15.455x – Mathematical Methods for Quantitative Finance

Recitation Notes #4

Let's look at examples of Itô's lemma in action. The simplest cases are where the stochastic variable X is just pure Brownian motion:

$$\mathrm{d}X_t = a\,\mathrm{d}t + b\,\mathrm{d}B_t = \mathrm{d}B_t.$$

So in terms of the general form of an Itô process, $a = 0, b = 1$.

> **Note on notation:** *With X_t, B_t, S_t, etc. the subscript t is just a reminder that these are time-dependent random variables. We will often drop the subscript to de-clutter the notation. So X and B have exactly the same meaning as X_t, B_t, etc.*

Exercises

The simplest cases are where the stochastic variable X is just pure Brownian motion:

$$\mathrm{d}X_t = a\,\mathrm{d}t + b\,\mathrm{d}B_t = \mathrm{d}B_t.$$

So in terms of the general form of an Itô process, $a = 0, b = 1$. For pure Brownian motion, therefore,

$$\mathrm{d}F = \frac{\partial F}{\partial t}\,\mathrm{d}t + \frac{\partial F}{\partial B}\,\mathrm{d}B + \frac{1}{2}\frac{\partial^2 F}{\partial B^2}\,\mathrm{d}t$$

Exercise: $F(t, B) = t^3 + B^3$. Find $\mathrm{d}F$.

Solution: Taking the required partial derivatives,

$$\begin{aligned}\mathrm{d}F &= 3t^2\,\mathrm{d}t + 3B^2\,\mathrm{d}B + 3B\,\mathrm{d}t\\ &= 3(t^2 + B)\,\mathrm{d}t + (3B^2)\,\mathrm{d}B.\end{aligned}$$

Exercise: $F(t, B) = e^{-rt}\sin(\theta B)$. Find $\mathrm{d}F$.

Solution: Taking partial derivatives,

$$\frac{\partial F}{\partial t} = -re^{-rt}\sin(\theta B), \quad \frac{\partial F}{\partial B} = \theta e^{-rt}\cos(\theta B), \quad \frac{\partial^2 F}{\partial B^2} = -\theta^2 e^{-rt}\sin(\theta B).$$

$$\begin{aligned}
\mathrm{d}F &= -re^{-rt}\sin(\theta B)\,\mathrm{d}t + \theta e^{-rt}\cos(\theta B)\,\mathrm{d}B - e^{-rt}\frac{\theta^2}{2}\sin(\theta B)\,\mathrm{d}t \\
&= e^{-rt}\Big[-\left(r+\frac{\theta^2}{2}\right)\sin(\theta B)\Big]\,\mathrm{d}t + e^{-rt}\big[\theta\cos(\theta B)\big]\,\mathrm{d}B.
\end{aligned}$$

Exercise: $F(t,B) = \log B$. Find $\mathrm{d}F$.

Solution:

$$\mathrm{d}F = \frac{1}{B}\,\mathrm{d}B + \frac{1}{2}\left(\frac{-1}{B^2}\right)\,\mathrm{d}t.$$

Exercise: $\mathrm{d}X/X = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}B$. Find a function $F(t,X)$ such that $\mathrm{d}F = a\,\mathrm{d}t + b\,\mathrm{d}B$ with constant coefficients a, b.

Solution: Let's use Itô's lemma in the form where we write the right-hand side in terms of $\mathrm{d}t$ and $\mathrm{d}B$ (rather than $\mathrm{d}X$):

$$\mathrm{d}F = \left[\frac{\partial F}{\partial t} + \frac{(\sigma X)^2}{2}\frac{\partial^2 F}{\partial X^2} + (\mu X)\frac{\partial F}{\partial X}\right]\,\mathrm{d}t + \left[(\sigma X)\frac{\partial F}{\partial X}\right]\,\mathrm{d}B.$$

Each of the expressions in square brackets must be constant, so there are two differential equations for F. Let's start with the second one, since it's much shorter:

$$(\sigma X)\frac{\partial F}{\partial X} = \text{constant} \implies X\frac{\partial F}{\partial X} = \frac{\text{constant}}{\sigma} = C.$$

The left-hand side is a logarithmic derivative, so let's try a very simple form where $F = F(X)$ and does not depend on time. Then

$$\begin{aligned}
X\frac{\mathrm{d}F}{\mathrm{d}X} &= C \implies \mathrm{d}F = C\frac{\mathrm{d}X}{X} \\
F(X) &= C\log X.
\end{aligned}$$

Now observe that this expression automatically makes the coefficient function of $\mathrm{d}t$ into a constant as well. In fact, if we choose $C = 1$, we find that

$$\mathrm{d}F = \left[\mu - \frac{\sigma^2}{2}\right]\,\mathrm{d}t + \sigma\,\mathrm{d}B.$$

We've shown that another route to the differential of $\mathrm{d}(\log S)$ seen in modeling stock prices comes from seeking a change of variables that makes the right-hand-side coefficients constant.

Exercise: An asset follow the Ornstein-Uhlenbeck process $\mathrm{d}S = \lambda(\bar{S} - S)\,\mathrm{d}t + \sigma\,\mathrm{d}B$. What PDE is satisfied by derivatives of the asset?

Solution: This case is actually no harder than the derivation of the usual Black-Scholes-Merton equation because the exact form of $\mathrm{d}S$ enters through one small feature, the coefficient of $\mathrm{d}B$. Given

$$
\begin{aligned}
\mathrm{d}S &= a(t,S)\,\mathrm{d}t + b(t,S)\,\mathrm{d}B, \\
\mathrm{d}V &= \left(\frac{\partial V}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t + \left(\frac{\partial V}{\partial S}\right)\mathrm{d}S, \\
\mathrm{d}(V - \Delta S) &= \left(\frac{\partial V}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V}{\partial S^2}\right)\mathrm{d}t + \left(\frac{\partial V}{\partial S} - \Delta\right)\mathrm{d}S \\
&= r\,\mathrm{d}t(V - \Delta S).
\end{aligned}
$$

Making the same choice of $\Delta = \partial V/\partial S$ in order to zero out the coefficient of $\mathrm{d}S$, we find the generalization

$$
\frac{\partial V}{\partial t} + \frac{b^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0,
$$

which is independent of $a(t,S)$ in the defining process for the underlying. This form gives us a direct route from a defining process to the PDE satisfied by derivatives on the underlying asset.

For the Ornstein-Uhlenbeck process,

$$
\frac{\partial V}{\partial t} + \frac{\sigma^2}{2}\frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.
$$