Set theory

1 Basic definitions

A **set** is a collection of objects. The set of all elements that we consider in a certain situation is called the **universe** and is usually denoted by Ω. If an object x in Ω belongs to set S, we say that x is an **element** of S and write $x \in S$. If x is not an element of S then we write $x \notin S$. The **empty set**, usually denoted by $\emptyset$, is a set such that $x \notin$ for all $x \in \Omega$ (i.e. it has no elements). If all the elements in a set B also belong to a set A then B is a **subset** of A, which we denote by $B \subseteq A$. If in addition there is at least one element of A that does not belong to B then B is a proper subset of A, denoted by $B \subset A$.

The elements of a set can be arbitrary objects and in particular they can be sets themselves. This is the case for the power set of a set, defined in the next section.

A useful way of defining a set is through a statement concerning its elements. Let S be the set of elements such that a certain statement $s(x)$ holds, to define S we write

$$S := \{x \mid s(x)\}.$$

For example, $A := \{x \mid 1 < x < 3\}$ is the set of all elements greater than 1 and smaller than 3. Let us define some important sets and set operations using this notation.

2 Basic operations

Definition 2.1 (Set operations)**.**

- *The **complement** S^c of a set S contains all elements that are not in S.*

$$S^c := \{x \mid x \notin S\}.$$

- *The **union** of two sets A and B contains the objects that belong to A or B.*

$$A \cup B := \{x \mid x \in A \text{ or } x \in B\}.$$

This can be generalized to a sequence of sets A_1, A_2, ...

$$\bigcup_n A_n := \{x \mid x \in A_n \text{ for some } n\},$$

where the sequence may be infinite.

- *The **intersection** of two sets A and B contains the objects that belong to A and B.*

$$A \cap B := \{x \mid x \in A \text{ and } x \in B\}.$$

  Again, this can be generalized to a sequence,

$$\bigcap_n A_n := \{x \mid x \in A_n \text{ for all } n\}.$$

- *The **difference** of two sets A and B contains the elements in A that are not in B.*

$$A/B := \{x \mid x \in A \text{ and } x \notin B\}.$$

- *The **power set** 2^S of a set S is the set of all possible subsets of S, including $\emptyset$ and S.*

$$2^S := \{S' \mid S' \subseteq S\}.$$

Two sets are equal if they have the same elements, i.e. $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. It is easy to verify for instance that $(A^c)^c = A$, $S \cup \Omega = \Omega$, $S \cap \Omega = S$ or the following identities which are known as *De Morgan's laws*.

Theorem 2.2 (De Morgan's laws)**.** *For any two sets A and B*

$$(A \cup B)^c = A^c \cap B^c,$$
$$(A \cap B)^c = A^c \cup B^c.$$

Proof. Let us prove the first identity; the proof of the second is almost identical.

First we prove that $(A \cup B)^c \subseteq A^c \cap B^c$. A standard way to prove the inclusion of a set in another set is to show that if an element belongs to the first set then it must also belong to the second. Any element x in $(A \cup B)^c$ (if the set is empty then the inclusion holds trivially, since $\emptyset \subseteq S$ for any set S) is in A^c; otherwise it would belong to A and consequently to $A \cup B$. Similarly, x also belongs to B^c. We conclude that x belongs to $A^c \cap B^c$, which proves the inclusion.

To complete the proof we establish $A^c \cap B^c \subseteq (A \cup B)^c$. If $x \in A^c \cap B^c$, then $x \notin A$ and $x \notin B$, so $x \notin A \cup B$ and consequently $x \in (A \cup B)^c$. □