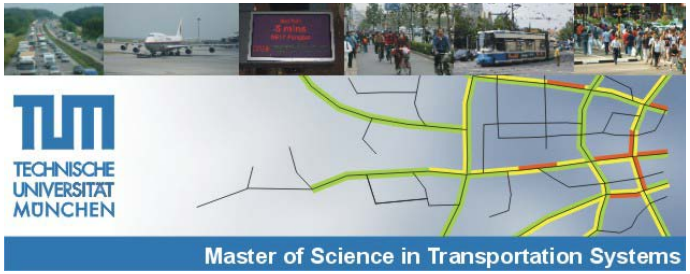
M.Sc. in ‚Transportation Systems‘



Applied Statistics in Transport Distributions

Regine Gerike

Technische Universität München, mobil.TUM

regine.gerike@tum.de

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Last Week: Probability, Descriptive Statistics

- Measures of Central Tendency
- Measures of Dispersion
- Measures of Position
- Measures of Shape

- R commands for descriptive analysis, graphics

Plan for Today's Lecture:
Discrete, Continuous Probability Distributions

- Probability Distributions - Definitions
- Probability Density Function, Cumulative Distribution Function
- Descriptive Measures: Mean and Variance
- Discrete Uniform Distribution
- Binomial Distribution

Distributions - Definitions

- A **random variable** is a function that assigns numerical value to each outcome in the sample space of a random experiment.
- A random variable is denoted by an uppercase letter such as X. After an experiment is conducted, the measured value of the random variable is denoted by a lowercase letter such as x = 70 metres.
- A **discrete random variable** is a random variable with a finite (or countably infinite) range.
- A **continuous random variable** is a random variable with an interval (either finite or infinite) of real numbers for its range.

Example: Discrete Versus Continuous Random Variables

Decide whether a discrete or continuous random variable is the best model for each of the following variables:

(a) The time you need to cover a certain distance.

(b) The outside diameter of a train axis.

(c) The Number of cars arriving at an intersection within a specific time period.

(d) The weight of an airplane.

(e) The PM10-immissions in a specific area.

(f) The current in an electronic circuit.

(g) Number of permanently delayed train connections in a network with 1,000 connections.

Probability Density Function (pdf)
Cumulative Distribution Function (cdf)

- A probability density function (pdf) is a function that gives the probability that a discrete random variable is exactly equal to some value.
- The probability density function satisfies the following conditions:

$$0 \leq p(x_i) \leq 1, \forall x_i$$

$$\sum_{i=1}^{k} p(x_i) = 1$$

- The cumulative distribution function (cdf), also probability distribution function or just distribution function, completely describes the probability distribution of a real-valued random variable X:

$$P(X \leq x) = \sum_{x_i \leq x} p(x_i)$$

Example pdf/cdf of a discrete random variable

- Consider the roll of a single die

Number observed (X)	Number of possible outcomes	f(X)
1		
2		
3		
4		
5		
6		
	Sum	

Example pdf/cdf of a discrete random variable

- Consider the roll of a single die

Number observed (X)	Number of possible outcomes	f(X)
1	1	1/6
2	1	1/6
3	1	1/6
4	1	1/6
5	1	1/6
6	1	1/6
	Sum	1

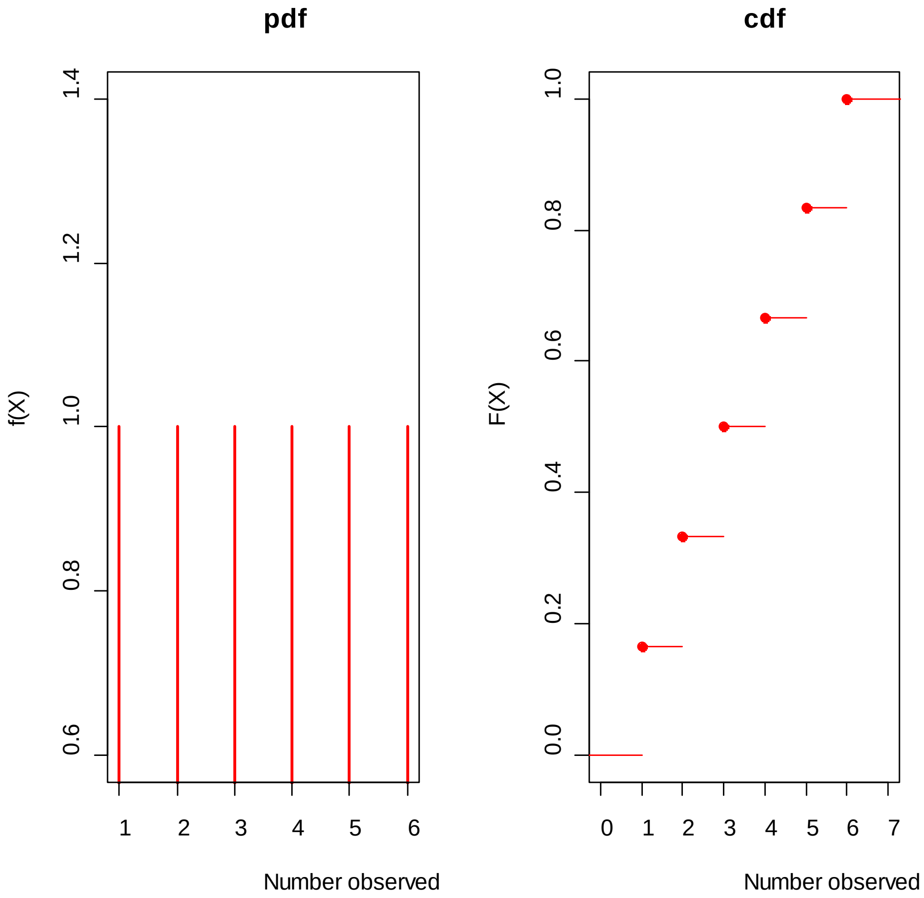
- Could you write down the cdf?

Example pdf/cdf of a discrete random variable

- Consider the roll of a single die

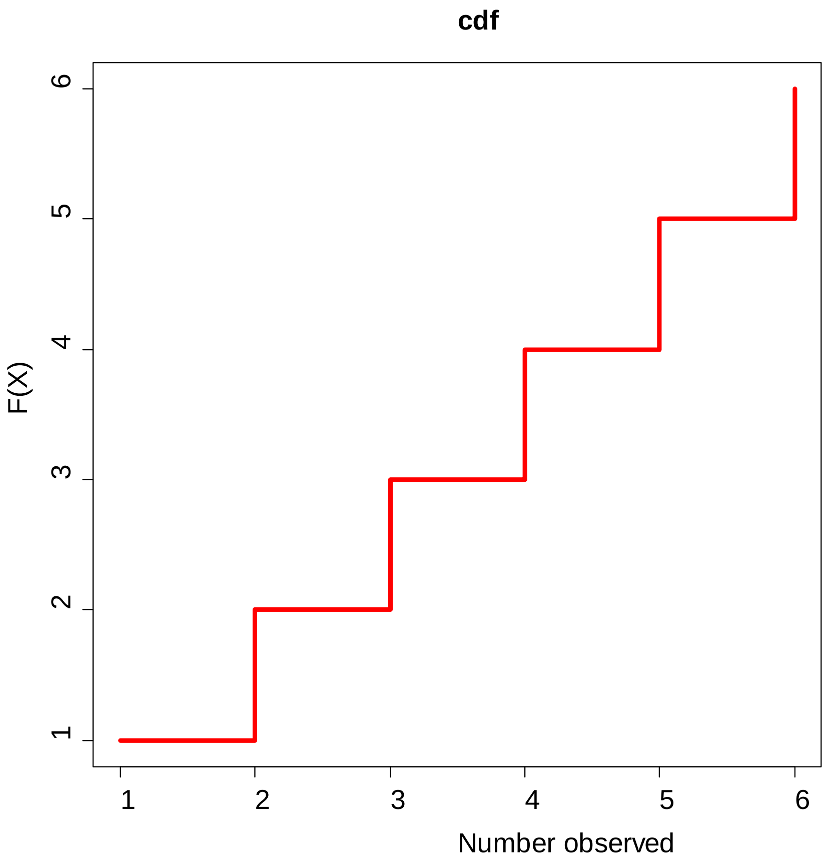
Number observed (X)	Value observed for pdf	f(X)	Value observed for cdf	F(X)
1	X=1	1/6	X=1	1/6
2	X=2	1/6	$X \leq 2$	2/6
3	X=3	1/6	$X \leq 3$	3/6
4	X=4	1/6	$X \leq 4$	4/6
5	X=5	1/6	$X \leq 5$	5/6
6	X=6	1/6	$X \leq 6$	6/6
	Sum	1		

Example pdf/cdf of a discrete random variable



... Or as a step function

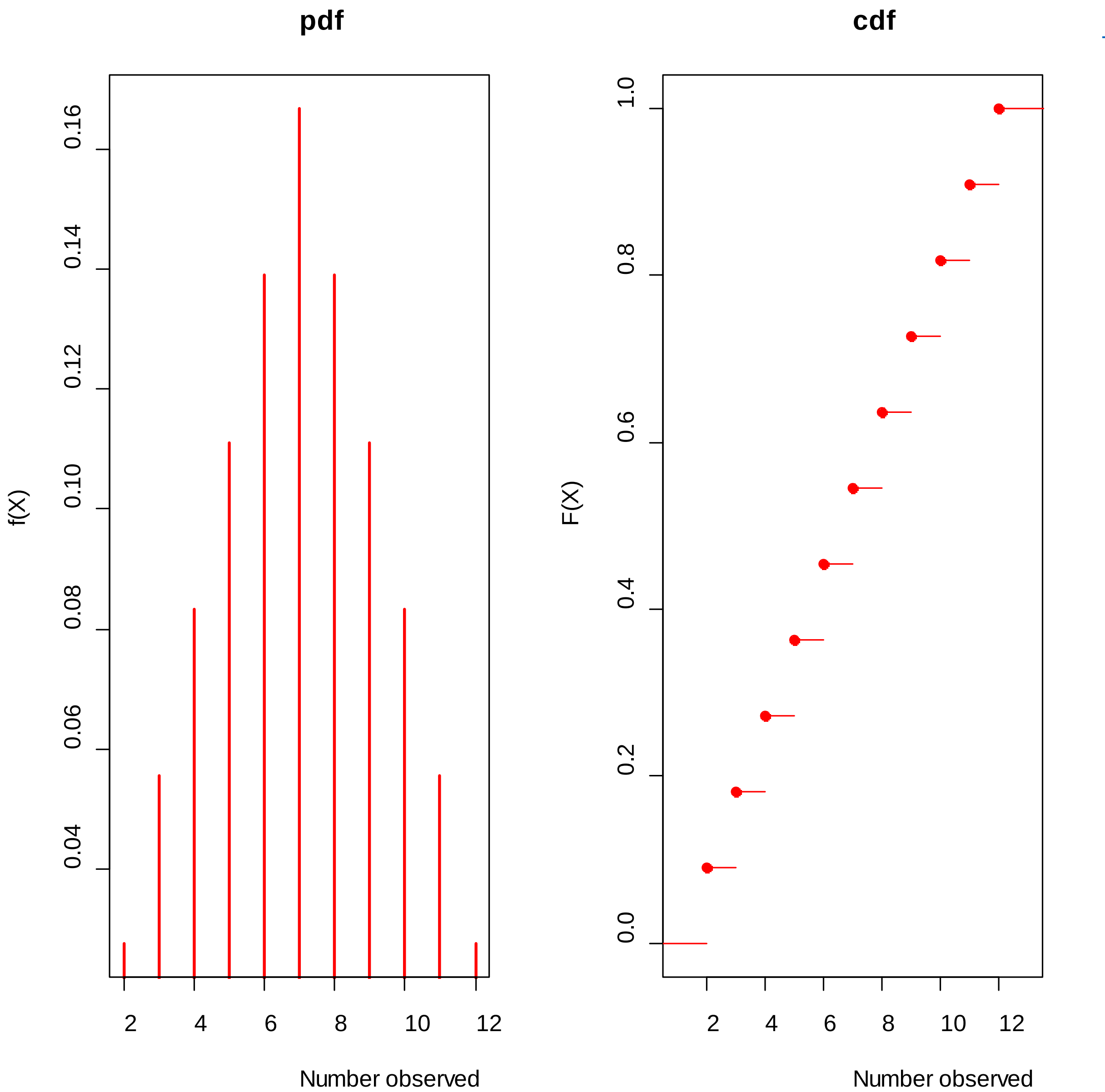
```
> par(mfrow = c(1, 1))
> (y1<-cumsum(table(x)))
1 2 3 4 5 6
1 2 3 4 5 6
> plot(x,y1,type="s",col="red",xlab = "Number observed", ylab = "F(X)",
+ main="cdf",cex.axis=1.5,lwd=4,cex.lab=1.5,cex.main=1.5)
```



Rolling two dice

Number observed (X)	Value die1, die2)	Value die1, die2)	Value die1, die2)	Value die1, die2)	Value die1, die2)	Value die1, die2)	Number of possible outcomes	f(X)
2	1,1						1	1/36
3	1,2	2,1					2	1/18
4	1,3	3,1	2,2				3	1/12
5	1,4	2,3	3,2	4,1			4	1/9
6	1,5	2,4	3,3	4,2	5,1		5	5/36
7	1,6	2,5	3,4	4,3	5,2	6,1	6	1/6
8	2,6	3,5	4,4	5,3	6,2		5	5/36
9	3,6	4,5	5,4,6,4	6,3			4	1/9
10	4,6	5,5					3	1/12
11	5,6	6,5					2	1/18
12	6,6						1	1/36
							36	1

Rolling two dice



Example Probability Density Function

For the following pdf:

x	-2	-1	0	1	2
f(x)	1/8	2/8	2/8	2/8	1/8

Determine the following probabilities:

a) P(X 2)

b) P(X>-2)

c) P(-1 X 1)

d) P(X -1 or X=2)

Example Cumulative Distribution Function

- Suppose that a day`s production of 850 manufactured parts contains 50 parts that do not conform to customer requirements. Two parts are selected at random, without replacement, from the batch. Let the random variable X equal the number of nonconforming parts in the sample. Compute the cumulative distribution function of X.

Descriptive Measures: Mean and Variance

- Mean μ (or expected value) of a discrete random variable X is the weighted average of the possible values of X, with weights equal to the probabilities.

$$\mu = \mathrm{E}(\mathrm{X}) = \sum_{\mathrm{x}} \mathrm{x}_{\mathrm{i}} * \mathrm{p}(\mathrm{x}_{\mathrm{i}})$$

- The variance of a discrete random variable X is the weighted squared deviation (x-μ)^2, with weights equal to the probabilities.

$$\sigma^2 = var(X) = \sum (x_i - \quad)^2 * p(x_i) = \sum x_i^2 p(x_i) - \quad {}^2$$

- Standard Deviation:

$$\sigma = \sqrt{\sigma^2}$$

Rolling two Dice: Expected Value

Number observed (X)	Value die1, die2)	Value die1, die2)	Value die1, die2)	Value die1, die2)	Value die1, die2)	Value die1, die2)	Number of possible outcomes	f(X)
2	1,1						1	1/36
3	1,2	2,1					2	1/18
4	1,3	3,1	2,2				3	1/12
5	1,4	2,3	3,2	4,1			4	1/9
6	1,5	2,4	3,3	4,2	5,1		5	5/36
7	1,6	2,5	3,4	4,3	5,2	6,1	6	1/6
8	2,6	3,5	4,4	5,3	6,2		5	5/36
9	3,6	4,5	5,4	6,3	6,4		4	1/9
10	4,6	5,5					3	1/12
11	5,6	6,5					2	1/18
12	6,6						1	1/36
							36	1

Expected Value: E(X)=7

Properties of expected values

1. The expected value of a constant is the same constant: $E(k)=k$.

2. The expected value of the sum of two random variables, X_1 and X_2, is equal to the sum of the expected values of X_1 and X_2:
   $E(X_1+X_2)=E(X_1)+E(X_2)$

$E(X_1 + X_2) = E(X_1) + E(X_2)$ - Example

- E(X) for rolling two dice was 7, we get the same result when considering each roll of the die as distinct random variable:

X_1	$f(X_1)$	$X*f(X_1)$	X_2	$f(X_2)$	$X*f(X_2)$
1	1/6	1/6	1	1/6	1/6
2	1/6	1/3	2	1/6	1/3
3	1/6	1/2	3	1/6	1/2
4	1/6	2/3	4	1/6	2/3
5	1/6	5/6	5	1/6	5/6
6	1/6	1	6	1/6	1
	$E(X_1)$	3.5		$E(X_2)$	3.5

Properties of expected values

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2. The expected value of the sum of two random variables, X_1 and X_2, is equal to the sum of the expected values of X_1 and X_2:
   $E(X_1+ X_2)=E(X_1)+E(X_2)$

3. When two random variables are statistically independent, the expected value of the product of two random variables is equal to the product of two expected values:
   $E(X_1 * X_2)=E(X_1)*E(X_2)$

4. The expected value of a random variable multiplied by a constant is equal to the constant times the expected value of the random variable:
   $E(k*X)=k*E(X)$

E(k*X)=k*E(X) - Example

X	f(X)	X*f(X)	k	K*X*f(X)
1	1/6	1/6	2	1/3
2	1/6	1/3	2	2/3
3	1/6	1/2	2	1
4	1/6	2/3	2	1 1/3
5	1/6	5/6	2	1 1/2
6	1/6	1	2	2
			E(k*X)	7

Properties of expected values

1. The expected value of a constant is the same constant: $E(k)=k$.

2. The expected value of the sum of two random variables, X_1 and X_2, is equal to the sum of the expected values of X_1 and X_2:
   $E(X_1+ X_2)=E(X_1)+E(X_2)$

3. When two random variables are statistically independent, the expected value of the product of two random variables is equal to the product of two expected values:
   $E(X_1 * X_2)=E(X_1)*E(X_2)$

4. The expected value of a random variable multiplied by a constant is equal to the constant times the expected value of the random variable:
   $E(k*X)=k*E(X)$

5. The last property is a composite of (1) and (4). If k and j are constants:
   $E(k*X+j)=k*E(X)+j$

Discrete Uniform Distribution

- Let X be a discrete random variable containing finitely many values: x_1, .., x_n. If:

$$P(X = x_i) = p(x_i) = \frac{1}{n}, \forall i = \{1, \dots, n\}$$

- Then X has discrete uniform distribution on $\{x_1, .., x_n\}$

- Mean: $\mu = \sum_{i=1}^{n} x_i * \frac{1}{n} = \frac{1}{n} * \sum x_i$

- Variance: $\sigma^2 = \sum {x_i}^2 * \frac{1}{n} - \left(\frac{1}{n} * \sum x_i\right)^2$

Example Discrete Uniform Distribution

- Let the random variable X have a discrete uniform distribution on the integers 1 x 3. Determine the mean and variance of X.

- Suppose the range of the discrete random variable X is the consecutive integers a, a+1,a+2,...,b, for a≤b. Then we can compute the mean of X and the variance of X with simplified formulas:

$$\mu = \frac{b+a}{2} \qquad \sigma^2 = \frac{(b-a+1)^2 - 1}{12}$$

Binomial Distribution

A random experiment consists of n Bernoulli trials such that:

- The trials are independent.
- Each trial results in only two possible outcomes, labelled as “success” and “failure”.
- The probability of a success in each trial, denoted as p, remains constant.

The random variable X that equals the number of trials that result in a success is a binomial random variable with parameters 0<p<1 and n=1, 2, ...

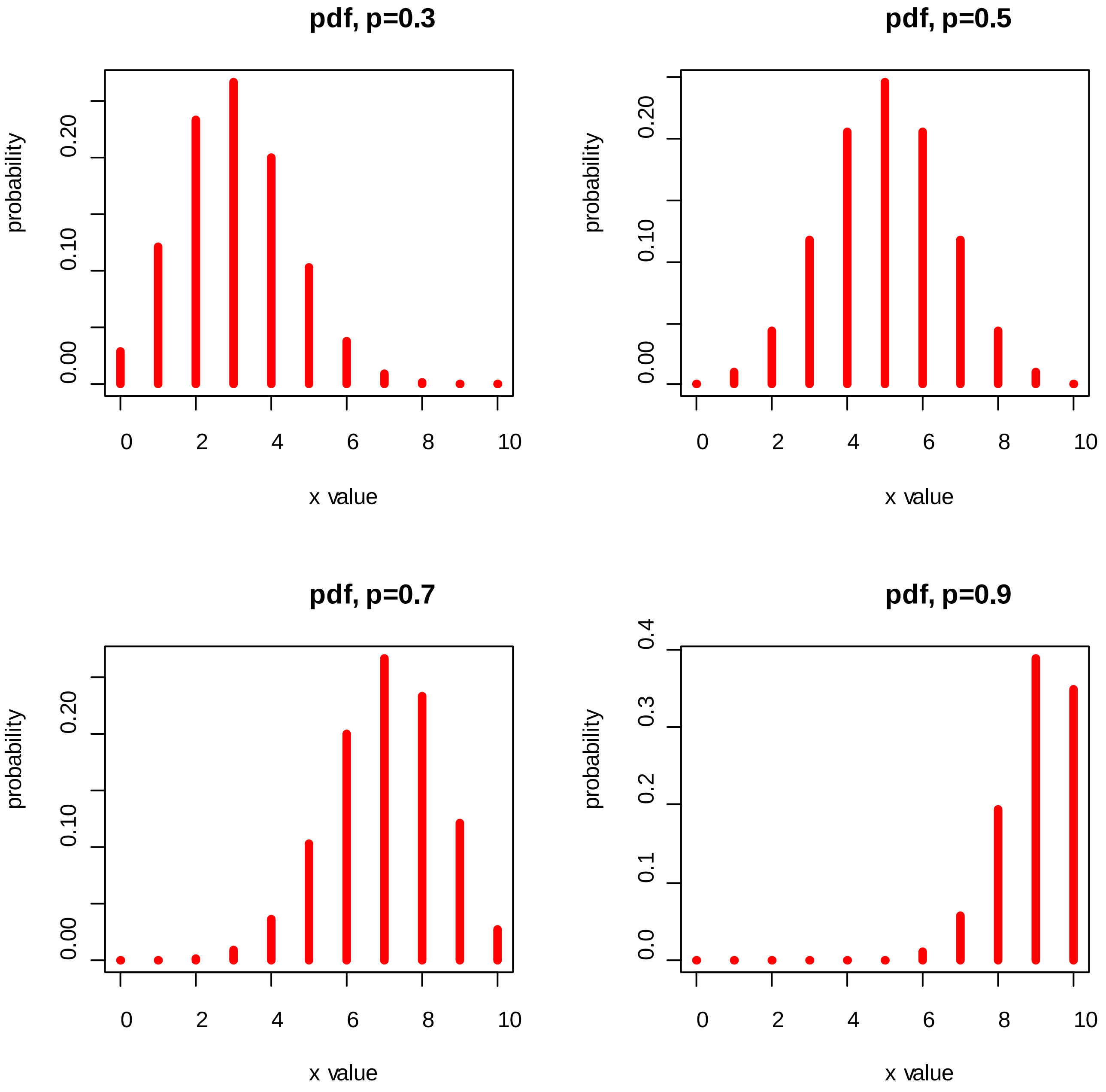
The probability density function is:

$$P(X = x) = \binom{n}{x} * \ p^{x} * (1 - p)^{n-x}, x = 0,1,2, \dots, n$$

- If X is a random variable with parameters p and n:

$$\mu = p * n \qquad \sigma^{2} = np(1 - p)$$

Binomial Distribution: Shape



Binomial Distribution: Example 1

- Determine the pdf for the number of heads when throwing a coin twice.

- It is binominal random variable with probability of success p=1/2 and number of trials 2:

$$P(X = x) = \binom{n}{x} * p^x * (1-p)^{n-x}, x = 0,1,2,\dots,n$$

$$P(X = 0) = \binom{2}{0} * p^0 * (1-p)^{2-0} = \frac{2!}{2!\,0!} * \left(\frac{1}{2}\right)^0 * \left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

$$P(X = 1) = \binom{2}{1} * p^1 * (1-p)^{2-1} = \frac{2!}{1!\,1!} * \left(\frac{1}{2}\right)^1 * \left(\frac{1}{2}\right)^1 = \frac{1}{2}$$

$$P(X = 2) = \binom{2}{2} * p^2 * (1-p)^{2-2} = \frac{2!}{0!\,2!} * \left(\frac{1}{2}\right)^2 * \left(\frac{1}{2}\right)^0 = \frac{1}{4}$$

Binomial Distribution: Example 2

Assume a company A is a small commuter airline.

Their planes hold only 15 people.

Records show that 20% of people making reservations do not show up for the flight. Suppose they decide to book 18 people for each flight.

1. Determine the probability that on a given flight at least one passenger will not have a seat.
2. What is the probability that there will be one or more empty seats on a flight?
3. What is the average number of passengers that book the flight and actually appear?
4. What is the variance?

Some important probability distributions in R

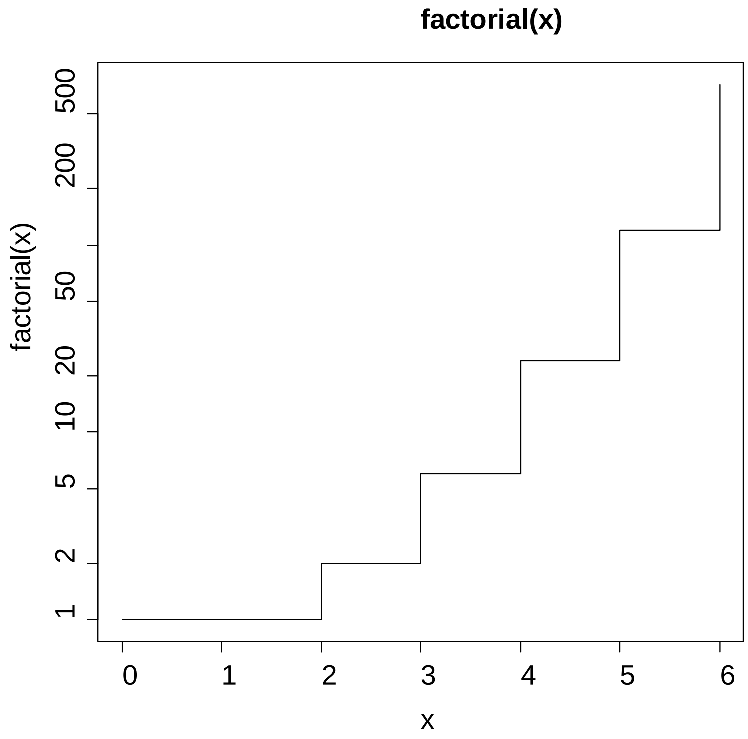
R Function	Distribution	Parameters
binom	Binomial	sample size n, probability p p(x) = choose(n,x) p^x (1-p)^(n-x)
hyper	Hypergeometric	m (number of white balls in the urn) n (number of black balls in the urn) k (number of balls drawn from the urn) p(x) = choose(m, x) choose(n, k-x) / choose(m+n, k)
pois	Poisson	mean p(x) = lambda^x exp(-lambda)/x!
normal	Normal	mean, standard deviation
t	Student's t	degrees of freedom

See http://cran.r-project.org/web/views/Distributions.html

Probability Functions

- number of permutations: factorial(x)
- Binomial coefficient: choose ()

```
> x<-0:6
> par(mfrow=c(1,1))
> plot(x,factorial(x),type="s",main="factorial(x)",
+ log="y",cex.lab = 1.5,cex.axis=1.5,cex.main=1.5)
```



Probability distributions in R

In R we can use the following functions when dealing with distributions:

- d_ (density or probability mass function, pdf)
- p_ (probability or cumulative probability/density function, cdf)
- q_ (quantile)
- r_ (random generation)

Here, the underscore "_" is a surrogate for the specific probability distribution (e.g. norm, binom, pois, hyper).

Example: the Binomial distribution in R

Having e.g. $X \sim Bin(7, 5)$ we can use

```
> dbinom(x = 5, size = 7, prob = 0.5)
[1] 0.164
```

to obtain $\mathbb{P}(X = 5)$, i.e. the value of the pmf for having $x = 5$ hits in $n = 7$ trials with a success probability of $p = 0.5$ in each trial.

Note that `x` can also be a vector, in which case each element of `x` gets evaluated separately.

```
> dbinom(x = c(3, 4, 5), size = 7, prob = 0.5)
[1] 0.273 0.273 0.164
```

Example: the Binomial distribution in R

The value of the cdf, $\mathbb{P}(X \leq 5)$, can be obtained by

```
> x1 <- pbinom(q = 5, size = 7, prob = 0.5,
+       lower.tail = TRUE)
```

```
[1] 0.938
```

Setting `lower.tail=FALSE` results in the probability $\mathbb{P}(X > 5)$ instead of $\mathbb{P}(X \leq 5)$:

```
> x2 <- pbinom(q = 5, size = 7, prob = 0.5,
+       lower.tail = FALSE)
```

```
[1] 0.0625
```

```
> x1 + x2
```

```
[1] 1
```

Example: the Binomial distribution in R

Furthermore, if we want the 0.4-quantile, i.e. the smallest value $\mathtt{x}$ for which $F(\mathtt{x}) \geq 0.4$, where F is the cumulative probability function, we can type

```
> qbinom(p = 0.4, size = 7, prob = 0.5)
```

```
[1] 3
```

Example: the Binomial distribution in R

Finally, we can draw e.g. 8 random samples by

```
> x <- rbinom(8, size = 7, prob = 0.5)
[1] 4 1 2 2 4 3 4 3
```

However, if we want to have reproducible results, we need to set a seed:

```
> set.seed(1)
> x <- rbinom(8, size = 7, prob = 0.5)
[1] 3 3 4 5 2 5 6 4
> set.seed(53)
> y <- rbinom(8, size = 7, prob = 0.5)
[1] 4 2 2 4 4 4 2 3
> set.seed(1)
> z <- rbinom(8, size = 7, prob = 0.5)
[1] 3 3 4 5 2 5 6 4
```

Binomial distribution in R – getting help

For additional information on `binom()` type

```
> help(pbinom)
```

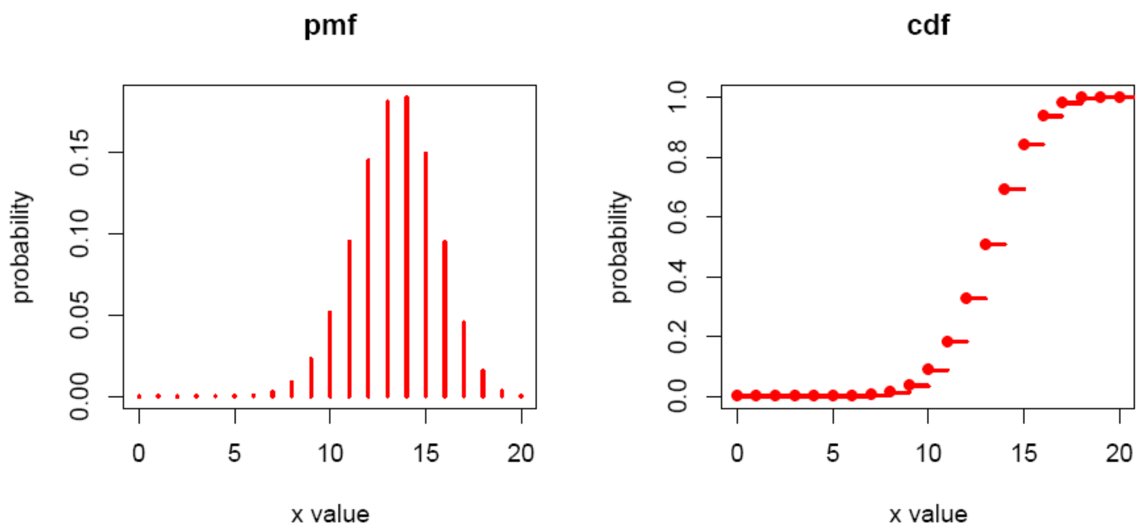
or for any other distribution, as for instance the normal distribution, type

```
> help(pnorm)
```

The Binomial distribution

Here an example for the pmf and cdf of a Binomial distribution: For $n = 20$ and $p = 2/3$ we get the following graphs

```
> x <- c(0:20)
> y <- dbinom(x, size = 20, prob = 0.67)
> z <- pbinom(x, size = 20, prob = 0.67)
> par(mfrow = c(1, 2))
> plot(x, y, type = "h", col = "red", lwd = 2, xlab = "x value", ylab = "probability",
+      main = "pmf")
> plot(x, z, pch = 16, col = "red", xlab = "x value", ylab = "probability", main = "cdf")
> segments(x, z, x + 1, z, col = "red", lwd = 2)
```



Hypergeometric Distribution

- A set of N objects contains
- K objects classified as successes
- N-K objects classified as failures
- A sample of size n objects is selected randomly (without replacement) from the N objects, where K N and n N.
- Let the random variable X denote the number of successes in the sample. Then X is a hypergeometric random variable and

$$f(x) = \frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$ x=max{0,n+K-N} to min{K,n}

- Mean and variance: $= p * n$ $\sigma^2 = np(1-p)\left(\frac{N-n}{N-1}\right)$

- Here p is interpreted as the proportion of successes in the set of N objects.

Example Cumulative Distribution Function

Suppose that a day`s production of 850 manufactured parts contains 50 parts that do not conform to customer requirements. Two parts are selected at random, without replacement, from the batch. Let the random variable X equal the number of nonconforming parts in the sample. Compute the cumulative distribution function of X.

$$P(X=0)=\frac{800}{850}*\frac{799}{849}=0.886$$

$$P(X\leq 1)=P(X=0)+P(X=1)=0.886+\frac{800}{850}*\frac{50}{849}+\frac{50}{850}*\frac{800}{849}=0.886+2*\frac{800}{850}*\frac{50}{849}=0.886+0.111=0.997$$

$$P(X\leq 2)=P(X=0)+P(X=1)+P(X=2)=0.886+0.111+\frac{50}{850}*\frac{49}{849}=0.886+0.111+0.003=1$$

$$f(x)=\frac{\binom{K}{x}\binom{N-K}{n-x}}{\binom{N}{n}}$$

$$P(X=0)=\frac{\binom{50}{0}\binom{800}{2}}{\binom{850}{2}}=\frac{319600}{360825}=0.886$$

$$P(X=1)=\frac{\binom{50}{1}\binom{800}{1}}{\binom{850}{2}}=\frac{40000}{360825}=0.111$$

$$P(X=2)=\frac{\binom{50}{2}\binom{800}{0}}{\binom{850}{2}}=\frac{1225}{360825}=0.003$$

Poisson Distribution

The Poisson distribution describes the probability that a random event will occur in a time or space interval under the conditions that

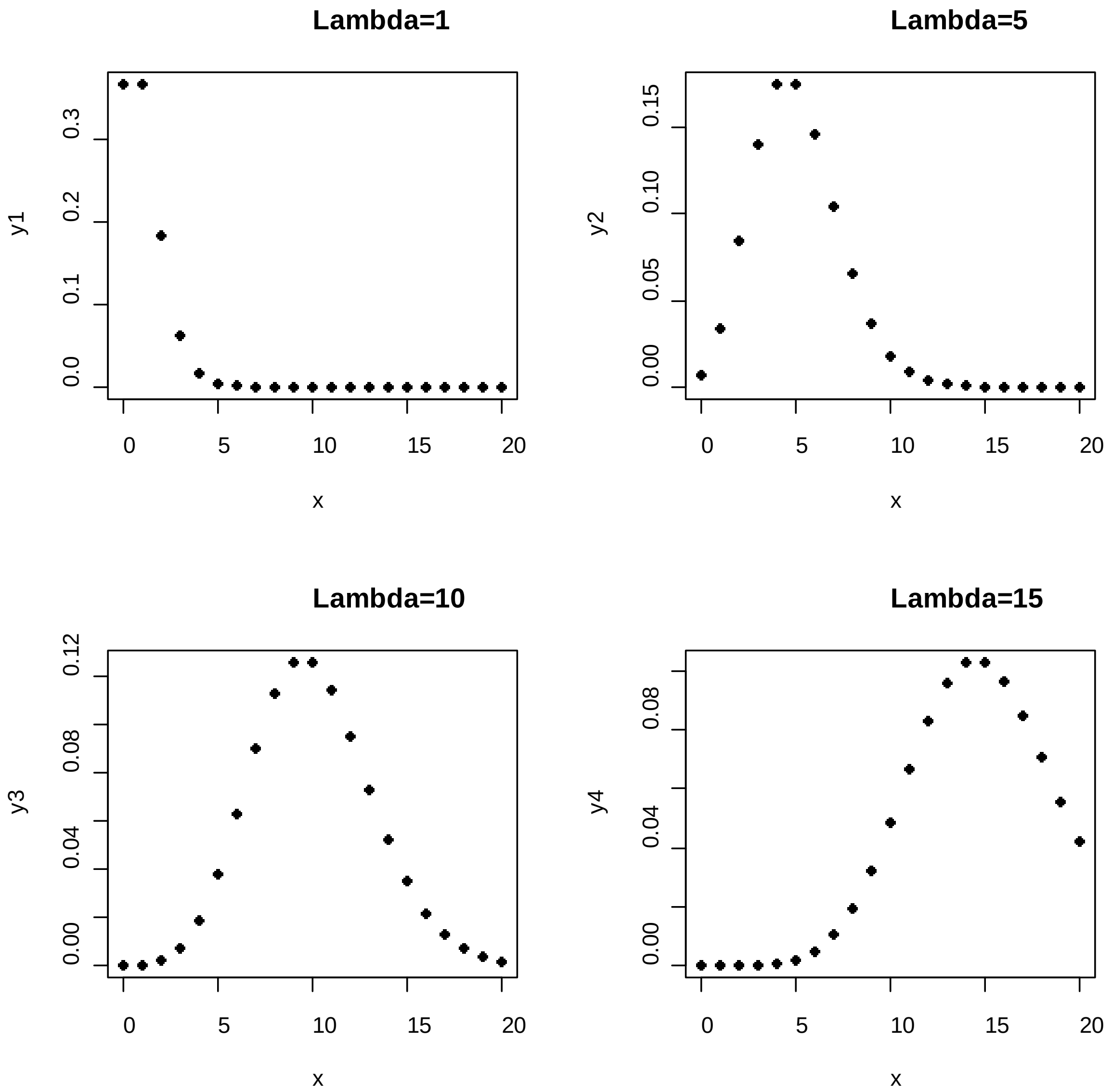
- The probability of the event occurring is very small,
- The number of trials is very large so that the event actually occurs a few times.
- The expected number of events occurring in the time interval is μ (often called λ).

Probability Density Function:

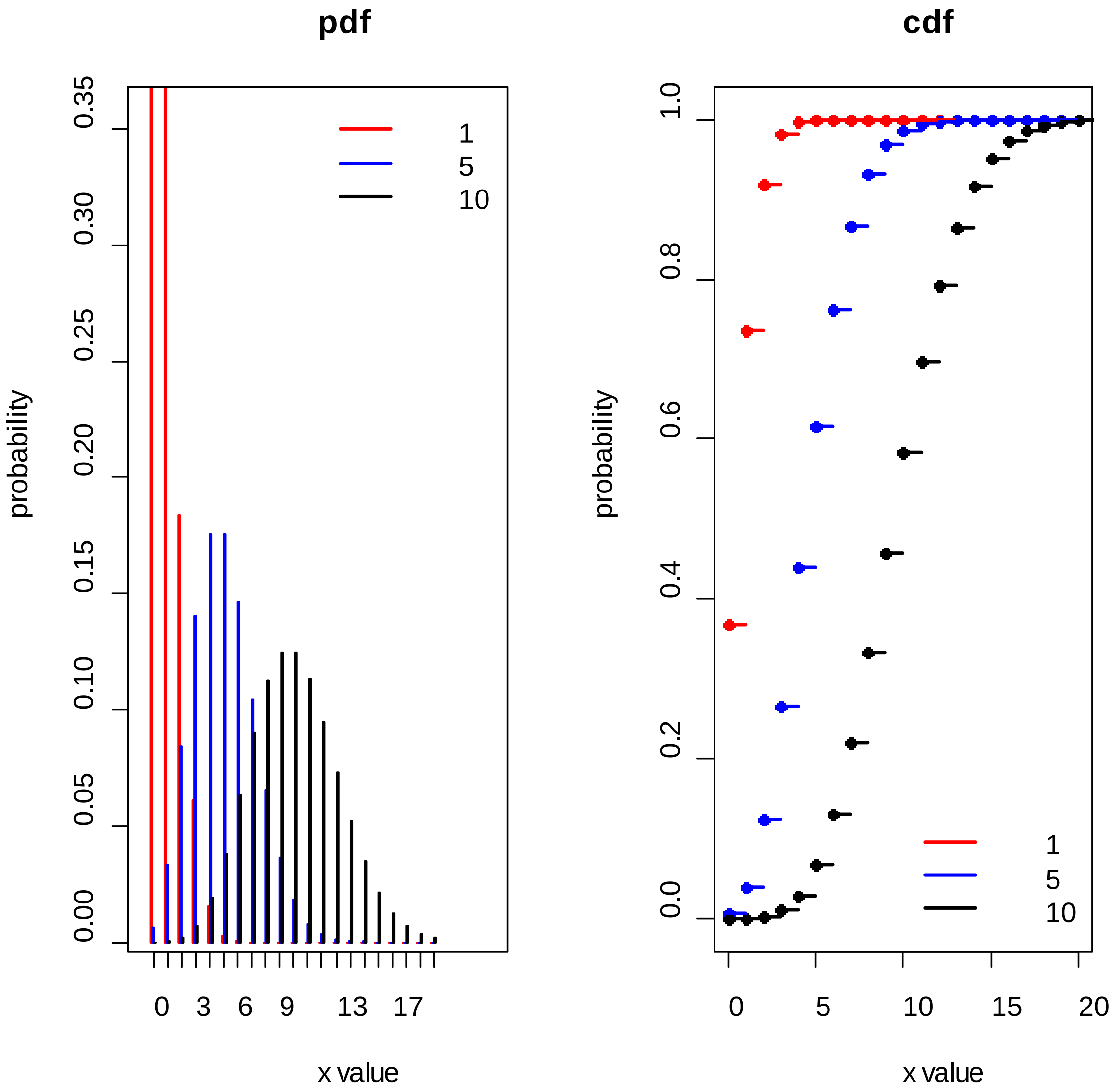
$$p(X = x) = \frac{\lambda^x e^{-\lambda}}{x!}, x = 1,2,3, \ldots, \lambda > 0 \textit{ is the parameter}$$

$$= \sigma^2 = \lambda = p * n$$

Poisson Distribution



The Poisson distribution



The Poisson distribution

```r
#Poisson distribution
x<-0:20
y1<-dpois(x,lambda=1)
y2<-dpois(x,lambda=5)
y3<-dpois(x,lambda=10)
z1<-ppois(x,lambda=1)
z2<-ppois(x,lambda=5)
z3<-ppois(x,lambda=10)
cols<-c("red","blue","black")
labs<-c(expression(lambda==1,lambda==5,lambda==10))
#expression: Creates or tests for objects of mode "expression".

par(mfrow=c(1,2))
barplot(rbind(y1,y2,y3),0.2,col=cols,border=cols,xlim=c(0,40),
space=c(0,5),axis.lty=1,names.arg=c(0:20),beside=T,xlab="x value",
ylab="probability",main="pdf")
box() #draws a box around the current plot in the given color and linetype
legend("topright",inset=0.01,labs,lwd=2,col=cols,box.col="white")
plot(rbind(x,x,x),rbind(z1,z2,z3),pch=16,col=cols,xlab="x value",
ylab="probability",main="cdf")
segments(x,z1,x+1,z1,col=cols[1],lwd=2)
#segments: Draw line segments between pairs of points.
segments(x,z2,x+1,z2,col=cols[2],lwd=2)
segments(x,z3,x+1,z3,col=cols[3],lwd=2)
legend("bottomright",inset=0.01,labs,lwd=2,col=cols,box.col="white")
```

Poisson Distribution: Example Auto Parts

The probability of a defective auto part is 0.001, in a shipment of 1,000 auto parts. What is the probability that there is one defective part? Please give the parameters and the formulas for the binomial and the Poisson distribution.

Binomial: n=1,000, p=0.1, P(X=1)=

$P(X=x)=\binom{n}{x} * \ p^x * (1-p)^{n-x} = P(X=1) = \binom{1{,}000}{1} * \ 0.001^1 * (1-0.001)^{999} =$
$dbinom(1{,}1000{,}0.001) = 0.3681$

This can be approximated by $\lambda = n * p = 1{,}000 * 0.001 = 1$.

The prob. that there is one defective part:

$$P(X=x)=\frac{\lambda^x e^{-\lambda}}{x!} = P(X=1) = \frac{1^1 e^{-1}}{1!} = \exp(-1) = 0.3679 = dpois(1{,}1).$$

The prob. that there is no defective part:

$$P(X=x)=\frac{\lambda^x e^{-\lambda}}{x!} = P(X=0) = \frac{1^0 e^{-1}}{0!} = \exp(-1) = 0.3679 = dpois(0{,}1).$$

The prob. that there is at least one defective part:

$$P(X \geq 1) = 1 - P(X < 1) = 1 - P(X < 0) = 1 - 0.3679 = 0.6321.$$

Continuous Random Variables

- For a continuous random variable X, a probability is a function such that

$$f(x) \geq 0$$

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

$$P(a \leq X \leq b) = \int_{a}^{b} f(x)dx$$

area under f(x) from a to b.

- Mean or expected value: $\mu = \int_{-\infty}^{\infty} xf(x)dx$

- Variance:

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$$

Continuous Random Variables: Example

Let the continuous random variable X denote the current measured in a thin copper wire in milliamperes. Assume that the range of X is [0,20 mA], and assume that the probability density function of X is f(x)=0.05 for 0 x 20. What is the probability that a current measurement is less than 10 milliamperes?

$$\mathrm{P(X < 10)} = \int_0^{10} xf(x)dx = \int_0^{10} 0.05dx = 0.05x|_0^{10} = 0.5$$

Mean:

$$\mu = \int_{-\infty}^{\infty} xf(x)dx = \int_0^{20} x0.05dx = \frac{0.05}{2}x^2\Big|_0^{20} = 10$$

Variance:

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2 = \frac{0.05}{3}x^3\Big|_0^{20} - 100$$

$$= 0.05 * 20 * \frac{20}{3} * 20 - 100 = 133.3333 - 100 = 33.3333, \sigma = 5.7735$$

Normal Distribution - Introduction

- The normal distribution, also called the Gaussian distribution, is an important family of continuous probability distributions, applicable in many fields.
- It is defined by two parameters, location and scale: the mean ("average", μ) and variance (standard deviation squared, σ^2) respectively.
- The standard normal distribution is the normal distribution with a mean of zero and a variance of one.
- The normal distribution is often called the bell curve because the graph of its probability density resembles a bell.

Normal Distribution – pdf, cdf

- Probability Mass Function / Probability Density Functio):

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left[\frac{x-\mu}{\sigma}\right]^2}$$

- Cumulative Probability Function:

$$F(X) = \int_{-\infty}^{+\infty} \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) dx = 1$$

Standard Normal Distribution

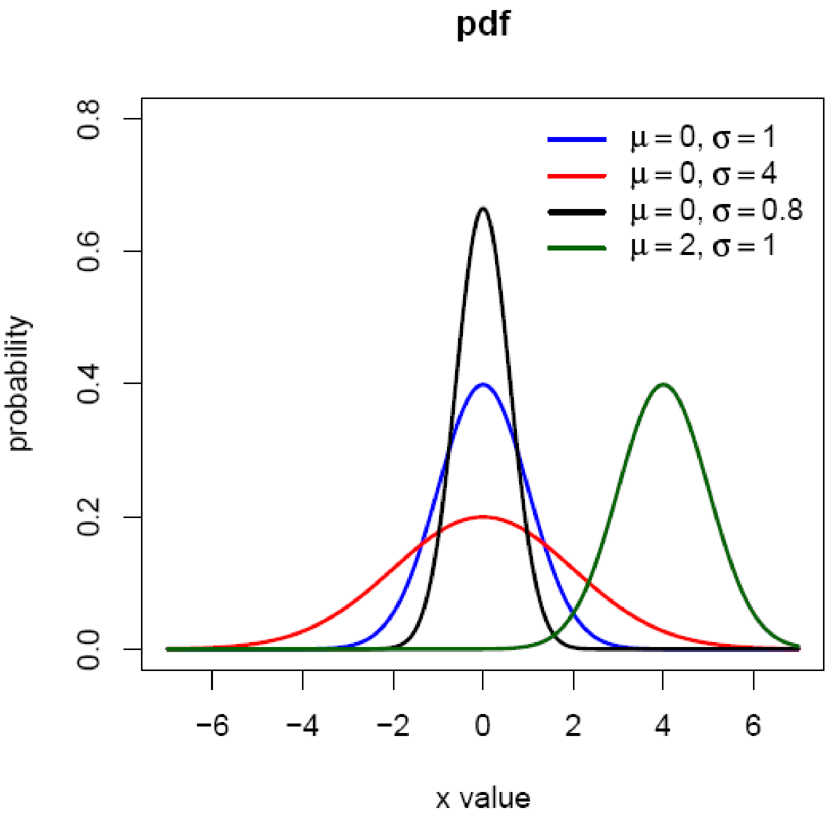
- The normal distribution with mean 0 and standard deviation 1: N(0,1), is called the standard normal distribution.

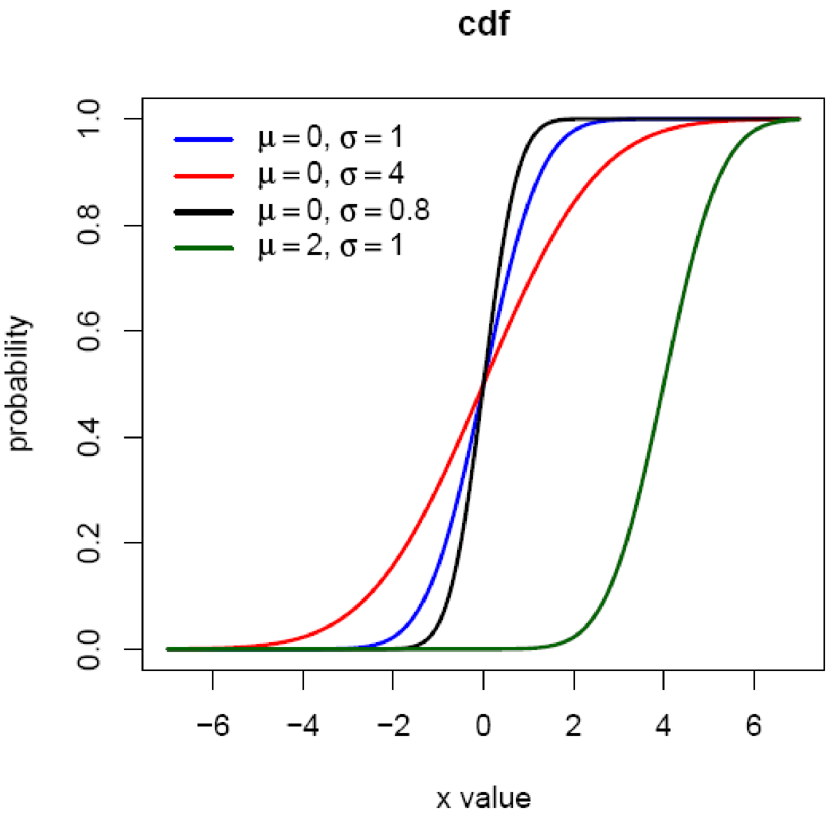
The following steps are necessary for standardising normal prob. distributions:

- Shift the axis of symmetry to the point of zero (subtract μ from each x), the shape of the distributions remains unchanged by this operation.
- Standardisation of the shape: Change the scale of the x-axis to a z-axis, by dividing the x-values by σ.
- As a result the original x-values were changed to standardised z-values.

$$Z = \frac{X - \mu}{\sigma}$$

The univariate Normal distribution





The univariate Normal distribution

```
> par(mfrow = c(1, 2))
> cols <- c("blue", "red", "black", "darkgreen")
> labs <- c(expression(paste(mu == 0, ", ", sigma == 1)), expression(paste(mu ==
+     0, ", ", sigma == 4)), expression(paste(mu == 0, ", ", sigma == 0.8)), expression(paste(mu ==
+     2, ", ", sigma == 1)))
> x <- seq(-7, 7, 0.05)
> plot(x, dnorm(x, mean = 0, sd = 1), type = "l", col = cols[1], lwd = 2, ylim = c(0,
+     0.8), xlab = "x value", ylab = "probability", main = "pdf")
> lines(x, dnorm(x, mean = 0, sd = 2), col = cols[2], lwd = 2)
> lines(x, dnorm(x, mean = 0, sd = 0.6), col = cols[3], lwd = 2)
> lines(x, dnorm(x, mean = 4, sd = 1), col = cols[4], lwd = 2)
> legend("topright", inset = 0.01, labs, lwd = 2, col = cols, box.col = "white")
> plot(x, pnorm(x, mean = 0, sd = 1), type = "l", col = cols[1], lwd = 2, xlab = "x value",
+     ylab = "probability", main = "cdf")
> lines(x, pnorm(x, mean = 0, sd = 2), col = cols[2], lwd = 2)
> lines(x, pnorm(x, mean = 0, sd = 0.6), col = cols[3], lwd = 2)
> lines(x, pnorm(x, mean = 4, sd = 1), col = cols[4], lwd = 2)
> legend("topleft", inset = 0.01, labs, lwd = 2, col = cols, box.col = "white")
```