Recitation 6

Question 1

Consider an investor who maximizes expected utility and has the utility function

$$u(x) = \sqrt{x}$$

(a) This investor is offered a choice between receiving \$1,000 for sure and lottery A, which pays either \$1,100 or \$900 with equal probabilities. Which of the two choices does the investor?

(b) The investor is offered a choice between lottery A, which pays either \$1,100 or \$900 with equal probabilities, and lottery B, which pays either \$1,250 or \$750 with equal probabilities. Which of the two choices does the investor prefer?

Solutions:

(a) This investor is offered a choice between receiving \$1,000 for sure and lottery A, which pays either \$1,100 or \$900 with equal probabilities. Which of the two choices does the investor?

Let us write down the investor's expected utility of the lottery A:

$$\begin{aligned} \mathrm{E}[\ u(\text{ Lottery A })\ ] = \frac{1}{2} \times u(\ \$1{,}100\ ) + \frac{1}{2} \times u(\ \$900\ ) = \\ \frac{1}{2} \times \sqrt{1,100} + \frac{1}{2} \times \sqrt{900} = 31.58 \end{aligned}$$

The investor's (expected[1]) utility of \$1,000 for sure is

$$u(\ \$1{,}000\ ) = 31.62.$$

The utility of the certain amount of cash is larger, hence that is the dominant choice of the investor.

[1] Note that for the certain case utility and expected utility are the same things.

(b) The investor is offered a choice between lottery A, which pays either \$1,100 or \$900 with equal probabilities, and lottery B, which pays either \$1,250 or \$750 with equal probabilities. Which of the two choices does the investor prefer?

The expected utility of the lottery B:

$$\begin{aligned}\mathrm{E}[\ u(\text{ Lottery B })\ ] = \frac{1}{2} \times u(\ \$1{,}250\ ) + \frac{1}{2} \times u(\ \$750\ ) = \\ \frac{1}{2} \times \sqrt{1,250} + \frac{1}{2} \times \sqrt{750} = 31.37\end{aligned}$$

Note that the expected payoffs of all three options we considered in the problem are the same. Indeed,

$$\mathrm{E}[\ \text{Payoff(Lottery A)}\ ] = \frac{1}{2} \times \text{Payoff(\$1,100)} + \frac{1}{2} \times \text{Payoff(\$900)} = \$1000$$

$$\mathrm{E}[\ \text{Payoff(Lottery B)}\ ] = \frac{1}{2} \times \text{Payoff(\$1,250)} + \frac{1}{2} \times \text{Payoff(\$750)} = \$1000$$

So, why does the investor prefer Lottery A over Lottery B, and \$1,000 for sure over Lottery A? The intuition behind is that assets A and B are risky, and asset B is riskier. The investor with concave utility function $u(x)$ does not like the uncertainty and risk preferring the least risky option.

Question 2

You are given the following lottery:

Probabilty	Payoff
0.1	$\$80,000$
0.3	$\$20,000$
0.6	$\$5,000$

(a) If the investor has utility function $u(x) = x$, what is the riskless payoff that would make her indifferent between choosing the lottery and the riskless payoff?

(b) What if investor has utility function $u(x) = x^{1/2}$

(c) What if investor has utility function $u(x) = x^{1/3}$

Solution:

In this question, we will learn to find Certainty equivalent for a given investment opportunity. Let us start with the definition.

Definition: Certainty equivalent (CE) is the smallest riskless payoff amount that makes an investor indifferent between taking a risky lottery or accepting this certain amount.

$$u(CE) = E[u(x)]$$

Let us discuss the main steps that will help us to find certainty equivalent for an investor with a given utility function and a given lottery:

Step 1: Find expected utility of investor from this lottery.

Step 2: Find CE by solving

$$u(CE) = E[u(x)]$$

with the given right hand side, or inverting the utility function and solving:

$$CE = u^{-1}(E[u(x)]).$$

We are already familiar with Step 1 from the previous problem. The second step requires us to find the inverse function of the utility function. Let us see how it works in the suggested examples.

(a) If the investor has utility function $u(x) = x$, what is the riskless payoff that would make her indifferent between choosing the lottery and the riskless payoff?

Step 1: Find expected utility of investor from this lottery.

$$\text{E}[\ u(\text{ Lottery })\ ] = 0.1 \times u(\ \$80{,}000\ ) + 0.3 \times u(\ \$20{,}000\ ) + 0.6 \times u(\ \$5{,}000\ ) =$$
$$0.1 \times\ 80{,}000\ + 0.3 \times\ 20{,}000\ + 0.6 \times\ 5{,}000\ = 17,000$$

Step 2: Find CE

$$u(CE) = 17,000.$$

Given the utility function $u(x) = x$ that equation means $CE = 17,000$.

Therefore, this investor is indifferent between taking \$17,000 and playing this lottery.

(b) What if investor has utility function $u(x) = x^{1/2}$

Step 1: Find expected utility of investor from this lottery.

$$\text{E}[\ u(\text{ Lottery })\ ] = 0.1 \times u(\ \$80{,}000\ ) + 0.3 \times u(\ \$20{,}000\ ) + 0.6 \times u(\ \$5{,}000\ ) =$$
$$0.1 \times (\ 80{,}000\ )^{1/2} + 0.3 \times (\ 20{,}000\ )^{1/2} + 0.6 \times (\ 5{,}000\ )^{1/2} = 113.14$$

Step 2: The inverse of utility function is $u^{-1}(y) = y^2$. Hence,

$$CE = u^{-1}(E[u(\text{Lottery})]) = (113.14)^2 = \$12,800$$

Hence, this investor is indifferent between taking \$12,800 and playing this lottery.

(c) What if investor has utility function $u(x) = x^{1/3}$

Step 1: Find expected utility of investor from this lottery.

$$\text{E}[\ u(\text{ Lottery })\ ] = 0.1 \times u(\text{ \$80,000 }) + 0.3 \times u(\text{ \$20,000 }) + 0.6 \times u(\text{ \$5,000 }) =$$
$$0.1 \times (\ 80{,}000\ )^{1/3} + 0.3 \times (\ 20{,}000\ )^{1/3} + 0.6 \times (\ 5{,}000\ )^{1/3} = 22.71$$

Step 2: The inverse of utility function is $u^{-1}(y) = y^3$. Hence,

$$CE = u^{-1}(E[u(\text{Lottery})]) = (22.71)^3 = \$11715.61$$

Let us summarize the problem. We were given three investors. Investor A was Risk-neutral, because she had a linear function. Both investors B and C are risk-averse since they had concave utility function. Investors C is more risk averse than investor B[2]. Therefore, if all three investors are presented to the same risky lottery, certainty equivalent for the lottery will be the highest for agent A and the lowest for agent C.

Question 3

Consider two investors: A and B. Investor A has a utility function $u(x) = x^{1/3}$ and investor B has a utility function of $u(x) = 1 - e^{-x/3}$ There is an investment opportunity that has the following property: With probability 0.5 it returns +25% and with probability 0.5 it returns −25%.

(a) Assume that both investors have initial wealth of \$1.

(1) Compute coefficient of relative risk aversion for both investors, evaluated at their initial wealth.

(2) For both investors, compute risk premium associated with the above investment opportunity.

(b) Assume that both investors have initial wealth of \$5.

(1) Compute coefficient of relative risk aversion for both investors, evaluated at their initial wealth.

(2) For both investors, compute risk premium associated with the above investment opportunity.

[2] Agent C has more concave utility function. That can be shown by taking the second derivative of the utility functions.

Solutions:

(a) Assume that both investors have initial wealth of \$1.

(1) Compute coefficient of relative risk aversion for both investors, evaluated at their initial wealth.

Let us start with the computation of the coefficient of relative risk aversion (RRA):

$$RRA(W) = -\frac{Wu''(W)}{u'(W)}$$

Using utility function of Investor A: $u(x) = x^{1/3}$, we can find its first and second derivatives:

$$u'(W) = \frac{1}{3}W^{-2/3} \qquad u''(W) = -\frac{1}{3}\frac{2}{3}W^{-5/3}$$

Therefore, the coefficient of relative risk aversion (RRA) is

$$RRA(W) = -\frac{Wu''(W)}{u'(W)} = \frac{W\frac{1}{3}\frac{2}{3}W^{-\frac{5}{3}}}{\frac{1}{3}W^{-\frac{2}{3}}} = \frac{2}{3}.$$

The coefficient of relative risk aversion of investor A is $2/3$. Repeating the same math for investor B with utility function $u(x) = 1 - e^{-x/3}$, we get

$$u'(W) = \frac{1}{3}e^{-W/3} \qquad u''(W) = -\frac{1}{3}\frac{1}{3}e^{-W/3}$$

RRA for Investor B:

$$RRA(W) = -\frac{Wu''(W)}{u'(W)} = \frac{W\frac{1}{3}\frac{1}{3}e^{-\frac{W}{3}}}{\frac{1}{3}e^{-\frac{W}{3}}} = \frac{1}{3}W.$$

Given $W = 1$, the coefficient of relative risk aversion for investor B is $1/3$. Hence, investor A has a higher RRA coefficient, $2/3$, compared to investor B, $1/3$, when their wealth is \$1.

(2) For both investors, compute risk premium associated with the above investment opportunity.

The risk premium π is the fraction of wealth that an investor is willing to lose that makes her indifferent between a risky zero expected return investment x and sure loss π

$$E[u(W(1+x))] = u(W(1-\pi))$$

Note that, an alternative definition of the risk premium would be π such that

$$CE = W(1-\pi) \qquad \pi = 1 - \frac{CE}{W}.$$

Recall the investment opportunity given to us:

$$\begin{cases} +25\%, & p = 1/2 \\ -25\%, & p = 1/2 \end{cases}$$

First, compute investors A's expected utility from investing in this project:

$$E[u(W(1+x))] = \frac{1}{2}u(\$1 \times (1+25\%)) + \frac{1}{2}u(\$1 \times (1-25\%))$$

$$E[u(W(1+x))] = \frac{1}{2}1.25^{1/3} + \frac{1}{2}0.75^{1/3} = 0.993$$

Second, compute risk premium:

$$\begin{gathered} 0.993 = u(W(1-\pi)) \\ 0.993 = (1-\pi)^{1/3} \\ \pi = 2.12\% \end{gathered}$$

Given $W = \$1$, and this particular investment opportunity, investor A's risk premium is 2.12%.

Doing the same for investor B:

$$E[u(W(1+x))] = \frac{1}{2}u(\$1 \times (1+25\%)) + \frac{1}{2}u(\$1 \times (1-25\%))$$

$$E[u(W(1+x))] = \frac{1}{2}\left(1 - e^{-1.25/3}\right) + \frac{1}{2}\left(1 - e^{-0.75/3}\right) = 0.281$$

Risk premium for investor B:

$$0.281 = u(W(1-\pi))$$

$$0.281 = 1 - e^{-(1-\pi)/3}$$

$$\pi = 1.04\%$$

Given $W = \$1$, and this particular investment opportunity, investor B's risk premium is 1.04%.

Let us summarize the results in one table.

Investor	A	B
RRA	2/3	1/3
Risk premium	2.12%	1.04%

The main takeaway is that the investor with a higher coefficient of RRA (investor A in the problem) requires a higher risk premium. That is the common feature of the models which we study.

(b) Assume that both investors have initial wealth of \$5.

(1) Compute coefficient of relative risk aversion for both investors, evaluated at their initial wealth.

Doing again calculation for investor A:

$$RRA(W) = \frac{2}{3}$$

The coefficient of relative risk aversion of investor A is 2/3 and as we it is independent of the initial wealth of investor. In the situation we say that the investor has **constant RRA** — it does not depend on her wealth. Next, RRA for investor B is

$$RRA(W) = \frac{1}{3}W$$

Given $W = 5$, RRA of B is 5/3. Generally, investor B's risk aversion increases as she gets richer.

(2) For both investors, compute risk premium associated with the above investment opportunity.

Let us repeat the calculation for investor A. The expected utility is

$$E[u(W(1+x))] = \frac{1}{2}u(\$5 \times (1+25\%)) + \frac{1}{2}u(\$5 \times (1-25\%))$$

$$E[u(W(1+x))] = \frac{1}{2}6.25^{1/3} + \frac{1}{2}3.75^{1/3} = 1.698$$

Risk premium for A is

$$1.698 = u(5 \times (1-\pi))$$

$$1.698 = [5 \times (1-\pi)]^{1/3}$$

$$\pi = 2.12\%$$

Given $W = \$5$ and this particular investment opportunity, investor A's risk premium is again 2.12%.

For investor B:

$$E[u(W(1+x))] = \frac{1}{2}u(\$5 \times (1+25\%)) + \frac{1}{2}u(\$5 \times (1-25\%))$$

$$E[u(W(1+x))] = \frac{1}{2}\left(1 - e^{-6.25/3}\right) + \frac{1}{2}\left(1 - e^{-3.75/3}\right) = 0.794$$

$$0.794 = u(W(1-\pi))$$

$$0.794 = 1 - e^{-5\times(1-\pi)/3}$$

$$\pi = 5.06\%$$

Given $W = \$5$ and this particular investment opportunity, investor B's risk premium is 5.06%. Let us summarize the results in one table.

Investor	**A**	**B**
Wealth \$1		
RRA	2/3	1/3
Risk premium	2.12%	1.04%
Wealth \$5		
RRA	2/3	5/3
Risk premium	2.12%	5.06%

Two main takeaways here. The first point is the same as before: the investor with a higher coefficient of RRA (investor A in (a) and investor B in (b)) requires a higher risk premium. The second point is that if an investor has a constant RRA utility then she will require the same risk premium independent of her initial wealth. Meanwhile, if an investor has an increasing in wealth RRA utility function then she will require the higher risk premium when she begins to be richer.

Question 4

Consider three ETFs:

(1) Vanguard Total Stock Market ETF (ticker symbol: VTI)

(2) iShares 20+ Year Treasury Bond ETF (ticker symbol: TLT)

(3) Vanguard FTSE Emerging Markets ETF (ticker symbol: WWO)

You are given monthly returns on these ETFs from January 2010 until December 2019. Answer the following questions:

(a) Compute the sample mean of monthly returns on each ETF: VTI, TLT, VWO.

(b) Compute the sample standard deviation of monthly returns on each ETF.

(c) Compute pairwise sample correlations between each ETF.

(d) If an investor wanted to diversify a pure stock investment (100% VTI portfolio), would you recommend using TLT or VWO?

Solutions:

This problem is solved in Excel and the solution is provided in the video.

Question 5

Consider two portfolios, P and Q.

Portfolio P has invested \$25,000 into SPDR S&P 500 ETF (ticker symbol: SPY), \$35,000 into Vanguard's short-term treasury ETF (ticker symbol: VGSH), and \$15,000 into emerging markets equities through iShares ETF (ticker symbol: IEMG).

Portfolio Q has invested \$10,000 into SPY, \$10,000 into VGSH, \$30,000 into IEMG, and \$20,000 into iShares mortgage-backed securities ETF (ticker symbol: MBB).

(a) Compute portfolio weights on each security in portfolios P and Q.

(b) Suppose that positions in portfolios P and Q get combined into portfolio R.

Compute portfolio weights of each security in the portfolio R.

Solutions:

(a) Compute portfolio weights on each security in portfolios P and Q.

The total value of portfolio P is

$$TV_P = \$25,000 + \$35,000 + \$15,000 = \$75,000.$$

The total value of portfolio Q is

$$TV_Q = \$10,000 + \$10,000 + \$30,000 + \$20,000 = \$70,000.$$

The weights on a security X in portfolio Y is

$$w_X^Y = \frac{\text{Value of } X \text{ in portfolio } Y}{TV_Y}.$$

Hence,

$$w_{SPY}^P = \frac{\$25,000}{\$75,000} = \frac{1}{3} = 0.33$$

$$w_{VGSH}^P = \frac{\$35,000}{\$75,000} = \frac{7}{15} = 0.47$$

$$w_{IEMG}^P = \frac{\$15,000}{\$75,000} = \frac{1}{5} = 0.2$$

Let us verify. that the weights sum up to one as the definition of weights require:

$$w_{SPY}^P + w_{VGSH}^P + w_{IEMG}^P = 0.33 + 0.47 + 0.2 = 1$$

Similar calculations for portfolio Q give us

$$w_{SPY}^Q = \frac{\$10,000}{\$70,000} = 0.14$$

$$w_{VGSH}^Q = \frac{\$10,000}{\$70,000} = 0.14$$

$$w_{IEMG}^Q = \frac{\$30,000}{\$70,000} = 0.43$$

$$w_{MBB}^Q = \frac{\$20,000}{\$70,000} = 0.29$$

$$w_{SPY}^Q + w_{VGSH}^Q + w_{IEMG}^Q + w_{MBB}^Q = 1$$

(b) Suppose that positions in portfolios P and Q get combined into portfolio R.

Compute portfolio weights of each security in the portfolio R.

To find the weights of every asset X in the portfolio R we will use the following formula,

$$w_X^R = \frac{\text{Value of } X \text{ in portfolio } P + \text{Value of } X \text{ in portfolio } Q}{TV_P + TV_Q}$$

Note that the total value of the portfolio R is

$$TV_R = TV_P + TV_Q = \$75,000 + \$70,000 = \$145,000.$$

The weights of the assets are

$$w^R_{SPY} = \frac{\$25,000 + \$10,000}{\$145,000} = 0.24$$

$$w^R_{VGSH} = \frac{\$35,000 + \$10,000}{\$145,000} = 0.31$$

$$w^R_{IEMG} = \frac{\$15,000 + \$30,000}{\$145,000} = 0.31$$

$$w^R_{MBB} = \frac{\$20,000}{\$145,000} = 0.14.$$

We can easily verify that the weights actually add up to one:

$$w^R_{SPY} + w^R_{VGSH} + w^R_{IEMG} + w^R_{MBB} = 1.$$

Question 6

Suppose that you have $20,000 that you would like to invest. You have very positive views on the future of clean energy, and therefore you would like to invest all $20,000 into iShares Global Clean Energy ETF (ticker symbol: ICLN). In addition, your broker allows you to borrow up to $45,000. You plan to borrow the full amount and invest it into ICLN.

(a) What is the weight of ICLN in your portfolio?

(b) What is the leverage ratio of your portfolio?

(c) If ICLN value increases by 25%, what is the return of your portfolio?

(d) If ICLN value declines by 25%, what is the return of your portfolio?

Solutions:

(a) What is the weight of ICLN in your portfolio?

By definition, the weight of the ICLN in your portfolio will be the sum of your initial wealth and borrowed amount divided by your total wealth, or net investment[3], often referred[4] as the equity of your portfolio.

$$w_{ICLN} = \frac{\$20,000 + \$45,000}{\$20,000} = 325\%$$

[3] the total assets net of the debt

[4] in the context

We can also compute the weight on your borrowed funds or your loan in the portfolio,

$$w_{Loan} = \frac{-\$45,000}{\$20,000} = -225\%$$

Note that the weights of your portfolio add up to 100% as they should.

(b) What is the leverage ratio of your portfolio?

Let us start with the definition of the leverage ratio. The leverage ratio as the ratio of your total assets to your net investment:

$$\text{Leverage Ratio} = \frac{\text{Asset Value}}{\text{Net Investments}} = \frac{\$65,000}{\$20,000} = 3.25$$

The net investment is your own wealth or equity of your portfolio. Note that the fact that the leverage is higher than 1 identifies that you borrow money to construct your portfolio.

(c) If ICLN value increases by 25%, what is the return of your portfolio?

First, let us compute the total asset value after the growth

$$\text{Asset Value} = \$65,000 \times (1 + 25\%) = \$81,250.$$

Now, your net investment is

$$\text{Net Investment} = \$81,250 - \$45,000 = \$36,250$$

Finally, the return of the portfolio is the percentage change for the net investment:

$$\text{Return} = \frac{\$36,250 - \$20,000}{\$20,000} = 81.25\%$$

Let us also discuss the alternative way to get the same result. The return, that can be called in the case **Leveraged Investment Return**, must be equal to return on the asset scaled by the leverage ratio:

$$\text{Leveraged Investment Return} = (\text{Leverage Ratio}) \times (\text{Asset Return}) = 3.25\% \times 25\% = 81.25\%.$$

(d) If ICLN value declines by 25%, what is the return of your portfolio?

Again, let us start with computing the total asset value after the growth

$$\text{Asset Value} = \$65,000 \times (1 - 25\%) = \$48,750.$$

Now, your net investment is

$$\text{Net Investment} = \$48,750 - \$45,000 = \$3,750$$

Therefore, the return of the portfolio is the percentage change for the net investment:

$$\text{Return} = \frac{\$3,750 - \$20,000}{\$20,000} = -81.25\%$$

We could also get the same result applying the formula,

$$\text{Leveraged Investment Return} = (\text{Leverage Ratio}) \times (\text{Asset Return}) = 3.25\% \times -25\% = -81.25\%.$$

The takeaway of the problem that leverage position allows us to get a higher return than what we would get if investing in the asset alone if the underlying appreciates in the value. Similarly, if the underlying's price decreases, then the leverage position can hurt, and the investment loses much more value than investing in the asset alone.

In finance, we also say that the leverage position allows us to **get higher exposure** to the return of the underlying asset.

Question 7

There are two stocks, A and B, with the following expected returns and standard deviations:

Stock	Expected return	Standard deviation
A	20%	20%
B	15%	25%

The correlation between the return on A and B is 0.2.

(a) Stock A has higher expected return and lower standard deviation. Would anyone ever invest in stock B?

(b) What is the expected return of a portfolio that invests 60% in A and 40% in B?

(c) What is the standard deviation of a portfolio that invests 60% in A and 40% in B?

(d) What happens to the expected return and standard deviation of 60%/40% portfolio if the correlation between the return of A and B increases to 0.6?

(e) Suppose you would like to construct a portfolio with an expected return of 19.5%. What weight would you put on stocks A and B?

(f) What is the standard deviation of the portfolio constructed in part (e)?

Solutions:

(a) Stock A has higher expected return and lower standard deviation. Would anyone ever invest in stock B?

At first glance, one can think that an investor should never invest in stock B since it seems that investing in stock A is a dominating strategy: it has both a higher expected return and lower volatility. That is actually the wrong logic. The answer is that some investors may prefer a combination of those two stocks. The reason is that stock B provides diversification benefits. We will discuss them in the next parts of the problem.

(b) What is the expected return of a portfolio that invests 60% in A and 40% in B? Let us denote the portfolio by P. To find the expected return of the

portfolio let us use the formula

$$E[R_P] = w_A E[R_A] + w_B E[R_B].$$

That says that the expected return of portfolio is just a weighted average of the expected returns of its components.

$$E[R_P] = 60\% \times 20\% + 40\% \times 15\% = 18\%.$$

Hence, the expected return of the portfolio is 18% which is between the expected returns of stocks A and B.

(c) What is the standard deviation of a portfolio that invests 60% in A and 40% in B?

To find the standard deviation of the portfolio, also called **volatility**, can be found by taking the square root of the portfolio's variance. To find the variance of the portfolio, we must use the following formula:

$$Var[R_P] = w_A^2 \times Var[R_A] + w_B^2 \times Var[R_B] + 2w_A w_B \times Cov(R_A, R_B).$$

or

$$Var[R_P] = w_A^2 \times Var[R_A] + w_B^2 \times Var[R_B] + 2w_A w_B \times SD[R_A] \times SD[R_B] \times Corr(R_A, R_B).$$

Making the calculations, we compute the variance,

$$Var[R_P] = 0.6^2 \times 0.2^2 + 0.4^2 \times 0.25^2 + 2 \times 0.6 \times 0.4 \times 0.2 \times 0.25 \times 0.2 = 0.0292.$$

Finally, to get the portfolio's volatility, we take the square root:

$$SD[R_P] = \sqrt{Var[R_P]} = \sqrt{0.0292} = 17.1\%$$

Note that though this portfolio has a lower return than stock A it also has lower volatility than the stock. Hence, investors with high risk-aversion might prefer portfolio P over stock A, and thus they will invest part of their wealth in stock B.

(d) What happens to the expected return and standard deviation of 60%/40% portfolio if the correlation between the return of A and B increases to 0.6?

Let us see how the correlation's change will affect the expected return and volatility of the portfolio. First, since the correlation does not enter the expected return formula, i.e., there is no dependence between the expected return and correlation, $E[R_p] = 18\%$ as before.

Next, the variance formula is actually affected by the correlation between the stocks. Repeating the calculations, we compute the new variance,

$$Var[R_P] = 0.6^2 \times 0.2^2 + 0.4^2 \times 0.25^2 + 2 \times 0.6 \times 0.4 \times 0.2 \times 0.25 \times \mathbf{0.6} = 0.0388.$$

$$SD[R_P] = \sqrt{Var[R_P]} = \sqrt{0.0388} = 19.7\%$$

Thus, once the stocks start being more correlated, the portfolio's standard deviation increases from 17.1% to 19.7%. The intuition behind is that stock B has a smaller diversification effect once the assets become more correlated.

(e) Suppose you would like to construct a portfolio with an expected return of 19.5%. What weight would you put on stocks A and B?

Let us call the portfolio Q. First, the requirement on the expected return pins down the left hand side of the following equation:

$$E[R_Q] = w_A^Q E[R_A] + w_B^Q E[R_B].$$

where w_A^Q and w_B^Q are weights of stocks A and B in the portfolio, respectively. Next, the weights of the portfolio must add up to 1. Hence, we can get

$$w_A^Q + w_B^Q = 1 \qquad w_B^Q = 1 - w_A^Q$$

Combining those two equations and plugging the known numbers we receive

$$19.5\% = w_A^Q \times 20\% + (1 - w_A^Q) \times 15\%.$$

That is a linear equation with one unknown which can be solved to

$$w_A^Q = 90\%$$

Hence, the weight of stock A in the portfolio is 90%, and the weight of stock B is

$$w_A^Q = 1 - 90\% = 10\%.$$

(f) What is the standard deviation of the portfolio constructed in Part (E)?

Here, we will repeat calculations from (b).

$$Var[R_Q] = w_A^{Q2} \times Var[R_A] + w_B^{Q2} \times Var[R_B] + 2 w_A^Q w_B^Q \times SD[R_A] \times SD[R_B] \times Corr(R_A, R_B).$$

$$Var[R_Q] = 0.9^2 \times 0.2^2 + 0.1^2 \times 0.25^2 + 2 \times 0.9 \times 0.1 \times 0.2 \times 0.25 \times 0.2 = 0.034825.$$

$$SD[R_Q] = \sqrt{Var[R_Q]} = 18.66\%$$

Question 8

Suppose that the average stock has a volatility of 35%, and that the correlation between pairs of stocks is 25%. Estimate the volatility of an equally weighted portfolio with

(a) 1 stock,

(b) 20 stocks,

(c) 500 stocks.

Solutions:

Let us first discuss the formula for the variance of the portfolio given the weights of stocks $\{w_i\}_{i=1}^n$, correlation between them $\{\rho_{i,j}\}_{i,j=1}^{n,n}$, and their volatilities $\{\sigma_i\}_{i=1}^n$:

$$Var[R_P] = \sum_{i=1}^n w_i^2 Var[R_i] + \sum_{i=1}^n \sum_{j\neq i}^n w_i w_j Cov[R_i, R_j].$$

or

$$Var[R_P] = \sum_{i=1}^n w_i^2 \sigma_i^2 + \sum_{i=1}^n \sum_{j\neq i}^n w_i w_j \rho_{ij} \sigma_i \sigma_j.$$

This formula comes from the product of the portfolio weights with the variance-covariance matrix of the assets (stocks),

$$\begin{bmatrix} Var[R_1] & Cov[R_1,R_2] & \cdots & Cov[R_1,R_n] \\ Cov[R_1,R_2] & Var[R_2] & \cdots & Cov[R_2,R_n] \\ \vdots & & & \\ Cov[R_1,R_n] & Cov[R_2,R_n] & \cdots & Var[R_n] \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho_{1,2}\sigma_1\sigma_2 & \cdots & \rho_{1,n}\sigma_1\sigma_n \\ \rho_{1,2}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2,n}\sigma_2\sigma_n \\ \vdots & & & \\ \rho_{1,n}\sigma_1\sigma_n & \rho_{2,n}\sigma_2\sigma_n & \cdots & \sigma_n^2 \end{bmatrix}$$

$$Var[R_P] = \begin{bmatrix} w_1 & w_2 & \cdot & \cdot & w_n \end{bmatrix} \begin{bmatrix} \sigma_1^2 & \rho_{1,2}\sigma_1\sigma_2 & \cdots & \rho_{1,n}\sigma_1\sigma_n \\ \rho_{1,2}\sigma_1\sigma_2 & \sigma_2^2 & \cdots & \rho_{2,n}\sigma_2\sigma_n \\ \vdots & & & \\ \rho_{1,n}\sigma_1\sigma_n & \rho_{2,n}\sigma_2\sigma_n & \cdots & \sigma_n^2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ w_n \end{bmatrix}.$$

Using that we have a very special structure of the returns with

1. the same average stock volatility, i.e, $Var[R_i] = \sigma_i^2 \equiv \sigma^2 = (35\%)^2$ for all i

2. the same average correlation between pairs of stock, i.e. $\rho_{i,j} = \rho = 25\%$ for all i and all $j \neq i$,

3. the same weight of all stocks in the portfolio, i.e. $w_i \equiv \frac{1}{n}$,

we get

$$Var[R_P] = \sum_{i=1}^{n} \left(\frac{1}{n}\right)^2 \times \sigma^2 + \sum_{i=1}^{n}\sum_{j\neq i}^{n} \left(\frac{1}{n}\right) \times \left(\frac{1}{n}\right) \times \rho \times \sigma \times \sigma,$$

i.e., the variance of the portfolio is just the sum of the same terms of average variance and average covariances for the stocks:

$$Var[R_P] = \frac{1}{n^2} \times \sum_{i=1}^{n} \underbrace{\sigma^2}_{\text{Avg Variance}} + \frac{1}{n^2} \times \sum_{i=1}^{n}\sum_{j\neq i}^{n} \underbrace{\rho \times \sigma^2}_{\text{Avg Covariance}}$$

The first consists of n similar terms. The second pair of sums consists of $n-1$ per each i terms, i.e., n times $n-1$ terms[5]:

$$Var[R_P] = \frac{1}{n^2} \times n \times \underbrace{\sigma^2}_{\text{Avg Variance}} + \frac{1}{n^2} \times n \times (n-1) \times \underbrace{\rho \times \sigma^2}_{\text{Avg Covariance}}$$

Finally, we get

$$Var[R_P] = \frac{1}{n} \times \underbrace{\sigma^2}_{\text{Avg Variance}} + \left(1 - \frac{1}{n}\right) \times \underbrace{\rho \times \sigma^2}_{\text{Avg Covariance}}$$

Notice that the portfolio's variance is the weighted average of average variance and average covariances for the stocks, where the weight of average variance, $1/n$, decreases when n grows. At the same time, the weight of average covariance, $1-1/n$, increases when n grows. In the limit, when n goes to infinity, the variance of the portfolio converges to the average covariance.

To finish the problem, we just need to plug the number of stocks in the formula.

$$Var[R_P] = \frac{1}{n} \times 0.35^2 + \left(1 - \frac{1}{n}\right) \times 0.25 \times 0.35^2$$

(a) $n = 1$ stock: $Var[R_P] = 0.35^2 = 0.1225,\ \ SD[R_P] = 35\%$

(b) $n = 20$ stocks:

$$Var[R_P] = \frac{1}{20} \times 0.35^2 + \left(1 - \frac{1}{20}\right) \times 0.25 \times 0.35^2 = 0.0352,$$

$$SD[R_P] = 18.77\%.$$

(c) $n = 500$ stocks: $Var[R_P] = 0.0308,\ \ SD[R_P] = 17.55\%$

[5] $n-1$ appears because for each fixed stock i there is $n-1$ other stocks, i.e., n stocks excluding one stock i.