14 Design of Experiments with Several Factors

CHAPTER OUTLINE

LEARNING OBJECTIVES

After careful study of this chapter, you should be able to do the following:

1. Design and conduct engineering experiments involving several factors using the factorial design approach
2. Know how to analyze and interpret main effects and interactions
3. Understand how the ANOVA is used to analyze the data from these experiments
4. Assess model adequacy with residual plots
5. Know how to use the two-level series of factorial designs

6. **Understand how two-level factorial designs can be run in blocks**
7. **Design and conduct two-level fractional factorial designs**

CD MATERIAL

8. **Incorporate random factors in factorial experiments.**
9. **Test for curvature in two-level factorial designs by using center points.**
10. **Use response surface methodology for process optimization experiments.**

Answers for most odd numbered exercises are at the end of the book. Answers to exercises whose numbers are surrounded by a box can be accessed in the e-text by clicking on the box. Complete worked solutions to certain exercises are also available in the e-Text. These are indicated in the Answers to Selected Exercises section by a box around the exercise number. Exercises are also available for some of the text sections that appear on CD only. These exercises may be found in the Mind-Expanding Exercises at the end of the chapter.

14-1 INTRODUCTION

An **experiment** is just a **test** or series of tests. Experiments are performed in all engineering and scientific disciplines and are an important part of the way we learn about how systems and processes work. The validity of the conclusions that are drawn from an experiment depends to a large extent on how the experiment was conducted. Therefore, the **design** of the experiment plays a major role in the eventual solution of the problem that initially motivated the experiment.

In this chapter we focus on experiments that include two or more factors that the experimenter thinks may be important. The **factorial experimental design** will be introduced as a powerful technique for this type of problem. Generally, in a factorial experimental design, experimental trials (or runs) are performed at all combinations of factor levels. For example, if a chemical engineer is interested in investigating the effects of reaction time and reaction temperature on the yield of a process, and if two levels of time (1 and 1.5 hours) and two levels of temperature (125 and 150°F) are considered important, a factorial experiment would consist of making experimental runs at each of the four possible combinations of these levels of reaction time and reaction temperature.

Most of the statistical concepts introduced in Chapter 13 for single-factor experiments can be extended to the factorial experiments of this chapter. The **analysis of variance** (**ANOVA**), in particular, will continue to be used as one of the primary tools for statistical data analysis. We will also introduce several graphical methods that are useful in analyzing the data from designed experiments.

14-2 SOME APPLICATIONS OF DESIGNED EXPERIMENTS (CD ONLY)

14-3 FACTORIAL EXPERIMENTS

When several factors are of interest in an experiment, a **factorial experimental design** should be used. As noted previously, in these experiments factors are varied together.

Definition

> By a **factorial experiment** we mean that in each complete trial or replicate of the experiment all possible combinations of the levels of the factors are investigated.

Thus, if there are two factors A and B with a levels of factor A and b levels of factor B, each replicate contains all ab treatment combinations.

The effect of a factor is defined as the change in response produced by a change in the level of the factor. It is called a **main effect** because it refers to the primary factors in the study. For example, consider the data in Table 14-1. This is a factorial experiment with two factors, A and B, each at two levels (A_{low}, A_{high}, and B_{low}, B_{high}). The main effect of factor A is the difference between the average response at the high level of A and the average response at the low level of A, or

$$A = \frac{30 + 40}{2} - \frac{10 + 20}{2} = 20$$

That is, changing factor A from the low level to the high level causes an average response increase of 20 units. Similarly, the main effect of B is

$$B = \frac{20 + 40}{2} - \frac{10 + 30}{2} = 10$$

In some experiments, the difference in response between the levels of one factor is not the same at all levels of the other factors. When this occurs, there is an **interaction** between the factors. For example, consider the data in Table 14-2. At the low level of factor B, the A effect is

$$A = 30 - 10 = 20$$

and at the high level of factor B, the A effect is

$$A = 0 - 20 = -20$$

Since the effect of A depends on the level chosen for factor B, there is interaction between A and B.

When an interaction is large, the corresponding main effects have very little practical meaning. For example, by using the data in Table 14-2, we find the main effect of A as

$$A = \frac{30 + 0}{2} - \frac{10 + 20}{2} = 0$$

and we would be tempted to conclude that there is no factor A effect. However, when we examined the effects of A at *different levels of factor B*, we saw that this was not the case. The effect of factor A depends on the levels of factor B. Thus, knowledge of the AB interaction is more useful than knowledge of the main effect. A significant interaction can mask the significance of main effects. Consequently, when interaction is present, the main effects of the factors involved in the interaction may not have much meaning.

Table 14-1 A Factorial Experiment with Two Factors

	Factor B	
Factor A	B_{low}	B_{high}
A_{low}	10	20
A_{high}	30	40

Table 14-2 A Factorial Experiment with Interaction

	Factor B	
Factor A	B_{low}	B_{high}
A_{low}	10	20
A_{high}	30	0

It is easy to estimate the interaction effect in factorial experiments such as those illustrated in Tables 14-1 and 14-2. In this type of experiment, when both factors have two levels, the AB interaction effect is the difference in the diagonal averages. This represents one-half the difference between the A effects at the two levels of B. For example, in Table 14-1, we find the AB interaction effect to be

$$AB = \frac{20 + 30}{2} - \frac{10 + 40}{2} = 0$$

Thus, there is no interaction between A and B. In Table 14-2, the AB interaction effect is

$$AB = \frac{20 + 30}{2} - \frac{10 + 0}{2} = 20$$

As we noted before, the interaction effect in these data is very large.

The concept of interaction can be illustrated graphically in several ways. Figure 14-1 plots the data in Table 14-1 against the levels of A for both levels of B. Note that the B_{low} and B_{high} lines are approximately parallel, indicating that factors A and B do not interact significantly. Figure 14-2 presents a similar plot for the data in Table 14-2. In this graph, the B_{low} and B_{high} lines are not parallel, indicating the interaction between factors A and B. Such graphical displays are called **two-factor interaction plots.** They are often useful in presenting the results of experiments, and many computer software programs used for analyzing data from designed experiments will construct these graphs automatically.

Figures 14-3 and 14-4 present another graphical illustration of the data from Tables 14-1 and 14-2. In Fig. 14-3 we have shown a **three-dimensional surface plot** of the data from Table 14-1. These data contain no interaction, and the surface plot is a plane lying above the A-B space. The slope of the plane in the A and B directions is proportional to the main effects of factors A and B, respectively. Figure 14-4 is a surface plot of the data from Table 14-2. Notice that the effect of the interaction in these data is to “twist” the plane, so that there is curvature in the response function. **Factorial experiments are the only way to discover interactions between variables.**

An alternative to the factorial design that is (unfortunately) used in practice is to change the factors ***one at a time*** rather than to vary them simultaneously. To illustrate this one-factor-at-a-time procedure, suppose that an engineer is interested in finding the values of temperature and pressure that maximize yield in a chemical process. Suppose that we fix temperature at 155°F (the current operating level) and perform five runs at different levels of time, say,

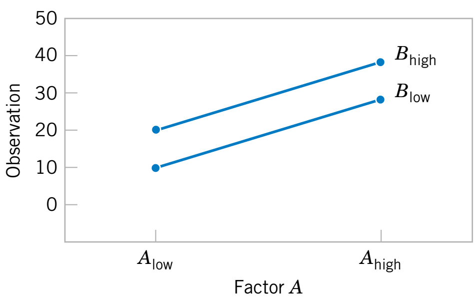


Figure 14-1 Factorial experiment, no interaction.

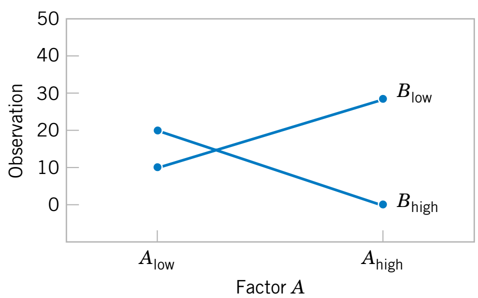


Figure 14-2 Factorial experiment, with interaction.

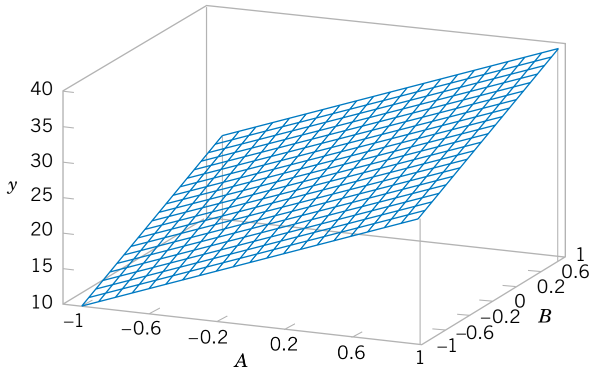


Figure 14-3 Three-dimensional surface plot of the data from Table 14-1, showing main effects of the two factors *A* and *B*.

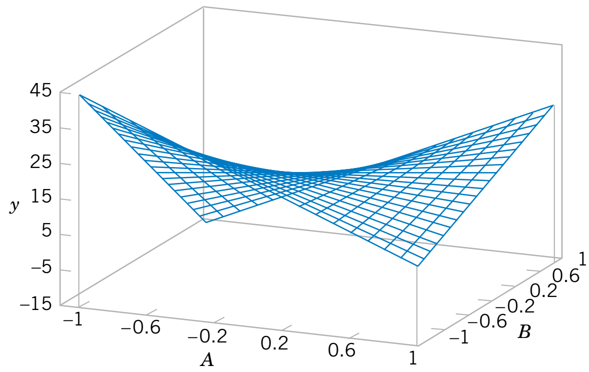


Figure 14-4 Three-dimensional surface plot of the data from Table 14-2 showing the effect of the *A* and *B* interaction.

0.5, 1.0, 1.5, 2.0, and 2.5 hours. The results of this series of runs are shown in Fig. 14-5. This figure indicates that maximum yield is achieved at about 1.7 hours of reaction time. To optimize temperature, the engineer then fixes time at 1.7 hours (the apparent optimum) and performs five runs at different temperatures, say, 140, 150, 160, 170, and 180°F. The results of this set of runs are plotted in Fig. 14-6. Maximum yield occurs at about 155°F. Therefore, we would conclude that running the process at 155°F and 1.7 hours is the best set of operating conditions, resulting in yields of around 75%.

Figure 14-7 displays the contour plot of actual process yield as a function of temperature and time with the one-factor-at-a-time experiments superimposed on the contours. Clearly, this one-factor-at-a-time approach has failed dramatically here, as the true optimum is at least 20 yield points higher and occurs at much lower reaction times and higher temperatures. The failure to discover the importance of the shorter reaction times is particularly important because this could have significant impact on production volume or capacity, production planning, manufacturing cost, and total productivity.

The one-factor-at-a-time approach has failed here because it cannot detect the interaction between temperature and time. Factorial experiments are the only way to detect interactions. Furthermore, the one-factor-at-a-time method is inefficient. It will require more

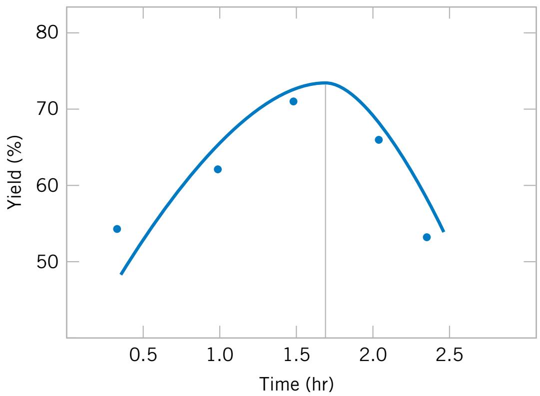


Figure 14-5 Yield versus reaction time with temperature constant at 155 °F.

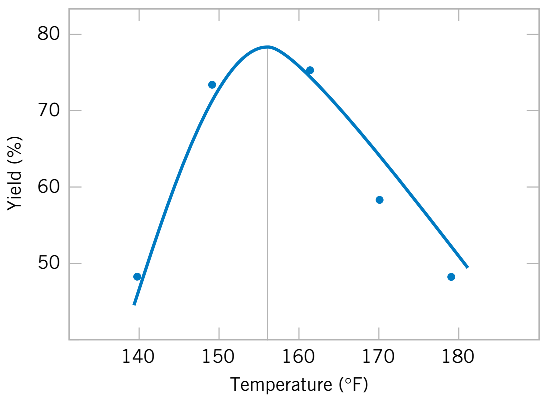


Figure 14-6 Yield versus temperature with reaction time constant at 1.7 hours.

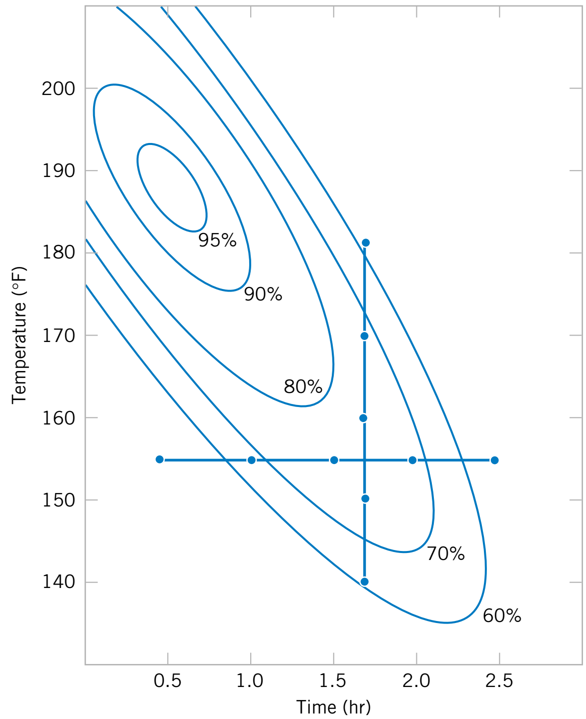


Figure 14-7 Optimization experiment using the one-factor-at-a-time method.

experimentation than a factorial, and as we have just seen, there is no assurance that it will produce the correct results.

14-4 TWO-FACTOR FACTORIAL EXPERIMENTS

The simplest type of factorial experiment involves only two factors, say, A and B. There are a levels of factor A and b levels of factor B. This two-factor factorial is shown in Table 14-3. The experiment has n **replicates,** and each replicate contains all ab treatment combinations.

Table 14-3 Data Arrangement for a Two-Factor Factorial Design

		Factor B					
		1	2	$\cdots$	b	Totals	Averages
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$	$y_{1..}$	$\bar{y}_{1..}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$	$y_{2..}$	$\bar{y}_{2..}$
	$\vdots$						
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$	$y_{a..}$	$\bar{y}_{a..}$
Totals		$y_{.1.}$	$y_{.2.}$		$y_{.b.}$	$y_{...}$	
Averages		$\bar{y}_{.1.}$	$\bar{y}_{.2.}$		$\bar{y}_{.b.}$		$\bar{y}_{...}$

The observation in the ijth cell for the kth replicate is denoted by y_{ijk}. In performing the experiment, the abn observations would be run in **random order.** Thus, like the single-factor experiment studied in Chapter 13, the two-factor factorial is a *completely randomized design*.

The observations may be described by the linear statistical model

$$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases} \tag{14-1}$$

where μ is the overall mean effect, τ_i is the effect of the ith level of factor A, β_j is the effect of the jth level of factor B, $(\tau\beta)_{ij}$ is the effect of the interaction between A and B, and ϵ_{ijk} is a random error component having a normal distribution with mean zero and variance σ^2. We are interested in testing the hypotheses of no main effect for factor A, no main effect for B, and no AB interaction effect. As with the single-factor experiments of Chapter 13, the analysis of variance (ANOVA) will be used to test these hypotheses. Since there are two factors in the experiment, the test procedure is sometimes called the two-way analysis of variance.

14-4.1 Statistical Analysis of the Fixed-Effects Model

Suppose that A and B are **fixed factors.** That is, the a levels of factor A and the b levels of factor B are specifically chosen by the experimenter, and inferences are confined to these levels only. In this model, it is customary to define the effects τ_i, β_j, and $(\tau\beta)_{ij}$ as deviations from the mean, so that $\sum_{i=1}^{a} \tau_i = 0, \sum_{j=1}^{b} \beta_j = 0, \sum_{i=1}^{a} (\tau\beta)_{ij} = 0$, and $\sum_{j=1}^{b} (\tau\beta)_{ij} = 0$.

The **analysis of variance** can be used to test hypotheses about the main factor effects of A and B and the AB interaction. To present the ANOVA, we will need some symbols, some of which are illustrated in Table 14-3. Let $y_{i\cdot\cdot}$ denote the total of the observations taken at the ith level of factor A; $y_{\cdot j\cdot}$ denote the total of the observations taken at the jth level of factor B; $y_{ij\cdot}$ denote the total of the observations in the ijth cell of Table 14-3; and $y_{\cdots}$ denote the grand total of all the observations. Define $\bar{y}_{i\cdot\cdot}, \bar{y}_{\cdot j\cdot}, \bar{y}_{ij\cdot}$, and $\bar{y}_{\cdots}$ as the corresponding row, column, cell, and grand averages. That is,

$$y_{i\cdot\cdot} = \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{i\cdot\cdot} = \frac{y_{i\cdot\cdot}}{bn} \qquad i = 1, 2, \ldots, a$$

$$y_{\cdot j\cdot} = \sum_{i=1}^{a} \sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{\cdot j\cdot} = \frac{y_{\cdot j\cdot}}{an} \qquad j = 1, 2, \ldots, b$$

$$y_{ij\cdot} = \sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{ij\cdot} = \frac{y_{ij\cdot}}{n} \qquad \begin{matrix} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{matrix}$$

$$y_{\cdots} = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{\cdots} = \frac{y_{\cdots}}{abn}$$

The hypotheses that we will test are as follows:

1. H_0: $\tau_1 = \tau_2 = \cdots = \tau_a = 0$ (no main effect of factor A)
H_1: at least one $\tau_i \neq 0$

2. H_0: $\beta_1 = \beta_2 = \cdots = \beta_b = 0$ (no main effect of factor B) (14-2)
H_1: at least one $\beta_j \neq 0$

3. H_0: $(\tau\beta)_{11} = (\tau\beta)_{12} = \cdots = (\tau\beta)_{ab} = 0$ (no interaction)
H_1: at least one $(\tau\beta)_{ij} \neq 0$

As before, the ANOVA tests these hypotheses by decomposing the total variability in the data into component parts and then comparing the various elements in this decomposition. Total variability is measured by the total sum of squares of the observations

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{\cdots})^2$$

and the sum of squares decomposition is defined below.

The **sum of squares identity for a two-factor ANOVA** is

$$\begin{aligned}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{\cdots})^2 &= bn\sum_{i=1}^{a}(\bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots})^2 \\ &+ an\sum_{j=1}^{b}(\bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots})^2 \\ &+ n\sum_{i=1}^{a}\sum_{j=1}^{b}(\bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots})^2 \\ &+ \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}(y_{ijk} - \bar{y}_{ij\cdot})^2 \end{aligned} \tag{14-3}$$

or symbolically,

$$SS_T = SS_A + SS_B + SS_{AB} + SS_E \tag{14-4}$$

Equations 14-3 and 14-4 state that the total sum of squares SS_T is partitioned into a sum of squares for the row factor A (SS_A), a sum of squares for the column factor B (SS_B), a sum of squares for the interaction between A and B (SS_{AB}), and an error sum of squares (SS_E). There are $abn - 1$ total degrees of freedom. The main effects A and B have $a - 1$ and $b - 1$ degrees of freedom, while the interaction effect AB has $(a - 1)(b - 1)$ degrees of freedom. Within each of the ab cells in Table 14-3, there are $n - 1$ degrees of freedom between the n replicates, and observations in the same cell can differ only because of random error. Therefore, there are $ab(n - 1)$ degrees of freedom for error. Therefore, the degrees of freedom are partitioned according to

$$abn - 1 = (a - 1) + (b - 1) + (a - 1)(b - 1) + ab(n - 1)$$

If we divide each of the sum of squares on the right-hand side of Equation 14-4 by the corresponding number of degrees of freedom, we obtain the **mean squares** for A, B, the interaction, and error:

$$MS_A = \frac{SS_A}{a-1} \qquad MS_B = \frac{SS_B}{b-1} \qquad MS_{AB} = \frac{SS_{AB}}{(a-1)(b-1)} \qquad MS_E = \frac{SS_E}{ab(n-1)}$$

Assuming that factors A and B are fixed factors, it is not difficult to show that the **expected values** of these mean squares are

$$E(MS_A) = E\left(\frac{SS_A}{a-1}\right) = \sigma^2 + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1} \qquad E(MS_B) = E\left(\frac{SS_B}{b-1}\right) = \sigma^2 + \frac{an\sum_{j=1}^{b}\beta_j^2}{b-1}$$

$$E(MS_{AB}) = E\left(\frac{SS_{AB}}{(a-1)(b-1)}\right) = \sigma^2 + \frac{n\sum_{i=1}^{a}\sum_{j=1}^{b}(\tau\beta)_{ij}^2}{(a-1)(b-1)}$$

$$E(MS_E) = E\left(\frac{SS_E}{ab(n-1)}\right) = \sigma^2$$

From examining these expected mean squares, it is clear that if the null hypotheses about main effects H_0: $\tau_i = 0$, H_0: $\beta_j = 0$, and the interaction hypothesis H_0: $(\tau\beta)_{ij} = 0$ are all true, all four mean squares are unbiased estimates of σ^2.

To test that the row factor effects are all equal to zero (H_0: $\tau_i = 0$), we would use the ratio

$$F_0 = \frac{MS_A}{MS_E}$$

which has an F-distribution with $a - 1$ and $ab(n - 1)$ degrees of freedom if H_0: $\tau_i = 0$ is true. This null hypothesis is rejected at the α level of significance if $f_0 > f_{\alpha,a-1,ab(n-1)}$. Similarly, to test the hypothesis that all the column factor effects are equal to zero (H_0: $\beta_j = 0$), we would use the ratio

$$F_0 = \frac{MS_B}{MS_E}$$

which has an F-distribution with $b - 1$ and $ab(n - 1)$ degrees of freedom if H_0: $\beta_j = 0$ is true. This null hypothesis is rejected at the α level of significance if $f_0 > f_{\alpha,b-1,ab(n-1)}$. Finally, to test the hypothesis H_0: $(\tau\beta)_{ij} = 0$, which is the hypothesis that all interaction effects are zero, we use the ratio

$$F_0 = \frac{MS_{AB}}{MS_E}$$

which has an F-distribution with $(a - 1)(b - 1)$ and $ab(n - 1)$ degrees of freedom if the null hypothesis H_0: $(\tau\beta)_{ij} = 0$. This hypothesis is rejected at the α level of significance if $f_0 > f_{\alpha,(a-1)(b-1),ab(n-1)}$.

It is usually best to conduct the test for interaction first and then to evaluate the main effects. If interaction is not significant, interpretation of the tests on the main effects is straightforward. However, as noted in Section 14-4, when interaction is significant, the main effects of the factors involved in the interaction may not have much practical interpretative value. Knowledge of the interaction is usually more important than knowledge about the main effects.

Computational formulas for the sums of squares are easily obtained.

Definition

Computing formulas for the sums of squares in a two-factor analysis of variance.

$$SS_T = \sum_{i=1}^{a} \sum_{j=1}^{b} \sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{\cdots}^2}{abn} \tag{14-5}$$

$$SS_A = \sum_{i=1}^{a} \frac{y_{i\cdot\cdot}^2}{bn} - \frac{y_{\cdots}^2}{abn} \tag{14-6}$$

$$SS_B = \sum_{j=1}^{b} \frac{y_{\cdot j\cdot}^2}{an} - \frac{y_{\cdots}^2}{abn} \tag{14-7}$$

$$SS_{AB} = \sum_{i=1}^{a} \sum_{j=1}^{b} \frac{y_{ij\cdot}^2}{n} - \frac{y_{\cdots}^2}{abn} - SS_A - SS_B \tag{14-8}$$

$$SS_E = SS_T - SS_{AB} - SS_A - SS_B \tag{14-9}$$

The computations are usually displayed in an ANOVA table, such as Table 14-4.

EXAMPLE 14-1

Aircraft primer paints are applied to aluminum surfaces by two methods: dipping and spraying. The purpose of the primer is to improve paint adhesion, and some parts can be primed using either application method. The process engineering group responsible for this operation is interested in learning whether three different primers differ in their adhesion properties.

Table 14-4 ANOVA Table for a Two-Factor Factorial, Fixed-Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A treatments	SS_A	$a - 1$	$MS_A = \frac{SS_A}{a - 1}$	$\frac{MS_A}{MS_E}$
B treatments	SS_B	$b - 1$	$MS_B = \frac{SS_B}{b - 1}$	$\frac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a - 1)(b - 1)$	$MS_{AB} = \frac{SS_{AB}}{(a - 1)(b - 1)}$	$\frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n - 1)$		
Total	SS_T	$abn - 1$	$MS_E = \frac{SS_E}{ab(n - 1)}$	

Table 14-5 Adhesion Force Data for Example 14-1

Primer Type	Dipping		Spraying		$y_{i\cdot\cdot}$
1	4.0, 4.5, 4.3	12.8	5.4, 4.9, 5.6	15.9	28.7
2	5.6, 4.9, 5.4	15.9	5.8, 6.1, 6.3	18.2	34.1
3	3.8, 3.7, 4.0	11.5	5.5, 5.0, 5.0	15.5	27.0
$y_{\cdot j\cdot}$	40.2		49.6		$89.8 = y_{\cdots}$

A factorial experiment was performed to investigate the effect of paint primer type and application method on paint adhesion. For each combination of primer type and application method, three specimens were painted, then a finish paint was applied, and the adhesion force was measured. The data from the experiment are shown in Table 14-5. The circled numbers in the cells are the cell totals $y_{ij\cdot}$. The sums of squares required to perform the ANOVA are computed as follows:

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{\cdots}^2}{abn}$$

$$= (4.0)^2 + (4.5)^2 + \cdots + (5.0)^2 - \frac{(89.8)^2}{18} = 10.72$$

$$SS_{\text{types}} = \sum_{i=1}^{a} \frac{y_{i\cdot\cdot}^2}{bn} - \frac{y_{\cdots}^2}{abn}$$

$$= \frac{(28.7)^2 + (34.1)^2 + (27.0)^2}{6} - \frac{(89.8)^2}{18} = 4.58$$

$$SS_{\text{methods}} = \sum_{j=1}^{b} \frac{y_{\cdot j\cdot}^2}{an} - \frac{y_{\cdots}^2}{abn}$$

$$= \frac{(40.2)^2 + (49.6)^2}{9} - \frac{(89.8)^2}{18} = 4.91$$

$$SS_{\text{interaction}} = \sum_{i=1}^{a}\sum_{j=1}^{b} \frac{y_{ij\cdot}^2}{n} - \frac{y_{\cdots}^2}{abn} - SS_{\text{types}} - SS_{\text{methods}}$$

$$= \frac{(12.8)^2 + (15.9)^2 + (11.5)^2 + (15.9)^2 + (18.2)^2 + (15.5)^2}{3}$$

$$- \frac{(89.8)^2}{18} - 4.58 - 4.91 = 0.24$$

and

$$SS_E = SS_T - SS_{\text{types}} - SS_{\text{methods}} - SS_{\text{interaction}}$$

$$= 10.72 - 4.58 - 4.91 - 0.24 = 0.99$$

The ANOVA is summarized in Table 14-6. The experimenter has decided to use $\alpha = 0.05$. Since $f_{0.05,2,12} = 3.89$ and $f_{0.05,1,12} = 4.75$, we conclude that the main effects of primer type and

Table 14-6 ANOVA for Example 14-1

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	P-Value
Primer types	4.58	2	2.29	28.63	$2.7 \times$ E-5
Application methods	4.91	1	4.91	61.38	$5.0 \times$ E-7
Interaction	0.24	2	0.12	1.50	0.2621
Error	0.99	12	0.08		
Total	10.72	17			

application method affect adhesion force. Furthermore, since $1.5 < f_{0.05,2,12}$, there is no indication of interaction between these factors. The last column of Table 14-6 shows the P-value for each F-ratio. Notice that the P-values for the two test statistics for the main effects are considerably less than 0.05, while the P-value for the test statistic for the interaction is greater than 0.05.

A graph of the cell adhesion force averages $\{\bar{y}_{ij\cdot}\}$ versus levels of primer type for each application method is shown in Fig. 14-8. The no-interaction conclusion is obvious in this graph, because the two lines are nearly parallel. Furthermore, since a large response indicates greater adhesion force, we conclude that spraying is the best application method and that primer type 2 is most effective.

Tests on Individual Means

When both factors are fixed, comparisons between the individual means of either factor may be made using any multiple comparison technique such as Fisher's LSD method (described in Chapter 13). When there is no interaction, these comparisons may be made using either the row averages $\bar{y}_{i\cdot\cdot}$ or the column averages $\bar{y}_{\cdot j\cdot}$. However, when interaction is significant, comparisons between the means of one factor (say, A) may be obscured by the AB interaction. In this case, we could apply a procedure such as Fisher's LSD method to the means of factor A, with factor B set at a particular level.

Minitab Output

Table 14-7 shows some of the output from the Minitab analysis of variance procedure for the aircraft primer paint experiment in Example 14-1. The upper portion of the table gives factor name and level information, and the lower portion of the table presents the analysis of variance for the adhesion force response. The results are identical to the manual calculations displayed in Table 14-6 apart from rounding.

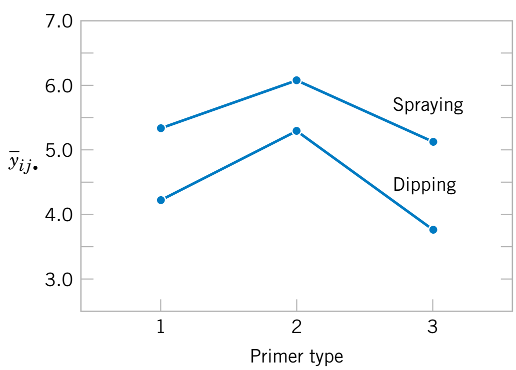


Figure 14-8 Graph of average adhesion force versus primer types for both application methods.

Table 14-7 Analysis of Variance From Minitab for Example 14-1

ANOVA (Balanced Designs)

Factor	Type	Levels	Values		
Primer	fixed	3	1	2	3
Method	fixed	2	Dip	Spray	

Analysis of Variance for Adhesion

Source	DF	SS	MS	F	P
Primer	2	4.5811	2.2906	27.86	0.000
Method	1	4.9089	4.9089	59.70	0.000
Primer *Method	2	0.2411	0.1206	1.47	0.269
Error	12	0.9867	0.0822		
Total	17	10.7178			

14-4.2 Model Adequacy Checking

Just as in the single-factor experiments discussed in Chapter 13, the **residuals** from a factorial experiment play an important role in assessing **model adequacy.** The residuals from a two-factor factorial are

$$e_{ijk} = y_{ijk} - \bar{y}_{ij\cdot}$$

That is, the residuals are just the difference between the observations and the corresponding cell averages.

Table 14-8 presents the residuals for the aircraft primer paint data in Example 14-1. The normal probability plot of these residuals is shown in Fig. 14-9. This plot has tails that do not fall exactly along a straight line passing through the center of the plot, indicating some potential problems with the normality assumption, but the deviation from normality does not appear severe. Figures 14-10 and, 14-11 plot the residuals versus the levels of primer types and application methods, respectively. There is some indication that primer type 3 results in slightly lower variability in adhesion force than the other two primers. The graph of residuals versus fitted values in Fig. 14-12 does not reveal any unusual or diagnostic pattern.

14-4.3 One Observation per Cell

In some cases involving a two-factor factorial experiment, we may have only one replicate—that is, only one observation per cell. In this situation, there are exactly as many parameters in the analysis of variance model as observations, and the error degrees of freedom are zero. Thus, we cannot test hypotheses about the main effects and interactions unless some additional

Table 14-8 Residuals for the Aircraft Primer Experiment in Example 14-1

	Application Method	
Primer Type	Dipping	Spraying
1	−0.27, 0.23, 0.03	0.10, −0.40, 0.30
2	0.30, −0.40, 0.10	−0.27, 0.03, 0.23
3	−0.03, −0.13, 0.17	0.33, −0.17, −0.17

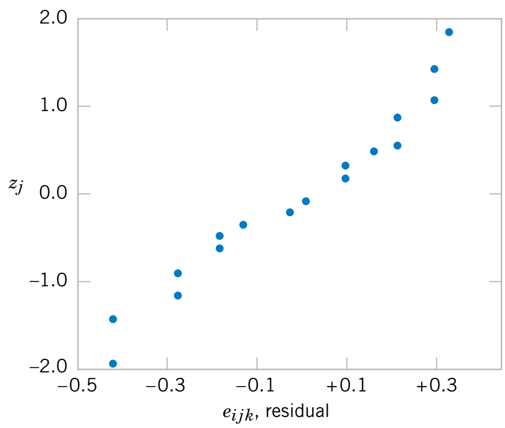


Figure 14-9 Normal probability plot of the residuals from Example 14-1.

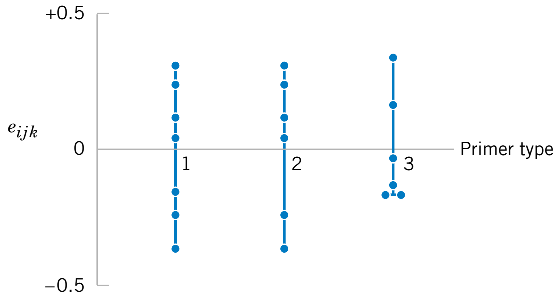


Figure 14-10 Plot of residuals versus primer type.

assumptions are made. One possible assumption is to assume the interaction effect is negligible and use the interaction mean square as an error mean square. Thus, the analysis is equivalent to the analysis used in the randomized block design. This no-interaction assumption can be dangerous, and the experimenter should carefully examine the data and the residuals for indications as to whether or not interaction is present. For more details, see Montgomery (2001).

14-4.4 Factorial Experiments with Random Factors: Overview

In Section 13-3 we introduced the concept of a **random factor.** This is, of course, a situation in which the factor of interest has a large number of possible levels and the experimenter chooses a subset of these levels at random from this population. Conclusions are then drawn about the population of factor levels.

Random factors can occur in factorial experiments. If all the factors are random, the analysis of variance model is called a **random-effects model.** If some factors are fixed and other factors are random, the analysis of variance model is called a **mixed model.** The statistical analysis of random and mixed models is very similar to that of the standard fixed-effects models that are the primary focus of this chapter. The primary differences are in the types of hypotheses that are tested, the construction of test statistics for these hypotheses, and the estimation of model parameters. Some additional details on these topics are presented in Section 14-6 on the CD. For a more in-depth presentation, refer to Montgomery (2001) and Neter, Wasserman, Nachtsheim, and Kutner (1996).

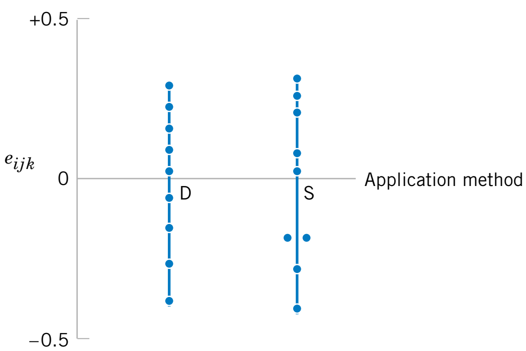


Figure 14-11 Plot of residuals versus application method.

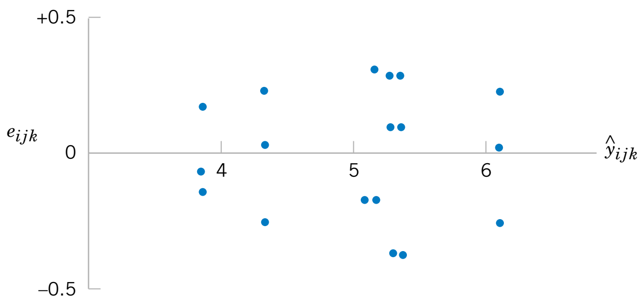


Figure 14-12 Plot of residuals versus predicted values $\hat{y}_{ijk}$.

EXERCISES FOR SECTION 14-4

14-1. In his book (*Design and Analysis of Experiments,* 5th edition, 2001 John Wiley & Sons), D. C. Montgomery presents the results of an experiment involving a storage battery used in the launching mechanism of a shoulder-fired ground-to-air missile. Three material types can be used to make the battery plates. The objective is to design a battery that is relatively unaffected by the ambient temperature. The output response from the battery is effective life in hours. Three temperature levels are selected, and a factorial experiment with four replicates is run. The data are as follows:

	Temperature (°F)					
Material	Low		Medium		High	
1	130	155	34	40	20	70
	74	180	80	75	82	58
2	150	188	136	122	25	70
	159	126	106	115	58	45
3	138	110	174	120	96	104
	168	160	150	139	82	60

(a) Test the appropriate hypotheses and draw conclusions using the analysis of variance with $\alpha = 0.05$.
(b) Graphically analyze the interaction.
(c) Analyze the residuals from this experiment.

14-2. An engineer who suspects that the surface finish of metal parts is influenced by the type of paint used and the drying time. He selected three drying times—20, 25, and 30 minutes—and used two types of paint. Three parts are tested with each combination of paint type and drying time. The data are as follows:

	Drying Time (min)		
Paint	20	25	30
1	74	73	78
	64	61	85
	50	44	92
2	92	98	66
	86	73	45
	68	88	85

(a) State and test the appropriate hypotheses using the analysis of variance with $\alpha = 0.05$.
(b) Analyze the residuals from this experiment.

14-3. An article in *Industrial Quality Control* (1956, pp. 5–8) describes an experiment to investigate the effect of two factors (glass type and phosphor type) on the brightness of a television tube. The response variable measured is the current (in microamps) necessary to obtain a specified brightness level. The data are shown in the following table:

Glass	Phosphor Type		
Type	1	2	3
1	280	300	290
	290	310	285
	285	295	290
2	230	260	220
	235	240	225
	240	235	230

(a) State the hypotheses of interest in this experiment.
(b) Test the above hypotheses and draw conclusions using the analysis of variance with $\alpha = 0.05$.
(c) Analyze the residuals from this experiment.

14-4. An experiment was conducted to determine whether either firing temperature or furnace position affects the baked density of a carbon anode. The data are as follows:

	Temperature (°C)		
Position	800	825	850
1	570	1063	565
	565	1080	510
	583	1043	590
2	528	988	526
	547	1026	538
	521	1004	532

(a) State the hypotheses of interest.
(b) Test the above hypotheses using the analysis of variance with $\alpha = 0.05$. What are your conclusions?
(c) Analyze the residuals from this experiment.

14-5. Continuation of Exercise 14-4. Using Fisher's LSD method, investigate the differences between the mean baked anode density at the three different levels of temperature in Exercise 14-4. Use $\alpha = 0.05$.

14-6. Johnson and Leone (*Statistics and Experimental Design in Engineering and the Physical Sciences,* John Wiley, 1977) describe an experiment conducted to investigate warping of copper plates. The two factors studied were temperature and the copper content of the plates. The response variable is the amount of warping. The data are as follows:

Temperature (°C)	**Copper Content (%)** 40	60	80	100
50	17, 20	16, 21	24, 22	28, 27
75	12, 9	18, 13	17, 12	27, 31
100	16, 12	18, 21	25, 23	30, 23
125	21, 17	23, 21	23, 22	29, 31

(a) Is there any indication that either factor affects the amount of warping? Is there any interaction between the factors? Use $\alpha = 0.05$.
(b) Analyze the residuals from this experiment.
(c) Plot the average warping at each level of copper content and compare the levels using Fisher's LSD method. Describe the differences in the effects of the different levels of copper content on warping. If low warping is desirable, what level of copper content would you specify?
(d) Suppose that temperature cannot be easily controlled in the environment in which the copper plates are to be used. Does this change your answer for part (c)?

14-7. Consider a two-factor factorial experiment. Develop a formula for finding a 100 $(1 - \alpha)$% confidence interval on the difference between any two means for either a row or column factor. Apply this formula to find a 95% CI on the difference in mean warping at the levels of copper content 60 and 80% in Exercise 14-6.

14-8. An article in the *Journal of Testing and Evaluation* (Vol. 16, no. 6, 1988, pp. 508–515) investigated the effects of cyclic loading frequency and environment conditions on fatigue crack growth at a constant 22 MPa stress for a particular material. The data from the experiment follow. The response variable is fatigue crack growth rate.

Frequency	**Environment** Air	H_2O	Salt H_2O
10	2.29	2.06	1.90
	2.47	2.05	1.93
	2.48	2.23	1.75
	2.12	2.03	2.06
1	2.65	3.20	3.10
	2.68	3.18	3.24
	2.06	3.96	3.98
	2.38	3.64	3.24
0.1	2.24	11.00	9.96
	2.71	11.00	10.01
	2.81	9.06	9.36
	2.08	11.30	10.40

(a) Is there indication that either factor affects crack growth rate? Is there any indication of interaction? Use $\alpha = 0.05$.
(b) Analyze the residuals from this experiment.
(c) Repeat the analysis in part (a) using $ln(y)$ as the response. Analyze the residuals from this new response variable and comment on the results.

14-9. An article in the *IEEE Transactions on Electron Devices* (November 1986, p. 1754) describes a study on the effects of two variables—polysilicon doping and anneal conditions (time and temperature)—on the base current of a bipolar transistor. The data from this experiment follows below Exercise 14-10.
(a) Is there any evidence to support the claim that either polysilicon doping level or anneal conditions affect base current? Do these variables interact? Use $\alpha = 0.05$.
(b) Graphically analyze the interaction.
(c) Analyze the residuals from this experiment.

14-10. Consider the experiment described in Exercise 14-9. Use Fisher's LSD method to isolate the effects of anneal conditions on base current, with $\alpha = 0.05$.

Polysilicon doping	**Anneal (temperature/time)** 900/60	900/180	950/60	1000/15	1000/30
1×10^{20}	4.40	8.30	10.15	10.29	11.01
	4.60	8.90	10.20	10.30	10.58
2×10^{20}	3.20	7.81	9.38	10.19	10.81
	3.50	7.75	10.02	10.10	10.60

14-5 GENERAL FACTORIAL EXPERIMENTS

Many experiments involve more than two factors. In this section we introduce the case where there are a levels of factor A, b levels of factor B, c levels of factor C, and so on, arranged in a factorial experiment. In general, there will be $abc \dots n$ total observations, if there are n replicates of the complete experiment.

Table 14-9 Analysis of Variance Table for the Three-Factor Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Expected Mean Squares	F_0
A	SS_A	$a - 1$	MS_A	$\sigma^2 + \frac{bcn\Sigma\tau_i^2}{a-1}$	$\frac{MS_A}{MS_E}$
B	SS_B	$b - 1$	MS_B	$\sigma^2 + \frac{acn\Sigma\beta_j^2}{b-1}$	$\frac{MS_B}{MS_E}$
C	SS_C	$c - 1$	MS_C	$\sigma^2 + \frac{abn\Sigma\gamma_k^2}{c-1}$	$\frac{MS_C}{MS_E}$
AB	SS_{AB}	$(a-1)(b-1)$	MS_{AB}	$\sigma^2 + \frac{cn\Sigma\Sigma(\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$\frac{MS_{AB}}{MS_E}$
AC	SS_{AC}	$(a-1)(c-1)$	MS_{AC}	$\sigma^2 + \frac{bn\Sigma\Sigma(\tau\gamma)_{ik}^2}{(a-1)(c-1)}$	$\frac{MS_{AC}}{MS_E}$
BC	SS_{BC}	$(b-1)(c-1)$	MS_{BC}	$\sigma^2 + \frac{an\Sigma\Sigma(\beta\gamma)_{jk}^2}{(b-1)(c-1)}$	$\frac{MS_{BC}}{MS_E}$
ABC	SS_{ABC}	$(a-1)(b-1)(c-1)$	MS_{ABC}	$\sigma^2 + \frac{n\Sigma\Sigma\Sigma(\tau\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$	$\frac{MS_{ABC}}{MS_E}$
Error	SS_E	$abc(n-1)$	MS_E	σ^2	
Total	SS_T	$abcn - 1$			

For example, consider the **three-factor-factorial experiment,** with underlying model

$$Y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \epsilon_{ijkl} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, c \\ l = 1, 2, \ldots, n \end{cases} \tag{14-10}$$

Notice that the model contains three main effects, three two-factor interactions, a three-factor interaction, and an error term. Assuming that A, B, and C are fixed factors, the analysis of variance is shown in Table 14-9. Note that there must be at least two replicates ($n \geq 2$) to compute an error sum of squares. The F-test on main effects and interactions follows directly from the expected mean squares. These ratios follow F distributions under the respective null hypotheses.

EXAMPLE 14-2

A mechanical engineer is studying the surface roughness of a part produced in a metal-cutting operation. Three factors, feed rate (A), depth of cut (B), and tool angle (C), are of interest. All three factors have been assigned two levels, and two replicates of a factorial design are run. The coded data are shown in Table 14-10.

The ANOVA is summarized in Table 14-11. Since manual ANOVA computions are tedious for three-factor experiments, we have used Minitab for the solution of this problem.

Table 14-10 Coded Surface Roughness Data for Example 14-2

	Depth of Cut (B)				
	0.025 inch		0.040 inch		
	Tool Angle (C)		Tool Angle (C)		
Feed Rate (A)	15°	25°	15°	25°	$y_{i\cdots}$
20 inches per minute	9 7	11 10	9 11	10 8	75
30 inches per minute	10 12	10 13	12 15	16 14	102

The F-ratios for all three main effects and the interactions are formed by dividing the mean square for the effect of interest by the error mean square. Since the experimenter has selected $\alpha = 0.05$, the critical value for each of these F-ratios is $f_{0.05,1,8} = 5.32$. Alternately, we could use the P-value approach. The P-values for all the test statistics are shown in the last column of Table 14-11. Inspection of these P-values is revealing. There is a strong main effect of feed rate, since the F-ratio is well into the critical region. However, there is some indication of an effect due to the depth of cut, since $P = 0.0710$ is not much greater than $\alpha = 0.05$. The next largest effect is the AB or feed rate $\times$ depth of cut interaction. Most likely, both feed rate and depth of cut are important process variables.

Obviously, factorial experiments with three or more factors can require many runs, particularly if some of the factors have several (more than two) levels. This point of view leads us to the class of factorial designs considered in Section 14-7 with all factors at two levels. These designs are easy to set up and analyze, and they may be used as the basis of many other useful experimental designs.

Table 14-11 Minitab ANOVA for Example 14-2

ANOVA (Balanced Designs)

Factor	Type	Levels	Values	
Feed	fixed	2	20	30
Depth	fixed	2	0.025	0.040
Angle	fixed	2	15	25

Analysis of Variance for Roughness

Source	DF	SS	MS	F	P
Feed	1	45.563	45.563	18.69	0.003
Depth	1	10.563	10.563	4.33	0.071
Angle	1	3.063	3.063	1.26	0.295
Feed*Depth	1	7.563	7.563	3.10	0.116
Feed*Angle	1	0.062	0.062	0.03	0.877
Depth*Angle	1	1.563	1.563	0.64	0.446
Feed*Depth*Angle	1	5.062	5.062	2.08	0.188
Error	8	19.500	2.437		
Total	15	92.938			

EXERCISES FOR SECTION 14-5

14-11. The percentage of hardwood concentration in raw pulp, the freeness, and the cooking time of the pulp are being investigated for their effects on the strength of paper. The data from a three-factor factorial experiment are shown in the following table.

(a) Analyze the data using the analysis of variance assuming that all factors are fixed. Use $\alpha = 0.05$.
(b) Find P-values for the F-ratios in part (a).
(c) The residuals are found by $e_{ijkl} = y_{ijkl} - \bar{y}_{ijk\cdot}$. Graphically analyze the residuals from this experiment.

Percentage of Hardwood Concentration	Cooking Time 1.5 hours			Cooking Time 2.0 hours		
	Freeness			Freeness		
	350	500	650	350	500	650
10	96.6	97.7	99.4	98.4	99.6	100.6
	96.0	96.0	99.8	98.6	100.4	100.9
15	98.5	96.0	98.4	97.5	98.7	99.6
	97.2	96.9	97.6	98.1	96.0	99.0
20	97.5	95.6	97.4	97.6	97.0	98.5
	96.6	96.2	98.1	98.4	97.8	99.8

14-12. The quality control department of a fabric finishing plant is studying the effects of several factors on dyeing for a blended cotton/synthetic cloth used to manufacture shirts. Three operators, three cycle times, and two temperatures were selected, and three small specimens of cloth were dyed under each set of conditions. The finished cloth was compared to a standard, and a numerical score was assigned. The results are shown in the following table.

(a) State and test the appropriate hypotheses using the analysis of variance with $\alpha = 0.05$.
(b) The residuals may be obtained from $e_{ijkl} = y_{ijkl} - \bar{y}_{ijk\cdot}$. Graphically analyze the residuals from this experiment.

	Temperature					
	300°			350°		
	Operator			Operator		
Cycle Time	1	2	3	1	2	3
	23	27	31	24	38	34
40	24	28	32	23	36	36
	25	26	28	28	35	39
	36	34	33	37	34	34
50	35	38	34	39	38	36
	36	39	35	35	36	31
	28	35	26	26	36	28
60	24	35	27	29	37	26
	27	34	25	25	34	34

14-6 FACTORIAL EXPERIMENTS WITH RANDOM FACTORS (CD ONLY)

14-7 2^k FACTORIAL DESIGNS

Factorial designs are frequently used in experiments involving several factors where it is necessary to study the joint effect of the factors on a response. However, several special cases of the general factorial design are important because they are widely employed in research work and because they form the basis of other designs of considerable practical value.

The most important of these special cases is that of k factors, each at only two levels. These levels may be quantitative, such as two values of temperature, pressure, or time; or they may be qualitative, such as two machines, two operators, the "high" and "low" levels of a factor, or perhaps the presence and absence of a factor. A complete replicate of such a design requires $2 \times 2 \times \cdots \times 2 = 2^k$ observations and is called a **2^k factorial design.**

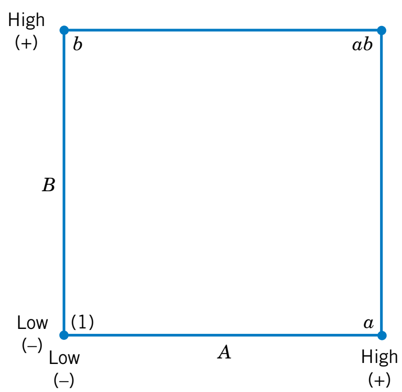
The 2^k design is particularly useful in the early stages of experimental work, when many factors are likely to be investigated. It provides the smallest number of runs for which k factors can be studied in a complete factorial design. Because there are only two levels for each factor, we must assume that the response is approximately linear over the range of the factor levels chosen.

14-7.1 2^2 Design

The simplest type of 2^k design is the 2^2—that is, two factors A and B, each at two levels. We usually think of these levels as the low and high levels of the factor. The 2^2 design is shown in Fig. 14-13. Note that the design can be represented geometrically as a square with the $2^2 = 4$ runs, or treatment combinations, forming the corners of the square. In the 2^2 design it is customary to denote the low and high levels of the factors A and B by the signs $-$ and $+$, respectively. This is sometimes called the **geometric notation** for the design.

A special notation is used to label the treatment combinations. In general, a treatment combination is represented by a series of lowercase letters. If a letter is present, the corresponding factor is run at the high level in that treatment combination; if it is absent, the factor is run at its low level. For example, treatment combination a indicates that factor A is at the high level and factor B is at the low level. The treatment combination with both factors at the low level is represented by (1). This notation is used throughout the 2^k design series. For example, the treatment combination in a 2^4 with A and C at the high level and B and D at the low level is denoted by ac.

The effects of interest in the 2^2 design are the main effects A and B and the two-factor interaction AB. Let the letters (1), a, b, and ab also represent the totals of all n observations taken at these design points. It is easy to estimate the effects of these factors. To estimate the main effect of A, we would average the observations on the right side of the square in Fig. 14-13 where A is at the high level, and subtract from this the average of the observations on the left side of the square, where A is at the low level, or



Treatment	A	B
(1)	−	−
a	+	−
b	−	+
ab	+	+

Figure 14-13 The 2^2 factorial design.

$$
\begin{aligned}
A &= \bar{y}_{A+} - \bar{y}_{A-} \\
&= \frac{a + ab}{2n} - \frac{b + (1)}{2n} \\
&= \frac{1}{2n}[a + ab - b - (1)] \qquad (14\text{-}11)
\end{aligned}
$$

Similarly, the main effect of B is found by averaging the observations on the top of the square, where B is at the high level, and subtracting the average of the observations on the bottom of the square, where B is at the low level:

$$
\begin{aligned}
B &= \bar{y}_{B+} - \bar{y}_{B-} \\
&= \frac{b + ab}{2n} - \frac{a + (1)}{2n} \\
&= \frac{1}{2n}[b + ab - a - (1)] \qquad (14\text{-}12)
\end{aligned}
$$

Finally, the AB interaction is estimated by taking the difference in the diagonal averages in Fig. 14-12, or

$$
\begin{aligned}
AB &= \frac{ab + (1)}{2n} - \frac{a + b}{2n} \\
&= \frac{1}{2n}[ab + (1) - a - b] \qquad (14\text{-}13)
\end{aligned}
$$

The quantities in brackets in Equations 14-11, 14-12, and 14-13 are called **contrasts.** For example, the A contrast is

$$\text{Contrast}_A = a + ab - b - (1)$$

In these equations, the contrast coefficients are always either $+1$ or -1. A table of plus and minus signs, such as Table 14-12, can be used to determine the sign on each treatment

Table 14-12 Signs for Effects in the 2^2 Design

Treatment Combination	Factorial Effect			
	I	A	B	AB
(1)	+	−	−	+
a	+	+	−	−
b	+	−	+	−
ab	+	+	+	+

combination for a particular contrast. The column headings for Table 14-12 are the main effects A and B, the AB interaction, and I, which represents the total. The row headings are the treatment combinations. Note that the signs in the AB column are the product of signs from columns A and B. To generate a contrast from this table, multiply the signs in the appropriate column of Table 14-12 by the treatment combinations listed in the rows and add. For example, $\text{contrast}_{AB} = [(1)] + [-a] + [-b] + [ab] = ab + (1) - a - b$.

Contrasts are used in calculating both the effect estimates and the sums of squares for A, B, and the AB interaction. The sums of squares formulas are

$$SS_A = \frac{[a + ab - b - (1)]^2}{4n}$$
$$SS_B = \frac{[b + ab - a - (1)]^2}{4n} \qquad (14\text{-}14)$$
$$SS_{AB} = \frac{[ab + (1) - a - b]^2}{4n}$$

The analysis of variance is completed by computing the total sum of squares SS_T (with $4n - 1$ degrees of freedom) as usual, and obtaining the error sum of squares SS_E [with $4(n - 1)$ degrees of freedom] by subtraction.

EXAMPLE 14-3

An article in the *AT&T Technical Journal* (Vol. 65, March/April 1986, pp. 39–50) describes the application of two-level factorial designs to integrated circuit manufacturing. A basic processing step in this industry is to grow an epitaxial layer on polished silicon wafers. The wafers are mounted on a susceptor and positioned inside a bell jar. Chemical vapors are introduced through nozzles near the top of the jar. The susceptor is rotated, and heat is applied. These conditions are maintained until the epitaxial layer is thick enough.

Table 14-13 presents the results of a 2^2 factorial design with $n = 4$ replicates using the factors A = deposition time and B = arsenic flow rate. The two levels of deposition time are $-$ =short and $+$ =long, and the two levels of arsenic flow rate are $-$ =55% and $+$ =59%. The response variable is epitaxial layer thickness (μm). We may find the estimates of the effects using Equations 14-11, 14-12, and 14-13 as follows:

$$A = \frac{1}{2n}[a + ab - b - (1)]$$
$$= \frac{1}{2(4)}[59.299 + 59.156 - 55.686 - 56.081] = 0.836$$

$$B = \frac{1}{2n}[b + ab - a - (1)]$$
$$= \frac{1}{2(4)}[55.686 + 59.156 - 59.299 - 56.081] = 0.067$$

$$AB = \frac{1}{2n}[ab + (1) - a - b]$$
$$AB = \frac{1}{2(4)}[59.156 + 56.081 - 59.299 - 55.686] = 0.032$$

Table 14-13 The 2^2 Design for the Epitaxial Process Experiment

Treatment Combination	Design Factors			Thickness (μm)				Thickness (μm)	
	A	B	AB					Total	Average
(1)	−	−	+	14.037	14.165	13.972	13.907	56.081	14.020
a	+	−	−	14.821	14.757	14.843	14.878	59.299	14.825
b	−	+	−	13.880	13.860	14.032	13.914	55.686	13.922
ab	+	+	+	14.888	14.921	14.415	14.932	59.156	14.789

The numerical estimates of the effects indicate that the effect of deposition time is large and has a positive direction (increasing deposition time increases thickness), since changing deposition time from low to high changes the mean epitaxial layer thickness by 0.836 μm. The effects of arsenic flow rate (B) and the AB interaction appear small.

The importance of these effects may be confirmed with the analysis of variance. The sums of squares for A, B, and AB are computed as follows:

$$SS_A = \frac{[a + ab - b - (1)]^2}{16} = \frac{[6.688]^2}{16} = 2.7956$$

$$SS_B = \frac{[b + ab - a - (1)]^2}{16} = \frac{[-0.538]^2}{16} = 0.0181$$

$$SS_{AB} = \frac{[ab + (1) - a - b]^2}{16} = \frac{[0.252]^2}{16} = 0.0040$$

$$SS_T = 14.037^2 + \cdots + 14.932^2 - \frac{(56.081 + \cdots + 59.156)^2}{16}$$
$$= 3.0672$$

The analysis of variance is summarized in Table 14-14 and confirms our conclusions obtained by examining the magnitude and direction of the effects. Deposition time is the only factor that significantly affects epitaxial layer thickness, and from the direction of the effect estimates we know that longer deposition times lead to thicker epitaxial layers.

Residual Analysis

It is easy to obtain the residuals from a 2^k design by fitting a **regression model** to the data. For the epitaxial process experiment, the regression model is

$$Y = \beta_0 + \beta_1 x_1 + \epsilon$$

Table 14-14 Analysis of Variance for the Epitaxial Process Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	P-Value
A (deposition time)	2.7956	1	2.7956	134.40	7.07 E-8
B (arsenic flow)	0.0181	1	0.0181	0.87	0.38
AB	0.0040	1	0.0040	0.19	0.67
Error	0.2495	12	0.0208		
Total	3.0672	15			

since the only active variable is deposition time, which is represented by a coded variable x_1. The low and high levels of deposition time are assigned values $x_1 = -1$ and $x_1 = +1$, respectively. The least squares fitted model is

$$\hat{y} = 14.389 + \left(\frac{0.836}{2}\right) x_1$$

where the intercept $\hat{\beta}_0$ is the grand average of all 16 observations ($\bar{y}$) and the slope $\hat{\beta}_1$is one-half the effect estimate for deposition time. (The regression coefficient is one-half the effect estimate because regression coefficients measure the effect of a unit change in x_1 on the mean of Y, and the effect estimate is based on a two-unit change from -1 to $+1$.)

This model can be used to obtain the predicted values at the four points that form the corners of the square in the design. For example, consider the point with low deposition time ($x_1 = -1$) and low arsenic flow rate. The predicted value is

$$\hat{y} = 14.389 + \left(\frac{0.836}{2}\right)(-1) = 13.971 \ \mu\text{m}$$

and the residuals for the four runs at that design point are

$$\begin{aligned} e_1 &= 14.037 - 13.971 = 0.066 \\ e_2 &= 14.165 - 13.971 = 0.194 \\ e_3 &= 13.972 - 13.971 = 0.001 \\ e_4 &= 13.907 - 13.971 = -0.064 \end{aligned}$$

The remaining predicted values and residuals at the other three design points are calculated in a similar manner.

A normal probability plot of these residuals is shown in Fig. 14-14. This plot indicates that one residual $e_{15} = -0.392$ is an **outlier.** Examining the four runs with high deposition time and high arsenic flow rate reveals that observation $y_{15} = 14.415$ is considerably smaller than the other three observations at that treatment combination. This adds some additional evidence to the tentative conclusion that observation 15 is an outlier. Another possibility is that some process variables affect the *variability* in epitaxial layer thickness. If we could discover which variables produce this effect, we could perhaps adjust these variables to levels that would minimize the variability in epitaxial layer thickness. This could have important implications in subsequent manufacturing stages. Figures 14-15 and 14-16 are plots of residuals versus deposition time and arsenic flow rate, respectively. Apart from that unusually large residual associated with y_{15}, there is no strong evidence that either deposition time or arsenic flow rate influences the variability in epitaxial layer thickness.

Figure 14-17 shows the standard deviation of epitaxial layer thickness at all four runs in the 2^2 design. These standard deviations were calculated using the data in Table 14-13. Notice that the standard deviation of the four observations with A and B at the high level is considerably larger than the standard deviations at any of the other three design points. Most of this difference is attributable to the unusually low thickness measurement associated with y_{15}. The standard deviation of the four observations with A and B at the low level is also somewhat larger than the standard deviations at the remaining two runs. This could indicate that other process variables not included in this experiment may affect the variability in epitaxial layer thickness. Another experiment to study this possibility, involving other process variables, could be designed and conducted. (The original paper in the *AT&T Technical Journal* shows that two additional factors, not considered in this example, affect process variability.)

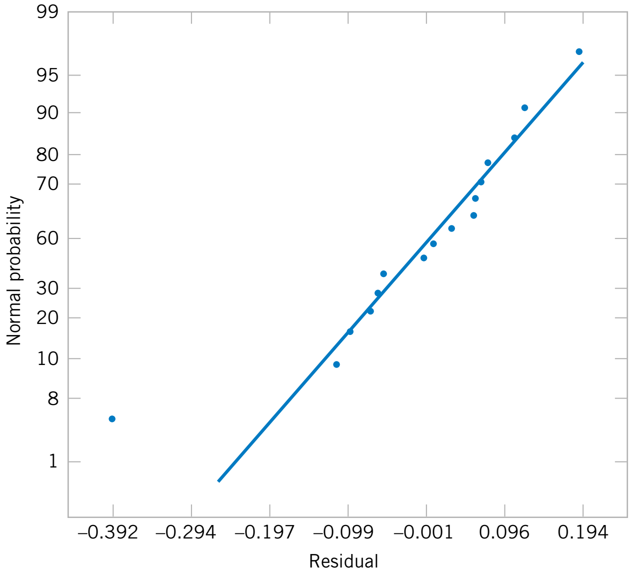


Figure 14-14 Normal probability plot of residuals for the epitaxial process experiment.

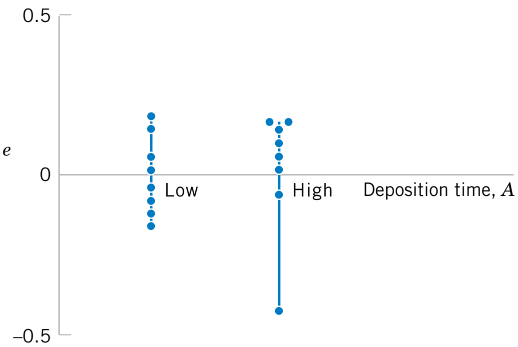


Figure 14-15 Plot of residuals versus deposition time.

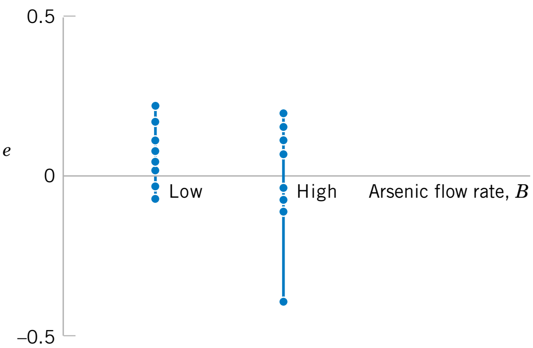


Figure 14-16 Plot of residuals versus arsenic flow rate.

14-7.2 2^k Design for $k \geq 3$ Factors

The methods presented in the previous section for factorial designs with $k = 2$ factors each at two levels can be easily extended to more than two factors. For example, consider $k = 3$ factors, each at two levels. This design is a 2^3 factorial design, and it has eight runs or treatment combinations. Geometrically, the design is a cube as shown in Fig. 14-18(a), with the eight runs forming the corners of the cube. Figure 14-18(b) lists the eight runs in a table, with each row representing one of the runs are the $-$ and $+$ settings indicating the low and high levels

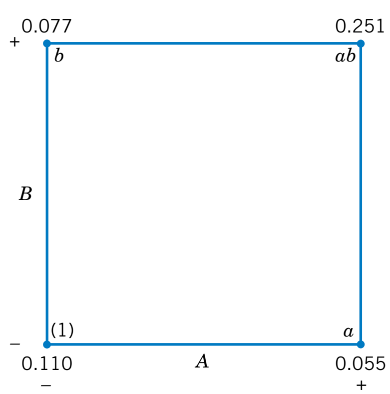
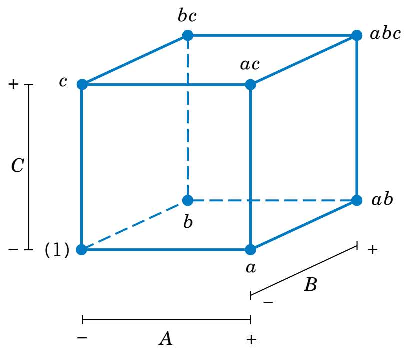


Figure 14-17 The standard deviation of epitaxial layer thickness at the four runs in the 2^2 design.



(a) Geometric view

Run	A	B	C
1	−	−	−
2	+	−	−
3	−	+	−
4	+	+	−
5	−	−	+
6	+	−	+
7	−	+	+
8	+	+	+

(b) The 2^3 design matrix

Figure 14-18 The 2^3 design.

for each of the three factors. This table is sometimes called the **design matrix.** This design allows three main effects to be estimated (A, B, and C) along with three two-factor interactions (AB, AC, and BC) and a three-factor interaction (ABC).

The main effects can easily be estimated. Remember that the lowercase letters (1), a, b, ab, c, ac, bc, and abc represent the total of all n replicates at each of the eight runs in the design. As seen in Fig. 14-19(a), the main effect of A can be estimated by averaging the four treatment combinations on the right-hand side of the cube, where A is at the high level, and by

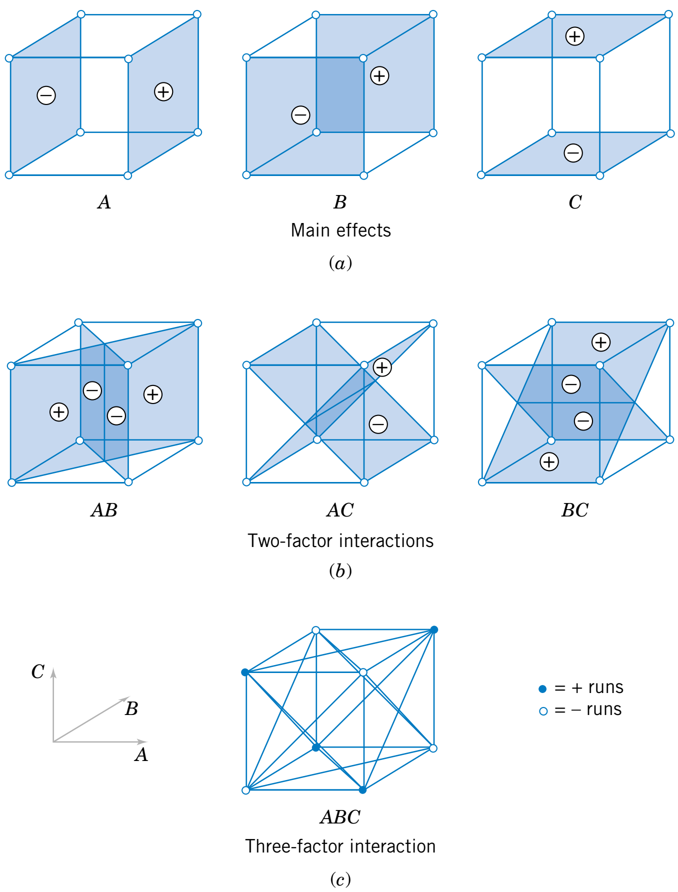


Figure 14-19 Geometric presentation of contrasts corresponding to the main effects and interaction in the 2^3 design. (a) Main effects. (b) Two-factor interactions. (c) Three-factor interaction.

subtracting from this quantity the average of the four treatment combinations on the left-hand side of the cube where A is at the low level. This gives

$$\begin{aligned} A &= \bar{y}_{A+} - \bar{y}_{A-} \\ &= \frac{a + ab + ac + abc}{4n} - \frac{(1) + b + c + bc}{4n} \end{aligned}$$

This equation can be rearranged as

$$A = \frac{1}{4n}[a + ab + ac + abc - (1) - b - c - bc] \qquad \text{(14-15)}$$

In a similar manner, the effect of B is the difference in averages between the four treatment combinations in the back face of the cube (Fig. 14-19a), and the four in the front. This yields

$$\begin{aligned} B &= \bar{y}_{B+} - \bar{y}_{B-} \\ &= \frac{1}{4n}[b + ab + bc + abc - (1) - a - c - ac] \end{aligned} \qquad \text{(14-16)}$$

The effect of C is the difference in average response between the four treatment combinations in the top face of the cube in Figure 14-19(a) and the four in the bottom, that is,

$$\begin{aligned} C &= \bar{y}_{C+} - \bar{y}_{C-} \\ &= \frac{1}{4n}[c + ac + bc + abc - (1) - a - b - ab] \end{aligned} \qquad \text{(14-17)}$$

The two-factor interaction effects may be computed easily. A measure of the AB interaction is the difference between the average A effects at the two levels of B. By convention, one-half of this difference is called the AB interaction. Symbolically,

B	**Average *A* Effect**
High (+)	$\dfrac{[(abc - bc) + (ab - b)]}{2n}$
Low (−)	$\dfrac{\{(ac - c) + [a - (1)]\}}{2n}$
Difference	$\dfrac{[abc - bc + ab - b - ac + c - a + (1)]}{2n}$

Since the AB interaction is one-half of this difference,

$$AB = \frac{1}{4n}[abc - bc + ab - b - ac + c - a + (1)] \qquad (14\text{-}18)$$

We could write Equation 14-18 as follows:

$$AB = \frac{abc + ab + c + (1)}{4n} - \frac{bc + b + ac + a}{4n}$$

In this form, the AB interaction is easily seen to be the difference in averages between runs on two diagonal planes in the cube in Fig. 14-19(b). Using similar logic and referring to Fig. 14-19(b), we find that the AC and BC interactions are

$$AC = \frac{1}{4n}[(1) - a + b - ab - c + ac - bc + abc] \qquad (14\text{-}19)$$

$$BC = \frac{1}{4n}[(1) + a - b - ab - c - ac + bc + abc] \qquad (14\text{-}20)$$

The ABC interaction is defined as the average difference between the AB interaction for the two different levels of C. Thus,

$$ABC = \frac{1}{4n}\{[abc - bc] - [ac - c] - [ab - b] + [a - (1)]\}$$

or

$$ABC = \frac{1}{4n}[abc - bc - ac + c - ab + b + a - (1)] \qquad (14\text{-}21)$$

As before, we can think of the ABC interaction as the difference in two averages. If the runs in the two averages are isolated, they define the vertices of the two tetrahedra that comprise the cube in Fig. 14-19(c).

In Equations 14-15 through 14-21, the quantities in brackets are **contrasts** in the treatment combinations. A table of plus and minus signs can be developed from the contrasts and is shown in Table 14-15. Signs for the main effects are determined directly from the test matrix in Figure 14-18(b). Once the signs for the main effect columns have been established, the signs for the remaining columns can be obtained by multiplying the appropriate

Table 14-15 Algebraic Signs for Calculating Effects in the 2^3 Design

Treatment Combination	Factorial Effect							
	I	A	B	AB	C	AC	BC	ABC
(1)	+	−	−	+	−	+	+	−
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
ab	+	+	+	+	−	−	−	−
c	+	−	−	+	+	−	−	+
ac	+	+	−	−	+	+	−	−
bc	+	−	+	−	+	−	+	−
abc	+	+	+	+	+	+	+	+

main effect row by row. For example, the signs in the AB column are the products of the A and B column signs in each row. The contrast for any effect can easily be obtained from this table.

Table 14-15 has several interesting properties:

1. Except for the identity column I, each column has an equal number of plus and minus signs.
2. The sum of products of signs in any two columns is zero; that is, the columns in the table are **orthogonal.**
3. Multiplying any column by column I leaves the column unchanged; that is, I is an **identity element.**
4. The product of any two columns yields a column in the table, for example $A \times B = AB$, and $AB \times ABC = A^2B^2C = C$, since any column multiplied by itself is the identity column.

The estimate of any main effect or interaction in a 2^k design is determined by multiplying the treatment combinations in the first column of the table by the signs in the corresponding main effect or interaction column, by adding the result to produce a contrast, and then by dividing the contrast by one-half the total number of runs in the experiment. For any 2^k design with n replicates, the effect estimates are computed from

$$\text{Effect} = \frac{\text{Contrast}}{n2^{k-1}} \tag{14-22}$$

and the sum of squares for any effect is

$$SS = \frac{(\text{Contrast})^2}{n2^k} \tag{14-23}$$

EXAMPLE 14-4

Consider the surface roughness experiment originally described in Example 14-2. This is a 2^3 factorial design in the factors feed rate (A), depth of cut (B), and tool angle (C), with $n = 2$ replicates. Table 14-16 presents the observed surface roughness data.

The main effects may be estimated using Equations 14-15 through 14-21. The effect of A, for example, is

$$\begin{aligned} A &= \frac{1}{4n}[a + ab + ac + abc - (1) - b - c - bc] \\ &= \frac{1}{4(2)}[22 + 27 + 23 + 40 - 16 - 20 - 21 - 18] \\ &= \frac{1}{8}[27] = 3.375 \end{aligned}$$

and the sum of squares for A is found using Equation 14-23:

$$SS_A = \frac{(\text{Contrast}_A)^2}{n2^k} = \frac{(27)^2}{2(8)} = 45.5625$$

It is easy to verify that the other effects are

$$\begin{aligned} B &= 1.625 \\ C &= 0.875 \\ AB &= 1.375 \\ AC &= 0.125 \\ BC &= -0.625 \\ ABC &= 1.125 \end{aligned}$$

Examining the magnitude of the effects clearly shows that feed rate (factor A) is dominant, followed by depth of cut (B) and the AB interaction, although the interaction effect is relatively small. The analysis of variance, summarized in Table 14-17, confirms our interpretation of the effect estimates.

Minitab will analyze 2^k factorial designs. The output from the Minitab DOE (Design of Experiments) module for this experiment is shown in Table 14-18. The upper portion of the table displays the effect estimates and regression coefficients for each factorial effect. However, a

Table 14-16 Surface Roughness Data for Example 14-4

Treatment Combinations	Design Factors			Surface Roughness	Totals
	A	B	C		
(1)	−1	−1	−1	9, 7	16
a	1	−1	−1	10, 12	22
b	−1	1	−1	9, 11	20
ab	1	1	−1	12, 15	27
c	−1	−1	1	11, 10	21
ac	1	−1	1	10, 13	23
bc	−1	1	1	10, 8	18
abc	1	1	1	16, 14	30

Table 14-17 Analysis of Variance for the Surface Finish Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	P-Value
A	45.5625	1	45.5625	18.69	0.0025
B	10.5625	1	10.5625	4.33	0.0709
C	3.0625	1	3.0625	1.26	0.2948
AB	7.5625	1	7.5625	3.10	0.1162
AC	0.0625	1	0.0625	0.03	0.8784
BC	1.5625	1	1.5625	0.64	0.4548
ABC	5.0625	1	5.0625	2.08	0.1875
Error	19.5000	8	2.4375		
Total	92.9375	15			

t-statistic is reported for each effect instead of the F-statistic used in Table 14-17. Now the square of a t random variable with d degrees of freedom is an F random variable with 1 numerator and d denominator degrees of freedom. Thus the square of the t-statistic reported by Minitab will be equal (apart from rounding errors) to the F-statistic in Table 14-17. To illustrate, for the main effect of feed Minitab reports $t = 4.32$ (with eight degrees of freedom), and $t^2 = (4.32)^2 = 18.66$, which is approximately equal to the F-ratio for feed reported in Table 14-17 ($F = 18.69$). This F-ratio has one numerator and eight denominator degrees of freedom.

The lower panel of the Minitab output in Table 14-18 is an analysis of variance summary focusing on the types of terms in the model. A regression model approach is used in the presentation. You might find it helpful to review Section 12-2.2, particularly the material on the partial F-test. The row entitled "main effects" under source refers to the three main effects

Table 14-18 Minitab Analysis for Example 14-4

Estimated Effects and Coefficients for Roughness

Term	Effect	Coef	StDev Coef	T	P
Constant		11.0625	0.3903	28.34	0.000
Feed	3.3750	1.6875	0.3903	4.32	0.003
Depth	1.6250	0.8125	0.3903	2.08	0.071
Angle	0.8750	0.4375	0.3903	1.12	0.295
Feed*Depth	1.3750	0.6875	0.3903	1.76	0.116
Feed*Angle	0.1250	0.0625	0.3903	0.16	0.877
Depth*Angle	−0.6250	−0.3125	0.3903	−0.80	0.446
Feed*Depth*Angle	1.1250	0.5625	0.3903	1.44	0.188

Analysis of Variance for Roughness

Source	DF	Seq SS	Adj SS	Adj MS	F	P
Main Effects	3	59.188	59.188	19.729	8.09	0.008
2-Way Interactions	3	9.187	9.187	3.062	1.26	0.352
3-Way Interactions	1	5.062	5.062	5.062	2.08	0.188
Residual Error	8	19.500	19.500	2.437		
Pure Error	8	19.500	19.500	2.437		
Total	15	92.938				

feed, depth, and angle, each having a single degree of freedom, giving the total 3 in the column headed "DF." The column headed "Seq SS" (an abbreviation for sequential sum of squares) reports how much the model sum of squares increases when each group of terms is added to a model that contains the terms listed *above* the groups. The first number in the "Seq SS" column presents the model sum of squares for fitting a model having only the three main effects. The row labeled "2-Way Interactions" refers to AB, AC, and BC, and the sequential sum of squares reported here is the increase in the model sum of squares if the interaction terms are added to a model containing only the main effects. Similarly, the sequential sum of squares for the three-way interaction is the increase in the model sum of squares that results from adding the term ABC to a model containing all other effects. The column headed "Adj SS" (an abbreviation for adjusted sum of squares) reports how much the model sum of squares increases when each group of terms is added to a model that contains *all* the other terms. Now since any 2^k design with an equal number of replicates in each cell is an orthogonal design, the adjusted sum of squares will equal the sequential sum of squares. Therefore, the F-tests for each row in the Minitab analysis of variance table are testing the significance of each group of terms (main effects, two-factor interactions, and three-factor interactions) as if they were the last terms to be included in the model. Clearly, only the main effect terms are significant. The t-tests on the individual factor effects indicate that feed rate and depth of cut have large main effects, and there may be some mild interaction between these two factors. Therefore, the Minitab output is in agreement with the results given previously.

Residual Analysis

We may obtain the residuals from a 2^k design by using the method demonstrated earlier for the 2^2 design. As an example, consider the surface roughness experiment. The three largest effects are A, B, and the AB interaction. The regression model used to obtain the predicted values is

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + \epsilon$$

where x_1 represents factor A, x_2 represents factor B, and x_1x_2 represents the AB interaction. The regression coefficients β_1, β_2, and β_{12} are estimated by one-half the corresponding effect estimates, and β_0 is the grand average. Thus

$$\begin{aligned}\hat{y} &= 11.0625 + \left(\frac{3.375}{2}\right)x_1 + \left(\frac{1.625}{2}\right)x_2 + \left(\frac{1.375}{2}\right)x_1x_2 \\ &= 11.0625 + 1.6875x_1 + 0.8125x_2 + 0.6875x_1x_2\end{aligned}$$

Note that the regression coefficients are presented by Minitab in the upper panel of Table 14-18. The predicted values would be obtained by substituting the low and high levels of A and B into this equation. To illustrate this, at the treatment combination where A, B, and C are all at the low level, the predicted value is

$$\hat{y} = 11.065 + 1.6875(-1) + 0.8125(-1) + 0.6875(-1)(-1) = 9.25$$

Since the observed values at this run are 9 and 7, the residuals are $9 - 9.25 = -0.25$ and $7 - 9.25 = -2.25$. Residuals for the other 14 runs are obtained similarly.

A normal probability plot of the residuals is shown in Fig. 14-20. Since the residuals lie approximately along a straight line, we do not suspect any problem with normality in the data. There are no indications of severe outliers. It would also be helpful to plot the residuals versus the predicted values and against each of the factors A, B, and C.

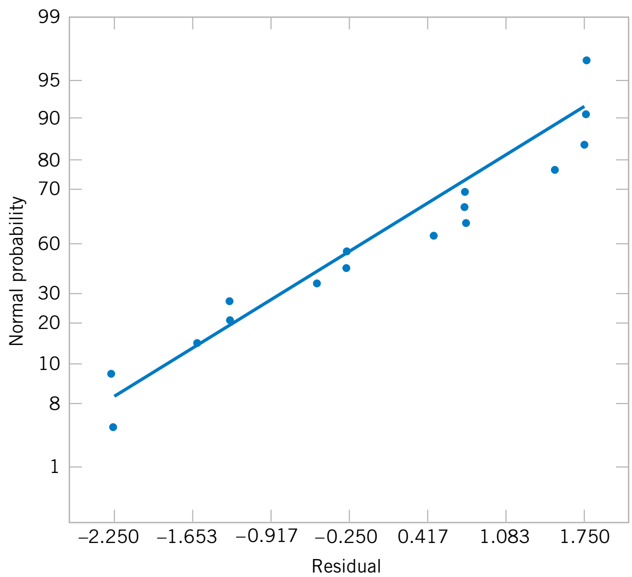


Figure 14-20 Normal probability plot of residuals from the surface roughness experiment.

Projection of 2^k Designs

Any 2^k design will collapse or project into another 2^k design in fewer variables if one or more of the original factors are dropped. Sometimes this can provide additional insight into the remaining factors. For example, consider the surface roughness experiment. Since factor C and all its interactions are negligible, we could eliminate factor C from the design. The result is to collapse the cube in Fig. 14-18 into a square in the $A - B$ plane; therefore, each of the four runs in the new design has four replicates. In general, if we delete h factors so that $r = k - h$ factors remain, the original 2^k design with n replicates will project into a 2^r design with $n2^h$ replicates.

14-7.3 Single Replicate of the 2^k Design

As the number of factors in a factorial experiment grows, the number of effects that can be estimated also grows. For example, a 2^4 experiment has 4 main effects, 6 two-factor interactions, 4 three-factor interactions, and 1 four-factor interaction, while a 2^6 experiment has 6 main effects, 15 two-factor interactions, 20 three-factor interactions, 15 four-factor interactions, 6 five-factor interactions, and 1 six-factor interaction. In most situations the **sparsity of effects principle** applies; that is, the system is usually dominated by the main effects and low-order interactions. The three-factor and higher order interactions are usually negligible. Therefore, when the number of factors is moderately large, say, $k \geq 4$ or 5, a common practice is to run only a single replicate of the 2^k design and then pool or combine the higher order interactions as an estimate of error. Sometimes a single replicate of a 2^k design is called an **unreplicated** 2^k factorial design.

When analyzing data from unreplicated factorial designs, occasionally real high-order interactions occur. The use of an error mean square obtained by pooling high-order interactions is inappropriate in these cases. A simple method of analysis can be used to overcome this problem. Construct a plot of the estimates of the effects on a normal probability scale. The effects that are negligible are normally distributed, with mean zero and variance σ^2 and will tend to fall along a straight line on this plot, whereas significant effects will have nonzero means and will not lie along the straight line. We will illustrate this method in the next example.

EXAMPLE 14-5

An article in *Solid State Technology* ("Orthogonal Design for Process Optimization and Its Application in Plasma Etching," May 1987, pp. 127–132) describes the application of factorial designs in developing a nitride etch process on a single-wafer plasma etcher. The process uses C_2F_6 as the reactant gas. It is possible to vary the gas flow, the power applied to the cathode, the pressure in the reactor chamber, and the spacing between the anode and the cathode (gap). Several response variables would usually be of interest in this process, but in this example we will concentrate on etch rate for silicon nitride.

We will use a single replicate of a 2^4 design to investigate this process. Since it is unlikely that the three- and four-factor interactions are significant, we will tentatively plan to combine them as an estimate of error. The factor levels used in the design are shown below:

Design Factor

Level	Gap (cm)	Pressure (mTorr)	C_2F_6 Flow (SCCM)	Power (w)
Low (−)	0.80	450	125	275
High (+)	1.20	550	200	325

Table 14-19 presents the data from the 16 runs of the 2^4 design. Table 14-20 is the table of plus and minus signs for the 2^4 design. The signs in the columns of this table can be used to estimate the factor effects. For example, the estimate of factor A is

$$\begin{aligned} A &= \frac{1}{8}[a + ab + ac + abc + ad + abd + acd + abcd - (1) - b \\ &\quad - c - bc - d - bd - cd - bcd] \\ &= \frac{1}{8}[669 + 650 + 642 + 635 + 749 + 868 + 860 + 729 \\ &\quad - 550 - 604 - 633 - 601 - 1037 - 1052 - 1075 - 1063] \\ &= -101.625 \end{aligned}$$

Table 14-19 The 2^4 Design for the Plasma Etch Experiment

A (Gap)	B (Pressure)	C (C_2F_6 Flow)	D (Power)	Etch Rate (Å/min)
−1	−1	−1	−1	550
1	−1	−1	−1	669
−1	1	−1	−1	604
1	1	−1	−1	650
−1	−1	1	−1	633
1	−1	1	−1	642
−1	1	1	−1	601
1	1	1	−1	635
−1	−1	−1	1	1037
1	−1	−1	1	749
−1	1	−1	1	1052
1	1	−1	1	868
−1	−1	1	1	1075
1	−1	1	1	860
−1	1	1	1	1063
1	1	1	1	729

Table 14-20 Contrast Constants for the 2^4 Design

	A	*B*	*AB*	*C*	*AC*	*BC*	*ABC*	*D*	*AD*	*BD*	*ABD*	*CD*	*ACD*	*BCD*	*ABCD*
(1)	−	−	+	−	+	+	−	−	+	+	−	+	−	−	+
a	+	−	−	−	−	+	+	−	−	+	+	+	+	−	−
b	−	+	−	−	+	−	+	−	+	−	+	+	−	+	−
ab	+	+	+	−	−	−	−	−	−	−	−	+	+	+	+
c	−	−	+	+	−	−	+	−	+	+	−	−	+	+	−
ac	+	−	−	+	+	−	−	−	−	+	+	−	−	+	+
bc	−	+	−	+	−	+	−	−	+	−	+	−	+	−	+
abc	+	+	+	+	+	+	+	−	−	−	−	−	−	−	−
d	−	−	+	−	+	+	−	+	−	−	+	−	+	+	−
ad	+	−	−	−	−	+	+	+	+	−	−	−	−	+	+
bd	−	+	−	−	+	−	+	+	−	+	−	−	+	−	+
abd	+	+	+	−	−	−	−	+	+	+	+	−	−	−	−
cd	−	−	+	+	−	−	+	+	−	−	+	+	−	−	+
acd	+	−	−	+	+	−	−	+	+	−	−	+	+	−	−
bcd	−	+	−	+	−	+	−	+	−	+	−	+	−	+	−
abcd	+	+	+	+	+	+	+	+	+	+	+	+	+	+	+

Thus, the effect of increasing the gap between the anode and the cathode from 0.80 to 1.20 centimeters is to decrease the etch rate by 101.625 angstroms per minute.

It is easy to verify (using Minitab, for example) that the complete set of effect estimates is

$$\begin{array}{lrlr}
A & = -101.625 & AD & = -153.625 \\
B & = \quad -1.625 & BD & = \quad -0.625 \\
AB & = \quad -7.875 & ABD & = \quad 4.125 \\
C & = \quad 7.375 & CD & = \quad -2.125 \\
AC & = \quad -24.875 & ACD & = \quad 5.625 \\
BC & = \quad -43.875 & BCD & = \quad -25.375 \\
ABC & = \quad -15.625 & ABCD & = \quad -40.125 \\
D & = \quad 306.125 & &
\end{array}$$

The normal probability plot of these effects from the plasma etch experiment is shown in Fig. 14-21. Clearly, the main effects of A and D and the AD interaction are significant, because they fall far from the line passing through the other points. The analysis of variance summarized in Table 14-21 confirms these findings. Notice that in the analysis of variance we have pooled the three- and four-factor interactions to form the error mean square. If the normal probability plot had indicated that any of these interactions were important, they would not have been included in the error term.

Since $A = -101.625$, the effect of increasing the gap between the cathode and anode is to decrease the etch rate. However, $D = 306.125$; thus, applying higher power levels will increase the etch rate. Figure 14-22 is a plot of the AD interaction. This plot indicates that the effect of changing the gap width at low power settings is small, but that increasing the gap at high power settings dramatically reduces the etch rate. High etch rates are obtained at high power settings and narrow gap widths.

The residuals from the experiment can be obtained from the regression model

$$\hat{y} = 776.0625 - \left(\frac{101.625}{2}\right) x_1 + \left(\frac{306.125}{2}\right) x_4 - \left(\frac{153.625}{2}\right) x_1 x_4$$

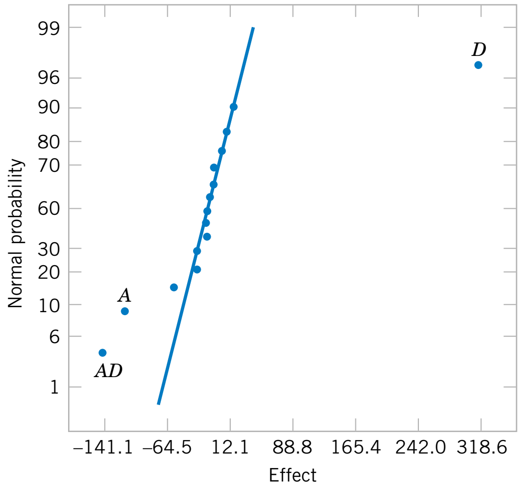


Figure 14-21 Normal probability plot of effects from the plasma etch experiment.

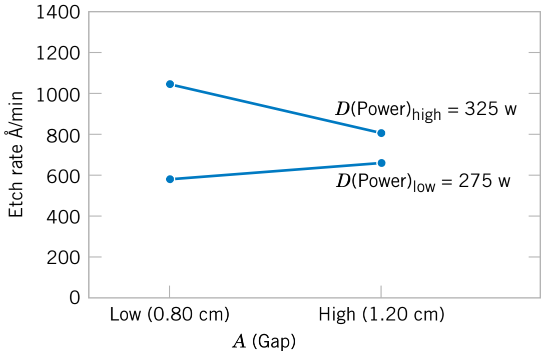


Figure 14-22 *AD* (Gap-Power) interaction from the plasma etch experiment.

For example, when both A and D are at the low level, the predicted value is

$$\hat{y} = 776.0625 - \left(\frac{101.625}{2}\right)(-1) + \left(\frac{306.125}{2}\right)(-1) - \left(\frac{153.625}{2}\right)(-1)(-1)$$

$$= 597$$

and the four residuals at this treatment combination are

$$e_1 = 550 - 597 = -47 \qquad e_2 = 604 - 597 = 7$$

$$e_3 = 633 - 597 = 36 \qquad e_4 = 601 - 597 = 4$$

The residuals at the other three treatment combinations (A high, D low), (A low, D high), and (A high, D high) are obtained similarly. A normal probability plot of the residuals is shown in Fig. 14-23. The plot is satisfactory.

Table 14-21 Analysis of Variance for the Plasma Etch Experiment

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	P-Value
A	41,310.563	1	41,310.563	20.28	0.0064
B	10.563	1	10.563	<1	—
C	217.563	1	217.563	<1	—
D	374,850.063	1	374,850.063	183.99	0.0000
AB	248.063	1	248.063	<1	—
AC	2,475.063	1	2,475.063	1.21	0.3206
AD	94,402.563	1	94,402.563	46.34	0.0010
BC	7,700.063	1	7,700.063	3.78	0.1095
BD	1.563	1	1.563	<1	—
CD	18.063	1	18.063	<1	—
Error	10,186.813	5	2,037.363		
Total	531,420.938	15			

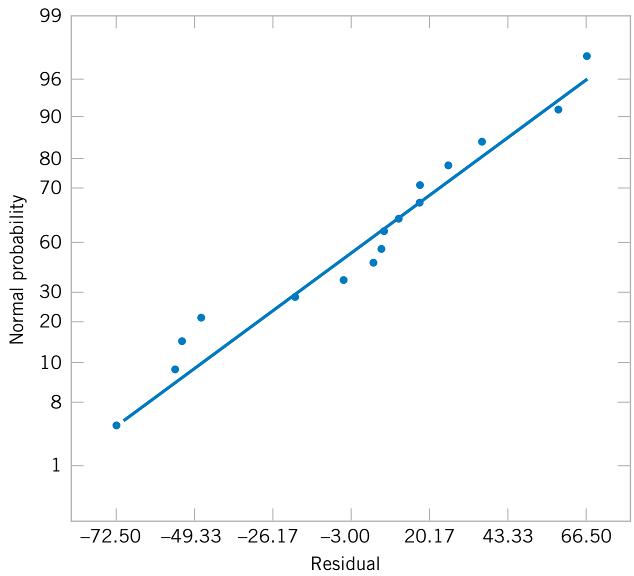


Figure 14-23 Normal probability plot of residuals from the plasma etch experiment.

14-7.4 Addition of Center Points to a 2^k Design (CD Only)

EXERCISES FOR SECTION 14-7

14-13. An engineer is interested in the effect of cutting speed (A), metal hardness (B), and cutting angle (C) on the life of a cutting tool. Two levels of each factor are chosen, and two replicates of a 2^3 factorial design are run. The tool life data (in hours) are shown in the following table:

Treatment Combination	Replicate I	Replicate II
(1)	221	311
a	325	435
b	354	348
ab	552	472
c	440	453
ac	406	377
bc	605	500
abc	392	419

(a) Analyze the data from this experiment.
(b) Find an appropriate regression model that explains tool life in terms of the variables used in the experiment.
(c) Analyze the residuals from this experiment.

14-14. Four factors are thought to influence the taste of a soft-drink beverage: type of sweetener (A), ratio of syrup to water (B), carbonation level (C), and temperature (D). Each factor can be run at two levels, producing a 2^4 design. At each run in the design, samples of the beverage are given to a test panel consisting of 20 people. Each tester assigns the beverage a point score from 1 to 10. Total score is the response variable, and the objective is to find a formulation that maximizes total score. Two replicates of this design are run, and the results are shown in the table. Analyze the data and draw conclusions. Use $\alpha = 0.05$ in the statistical tests.

Treatment Combination	Replicate I	Replicate II
(1)	159	163
a	168	175
b	158	163
ab	166	168
c	175	178
ac	179	183
bc	173	168
abc	179	182
d	164	159
ad	187	189
bd	163	159
abd	185	191
cd	168	174
acd	197	199
bcd	170	174
abcd	194	198

14-15. Consider the experiment in Exercise 14-14. Determine an appropriate model and plot the residuals against the levels of factors *A*, *B*, *C*, and *D*. Also construct a normal probability plot of the residuals. Comment on these plots.

14-16. The data shown here represent a single replicate of a 2^5 design that is used in an experiment to study the compressive strength of concrete. The factors are mix (*A*), time (*B*), laboratory (*C*), temperature (*D*), and drying time (*E*).

(1)	= 700	*e*	= 800
a	= 900	*ae*	= 1200
b	= 3400	*be*	= 3500
ab	= 5500	*abe*	= 6200
c	= 600	*ce*	= 600
ac	= 1000	*ace*	= 1200
bc	= 3000	*bce*	= 3006
abc	= 5300	*abce*	= 5500
d	= 1000	*de*	= 1900
ad	= 1100	*ade*	= 1500
bd	= 3000	*bde*	= 4000
abd	= 6100	*abde*	= 6500
cd	= 800	*cde*	= 1500
acd	= 1100	*acde*	= 2000
bcd	= 3300	*bcde*	= 3400
abcd	= 6000	*abcde*	= 6800

(a) Estimate the factor effects.
(b) Which effects appear important? Use a normal probability plot.
(c) If it is desirable to maximize the strength, in which direction would you adjust the process variables?
(d) Analyze the residuals from this experiment.

14-17. An article in the *IEEE Transactions on Semiconductor Manufacturing* (Vol. 5, no. 3, 1992, pp. 214–222) describes an experiment to investigate the surface charge on a silicon wafer. The factors thought to influence induced surface charge are cleaning method (spin rinse dry or SRD and spin dry or SD) and the position on the wafer where the charge was measured. The surface charge ($\times 10^{11}$ q/cm^3) response data are as shown.

		Test Position L	Test Position R
Cleaning Method	SD	1.66	1.84
		1.90	1.84
		1.92	1.62
	SRD	−4.21	−7.58
		−1.35	−2.20
		−2.08	−5.36

(a) Estimate the factor effects.
(b) Which factors appear important? Use $\alpha = 0.05$.
(c) Analyze the residuals from this experiment.

14-18. An experiment described by M. G. Natrella in the National Bureau of Standards *Handbook of Experimental Statistics* (No. 91, 1963) involves flame testing fabrics after applying fire-retardant treatments. The four factors considered are type of fabric (*A*), type of fire-retardant treatment (*B*), laundering condition (*C*—the low level is no laundering, the high level is after one laundering), and method of conducting the flame test (*D*). All factors are run at two levels, and the response variable is the inches of fabric burned on a standard size test sample. The data are:

(1)	= 42	*d*	= 40
a	= 31	*ad*	= 30
b	= 45	*bd*	= 50
ab	= 29	*abd*	= 25
c	= 39	*cd*	= 40
ac	= 28	*acd*	= 25
bc	= 46	*bcd*	= 50
abc	= 32	*abcd*	= 23

(a) Estimate the effects and prepare a normal plot of the effects.
(b) Construct an analysis of variance table based on the model tentatively identified in part (a).
(c) Construct a normal probability plot of the residuals and comment on the results.

14-19. An experiment was run in a semiconductor fabrication plant in an effort to increase yield. Five factors, each at two levels, were studied. The factors (and levels) were A = aperture setting (small, large), B = exposure time (20% below nominal, 20% above nominal), C = development time (30 and 45 seconds), D = mask dimension (small, large), and E = etch time (14.5 and 15.5 minutes). The following unreplicated 2^5 design was run:

(1)	= 7	*e*	= 8
a	= 9	*ae*	= 12
b	= 34	*be*	= 35
ab	= 55	*abe*	= 52
c	= 16	*ce*	= 15
ac	= 20	*ace*	= 22
bc	= 40	*bce*	= 45
abc	= 60	*abce*	= 65
d	= 8	*de*	= 6
ad	= 10	*ade*	= 10
bd	= 32	*bde*	= 30
abd	= 50	*abde*	= 53
cd	= 18	*cde*	= 15
acd	= 21	*acde*	= 20
bcd	= 44	*bcde*	= 41
abcd	= 61	*abcde*	= 63

(a) Construct a normal probability plot of the effect estimates. Which effects appear to be large?

(b) Conduct an analysis of variance to confirm your findings for part (a).
(c) Construct a normal probability plot of the residuals. Is the plot satisfactory?
(d) Plot the residuals versus the predicted yields and versus each of the five factors. Comment on the plots.
(e) Interpret any significant interactions.
(f) What are your recommendations regarding process operating conditions?
(g) Project the 2^5 design in this problem into a 2^r for $r < 5$ design in the important factors. Sketch the design and show the average and range of yields at each run. Does this sketch aid in data interpretation?

14-20. Consider the data from Exercise 14-13. I suppose that the data from the second replicate was not available. Analyze the data from replicate I only and comment on your findings.

14-21. An experiment has run a single replicate of a 2^4 design and calculated the following factor effects:

$A = 80.25$	$AB = 53.25$	$ABC = -2.95$
$B = -65.50$	$AC = 11.00$	$ABD = -8.00$
$C = -9.25$	$AD = 9.75$	$ACD = 10.25$
$D = -20.50$	$BC = 18.36$	$BCD = -7.95$
	$BD = 15.10$	$ABCD = -6.25$
	$CD = -1.25$	

(a) Construct a normal probability plot of the effects.
(b) Identify a tentative model, based on the plot of effects in part (a).
(c) Estimate the regression coefficients in this model, assuming that $\bar{y} = 400$.

14-22. A 2^4 factorial design was run in a chemical process. The design factors are A = time, B = concentration, C = pressure, and D = temperature. The response variable is yield. The data follows:

Run	A	B	C	D	Yield (pounds)
1	−	−	−	−	12
2	+	−	−	−	18
3	−	+	−	−	13
4	+	+	−	−	16
5	−	−	+	−	17
6	+	−	+	−	15
7	−	+	+	−	20
8	+	+	+	−	15
9	−	−	−	+	10
10	+	−	−	+	25
11	−	+	−	+	13
12	+	+	−	+	24
13	−	−	+	+	19
14	+	−	+	+	21
15	−	+	+	+	17
16	+	+	+	+	23

Factor Levels	−	+
A (hours)	25	3
B (%)	14	18
C (psi)	60	80
D (°C)	200	250

(a) Estimate the factor effects. Based on a normal probability plot of the effect estimates, identify a model for the data from this experiment.
(b) Conduct an ANOVA based on the model identified in part (a). What are your conclusions?
(c) Analyze the residuals and comment on model adequacy.
(d) Find a regression model to predict yield in terms of the actual factor levels.
(e) Can this design be projected into a 2^3 design with two replicates? If so, sketch the design and show the average and range of the two yield values at each cube corner. Discuss the practical value of this plot.

14-8 BLOCKING AND CONFOUNDING IN THE 2^k DESIGN

It is often impossible to run all the observations in a 2^k factorial design under homogeneous conditions. Blocking is the design technique that is appropriate for this general situation. However, in many situations the block size is smaller than the number of runs in the complete replicate. In these cases, **confounding** is a useful procedure for running the 2^k design in 2^p blocks where the number of runs in a block is less than the number of treatment combinations in one complete replicate. The technique causes certain interaction effects to be indistinguishable from blocks or **confounded with blocks.** We will illustrate confounding in the 2^k factorial design in 2^p blocks, where $p < k$.

Consider a 2^2 design. Suppose that each of the $2^2 = 4$ treatment combinations requires four hours of laboratory analysis. Thus, two days are required to perform the experiment. If days are considered as blocks, we must assign two of the four treatment combinations to each day.

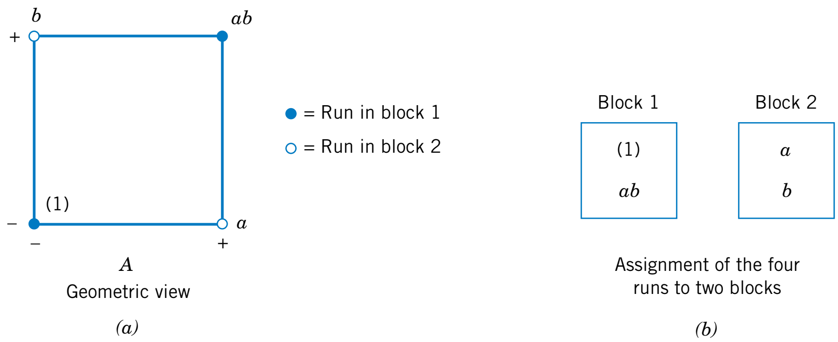


Figure 14-24 A 2^2 design in two blocks. (a) Geometric view. (b) Assignment of the four runs to two blocks.

This design is shown in Fig. 14-24. Notice that block 1 contains the treatment combinations (1) and ab and that block 2 contains a and b. The contrasts for estimating the main effects of factors A and B are

$$\text{Contrast}_A = ab + a - b - (1)$$
$$\text{Contrast}_B = ab + b - a - (1)$$

Note that these contrasts are unaffected by blocking since in each contrast there is one plus and one minus treatment combination from each block. That is, any difference between block 1 and block 2 that increases the readings in one block by an additive constant cancels out. The contrast for the AB interaction is

$$\text{Contrast}_{AB} = ab + (1) - a - b$$

Since the two treatment combinations with the plus signs, ab and (1), are in block 1 and the two with the minus signs, a and b, are in block 2, the block effect and the AB interaction are identical. That is, the AB interaction is confounded with blocks.

The reason for this is apparent from the table of plus and minus signs for the 2^2 design shown in Table 14-12. From the table we see that all treatment combinations that have a plus on AB are assigned to block 1, whereas all treatment combinations that have a minus sign on AB are assigned to block 2.

This scheme can be used to confound any 2^k design in two blocks. As a second example, consider a 2^3 design, run in two blocks. From the table of plus and minus signs, shown in Table 14-15, we assign the treatment combinations that are minus in the ABC column to block 1 and those that are plus in the ABC column to block 2. The resulting design is shown in Fig. 14-25.

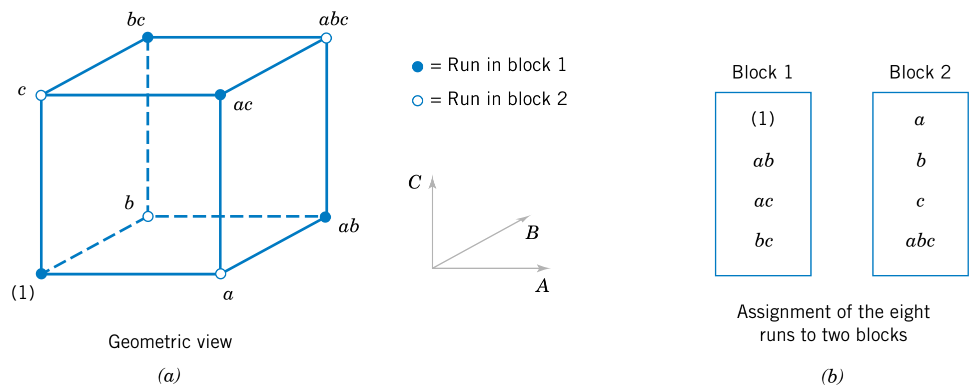


Figure 14-25 The 2^3 design in two blocks with ABC confounded. (a) Geometric View. (b) Assignment of the eight runs to two blocks.

There is a more general method of constructing the blocks. The method employs a **defining contrast,** say

$$L = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k \tag{14-24}$$

where x_i is the level of the ith factor appearing in a treatment combination and α_i is the exponent appearing on the ith factor in the effect that is to be confounded with blocks. For the 2^k system, we have either $\alpha_i = 0$ or 1, and either $x_i = 0$ (low level) or $x_i = 1$ (high level). Treatment combinations that produce the same value of L (modulus 2) will be placed in the same block. Since the only possible values of L (mod 2) are 0 and 1, this will assign the 2^k treatment combinations to exactly two blocks.

As an example, consider the 2^3 design with ABC confounded with blocks. Here x_1 corresponds to A, x_2 to B, x_3 to C, and $\alpha_1 = \alpha_2 = \alpha_3 = 1$. Thus, the defining contrast that would be used to confound ABC with blocks is

$$L = x_1 + x_2 + x_3$$

To assign the treatment combinations to the two blocks, we substitute the treatment combinations into the defining contrast as follows:

$$\begin{aligned}
(1)&: L = 1(0) + 1(0) + 1(0) = 0 = 0 \pmod 2\\
a&: L = 1(1) + 1(0) + 1(0) = 1 = 1 \pmod 2\\
b&: L = 1(0) + 1(1) + 1(0) = 1 = 1 \pmod 2\\
ab&: L = 1(1) + 1(1) + 1(0) = 2 = 0 \pmod 2\\
c&: L = 1(0) + 1(0) + 1(1) = 1 = 1 \pmod 2\\
ac&: L = 1(1) + 1(0) + 1(1) = 2 = 0 \pmod 2\\
bc&: L = 1(0) + 1(1) + 1(1) = 2 = 0 \pmod 2\\
abc&: L = 1(1) + 1(1) + 1(1) = 3 = 1 \pmod 2
\end{aligned}$$

Thus (1), ab, ac, and bc are run in block 1, and a, b, c, and abc are run in block 2. This same design is shown in Fig. 14-25.

A shortcut method is useful in constructing these designs. The block containing the treatment combination (1) is called the **principal block.** Any element [except (1)] in the principal block may be generated by multiplying two other elements in the principal block modulus 2 on the exponents. For example, consider the principal block of the 2^3 design with ABC confounded, shown in Fig. 14-25. Note that

$$\begin{aligned}
ab \cdot ac &= a^2bc = bc\\
ab \cdot bc &= ab^2c = ac\\
ac \cdot bc &= abc^2 = ab
\end{aligned}$$

Treatment combinations in the other block (or blocks) may be generated by multiplying one element in the new block by each element in the principal block modulus 2 on the exponents. For the 2^3 with ABC confounded, since the principal block is (1), ab, ac, and bc, we know that the treatment combination b is in the other block. Thus, elements of this second block are

$$
\begin{aligned}
b \cdot (1) &= b \\
b \cdot ab = ab^2 &= a \\
b \cdot ac &= abc \\
b \cdot bc = b^2c &= c
\end{aligned}
$$

EXAMPLE 14-6

An experiment is performed to investigate the effect of four factors on the terminal miss distance of a shoulder-fired ground-to-air-missile. The four factors are target type (A), seeker type (B), target altitude (C), and target range (D). Each factor may be conveniently run at two levels, and the optical tracking system will allow terminal miss distance to be measured to the nearest foot. Two different operators or gunners are used in the flight test and, since there may be differences between operators, the test engineers decided to conduct the 2^4 design in two blocks with $ABCD$ confounded. Thus, the defining contrast is

$$L = x_1 + x_2 + x_3 + x_4$$

The experimental design and the resulting data are shown in Fig. 14-26. The effect estimates obtained from Minitab are shown in Table 14-22. A normal probability plot of the effects in Fig. 14-27 reveals that A (target type), D (target range), AD, and AC have large effects. A confirming analysis of variance, pooling the three-factor interactions as error, is shown in Table 14-23. Since the AC and AD interactions are significant, it is logical to conclude that A (target type), C (target altitude), and D (target range) all have important effects on the miss distance and that there are interactions between target type and altitude and target type and range. Notice that the $ABCD$ effect is treated as blocks in this analysis.

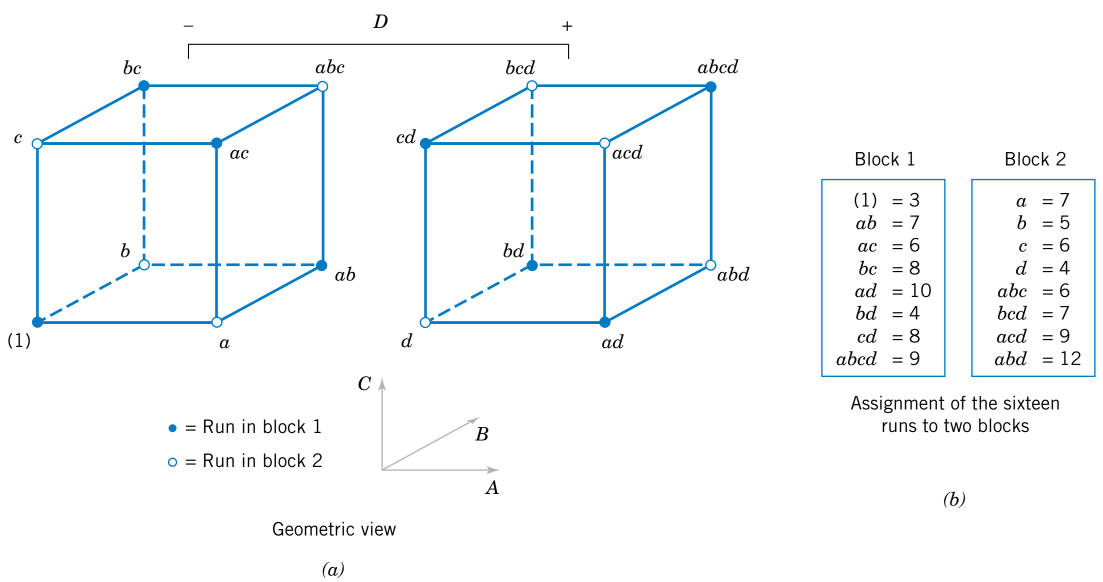


Figure 14-26 The 2^4 design in two blocks for Example 14-6. (a) Geometric view. (b) Assignment of the 16 runs to two blocks.

Table 14-22 Minitab Effect Estimates for Example 14-6

Estimated Effects and Coefficients for Distance		
Term	Effect	Coef
Constant		6.938
Block		0.063
A	2.625	1.312
B	0.625	0.313
C	0.875	0.438
D	1.875	0.938
AB	−0.125	−0.063
AC	−2.375	−1.187
AD	1.625	0.813
BC	−0.375	−0.188
BD	−0.375	−0.187
CD	−0.125	−0.062
ABC	−0.125	−0.063
ABD	0.875	0.438
ACD	−0.375	−0.187
BCD	−0.375	−0.187

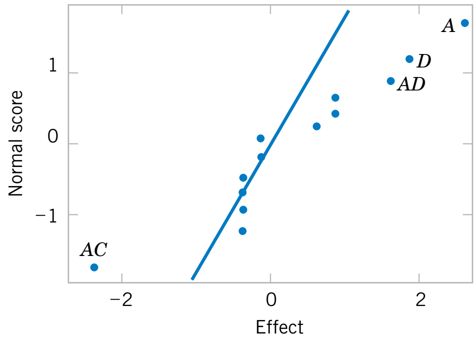


Figure 14-27 Normal probability plot of the effects from Minitab, Example 14-6.

It is possible to confound the 2^k design in four blocks of 2^{k-2} observations each. To construct the design, two effects are chosen to confound with blocks, and their defining contrasts are obtained. A third effect, the **generalized interaction** of the two effects initially chosen, is also confounded with blocks. The generalized interaction of two effects is found by multiplying their respective letters and reducing the exponents modulus 2.

For example, consider the 2^4 design in four blocks. If AC and BD are confounded with blocks, their generalized interaction is $(AC)(BD) = ABCD$. The design is constructed by using

Table 14-23 Analysis of Variance for Example 14-6

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	P-Value
Blocks ($ABCD$)	0.0625	1	0.0625	0.06	—
A	27.5625	1	27.5625	25.94	0.0070
B	1.5625	1	1.5625	1.47	0.2920
C	3.0625	1	3.0625	2.88	0.1648
D	14.0625	1	14.0625	13.24	0.0220
AB	0.0625	1	0.0625	0.06	—
AC	22.5625	1	22.5625	21.24	0.0100
AD	10.5625	1	10.5625	9.94	0.0344
BC	0.5625	1	0.5625	0.53	—
BD	0.5625	1	0.5625	0.53	—
CD	0.0625	1	0.0625	0.06	—
Error ($ABC + ABD + ACD + BCD$)	4.2500	4	1.0625		
Total	84.9375	15			

the defining contrasts for AC and BD:

$$L_1 = x_1 + x_3$$
$$L_2 = x_2 + x_4$$

It is easy to verify that the four blocks are

Block 1	Block 2	Block 3	Block 4
$L_1 = 0, L_2 = 0$	$L_1 = 1, L_2 = 0$	$L_1 = 0, L_2 = 1$	$L_1 = 1, L_2 = 1$
(1)	a	b	ab
ac	c	abc	bc
bd	abd	d	ad
$abcd$	bcd	acd	cd

This general procedure can be extended to confounding the 2^k design in 2^p blocks, where $p < k$. Start by selecting p effects to be confounded, such that no effect chosen is a generalized interaction of the others. Then the blocks can be constructed from the p defining contrasts $L_1, L_2, \ldots, L_p$ that are associated with these effects. In addition to the p effects chosen to be confounded, exactly $2^p - p - 1$ additional effects are confounded with blocks; these are the generalized interactions of the original p effects chosen. Care should be taken so as not to confound effects of potential interest.

For more information on confounding in the 2^k factorial design, refer to Montgomery (2001, Chapter 7). This book contains guidelines for selecting factors to confound with blocks so that main effects and low-order interactions are not confounded. In particular, the book contains a table of suggested confounding schemes for designs with up to seven factors and a range of block sizes, some of which are as small as two runs.

EXERCISES FOR SECTION 14-8

14-23. Consider the data from the first replicate of Exercise 14-13. Suppose that these observations could not all be run under the same conditions. Set up a design to run these observations in two blocks of four observations each, with ABC confounded. Analyze the data.

14-24. Consider the data from the first replicate of Exercise 14-14. Construct a design with two blocks of eight observations each, with $ABCD$ confounded. Analyze the data.

14-25. Repeat Exercise 14-24 assuming that four blocks are required. Confound ABD and ABC (and consequently CD) with blocks.

14-26. Construct a 2^5 design in two blocks. Select the $ABCDE$ interaction to be confounded with blocks.

14-27. Construct a 2^5 design in four blocks. Select the appropriate effects to confound so that the highest possible interactions are confounded with blocks.

14-28. Consider the data from Exercise 14-18. Construct the design that would have been used to run this experiment in two blocks of eight runs each. Analyze the data and draw conclusions.

14-29. An article in *Industrial and Engineering Chemistry* ("Factorial Experiments in Pilot Plant Studies," 1951, pp. 1300–1306) reports on an experiment to investigate the effect of temperature (A), gas throughput (B), and concentration (C) on the strength of product solution in a recirculation unit. Two blocks were used with ABC confounded, and the experiment was replicated twice. The data are as follows:

Replicate 1

Block 1	Block 2
(1) = 99	a = 18
ab = 52	b = 51
ac = 42	c = 108
bc = 95	abc = 35

Replicate 2

Block 3	Block 4
(1) = 46	a = 18
ab = 47	b = 62
ac = 22	c = 104
bc = 67	abc = 36

(a) Analyze the data from this experiment.
(b) Analyze the residuals and comment on model adequacy.
(c) Comment on the efficiency of this design. Note that we have replicated the experiment twice, yet we have no information on the *ABC* interaction.
(d) Suggest a better design, specifically, one that would provide some information on *all* interactions.

14-30. Consider the 2^6 factorial design. Set up a design to be run in four blocks of 16 runs each. Show that a design that confounds three of the four-factor interactions with blocks is the best possible blocking arrangement.

14-9 FRACTIONAL REPLICATION OF THE 2^k DESIGN

As the number of factors in a 2^k factorial design increases, the number of runs required increases rapidly. For example, a 2^5 requires 32 runs. In this design, only 5 degrees of freedom correspond to main effects, and 10 degrees of freedom correspond to two-factor interactions. Sixteen of the 31 degrees of freedom are used to estimate high-order interactions—that is, three-factor and higher order interactions. Often there is little interest in these high-order interactions, particularly when we first begin to study a process or system. If we can assume that certain high-order interactions are negligible, a **fractional factorial design** involving fewer than the complete set of 2^k runs can be used to obtain information on the main effects and low-order interactions. In this section, we will introduce fractional replications of the 2^k design.

A major use of fractional factorials is in **screening experiments.** These are experiments in which many factors are considered with the purpose of identifying those factors (if any) that have large effects. Screening experiments are usually performed in the early stages of a project when it is likely that many of the factors initially considered have little or no effect on the response. The factors that are identified as important are then investigated more thoroughly in subsequent experiments.

14-9.1 One-Half Fraction of the 2^k Design

A one-half fraction of the 2^k design contains 2^{k-1} runs and is often called a 2^{k-1} fractional factorial design. As an example, consider the 2^{3-1} design—that is, a one-half fraction of the 2^3. This design has only four runs, in contrast to the full factorial that would require eight runs. The table of plus and minus signs for the 2^3 design is shown in Table 14-24. Suppose we select the four treatment combinations a, b, c, and abc, as our one-half fraction. These treatment combinations are shown in the top half of Table 14-24 and in Fig. 14-28(a).

Notice that the 2^{3-1} design is formed by selecting only those treatment combinations that yield a plus on the *ABC* effect. Thus, *ABC* is called the **generator** of this particular fraction.

Table 14-24 Plus and Minus Signs for the 2^3 Factorial Design

Treatment Combination	Factorial Effect							
	I	*A*	*B*	*C*	*AB*	*AC*	*BC*	*ABC*
a	+	+	−	−	−	−	+	+
b	+	−	+	−	−	+	−	+
c	+	−	−	+	+	−	−	+
abc	+	+	+	+	+	+	+	+
ab	+	+	+	−	+	−	−	−
ac	+	+	−	+	−	+	−	−
bc	+	−	+	+	−	−	+	−
(1)	+	−	−	−	+	+	+	−

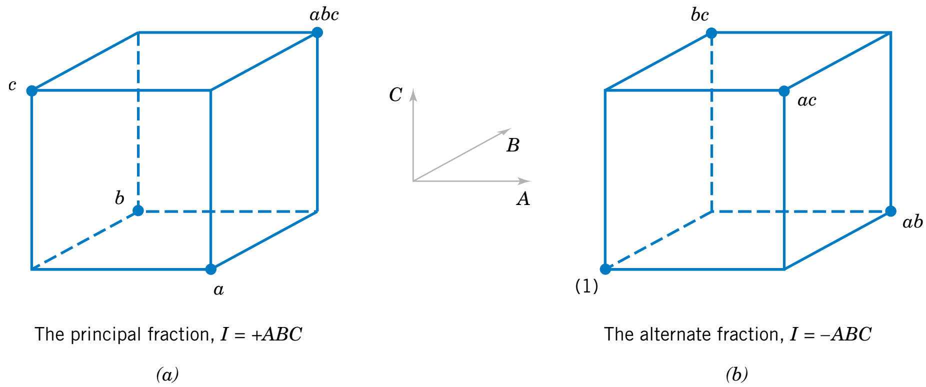


Figure 14-28 The one-half fractions of the 2^3 design. (a) The principal fraction, $I = +ABC$. (b) The alternate fraction, $I = -ABC$.

Furthermore, the identity element I is also plus for the four runs, so we call

$$I = ABC$$

the defining relation for the design.

The treatment combinations in the 2^{3-1} design yields three degrees of freedom associated with the main effects. From the upper half of Table 14-24, we obtain the estimates of the main effects as linear combinations of the observations, say,

$$\begin{aligned} A &= \tfrac{1}{2}[a - b - c + abc] \\ B &= \tfrac{1}{2}[-a + b - c + abc] \\ C &= \tfrac{1}{2}[-a - b + c + abc] \end{aligned}$$

It is also easy to verify that the estimates of the two-factor interactions should be the following linear combinations of the observations:

$$\begin{aligned} BC &= \tfrac{1}{2}[a - b - c + abc] \\ AC &= \tfrac{1}{2}[-a + b - c + abc] \\ AB &= \tfrac{1}{2}[-a - b + c + abc] \end{aligned}$$

Thus, the linear combination of observations in column A, ℓ_A, estimates both the main effect of A and the BC interaction. That is, the linear combination ℓ_A estimates the sum of these two effects $A + BC$. Similarly, ℓ_B estimates $B + AC$, and ℓ_C estimates $C + AB$. Two or more effects that have this property are called **aliases.** In our 2^{3-1} design, A and BC are aliases, B and AC are aliases, and C and AB are aliases. Aliasing is the direct result of fractional replication. In many practical situations, it will be possible to select the fraction so that the main effects and low-order interactions that are of interest will be aliased only with high-order interactions (which are probably negligible).

The alias structure for this design is found by using the defining relation $I = ABC$. Multiplying any effect by the defining relation yields the aliases for that effect. In our example, the alias of A is

$$A = A \cdot ABC = A^2BC = BC$$

since $A \cdot I = A$ and $A^2 = I$. The aliases of B and C are

$$B = B \cdot ABC = AB^2C = AC$$

and

$$C = C \cdot ABC = ABC^2 = AB$$

Now suppose that we had chosen the other one-half fraction, that is, the treatment combinations in Table 14-24 associated with minus on ABC. These four runs are shown in the lower half of Table 14-24 and in Fig. 14-28(b). The defining relation for this design is $I = -ABC$. The aliases are $A = -BC$, $B = -AC$, and $C = -AB$. Thus, estimates of A, B, and C that result from this fraction really estimate $A - BC$, $B - AC$, and $C - AB$. In practice, it usually does not matter which one-half fraction we select. The fraction with the plus sign in the defining relation is usually called the **principal fraction,** and the other fraction is usually called the **alternate fraction.**

Note that if we had chosen AB as the generator for the fractional factorial,

$$A = A \cdot AB = B$$

and the two main effects of A and B would be aliased. This typically loses important information.

Sometimes we use **sequences** of fractional factorial designs to estimate effects. For example, suppose we had run the principal fraction of the 2^{3-1} design with generator ABC. From this design we have the following effect estimates:

$$\ell_A = A + BC$$

$$\ell_B = B + AC$$

$$\ell_C = C + AB$$

Suppose that we are willing to assume at this point that the two-factor interactions are negligible. If they are, the 2^{3-1} design has produced estimates of the three main effects A, B, and C. However, if after running the principal fraction we are uncertain about the interactions, it is possible to estimate them by running the *alternate* fraction. The alternate fraction produces the following effect estimates:

$$\ell'_A = A - BC$$

$$\ell'_B = B - AC$$

$$\ell'_C = C - AB$$

We may now obtain de-aliased estimates of the main effects and two-factor interactions by adding and subtracting the linear combinations of effects estimated in the two individual fractions. For example, suppose we want to de-alias A from the two-factor interaction BC. Since $\ell_A = A + BC$ and $\ell'_A = A - BC$, we can combine these effect estimates as follows:

$$\frac{1}{2}(\ell_A + \ell'_A) = \frac{1}{2}(A + BC + A - BC) = A$$

and

$$\frac{1}{2}(\ell_A - \ell'_A) = \frac{1}{2}(A + BC - A + BC) = BC$$

For all three pairs of effect estimates, we would obtain the following results:

Effect, i	**from $\frac{1}{2}(l_i + l_i')$**	**from $\frac{1}{2}(l_i - l_i')$**
$i = A$	$\frac{1}{2}(A + BC + A - BC) = A$	$\frac{1}{2}[A + BC - (A - BC)] = BC$
$i = B$	$\frac{1}{2}(B + AC + B - AC) = B$	$\frac{1}{2}[B + AC - (B - AC)] = AC$
$i = C$	$\frac{1}{2}(C + AB + C - AB) = C$	$\frac{1}{2}[C + AB - (C - AB)] = AB$

Thus, by combining a sequence of two fractional factorial designs, we can isolate both the main effects and the two-factor interactions. This property makes the fractional factorial design highly useful in experimental problems since we can run sequences of small, efficient experiments, combine information across *several* experiments, and take advantage of learning about the process we are experimenting with as we go along. This is an illustration of the concept of sequential experimentation.

A 2^{k-1} design may be constructed by writing down the treatment combinations for a full factorial with $k - 1$ factors, called the **basic design,** and then adding the kth factor by identifying its plus and minus levels with the plus and minus signs of the highest order interaction. Therefore, a 2^{3-1} fractional factorial is constructed by writing down the basic design as a full 2^2 factorial and then equating factor C with the $\pm AB$ interaction. Thus, to construct the principal fraction, we would use $C = +AB$ as follows:

Basic Design		**Fractional Design**		
Full 2^2		**2^{3-1}, $I = +ABC$**		
A	B	A	B	$C = AB$
−	−	−	−	+
+	−	+	−	−
−	+	−	+	−
+	+	+	+	+

To obtain the alternate fraction we would equate the last column to $C = -AB$.

EXAMPLE 14-7

To illustrate the use of a one-half fraction, consider the plasma etch experiment described in Example 14-5. Suppose that we decide to use a 2^{4-1} design with $I = ABCD$ to investigate the four factors gap (A), pressure (B), C_2F_6 flow rate (C), and power setting (D). This design would be constructed by writing down as the basic design a 2^3 in the factors A, B, and C and then setting the levels of the fourth factor $D = ABC$. The design and the resulting etch rates are shown in Table 14-25. The design is shown graphically in Fig. 14-29.

Table 14-25 The 2^{4-1} Design with Defining Relation $I = ABCD$

A	B	C	$D = ABC$	Treatment Combination	Etch Rate
−	−	−	−	(1)	550
+	−	−	+	*ad*	749
−	+	−	+	*bd*	1052
+	+	−	−	*ab*	650
−	−	+	+	*cd*	1075
+	−	+	−	*ac*	642
−	+	+	−	*bc*	601
+	+	+	+	*abcd*	729

In this design, the main effects are aliased with the three-factor interactions; note that the alias of A is

$$A \cdot I = A \cdot ABCD \qquad \text{or} \qquad A = A^2BCD = BCD$$

and similarly $B = ACD$, $C = ABD$, and $D = ABC$.

The two-factor interactions are aliased with each other. For example, the alias of AB is CD:

$$AB \cdot I = AB \cdot ABCD \qquad \text{or} \qquad AB = A^2B^2CD = CD$$

The other aliases are $AC = BD$ and $AD = BC$.

The estimates of the main effects and their aliases are found using the four columns of signs in Table 14-25. For example, from column A we obtain the estimated effect

$$\ell_A = A + BCD = \tfrac{1}{4}(-550 + 749 - 1052 + 650 - 1075 + 642 - 601 + 729)$$
$$= -127.00$$

The other columns produce

$$\ell_B = B + ACD = 4.00 \quad \ell_C = C + ABD = 11.50 \quad \text{and} \quad \ell_D = D + ABC = 290.50$$

Clearly, ℓ_A and ℓ_D are large, and if we believe that the three-factor interactions are negligible, the main effects A (gap) and D (power setting) significantly affect etch rate.

The interactions are estimated by forming the AB, AC, and AD columns and adding them to the table. For example, the signs in the AB column are $+, -, -, +, +, -, -, +$, and this column produces the estimate

$$\ell_{AB} = AB + CD = \tfrac{1}{4}(550 - 749 - 1052 + 650 + 1075 - 642 - 601 + 729) = -10$$

From the AC and AD columns we find

$$\ell_{AC} = AC + BD = -25.50 \qquad \text{and} \qquad \ell_{AD} = AD + BC = -197.50$$

The ℓ_{AD} estimate is large; the most straightforward interpretation of the results is that since A and D are large, this is the AD interaction. Thus, the results obtained from the 2^{4-1} design agree with the full factorial results in Example 14-5.

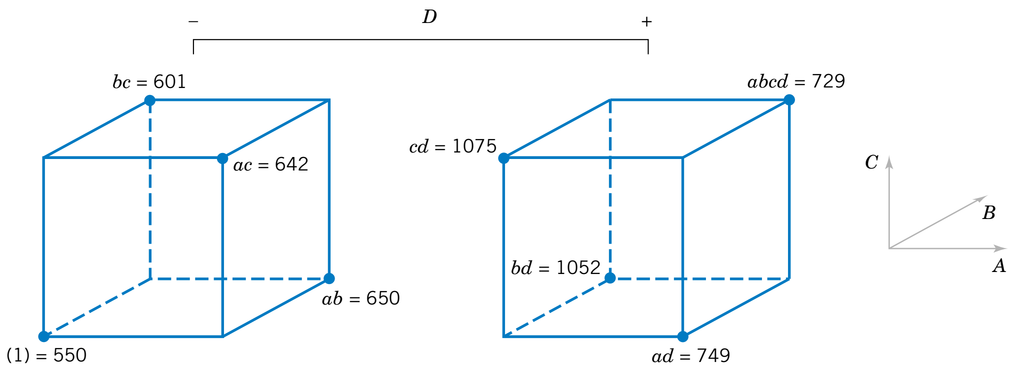


Figure 14-29 The 2^{4-1} design for the experiment of Example 14-7.

Table 14-26 Effect Estimates from Minitab, Example 14-7

Fractional Factorial Fit		
Estimated Effects and Coefficients for Etch Rt		
Term	Effect	Coef
Constant		756.00
Gap	−127.00	−63.50
Pressure	4.00	2.00
F.	11.50	5.75
Power	290.50	145.25
Gap*Pressure	−10.00	−5.00
Gap*F.	−25.50	−12.75
Gap*Power	−197.50	−98.75

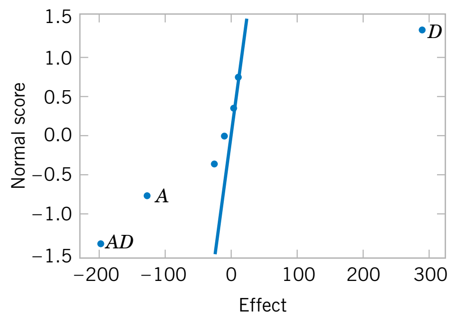


Figure 14-30 Normal probability plot of the effects from Minitab, Example 14-7.

Computer Solution

Fractional factorial designs are usually analyzed with a software package. Table 14-26 shows the effect estimates obtained from Minitab for Example 14-7. They are in agreement with the hand calculation reported earlier.

Normal Probability Plot of Effects

The normal probability plot is very useful in assessing the significance of effects from a fractional factorial design, particularly when many effects are to be estimated. We strongly recommend examining this plot. Figure 14-30 presents the normal probability plot of the effects from Example 14-7. This plot was obtained from Minitab. Notice that the A, D, and AD interaction effects stand out clearly in this graph.

Residual Analysis

The residuals can be obtained from a fractional factorial by the regression model method shown previously. Note that the Minitab output for Example 14-7 in Table 14-26 shows the regression coefficients. The residuals should be graphically analyzed as we have discussed before, both to assess the validity of the underlying model assumptions and to gain additional insight into the experimental situation.

Projection of the 2^{k-1} Design

If one or more factors from a one-half fraction of a 2^k can be dropped, the design will project into a full factorial design. For example, Fig. 14-31 presents a 2^{3-1} design. Notice that this design will project into a full factorial in any two of the three original factors. Thus, if we think that at most two of the three factors are important, the 2^{3-1} design is an excellent design for identifying the significant factors. This **projection property** is highly useful in factor screening, because it allows negligible factors to be eliminated, resulting in a stronger experiment in the active factors that remain.

In the 2^{4-1} design used in the plasma etch experiment in Example 14-7, we found that two of the four factors (B and C) could be dropped. If we eliminate these two factors, the remaining columns in Table 14-25 form a 2^2 design in the factors A and D, with two replicates. This design is shown in Fig. 14-32. The main effects of A and D and the strong two-factor AD interaction are clearly evident from this graph.

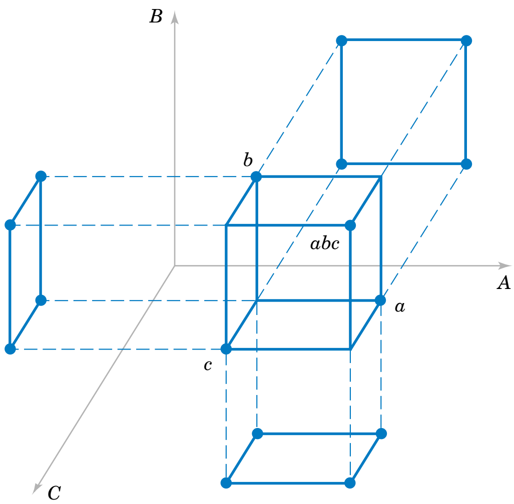


Figure 14-31 Projection of a 2^{3-1} design into three 2^2 designs.

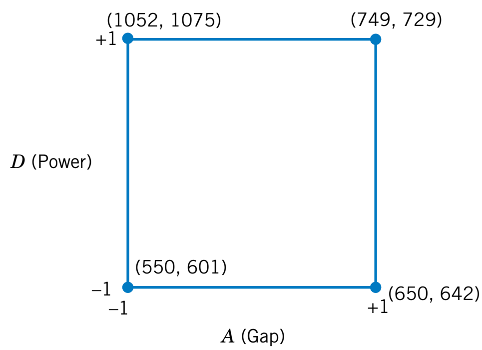


Figure 14-32 The 2^2 design obtained by dropping factors B and C from the plasma etch experiment in Example 14-7.

Design Resolution

The concept of design resolution is a useful way to catalog fractional factorial designs according to the alias patterns they produce. Designs of resolution III, IV, and V are particularly important. The definitions of these terms and an example of each follow.

1. **Resolution III Designs.** These are designs in which no main effects are aliased with any other main effect, but main effects are aliased with two-factor interactions and some two-factor interactions may be aliased with each other. The 2^{3-1} design with $I = ABC$ is a resolution III design. We usually employ a Roman numeral subscript to indicate design resolution; thus, this one-half fraction is a 2^{3-1}_{III} design.
2. **Resolution IV Designs.** These are designs in which no main effect is aliased with any other main effect or two-factor interactions, but two-factor interactions are aliased with each other. The 2^{4-1} design with $I = ABCD$ used in Example 14-7 is a resolution IV design (2^{4-1}_{IV}).
3. **Resolution V Designs.** These are designs in which no main effect or two-factor interaction is aliased with any other main effect or two-factor interaction, but two-factor interactions are aliased with three-factor interactions. The 2^{5-1} design with $I = ABCDE$ is a resolution V design (2^{5-1}_{V}).

Resolution III and IV designs are particularly useful in factor screening experiments. A resolution IV design provides good information about main effects and will provide some information about all two-factor interactions.

14-9.2 Smaller Fractions: The 2^{k-p} Fractional Factorial

Although the 2^{k-1} design is valuable in reducing the number of runs required for an experiment, we frequently find that smaller fractions will provide almost as much useful information at even greater economy. In general, a 2^k design may be run in a $1/2^p$ fraction called a 2^{k-p} fractional factorial design. Thus, a 1/4 fraction is called a 2^{k-2} design, a 1/8 fraction is called a 2^{k-3} design, a 1/16 fraction a 2^{k-4} design, and so on.

To illustrate the 1/4 fraction, consider an experiment with six factors and suppose that the engineer is primarily interested in main effects but would also like to get some information about the two-factor interactions. A 2^{6-1} design would require 32 runs and would have 31 degrees of freedom for estimating effects. Since there are only six main effects and 15 two-factor interactions, the one-half fraction is inefficient—it requires too many runs. Suppose we consider a 1/4 fraction, or a 2^{6-2} design. This design contains 16 runs and, with 15 degrees of freedom, will allow all six main effects to be estimated, with some capability for examining the two-factor interactions.

To generate this design, we would write down a 2^4 design in the factors A, B, C, and D as the basic design and then add two columns for E and F. To find the new columns we could select the two **design generators** $I = ABCE$ and $I = BCDF$. Thus, column E would be found from $E = ABC$, and column F would be $F = BCD$. That is, columns $ABCE$ and $BCDF$ are equal to the identity column. However, we know that the product of any two columns in the table of plus and minus signs for a 2^k design is just another column in the table; therefore, the product of $ABCE$ and $BCDF$ or $ABCE(BCDF) = AB^2C^2DEF = ADEF$ is also an identity column. Consequently, the **complete defining relation** for the 2^{6-2} design is

$$I = ABCE = BCDF = ADEF$$

We refer to each term in a defining relation (such as $ABCE$ above) as a **word.** To find the alias of any effect, simply multiply the effect by each word in the foregoing defining relation. For example, the alias of A is

$$A = BCE = ABCDF = DEF$$

The complete alias relationships for this design are shown in Table 14-27. In general, the resolution of a 2^{k-p} design is equal to the number of letters in the shortest word in the complete defining relation. Therefore, this is a resolution IV design; main effects are aliased with three-factor and higher interactions, and two-factor interactions are aliased with each other. This design would provide good information on the main effects and would give some idea about the strength of the two-factor interactions. The construction and analysis of the design are illustrated in Example 14-8.

EXAMPLE 14-8

Parts manufactured in an injection-molding process are showing excessive shrinkage, which is causing problems in assembly operations upstream from the injection-molding area. In an effort to reduce the shrinkage, a quality-improvement team has decided to use a designed experiment to study the injection-molding process. The team investigates six factors—mold temperature (A), screw speed (B), holding time (C), cycle time (D), gate size (E), and holding

Table 14-27 Alias Structure for the 2^{6-2}_{IV} Design with $I = ABCE = BCDF = ADEF$

$A = BCE = DEF = ABCDF$	$AB = CE = ACDF = BDEF$
$B = ACE = CDF = ABDEF$	$AC = BE = ABDF = CDEF$
$C = ABE = BDF = ACDEF$	$AD = EF = BCDE = ABCF$
$D = BCF = AEF = ABCDE$	$AE = BC = DF = ABCDEF$
$E = ABC = ADF = BCDEF$	$AF = DE = BCEF = ABCD$
$F = BCD = ADE = ABCEF$	$BD = CF = ACDE = ABEF$
$ABD = CDE = ACF = BEF$	$BF = CD = ACEF = ABDE$
$ACD = BDE = ABF = CEF$	

Table 14-28 A 2_{IV}^{6-2} Design for the Injection-Molding Experiment in Example 14-8

Run	A	B	C	D	$E = ABC$	$F = BCD$	Observed Shrinkage (×10)
1	−	−	−	−	−	−	6
2	+	−	−	−	+	−	10
3	−	+	−	−	+	+	32
4	+	+	−	−	−	+	60
5	−	−	+	−	+	+	4
6	+	−	+	−	−	+	15
7	−	+	+	−	−	−	26
8	+	+	+	−	+	−	60
9	−	−	−	+	−	+	8
10	+	−	−	+	+	+	12
11	−	+	−	+	+	−	34
12	+	+	−	+	−	−	60
13	−	−	+	+	+	−	16
14	+	−	+	+	−	−	5
15	−	+	+	+	−	+	37
16	+	+	+	+	+	+	52

pressure (F)—each at two levels, with the objective of learning how each factor affects shrinkage and obtaining preliminary information about how the factors interact.

The team decides to use a 16-run two-level fractional factorial design for these six factors. The design is constructed by writing down a 2^4 as the basic design in the factors A, B, C, and D and then setting $E = ABC$ and $F = BCD$ as discussed above. Table 14-28 shows the design, along with the observed shrinkage (×10) for the test part produced at each of the 16 runs in the design.

A normal probability plot of the effect estimates from this experiment is shown in Fig. 14-33. The only large effects are A (mold temperature), B (screw speed), and the AB interaction. In light of the alias relationship in Table 14-27, it seems reasonable to tentatively adopt these conclusions. The plot of the AB interaction in Fig. 14-34 shows that the process is

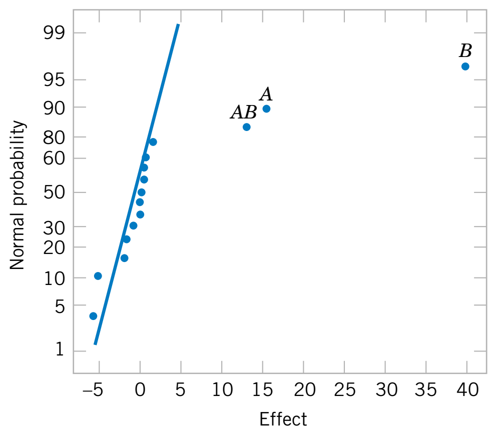


Figure 14-33 Normal probability plot of effects for Example 14-8.

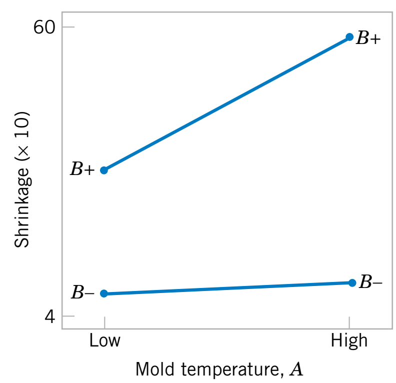


Figure 14-34 Plot of AB (mold temperature–screw speed) interaction for Example 14-8.

insensitive to temperature if the screw speed is at the low level but sensitive to temperature if the screw speed is at the high level. With the screw speed at a low level, the process should produce an average shrinkage of around 10% regardless of the temperature level chosen.

Based on this initial analysis, the team decides to set both the mold temperature and the screw speed at the low level. This set of conditions should reduce the mean shrinkage of parts to around 10%. However, the variability in shrinkage from part to part is still a potential problem. In effect, the mean shrinkage can be adequately reduced by the above modifications; however, the part-to-part variability in shrinkage over a production run could still cause problems in assembly. One way to address this issue is to see if any of the process factors affect the variability in parts shrinkage.

Figure 14-35 presents the normal probability plot of the residuals. This plot appears satisfactory. The plots of residuals versus each factor were then constructed. One of these plots, that for residuals versus factor C (holding time), is shown in Fig. 14-36. The plot reveals much less scatter in the residuals at the low holding time than at the high holding time. These residuals were obtained in the usual way from a model for predicted shrinkage

$$\begin{aligned}\hat{y} &= \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \hat{\beta}_{12} x_1 x_2 \\ &= 27.3125 + 6.9375 x_1 + 17.8125 x_2 + 5.9375 x_1 x_2\end{aligned}$$

where x_1, x_2, and x_1x_2 are coded variables that correspond to the factors A and B and the AB interaction. The residuals are then

$$e = y - \hat{y}$$

The regression model used to produce the residuals essentially removes the location effects of A, B, and AB from the data; the residuals therefore contain information about unexplained variability. Figure 14-36 indicates that there is a pattern in the variability and

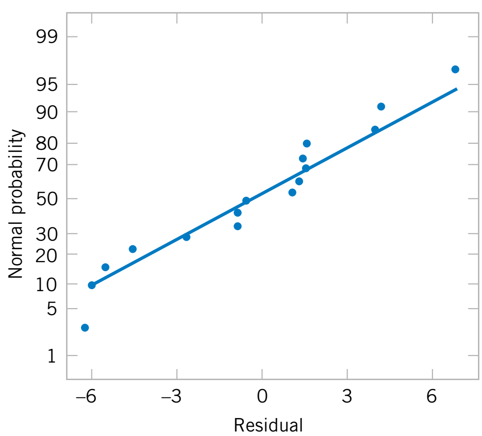


Figure 14-35 Normal probability plot of residuals for Example 14-8.

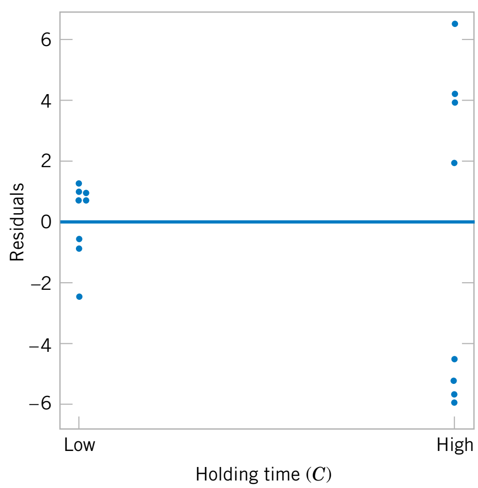


Figure 14-36 Residuals versus holding time (C) for Example 14-8.

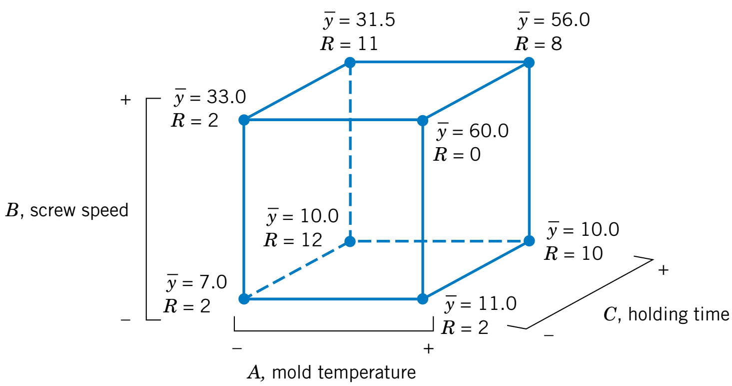


Figure 14-37 Average shrinkage and range of shrinkage in factors A, B, and C for Example 14-8.

that the variability in the shrinkage of parts may be smaller when the holding time is at the low level.

Figure 14-37 shows the data from this experiment projected onto a cube in the factors A, B, and C. The average observed shrinkage and the range of observed shrinkage are shown at each corner of the cube. From inspection of this figure, we see that running the process with the screw speed (B) at the low level is the key to reducing average parts shrinkage. If B is low, virtually any combination of temperature (A) and holding time (C) will result in low values of average parts shrinkage. However, from examining the ranges of the shrinkage values at each corner of the cube, it is immediately clear that setting the holding time (C) at the low level is the most appropriate choice if we wish to keep the part-to-part variability in shrinkage low during a production run.

The concepts used in constructing the 2^{6-2} fractional factorial design in Example 14-8 can be extended to the construction of any 2^{k-p} fractional factorial design. In general, a 2^k fractional factorial design containing 2^{k-p} runs is called a $1/2^p$ fraction of the 2^k design or, more simply, a 2^{k-p} fractional factorial design. These designs require the selection of p independent generators. The defining relation for the design consists of the p generators initially chosen and their $2^p - p - 1$ generalized interactions.

The alias structure may be found by multiplying each effect column by the defining relation. Care should be exercised in choosing the generators so that effects of potential interest are not aliased with each other. Each effect has $2^p - 1$ aliases. For moderately large values of k, we usually assume higher order interactions (say, third- or fourth-order or higher) to be negligible, and this greatly simplifies the alias structure.

It is important to select the p generators for the 2^{k-p} fractional factorial design in such a way that we obtain the best possible alias relationships. A reasonable criterion is to select the generators so that the resulting 2^{k-p} design has the highest possible design resolution. Montgomery (2001) presents a table of recommended generators for 2^{k-p} fractional factorial designs for $k \leq 15$ factors and up to as many as $n \leq 128$ runs. A portion of his table is reproduced here as Table 14-29. In this table, the generators are shown with either $+$ or $-$ choices; selection of all generators as $+$ will give a principal fraction, while if any generators are $-$ choices, the design will be one of the alternate fractions for the same family. The suggested generators in this table will result in a design of the highest possible resolution. Montgomery (2001) also gives a table of alias relationships for these designs.

Table 14-29 Selected 2^{k-p} Fractional Factorial Designs

Number of Factors k	Fraction	Number of Runs	Design Generators	Number of Factors k	Fraction	Number of Runs	Design Generators
3	2_{III}^{3-1}	4	$C = \pm AB$	10			$H = \pm ABCG$
4	2_{IV}^{4-1}	8	$D = \pm ABC$				$J = \pm ACDE$
5	2_{V}^{5-1}	16	$E = \pm ABCD$		2_{V}^{10-3}	128	$K = \pm ACDF$
	2_{III}^{5-2}	8	$D = \pm AB$				$G = \pm BCDF$
			$E = \pm AC$				$H = \pm ACDF$
6	2_{VI}^{6-1}	32	$F = \pm ABCDE$				$J = \pm ABDE$
	2_{IV}^{6-2}	16	$E = \pm ABC$		2_{IV}^{10-4}	64	$K = \pm ABCE$
			$F = \pm BCD$				$F = \pm ABCD$
	2_{III}^{6-3}	8	$D = \pm AB$				$G = \pm ABCE$
			$E = \pm AC$				$H = \pm ABDE$
			$F = \pm BC$				$J = \pm ACDE$
7	2_{VII}^{7-1}	64	$G = \pm ABCDEF$		2_{IV}^{10-5}	32	$K = \pm BCDE$
	2_{IV}^{7-2}	32	$E = \pm ABC$				$E = \pm ABC$
			$G = \pm ABDE$				$F = \pm BCD$
	2_{IV}^{7-3}	16	$E = \pm ABC$				$G = \pm ACD$
			$F = \pm BCD$				$H = \pm ABD$
			$G = \pm ACD$				$J = \pm ABCD$
	2_{III}^{7-4}	8	$D = \pm AB$		2_{III}^{10-6}	16	$K = \pm AB$
			$E = \pm AC$	11			$G = \pm CDE$
			$F = \pm BC$				$H = \pm ABCD$
			$G = \pm ABC$				$J = \pm ABF$
8	2_{V}^{8-2}	64	$G = \pm ABCD$				$K = \pm BDEF$
			$H = \pm ABEF$		2_{IV}^{11-5}	64	$L = \pm ADEF$
	2_{IV}^{8-3}	32	$F = \pm ABC$				$F = \pm ABC$
			$G = \pm ABD$				$G = \pm BCD$
			$H = \pm BCDE$				$H = \pm CDE$
	2_{IV}^{8-4}	16	$E = \pm BCD$				$J = \pm ACD$
			$F = \pm ACD$				$K = \pm ADE$
			$G = \pm ABC$		2_{IV}^{11-6}	32	$L = \pm BDE$
			$H = \pm ABD$				$E = \pm ABC$
9	2_{VI}^{9-2}	128	$H = \pm ACDFG$				$F = \pm BCD$
			$J = \pm BCEFG$				$G = \pm ACD$
	2_{IV}^{9-3}	64	$G = \pm ABCD$				$H = \pm ABD$
			$H = \pm ACEF$				$J = \pm ABCD$
			$J = \pm CDEF$				$K = \pm AB$
	2_{IV}^{9-4}	32	$F = \pm BCDE$		2_{III}^{11-7}	16	$L = \pm AC$
			$G = \pm ACDE$				
			$H = \pm ABDE$				
			$J = \pm ABCE$				
	2_{III}^{9-5}	16	$E = \pm ABC$				
			$F = \pm BCD$				
			$G = \pm ACD$				
			$H = \pm ABD$				
			$J = \pm ABCD$				

Source: Montgomery (2001)

EXAMPLE 14-9 To illustrate the use of Table 14-29, suppose that we have seven factors and that we are interested in estimating the seven main effects and obtaining some insight regarding the two-factor interactions. We are willing to assume that three-factor and higher interactions are negligible. This information suggests that a resolution IV design would be appropriate.

Table 14-29 shows that two resolution IV fractions are available: the 2_{IV}^{7-2} with 32 runs and the 2_{IV}^{7-3} with 16 runs. The aliases involving main effects and two- and three-factor interactions for the 16-run design are presented in Table 14-30. Notice that all seven main effects are aliased with three-factor interactions. All the two-factor interactions are aliased in groups of three. Therefore, this design will satisfy our objectives; that is, it will allow the estimation of the main effects, and it will give some insight regarding two-factor interactions. It is not necessary to run the 2_{IV}^{7-2} design, which would require 32 runs. The construction of the 2_{IV}^{7-3} design is shown in Table 14-31. Notice that it was constructed by starting with the 16-run 2^4 design in A, B, C, and D as the basic design and then adding the three columns $E = ABC$, $F = BCD$, and $G = ACD$ as suggested in Table 14-29. Thus, the generators for this design are $I = ABCE$, $I = BCDF$, and $I = ACDG$. The complete defining relation is $I = ABCE = BCDF = ADEF = ACDG = BDEG = CEFG = ABFG$. This defining relation was used to produce the aliases in Table 14-30. For example, the alias relationship of A is

$$A = BCE = ABCDF = DEF = CDG = ABDEG = ACEFG = BFG$$

which, if we ignore interactions higher than three factors, agrees with Table 14-30.

For seven factors, we can reduce the number of runs even further. The 2^{7-4} design is an eight-run experiment accommodating seven variables. This is a 1/16th fraction and is obtained by first writing down a 2^3 design as the basic design in the factors A, B, and C, and then forming the four new columns from $I = ABD$, $I = ACE$, $I = BCF$, and $I = ABCG$, as suggested in Table 14-29. The design is shown in Table 14-32.

The complete defining relation is found by multiplying the generators together two, three, and finally four at a time, producing

$$\begin{aligned} I &= ABD = ACE = BCF = ABCG = BCDE = ACDF = CDG = ABEF \\ &= BEG = AFG = DEF = ADEG = CEFG = BDFG = ABCDEFG \end{aligned}$$

The alias of any main effect is found by multiplying that effect through each term in the

Table 14-30 Generators, Defining Relation, and Aliases for the 2_{IV}^{7-3} Fractional Factorial Design

Generators and Defining Relation	
$E = ABC$, $F = BCD$, $G = ACD$	
$I = ABCE = BCDF = ADEF = ACDG = BDEG = ABFG = CEFG$	
Aliases	
$A = BCE = DEF = CDG = BFG$	$AB = CE = FG$
$B = ACE = CDF = DEG = AFG$	$AC = BE = DG$
$C = ABE = BDF = ADG = EFG$	$AD = EF = CG$
$D = BCF = AEF = ACG = BEG$	$AE = BC = DF$
$E = ABC = ADF = BDG = CFG$	$AF = DE = BG$
$F = BCD = ADE = ABG = CEG$	$AG = CD = BF$
$G = ACD = BDE = ABF = CEF$	$BD = CF = EG$
$ABD = CDE = ACF = BEF = BCG = AEG = DFG$	

Table 14-31 A 2^{7-3}_{IV} Fractional Factorial Design

	Basic Design						
Run	A	B	C	D	$E = ABC$	$F = BCD$	$G = ACD$
1	−	−	−	−	−	−	−
2	+	−	−	−	+	−	+
3	−	+	−	−	+	+	−
4	+	+	−	−	−	+	+
5	−	−	+	−	+	+	+
6	+	−	+	−	−	+	−
7	−	+	+	−	−	−	+
8	+	+	+	−	+	−	−
9	−	−	−	+	−	+	−
10	+	−	−	+	+	+	−
11	−	+	−	+	+	−	+
12	+	+	−	+	−	−	−
13	−	−	+	+	+	−	−
14	+	−	+	+	−	−	+
15	−	+	+	+	−	+	−
16	+	+	+	+	+	+	+

defining relation. For example, the alias of A is

$$\begin{aligned} A &= BD = CE = ABCF = BCG = ABCDE = CDF = ACDG \\ &= BEF = ABEG = FG = ADEF = DEG = ACEFG = ABDFG = BCDEFG \end{aligned}$$

This design is of resolution III, since the main effect is aliased with two-factor interactions. If we assume that all three-factor and higher interactions are negligible, the aliases of the seven main effects are

$$\ell_A = A + BD + CE + FG$$

$$\ell_B = B + AD + CF + EG$$

$$\ell_C = C + AE + BF + DG$$

$$\ell_D = D + AB + CG + EF$$

$$\ell_E = E + AC + BG + DF$$

$$\ell_F = F + BC + AG + DE$$

$$\ell_G = G + CD + BE + AF$$

Table 14-32 A 2^{7-4}_{III} Fractional Factorial Design

A	B	C	$D = AB$	$E = AC$	$F = BC$	$G = ABC$
−	−	−	+	+	+	−
+	−	−	−	−	+	+
−	+	−	−	+	−	+
+	+	−	+	−	−	−
−	−	+	+	−	−	+
+	−	+	−	+	−	−
−	+	+	−	−	+	−
+	+	+	+	+	+	+

This 2^{7-4}_{III} design is called a **saturated fractional factorial,** because all the available degrees of freedom are used to estimate main effects. It is possible to combine sequences of these resolution III fractional factorials to separate the main effects from the two-factor interactions. The procedure is illustrated in Montgomery (2001) and in Box, Hunter, and Hunter (1978).

EXERCISES FOR SECTION 14-9

14-31. R. D. Snee ("Experimenting with a Large Number of Variables," in *Experiments in Industry: Design, Analysis and Interpretation of Results,* by R. D. Snee, L. D. Hare, and J. B. Trout, eds., ASQC, 1985) describes an experiment in which a 2^{5-1} design with $I = ABCDE$ was used to investigate the effects of five factors on the color of a chemical product. The factors are A = solvent/reactant, B = catalyst/reactant, C = temperature, D = reactant purity, and E = reactant pH. The results obtained are as follows:

$$\begin{array}{rlrl}
e = & -0.63 & d = & 6.79 \\
a = & 2.51 & ade = & 6.47 \\
b = & -2.68 & bde = & 3.45 \\
abe = & 1.66 & abd = & 5.68 \\
c = & 2.06 & cde = & 5.22 \\
ace = & 1.22 & acd = & 4.38 \\
bce = & -2.09 & bcd = & 4.30 \\
abc = & 1.93 & abcde = & 4.05
\end{array}$$

(a) Prepare a normal probability plot of the effects. Which factors are active?
(b) Calculate the residuals. Construct a normal probability plot of the residuals and plot the residuals versus the fitted values. Comment on the plots.
(c) If any factors are negligible, collapse the 2^{5-1} design into a full factorial in the active factors. Comment on the resulting design, and interpret the results.

14-32. Montgomery (2001) describes a 2^{4-1} fractional factorial design used to study four factors in a chemical process. The factors are A = temperature, B = pressure, C = concentration, and D = stirring rate, and the response is filtration rate. The design and the data are as follows:

Run	A	B	C	$D = ABC$	Treatment Combination	Filtration Rate
1	−	−	−	−	(1)	45
2	+	−	−	+	*ad*	100
3	−	+	−	+	*bd*	45
4	+	+	−	−	*ab*	65
5	−	−	+	+	*cd*	75
6	+	−	+	−	*ac*	60
7	−	+	+	−	*bc*	80
8	+	+	+	+	*abcd*	96

(a) Write down the alias relationships.
(b) Estimate the factor effects. Which factor effects appear large?
(c) Project this design into a full factorial in the three apparently important factors and provide a practical interpretation of the results.

14-33. An article in *Industrial and Engineering Chemistry* ("More on Planning Experiments to Increase Research Efficiency," 1970, pp. 60–65) uses a 2^{5-2} design to investigate the effect on process yield of A = condensation temperature, B = amount of material 1, C = solvent volume, D = condensation time, and E = amount of material 2. The results obtained are as follows:

$$\begin{array}{rlrl}
e = & 23.2 & cd = & 23.8 \\
ab = & 15.5 & ace = & 23.4 \\
ad = & 16.9 & bde = & 16.8 \\
bc = & 16.2 & abcde = & 18.1
\end{array}$$

(a) Verify that the design generators used were $I = ACE$ and $I = BDE$.
(b) Write down the complete defining relation and the aliases from the design.
(c) Estimate the main effects.
(d) Prepare an analysis of variance table. Verify that the AB and AD interactions are available to use as error.
(e) Plot the residuals versus the fitted values. Also construct a normal probability plot of the residuals. Comment on the results.

14-34. Consider the 2^{6-2} design in Table 14-28. Suppose that after analyzing the original data, we find that factors C and E can be dropped. What type of 2^k design is left in the remaining variables?

14-35. Consider the 2^{6-2} design in Table 14-28. Suppose that after the original data analysis, we find that factors D and F can be dropped. What type of 2^k design is left in the remaining variables? Compare the results with Exercise 14-34. Can you explain why the answers are different?

14-36. Suppose that in Exercise 14-22 it was possible to run only a $\frac{1}{2}$ fraction of the 2^4 design. Construct the design and use only the data from the eight runs you have generated to perform the analysis.

14-37. Suppose that in Exercise 14-16 only a ¼ fraction of the 2^5 design could be run. Construct the design and analyze the data that are obtained by selecting only the response for the eight runs in your design.

14-38. Construct the 2_{IV}^{8-4} design recommended in Table 13-29. What are the aliases of the main effects and two-factor interactions?

14-39. Construct a 2_{III}^{6-3} fractional factorial design. Write down the aliases, assuming that only main effects and two-factor interactions are of interest.

14-40. Consider the problem in Exercise 14-19. Suppose that only half of the 32 runs could be made.
(a) Choose the half that you think should be run.
(b) Write out the alias relationships for your design.
(c) Estimate the factor effects.
(d) Plot the effect estimates on normal probability paper and interpret the results.
(e) Set up an analysis of variance for the factors identified as potentially interesting from the normal probability plot in part (d).
(f) Analyze the residuals from the model.
(g) Provide a practical interpretation of the results.

14-10 RESPONSE SURFACE METHODS AND DESIGNS (CD ONLY)

Supplemental Exercises

14-41. An article in *Process Engineering* (No. 71, 1992, pp. 46–47) presents a two-factor factorial experiment used to investigate the effect of pH and catalyst concentration on product viscosity (cSt). The data are as follows:

		Catalyst Concentration	
		2.5	2.7
pH	5.6	192, 199, 189, 198	178, 186, 179, 188
	5.9	185, 193, 185, 192	197, 196, 204, 204

(a) Test for main effects and interactions using $\alpha = 0.05$. What are your conclusions?
(b) Graph the interaction and discuss the information provided by this plot.
(c) Analyze the residuals from this experiment.

14-42. Heat treating of metal parts is a widely used manufacturing process. An article in the *Journal of Metals* (Vol. 41, 1989) describes an experiment to investigate flatness distortion from heat treating for three types of gears and two heat-treating times. The data are as follows:

	Time (minutes)	
Gear Type	90	120
20-tooth	0.0265	0.0560
	0.0340	0.0650
24-tooth	0.0430	0.0720
	0.0510	0.0880
28-tooth	0.0405	0.0620
	0.0575	0.0825

(a) Is there any evidence that flatness distortion is different for the different gear types? Is there any indication that heat treating time affects the flatness distortion? Do these factors interact? Use $\alpha = 0.05$.
(b) Construct graphs of the factor effects that aid in drawing conclusions from this experiment.
(c) Analyze the residuals from this experiment. Comment on the validity of the underlying assumptions.

14-43. An article in the *Textile Research Institute Journal* (Vol. 54, 1984, pp. 171–179) reported the results of an experiment that studied the effects of treating fabric with selected inorganic salts on the flammability of the material. Two application levels of each salt were used, and a vertical burn test was used on each sample. (This finds the temperature at which each sample ignites.) The burn test data follow.

	Salt					
Level	Untreated	$MgCl_2$	NaCl	$CaCO_3$	$CaCl_2$	Na_2CO_3
1	812	752	739	733	725	751
	827	728	731	728	727	761
	876	764	726	720	719	755
2	945	794	741	786	756	910
	881	760	744	771	781	854
	919	757	727	779	814	848

(a) Test for differences between salts, application levels, and interactions. Use $\alpha = 0.01$.
(b) Draw a graph of the interaction between salt and application level. What conclusions can you draw from this graph?
(c) Analyze the residuals from this experiment.

14-44. An article in the *IEEE Transactions on Components, Hybrids, and Manufacturing Technology* (Vol. 15, 1992) describes an experiment for investigating a method for aligning optical chips onto circuit boards. The method involves placing solder bumps onto the bottom of the chip. The experiment used three solder bump sizes and three alignment methods. The response variable is alignment accuracy (in micrometers). The data are as follows:

Solder Bump Size (diameter in μm)	**Alignment Method** 1	2	3
75	4.60	1.55	1.05
	4.53	1.45	1.00
130	2.33	1.72	0.82
	2.44	1.76	0.95
260	4.95	2.73	2.36
	4.55	2.60	2.46

(a) Is there any indication that either solder bump size or alignment method affects the alignment accuracy? Is there any evidence of interaction between these factors? Use $\alpha = 0.05$.
(b) What recommendations would you make about this process?
(c) Analyze the residuals from this experiment. Comment on model adequacy.

14-45. An article in *Solid State Technology* (Vol. 29, 1984, pp. 281–284) describes the use of factorial experiments in photolithography, an important step in the process of manufacturing integrated circuits. The variables in this experiment (all at two levels) are prebake temperature (A), prebake time (B), and exposure energy (C), and the response variable is delta line width, the difference between the line on the mask and the printed line on the device. The data are as follows: $(1) = -2.30$, $a = -9.87$, $b = -18.20$, $ab = -30.20$, $c = -23.80$, $ac = -4.30$, $bc = -3.80$, and $abc = -14.70$.
(a) Estimate the factor effects.
(b) Use a normal probability plot of the effect estimates to identity factors that may be important.
(c) What model would you recommend for predicting the delta line width response, based on the results of this experiment?
(d) Analyze the residuals from this experiment, and comment on model adequacy.

14-46. An article in the *Journal of Coatings Technology* (Vol. 60, 1988, pp. 27–32) describes a 2^4 factorial design used for studying a silver automobile basecoat. The response variable is distinctness of image (DOI). The variables used in the experiment are

A = Percentage of polyester by weight of polyester/melamine (low value = 50%, high value = 70%)
B = Percentage of cellulose acetate butyrate carboxylate (low value = 15%, high value = 30%)
C = Percentage of aluminum stearate (low value = 1%, high value = 3%)
D = Percentage of acid catalyst (low value = 0.25%, high value = 0.50%)

The responses are $(1) = 63.8$, $a = 77.6$, $b = 68.8$, $ab = 76.5$, $c = 72.5$, $ac = 77.2$, $bc = 77.7$, $abc = 84.5$, $d = 60.6$, $ad = 64.9$, $bd = 72.7$, $abd = 73.3$, $cd = 68.0$, $acd = 76.3$, $bcd = 76.0$, and $abcd = 75.9$.
(a) Estimate the factor effects.
(b) From a normal probability plot of the effects, identify a tentative model for the data from this experiment.
(c) Using the apparently negligible factors as an estimate of error, test for significance of the factors identified in part (b). Use $\alpha = 0.05$.
(d) What model would you use to describe the process, based on this experiment? Interpret the model.
(e) Analyze the residuals from the model in part (d) and comment on your findings.

14-47. An article in the *Journal of Manufacturing Systems* (Vol. 10, 1991, pp. 32–40) describes an experiment to investigate the effect of four factors P = waterjet pressure, F = abrasive flow rate, G = abrasive grain size, and V = jet traverse speed on the surface roughness of a waterjet cutter. A 2^4 design follows.

Run	**Factors** V (in/min)	F (lb/min)	P (kpsi)	G (Mesh No.)	Surface Roughness (μm)
1	6	2.0	38	80	104
2	2	2.0	38	80	98
3	6	2.0	30	80	103
4	2	2.0	30	80	96
5	6	1.0	38	80	137
6	2	1.0	38	80	112
7	6	1.0	30	80	143
8	2	1.0	30	80	129
9	6	2.0	38	170	88
10	2	2.0	38	170	70
11	6	2.0	30	170	110
12	2	2.0	30	170	110
13	6	1.0	38	170	102
14	2	1.0	38	170	76
15	6	1.0	30	170	98
16	2	1.0	30	170	68

(a) Estimate the factor effects.
(b) Form a tentative model by examining a normal probability plot of the effects.
(c) Is the model in part (b) a reasonable description of the process? Is lack of fit significant? Use $\alpha = 0.05$.
(d) Interpret the results of this experiment.
(e) Analyze the residuals from this experiment.

14-48. Construct a 2^{4-1}_{IV} design for the problem in Exercise 14-46. Select the data for the eight runs that would have been required for this design. Analyze these runs and compare your conclusions to those obtained in Exercise 14-46 for the full factorial.

14-49. Construct a 2^{4-1}_{IV} design for the problem in Exercise 14-47. Select the data for the eight runs that would have been required for this design, plus the center points. Analyze these data and compare your conclusions to those obtained in Exercise 14-47 for the full factorial.

14-50. Construct a 2^{8-4}_{IV} design in 16 runs. What are the alias relationships in this design?

14-51. Construct a 2^{5-2}_{III} design in eight runs. What are the alias relationships in this design?

14-52. In a process development study on yield, four factors were studied, each at two levels: time (A), concentration (B), pressure (C), and temperature (D). A single replicate of a 2^4 design was run, and the data are shown in the table below.

(a) Plot the effect estimates on a normal probability scale. Which factors appear to have large effects?
(b) Conduct an analysis of variance using the normal probability plot in part (a) for guidance in forming an error term. What are your conclusions?
(c) Analyze the residuals from this experiment. Does your analysis indicate any potential problems?
(d) Can this design be collapsed into a 2^3 design with two replicates? If so, sketch the design with the average and range of yield shown at each point in the cube. Interpret the results.

14-53. An article in the *Journal of Quality Technology* (Vol. 17, 1985, pp. 198–206) describes the use of a replicated fractional factorial to investigate the effect of five factors on the free height of leaf springs used in an automotive application. The factors are A = furnace temperature, B = heating time, C = transfer time, D = hold down time, and E = quench oil temperature. The data are shown in the following table.
(a) What is the generator for this fraction? Write out the alias structure.
(b) Analyze the data. What factors influence mean free height?
(c) Calculate the range of free height for each run. Is there any indication that any of these factors affect variability in free height?
(d) Analyze the residuals from this experiment and comment on your findings.

Run Number	Actual Run Order	A	B	C	D	Yield (lbs)
1	5	−	−	−	−	12
2	9	+	−	−	−	18
3	8	−	+	−	−	13
4	13	+	+	−	−	16
5	3	−	−	+	−	17
6	7	+	−	+	−	15
7	14	−	+	+	−	20
8	1	+	+	+	−	15
9	6	−	−	−	+	10
10	11	+	−	−	+	25
11	2	−	+	−	+	13
12	15	+	+	−	+	24
13	4	−	−	+	+	19
14	16	+	−	+	+	21
15	10	−	+	+	+	17
16	12	+	+	+	+	23

Factor Levels	Low (−)	High (+)
A (h)	2.5	3
B (%)	14	18
C (psi)	60	80
D (°C)	225	250

A	*B*	*C*	*D*	*E*	Free Height		
−	−	−	−	−	7.78	7.78	7.81
+	−	−	+	−	8.15	8.18	7.88
−	+	−	+	−	7.50	7.56	7.50
+	+	−	−	−	7.59	7.56	7.75
−	−	+	+	−	7.54	8.00	7.88
+	−	+	−	−	7.69	8.09	8.06
−	+	+	−	−	7.56	7.52	7.44
+	+	+	+	−	7.56	7.81	7.69
−	−	−	−	+	7.50	7.56	7.50
+	−	−	+	+	7.88	7.88	7.44
−	+	−	+	+	7.50	7.56	7.50
+	+	−	−	+	7.63	7.75	7.56
−	−	+	+	+	7.32	7.44	7.44
+	−	+	−	+	7.56	7.69	7.62
−	+	+	−	+	7.18	7.18	7.25
+	+	+	+	+	7.81	7.50	7.59

14-54. An article in *Rubber Chemistry and Technology* (Vol. 47, 1974, pp. 825–836) describes an experiment that studies the Mooney viscosity of rubber to several variables, including silica filler (parts per hundred) and oil filler (parts per hundred). Data typical of that reported in this experiment follow, where

$$x_1 = \frac{\text{silica} - 60}{15}, \qquad x_2 = \frac{\text{oil} - 21}{15}$$

Coded levels		
x_1	x_2	y
−1	−1	13.71
1	−1	14.15
−1	1	12.87
1	1	13.53
−1	−1	13.90
1	−1	14.88
−1	1	12.25
−1	1	13.35

(a) What type of experimental design has been used?
(b) Analyze the data and draw appropriate conclusions.

MIND-EXPANDING EXERCISES

14-55. Consider an unreplicated 2^k factorial, and suppose that one of the treatment combinations is missing. One logical approach to this problem is to estimate the missing value with a number that makes the highest order interaction estimate zero. Apply this technique to the data in Example 14-5, assuming that *ab* is missing. Compare the results of the analysis of these data with the results in Example 14-5.

14-56. What blocking scheme would you recommend if it were necessary to run a 2^4 design in four blocks of four runs each?

14-57. Consider a 2^2 design in two blocks with AB confounded with blocks. Prove algebraically that $SS_{AB} = SS_{\text{Blocks}}$.

14-58. Consider a 2^3 design. Suppose that the largest number of runs that can be made in one block is four, but we can afford to perform a total of 32 observations.

(a) Suggest a blocking scheme that will provide some information on all interactions.
(b) Show an outline (source of variability, degrees of freedom only) for the analysis of variance for this design.

14-59. Construct a 2^{5-1} design. Suppose that it is necessary to run this design in two blocks of eight runs each. Show how this can be done by confounding a two-factor interaction (and its aliased three-factor interaction) with blocks.

14-60. Construct a 2_{IV}^{7-2} design. Show how this design may be confounded in four blocks of eight runs each. Are any two-factor interactions confounded with blocks?

14-61. Construct a 2_{IV}^{7-3} design. Show how this design can be confounded in two blocks of eight runs each without losing information on any of the two-factor interactions.

14-62. Set up a 2_{III}^{7-4} design using $D = AB$, $E = AC$, $F = BC$, and $G = ABC$ as the design generators. Ignore all interaction above the two factors.

(a) Verify that each main effect is aliased with three two-factor interactions.
(b) Suppose that a second 2_{III}^{7-4} design with generators $D = -AB$, $E = -AC$, $F = -BC$, and $G = ABC$ is run. What are the aliases of the main effects in this design?
(c) What factors may be estimated if the two sets of factor effect estimates above are combined?

To work Exercises 14-63 through 14-67 you will need to read Section 14.6 on the CD.

14-63. Consider the experiment described in Example 14-4. Suppose that both factors were random.

(a) Analyze the data and draw appropriate conclusions.
(b) Estimate the variance components.

14-64. For the breaking strength data in Table S14-1, suppose that the operators were chosen at random, but machines were a fixed factor. Does this influence the analysis or your conclusions?

14-65. A company employs two time-study engineers. Their supervisor wishes to determine whether the standards set by them are influenced by an interaction between engineers and operators. She selects three operators at random and conducts an experiment in which the engineers set standard times for the same job. She obtains the data shown here:

	Operator		
Engineer	1	2	3
1	2.59	2.38	2.40
	2.78	2.49	2.72
2	2.15	2.85	2.66
	2.86	2.72	2.87

(a) State the appropriate hypotheses.
(b) Use the analysis of variance to test these hypotheses with $\alpha = 0.05$.
(c) Graphically analyze the residuals from this experiment.
(d) Estimate the appropriate variance components.

14-66. Consider the experiment on baked anode density described in Exercise 14-4. Suppose that positions on the furnace were chosen at random and temperature is a fixed factor.

(a) State the appropriate hypotheses.
(b) Use the analysis of variance to test these hypotheses with $\alpha = 0.05$.
(c) Estimate the variance components.

14-67. Consider the experiment described in Exercise 14-63. How does the analysis (and conclusions) change if both factors are random? Use $\alpha = 0.05$.

To work Exercises 14-68 and 14-69 you will need to read Section 14-7.4 on the CD.

MIND-EXPANDING EXERCISES

14-68. Consider the experiment in Exercise 14-19. Suppose that a center point had been run (replicated five times) and the responses were 45, 40, 41, 47, and 43.

(a) Estimate the experimental error using the center points. Compare this to the estimate obtained originally in Exercise 14-19 by pooling apparently nonsignificant effects.

(b) Test for look of fit, using $\alpha = 0.05$.

14-69. Consider the data from Exercise 14-13, replicate 1 only. Suppose that a center point with four replicates is added to these eight factorial runs and the responses are 425, 400, 437, and 418.

(a) Estimate the facter effects.

(b) Test for lack of fit using $\alpha = 0.05$

(c) Test for main effects and interactions using $\alpha = 0.05$.

(d) Analyze residuals and draw conclusions.

To work problem 14-70 through 14-74 you will need to read Section 14-10 on the CD.

14-70. An article in *Rubber Age* (Vol. 89, 1961, pp. 453–458) describes an experiment on the manufacture of a product in which two factors were varied. The factors are reaction time (hr) and temperature (°C). These factors are coded as $x_1 = (\text{time} - 12)/8$ and $x_2 = (\text{temperature} - 250)/30$. The following data were observed where y is the yield (in percent):

Run Number	x_1	x_2	y
1	−1	0	83.8
2	1	0	81.7
3	0	0	82.4
4	0	0	82.9
5	0	−1	84.7
6	0	1	75.9
7	0	0	81.2
8	−1.414	−0.414	81.3
9	−1.414	1.414	83.1
10	1.414	−1.414	85.3
11	1.414	1.414	72.7
12	0	0	82.0

(a) Plot the points at which the experimental runs were made.

(b) Fit a second-order model to the data. Is the second-order model adequate?

(c) Plot the yield response surface. What recommendations would you make about the operating conditions for this process?

14-71. Consider the first-order model

$$\hat{y} = 50 + 1.5x_1 - 0.8x_2$$

where $-1 \leq x_i \leq 1$. Find the direction of steepest ascent.

14-72. A manufacturer of cutting tools has developed two empirical equations for tool life (y_1) and tool cost (y_2). Both models are functions of tool hardness (x_1) and manufacturing time (x_2). The equations are

$$\hat{y}_1 = 10 + 5x_1 + 2x_2$$

$$\hat{y}_2 = 23 + 3x_1 + 4x_2$$

and both equations are valid over the range $-1.5 \leq x_i \leq 1.5$. Suppose that tool life must exceed 12 hours and cost must be below \$27.50.

(a) Is there a feasible set of operating conditions?

(b) Where would you run this process?

14-73. An article in *Tappi* (Vol. 43, 1960, pp. 38–44) describes an experiment that investigated the ash value of paper pulp (a measure of inorganic impurities). Two variables, temperature T in degrees Celsius and time t in hours, were studied, and some of the results are shown in the following table. The coded predictor variables shown are

$$x_1 = \frac{(T - 775)}{115}, \qquad x_2 = \frac{(t - 3)}{1.5}$$

and the response y is (dry ash value in %) $\times 10^3$.

x_1	x_2	y	x_1	x_2	y
−1	−1	211	0	−1.5	168
1	−1	92	0	1.5	179
−1	1	216	0	0	122
1	1	99	0	0	175
−1.5	0	222	0	0	157
1.5	0	48	0	0	146

(a) What type of design has been used in this study? Is the design rotatable?

MIND-EXPANDING EXERCISES

(b) Fit a quadratic model to the data. Is this model satisfactory?

(c) If it is important to minimize the ash value, where would you run the process?

14-74. In their book *Empirical Model Building and Response Surfaces* (John Wiley, 1987), G. E. P. Box and N. R. Draper describe an experiment with three factors. The data shown in the following table are a variation of the original experiment on page 247 of their book. Suppose that these data were collected in a semiconductor manufacturing process.

(a) The response y_1 is the average of three readings on resistivity for a single wafer. Fit a quadratic model to this response.

(b) The response y_2 is the standard deviation of the three resistivity measurements. Fit a linear model to this response.

(c) Where would you recommend that we set x_1, x_2, and x_3 if the objective is to hold mean resistivity at 500 and minimize the standard deviation?

x_1	x_2	x_3	y_1	y_2	x_1	x_2	x_3	y_1	y_2
−1	−1	−1	24.00	12.49	1	0	0	501.67	92.50
0	−1	−1	120.33	8.39	−1	1	0	264.00	63.50
1	−1	−1	213.67	42.83	0	1	0	427.00	88.61
−1	0	−1	86.00	3.46	1	1	0	730.67	21.08
0	0	−1	136.63	80.41	−1	−1	1	220.67	133.82
1	0	−1	340.67	16.17	0	−1	1	239.67	23.46
−1	1	−1	112.33	27.57	1	−1	1	422.00	18.52
0	1	−1	256.33	4.62	−1	0	1	199.00	29.44
1	1	−1	271.67	23.63	0	0	1	485.33	44.67
−1	−1	0	81.00	0.00	1	0	1	673.67	158.21
0	−1	0	101.67	17.67	−1	1	1	176.67	55.51
1	−1	0	357.00	32.91	0	1	1	501.00	138.94
−1	0	0	171.33	15.01	1	1	1	1010.00	142.45
0	0	0	372.00	0.00					

IMPORTANT TERMS AND CONCEPTS

In the E-book, click on any term or concept below to go to that subject.

Analysis of variance (ANOVA)
Blocking and nuisance factors
Confounding
Factorial Experiment
Fractional factorial design
Interaction
Main effect
Normal probability plot of factor effects
Orthogonal design
Regression model
Residual analysis
Two-level factorial design

CD MATERIAL

Center points in a factorial
Central composite design
Mixed model
Random model

Response surface
Steepest ascent
Variance components

14-2 SOME APPLICATIONS OF EXPERIMENTAL DESIGN TECHNIQUES (CD ONLY)

Experimental design is an extremely important tool for engineers and scientists who are interested in improving the performance of a manufacturing process. It also has extensive application in the development of new processes and in new product design. We now give some examples.

A Process Characterization Experiment

A team of development engineers is working on a new process for soldering electronic components to printed circuit boards. Specifically, the team is working with a new type of flow solder machine that should reduce the number of defective solder joints. (A flow solder machine preheats printed circuit boards and then moves them into contact with a wave of liquid solder. This machine makes all the electrical and most of the mechanical connections of the components to the printed circuit board. Solder defects require touchup or rework, which adds cost and often damages the boards.) The process will have several (perhaps many) variables, and all of them are not equally important. The initial list of candidate variables to be included in the experiment is constructed by combining the knowledge and information about the process from all team members. In this example, the engineers conducted a **brainstorming** session and invited manufacturing personnel with experience using various types of flow soldering equipment to participate. The team determined that the flow solder machine has several variables that can be controlled. They are

1. Solder temperature
2. Preheat temperature
3. Conveyor speed
4. Flux type
5. Flux-specific gravity
6. Solder wave depth
7. Conveyor angle

In addition to these controllable factors, there are several other factors that cannot be easily controlled, once the machine enters routine manufacturing, including

1. Thickness of the printed circuit board
2. Types of components used on the board
3. Layout of the components on the board
4. Operator
5. Environmental factors
6. Production rate

Sometimes we call the uncontrollable factors *noise* factors. A schematic representation of the process is shown in Fig. S14-1.

In this situation the engineer is interested in **characterizing** the flow solder machine; that is, he is interested in determining which factors (both controllable and uncontrollable) affect the occurrence of defects on the printed circuit boards. To determine these factors, he can design an experiment that will enable him to estimate the magnitude and direction of the factor effects. Sometimes we call such an experiment a **screening experiment.** The information from this characterization study or screening experiment can help determine the critical process variables, as well as the direction of adjustment for these factors in order to reduce the

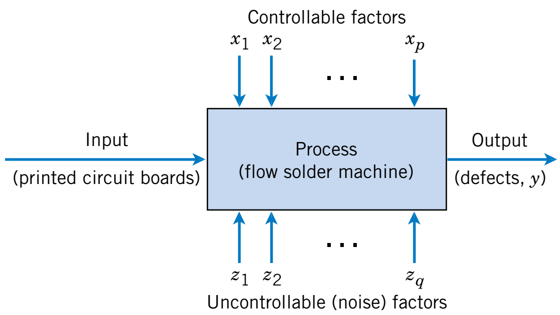


Figure S14-1 The flow solder experiment.

number of defects, and assist in determining which process variables should be carefully controlled during manufacturing to prevent high defect levels and erratic process performance.

An Optimization Experiment

In a characterization experiment, we are interested in determining which factors affect the response. A logical next step is to determine the region in the important factors that leads to an **optimum response.** For example, if the response is cost, we will look for a region of minimum cost.

As an illustration, suppose that the yield of a chemical process is influenced by the operating temperature and the reaction time. We are currently operating the process at 155°F and 1.7 hours of reaction time, and the current process yield is around 75%. Figure S14-2 shows a view of the time–temperature space from above. In this graph we have connected points of constant yield with lines. These lines are yield **contours,** and we have shown the contours at 60, 70, 80, 90, and 95% yield. To locate the optimum, we might begin with a factorial experiment such as we described below, with the two factors, time and temperature,

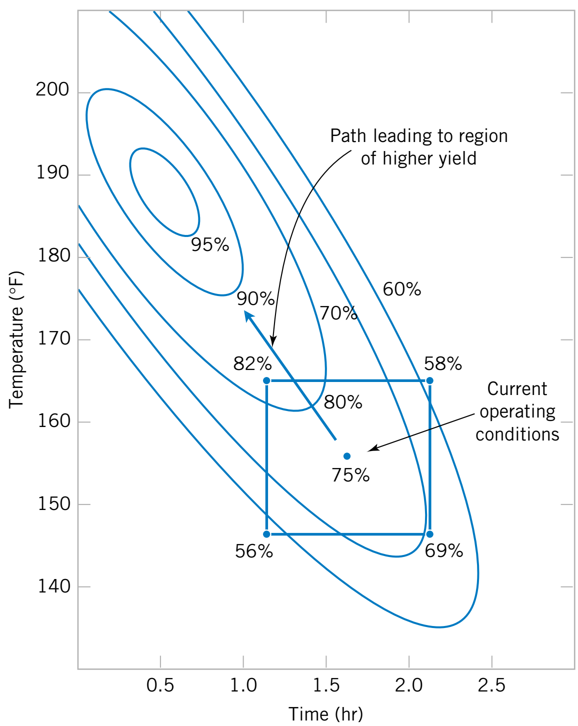


Figure S14-2 Contour plot of yield as a function of reaction time and reaction temperature, illustrating an optimization experiment.

run at two levels each at 10°F and 0.5 hours above and below the current operating conditions. This two-factor factorial design is shown in Fig. S14-2. The average responses observed at the four points in the experiment (145°F, 1.2 hours; 145°F, 2.2 hours; 165°F, 1.2 hours; and 165°F, 2.2 hours) indicate that we should move in the general direction of increased temperature and lower reaction time to increase yield. A few additional runs could be performed in this direction to locate the region of maximum yield.

A Product Design Example

We can also use experimental design in the development of new products. For example, suppose that a group of engineers are designing a door hinge for an automobile. The product characteristic is the check effort, or the holding ability, of the latch that prevents the door from swinging closed when the vehicle is parked on a hill. The check mechanism consists of a leaf spring and a roller. When the door is opened, the roller travels through an arc causing the leaf spring to be compressed. To close the door, the spring must be forced aside, and this creates the check effort. The engineering team thinks that check effort is a function of the following factors:

1. Roller travel distance
2. Spring height from pivot to base
3. Horizontal distance from pivot to spring
4. Free height of the reinforcement spring
5. Free height of the main spring

The engineers can build a prototype hinge mechanism in which all these factors can be varied over certain ranges. Once appropriate levels for these five factors have been identified, an experiment can be designed consisting of various combinations of the factor levels, and the prototype can be tested at these combinations. This will produce information concerning which factors are most influential on the latch check effort, and through analysis of this information, the latch design can be improved.

These examples illustrate only three applications of experimental design methods. In the engineering environment, experimental design applications are numerous. Some potential areas of use are

1. Process troubleshooting
2. Process development and optimization
3. Evaluation of material and alternatives
4. Reliability and life testing
5. Performance testing
6. Product design configuration
7. Component tolerance determination

Experimental design methods allow these problems to be solved efficiently during the early stages of the product cycle. This has the potential to dramatically lower overall product cost and reduce development lead time.

14-6 FACTORIAL EXPERIMENTS WITH RANDOM FACTORS (CD ONLY)

In this chapter, we focus primarily on the case where all the factors are fixed; that is, the experimenter specifically chose the levels, and the conclusions from the experiment are confined to those specific levels. Now we briefly consider the situation where one or more of the factors in a factorial experiment are random, using the two-factor factorial design as an illustration.

The Random-Effects Model

Suppose that we have two factors A and B arranged in a factorial experiment in which the levels of both factors are selected at random from larger populations of factor levels, and we wish to extend our conclusion to the entire population of factor levels. The observations are represented by the model

$$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases} \tag{S14-1}$$

where the parameters τ_i, β_j, $(\tau\beta)_{ij}$, and ϵ_{ijk} are normally and independently distributed random variables with means zero and variances σ_τ^2, σ_β^2, $\sigma_{\tau\beta}^2$, and σ^2, respectively. As a result of these assumptions, the variance of any observation Y_{ijk} is

$$V(Y_{ijk}) = \sigma_\tau^2 + \sigma_\beta^2 + \sigma_{\tau\beta}^2 + \sigma^2$$

and σ_τ^2, σ_β^2, $\sigma_{\tau\beta}^2$, and σ^2 are called **variance components.** The hypotheses that we are interested in testing are H_0: $\sigma_\tau^2 = 0$, H_0: $\sigma_\beta^2 = 0$, and H_0: $\sigma_{\tau\beta}^2 = 0$. Notice the similarity to the single-factor experiment random-effects model discussed in Chapter 13.

The basic analysis of variance remains unchanged; that is, SS_A, SS_B, SS_{AB}, SS_T, and SS_E are all calculated as in the fixed-effects case. To construct the test statistics, we must examine the expected mean squares. They are

$$\begin{aligned} E(MS_A) &= \sigma^2 + n\sigma_{\tau\beta}^2 + bn\sigma_\tau^2 \\ E(MS_B) &= \sigma^2 + n\sigma_{\tau\beta}^2 + an\sigma_\beta^2 \\ E(MS_{AB}) &= \sigma^2 + n\sigma_{\tau\beta}^2 \end{aligned}$$

and

$$E(MS_E) = \sigma^2 \tag{S14-2}$$

Note from the expected mean squares that the appropriate statistic for testing the no-interaction hypothesis H_0: $\sigma_{\tau\beta}^2 = 0$ is

$$F_0 = \frac{MS_{AB}}{MS_E} \tag{S14-3}$$

since if H_0 is true, both numerator and denominator of F_0 have expectation σ^2, and only if H_0 is false is $E(MS_{AB})$ greater than $E(MS_E)$. The ratio F_0 is distributed as $F_{(a-1)(b-1),ab(n-1)}$. Similarly, for testing that there is no main effect of factor A, or H_0: $\sigma_\tau^2 = 0$, we would use

$$F_0 = \frac{MS_A}{MS_{AB}} \tag{S14-4}$$

which is distributed as $F_{a-1,(a-1)(b-1)}$, and for testing H_0: $\sigma_\beta^2 = 0$ the test statistic is

$$F_0 = \frac{MS_B}{MS_{AB}} \tag{S14-5}$$

which is distributed as $F_{b-1,(a-1)(b-1)}$. These are all upper-tail, one-tail tests. Thus, the null hypotheses above would be rejected at the α level of significance if the calculated value of f_0 exceeds the upper α percentage point of the F-distribution. Notice that these test statistics are not the same as those used if both factors A and B are fixed. The expected mean squares are always used as a guide to test statistic construction.

The variance components may be estimated by equating the observed mean squares to their expected values and solving for the variance components. This yields

$$\begin{aligned} \hat{\sigma}^2 &= MS_E \\ \hat{\sigma}_{\tau\beta}^2 &= \frac{MS_{AB} - MS_E}{n} \\ \hat{\sigma}_\beta^2 &= \frac{MS_B - MS_{AB}}{an} \\ \hat{\sigma}_\tau^2 &= \frac{MS_A - MS_{AB}}{bn} \end{aligned} \tag{S14-6}$$

EXAMPLE S14-1 Two factors that may influence the breaking strength of cloth are being studied. Four test machines and three operators are chosen at random, and an experiment is run using cloth from the same production segment. The data are shown in Table S14-1, and the analysis of variance is in Table S14-2. Notice that the first four columns in Table S14-2. are computed as in a standard (fixed-effects model) analysis. The test statistics are computed using Equations (S14-3) through S14-5. We will use $\alpha = 0.05$. The test statistic for the no-interaction hypothesis H_0: $\sigma_{\tau\beta}^2 = 0$ is

$$f_0 = \frac{MS_{AB}}{MS_E} = \frac{5.94}{3.75} = 1.584$$

Table S14-1 Breaking Strength Data for Example S14-1

	Machine			
Operator	1	2	3	4
A	113	113	111	113
	112	118	111	119
B	111	110	111	114
	112	111	109	112
C	109	112	114	111
	111	115	112	112

Table S14-2 Analysis of Variance for Example 14-2A

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	P-Value
Operators	26.33	2	13.17	2.217	0.1904
Machines	24.33	3	8.11	1.365	0.3403
Interaction	35.67	6	5.94	1.584	0.2338
Error	45.00	12	3.75		
Total	131.33	23			

which would be compared to $f_{0.05,6,12} = 3.00$, and so H_0: $\sigma^2_{\tau\beta} = 0$ cannot be rejected. The P-value for this ratio is $P = 0.2338$. To test for no machine effect (H_0: $\sigma^2_{\beta} = 0$), we compute

$$f_0 = \frac{MS_B}{MS_{AB}} = \frac{8.11}{5.94} = 1.365$$

which would be compared to $f_{0.05,3,6} = 4.76$. The P-value for this ratio is $P = 0.3403$. Therefore, we conclude that machines do not significantly affect the breaking strength test results. To test for no operator effect (H_0: $\sigma^2_{\tau} = 0$), we compute

$$f_0 = \frac{MS_A}{MS_{AB}} = \frac{13.17}{5.94} = 2.217$$

which would be compared to $f_{0.05,2,6} = 5.14$. The P-value for this ratio is $P = 0.1904$. We conclude that the operators do not affect the breaking strength test results. The variance components may be estimated using Equations S14-6 as follows:

$$\begin{aligned}
\hat{\sigma}^2 &= 3.75 \\
\hat{\sigma}^2_{\tau\beta} &= \frac{5.94 - 3.75}{2} = 1.10 \\
\hat{\sigma}^2_{\tau} &= \frac{13.17 - 5.94}{8} = 0.90 \\
\hat{\sigma}^2_{\beta} &= \frac{8.11 - 5.94}{6} = 0.36
\end{aligned}$$

All of the variance components $\hat{\sigma}^2_{\tau}$, $\hat{\sigma}^2_{\beta}$, and $\hat{\sigma}^2_{\tau\beta}$ are small; in fact, they are not significantly different from zero.

The Mixed Model

Now suppose that one of the factors, A, is fixed and the other, B, is random. This is called the **mixed model** analysis of variance. The linear model is

$$Y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk} \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases} \tag{S14-7}$$

In this model, τ_i is a fixed effect defined such that $\sum_{i=1}^{a} \tau_i = 0$, β_j, is a random effect, the interaction term $(\tau\beta)_{ij}$ is a random effect, and ϵ_{ijk} is a normally and independently distributed random variable with mean zero and variance σ^2. The interaction elements $(\tau\beta)_{ij}$ are normal random variables with mean zero and variance $[(a-1)/a]\sigma^2_{\tau\beta}$, and $\sum_{i=1}^{a}(\tau\beta)_{ij} = 0$. Because the sum of the interaction effects over the levels of the fixed factor equals zero, this version of the mixed model is called the **restricted model.** The interaction elements are not all independent. For more details, see Montgomery (2001).

The expected mean squares in this case are

$$\begin{aligned} E(MS_A) &= \sigma^2 + n\sigma^2_{\tau\beta} + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1} \\ E(MS_B) &= \sigma^2 + an\sigma^2_{\beta} \\ E(MS_{AB}) &= \sigma^2 + n\sigma^2_{\tau\beta} \\ E(MS_E) &= \sigma^2 \end{aligned} \tag{S14-8}$$

Therefore, the appropriate test statistic for testing H_0: $\tau_i = 0$ is

$$F_0 = \frac{MS_A}{MS_{AB}} \tag{S14-9}$$

for which the reference distribution is $F_{a-1,(a-1)(b-1)}$. For testing H_0: $\sigma^2_\beta = 0$, the test statistic is

$$F_0 = \frac{MS_B}{MS_E} \tag{S14-10}$$

for which the reference distribution is $F_{b-1,ab(n-1)}$. Finally, for testing H_0: $\sigma^2_{\tau\beta} = 0$ we would use

$$F_0 = \frac{MS_{AB}}{MS_E} \tag{S14-11}$$

for which the reference distribution is $F_{(a-1)(b-1),ab(n-1)}$.

Table S14-3 Analysis of Variance for the Two-Factor Mixed Model (A fixed, B random)

Source of Variation	Sum of Squares	Degrees of Freedom	Expected Mean Square	Mean Square	F_0
Rows (A)	SS_A	$a-1$	$\sigma^2 + n\sigma^2_{\tau\beta} + bn\sum\tau_i^2/(a-1)$	MS_A	MS_A/MS_{AB}
Columns (B)	SS_B	$b-1$	$\sigma^2 + an\sigma^2_{\beta}$	MS_B	MS_B/MS_E
Interaction	SS_{AB}	$(a-1)(b-1)$	$\sigma^2 + n\sigma^2_{\tau\beta}$	MS_{AB}	MS_{AB}/MS_E
Error	SS_E	$ab(n-1)$	σ^2	MS_E	
Total	SS_T	$abn-1$			

The variance components σ^2_β, $\sigma^2_{\tau\beta}$, and σ^2 may be estimated by eliminating the first equation from Equations S14-8, leaving three equations in three unknowns, whose solution is

$$\hat{\sigma}^2_\beta = \frac{MS_B - MS_E}{an}$$

$$\hat{\sigma}^2_{\tau\beta} = \frac{MS_{AB} - MS_E}{n}$$

and

$$\hat{\sigma}^2 = MS_E$$

This general approach can be used to estimate the variance components in *any* mixed model. After eliminating the mean squares containing fixed factors, there will always be a set of equations remaining that can be solved for the variance components. Table S14-3 summarizes the analysis of variance for the two-factor mixed model.

Computer Output

Some software packages have the capability to handle random factors in the analysis of variance. Table S14-4 is the computer solution to Example S14-1 from Minitab. Notice that the F-tests on the main effects have been constructed using MS_{AB} in the denominator. The results closely match those in Table S14-2.

Table S14-5 is the Minitab output for the breaking strength data assuming a mixed model, where operators are fixed and machines are random. Minitab allowed us to assume that restricted form of the mixed model [not all software packages make this easy, and some automatically use an unrestricted model, that is, the term $\sum_{i=1}^{a} (\tau\beta)_{ij} \neq 0$ in general]. Notice that

Table S14-4 Minitab Output for Example S14-1

Analysis of Variance (Balanced Designs)

Factor	Type	Levels	Values			
Operator	random	3	A	B	C	
Machine	random	4	1	2	3	4

Analysis of Variance for Strength

Source	DF	SS	MS	F	P
Operator	2	26.333	13.167	2.21	0.190
Machine	3	24.333	8.111	1.36	0.340
Operator*Machine	6	35.667	5.944	1.59	0.234
Error	12	45.000	3.750		
Total	23	131.333			

Table S14-5 Minitab Analysis of Variance for the Breaking Strength Data in Table S14-1 Where Operators Are Fixed and Machines Are Random (Mixed Model)

Analysis of Variance (Balanced Designs)

Factor	Type	Levels	Values			
Operator	fixed	3	A	B	C	
Machine	random	4	1	2	3	4

Analysis of Variance for Strength

Source	DF	SS	MS	F	P
Operator	2	26.333	13.167	2.21	0.190
Machine	3	24.333	8.111	2.16	0.145
Operator*Machine	6	35.667	5.944	1.59	0.234
Error	12	45.000	3.750		
Total	23	131.333			

the test statistics for main effects and interactions have been constructed properly using Equations S14-9 through S14-11.

14-7.4 Addition of Center Points to a 2^k Design (CD Only)

A potential concern in the use of two-level factorial designs is the assumption of linearity in the factor effects. Of course, perfect linearity is unnecessary, and the 2^k system will work quite well even when the linearity assumption holds only approximately. However, there is a method of replicating certain points in the 2^k factorial that will provide protection against curvature as well as allow an independent estimate of error to be obtained. The method consists of adding **center points** to the 2^k design. These consist of n_C replicates run at the point $x_i = 0$ $(i = 1, 2, \ldots, k)$. One important reason for adding the replicate runs at the design center is that center points do not affect the usual effects estimates in a 2^k design. We assume that the k factors are quantitative.

To illustrate the approach, consider a 2^2 design with one observation at each of the factorial points $(-, -)$, $(+, -)$, $(-, +)$, and $(+, +)$ and n_C observations at the center points (0, 0). Figure S14-3 illustrates the situation. Let $\bar{y}_F$ be the average of the four runs at the four

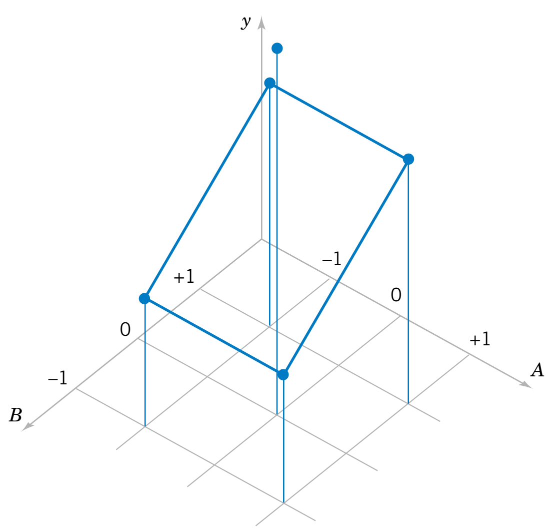


Figure S14-3 A 2^2 design with center points.

factorial points, and let $\bar{y}_C$ be the average of the n_C run at the center point. If the difference $\bar{y}_F - \bar{y}_C$ is small, the center points lie on or near the plane passing through the factorial points, and there is no curvature. On the other hand, if $\bar{y}_F - \bar{y}_C$ is large, curvature is present. A single-degree-of-freedom sum of squares for curvature is given by

$$SS_{\text{Curvature}} = \frac{n_F n_C(\bar{y}_F - \bar{y}_C)^2}{n_F + n_C} = \left(\frac{\bar{y}_F - \bar{y}_C}{\sqrt{\dfrac{1}{n_F} + \dfrac{1}{n_C}}} \right)^2 \tag{S14-12}$$

where, in general, n_F is the number of factorial design points. This quantity may be compared to the error mean square to test for curvature. Notice that when Equation S14-12 is divided by $\hat{\sigma}^2 = MS_E$, the result is similar to the square of the t statistic used to compare two means.

More specifically, when points are added to the center of the 2^k design, the model we may entertain is

$$Y = \beta_0 + \sum_{j=1}^{k} \beta_j x_j + \sum_{i<j}\sum \beta_{ij} x_i x_j + \sum_{j=1}^{k} \beta_{jj} x_j^2 + \epsilon$$

where the β_{jj} are pure quadratic effects. The test for curvature actually tests the hypotheses

$$H_0\colon \sum_{j=1}^{k} \beta_{jj} = 0$$

$$H_1\colon \sum_{j=1}^{k} \beta_{jj} \neq 0$$

Furthermore, if the factorial points in the design are unreplicated, we may use the n_C center points to construct an estimate of error with $n_C - 1$ degrees of freedom.

EXAMPLE S14-2

A chemical engineer is studying the percentage of conversion or yield of a process. There are two variables of interest, reaction time and reaction temperature. Because she is uncertain about the assumption of linearity over the region of exploration, the engineer decides to conduct a 2^2

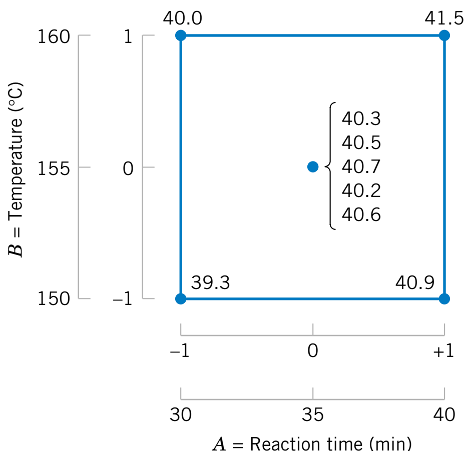


Figure S14-4 The 2^2 design with five center points for Example S14-2.

Table S14-6 Analysis of Variance for Example S14-2

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	P-Value
A (Time)	2.4025	1	2.4025	55.87	0.0017
B (Temperature)	0.4225	1	0.4225	9.83	0.0350
AB	0.0025	1	0.0025	0.06	0.8237
Curvature	0.0027	1	0.0027	0.06	0.8163
Error	0.1720	4	0.0430		
Total	3.0022	8			

design (with a single replicate of each factorial run) augmented with five center points. The design and the yield data are shown in Fig. S14-4.

Table S14-6 summarizes the analysis of variance for this experiment. The mean square error is calculated from the center points as follows:

$$MS_E = \frac{SS_E}{n_C - 1} = \frac{\sum\limits_{\text{Center points}} (y_i - \bar{y}_C)^2}{n_C - 1}$$

$$= \frac{\sum\limits_{i=1}^{5} (y_i - 40.46)^2}{4} = \frac{0.1720}{4} = 0.0430$$

The average of the points in the factorial portion of the design is $\bar{y}_F = 40.425$, and the average of the points at the center is $\bar{y}_C = 40.46$. The difference $\bar{y}_F - \bar{y}_C = 40.425 - 40.46 = -0.035$ appears to be small. The curvature sum of squares in the analysis of variance table is computed from Equation S14-12 as follows:

$$SS_{\text{Curvature}} = \frac{n_F n_C (\bar{y}_F - \bar{y}_C)^2}{n_F + n_C}$$

$$= \frac{(4)(5)(-0.035)^2}{4 + 5} = 0.0027$$

The analysis of variance indicates that both factors exhibit significant main effects, that there is no interaction, and that there is no evidence of curvature in the response over the region of exploration. That is, the null hypothesis $H_0\colon \sum_{j=1}^{k} \beta_{jj} = 0$ cannot be rejected.

14-10 RESPONSE SURFACE METHODS AND DESIGNS (CD ONLY)

Response surface methodology, or RSM, is a collection of mathematical and statistical techniques that are useful for modeling and analysis in applications where a response of interest is influenced by several variables and the objective is to **optimize this response.** For example, suppose that a chemical engineer wishes to find the levels of temperature (x_1) and feed concentration (x_2) that maximize the yield (y) of a process. The process yield is a function of the levels of temperature and feed concentration, say,

$$Y = f(x_1, x_2) + \epsilon$$

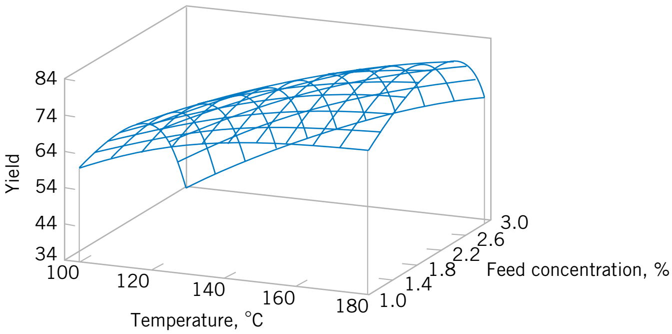


Figure S14-5 A three-dimensional response surface showing the expected yield as a function of temperature and feed concentration.

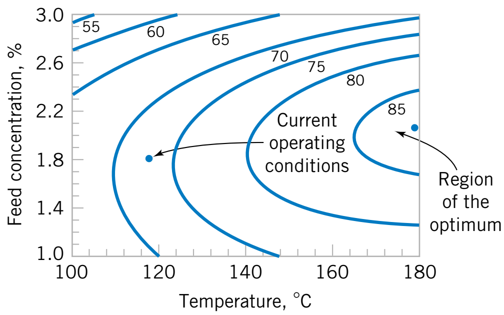


Figure S14-6 A contour plot of the yield response surface in Figure S14-5.

where ϵ represents the noise or error observed in the response Y. If we denote the expected response by $E(Y) = f(x_1, x_2) = \eta$, then the surface represented by

$$\eta = f(x_1, x_2)$$

is called a **response surface.**

We may represent the response surface graphically as shown in Fig. S14-5, where η is plotted versus the levels of x_1 and x_2. Notice that the response is represented as a surface plot in a three-dimensional space. To help visualize the shape of a response surface, we often plot the contours of the response surface as shown in Fig. S14-6. In the contour plot, lines of constant response are drawn in the x_1, x_2 plane. Each contour corresponds to a particular height of the response surface. The contour plot is helpful in studying the levels of x_1 and x_2 that result in changes in the shape or height of the response surface.

In most RSM problems, the form of the relationship between the response and the independent variables is unknown. Thus, the first step in RSM is to find a suitable approximation for the true relationship between Y and the independent variables. Usually, a low-order polynomial in some region of the independent variables is employed. If the response is well modeled by a linear function of the independent variables, the approximating function is the **first-order model**

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \cdots + \beta_k x_k + \epsilon \tag{S14-13}$$

If there is curvature in the system, then a polynomial of higher degree must be used, such as the **second-order model**

$$Y = \beta_0 + \sum_{i=1}^{k} \beta_i x_i + \sum_{i=1}^{k} \beta_{ii} x_i^2 + \sum_{i<j}\sum \beta_{ij} x_i x_j + \epsilon \tag{S14-14}$$

Many RSM problems use one or both of these approximating polynomials. Of course, it is unlikely that a polynomial model will be a reasonable approximation of the true functional relationship over the entire space of the independent variables, but for a relatively small region they usually work quite well.

The method of least squares, discussed in Chapters 11 and 12, is used to estimate the parameters in the approximating polynomials. The response surface analysis is then done in terms of the fitted surface. If the fitted surface is an adequate approximation of the true response function, analysis of the fitted surface will be approximately equivalent to analysis of the actual system.

RSM is a **sequential** procedure. Often, when we are at a point on the response surface that is remote from the optimum, such as the current operating conditions in Fig. S14-6, there is little curvature in the system and the first-order model will be appropriate. Our objective here is to lead the experimenter rapidly and efficiently to the general vicinity of the optimum. Once the region of the optimum has been found, a more elaborate model such as the second-order model may be employed, and an analysis may be performed to locate the optimum. From Fig. S14-6, we see that the analysis of a response surface can be thought of as "climbing a hill," where the top of the hill represents the point of maximum response. If the true optimum is a point of minimum response, we may think of "descending into a valley."

The eventual objective of RSM is to determine the optimum operating conditions for the system or to determine a region of the factor space in which operating specifications are satisfied. Also, note that the word "optimum" in RSM is used in a special sense. The "hill climbing" procedures of RSM guarantee convergence to a local optimum only.

Method of Steepest Ascent

Frequently, the initial estimate of the optimum operating conditions for the system will be far from the actual optimum. In such circumstances, the objective of the experimenter is to move rapidly to the general vicinity of the optimum. We wish to use a simple and economically efficient experimental procedure. When we are remote from the optimum, we usually assume that a first-order model is an adequate approximation to the true surface in a small region of the x's.

The **method of steepest ascent** is a procedure for moving sequentially along the path of steepest ascent, that is, in the direction of the maximum increase in the response. Of course, if **minimization** is desired, we are talking about the **method of steepest descent.** The fitted first-order model is

$$\hat{y} = \hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i \tag{S14-15}$$

and the first-order response surface, that is, the contours of $\hat{y}$, is a series of parallel lines such as that shown in Fig. S14-7. The direction of steepest ascent is the direction in which

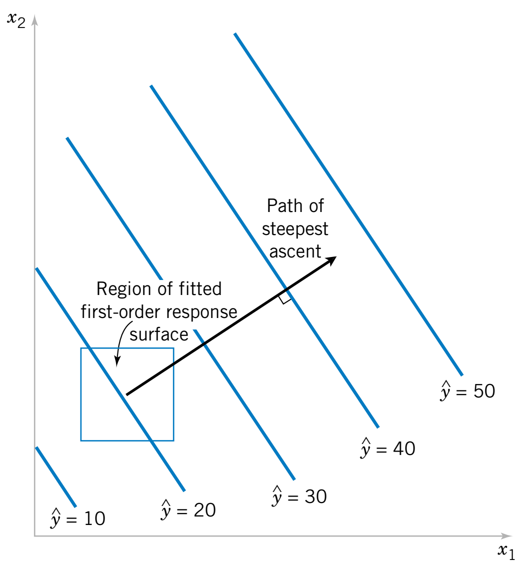
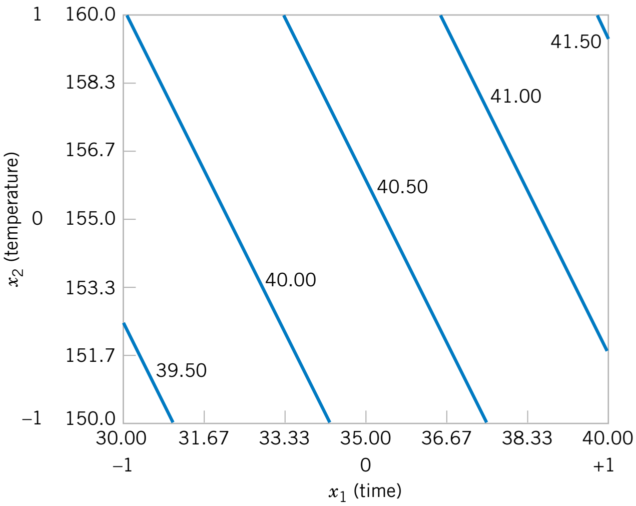


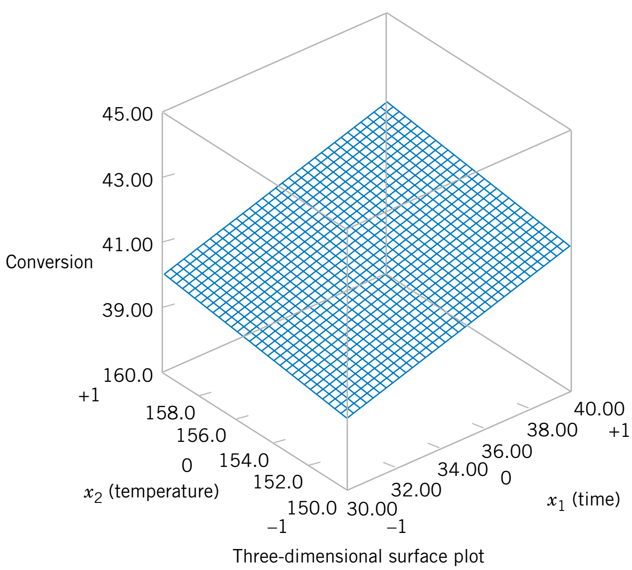
Figure S14-7 First-order response surface and path of steepest ascent.

$\hat{y}$ increases most rapidly. This direction is normal to the fitted response surface contours. We usually take as the **path of steepest ascent** the line through the center of the region of interest and normal to the fitted surface contours. Thus, the steps along the path are proportional to the regression coefficients $\{\hat{\beta}_i\}$. The experimenter determines the actual step size based on process knowledge or other practical considerations.

Experiments are conducted along the path of steepest ascent until no further increase in response is observed. Then a new first-order model may be fit, a new direction of steepest ascent determined, and further experiments conducted in that direction until the experimenter feels that the process is near the optimum.



(a)



(b)

Figure S14-8 Response surface plots for the first-order model in Example S14-3.

EXAMPLE S14-3

In Example S14-2 we described an experiment on a chemical process in which two factors, reaction time (x_1) and reaction temperature (x_2), affect the percent conversion or yield (Y). Figure S14-3 shows the 2^2 design plus five center points used in this study. The engineer found that both factors were important, there was no interaction, and there was no curvature in the response surface. Therefore, the first-order model

$$Y = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \epsilon$$

should be appropriate. Now the effect estimate of time is 1.55 hours and the effect estimate of temperature is 0.65°F, and since the regression coefficients $\hat{\beta}_1$ and $\hat{\beta}_2$ are one-half of the corresponding effect estimates, the fitted first-order model is

$$\hat{y} = 40.44 + 0.775x_1 + 0.325x_2$$

Figure S14-8(a) and (b) show the contour plot and three-dimensional surface plot of this model. Figure S14-8 also shows the relationship between the **coded variables** x_1 and x_2 (that defined the high and low levels of the factors) and the original variables, time (in minutes) and temperature (in °F).

From examining these plots (or the fitted model), we see that to move away from the design center—the point ($x_1 = 0, x_2 = 0$)—along the path of steepest ascent, we would move 0.775 unit in the x_1 direction for every 0.325 unit in the x_2 direction. Thus, the path of steepest ascent passes through the point ($x_1 = 0, x_2 = 0$) and has a slope 0.325/0.775. The engineer decides to use 5 minutes of reaction time as the basic step size. Now, 5 minutes of reaction time is equivalent to a step in the *coded* variable x_1 of $\Delta x_1 = 1$. Therefore, the steps along the path of steepest ascent are $\Delta x_1 = 1.0000$ and $\Delta x_2 = (0.325/0.775)\Delta x_1 = 0.42$. A change of $\Delta x_2 = 0.42$ in the coded variable x_2 is equivalent to about 2°F in the original variable temperature. Therefore, the engineer will move along the path of steepest ascent by increasing reaction time by 5 minutes and temperature by 2°F. An actual observation on yield will be determined at each point.

Figure S14-9 shows several points along this path of steepest ascent and the yields actually observed from the process at those points. At points *A*–*D* the observed yield increases steadily, but beyond point *D*, the yield decreases. Therefore, steepest ascent would terminate in the vicinity of 55 minutes of reaction time and 163°F with an observed percent conversion of 67%.

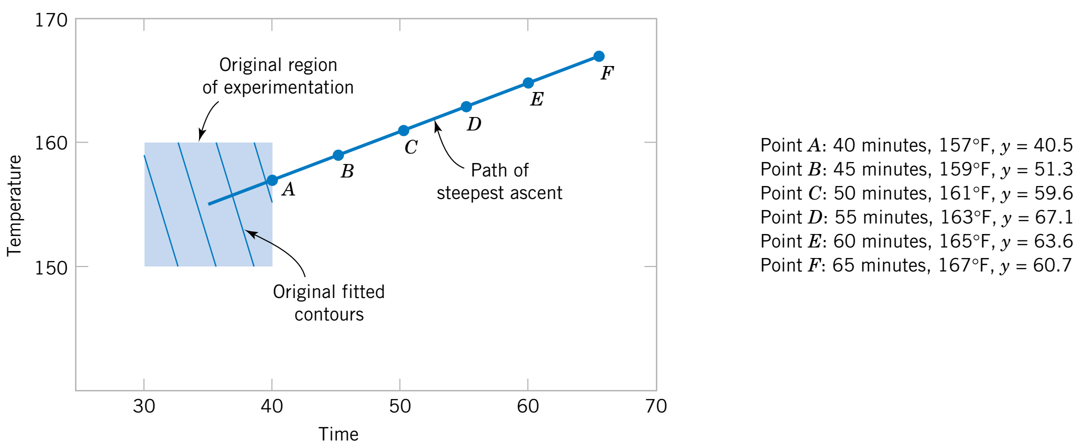


Figure S14-9 Steepest ascent experiment for Example S14-3.

Analysis of a Second-Order Response Surface

When the experimenter is relatively close to the optimum, a second-order model is usually required to approximate the response because of curvature in the true response surface. The fitted second-order model is

$$\hat{y} = \hat{\beta}_0 + \sum_{i=1}^{k} \hat{\beta}_i x_i + \sum_{i=1}^{k} \hat{\beta}_{ii} x_i^2 + \sum_{i<j}\sum \hat{\beta}_{ij} x_i x_j$$

where $\hat{\beta}$ denotes the least squares estimate of β. In this section we show how to use this fitted model to find the optimum set of operating conditions for the x's and to characterize the nature of the response surface.

EXAMPLE S14-4

Continuation of Example S14-3

Consider the chemical process from Example S14-3, where the method of steepest ascent terminated at a reaction time of 55 minutes and a temperature of 163°F. The experimenter decides to fit a second-order model in this region. Table S14-7 and Fig. S14-10 show the experimental design, which consists of a 2^2 design centered at 55 minutes and 165°F, five center points, and four runs along the coordinate axes called axial runs. This type of design is called a **central composite design,** and it is a very popular design for fitting second-order response surfaces.

Two response variables were measured during this phase of the experiment: percentage conversion (yield) and viscosity. The least-squares quadratic model for the yield response is

$$\hat{y}_1 = 69.1 + 1.633x_1 + 1.083x_2 - 0.969x_1^2 - 1.219x_2^2 + 0.225x_1x_2$$

The analysis of variance for this model is shown in Table S14-8.

Figure S14-11 shows the response surface contour plot and the three-dimensional surface plot for this model. From examination of these plots, the maximum yield is about 70%, obtained at approximately 60 minutes of reaction time and 167°F.

The viscosity response is adequately described by the first-order model

$$\hat{y}_2 = 37.08 + 3.85x_1 + 3.10x_2$$

Table S14-7 Central Composite Design for Example S14-4

Observation Number	Time (minutes)	Temperature (°F)	Coded Variables x_1	x_2	Conversion (percent) Response 1	Viscosity (mPa-sec) Response 2
1	50	160	−1	−1	65.3	35
2	60	160	1	−1	68.2	39
3	50	170	−1	1	66	36
4	60	170	1	1	69.8	43
5	48	165	−1.414	0	64.5	30
6	62	165	1.414	0	69	44
7	55	158	0	−1.414	64	31
8	55	172	0	1.414	68.5	45
9	55	165	0	0	68.9	37
10	55	165	0	0	69.7	34
11	55	165	0	0	68.5	35
12	55	165	0	0	69.4	36
13	55	165	0	0	69	37

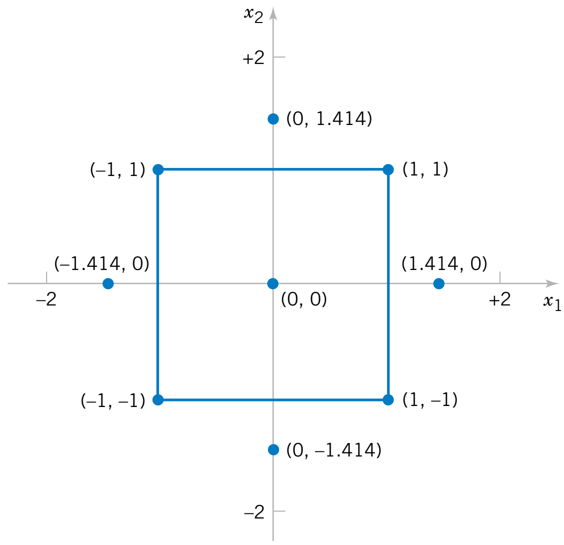


Figure S14-10 Central composite design for Example S14-4.

Table S14-9 summarizes the analysis of variance for this model. The response surface is shown graphically in Fig. S14-12. Notice that viscosity increases as both time and temperature increase.

As in most response surface problems, the experimenter in this example had conflicting objectives regarding the two responses. The objective was to maximize yield, but the acceptable range for viscosity was $38 \leq y_2 \leq 42$. When there are only a few independent variables, an easy way to solve this problem is to overlay the response surfaces to find the optimum. Figure S14-13 shows the overlay plot of both responses, with the contours $y_1 = 69\%$ conversion, $y_2 = 38$, and $y_2 = 42$ highlighted. The shaded areas on this plot identify unfeasible combinations of time and temperature. This graph shows that several combinations of time and temperature will be satisfactory.

Example S14-4 illustrates the use of a **central composite design** (CCD) for fitting a second-order response surface model. These designs are widely used in practice because they are relatively efficient with respect to the number of runs required. In general, a CCD in k

Table S14-8 Analysis of Variance for the Quadratic Model, Yield Response

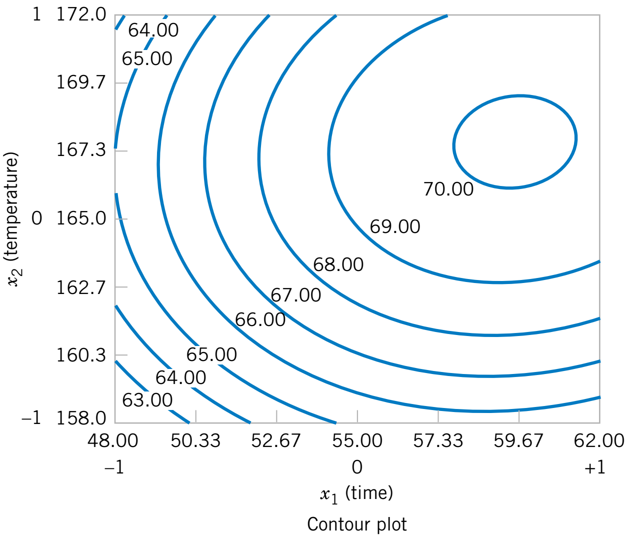
Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	P-Value
Model	45.89	5	9.178	14.93	0.0013
Residual	4.30	7	0.615		
Total	50.19	12			

Independent Variable	Coefficient Estimate	Standard Error	t for H_0 Coefficient = 0	P-Value
Intercept	69.100	0.351	197.1	
x_1	1.633	0.277	5.891	0.0006
x_2	1.083	0.277	3.907	0.0058
x_1^2	−0.969	0.297	−3.259	0.0139
x_2^2	−1.219	0.297	−4.100	0.0046
x_1x_2	0.225	0.392	0.5740	0.5839

Table S14-9 Analysis of Variance for the First-Order Model, Viscosity Response

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	f_0	*P*-Value
Model	195.4	2	97.72	15.89	0.0008
Residual	61.5	10	6.15		
Total	256.9	12			

Independent Variable	Coefficient Estimate	Degrees of Freedom	Standard Error	t for H_0 Coefficient = 0	*P*-Value
Intercept	37.08	1	0.69	53.91	
x_1	3.85	1	0.88	4.391	0.0014
x_2	3.10	1	0.88	3.536	0.0054



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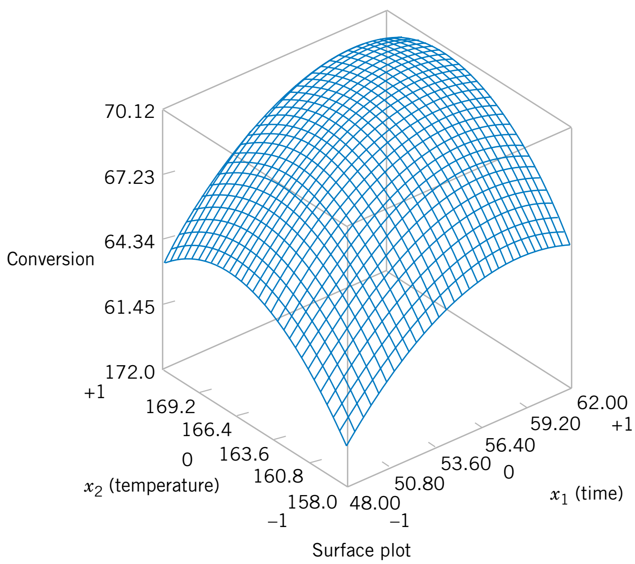

(*b*)

Figure S14-11 Response surface plots in the yield response, Example S14-4.

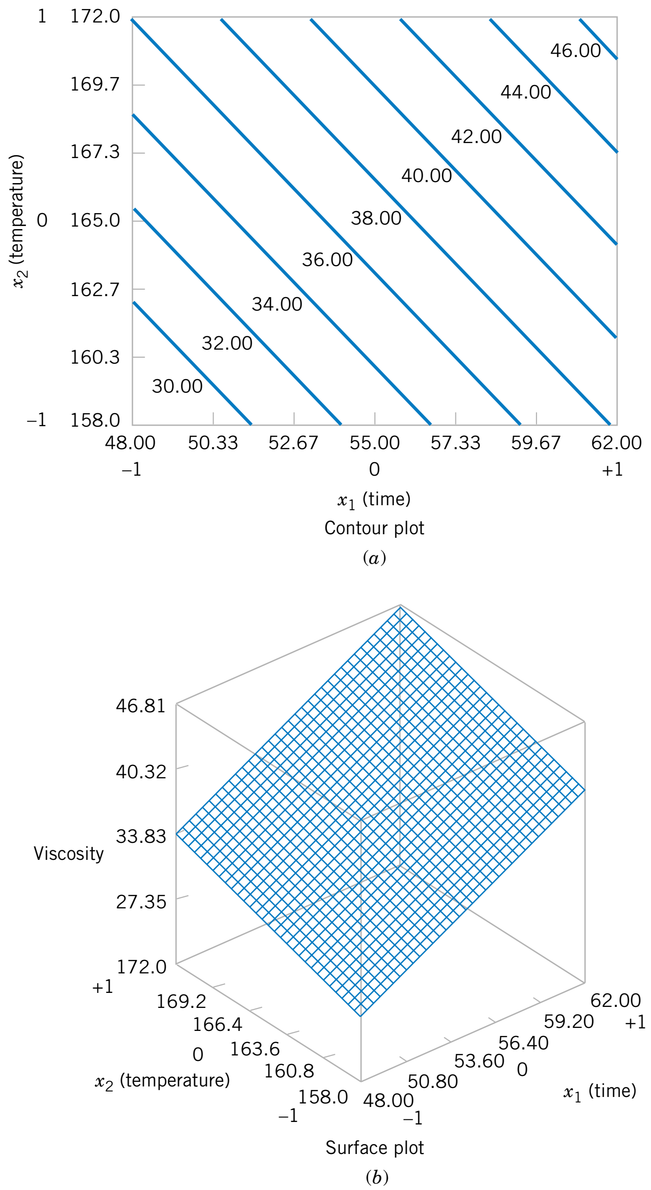

Figure S14-12 Response surface plots for the viscosity response, Example S14-4.

factors requires 2^k factorial runs, $2k$ axial runs, and at least one center point (three to five center points are typically used). Designs for $k = 2$ and $k = 3$ factors are shown in Fig. S14-14.

The central composite design may be made **rotatable** by proper choice of the axial spacing α in Fig. S14-14. If the design is rotatable, the standard deviation of predicted response $\hat{y}$ is constant at all points that are the same distance from the center of the design. For rotatability, choose $\alpha = (F)^{1/4}$, where F is the number of points in the factorial part of the design (usually $F = 2^k$). For the case of $k = 2$ factors, $\alpha = (2^2)^{1/4} = 1.414$, as was used in the design in Example S14-4. Figure S14-15 presents a contour plot and a surface plot of the standard deviation of prediction for the quadratic model used for the yield response. Notice that the contours are concentric circles, implying that yield is predicted with equal precision for all points that are the same distance from the center of the design. Also, as one would expect, the precision decreases with increasing distance from the design center.

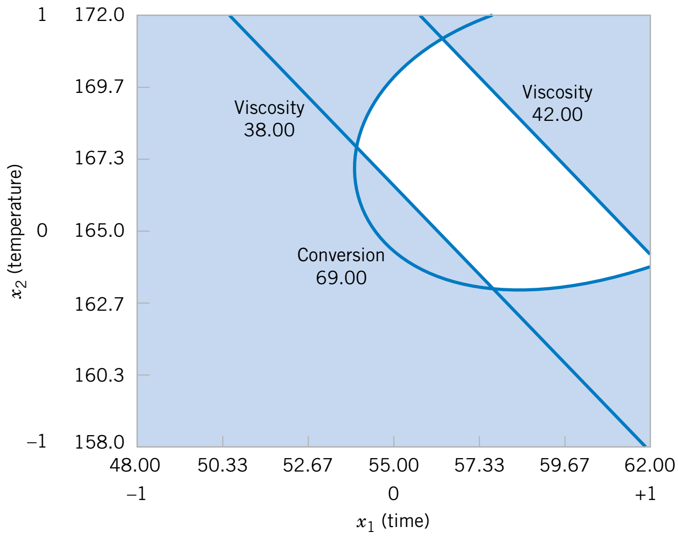

Figure S14-13 Overlay of yield and viscosity response surfaces, Example S14-4.

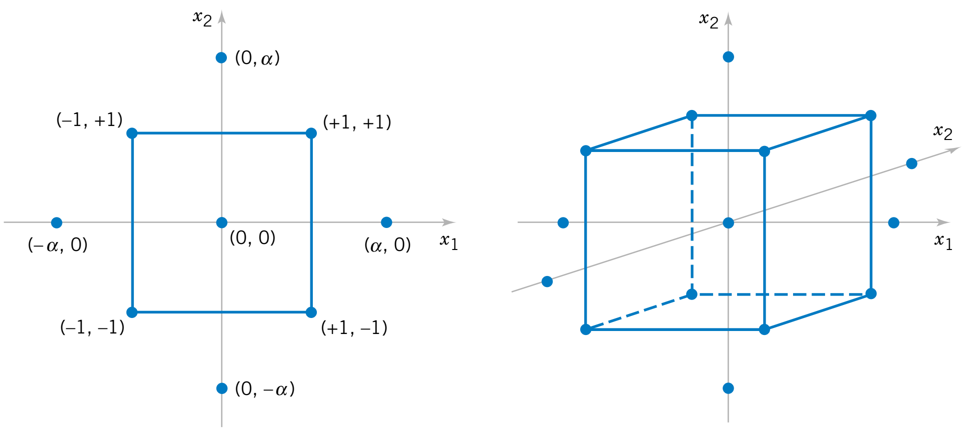

Figure S14-14 Central composite designs for $k = 2$ and $k = 3$.

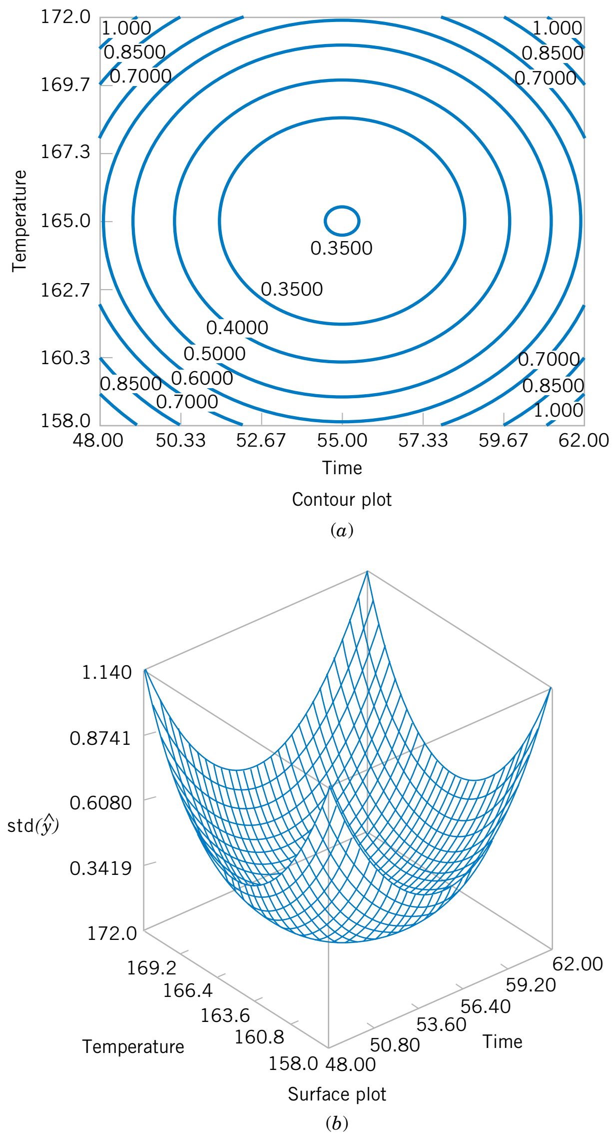

Figure S14-15 Plots of constant $\sqrt{V(\hat{y})}$ for a rotatable central composite design.