Recitation 6

Implied Volatility

In the Week 6 lecture, we saw how the Black-Scholes-Merton (BSM) model can be used to infer the volatilities of underlying stock prices. These so-called **implied volatilities**—i.e., volatilities that are "reverse-engineered" from the BSM model using observed options prices—are forward-looking measures of market uncertainty about future returns, and are typically less variable than options prices themselves.

For example, consider a European call option on a non-dividend-paying stock with spot price $S = 21$, strike price $K = 20$, risk-free rate $r = 10\%$, and time-to-expiration $T - t = 0.25$. Recall the BSM pricing formula for a European call option:

$$C(S, K, T-t, r, \sigma) = S\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2) \tag{1}$$

where $\mathcal{N}(\cdot)$ is the cumulative density function of a standard normal random variable, with $d_1 = \frac{\ln\left(\frac{S}{K}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$ and $d_2 = d_1 - \sigma\sqrt{T-t}$.

Suppose the market price of the call option is $C = 1.875$. How can we use Equation (1) to find σ, the volatility of the underlying stock price?

While it's impossible to invert Equation (1) in order to solve for σ as a function of S, K, r, T, and C, we can numerically iterate on σ using the BSM & Black's Model.xls spreadsheet.

Begin by guessing an initial value for σ, say 0.3. Plugging these parameters into the "BSM" tab of the spreadsheet, we calculate a call price of 2.10.

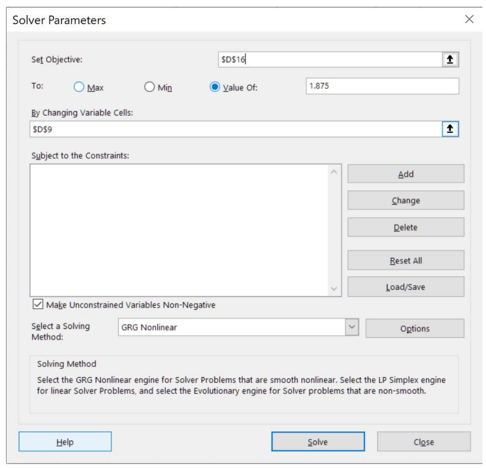
Black-Scholes-Merton Model				
	Inputs (yellow)			
X	$20.00000			strike price
S	$21.000000			current stock price
r	10.00%			spot yield for maturity T
sigma	0.3			volatility
T	0.25			time to option expiration
delta	0.00%			dividend yield
d1	0.566934428			
N(d1)	0.714620634	N(-d1)	0.29	
d2	0.416934428			
N(d2)	0.661636815	N(-d2)	0.34	
C	$2.101014			call value
P	$0.6072			put value

Clearly, the call price of 2.10 is too high relative to the market price of 1.875, so what value of σ should we try next? Since we know that the price of a European call option is increasing in the volatility of the underlying stock price, we should try a smaller value of σ, say 0.2. Doing so yields a call price of 1.76, which is now too low relative to the market price of 1.875.

We can repeat the above procedure, plugging in a value of σ somewhere between 0.2 and 0.3, say 0.25. Depending on whether the calculated call price is above or below the market price, we adjust our input of σ upward or downward. Iterate until the calculated call price is "close enough" to the market price, and the resulting value of σ is the implied volatility of the option.

This iterative method can be tedious to implement by hand, though! Is there a faster way to find the optimal input of σ? Yes! One way to efficiently calculate the implied volatility of the call option is to use Excel's Solver tool.

After loading the Solver add-in, open Solver under the "Analysis" group in the "Data" tab. In the pop-up menu, define the objective to be the cell corresponding to the value of the call option. We want to change the variable in the cell corresponding to σ such that the call option has a value equal to its market price of 1.875. The correct inputs for Solver are displayed in the figure below:



Solver finds that $\sigma = 0.235$ yields a value of the call option that is approximately equal to the market price of 1.875. Thus, the implied volatility of the call option is 23.5% per annum.

Black-Scholes-Merton Model				
	Inputs (yellow)			
X	$20.00000			strike price
S	$21.000000			current stock price
r	10.00%			spot yield for maturity T
sigma	0.234512509			volatility
T	0.25			time to option expiration
delta	0.00%			dividend yield
d1	0.687934976			
N(d1)	0.754253135	N(-d1)	0.25	
d2	0.570678722			
N(d2)	0.715891277	N(-d2)	0.28	
C	$1.874999			call value
P	$0.3812			put value

In principle, one could use a similar procedure with binomial trees to compute the implied volatilities of American options.

Black's Model

Recall from the Week 6 lecture that we can price European options on futures contracts using **Black's model**, a version of the BSM model that assumes that futures prices are log-normally distributed. Let's see how we can use Black's model to price a European call option on a bond.

Assume that a European call option with 10 months to expiration is written on a bond with 9.75 years to maturity, a face value of $1,000, and a semiannual coupon of $50. Furthermore, assume that the futures price of the bond is $939.68, the strike price of the call option is $1,000, and the risk-free rate for the next 10 months is 10%.

If the volatility of the futures bond price is 9% annually, what is the price of the call option according to Black's model?

Solution: Recall that, according to Black's model, the price of a European call option on a futures contract with strike price K, futures price F, risk-free rate r, and volatility of the futures price σ is equal to:

$$c = e^{-rT}\left[F\mathcal{N}\left(d_1\right) - K\mathcal{N}\left(d_2\right)\right]$$

where

$$d_1 = \frac{\ln(F/K) + (\sigma^2/2)T}{\sigma\sqrt{T}}; \quad d_2 = d_1 - \sigma\sqrt{T}.$$

As noted in the Week 6 lecture, this pricing formula is identical to that for a European call option on a stock according to the BSM model, except that we substitute the futures price for the stock price and set the "dividend yield" equal to the risk-free rate.

Plugging in $K = 1,000$, $F = 939.68$, $r = 0.1$, $\sigma = 0.09$, and $T = 10/12 = 0.8333$ into the expressions for d_1, d_2, and c above, we get that:

$$\begin{aligned}
d_1 &= \frac{\ln(939.68/1000) + (0.09^2/2) \times 0.8333}{0.09\sqrt{0.8333}} \\
d_2 &= d_1 - 0.09\sqrt{0.8333} \\
c &= e^{-0.1(0.8333)}\left[939.68\mathcal{N}\left(d_1\right) - 1000\mathcal{N}\left(d_2\right)\right] = 9.49.
\end{aligned}$$

Alternatively, we can input the parameters into the corresponding cells in the "Black's Model" tab of the BSM & Black's Model.xls spreadsheet to find the price of the European call option directly.

	Inputs (yellow)			
X	1000			strike price
F	$939.68			current forward price of the bond
r	10.00%			spot yield for maturity T
sigma	0.09			volatility of forward bond price
T	0.8333			time to option expiration
d1	-0.716204439			
N(d1)	0.236932561	N(-d1)	0.763067	
d2	-0.79836118			
N(d2)	0.212330462	N(-d2)	0.78767	
C	$9.486			call value
P	$64.98			put value

Currency Options

Currency options are conceptually quite similar to options on stocks and futures that we've seen in lecture. A **currency option** gives the holder the right, but not the obligation, to buy or sell a certain currency at a predetermined exchange rate on or before a specified date. Are there BSM formulae analogous to those used to price call and put options on stocks for European currency options?

Recall from the Week 5 lecture that the BSM pricing formulae for European call and put options on a stock with a *known dividend yield* δ are as follows:

$$c = Se^{-\delta T}\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2); \quad p = Ke^{-rT}\mathcal{N}(-d_2) - Se^{-\delta T}\mathcal{N}(-d_1)$$

with $d_1 = \frac{\ln(S/K)+\left(r-\delta+\sigma^2/2\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$.

The key to pricing currency options is to realize that foreign currency is analogous to a stock paying a known dividend yield: the holder of the foreign currency receives a "dividend yield" equal to the risk-free rate, r_f, in the foreign currency.

To keep the notation consistent, let's define S to be the spot exchange rate: in particular, S is the value of one unit of a foreign currency in U.S. dollars. Then, the prices of European call and put currency options are the same as those for a stock with a known dividend yield, except we replace δ with the foreign risk-free rate r_f:

$$c = Se^{-r_f T}\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2); \quad p = Ke^{-rT}\mathcal{N}(-d_2) - Se^{-r_f T}\mathcal{N}(-d_1)$$

with $d_1 = \frac{\ln(S/K)+\left(r-r_f+\sigma^2/2\right)T}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$. Both the domestic risk-free rate, r, and the foreign risk-free rate, r_f, are for maturity T.

As an example, consider a 6-month European call option on the British pound. Suppose that the spot exchange rate is 1.50, the strike price is 1.50, the risk-free rate in the United States is 5% per annum, the risk-free rate in the United Kingdom is 7% per annum, and the pound/dollar exchange rate has a volatility of 12% per annum. What is the price of the call option (in dollars)?

Solution: Using the "BSM" tab of the BSM & Black's Model.xls spreadsheet, we can calculate the price of the European call option with $S = 1.50$, $K = 1.50$, $r = 0.05$, $\sigma = 0.12$, δ or $r_f = 0.07$, and $T = 0.5$ to be 4.2 cents.

Black-Scholes-Merton Model				
	Inputs (yellow)			
X	$1.50000			strike price
S	$1.500000			current stock price
r	5.00%			spot yield for maturity T
sigma	0.12			volatility
T	0.5			time to option expiration
delta	7.00%			dividend yield
d1	-0.075424723			
N(d1)	0.469938394	N(-d1)	0.53	
d2	-0.160277537			
N(d2)	0.436331226	N(-d2)	0.56	
C	$0.042325			call value
P	$0.0569			put value

Finally, assume that the 6-month forward exchange rate F between pounds and dollars is equal to the strike price K of the European call option. What would be the price of a 6-month European put option on the pound?

Solution: We don't need any other information to solve this, actually! From put-call parity, note that we can relate European call and put prices on currency options as:

$$c + Ke^{-rT} = p + Se^{-r_f T}.$$

Back in the Week 1 lecture, we learned the following relationship between the forward exchange rate, F, and the spot exchange rate, S:

$$F = Se^{\left(r-r_f\right)T}.$$

Combining these two equations, we have that:

$$c + Ke^{-rT} = p + Fe^{-rT}.$$

So, if $K = F$, then $c = p$, and the price of a European put option is the same as that of a European call option! This result holds generically for options on any underlying asset as long as the strike price is equal to the forward price.