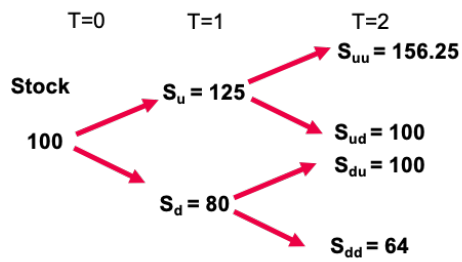
Recitation 13

Spring 2021

Question 1

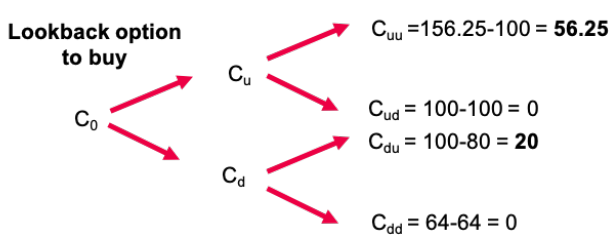
Suppose that there is a stock with a current price of \$100. The graph below shows the evolution of the share price over the next two years:



The stock does not pay dividends. The risk-free rate (EAR) is 5%. Assume that there is no arbitrage and that investors can borrow and lend at the risk free rate.
Consider an exotic option, called a European lookback option, which allows you to buy a stock 2 years from now for the lowest price of the stock during the coming two years. Price this option using risk-neutral probabilities.

Solutions:

We start by creating a tree with the option payoff in different states. In the uu and du states, the lookback option gets exercised, since the $T = 2$ price is greater than the lowest price of the stock over the prior 2 years. In the ud and dd states the option does not get exercised.



We now find the risk-neutral probabilities of the "up" and "down" moves. First, we find the multiples u and d by which the price goes either up or down each year. Recall that $S_u = uS_0$ and $S_d = dS_0$. So, $u = 1.25$ and $d = 0.8$. We check that these are consistent with the stock prices in year 2 as well:

$$
\begin{aligned}
S_{uu} &= uS_u = 1.25 \times 125 = 156.25 \\
S_{ud} &= dS_u = 0.8 \times 125 = 100 \\
S_{du} &= uS_d = 1.25 \times 80 = 100 \\
S_{uu} &= uS_u = 0.8 \times 80 = 64
\end{aligned}
$$

So, the risk neutral probabilities are:

$$
\begin{aligned}
q_u &= \frac{(1+r)-d}{u-d} = \frac{1+0.05-0.8}{1.25-0.8} = 0.556 \\
q_d &= \frac{u-(1+r)}{u-d} = \frac{1.25-(1+0.05)}{1.25-0.8} = 0.444
\end{aligned}
$$

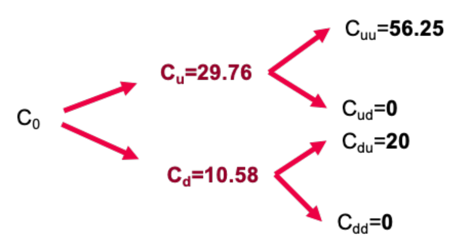
We can now price the options at each stage by taking the expected payoff of the option and discounting it to present value.
It follows that the price of the option at $T = 1$ in the up state is:

$$C_u = \frac{q_u C_{uu} + q_d C_{ud}}{1+r} = \frac{0.556 \times 56.25}{1+0.05} = 29.76$$

The price of the option at $T = 1$ in the down state is:

$$C_d = \frac{q_u C_{du} + q_d C_{dd}}{1+r} = \frac{0.556 \times 20}{1+0.05} = 10.58$$

The final step is to price the option in Year 0:



The price of the lookback option at $T = 0$ is:

$$C_0 = \frac{q_u C_u + q_d C_d}{1+r} = \frac{0.556 \times 29.76 + 0.444 \times 10.58}{1+0.05} = 20.23$$

Alternatively, with the risk-neutral pricing approach, we can directly find the value of the option at $T = 0$:

$$
\begin{aligned}
C_0 &= \frac{q_u^2 C_{uu} + q_u q_d C_{ud} + q_d q_u C_{du} + q_d^2 C_{dd}}{(1+r)^2} \\
&= \frac{0.556^2 \times 56.25 + 0.556 \times 0.444 \times 20}{(1.05)^2} = 20.23
\end{aligned}
$$

The numerator here is the expected payoff from this option in year 2 under the risk-neutral probabilities q. To find the present value, we discount it to present at the risk-free rate, r.

Question 2

Consider the lookback option covered in question 1. Construct the following replicating portfolios:

(a) A portfolio at time $T = 0$, which replicates the payoff from the option at time $T = 1$

(b) A portfolio at time $T = 1$ in the "up" node, which replicates the payoff from the option at time $T = 2$

(c) A portfolio at time $T = 1$ in the "down" node, which replicates the payoff from the option at time $T = 2$

Assume the same stock price evolution as in Question 1.

Solutions:

Suppose that the replicating portfolio has:

- δ shares of stock
- b dollars invested in the riskless bond

Recall the following:

$$\delta = \frac{C_u - C_d}{(u-d)S}$$
$$b = \frac{1}{1+r}\frac{uC_d - dC_u}{u-d}$$

We'll apply this to the problem:

(a) Replicating portfolio at $T = 0$. We have C_u and C_d from solving question 1. Plugging in yields:

$$\delta_0 = \frac{C_u - C_d}{(u-d)S_0} = \frac{29.76 - 10.58}{(1.25 - 0.80) \times 100} = 0.4262$$
$$b_0 = \frac{1}{1+r}\frac{uC_d - dC_u}{u-d} = -22.396$$

We can check that price the price of the replicating portfolio matches the option price found in question 1:

$$\delta_0 S_0 + b_0 = 0.4262 \times 100 - 22.396 = 20.23$$

(b) Replicating portfolio at $T = 1$ in the "up" node:

$$\delta_{1,u} = \frac{C_{uu} - C_{ud}}{(u-d)S_u} = \frac{56.25 - 0}{(1.25 - 0.80) \times 125} = 1$$
$$b_{1,u} = \frac{1}{1+r}\frac{uC_{ud} - dC_{uu}}{u-d} = -95.238$$

We can check that price the price of the replicating portfolio matches the option price C_u in question 1:

$$\delta_{1,u} S_u + b_{1,u} = 1 \times 125 - 92.238 = 29.76$$

(c) Replicating portfolio at $T = 1$ in the "down" node:

$$\delta_{1,d} = \frac{C_{du} - C_{dd}}{(u-d)S_d} = \frac{20 - 0}{(1.25 - 0.80) \times 80} = 0.556$$
$$b_{1,d} = \frac{1}{1+r}\frac{uC_{dd} - dC_{du}}{u-d} = -33.862$$

We can check that price the price of the replicating portfolio matches the option price C_d in question 1:

$$\delta_{1,d}S_d + b_{1,d} = 0.556 \times 80 - 33.862 = 10.58$$

Altogether, we have:

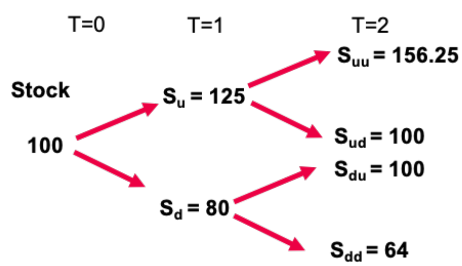
(a) $T = 0$:
$$\delta_0 = 0.4262, b_0 = -22.396$$

(b) $T = 1$ in "up" node:
$$\delta_{1,u} = 1, b_{1,u} = -95.238$$

(c) $T = 1$ in "down" node:
$$\delta_{1,d} = 0.556, b_{1,d} = -33.862$$

Question 3

Consider an exotic option, called a European lookback option, which allows you to buy a stock 2 years from now for the lowest price of the stock during the coming 2 years.
Suppose there is a stock with a current price of \$100. Over the next 2 years:



The stock does not pay dividends. The risk-free rate (EAR) is 5%. Assume that there is no arbitrage and that investors can borrow and lend at the risk free rate.
Price this option using state prices.

Solutions:

By definition, state prices are as follows.

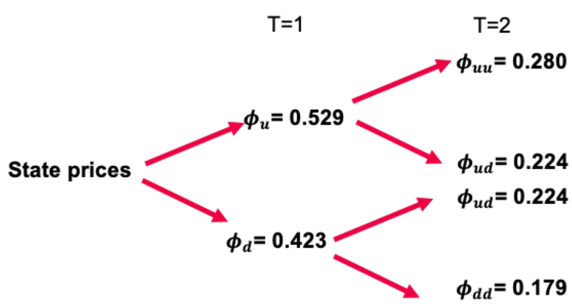
In year $T = 1$:

$$\phi_u = \frac{q_u}{1+r} = \frac{0.556}{1+5\%} = 0.529$$
$$\phi_d = \frac{q_d}{1+r} = \frac{0.444}{1+5\%} = 0.423$$

In year $T = 2$:

$$\phi_{uu} = \frac{q_u^2}{(1+r)^2} = \frac{0.556^2}{(1+5\%)^2} = 0.280$$
$$\phi_{ud} = \phi_{du} = \frac{q_u q_d}{(1+r)^2} = \frac{0.556 \times 0.444}{(1+5\%)^2} = 0.224$$
$$\phi_{dd} = \frac{q_d^2}{(1+r)^2} = \frac{0.444^2}{(1+5\%)^2} = 0.179$$

Below is a summary of the state prices in this economy:



From the state prices, we can price any security, including the lookback option.
We'll start with the price of the option in Year 1 in the "up" state:

$$C_u = \phi_u C_{uu} + \phi_d C_{ud} = 0.529 \times 56.25 + 0 = 29.76$$

The price of the option in Year 1 in the "down" state is:

$$C_d = \phi_u C_{du} + \phi_d C_{dd} = 0.529 \times 20 + 0 = 10.58$$

The price of the option in Year 0 is:

$$\begin{aligned} C_0 &= \phi_{uu} C_{uu} + \phi_{ud} C_{ud} + \phi_{du} C_{du} + \phi_{dd} C_{dd} \\ &= 0.280 \times 56.25 + 0.224 \times 0 + 0.224 \times 20 + 0.179 \times 0 \\ &= 20.23 \end{aligned}$$

Furthermore, while we know the price of the stock in year 0, we can use state prices to compute it to show that state prices and stock prices are consistent:

$$\begin{aligned} S_0 &= \phi_{uu} S_{uu} + \phi_{ud} S_u d + \phi_{du} S_{du} + \phi dd S_{dd} \\ &= 0.280 \times 156.25 + 0.224 \times 100 + 0.224 \times 100 + 0.179 \times 64 \\ &= 100 \end{aligned}$$

Question 4

Suppose that all assumptions of the Black-Scholes-Merton (BSM) formula hold. There is a stock, XYZ, with the current price of $S = \$50$. It has annual volatility $\sigma = 35\%$.
The annualized continuously-compounded risk-free interest rate is 1.25%.

(a) What is the price of the call option with a strike price $K = \$55$ and 6 months to maturity?

(b) What is the price of the put option with a strike price $K = \$55$ and 6 months to maturity?

Solutions:

Recall the Black-Scholes-Merton pricing formula:

$$C_0 = C(S_0, K, T) = S_0 N(x) - Ke^{-rT} N(x - \sigma\sqrt{T})$$

where

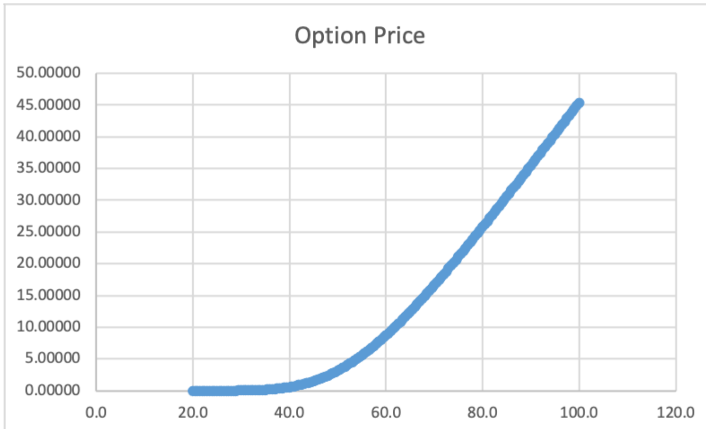
$$x = \frac{\ln\left(\frac{S_0}{Ke^{-rT}}\right)}{\sigma\sqrt{T}} + \frac{1}{2}\sigma\sqrt{T}$$

Other components of the expressions are defined as follows:

- S_0 is the current share price
- N is the cumulative distribution function of a standard normal distribution
- K is the strike price of the option
- r is the continuously compounded risk-free interest rate
- σ is the annualized volatility of the stock price
- T is the time to maturity in years

For the call option in this question, $T = 6/12 = 0.5$.
Plugging in S_0, σ, r, K, and T yields a call price $\mathbf{C} = \mathbf{\$3.15}$. This can be computed in Excel, using the NORM.S.DIST function to find $N(x)$.

You can plot the change in option price as you adjust various parameters, such as original share price. A plot of option price vs. share price is shown below, confirming that the option price increases with stock price:



The slope of this line, which is the delta of this option, starts at 0 and gets closer to 1 as stock price rises. You can also see that this function is convex, which means that the γ of the call option, defined as the second derivative of the option price with respect to the stock price, is positive.

To compute the price of the put option, we will use Put-Call Parity:

$$C + e^{-rT}K = S + P$$

This yields a put option price $\mathbf{P} = \mathbf{\$7.81}$.
In general, to compute the price of a European put option:

1. Compute the price of the corresponding call option (same maturity, same strike) using the Black-Scholes-Merton formula
2. Use put-call parity to compute the put price

Question 5

Suppose that all assumptions of the Black-Scholes-Merton (BSM) formula hold. There is a stock, XYZ, with a current price of \$50. It has annual volatility $\sigma = 35\%$. The annualized continuously-compounded risk-free interest rate is 1.25%.
Consider a put option with a strike price $K = \$55$ and 6 months to maturity.

(a) Construct a replicating portfolio for this put option

(b) Suppose that the stock price decreases to \$47. What is the change in the option price and what is the change in the value of replicating portfolio?

(c) Suppose that the stock price increases to \$53. What is the change in the option price and what is the change in the value of replicating portfolio?

Solutions:

We know the replicating portfolio for a call option:

- Buy $N(x)$ shares of stock
- Borrow $Ke^{-rT}N(x - \sigma\sqrt{T})$ dollars

What about for a put option? Recall put-call parity: $P = C + e^{-rT}K - S$. We will plug the BSM formula for the value of the call option in this equation:

$$P = SN(x) - Ke^{-rT}N(x - \sigma\sqrt{T}) + e^{-rT}K - S$$
$$P = -S(1 - N(x)) + Ke^{-rT}(1 - N(x - \sigma\sqrt{T}))$$

Interpreting the coefficients of S and Ke^{-rT}:

- Short $1 - N(x)$ shares of the stock
- Invest $Ke^{-rT}(1 - N(x - \sigma\sqrt{T}))$ dollars at the risk-free rate

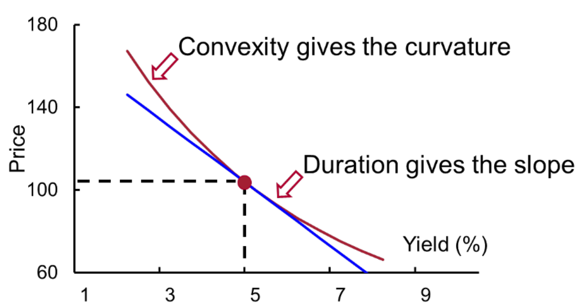
By construction, the value of the replicating portfolio matches the value of thee option.
Now, what happens when we change the value of the underlying stock price?
By plugging into BSM, we see the impact on the **put option price**:

- S_0 decreases to \$47 $\rightarrow$ put option price increases by \$1.92 to \$9.73.
- S_0 increases to \$53 $\rightarrow$ put option price decreases by \$1.64 to \$6.17.

Now, we see what happens to the **replicating portfolio**, shorting the same number of stocks and investing the same amount of dollars, when the share price changes:

- S_0 decreases to \$47 $\rightarrow$ replicating portfolio increases by \$1.78 to \$9.59.
- S_0 increases to \$53 $\rightarrow$ put option price decreases by \$1.78 to \$6.03.

We see that the value of the replicating portfolio does not increase as much as the option value when the stock price declines. On the other hand, the value of replicating portfolio decreases more than the value of the option when the stock price increases.
The reason behind this result is that the value of the put option, as shown on the below graph, is a convex function of the stock price:



In other words, the put option value has a positive γ. On the other hand, the value of our replicating portfolio is a simple linear function of the underlying stock price. Therefore, when the stock price increases, the value of replicating portfolio overshoots, and when the stock price decreases, it undershoots.