Recitation 5

The Greeks

Recall from the Week 5 lecture that the "Greeks"—delta, gamma, theta, rho, and vega—are widely-used measures of the sensitivity of an option's price to various factors. In particular, **delta** measures the sensitivity of an option's price to changes in the price of the underlying security.

According to the Black-Scholes-Merton (BSM) model, the delta of a European option on a non-dividend-paying stock has the following analytical expression:

$$\Delta = \frac{dP}{dS} = \begin{cases} \mathcal{N}(d_1) & \text{for Calls} \\ -\mathcal{N}(-d_1) & \text{for Puts} \end{cases}$$

where P is the price of the option, $\mathcal{N}(\cdot)$ is the cumulative density function of a standard normal random variable, and $d_1 = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$. It is true more generally that call options will have positive deltas and put options will have negative deltas.

But what if we want to compute the delta of an option without invoking the BSM model? In the Week 5 lecture, we saw how to calculate the price of a European call option numerically using the binomial tree simulator in the BinomialTree.xls spreadsheet.

By varying the price of the underlying stock and recalculating the option price, we can numerically approximate the derivative $\frac{dP}{dS}$ using the formula:

$$\frac{dP}{dS} \approx \frac{\text{New Option Price} \; - \; \text{Original Option Price}}{\text{New Stock Price} \; - \; \text{Original Stock Price}}. \tag{1}$$

For example, let's use BinomialTree.xls to find the price of a European call option on a stock with the following parameters: $\mu = 0.1$, $\sigma = 0.3$, $r = 2\%$, dividend yield $= 0\%$, $S_0 = 100$, $K = 100$, $T = 0.1$, and $n = 10$:

BINOMIAL TREE MODEL

Stock Assumption		Option Assumption			Tree		Risk Neutral Prob	
mu	0.1	T	0.1		n	10	q*	0.495834
sigma	0.3	K	100		h	0.01		
r	0.02	Call or Put	1	(=1 for call	u	1.030455	Price Binomial	3.787
div	0				d	0.970446	Delta Binomial	0.527
S0	100				q	0.509173		

time ==>	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
i==>	0	1	2	3	4	5	6	7	8	9	10
0	100.000	103.045	106.184	109.417	112.750	116.183	119.722	123.368	127.125	130.996	134.986
1		97.045	100.000	103.045	106.184	109.417	112.750	116.183	119.722	123.368	127.125
2			94.176	97.045	100.000	103.045	106.184	109.417	112.750	116.183	119.722
3				91.393	94.176	97.045	100.000	103.045	106.184	109.417	112.750
4					88.692	91.393	94.176	97.045	100.000	103.045	106.184
5						86.071	88.692	91.393	94.176	97.045	100.000
6							83.527	86.071	88.692	91.393	94.176
7								81.058	83.527	86.071	88.692
8									78.663	81.058	83.527
9										76.338	78.663
10											74.082

The price of the European call option is 3.787. In the spreadsheet, we see that the price of the underlying stock in the "up" node at step $i = 1$ is $S_1^u = 103.045$, and the price in the "down" node at $i = 1$ is $S_1^d = 97.045$. The price of the call option at each step and node is displayed in the table below:

time ==>	0.000	0.010	0.020	0.030	0.040	0.050	0.060	0.070	0.080	0.090
i==>	0.0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0
0	3.787	5.381	7.440	9.982	12.965	16.283	19.802	23.428	27.165	31.016
1		2.221	3.358	4.943	7.053	9.707	12.830	16.243	19.762	23.388
2			1.103	1.801	2.871	4.445	6.640	9.477	12.790	16.203
3				0.418	0.749	1.323	2.289	3.851	6.224	9.437
4					0.092	0.185	0.373	0.753	1.520	3.065
5						0.000	0.000	0.000	0.000	0.000
6							0.000	0.000	0.000	0.000
7								0.000	0.000	0.000
8									0.000	0.000
9										0.000
10										

The price of the European call option in the "up" node at step $i = 1$ is $c_1^u = 5.381$, and the price in the "down" node at $i = 1$ is $c_1^d = 2.221$.

We can plug S_1^u and S_1^d into the numerator and c_1^u and c_1^d into the denominator of Equation (1) to numerically approximate delta as:

$$\frac{dP}{dS} \approx \frac{c_1^u - c_1^d}{S_1^u - S_1^d} = \frac{5.381 - 2.221}{103.045 - 97.045} = 0.527.$$

This is the same output as that reported by BinomialTree.xls in the "Delta Binominal" cell! Essentially, the binomial tree simulator calculates the delta of an option using the u and d implied by the parameterization provided, where $u = e^{\sigma \times \sqrt{h}}$ and $h = T/n$. Since $d = 1/u$, this approximation is akin to a *symmetric difference quotient*.

Alternatively, you can numerically approximate delta by using a baseline parameterization of the binomial tree to get the "original" stock and option prices in Equation (1), and then vary the stock price by a given amount in one direction—e.g., a 6% increase—to get the "new" stock and option prices. This is akin to a *one-sided difference quotient*.

As discussed in the Week 5 lecture, **gamma** measures the sensitivity of delta to changes in the price of the underlying security. According to the BSM model, the gamma of a European option on a non-dividend-paying stock has the following analytical expression:

$$\Gamma = \frac{d^2P}{dS^2} = \frac{d\Delta}{dS} = \frac{N'(d_1)}{S\sigma\sqrt{T}}$$

with $\mathcal{N}'(x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$. Mathematically, gamma measures the curvature of an option's price with respect to the price of the underlying security.

Can we numerically approximate gamma like we did for delta? Of course! Our numerical approximation formula now becomes:

$$\frac{d^2P}{dS^2} \approx \frac{\text{New Delta} - \text{Original Delta}}{\text{New Stock Price} - \text{Original Stock Price}}. \tag{2}$$

For example, assume the same parameterization for the European call option as above, with a current underlying stock price of $S_0 = 100$. Using BinomialTree.xls, we can calculate delta for a symmetric 3% increase and decrease in the stock price. For the 3% increase, we plug in $S_0 = 100 \times 1.03 = 103$ and find the delta of the call option to be 0.647. For the 3% decrease, we plug in $S_0 = 100 \times 0.97 = 97$ and find the delta of the call option to be 0.401.

Applying Equation (2) with the "original" values of delta and the stock price corresponding to the 3% decrease and the "new" values corresponding to the 3% increase, we have that

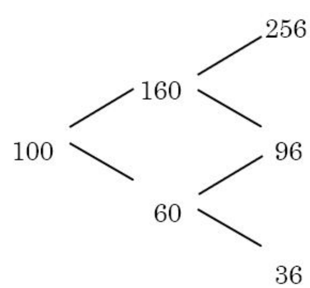
$$\frac{d^2P}{dS^2} \approx \frac{0.647 - 0.401}{103 - 97} = 0.041.$$

Exotic Option Example

In the Week 5 lecture, we saw how the binomial model can be used to price European call and put options. However, it can also be used to price more "exotic" options with complex payment structures.

For example, assume that a firm has decided to offer a new type of exotic equity option. The expiration date of the option is two periods from now. The option is not initially specified to be a put or a call. Instead, the owner makes this choice after one period. Once the choice is made, the option can be exercised at any time. For example, if after one period the owner chose for the new exotic option to be a put, it would at that time become identical to an ordinary American put with one period remaining until expiration.

A customer has asked for a quote for an option of this type on FA Inc. stock with a strike price of 100. The current price of FA Inc. stock is 100 per share, and over each period the stock price evolves as shown on the tree diagram below. The risk-neutral probability of an "up" move is 0.5. The stock does not pay dividends, and the interest rate is 10% per period. What is the lowest price the firm could charge and still break even?



Solution: Let $C(S, n)$ and $P(S, n)$ be the values of American call and put options when the underlying stock price is S and there are n periods remaining until maturity.

As always, we begin by finding the payoffs of the call and put options at each node by moving backwards through the tree. Two periods from now—i.e., when $n = 0$—the payoff of a call option will be $256 - 100 = 156$ in the "up-up" node and 0 in either the "up-down"/"down-up" or "down-down" nodes. Similarly, the payoff of a put option will be 0 in the "up-up" node, $100 - 96 = 4$ in the "up-down"/"down-up" nodes, and $100 - 36 = 64$ in the "down-down" node.

What would be the values of the call and put options in each node one period from now—i.e., when $n = 1$—given the payoffs in each node at $n = 0$, the risk-neutral probability of an "up" move of 0.5, and an interest rate of 10% per period?

Assume we are in the "up" node at $n = 1$ and the stock price is 160. If we choose to exercise the call option at this node, then its payoff will be $160 - 100 = 60$. Alternatively, if we choose to wait to exercise the call option at $n = 0$, then its value is equal to $(0.5 \times 156 + 0.5 \times 0)/1.1 = 70.9091$. Thus, the value of the call option in the "up" node at $n = 1$ is $C(160, 1) = \max[60, 70.9091] = 70.9091$. A similar calculation for the put option yields a value of $P(160, 1) = \max[100 - 160, (0.5 \times 0 + 0.5 \times 4)/1.1] = \max[-60, 1.8182] = 1.8182$.

Now, assume we are in the "down" node at $n = 1$ and the stock price is 60. If we choose to exercise the call option at this node, then its payoff will be $60 - 100 = -40$. Alternatively, if we choose to wait to exercise the call option at $n = 0$, then its value is equal to $(0.5 \times 0 + 0.5 \times 0)/1.1 = 0$. Thus, the value of the call option in the "down" node at $n = 1$ is $C(60, 1) = \max[-40, 0] = 0$. A similar calculation for the put option yields a value of $P(60, 1) = \max[100 - 60, (0.5 \times 4 + 0.5 \times 64)/1.1] = \max[40, 30.9091] = 40$.

To summarize, the values of the call and put options at $n = 1$ are:

$$
\begin{aligned}
C(160, 1) &= 70.9091 \\
P(160, 1) &= 1.8182 \\
C(60, 1) &= 0 \\
P(60, 1) &= 40
\end{aligned}
$$

Thus, if the stock price goes down to 60 at $n = 1$, the buyer will choose for the option to be a put, and the put will be exercised immediately. If the stock price instead goes up to 160 at $n = 1$, the buyer will choose for the option to be a call, and the call will be held until its expiration.

Given the optimal exercise policy above, we can find the current value of the option—i.e., at $n = 2$—by discounting the expected payoffs from exercising the put option in the "down" node at $n = 1$ and holding the call option in the "up" node at $n = 1$. The current value of the option is:

$$
\begin{aligned}
&[0.5 \times C(160, 1) + 0.5 \times P(60, 1)]/1.1 = \\
&[0.5 \times 70.9091 + 0.5 \times 40]/1.1 = 50.4132.
\end{aligned}
$$