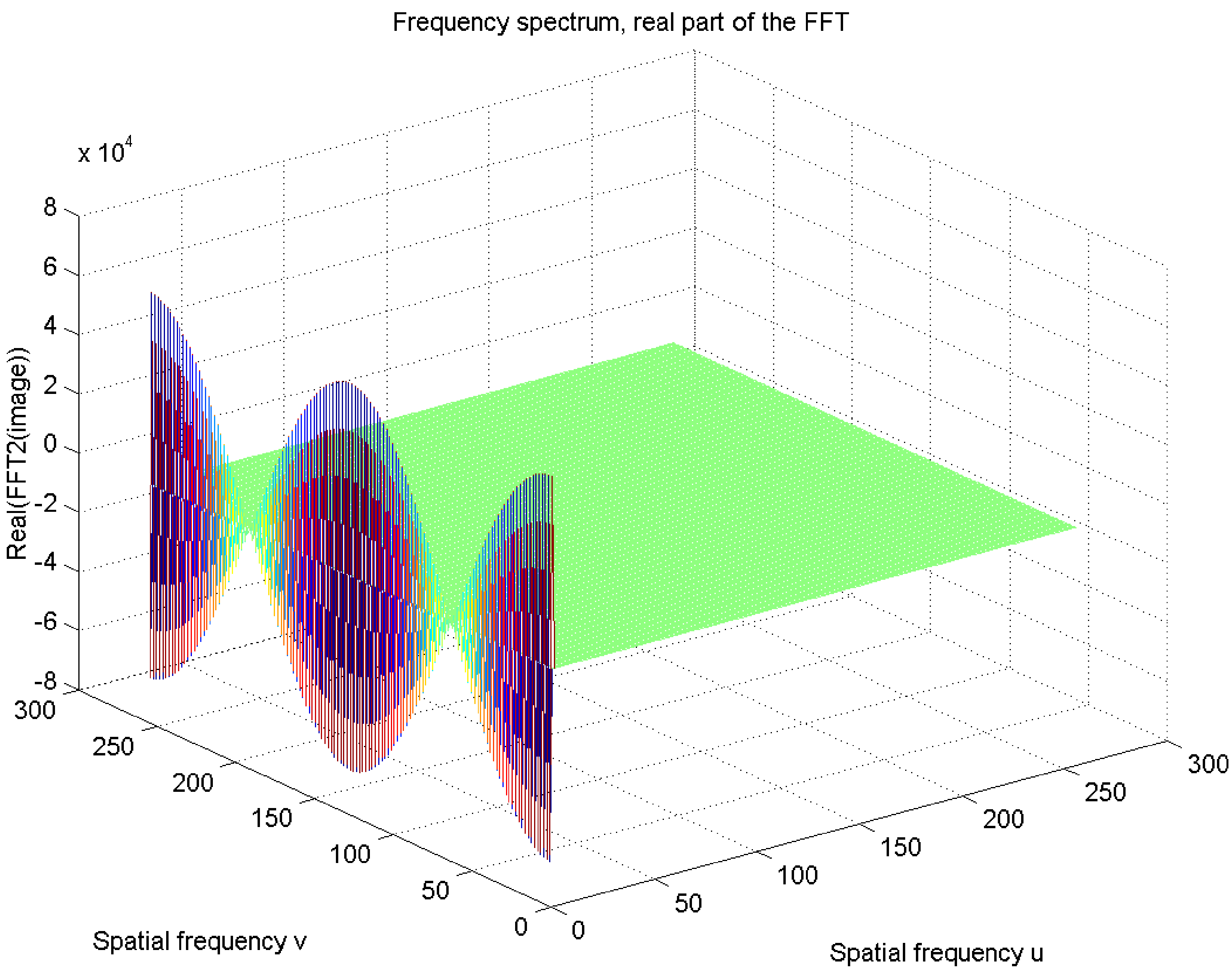


image

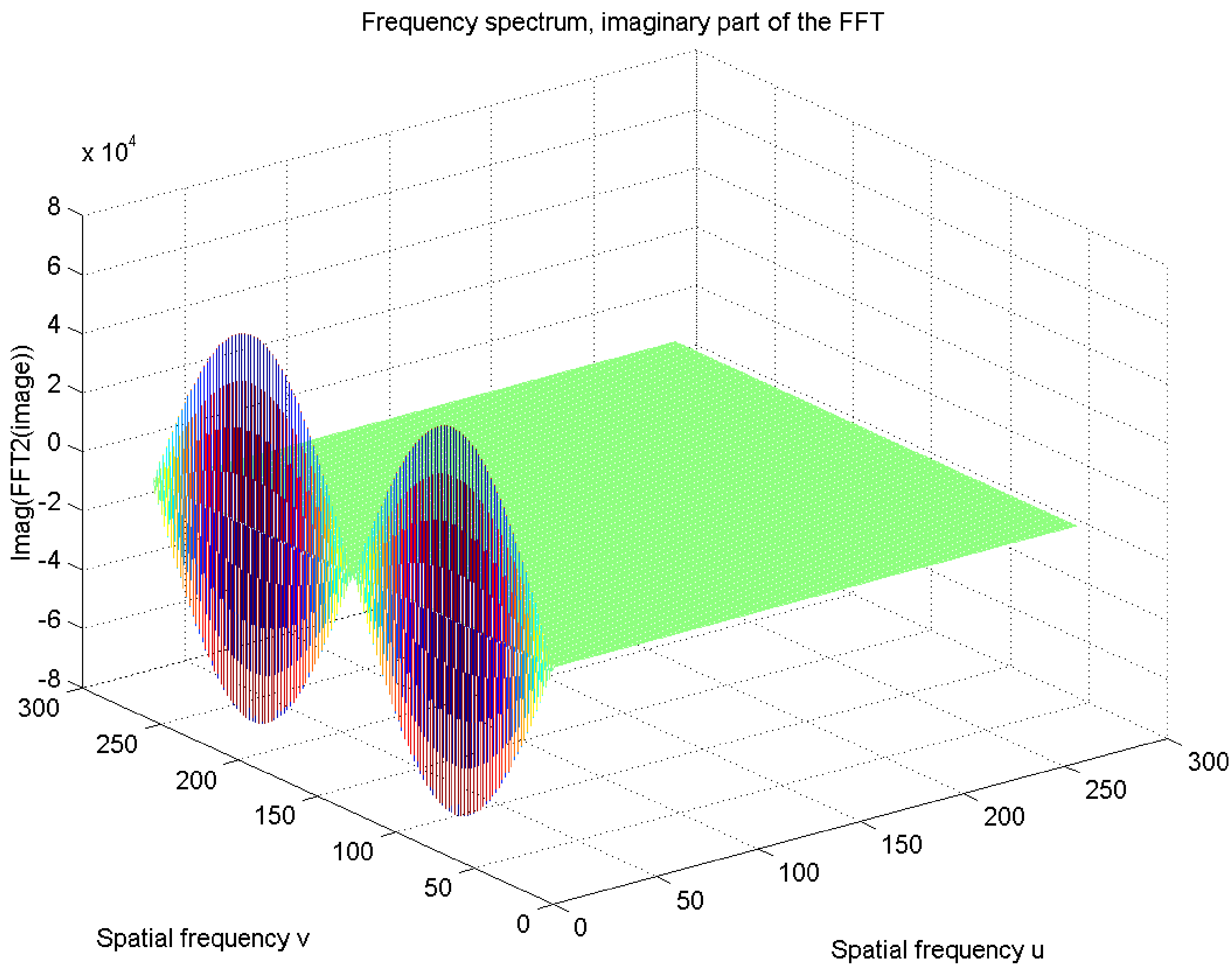
mesh

Horizontal line example, input image 265×256

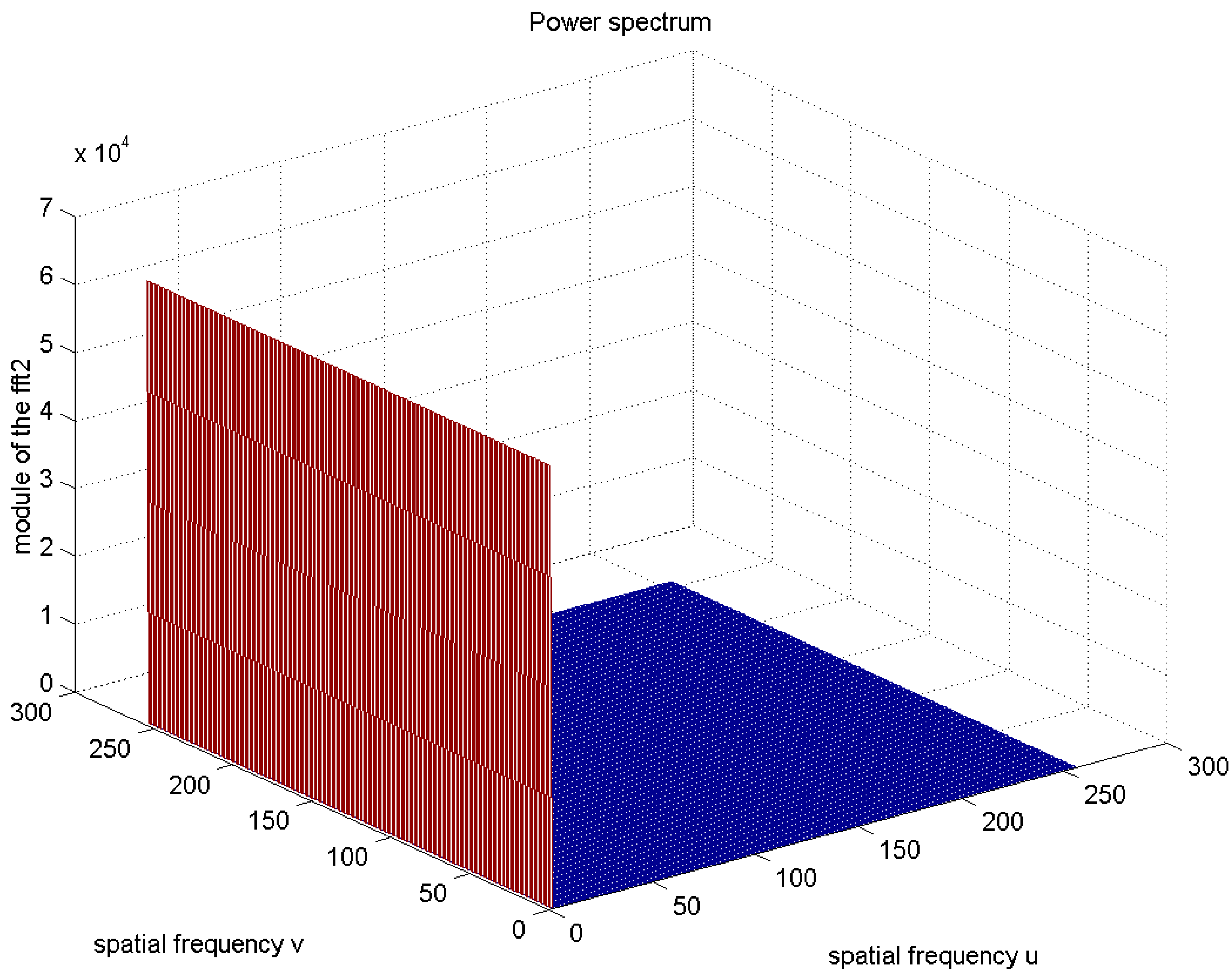
Horizontal line example, real part of the spectrum



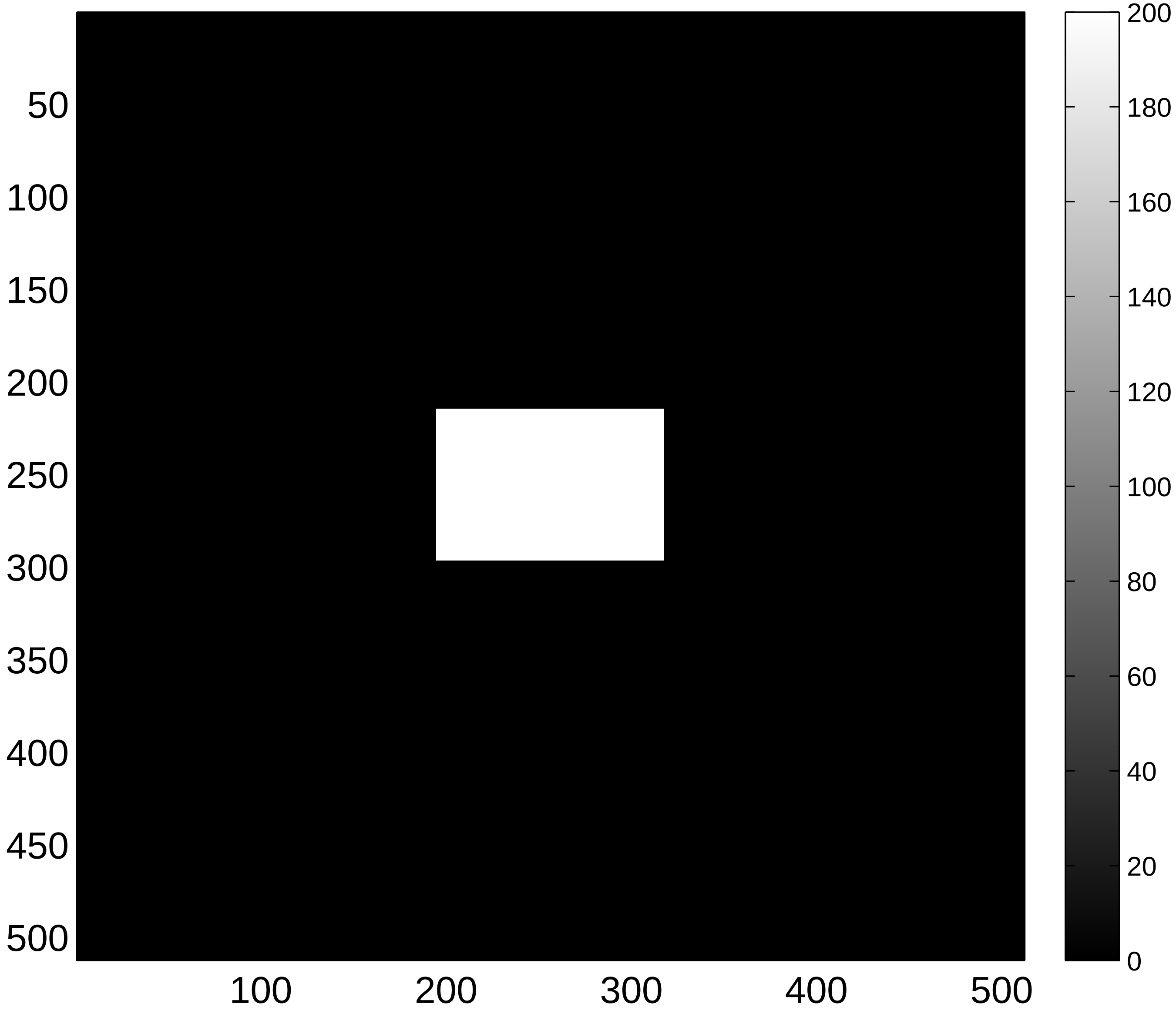
Horizontal line example, imaginary part of the spectrum



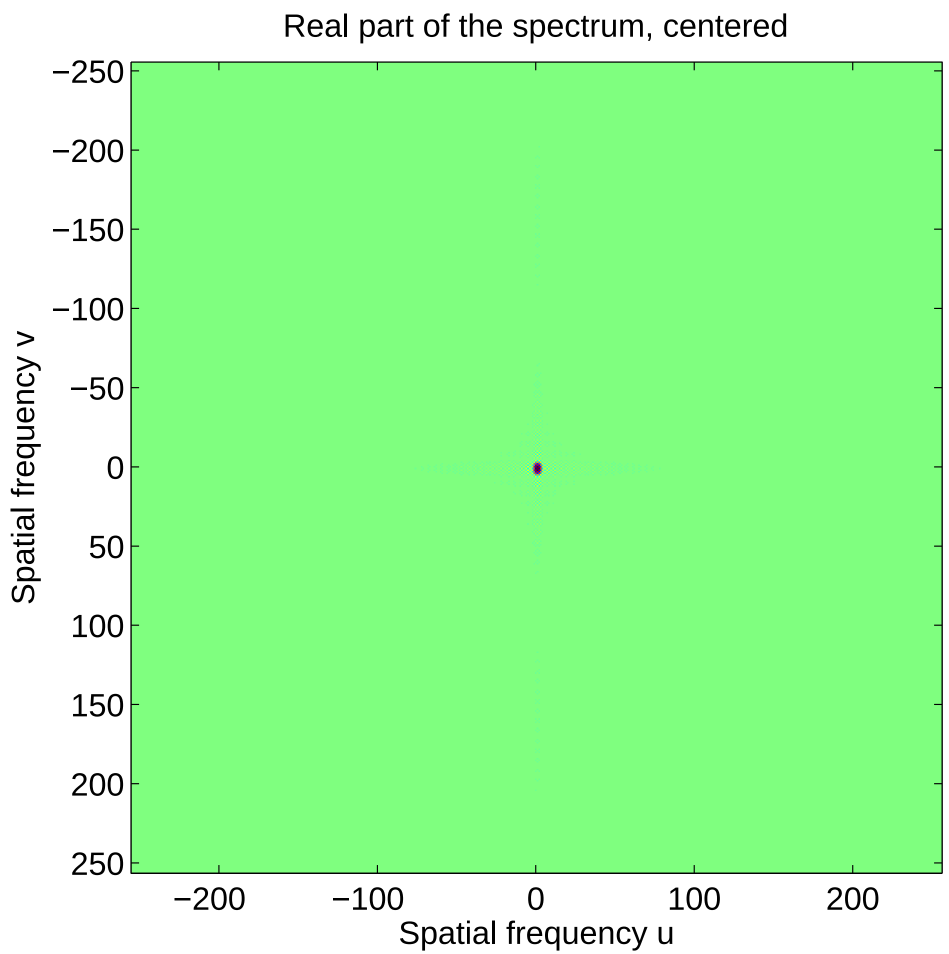
Horizontal line example, power spectrum

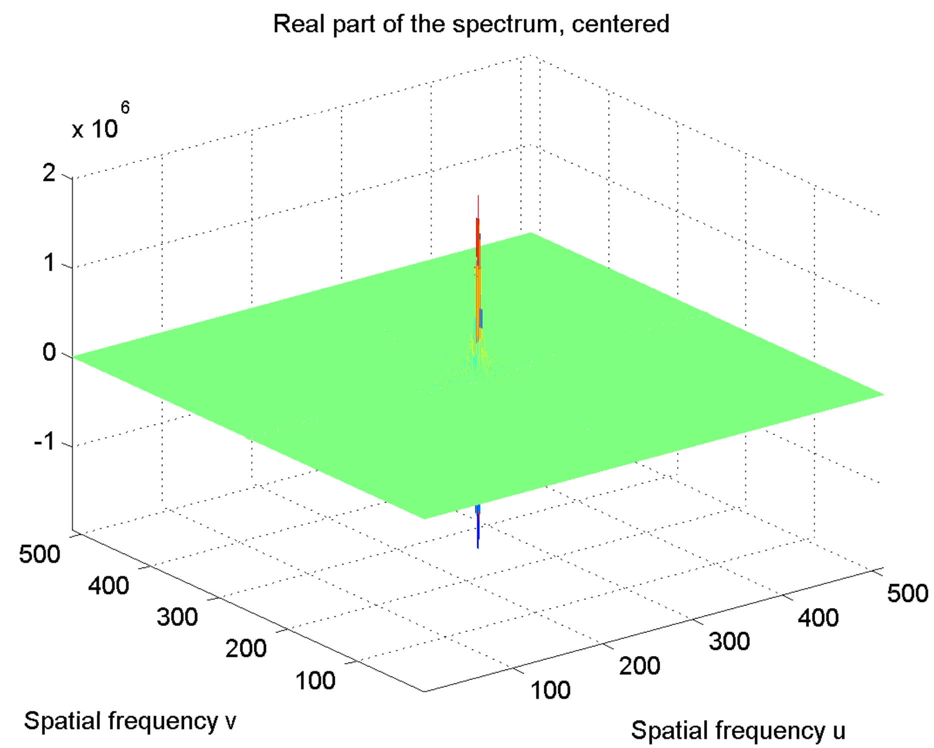


Rectangle example, input image 512×512



Real part of the centered spectrum, image and mesh

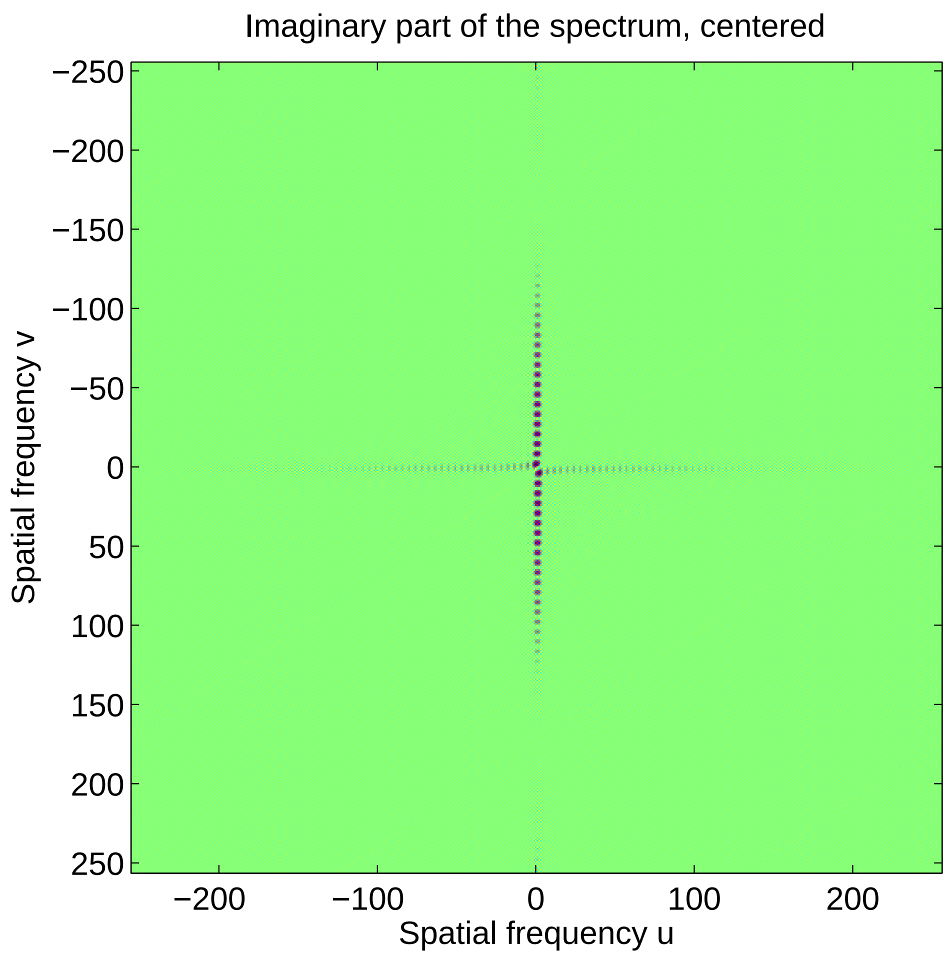




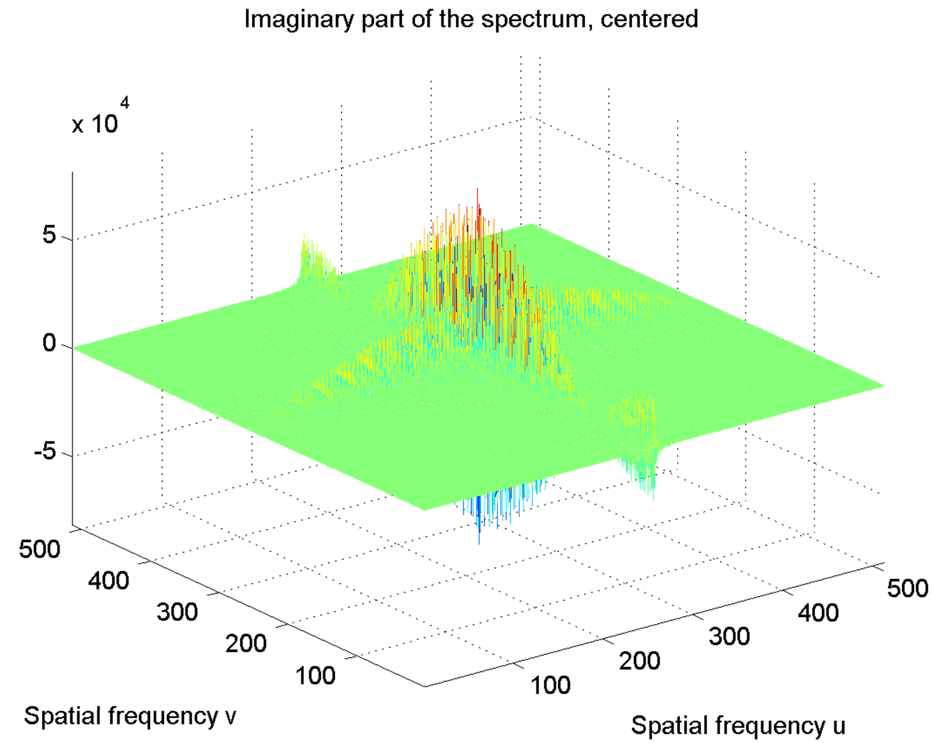
real part, image

real part, mesh

Imaginary part of the centered spectrum image and mesh

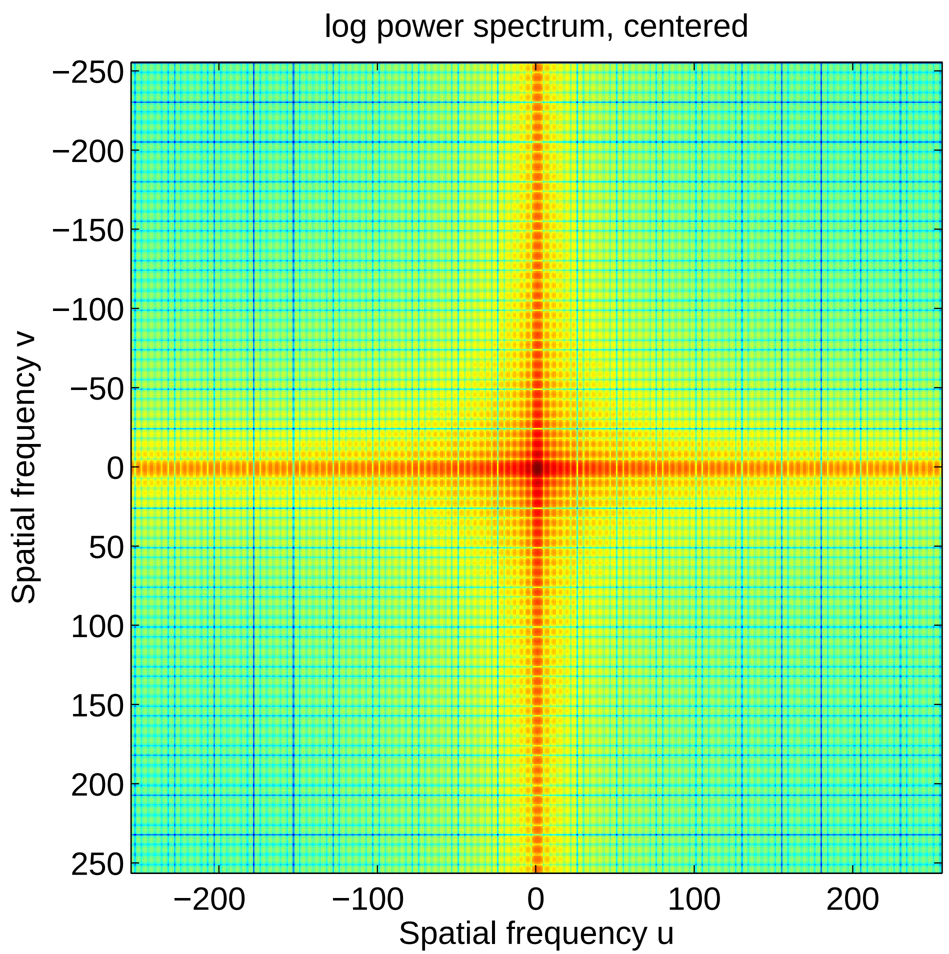


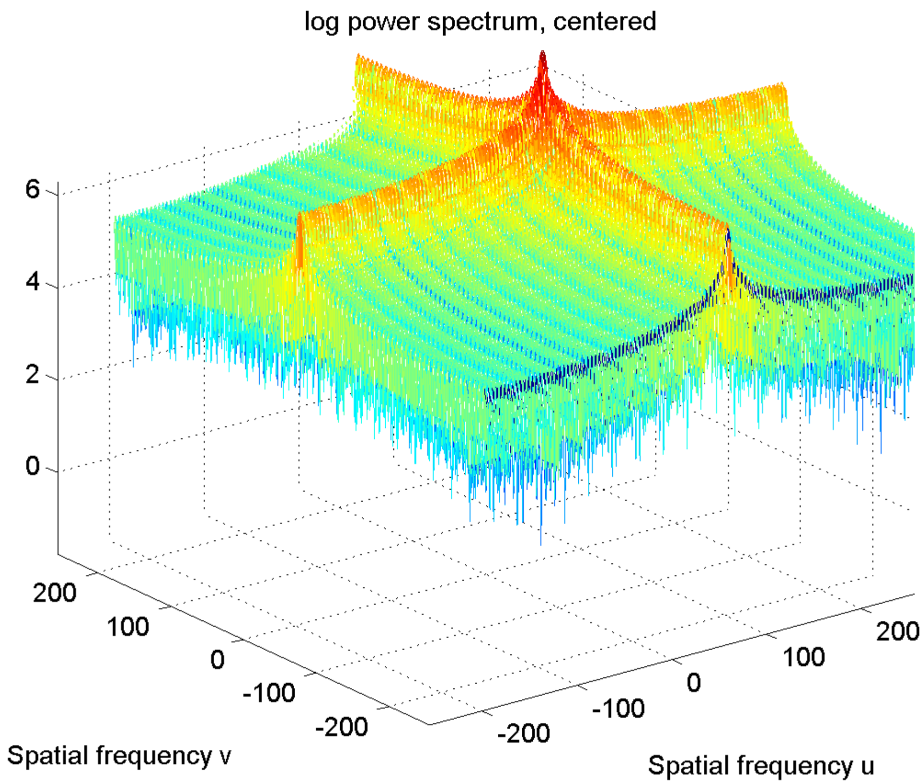
imaginary part, image



imaginary part, mesh

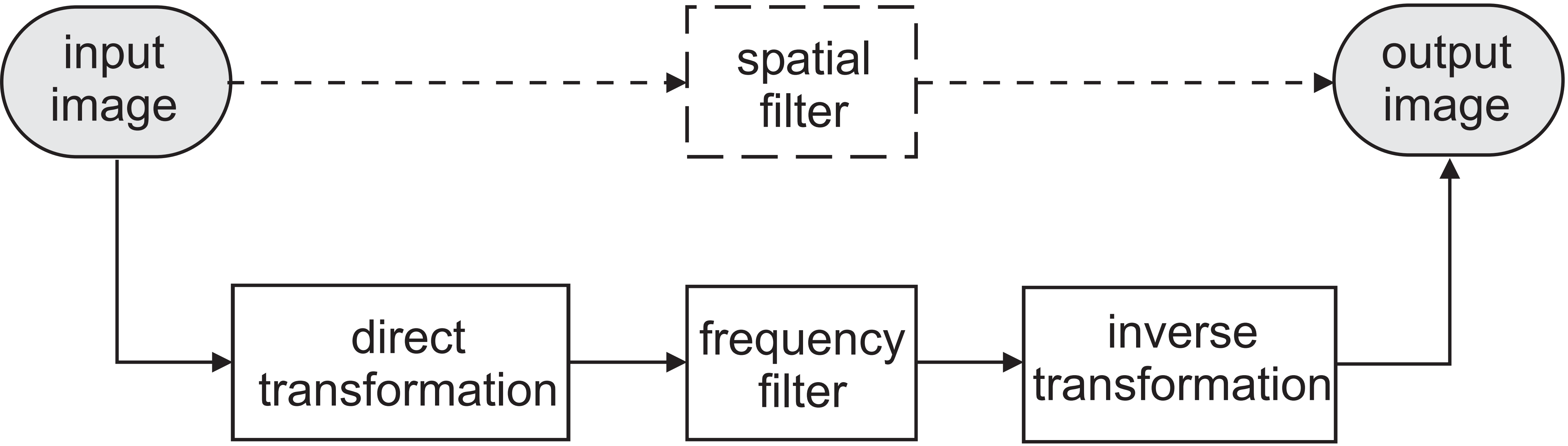
Log power of the centered spectrum image and mesh

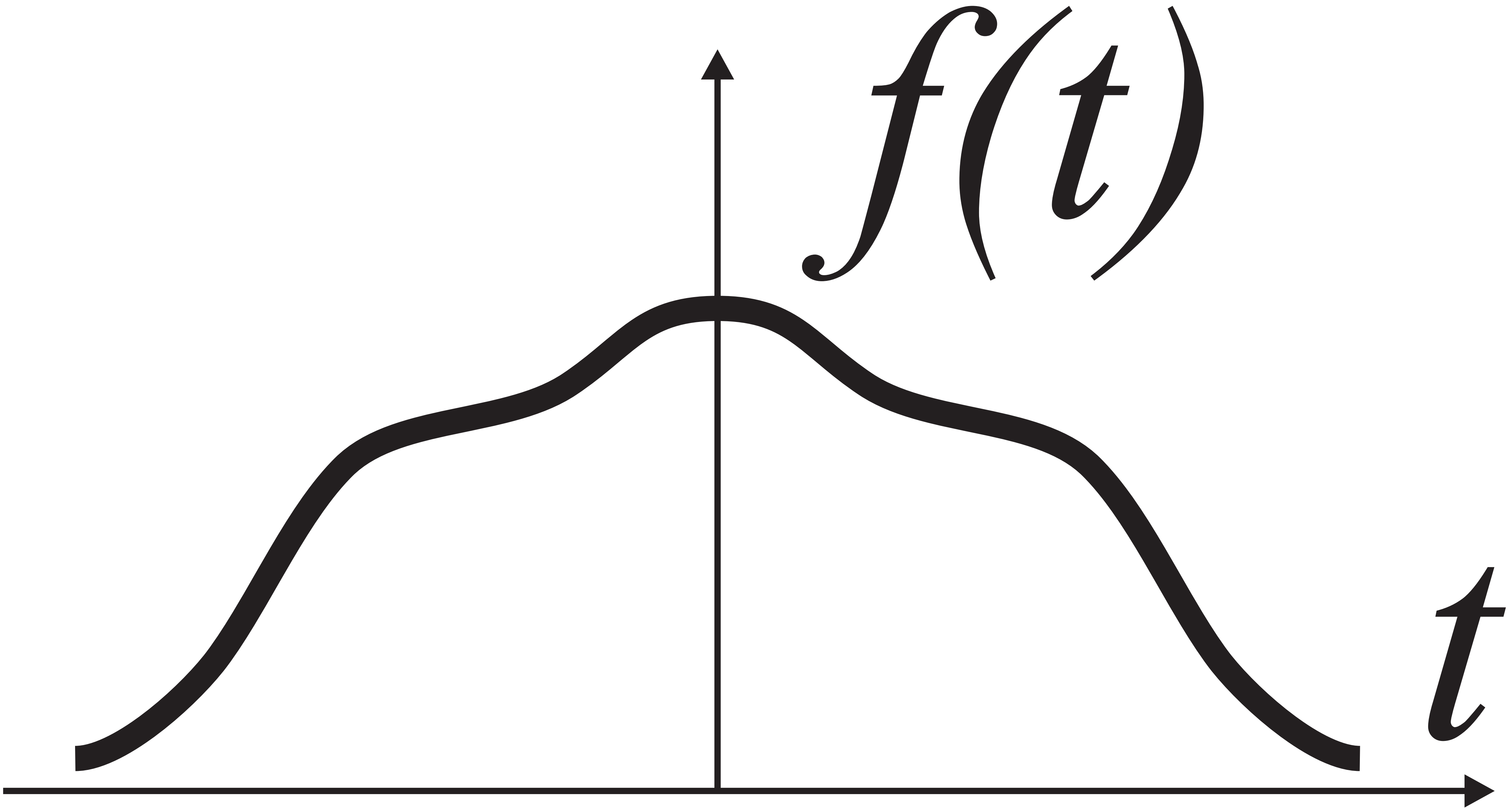




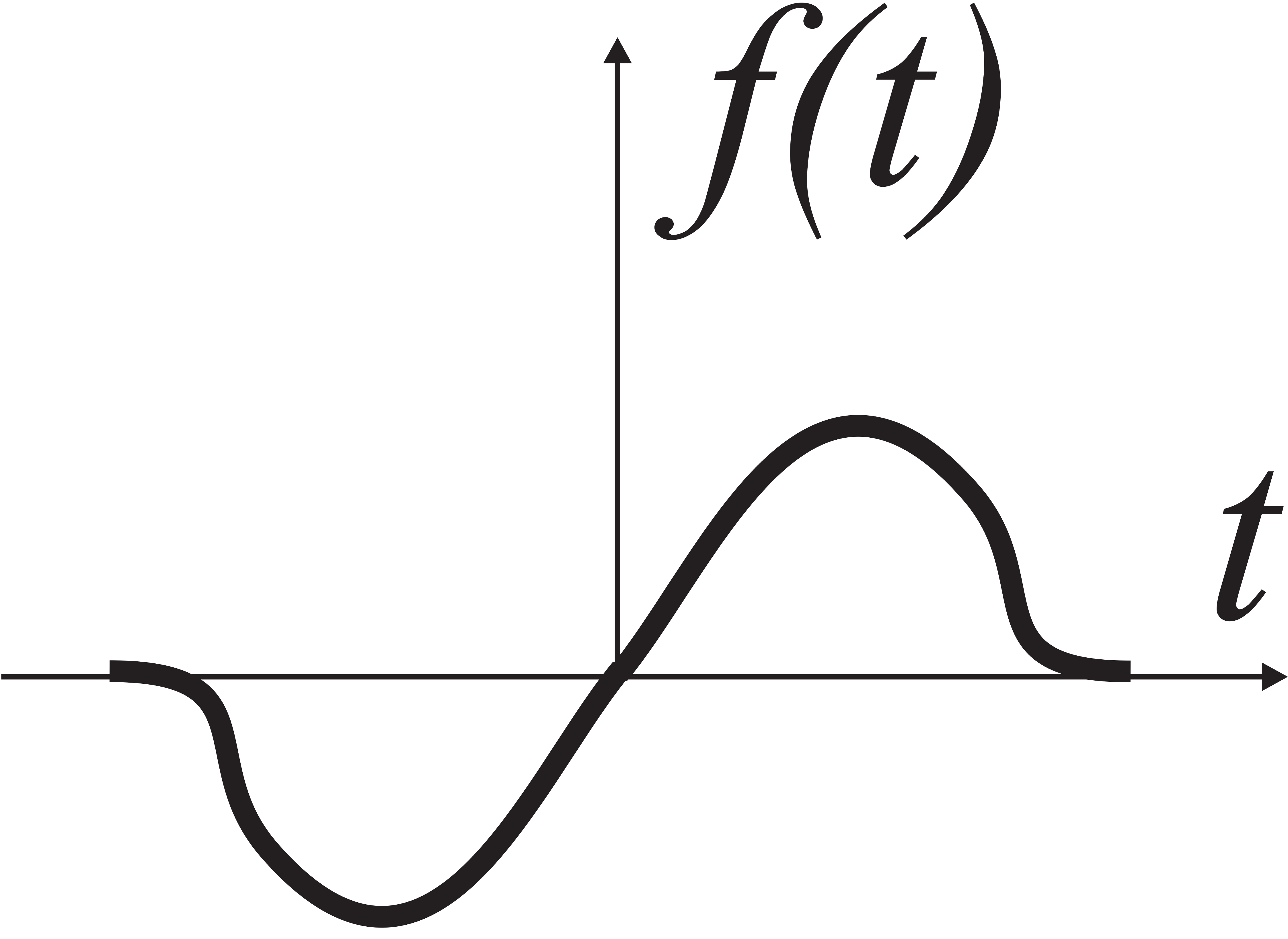
image

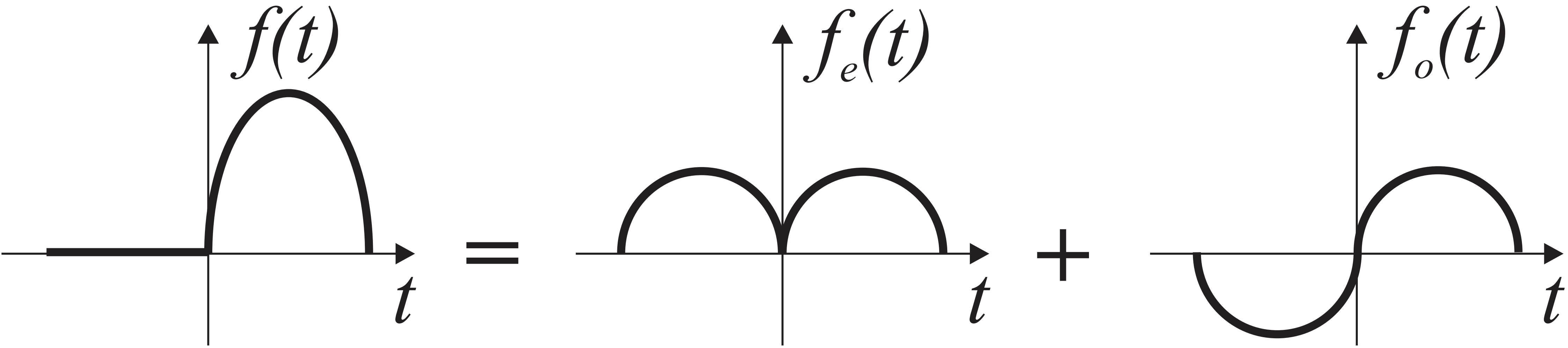
mesh


input image
spatial filter
output image
direct transformation
frequency filter
inverse transformation

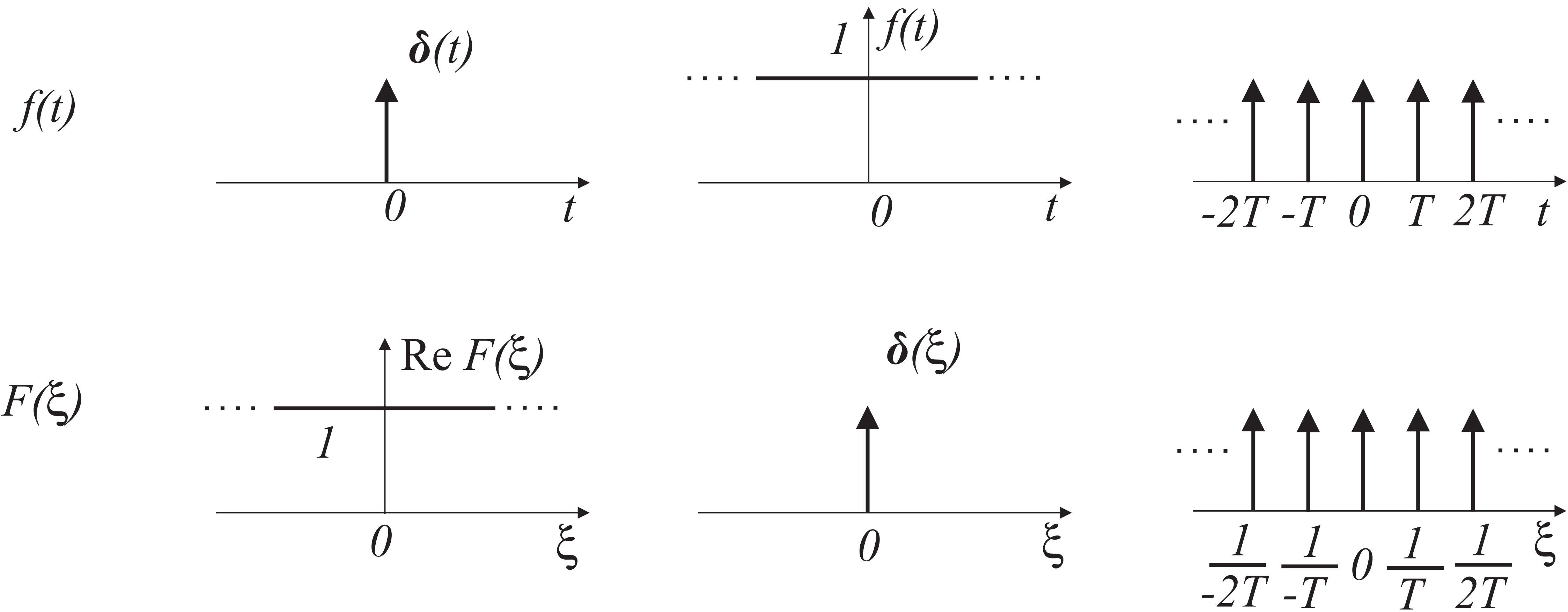


f(t)
t

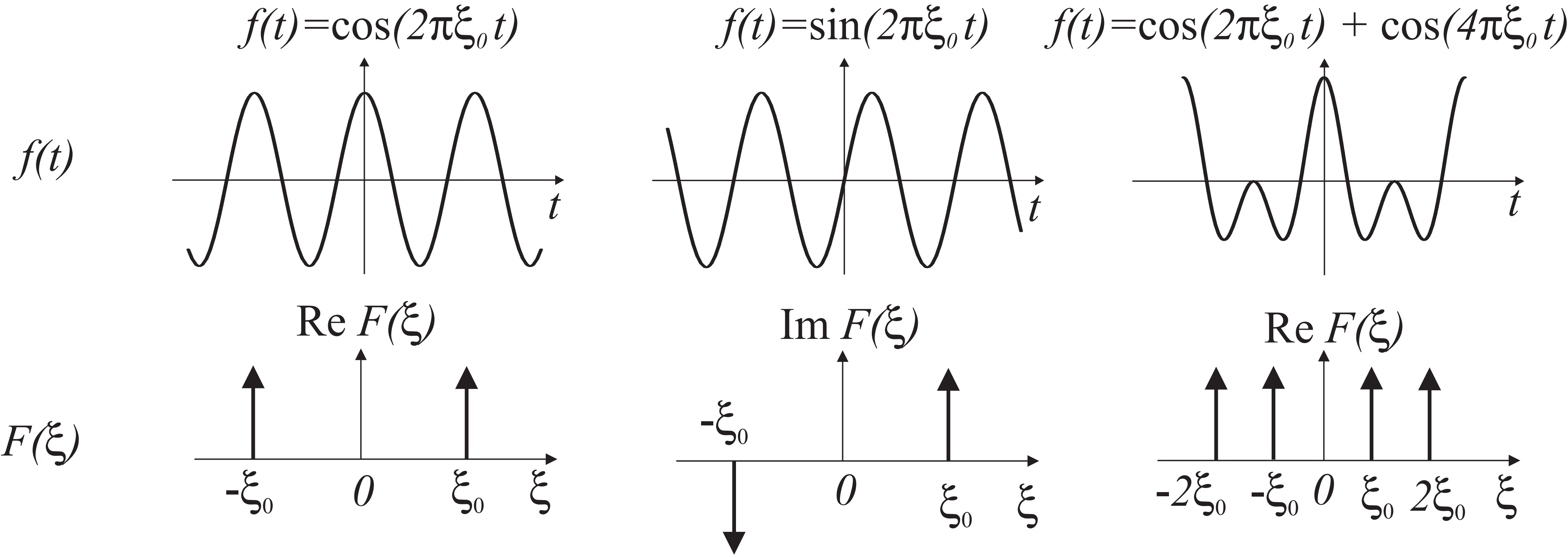

f(t)
t

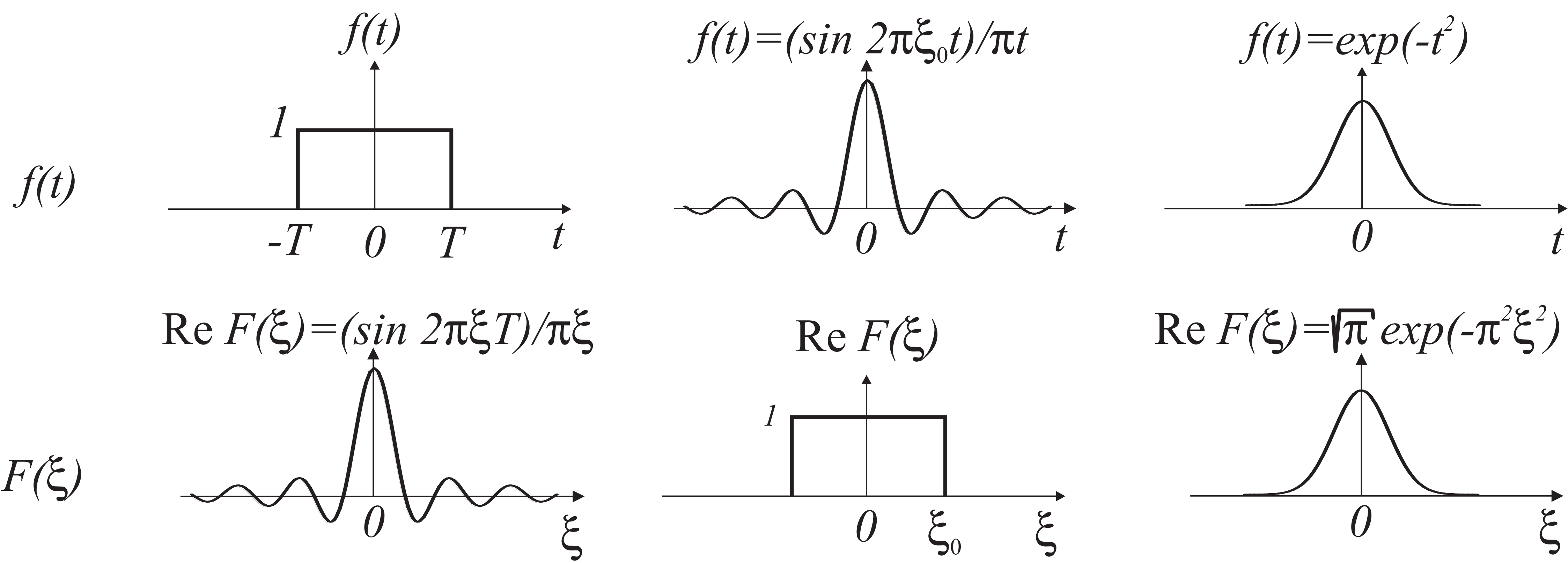


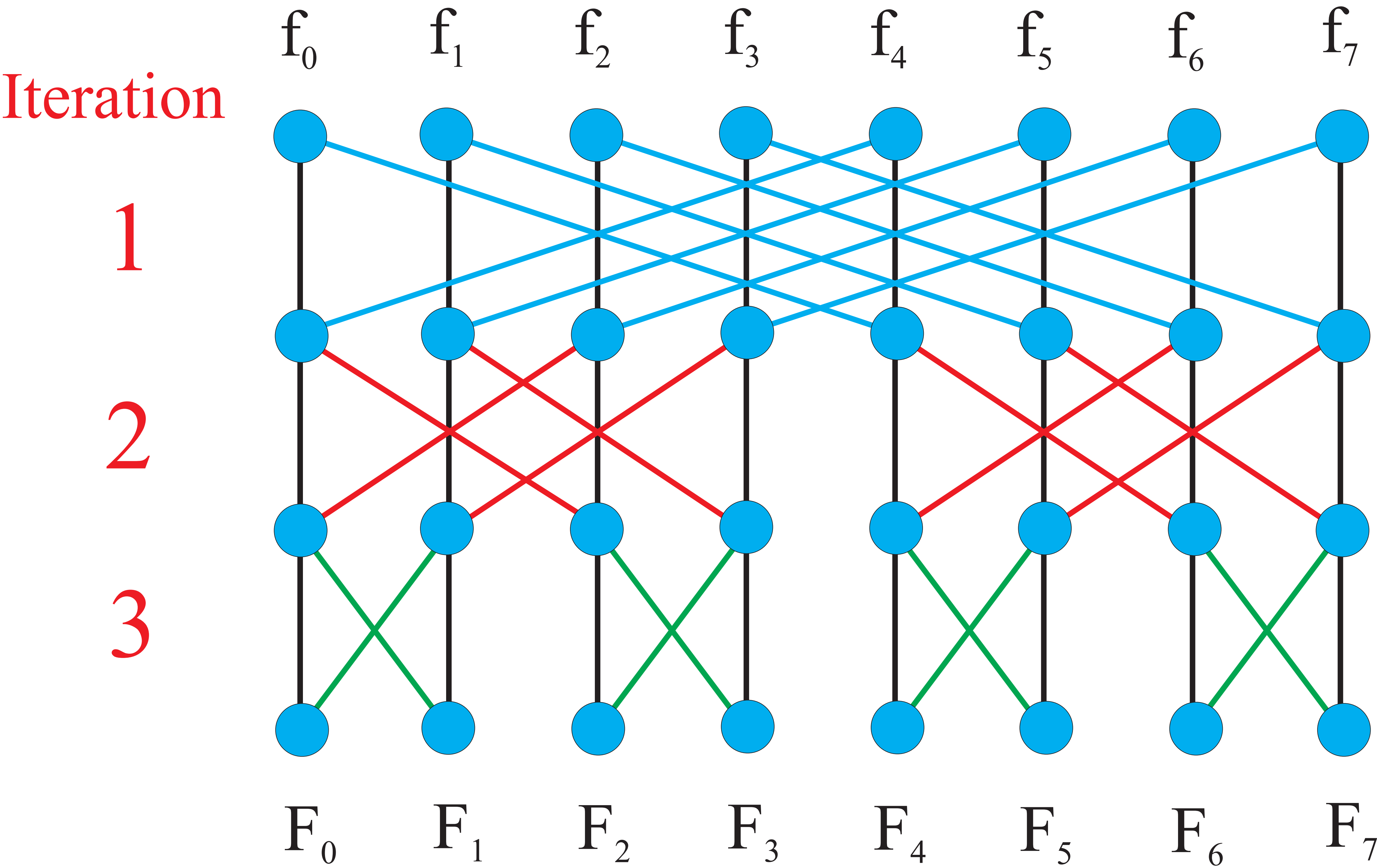
$f(t)$
t
=
$f_e(t)$
t
+
$f_o(t)$
t



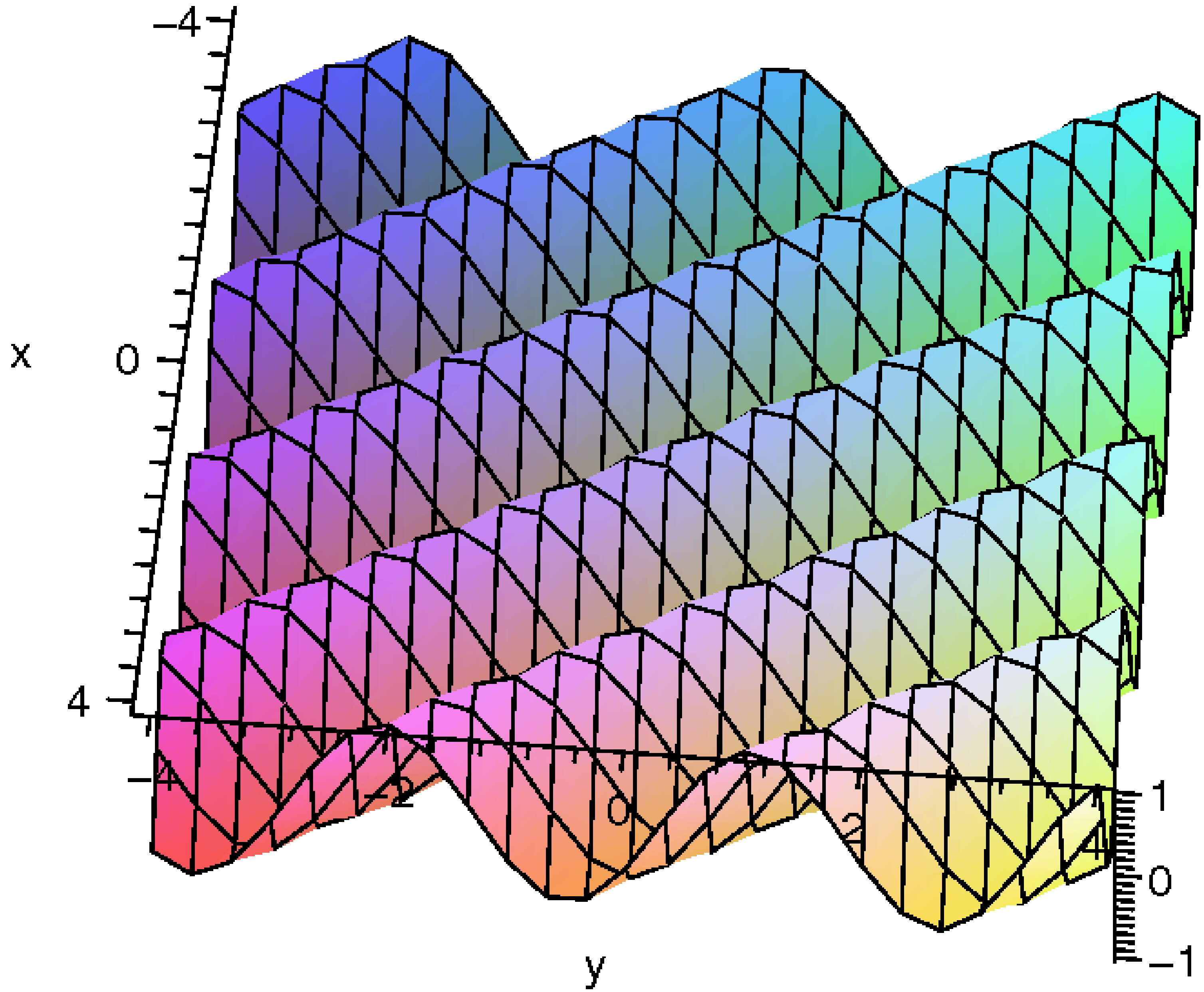
f(t)
δ(t)
0
t
1
f(t)
0
t
-2T -T 0 T 2T
t
F(ξ)
Re F(ξ)
1
0
ξ
δ(ξ)
0
ξ
1/-2T 1/-T 0 1/T 1/2T
ξ


$f(t)$
$F(\xi)$
$f(t)=\cos(2\pi\xi_0 t)$
t
Re $F(\xi)$
$-\xi_0$
0
ξ_0
ξ
$f(t)=\sin(2\pi\xi_0 t)$
t
Im $F(\xi)$
$-\xi_0$
0
ξ_0
ξ
$f(t)=\cos(2\pi\xi_0 t) + \cos(4\pi\xi_0 t)$
t
Re $F(\xi)$
$-2\xi_0$
$-\xi_0$
0
ξ_0
$2\xi_0$
ξ

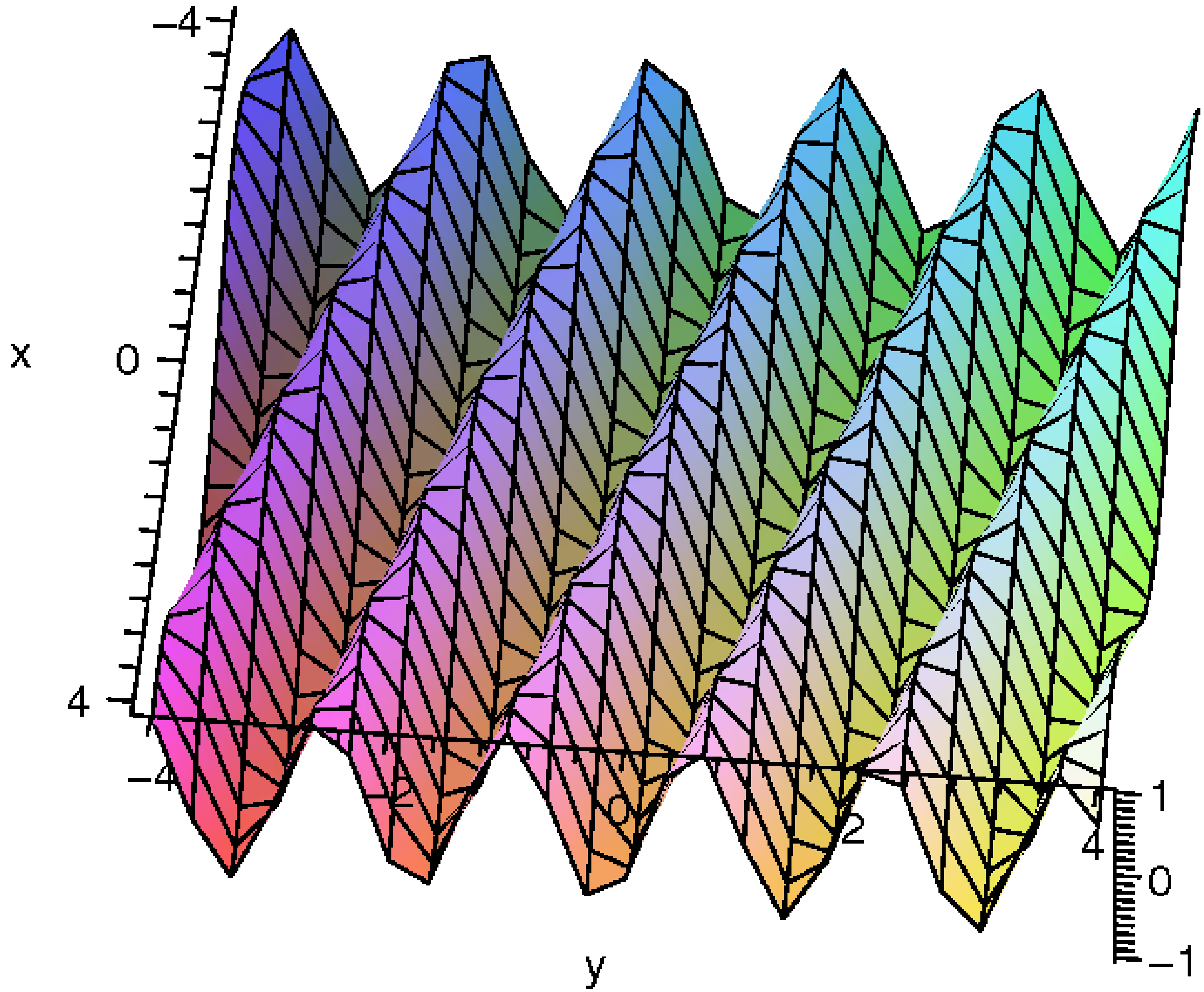

$f(t)$
$f(t)$
1
$-T$
0
T
t
$f(t)=(\sin 2\pi\xi_0 t)/\pi t$
0
t
$f(t)=\exp(-t^2)$
0
t
$F(\xi)$
$\mathrm{Re}\, F(\xi)=(\sin 2\pi\xi T)/\pi\xi$
0
ξ
$\mathrm{Re}\, F(\xi)$
1
0
ξ_0
ξ
$\mathrm{Re}\, F(\xi)=\sqrt{\pi}\, \exp(-\pi^2\xi^2)$
0
ξ



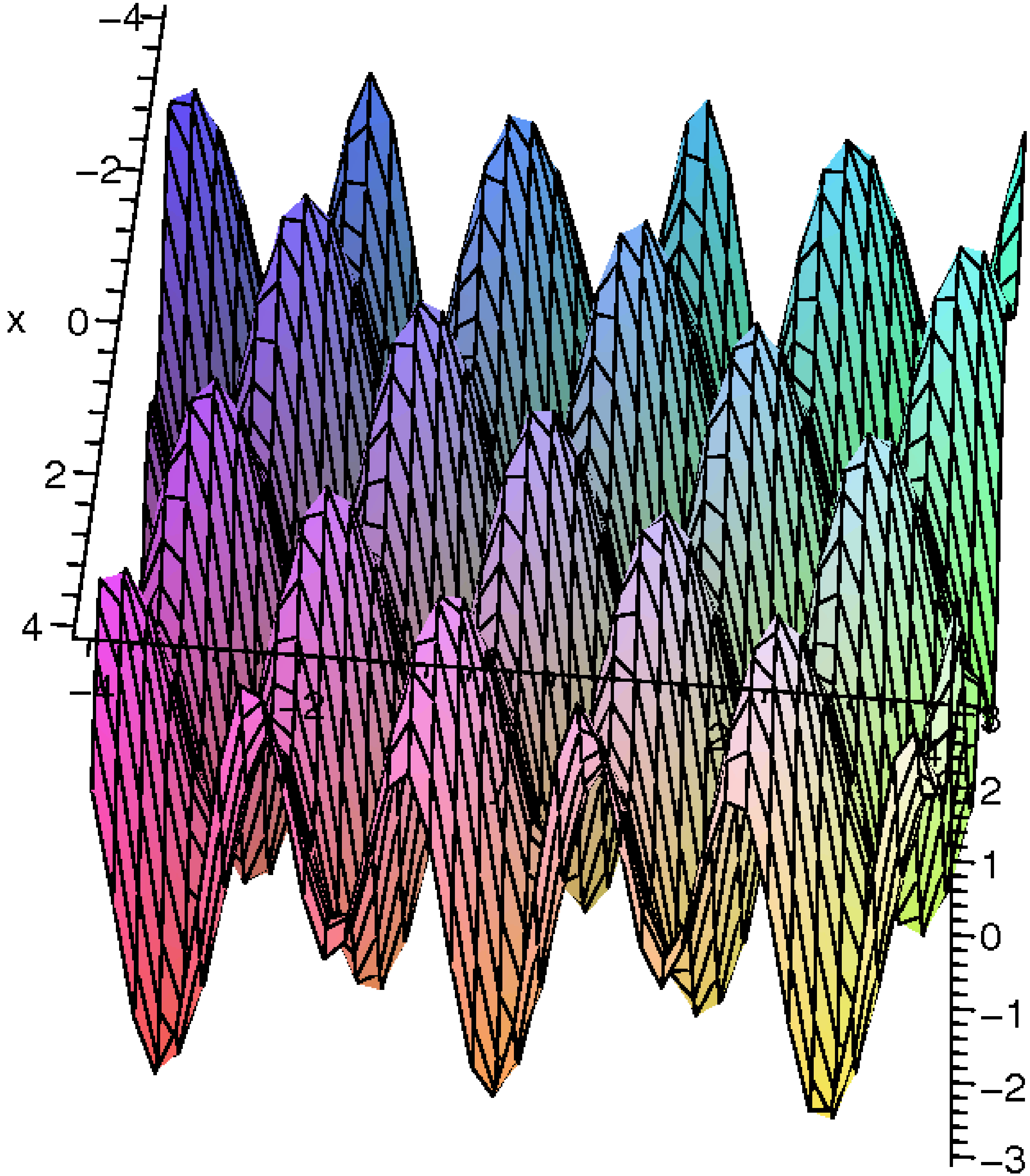
f_0 f_1 f_2 f_3 f_4 f_5 f_6 f_7
Iteration
1
2
3
F_0 F_1 F_2 F_3 F_4 F_5 F_6 F_7



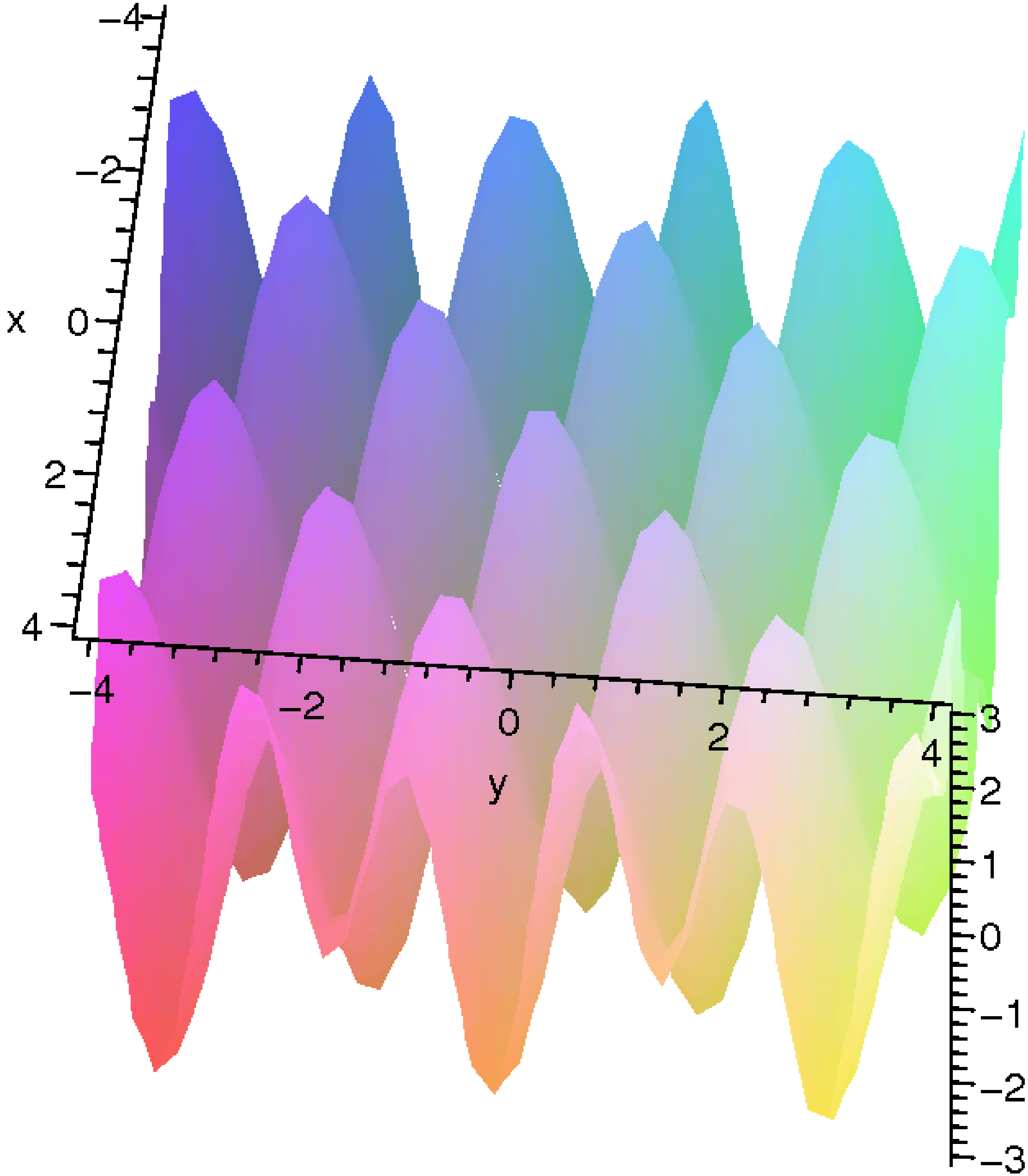
−4
x
0
4
−4
−2
0
2
4
y
1
0
−1

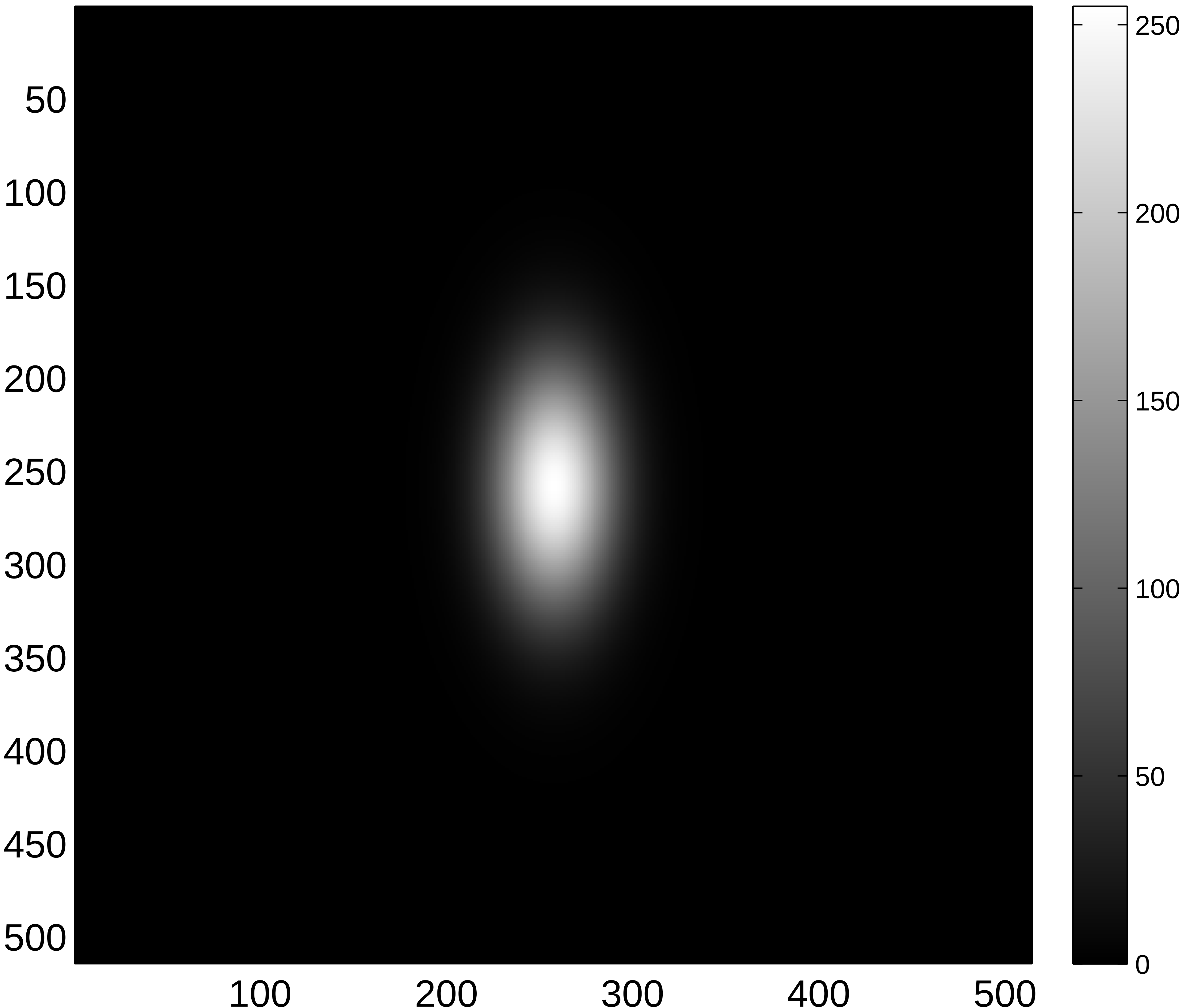


x
y
−4
0
4
−4
−2
0
2
4
1
0
−1

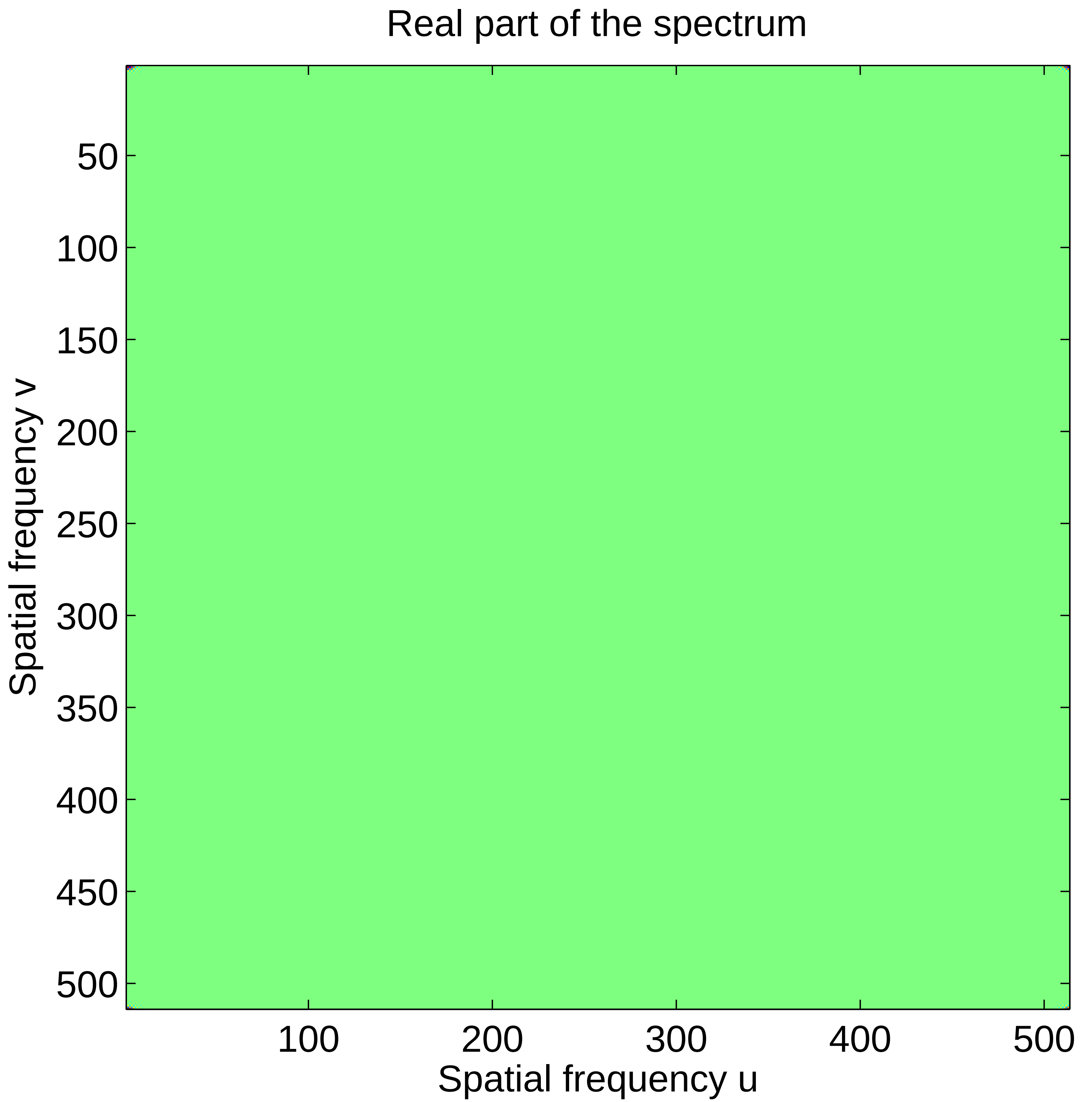


x
–4
–2
0
2
4
–4
–2
2
1
0
–1
–2
–3

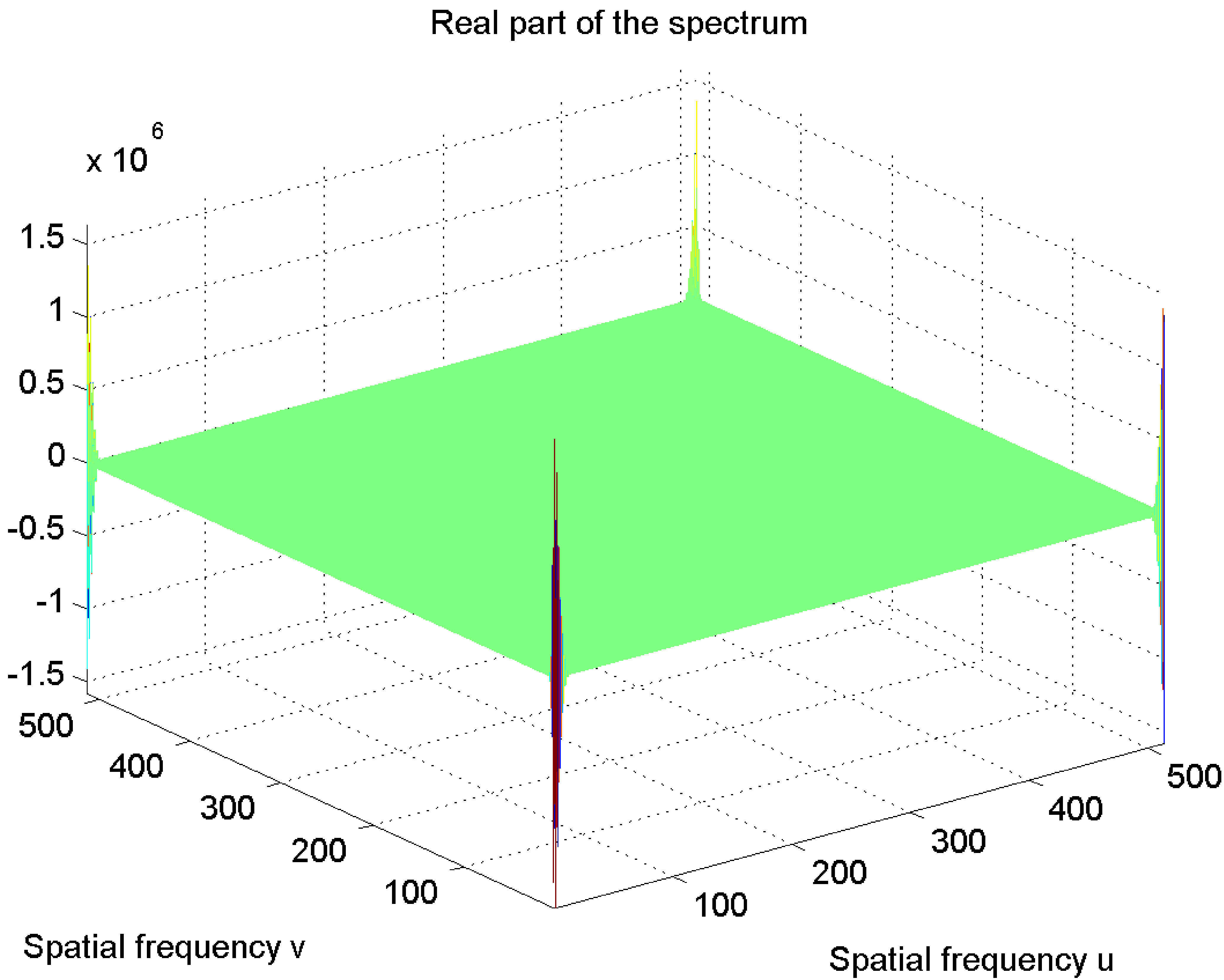

x
y
−4
−2
0
2
4
3
2
1
0
−1
−2
−3

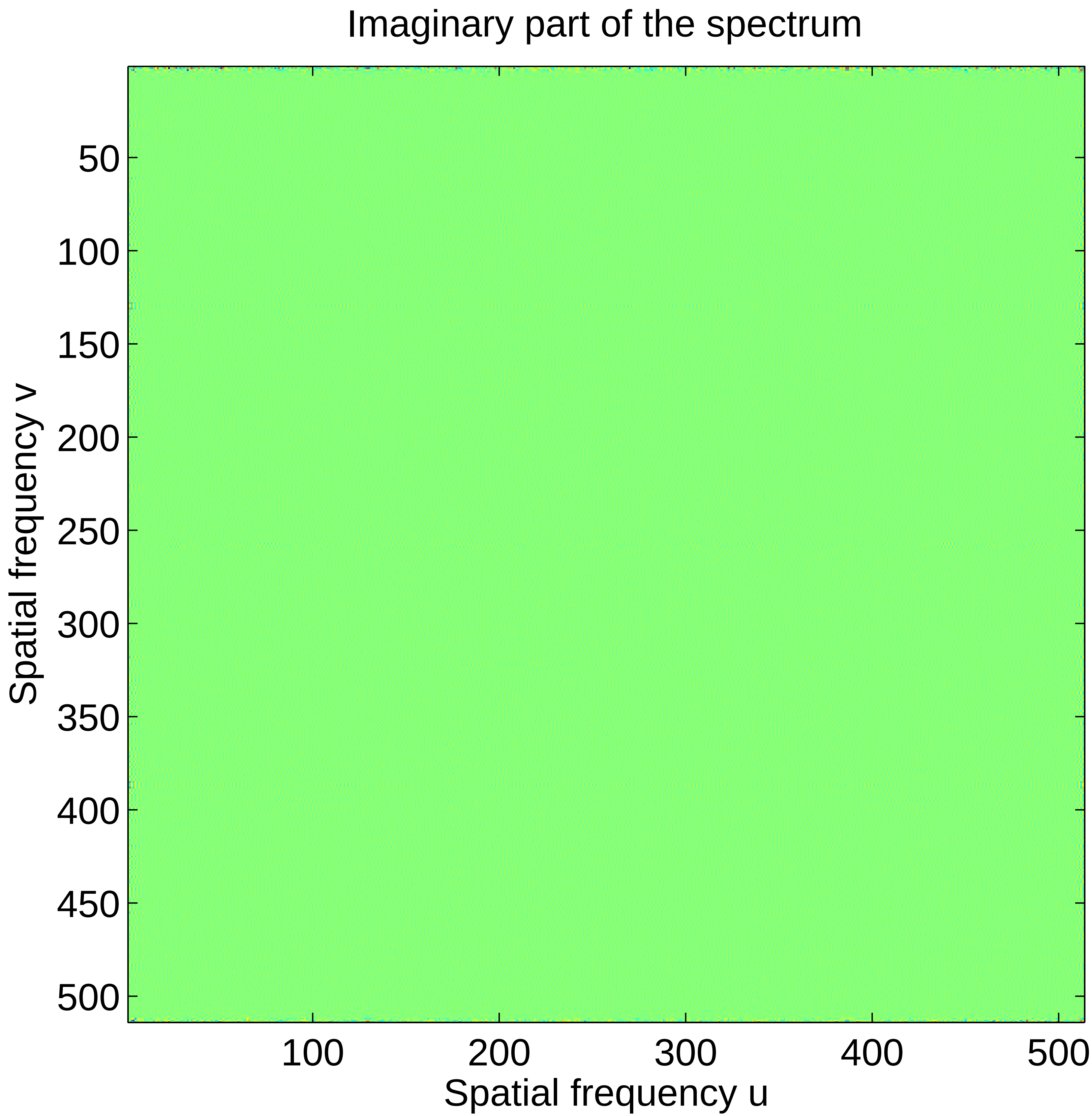


50
100
150
200
250
300
350
400
450
500
100
200
300
400
500
250
200
150
100
50
0

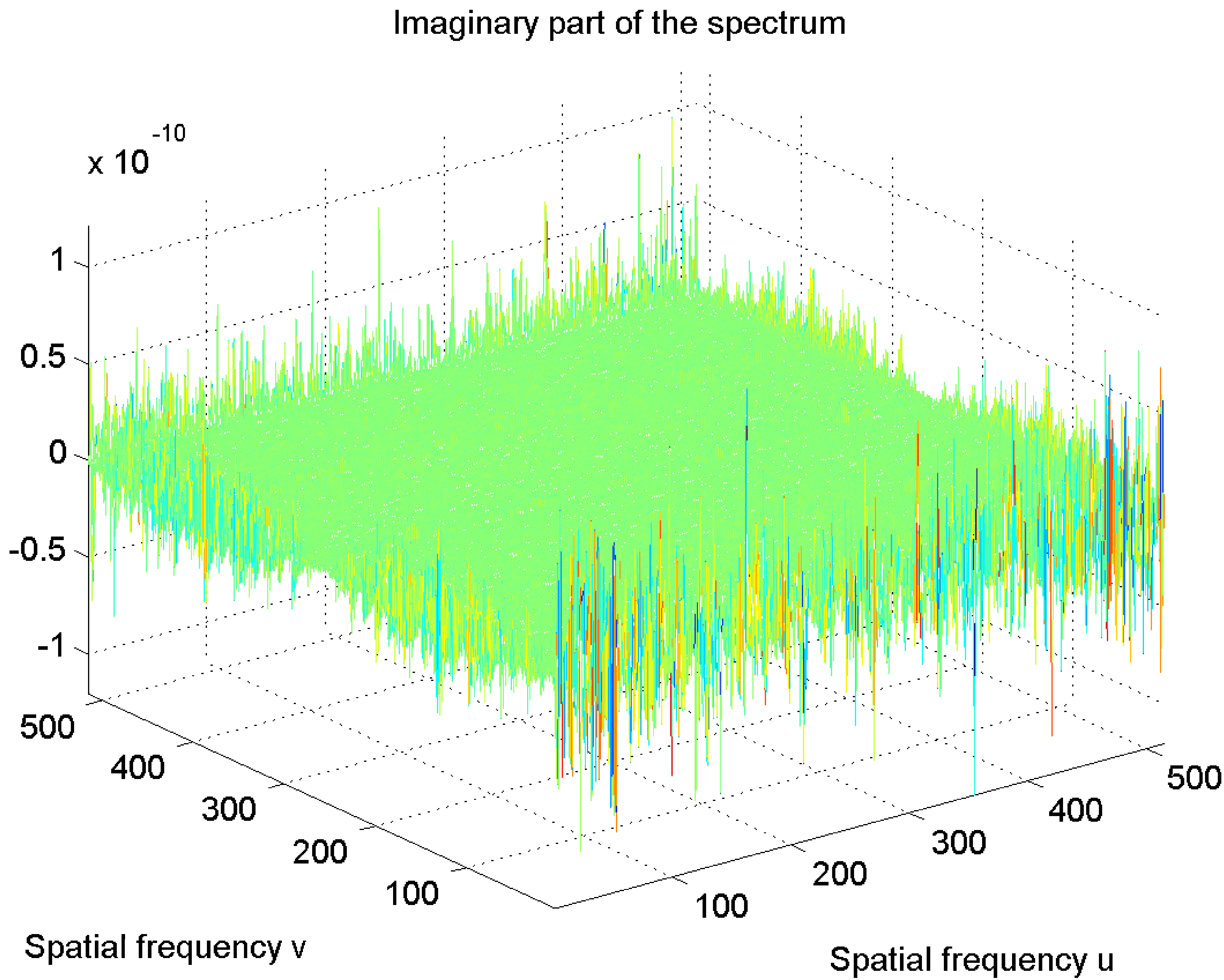


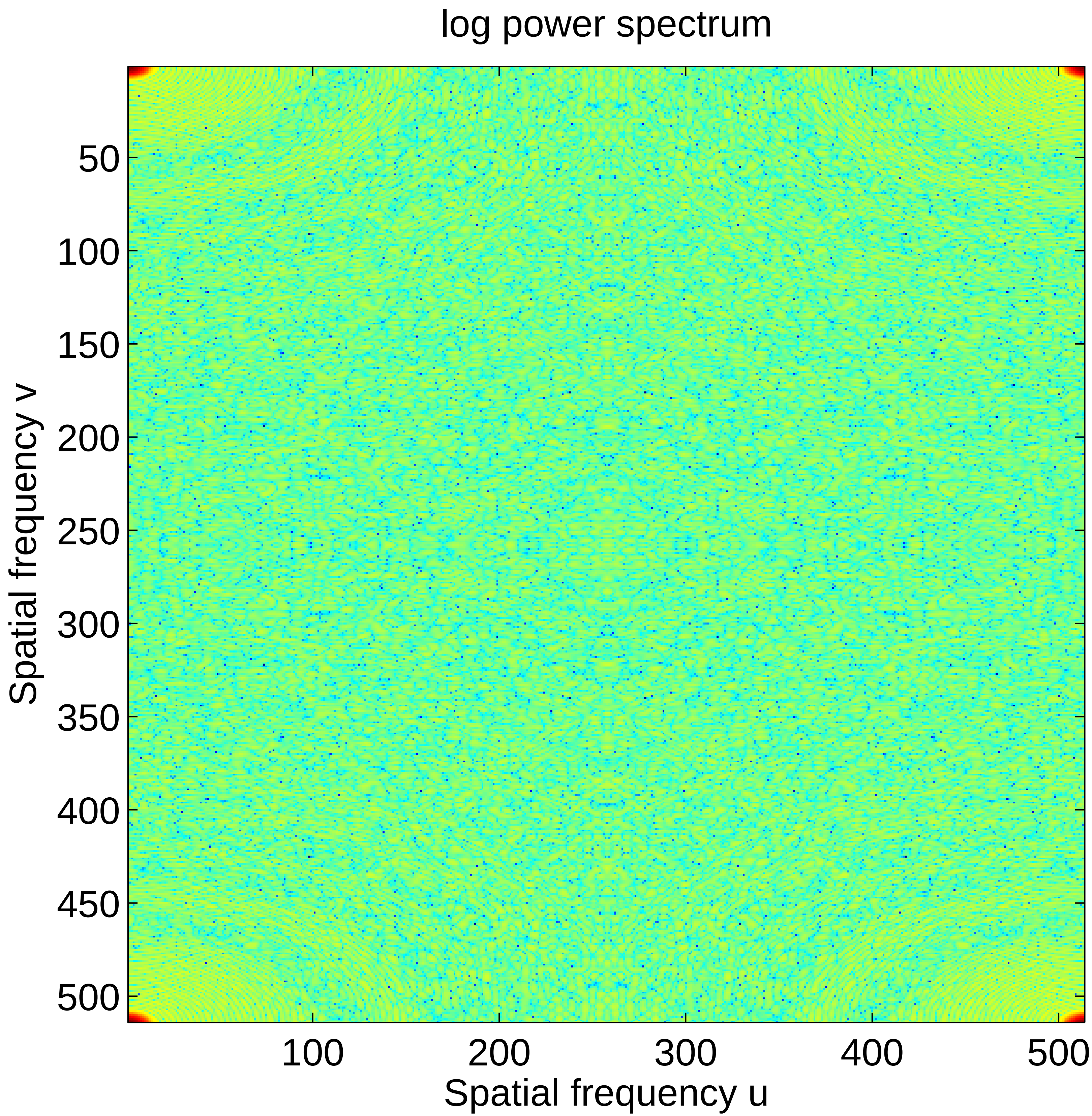
Real part of the spectrum
50
100
150
200
250
300
350
400
450
500
Spatial frequency v
100
200
300
400
500
Spatial frequency u


Real part of the spectrum
x 10^6
1.5
1
0.5
0
-0.5
-1
-1.5
500
400
300
200
100
Spatial frequency v
100
200
300
400
500
Spatial frequency u

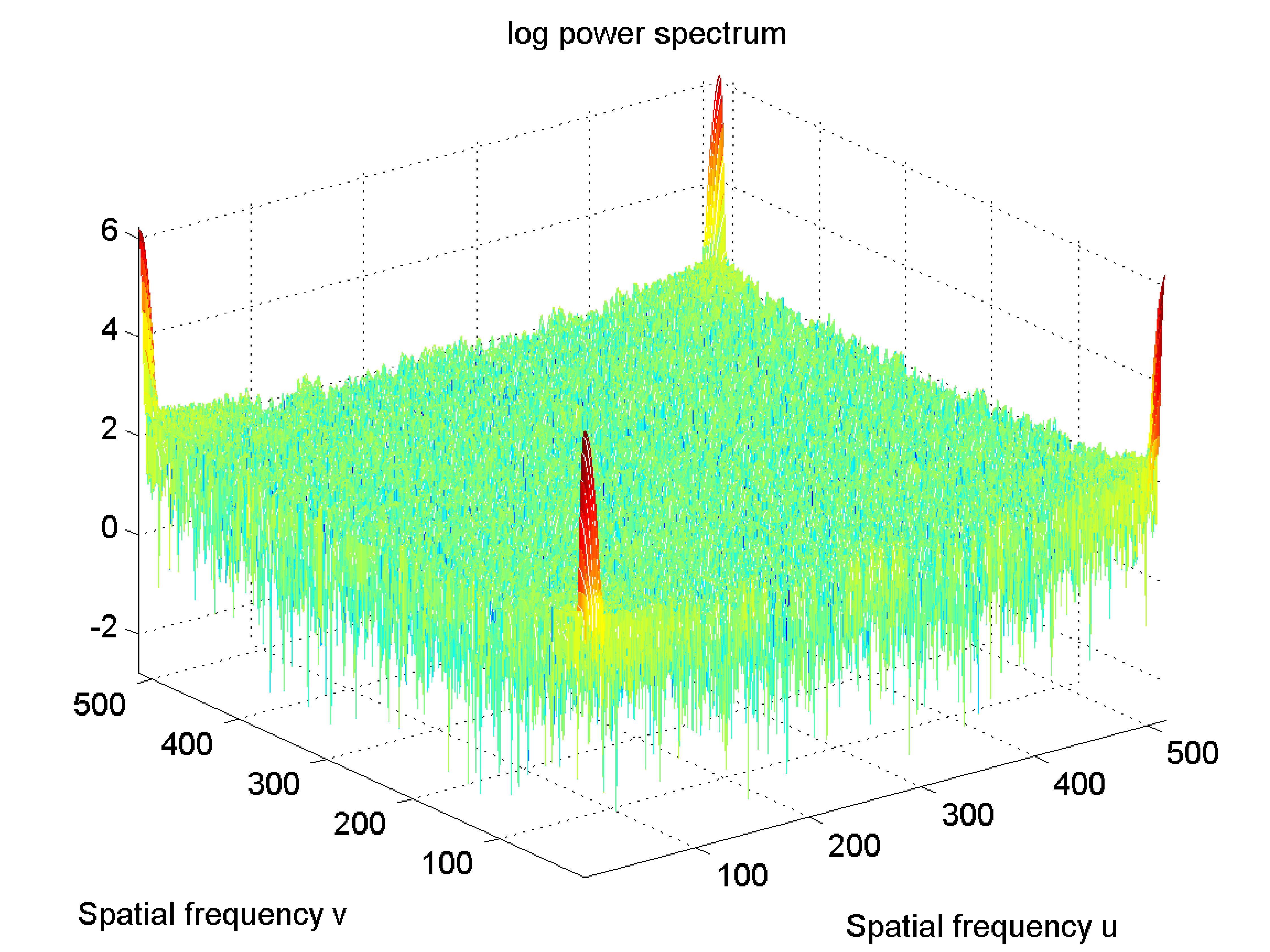


Imaginary part of the spectrum
50
100
150
200
250
300
350
400
450
500
Spatial frequency v
100
200
300
400
500
Spatial frequency u

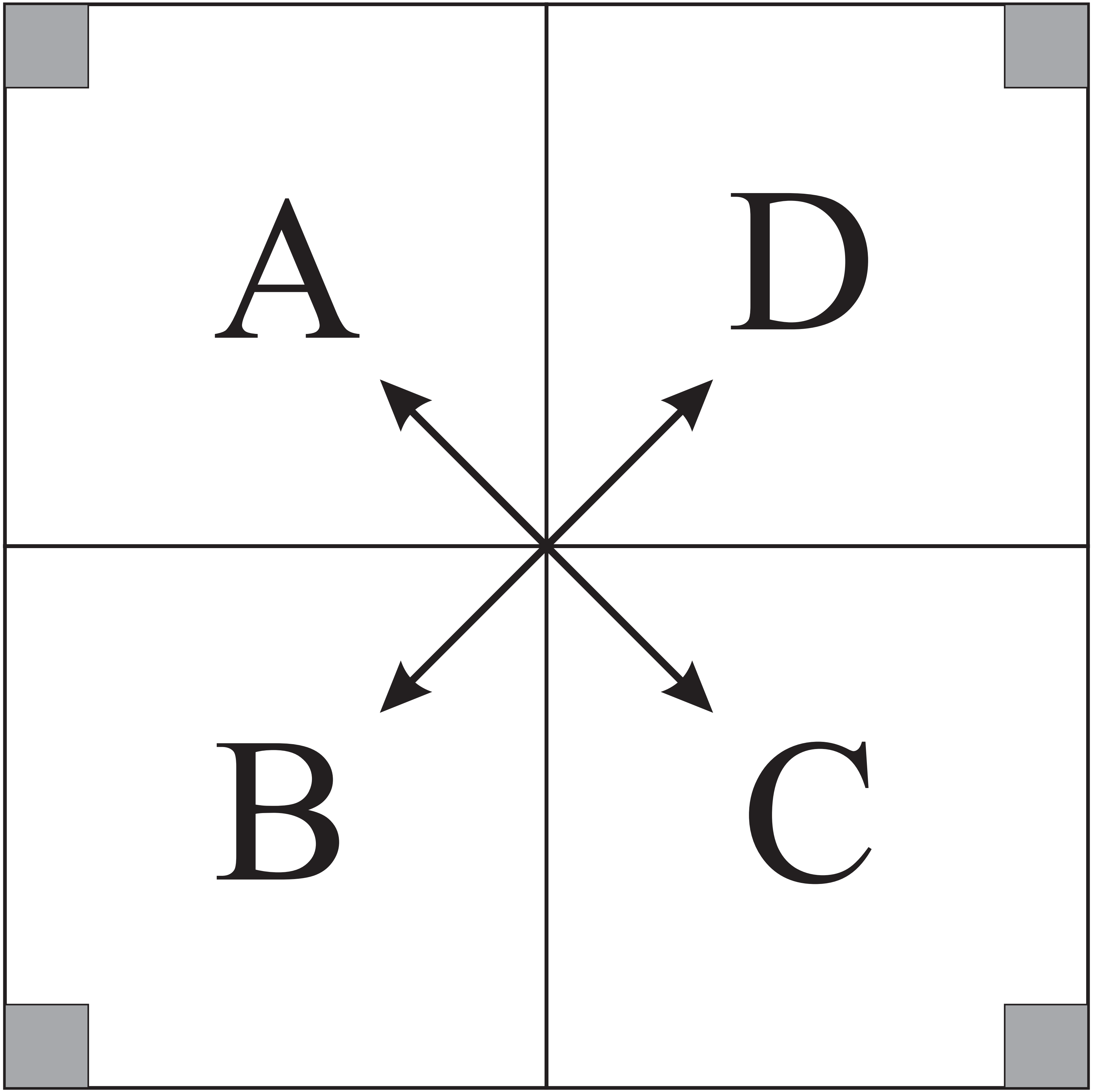

Imaginary part of the spectrum
x 10^-10
1
0.5
0
-0.5
-1
500
400
300
200
100
Spatial frequency v
100
200
300
400
500
Spatial frequency u

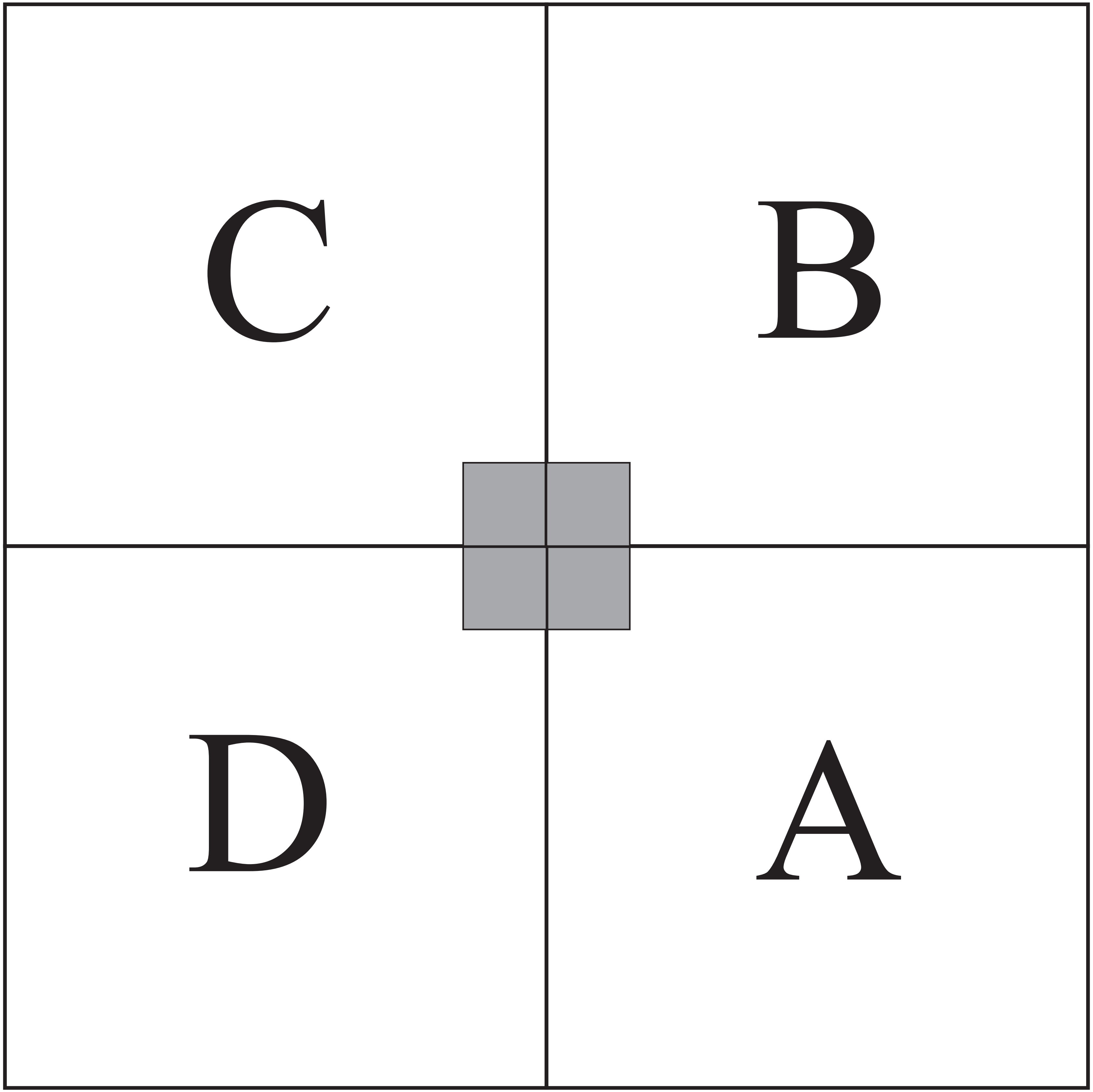


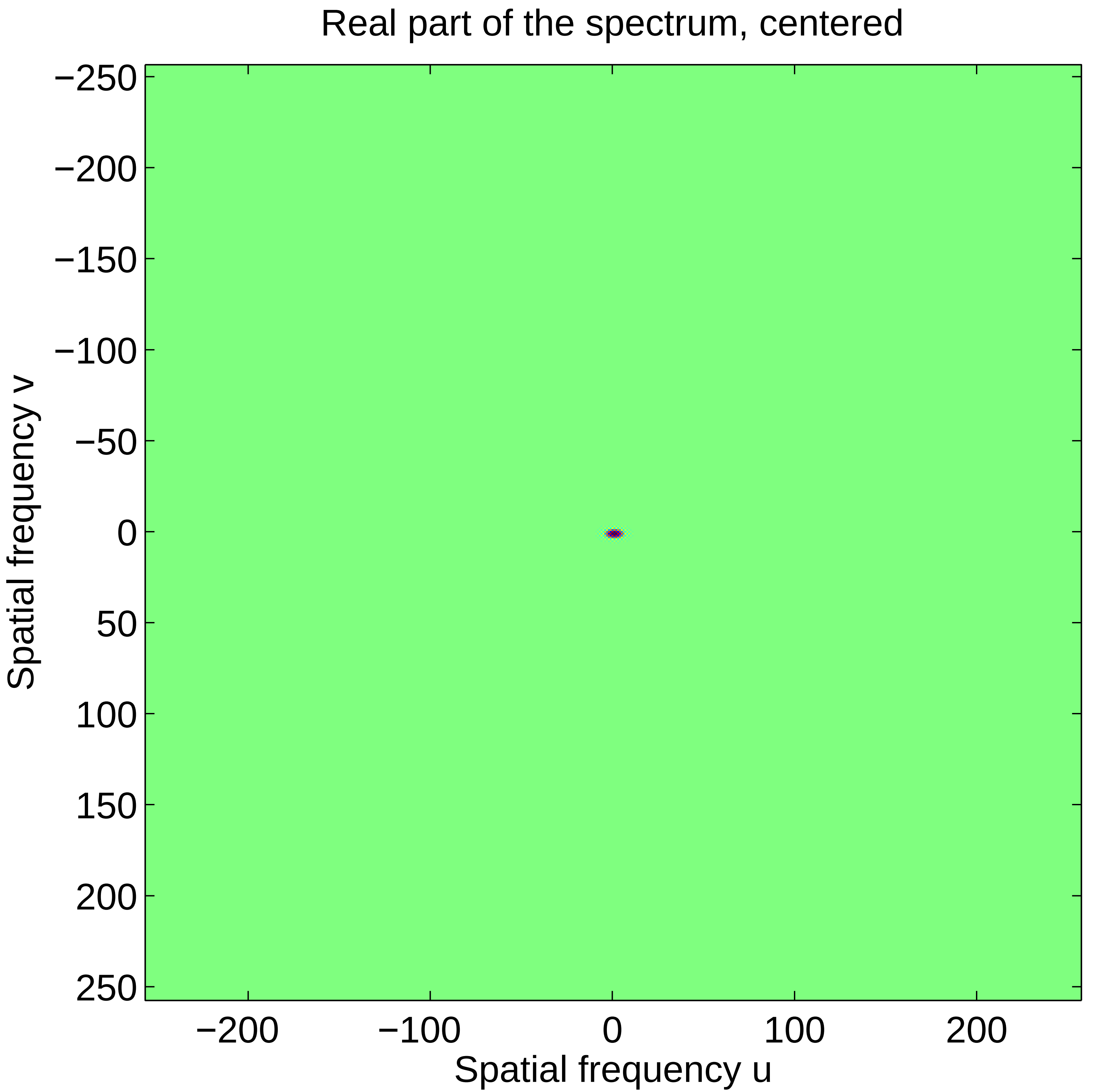
log power spectrum
Spatial frequency v
50
100
150
200
250
300
350
400
450
500
100
200
300
400
500
Spatial frequency u

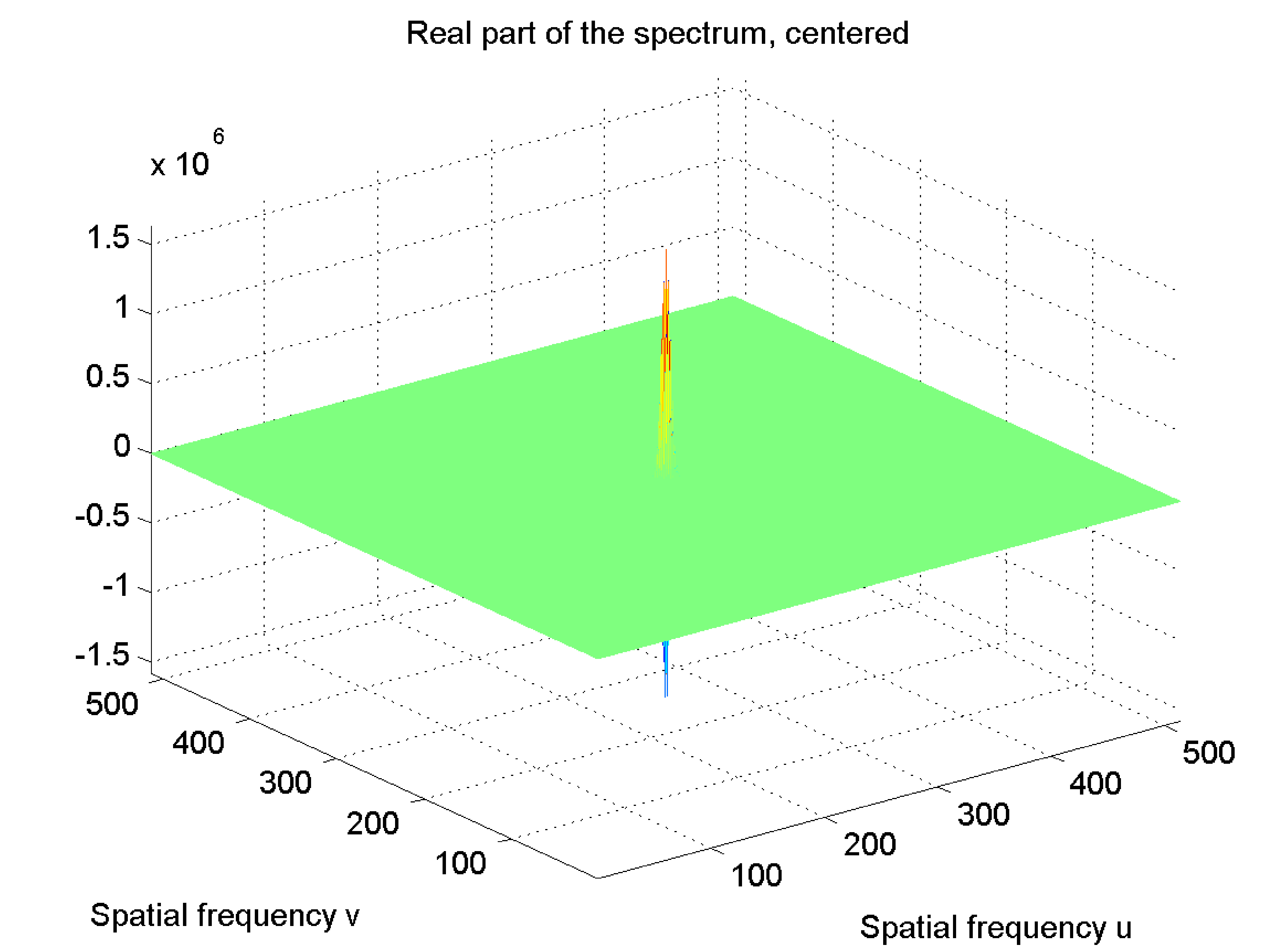


log power spectrum
6
4
2
0
-2
500
400
300
200
100
Spatial frequency v
100
200
300
400
500
Spatial frequency u

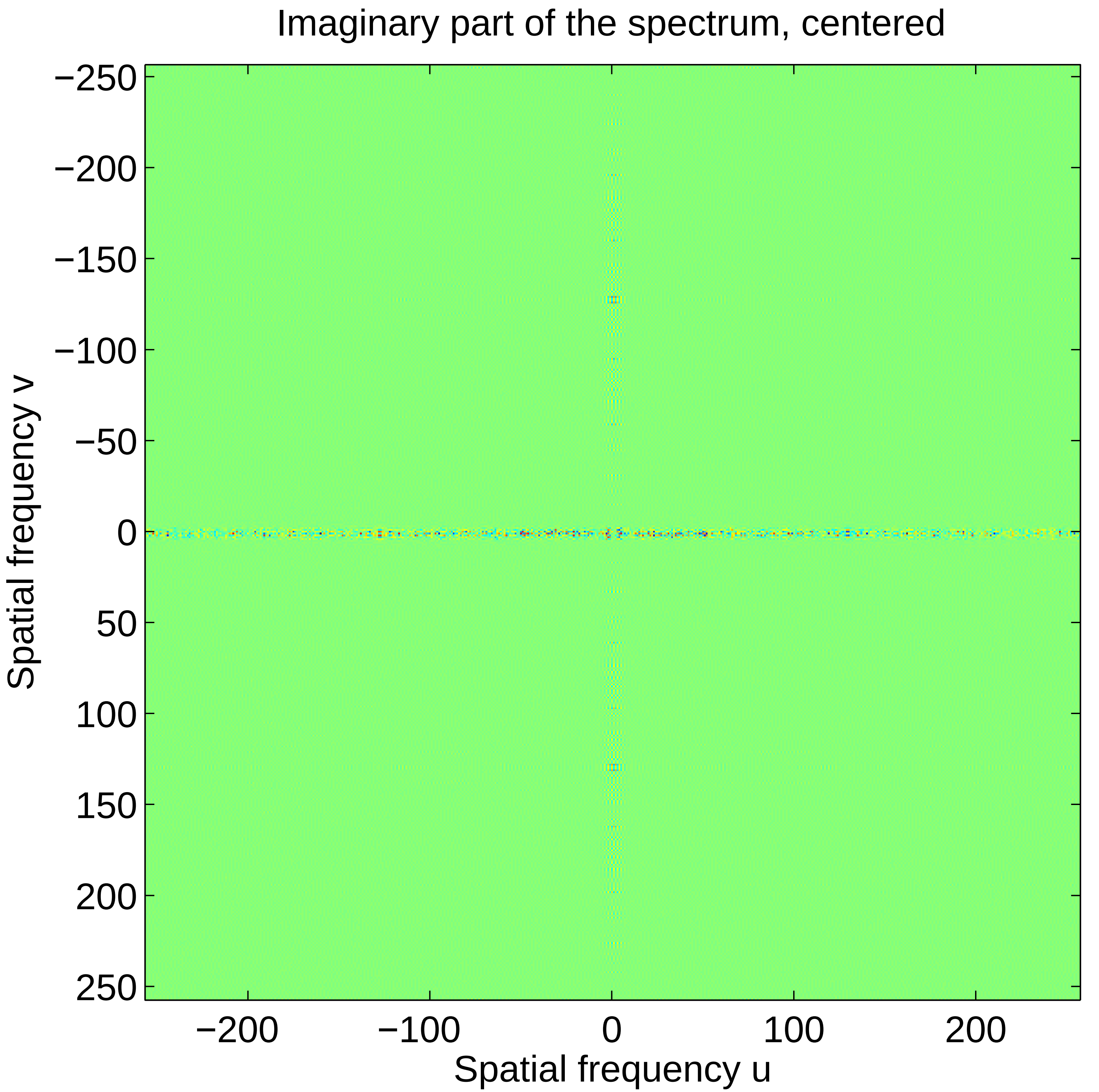

A
D
B
C

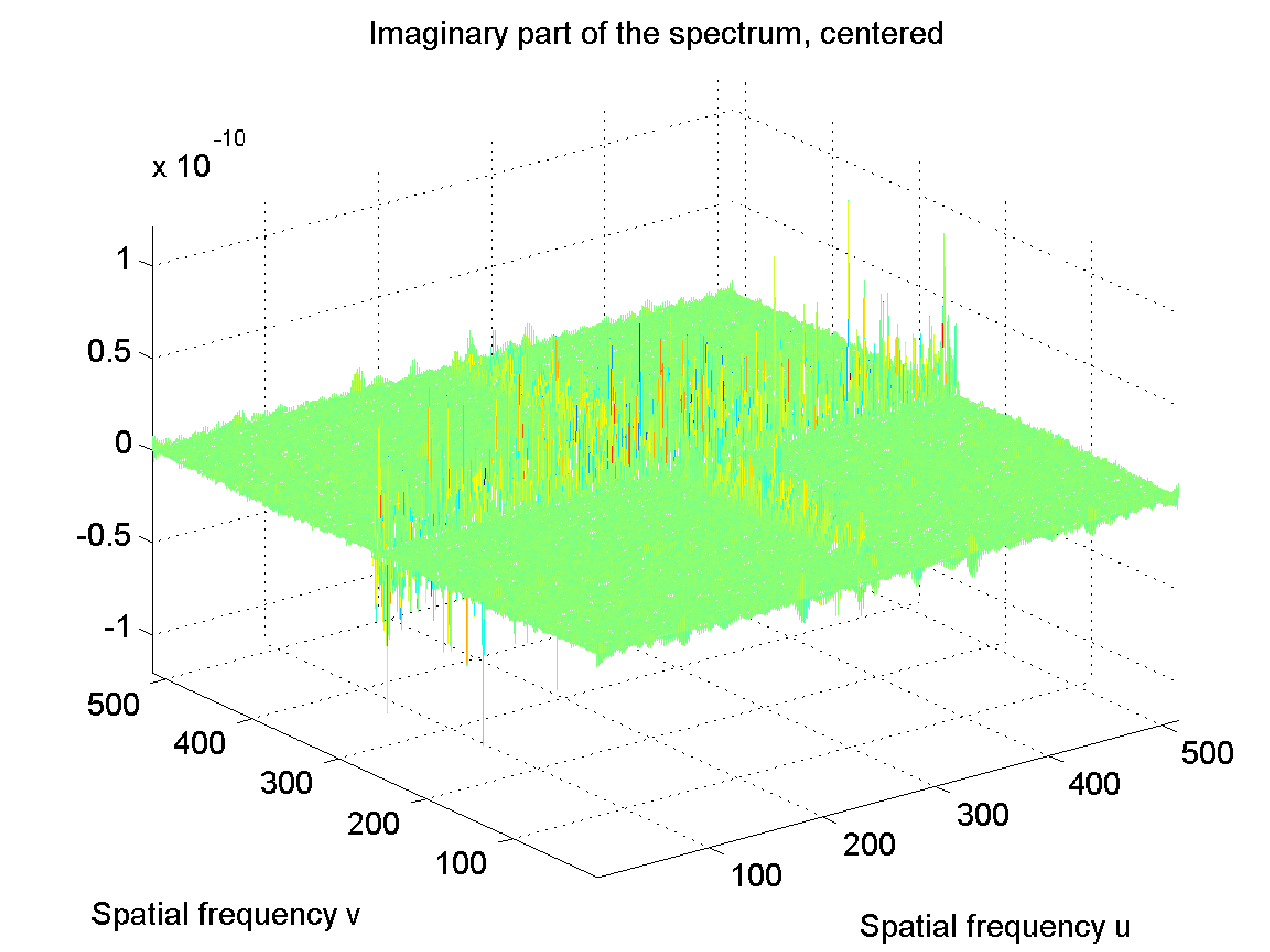

C
B
D
A


Real part of the spectrum, centered
−250
−200
−150
−100
−50
0
50
100
150
200
250
Spatial frequency v
−200
−100
0
100
200
Spatial frequency u

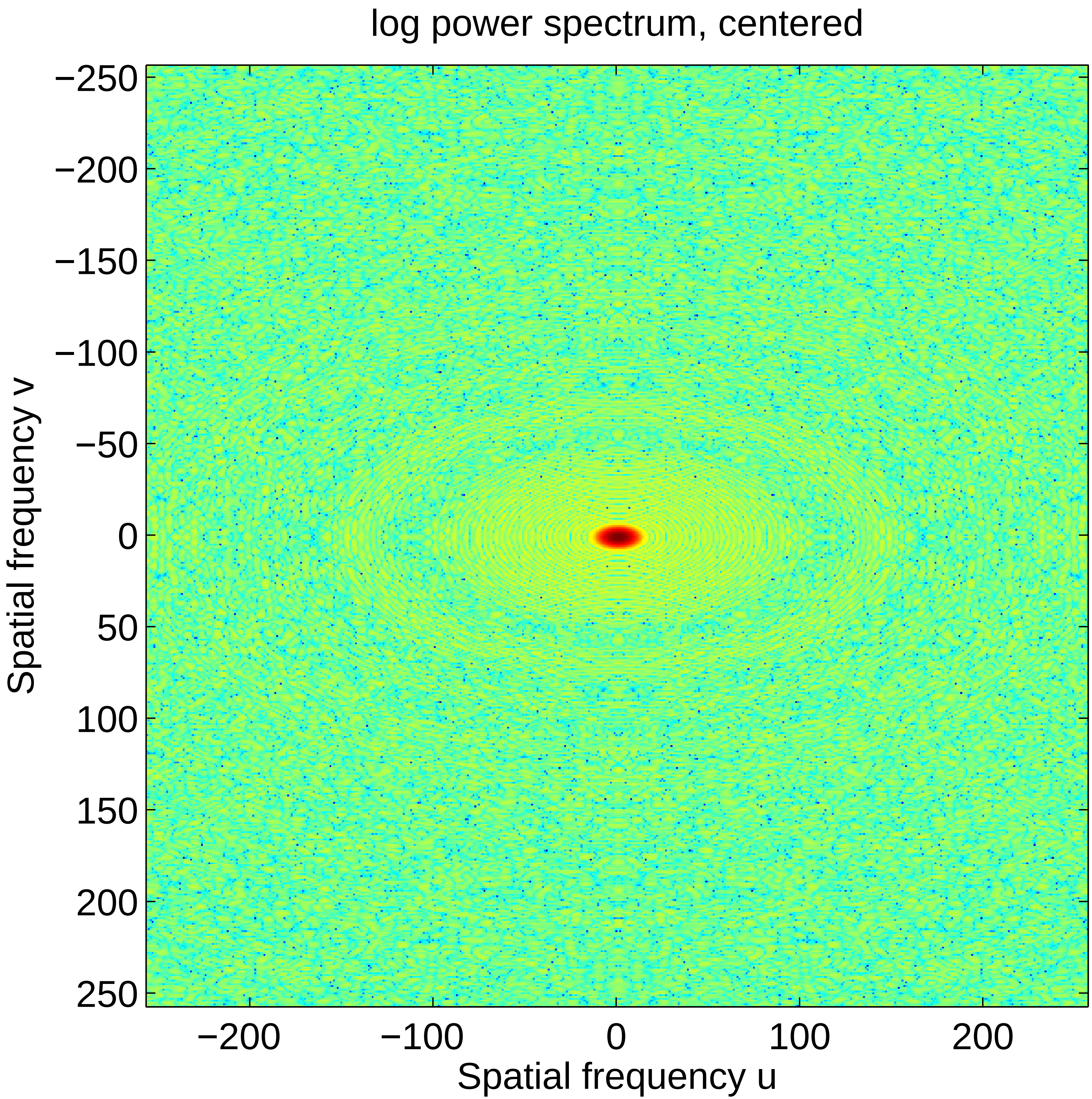


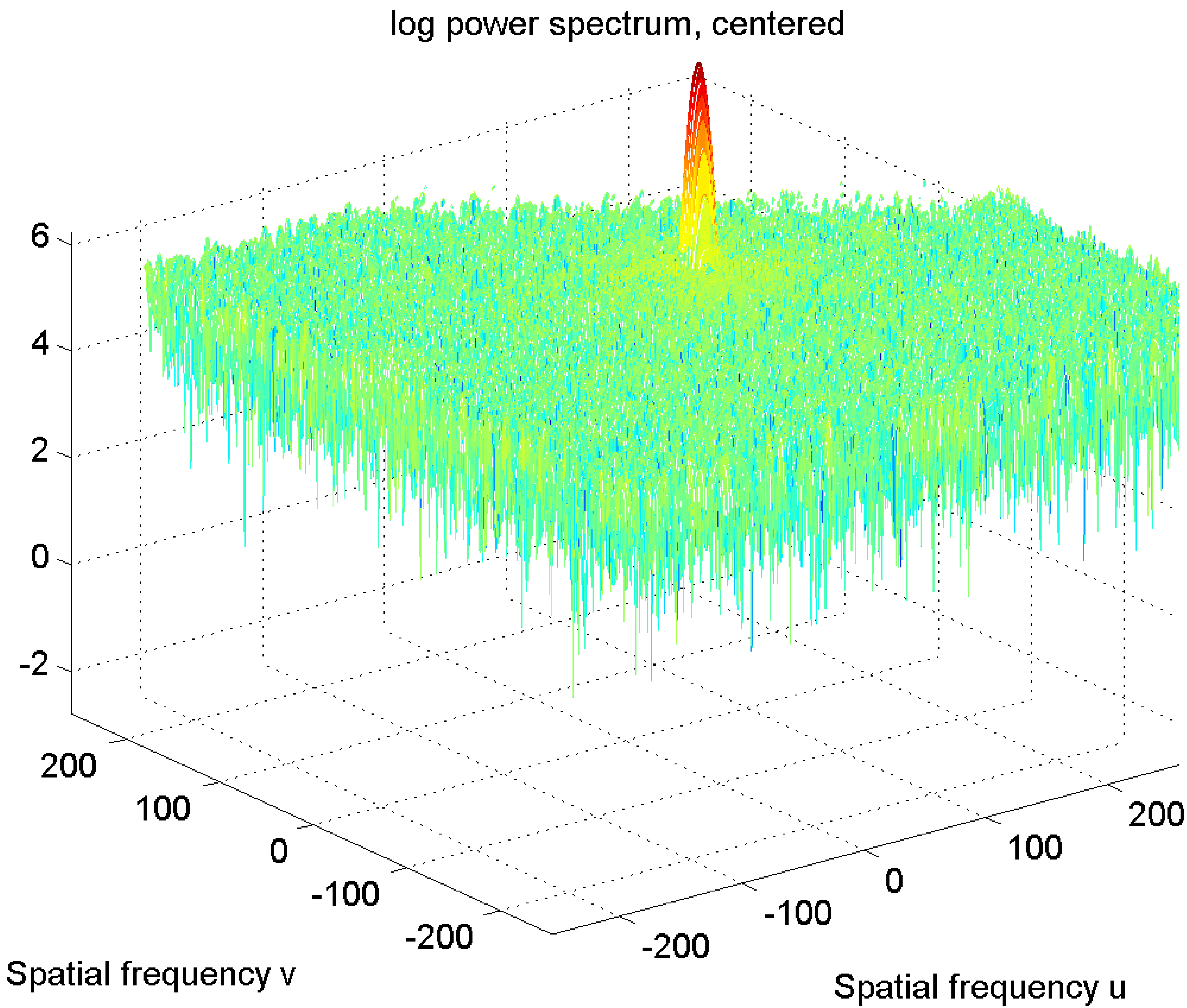
Real part of the spectrum, centered
x 10^6
1.5
1
0.5
0
-0.5
-1
-1.5
500
400
300
200
100
Spatial frequency v
100
200
300
400
500
Spatial frequency u


Imaginary part of the spectrum, centered
Spatial frequency v
Spatial frequency u
−250
−200
−150
−100
−50
0
50
100
150
200
250
−200
−100
0
100
200

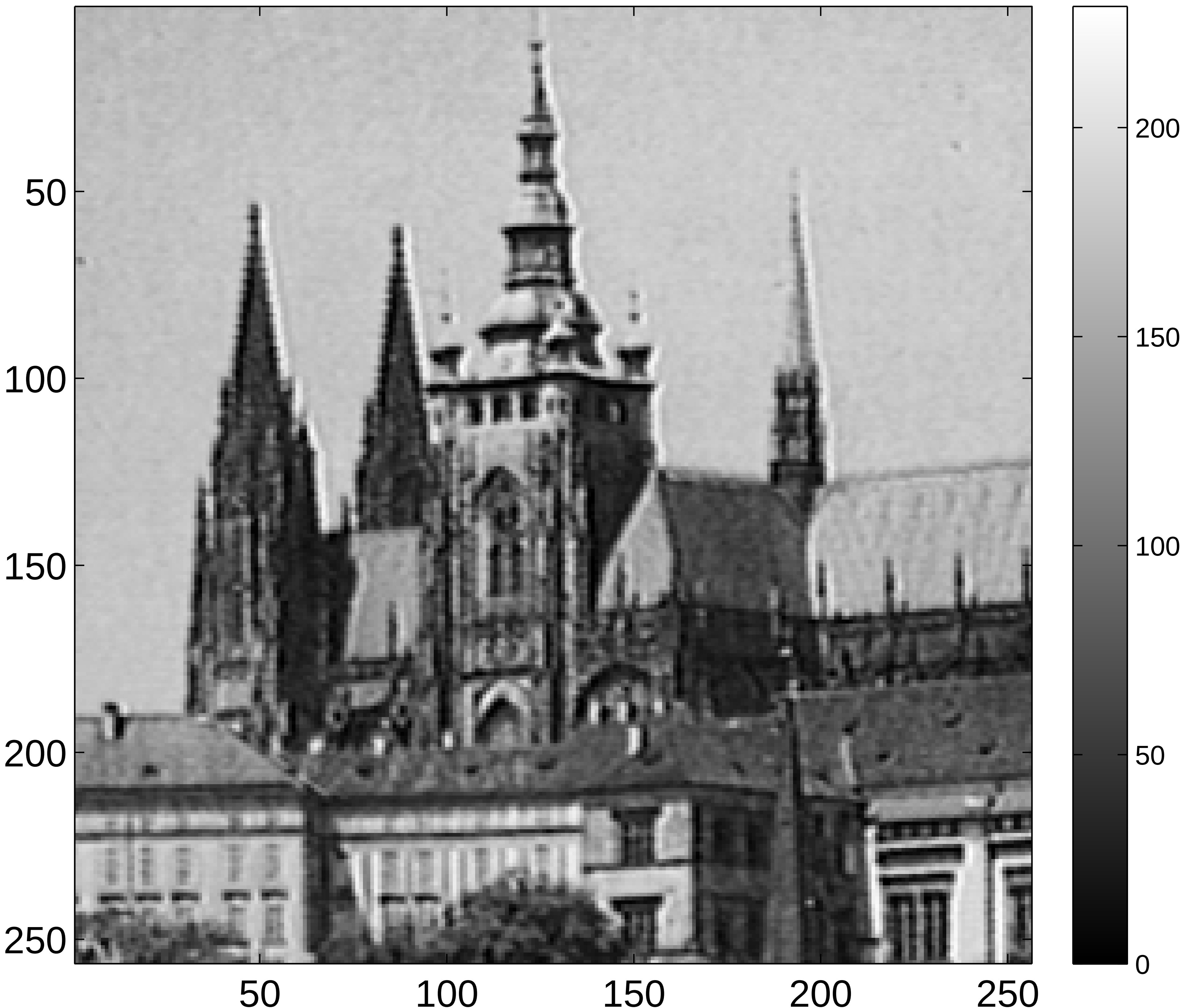


Imaginary part of the spectrum, centered
x 10^-10
1
0.5
0
-0.5
-1
500
400
300
200
100
Spatial frequency v
100
200
300
400
500
Spatial frequency u

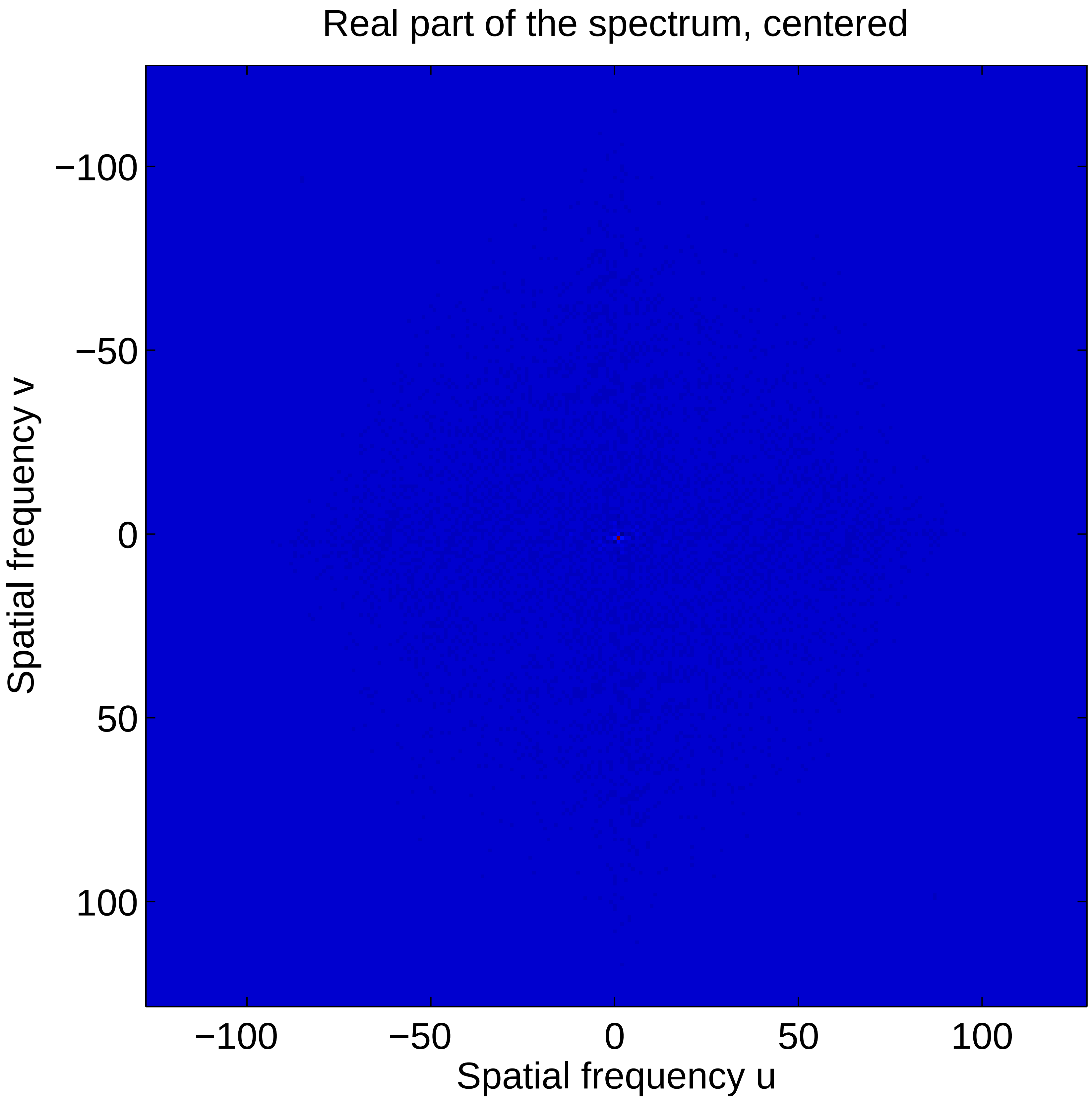

log power spectrum, centered
Spatial frequency v
Spatial frequency u
−250
−200
−150
−100
−50
0
50
100
150
200
250
−200
−100
0
100
200

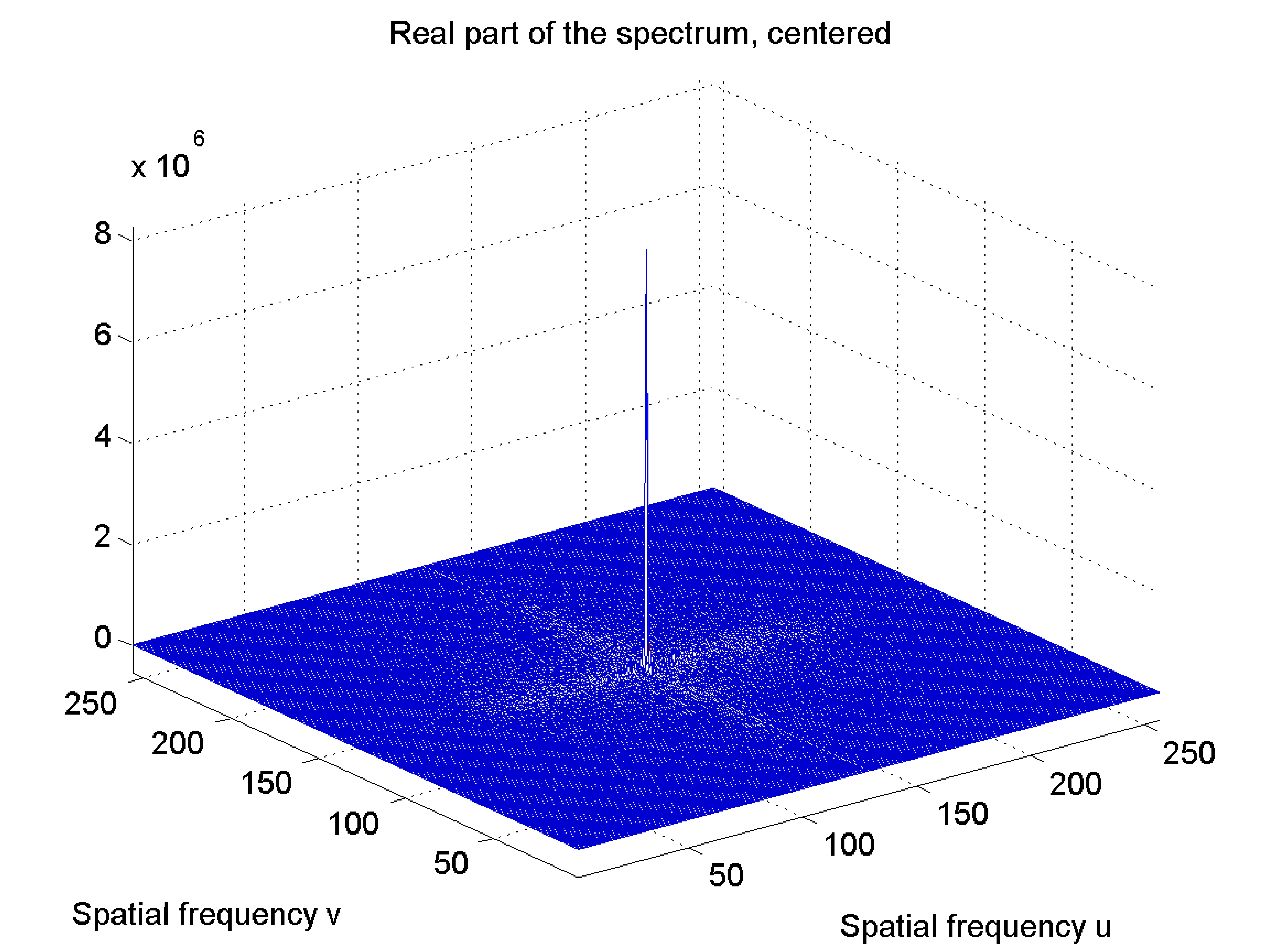


log power spectrum, centered
6
4
2
0
-2
200
100
0
-100
-200
Spatial frequency v
200
100
0
-100
-200
Spatial frequency u

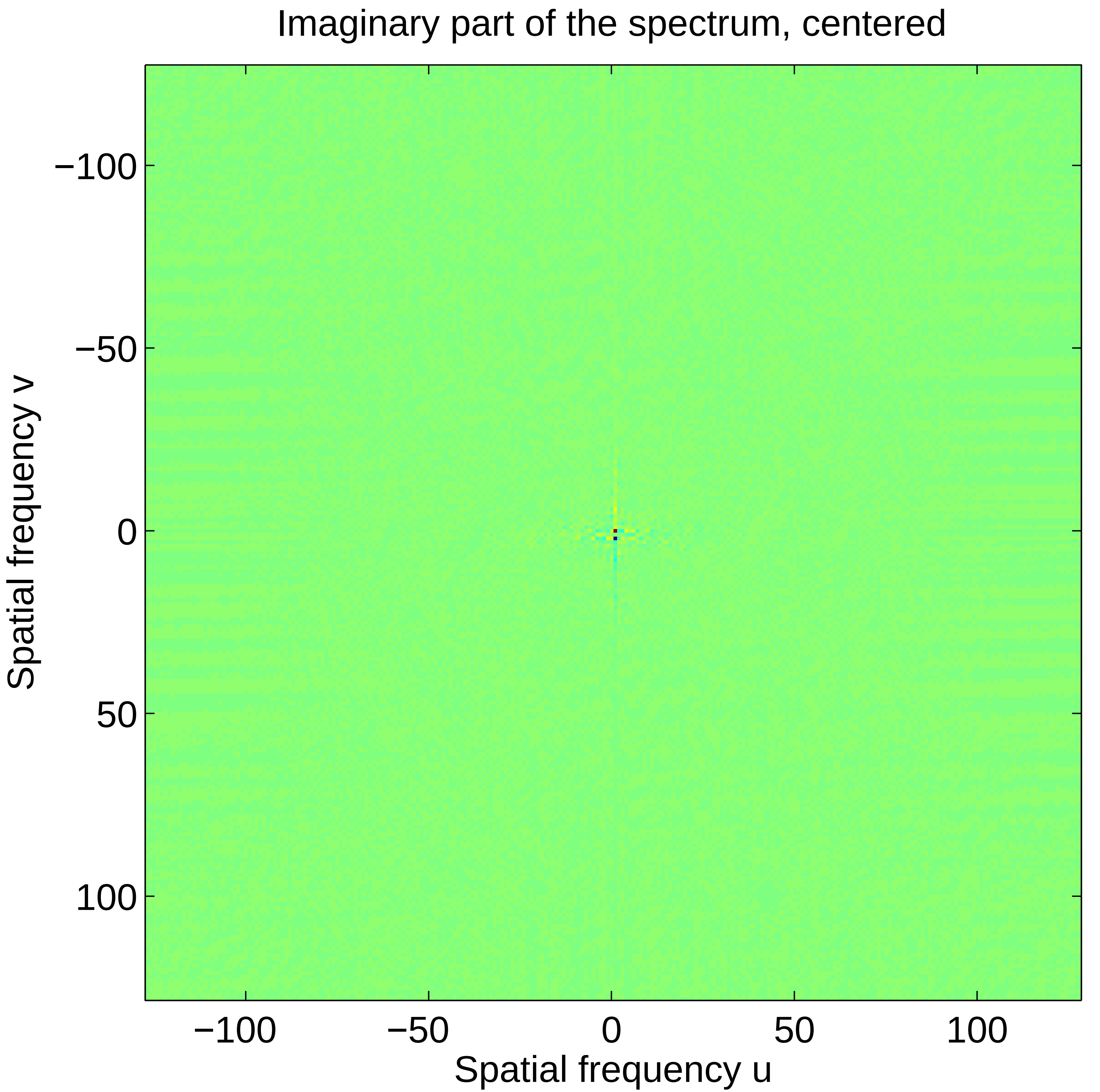


50
100
150
200
250
50
100
150
200
250
0
50
100
150
200

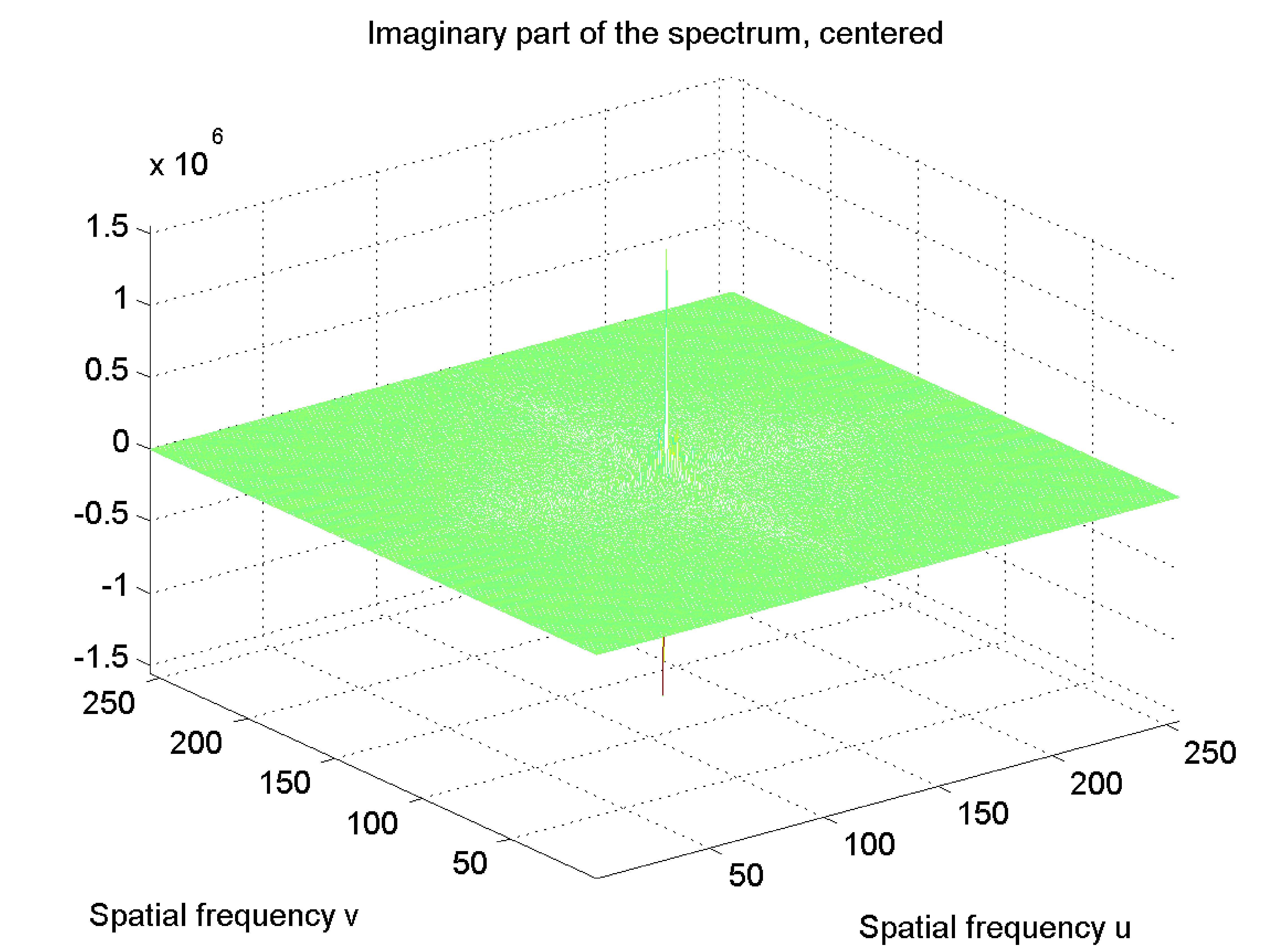

Real part of the spectrum, centered
−100
−50
0
50
100
Spatial frequency v
−100
−50
0
50
100
Spatial frequency u

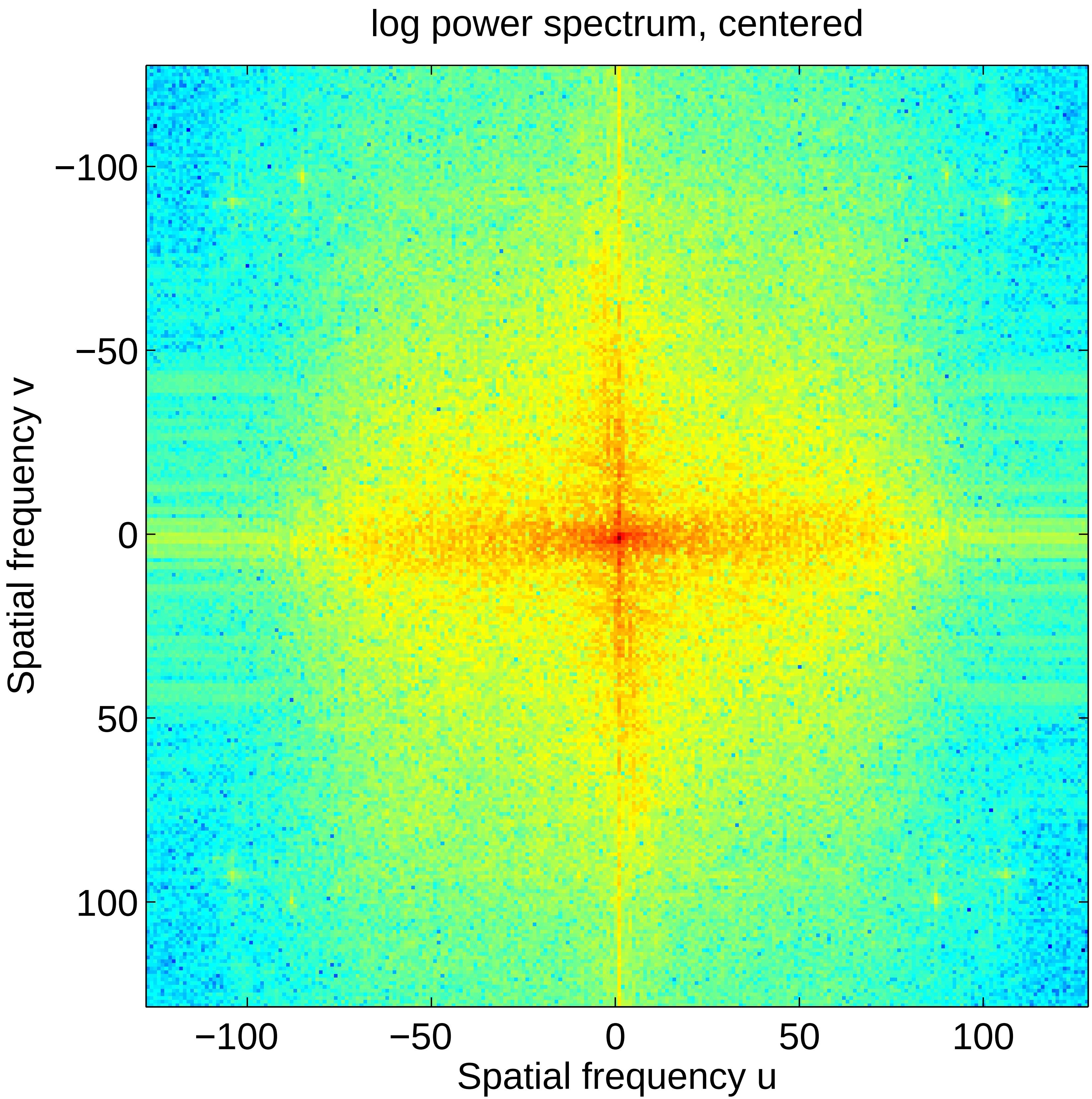


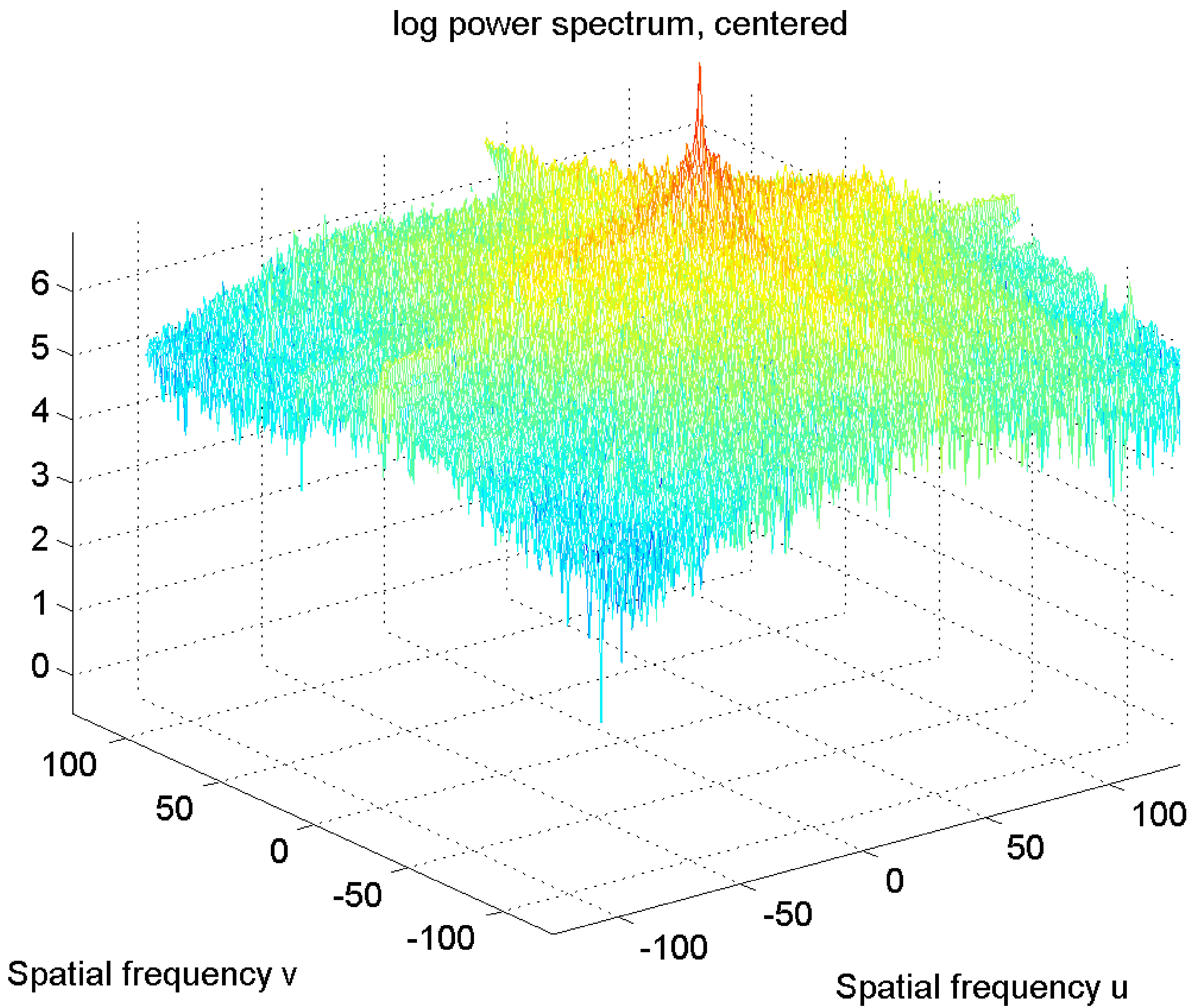
Real part of the spectrum, centered
x 10^6
8
6
4
2
0
250
200
150
100
50
250
200
150
100
50
Spatial frequency v
Spatial frequency u

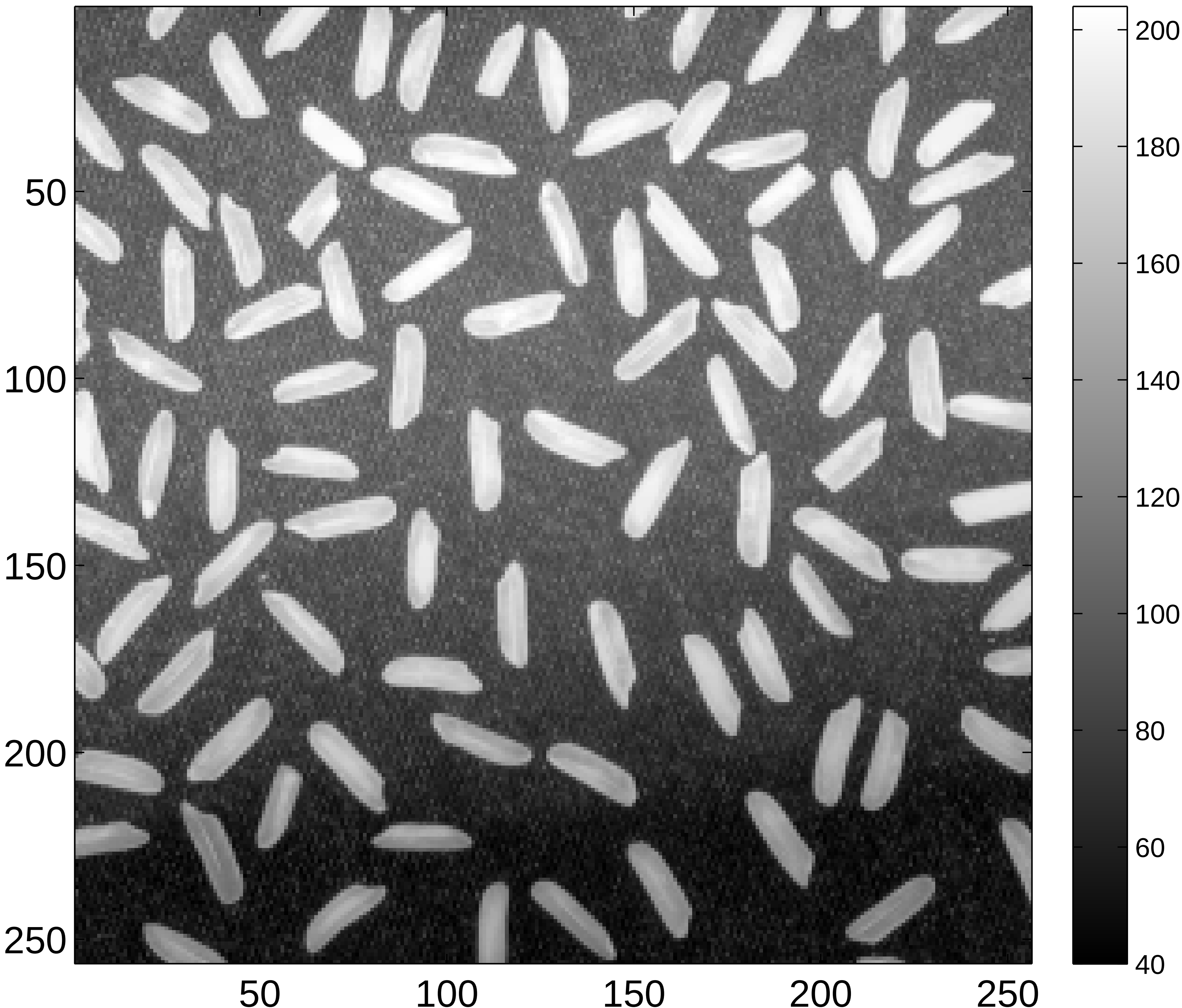


Imaginary part of the spectrum, centered
−100
−50
0
50
100
Spatial frequency v
−100
−50
0
50
100
Spatial frequency u

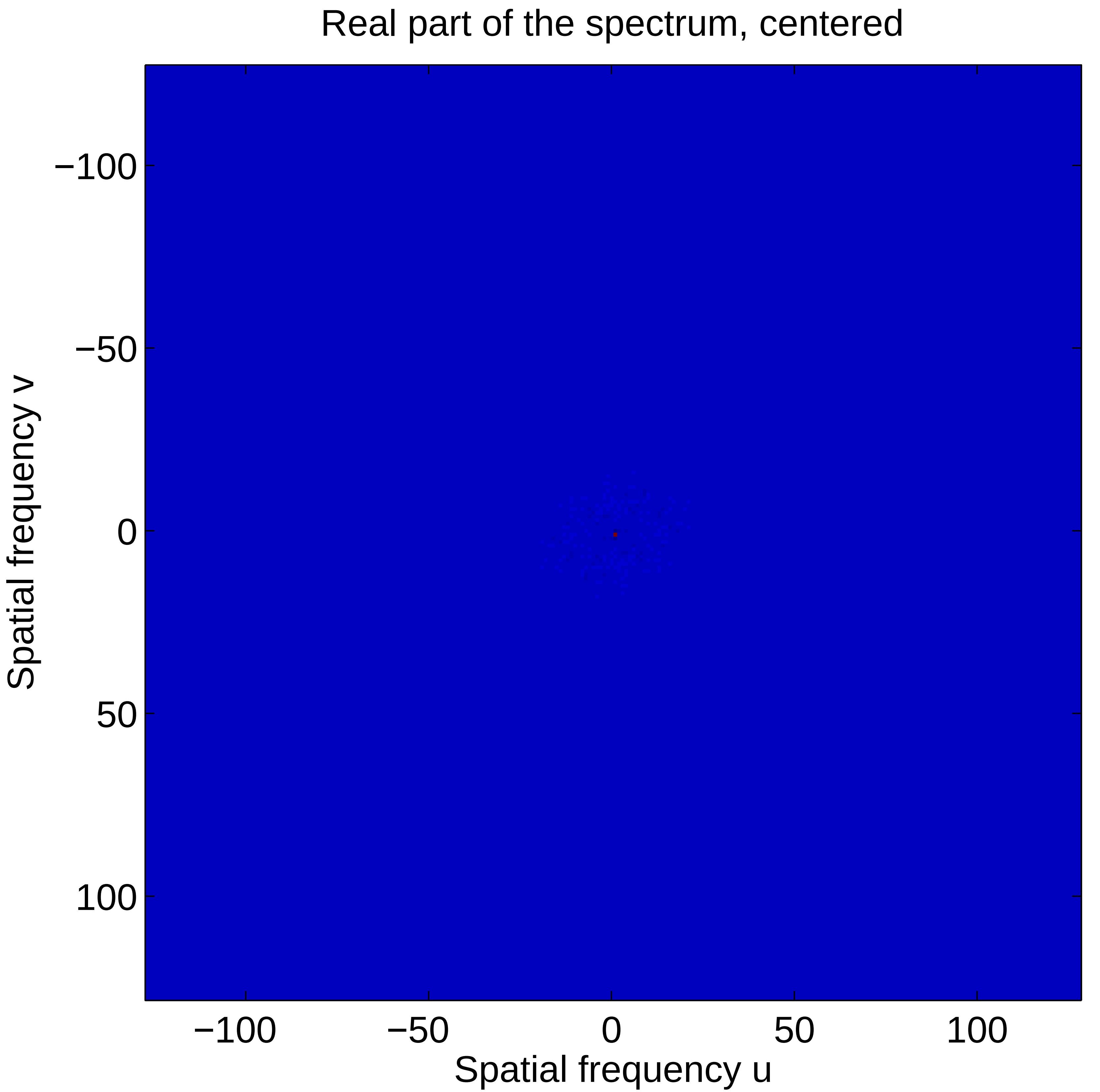

Imaginary part of the spectrum, centered
x 10^6
1.5
1
0.5
0
-0.5
-1
-1.5
250
200
150
100
50
Spatial frequency v
250
200
150
100
50
Spatial frequency u

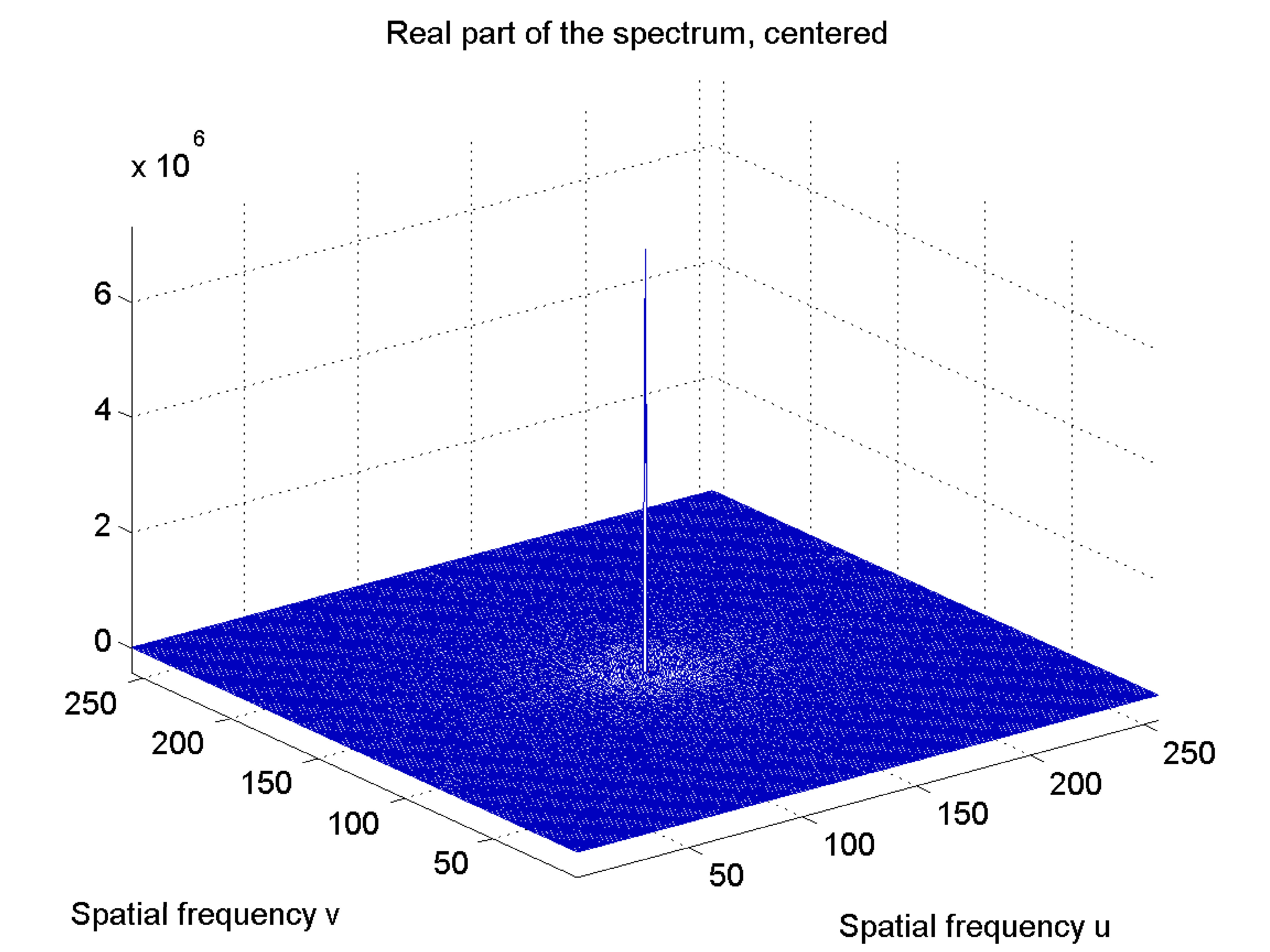

log power spectrum, centered
−100
−50
0
50
100
Spatial frequency v
−100
−50
0
50
100
Spatial frequency u


log power spectrum, centered
6
5
4
3
2
1
0
100
50
0
-50
-100
Spatial frequency v
-100
-50
0
50
100
Spatial frequency u

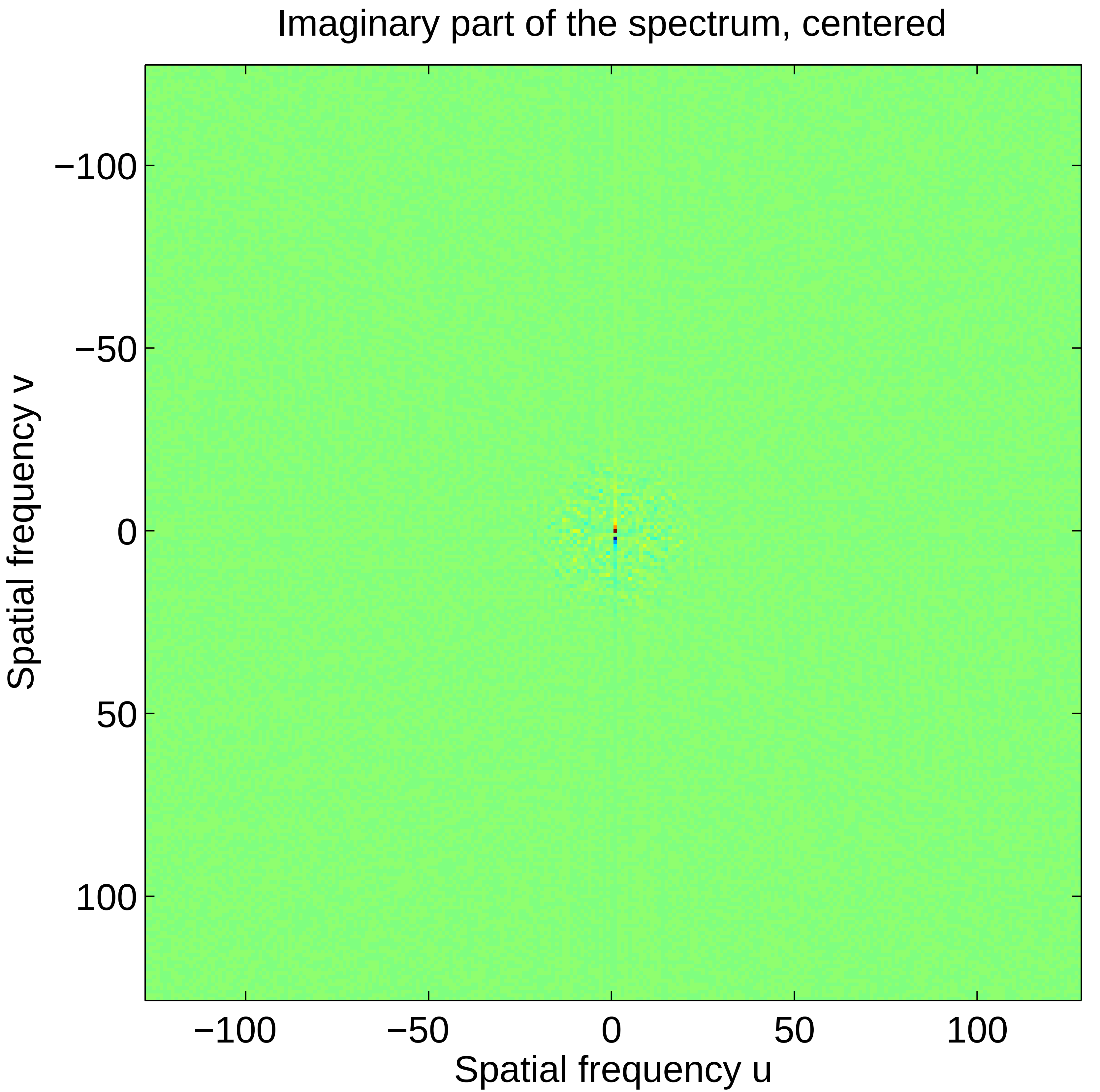


50
100
150
200
250
50
100
150
200
250
200
180
160
140
120
100
80
60
40

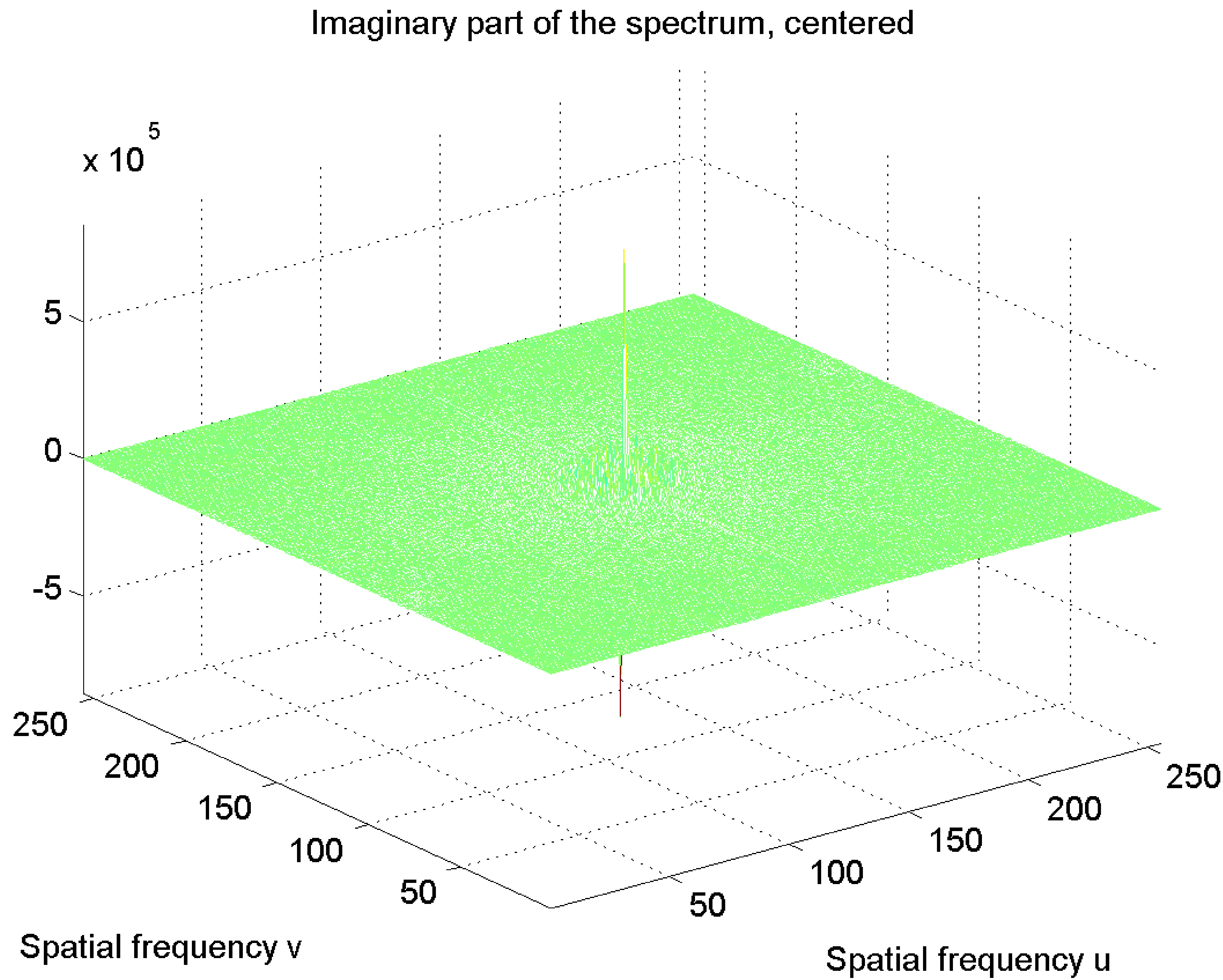

Real part of the spectrum, centered
Spatial frequency v
−100
−50
0
50
100
Spatial frequency u
−100
−50
0
50
100

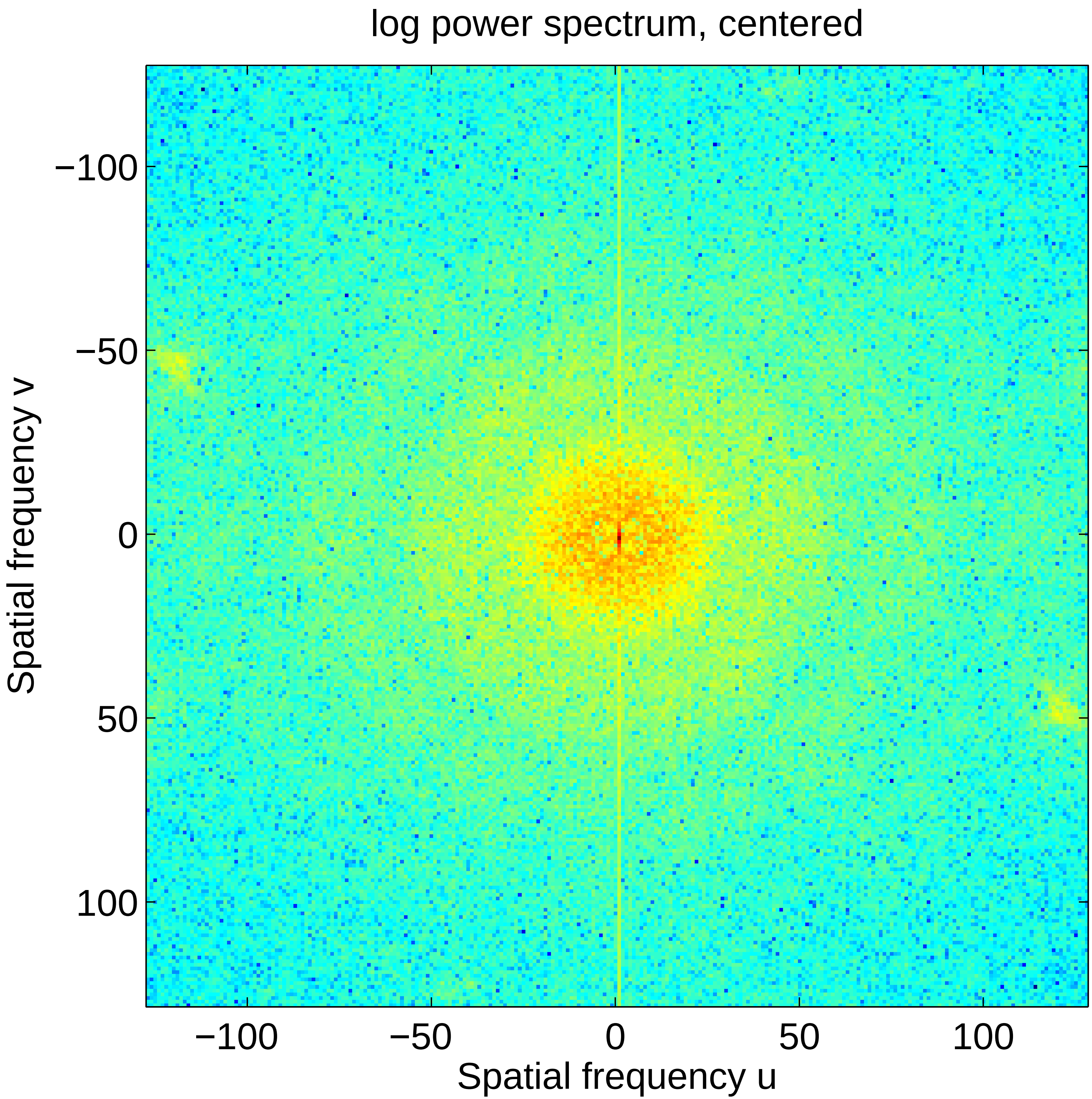


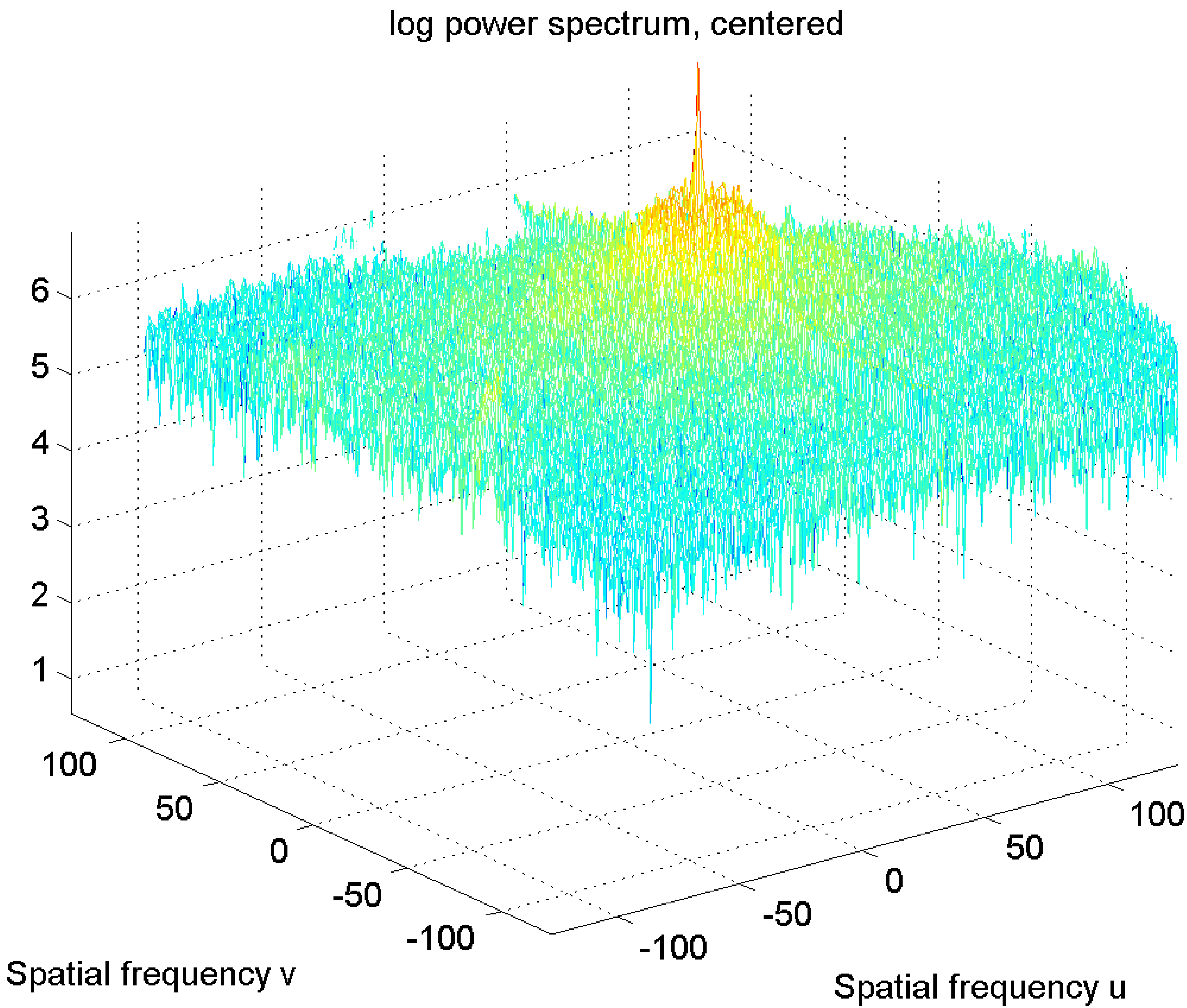
Real part of the spectrum, centered
x 10^6
6
4
2
0
250
200
150
100
50
Spatial frequency v
250
200
150
100
50
Spatial frequency u



Imaginary part of the spectrum, centered
−100
−50
0
50
100
Spatial frequency v
−100
−50
0
50
100
Spatial frequency u


Imaginary part of the spectrum, centered
x 10^5
5
0
-5
250
200
150
100
50
Spatial frequency v
50
100
150
200
250
Spatial frequency u

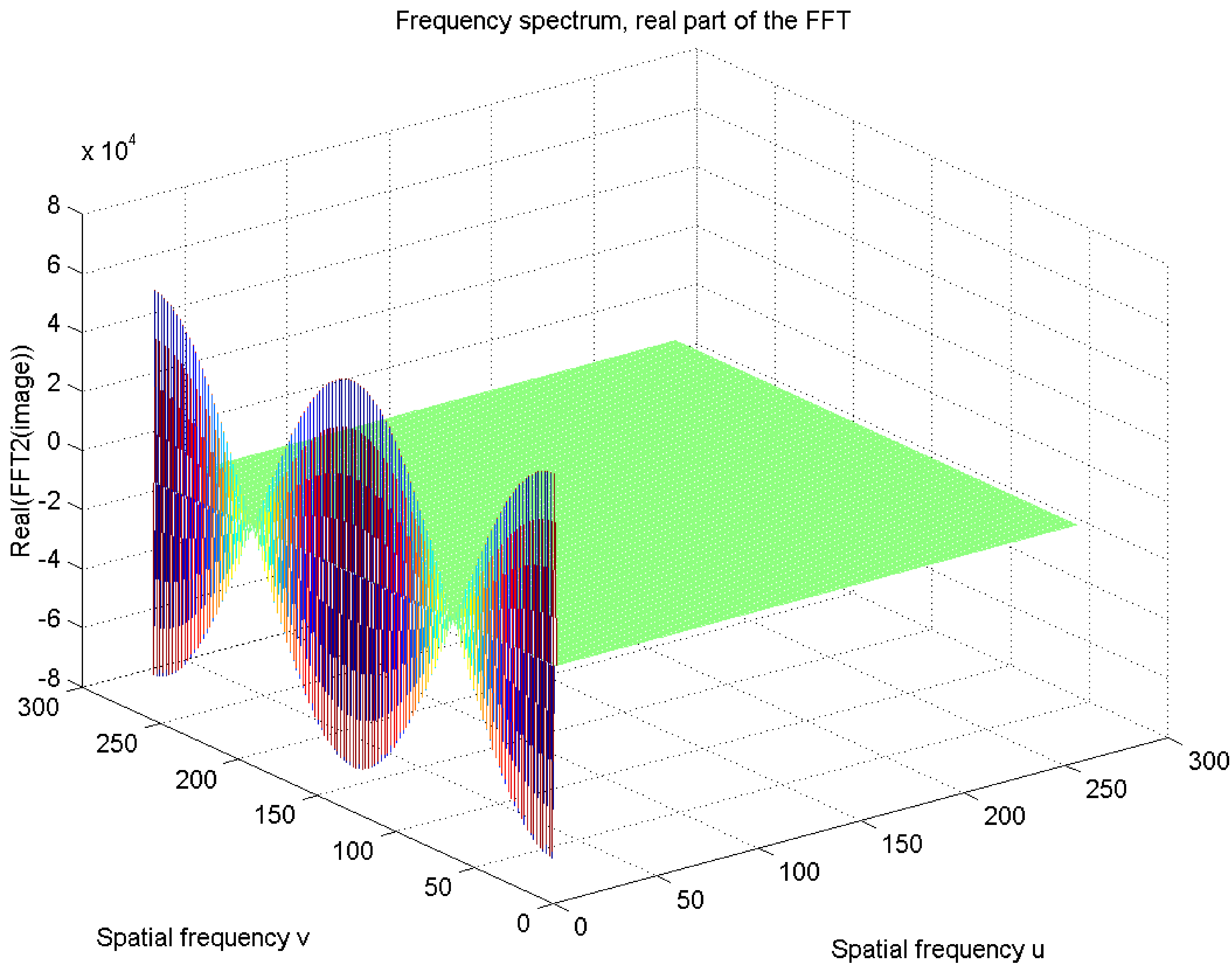

log power spectrum, centered
−100
−50
0
50
100
Spatial frequency v
−100
−50
0
50
100
Spatial frequency u



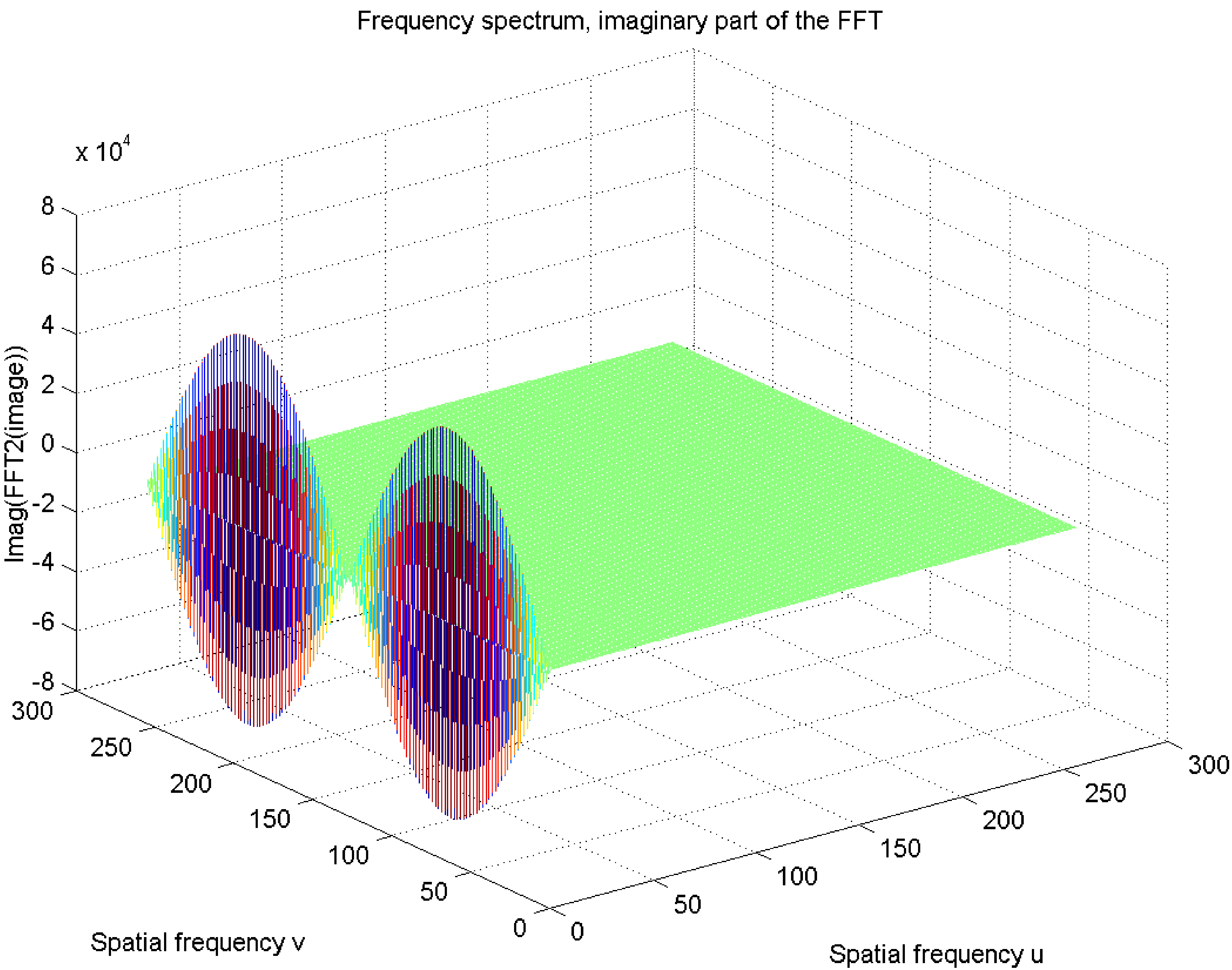
log power spectrum, centered
6
5
4
3
2
1
100
50
0
-50
-100
Spatial frequency v
-100
-50
0
50
100
Spatial frequency u

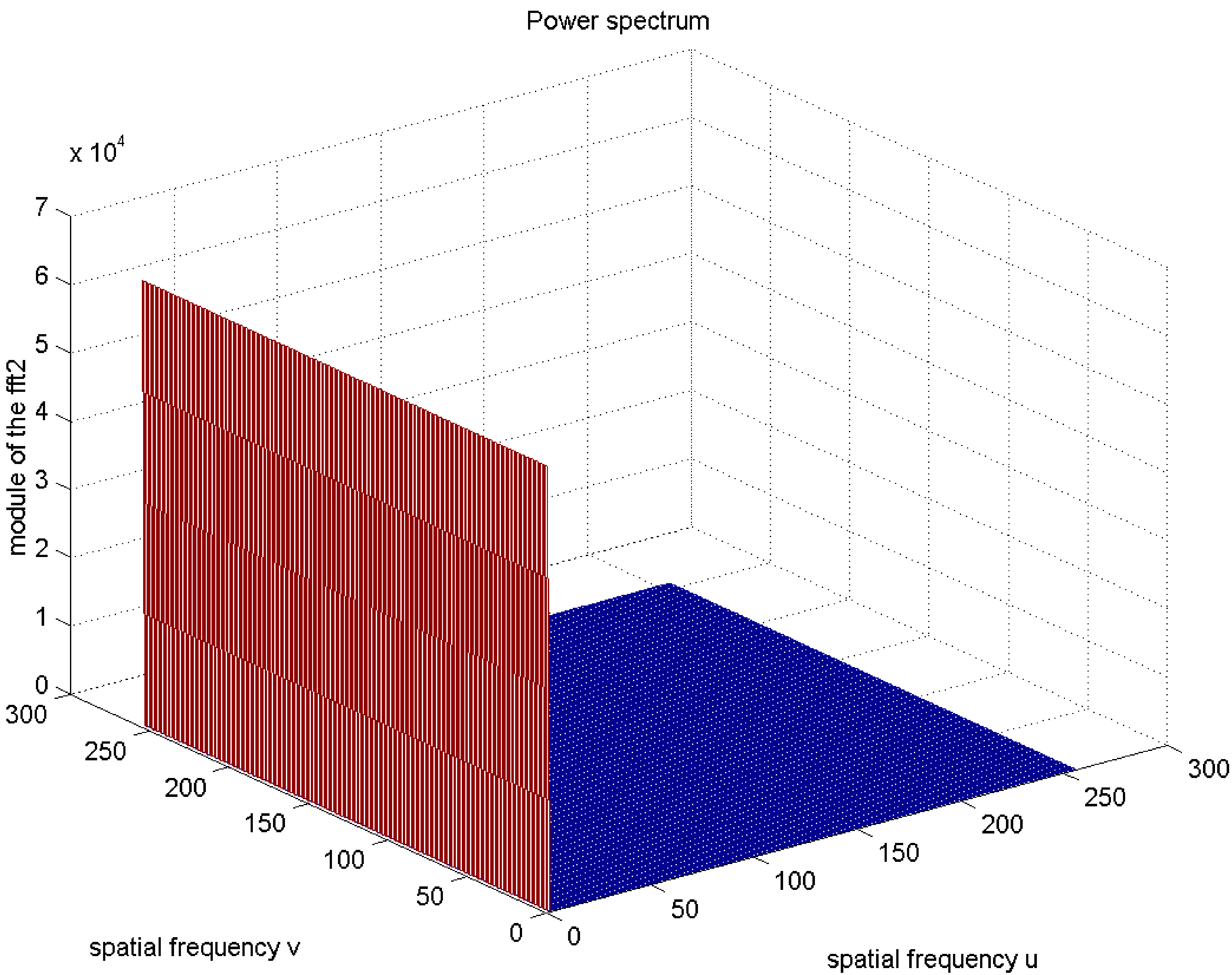
[redacted]

[redacted]

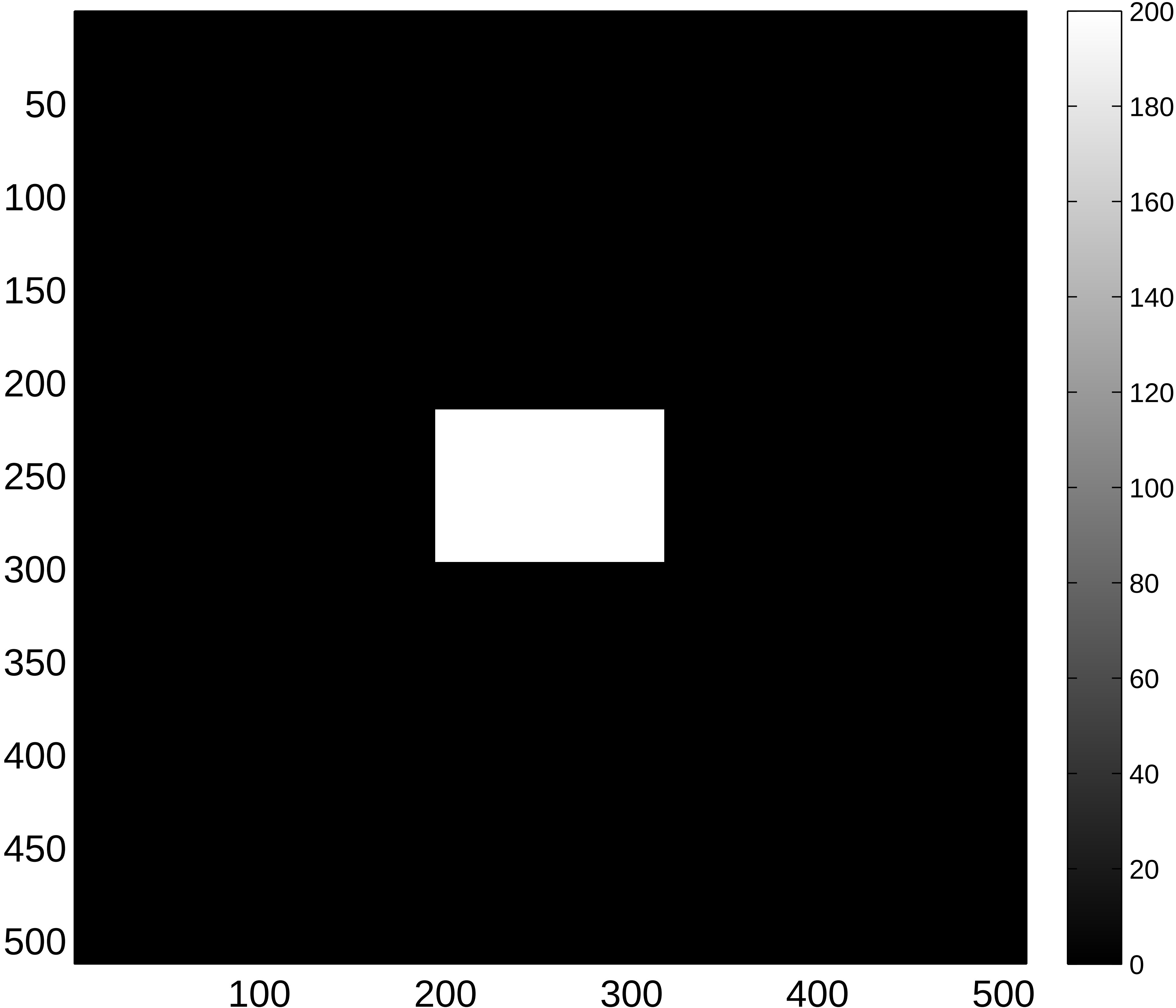


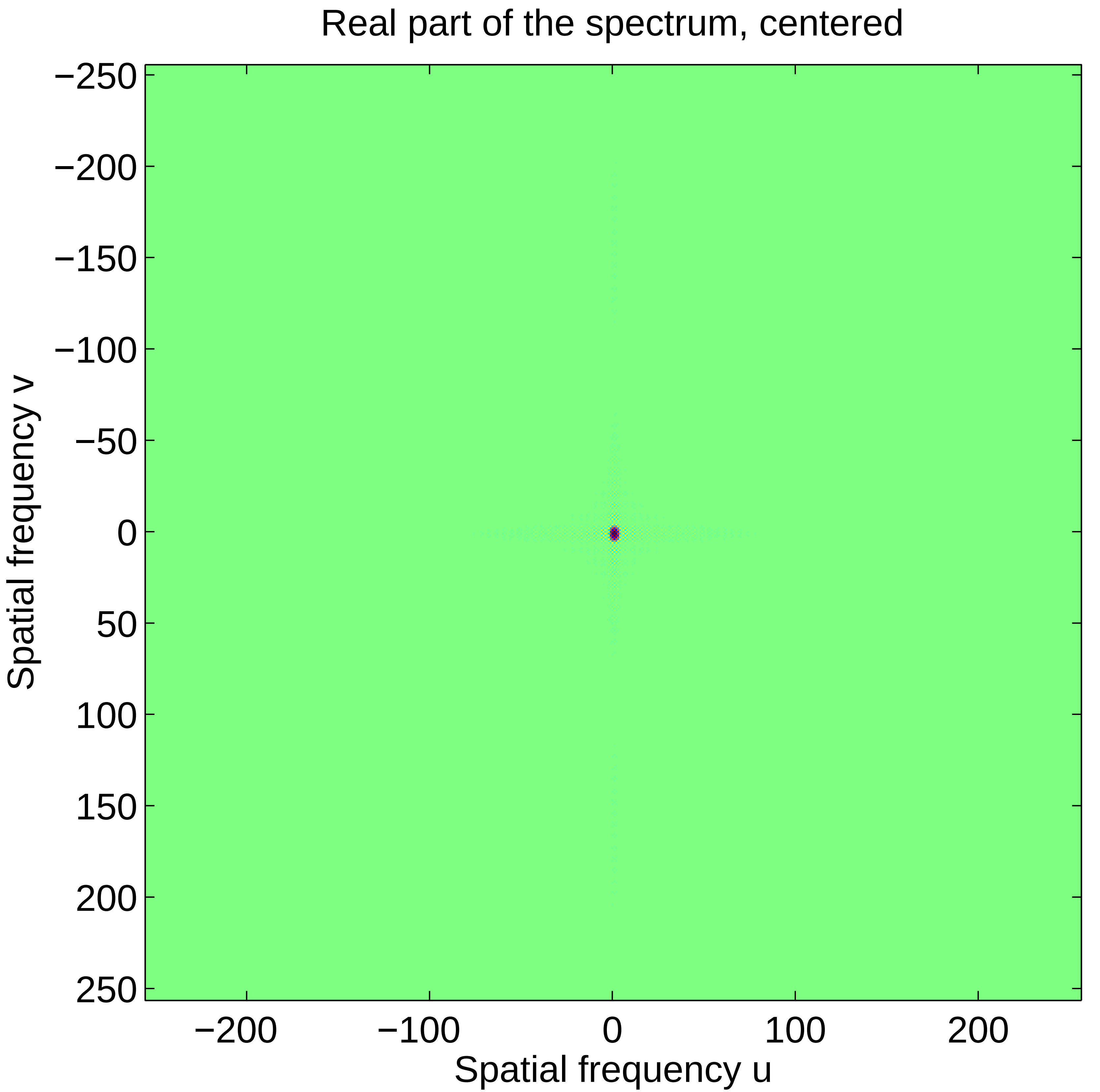
Frequency spectrum, real part of the FFT
x 10^4
8
6
4
2
0
-2
-4
-6
-8
Real(FFT2(image))
300
250
200
150
100
50
0
Spatial frequency v
0
50
100
150
200
250
300
Spatial frequency u

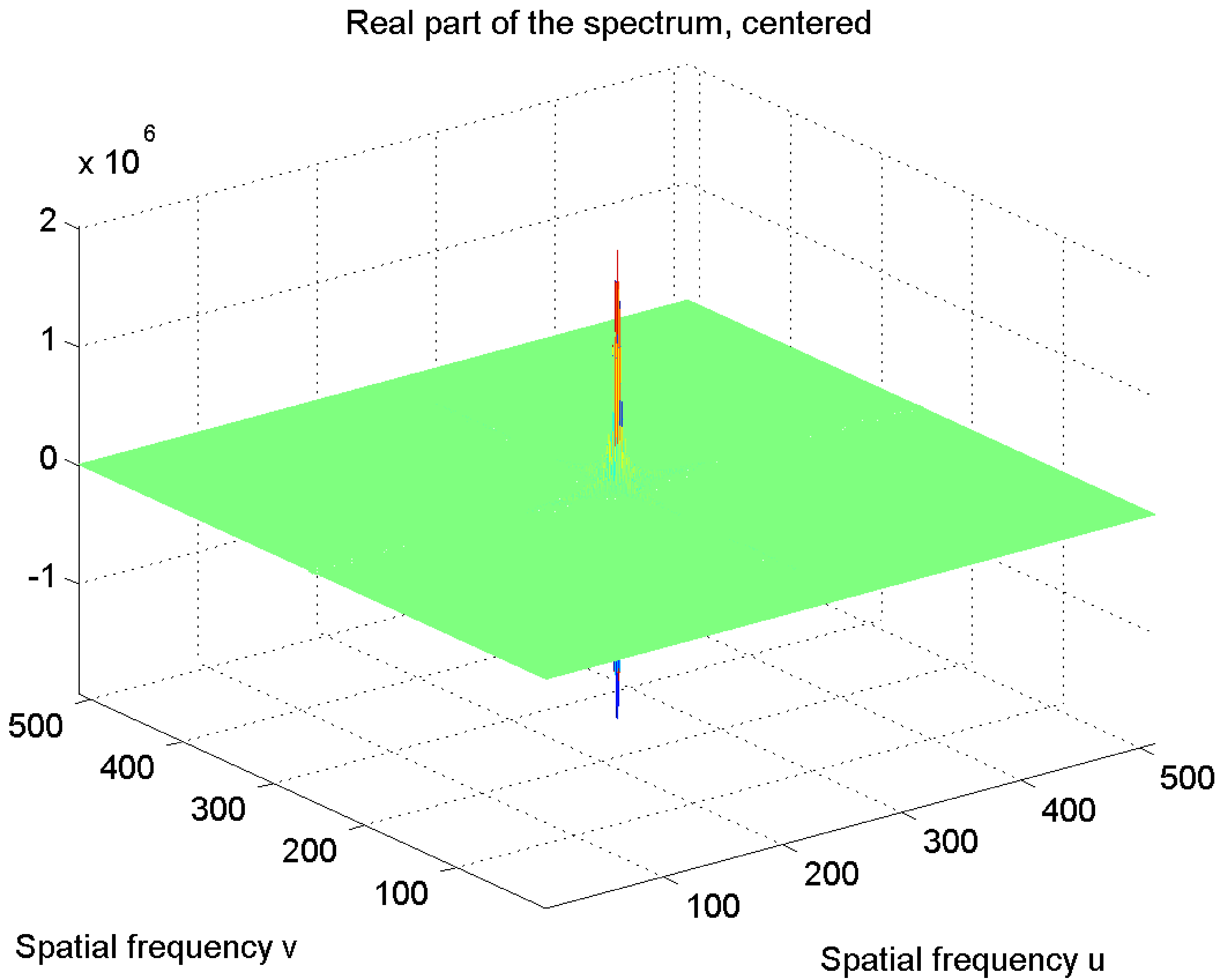

Frequency spectrum, imaginary part of the FFT
x 10^4
8
6
4
2
0
-2
-4
-6
-8
Imag(FFT2(image))
300
250
200
150
100
50
0
Spatial frequency v
0
50
100
150
200
250
300
Spatial frequency u

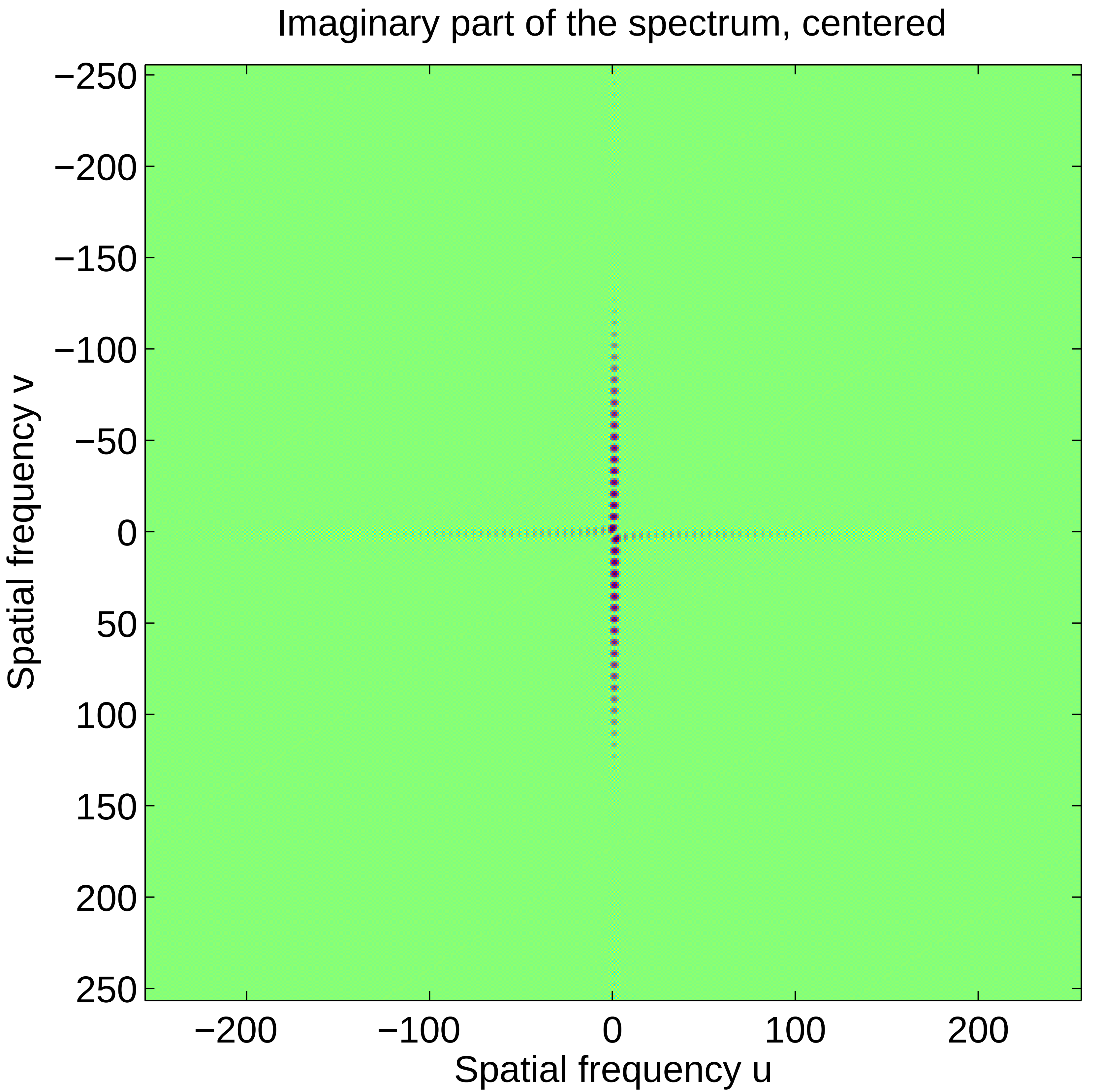


Power spectrum
x 10^4
7
6
5
4
3
2
1
0
module of the fft2
300
250
200
150
100
50
0
spatial frequency v
0
50
100
150
200
250
300
spatial frequency u

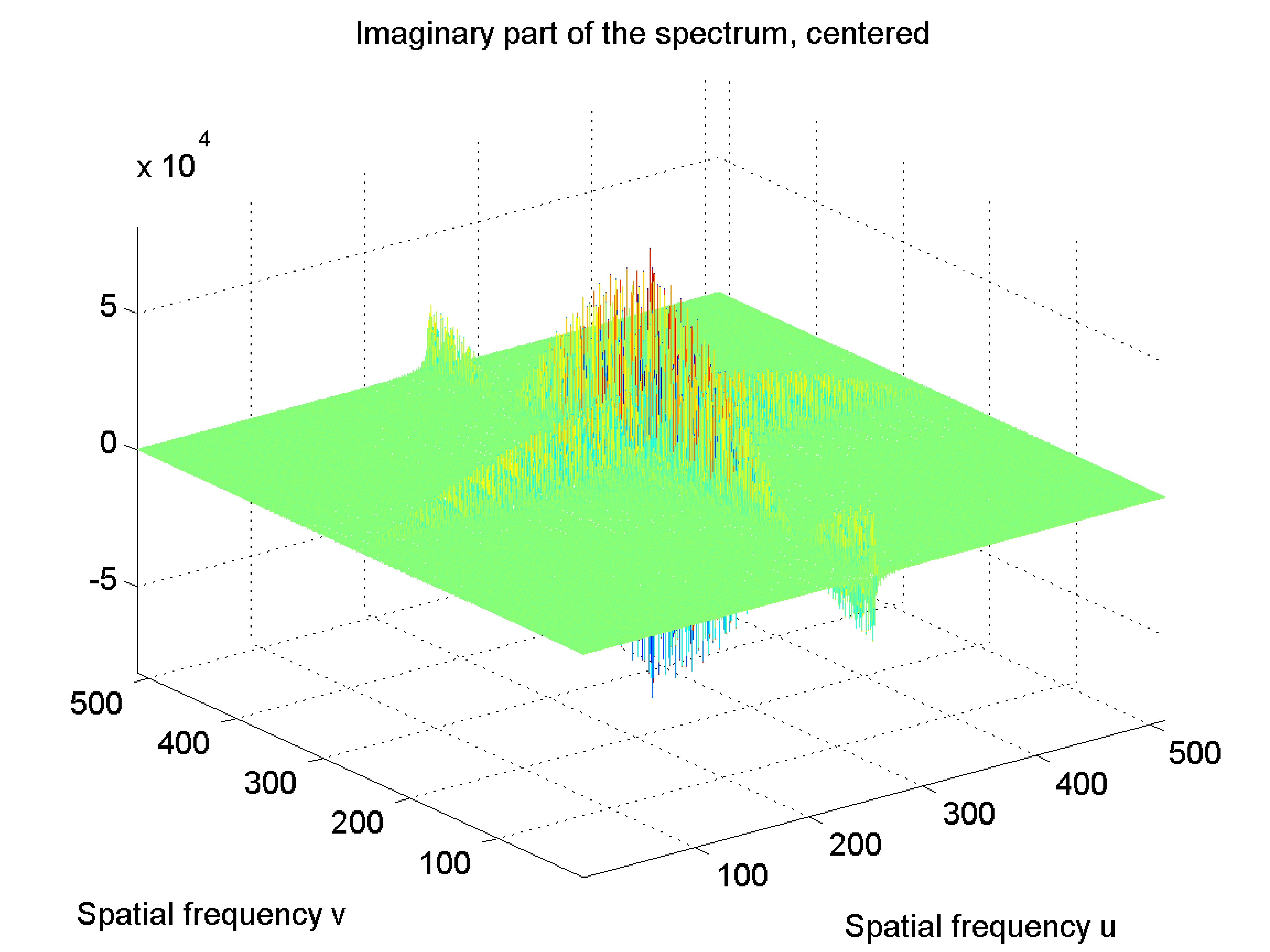

50
100
150
200
250
300
350
400
450
500
100
200
300
400
500
200
180
160
140
120
100
80
60
40
20
0

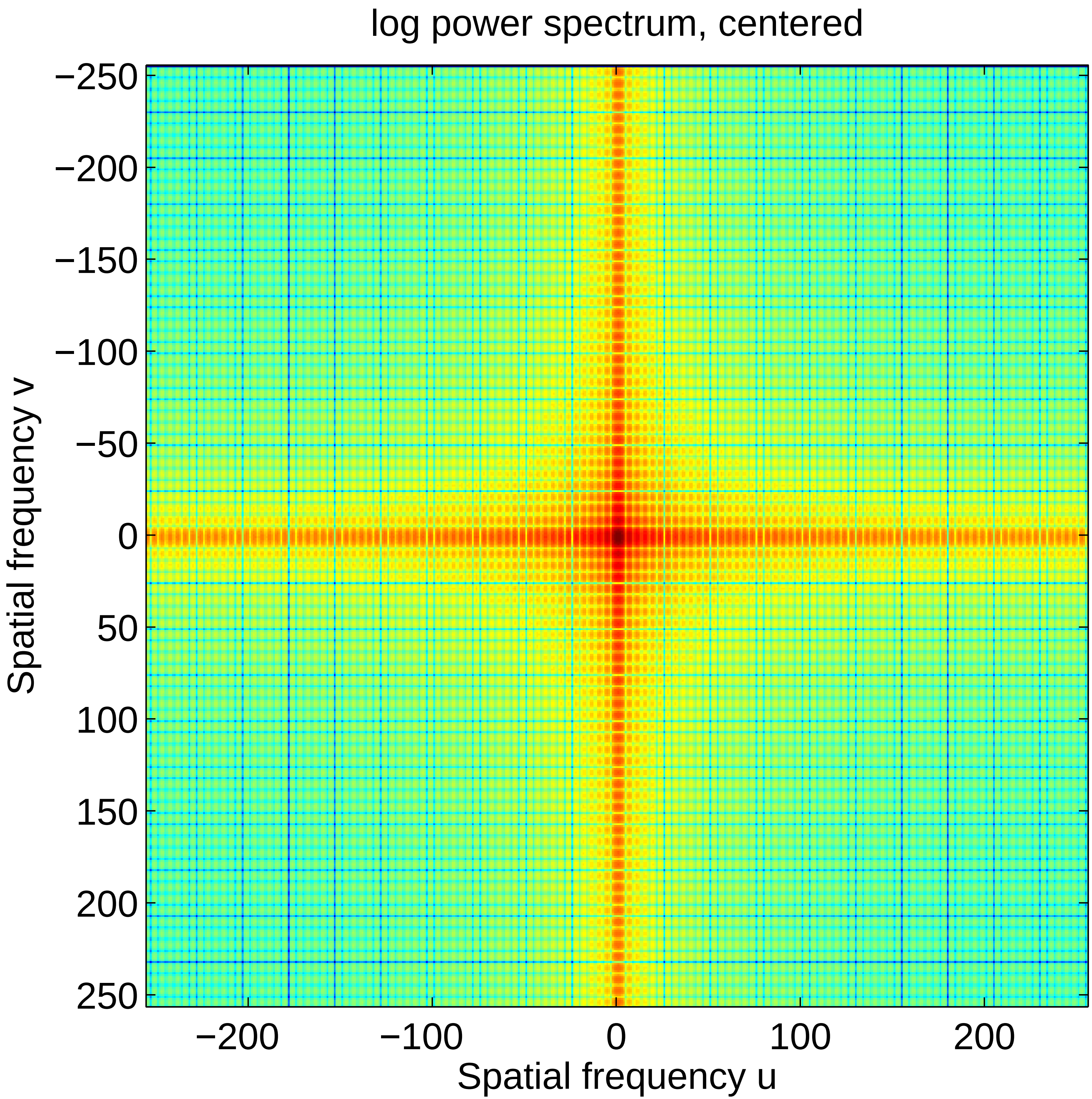

Real part of the spectrum, centered
Spatial frequency v
Spatial frequency u
−250
−200
−150
−100
−50
0
50
100
150
200
250
−200
−100
0
100
200


Real part of the spectrum, centered
x 10^6
2
1
0
-1
500
400
300
200
100
Spatial frequency v
100
200
300
400
500
Spatial frequency u

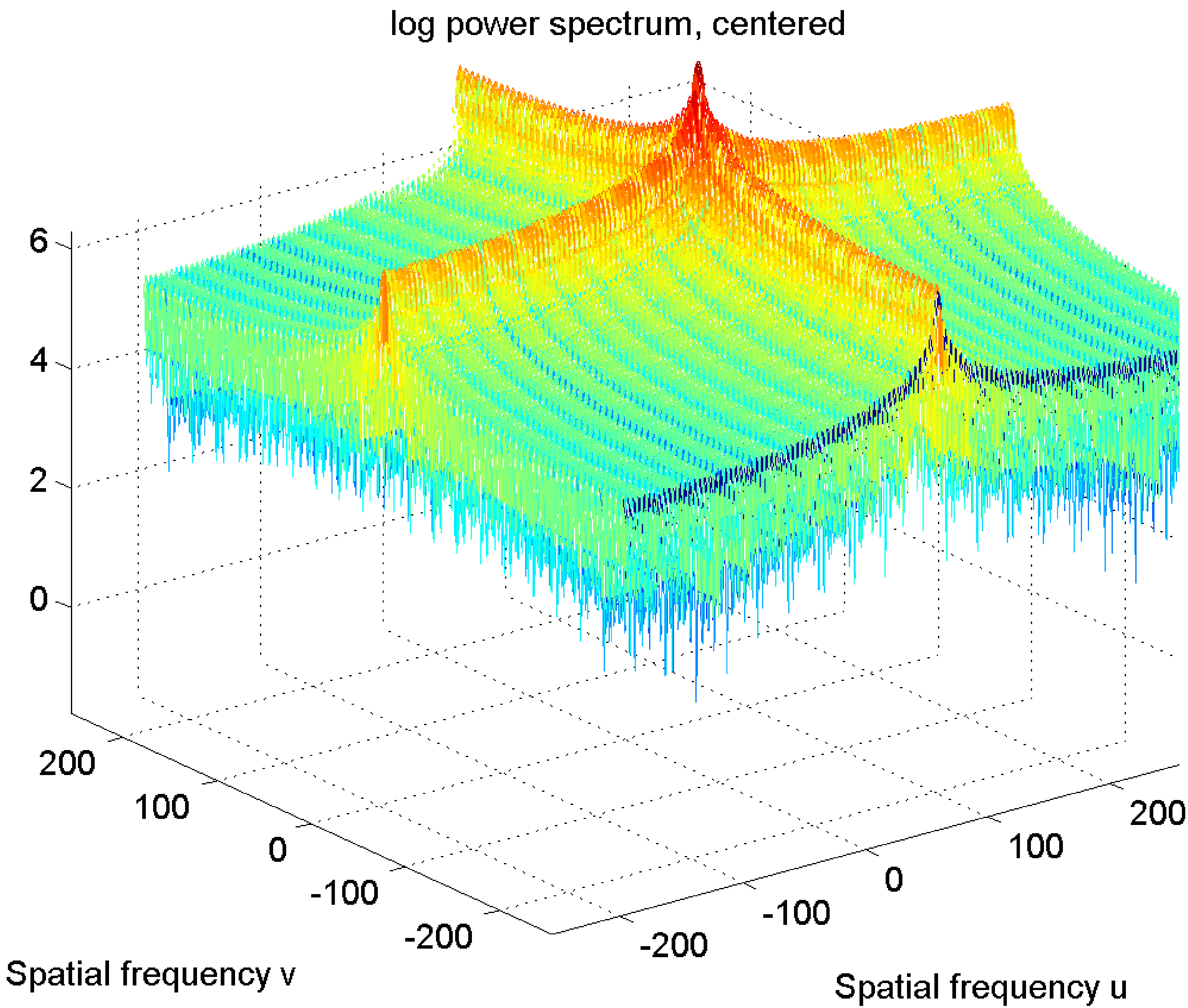


Imaginary part of the spectrum, centered
−250
−200
−150
−100
−50
0
50
100
150
200
250
Spatial frequency v
−200
−100
0
100
200
Spatial frequency u


Imaginary part of the spectrum, centered
x 10^4
5
0
-5
500
400
300
200
100
Spatial frequency v
100
200
300
400
500
Spatial frequency u



log power spectrum, centered
Spatial frequency v
−250
−200
−150
−100
−50
0
50
100
150
200
250
Spatial frequency u
−200
−100
0
100
200


log power spectrum, centered
6
4
2
0
200
100
0
-100
-200
Spatial frequency v
200
100
0
-100
-200
Spatial frequency u