Week 4 – Options strategies and pricing basics

MIT Sloan School of Management

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Topic outline

- Review of option basics
- Popular options trading strategies
- Binomial pricing
  - No arbitrage pricing on binomial tree and “risk-neutral” probabilities
- Pricing American options

Review of definitions

A European option gives its holder **the right, but not the obligation**, to buy (call) or sell (put) a pre-specified security or commodity ("underlying") for a pre-specified price ("strike" or "exercise") on a pre-specified date ("maturity" or "expiration")

An American option is the same except exercise can occur any time before maturity.

- Using the option is called "exercising" the option
- Selling an option is called "writing" an option

Since the holder of the option has a right, but not the obligation to exercise it, the option always has a non-negative value to the option holder

At the beginning of the contract, the option buyer pays the *option premium* to the option seller (sometimes called "option writer")

In exchange of the option premium, the seller of the option has the obligation to deliver the underlying security at maturity T, if requested by the option buyer

Options

Underlying Symbol Lookup | Chain Type | Range | Expiration | Strike
AAPL | Calls & Puts | Near The Money | All + Leaps | View Chain

Index list

APPLE INC COM | S&P Options Report What's This?

Apply for or upgrade options trading

Symbol	Bid	Ask	Last	Change	Change %	B/A Size	High	Low	Volume
AAPL	378.02	378.34	377.37	-15.20	-3.87	100X100	391.32	377.35	31,057,117

Thu Aug 04 2011 5:46:13 PM EDT

Calls and Puts | Learn more

Expand all expirations

AAPL Aug 5 2011	1 Days to Expiration (Weeklys)
AAPL Aug 12 2011	8 Days to Expiration (Weeklys)
AAPL Aug 20 2011	16 Days to Expiration
AAPL Sep 17 2011	44 Days to Expiration
AAPL Oct 22 2011	79 Days to Expiration

Collapse

Calls	Bid	Ask	Last	Change	Vol	Op Int	Strike	Puts	Bid	Ask	Last	Change	Vol	Op Int
365.0 Call	31.80	32.95	34.00	-5.15	195	2,316	365.00	365.0 Put	18.45	19.25	17.55	6.10	418	1,939
370.0 Call	28.75	29.75	29.10	-6.60	590	4,272	370.00	370.0 Put	20.40	21.30	20.00	7.05	159	2,034
375.0 Call	26.10	26.90	26.80	-5.85	281	3,048	375.00	375.0 Put	22.75	23.50	22.55	7.95	641	1,249
380.0 Call	23.75	24.20	24.30	-5.05	1,012	9,944	380.00	380.0 Put	25.15	25.90	25.00	8.56	751	2,586
385.0 Call	20.95	21.75	21.65	-4.75	602	2,928	385.00	385.0 Put	27.10	28.45	28.00	9.25	182	2,156
390.0 Call	18.75	19.50	19.25	-4.44	1,371	5,563	390.00	390.0 Put	30.05	31.15	30.06	9.38	697	1,706

Color Indicates options that are in-the-money NS Indicates non-standard option

Example: Apple Oct 380 Call

Underlying: Apple stock price $= S_t = 377.37$; Strike price: $K = 380$; Maturity date: $T = 79 days$.

Common types of exchange-traded options

Stock options

- Exchanges: CBOE, AMEX, PHLX, ISE (Nasdaq) etc.
- Over 1000 different stocks

Index options

- DJ Industrial, S&P100, S&P500, Nasdaq 100, Russell 2000 (on CBOE), etc.

Foreign currency options (traded mainly on PHLX)

Options on futures

- interest rates, agricultural, oil, livestock, metals, currencies, cryptocurrencies, and stock indices
- Typically traded on same exchange as underlying futures contracts

As for futures, exchange-traded options require margin to ensure performance.

Call option payoff

- How is the payoff of a European call option determined?

If at maturity T, $\boldsymbol{S_T < K}$, the option buyer has the choice between:

buy the stock in the market at S_T

buy the stock from the option seller for $K > S_T$

$\Rightarrow$ **Walk away from the option. Payoff = 0**

If at T, $\boldsymbol{S_T > K}$, the option buyer has the choice between:

buy the stock in the market at S_T

buy the stock from the option seller for $K < S_T$

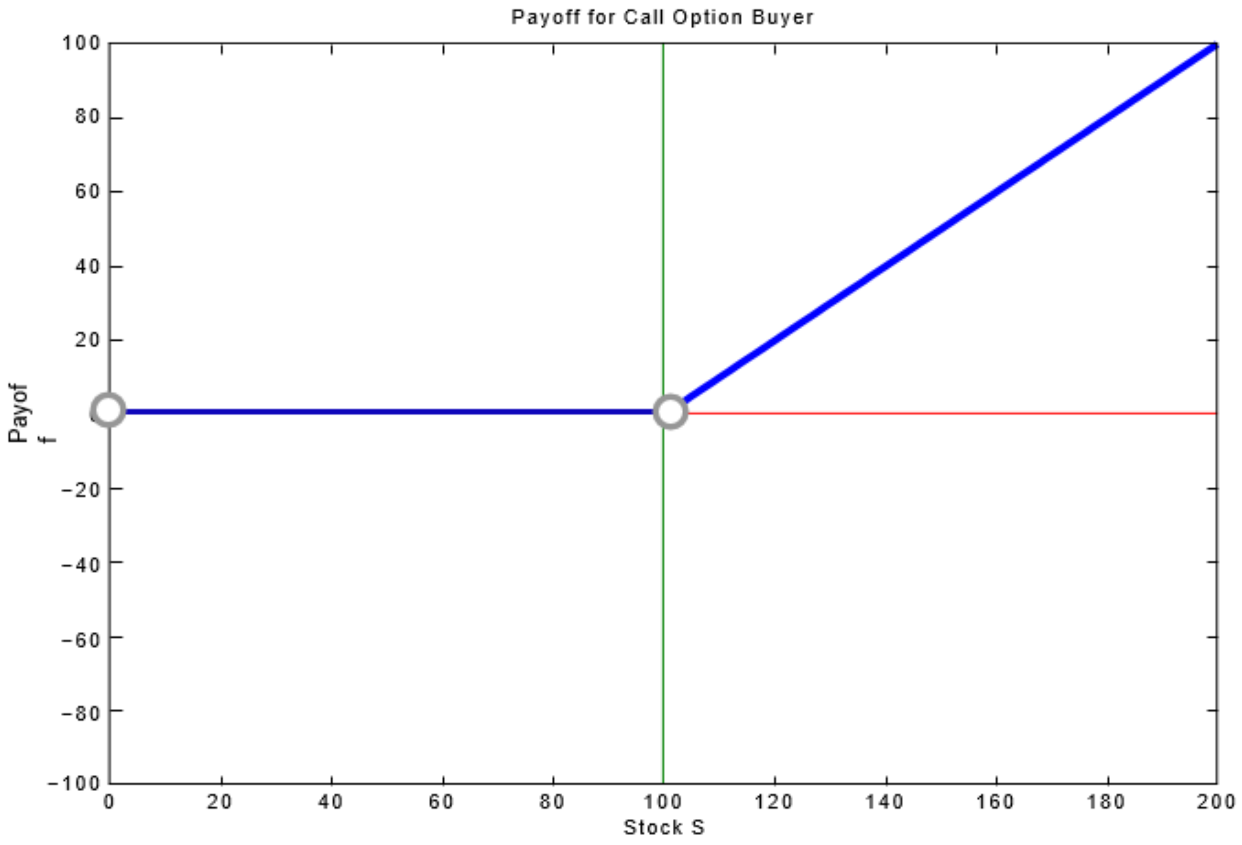
$\Rightarrow$ Exercise the option. **Payoff =** $\boldsymbol{S_T - K}$

The option seller *must* deliver the stock in exchange for a price K

Thus, the payoff to the call buyer is

Payoff of a Call = $\max(S_T - K, 0)$

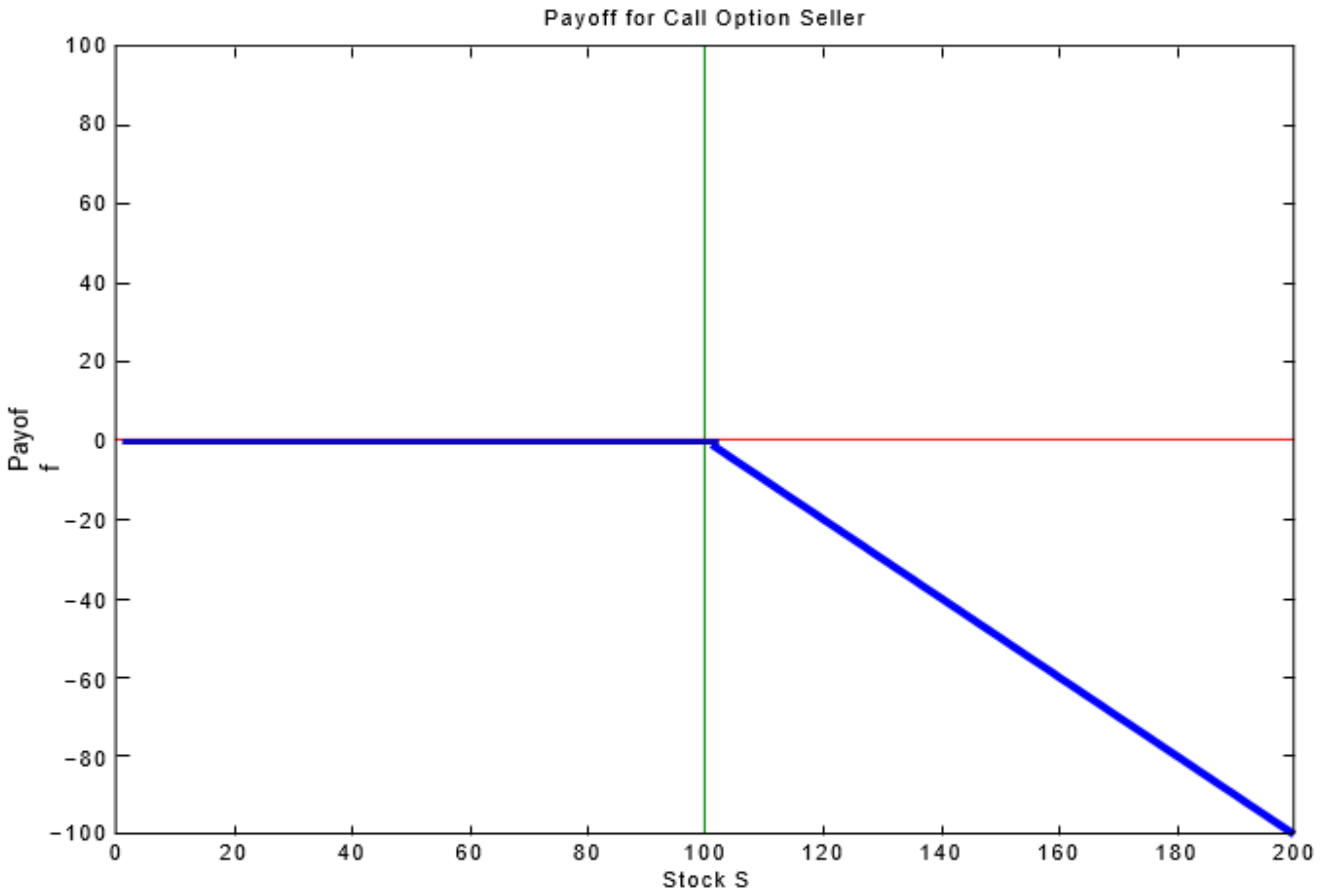
Call option payoff diagram (long)



Out of the Money (OTM): For a stock price below $K = 100$, the call option expires worthless.
In the Money (ITM): For stock price above $K = 100$, the call option payoff increases with stock price one-for-one.

At the Money (ATM): Stock price is equal to strike price.

Call option payoff diagram (short or written)



Zero sum in payoffs: Gains to buyer = losses to writer

Put option payoff

- How is the payoff of a put option determined?

If at T, $S_T < K$ the put option buyer has the choice between:

sell the stock in the market at S_T;
sell the stock to the option writer for $K > S_T$
$\Rightarrow$ Exercise the put option to sell. Payoff $= K - S_T$

The option seller *must* purchase the stock in exchange of a price K.

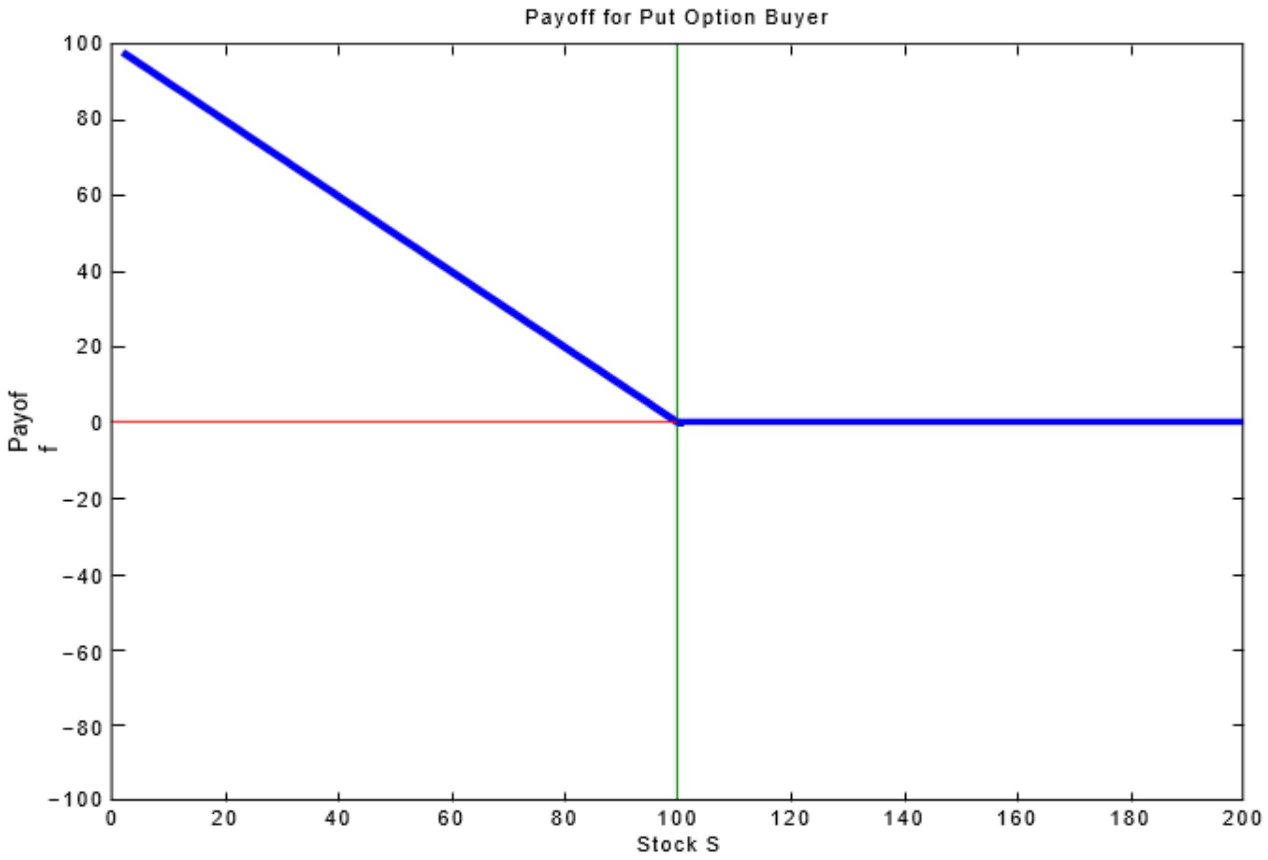
If at T, $S_T > K$ the option buyer has the choice between:

sell the stock in the market at S_T;
sell the stock to the option writer for $K < S_T$
$\Rightarrow$ Walk away from the option. Payoff $= 0$

Thus, the payoff to a put buyer is

Payoff of a Put $= \max(K - S_T, 0)$

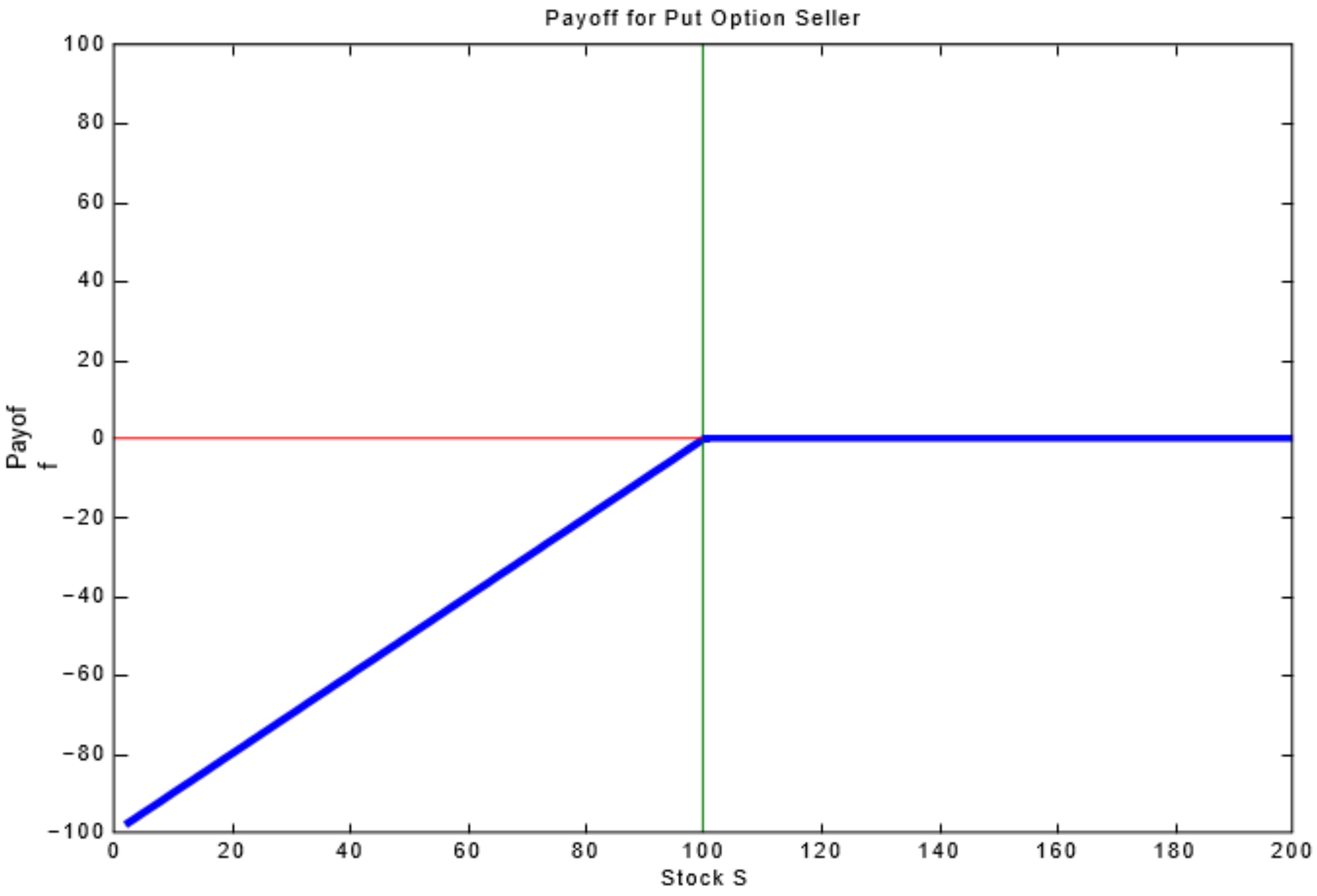
Put option payoff diagram (long)



For a stock price above K = 100 the put option expires worthless (OTM)

For a stock price less than K, payoff decreases one-for-one with stock price (ITM)

Put option payoff diagram (short or written)



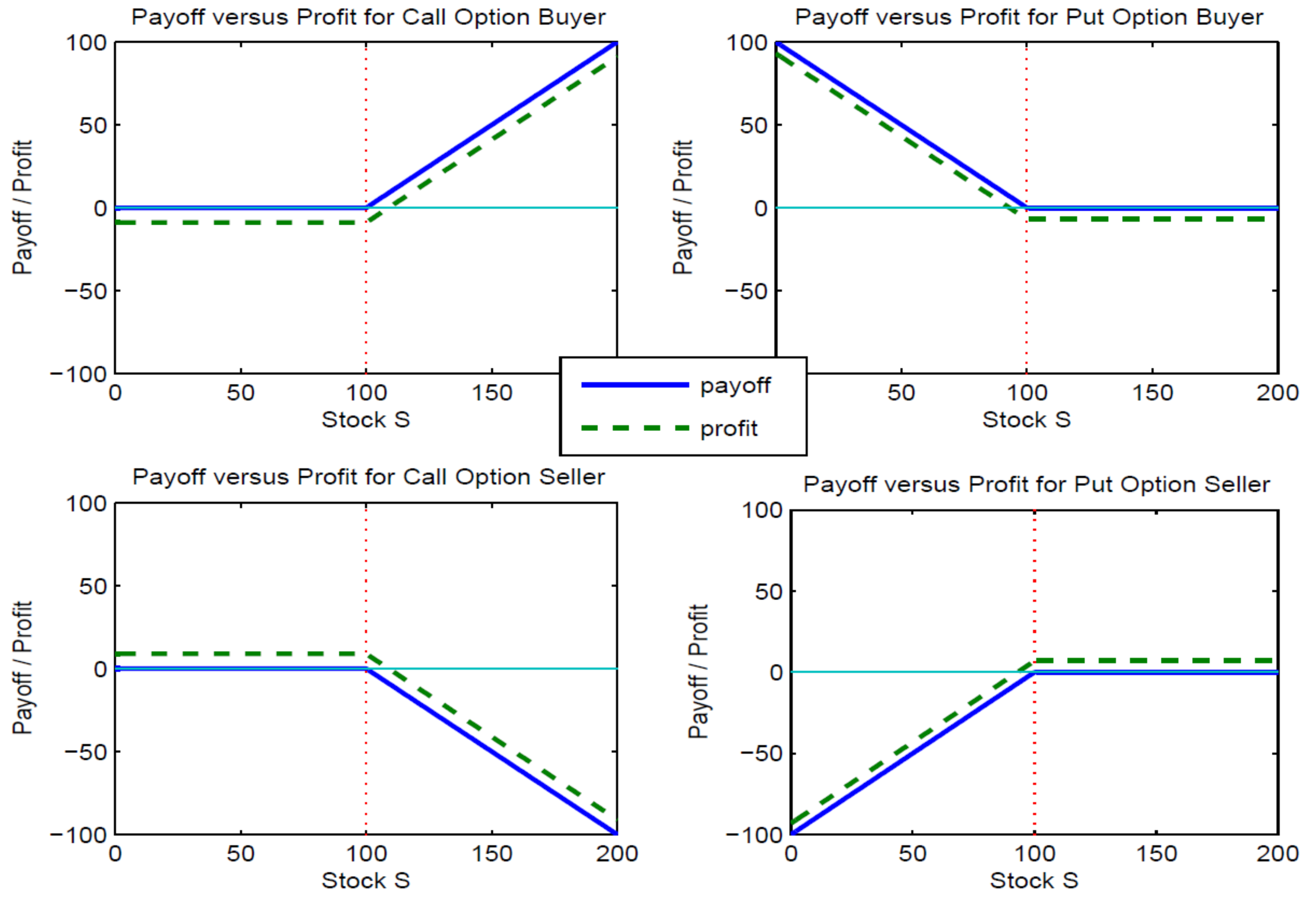
Payoff versus profit

- An option buyer has to pay a premium to seller. Thus, his/her profit is approximately equal to the payoff in previous pictures, minus the option premium.

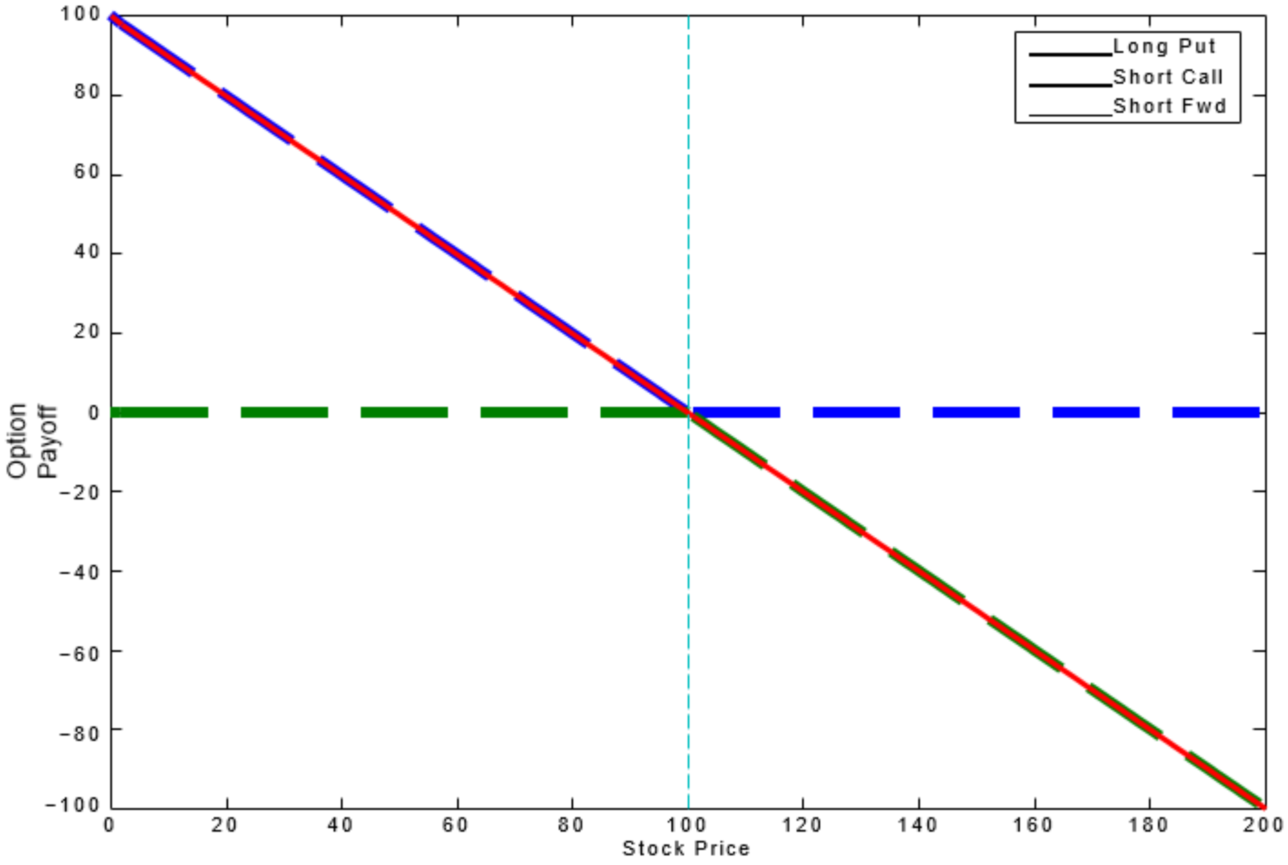
- Similarly, an option seller receives the premium from the buyer. Thus, his/her profit is approximately equal to payoff in previous pictures, plus the option premium.

- Note that in profit diagrams, it is standard not to adjust for the time value of the upfront premium payment.

Payoff versus profit



Put-Call Parity



The combination "Long Put + Short Call" equals a short forward contract with delivery price K.

We found the value of a short forward earlier, which implies the **Put Call Parity** relationship:

$$\text{Put} - \text{Call} = e^{-r \times T} \times (K - F_{0,T})$$

Put-Call Parity

- Given the price of a call, we can always get the price of the put (and vice versa)
  - Useful for seeking arbitrage opportunities & inferring value of puts from formulas for value of calls
  - Useful for interpretation, e.g., value & properties of stock prices for leveraged firms and the value of risky corporate debt
  - *Important: This formula only holds for European options!*

- For a non-dividend paying stock, we can rewrite this in terms of the current stock price:

$$\begin{aligned}
\text{Put} &= \text{Call} + e^{-r\times T} \times (K - F_{0,T}) \\
&= \text{Call} + e^{-r\times T} \times K - e^{-r\times T} \times F_{0,T} \\
&= \text{Call} + e^{-r\times T} \times K - e^{-r\times T} \times S_0 \times e^{r\times T} \\
&= \text{Call} + e^{-r\times T} \times K - S_0
\end{aligned}$$

- For instance, if $S_0 = \$100$, $T = 1$, $r = 5\%$ *K =$100, and Call* $= \$10.4506$, then

$$\begin{aligned}
\text{Put} &= \$10.4506 + e^{-5\%\times 1} \times \$100 - \$100 \\
&= \$10.4506 + \$95.1229 - \$100 = \$5.5735
\end{aligned}$$

Popular options strategies

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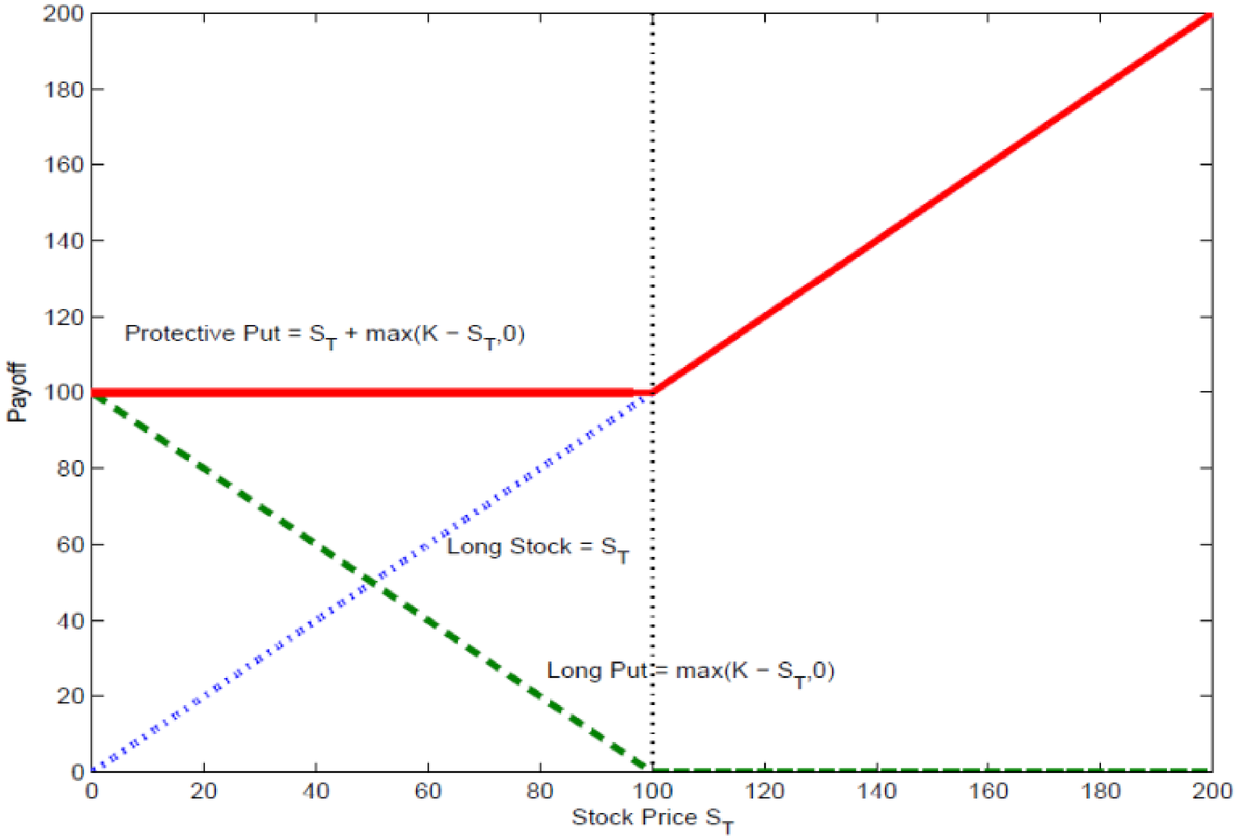
Option strategies: Protective put

Combining options with various strike prices allows a trader to speculate on particular views, or a hedger to construct customized portfolio hedges

Protective Put: Buy a put to hedge the downside of a long position in stock

- Portfolio payoff at T = $S_T + \max(K - S_T, 0)$

$K = 100$



This is like portfolio insurance

Hedging with options vs. forwards

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<u>Example</u>: A fund manager is long a stock with price S and it is worried about S declining.

Consider 2 strategies:

(1) It can hedge the risk by shorting a forward or futures contract on S with delivery price K

$$\text{Payoff at } T = S_T + (K - S_T) = K$$

(2) It can insure against the risk by buying a put option

$$\text{Payoff at } T = S_T + \max(K - S_T, 0) = K \text{ when } S_T < K$$
$$= S_T \text{ when } S_T \geq K$$

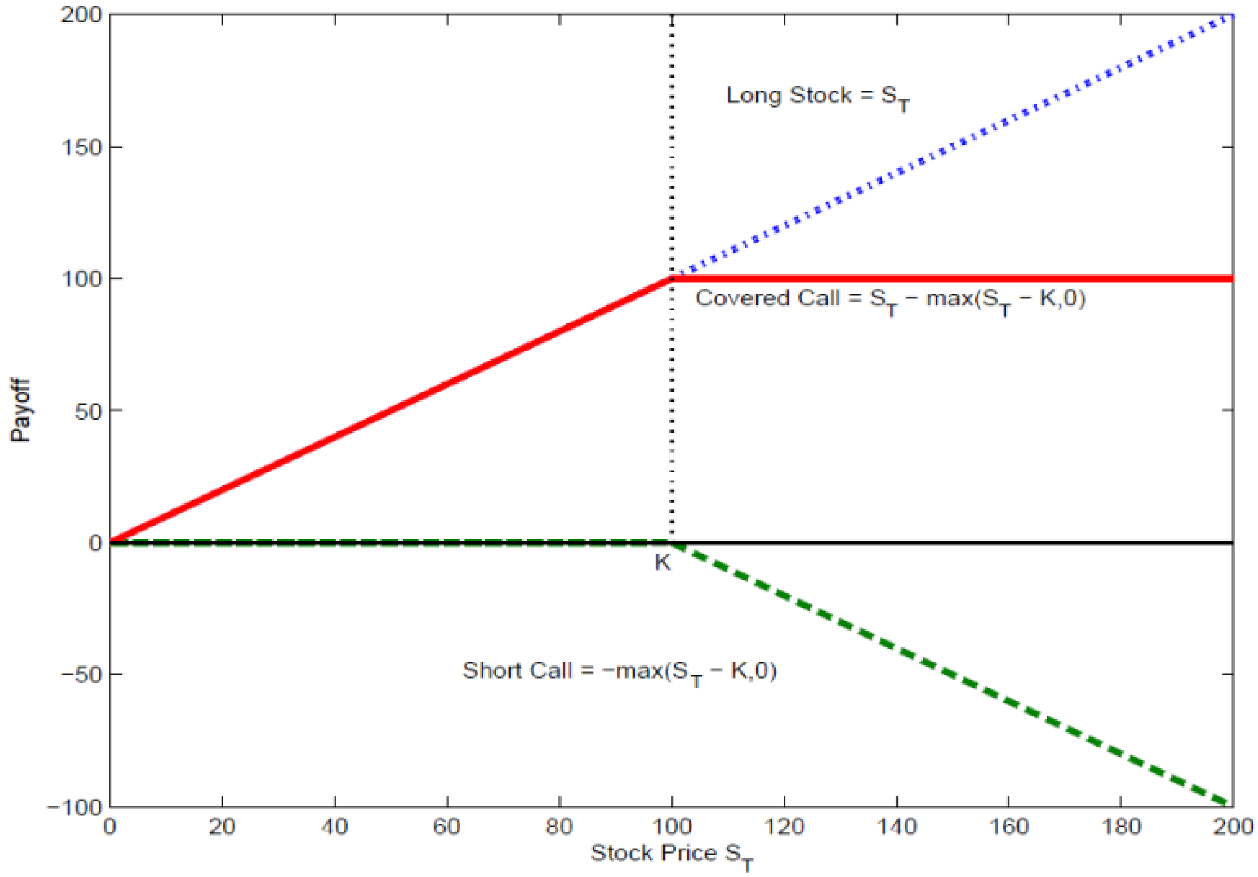
- In this case, the option is really an insurance contract
  - As with any insurance contract, it costs money upfront to purchase options (the option premium)
- By contrast, it costs nothing to enter the forward contract

Q: *Which strategy is better?*

Options strategies: Covered call

Covered Call: A short call gives unlimited liability. Holding the stock provides a hedge for the selling of a call option

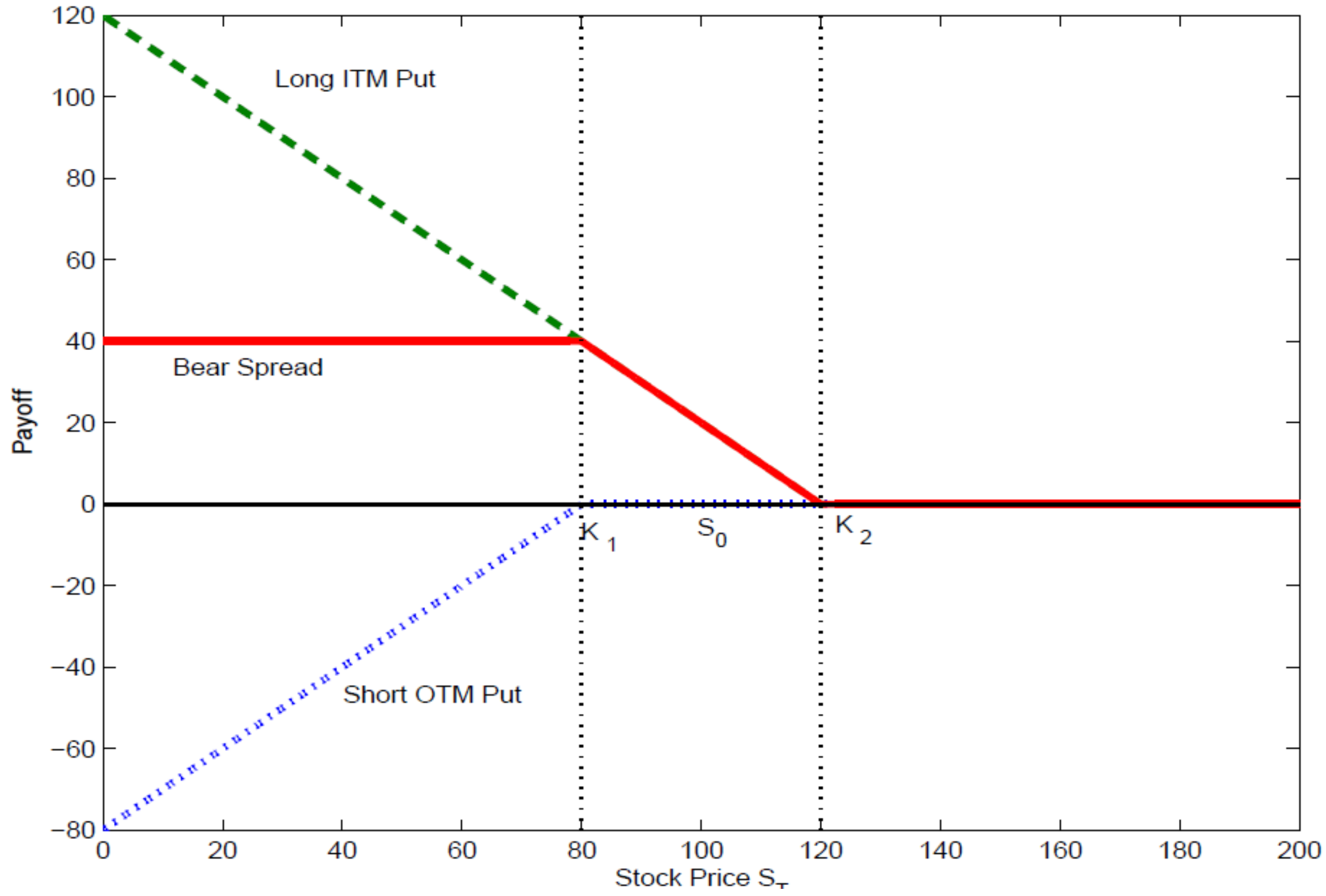
- Payoff at T = $S_T - \max(S_T - K, 0)$



Options strategies: Bear spread

Bear Spread: Bet on a decrease in the stock price.

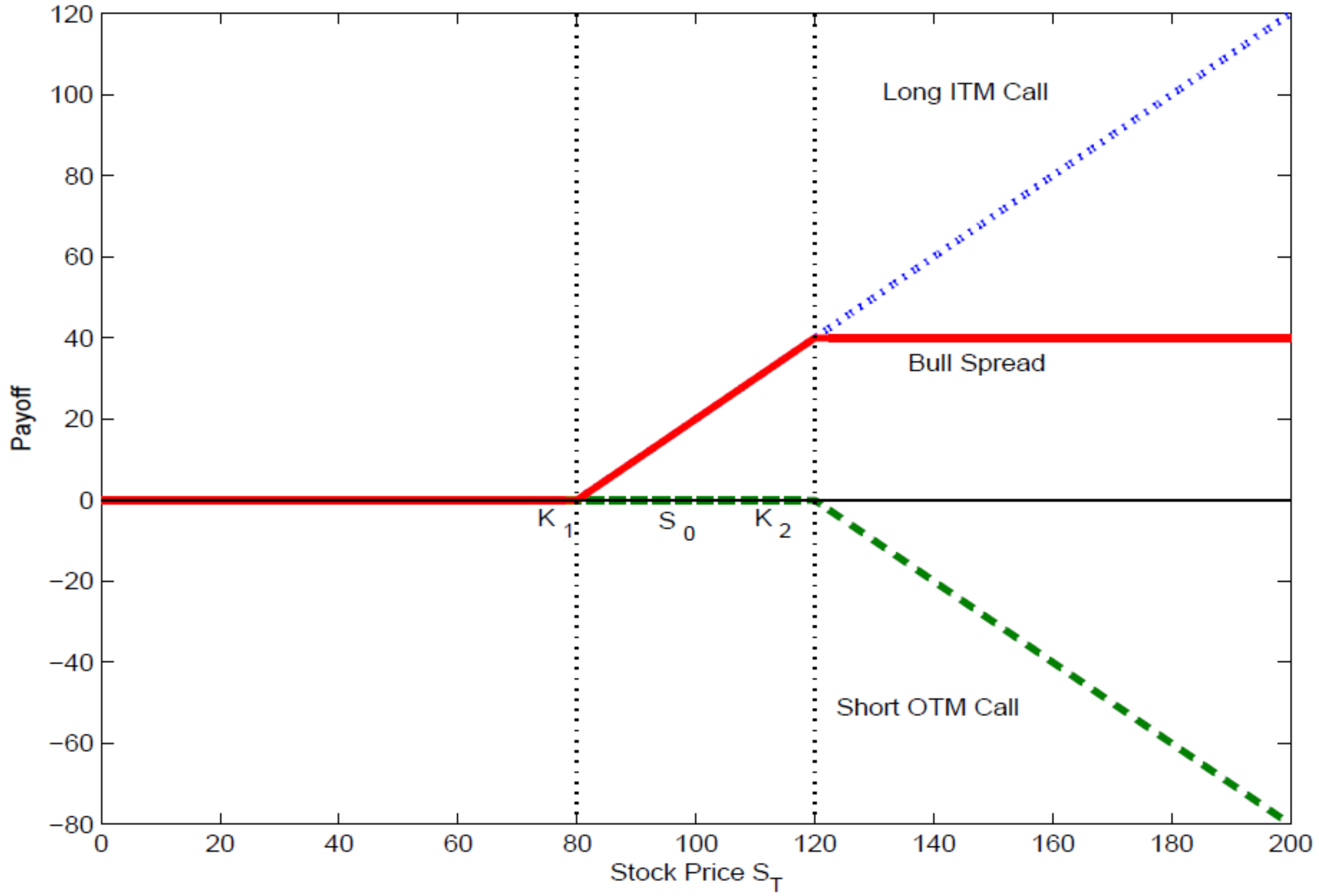
- Short OTM put (strike K_1) and long ITM put (strike $K_2 > K_1$)



Options strategies: Bull spreads

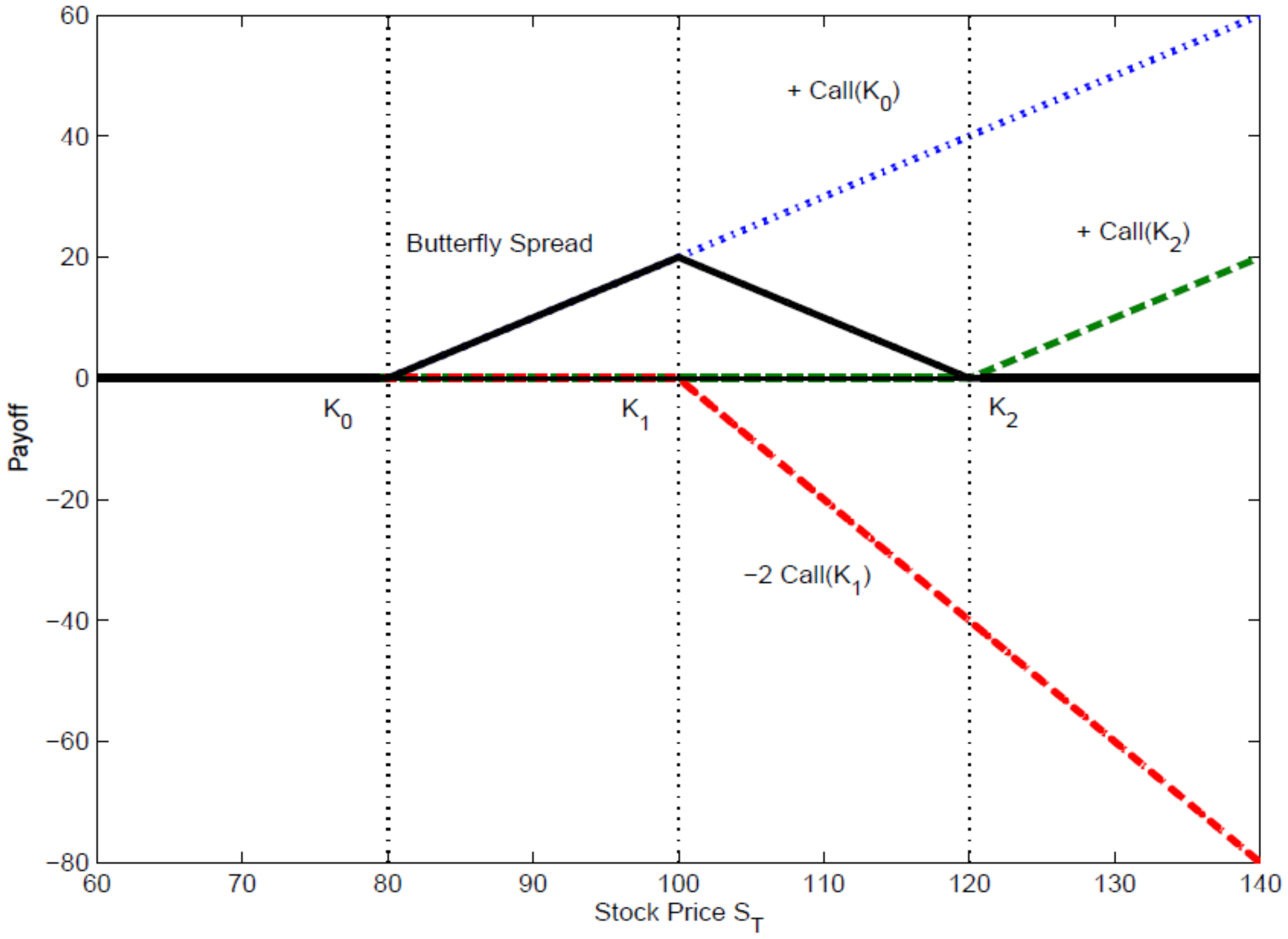
Bet on an increase in the stock price.

Long ITM call (strike K_1) and short OTM call (strike $K_2 > K_1$)



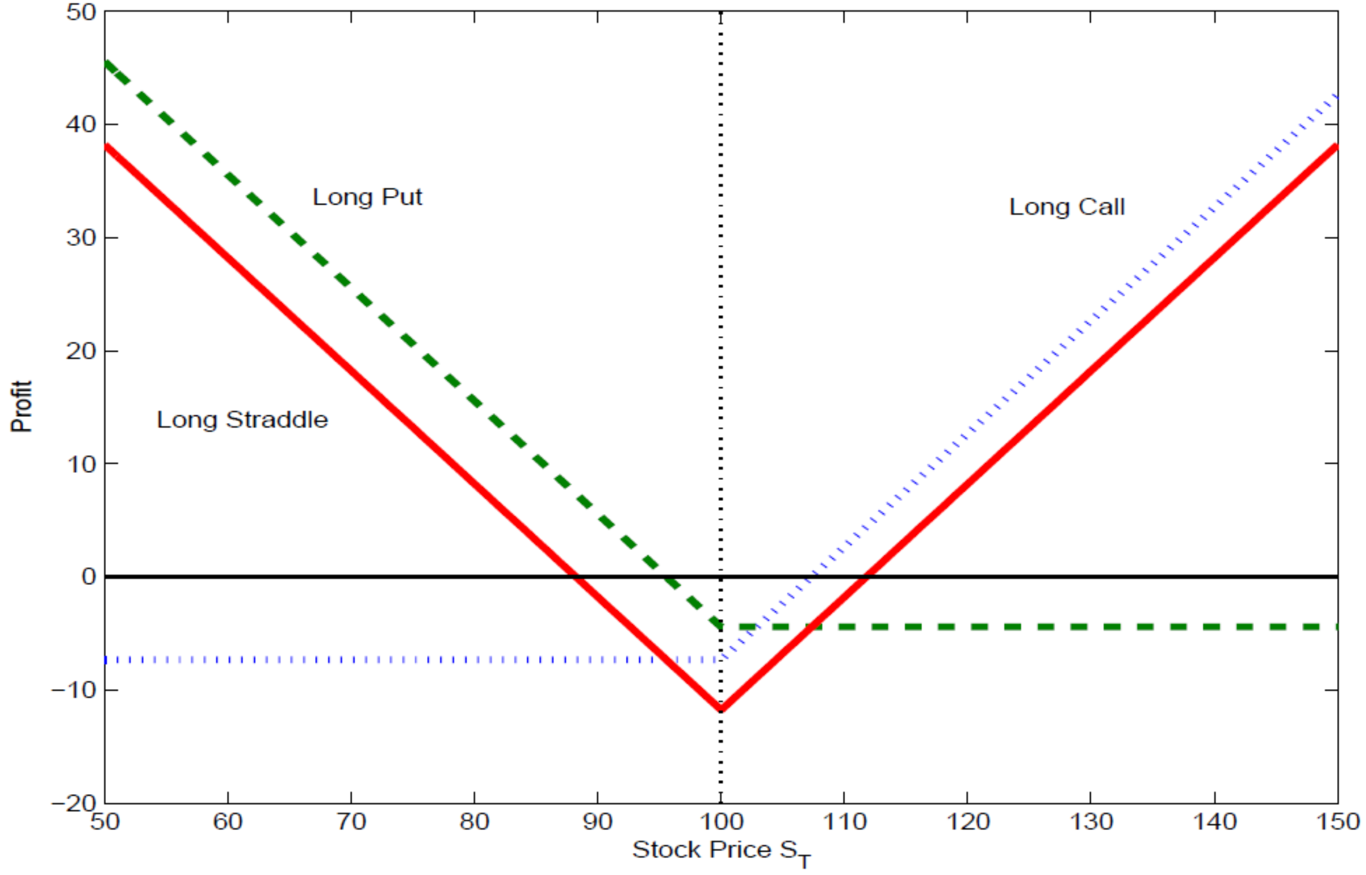
Options strategies: Butterfly spread

Butterfly Spread: Long 1 call with strike K_0, short 2 calls with strike K_1, and long 1 call with strike K_2, with $K_0 < K_1 < K_2$ and $K_1 = (K_0 + K_2)/2$



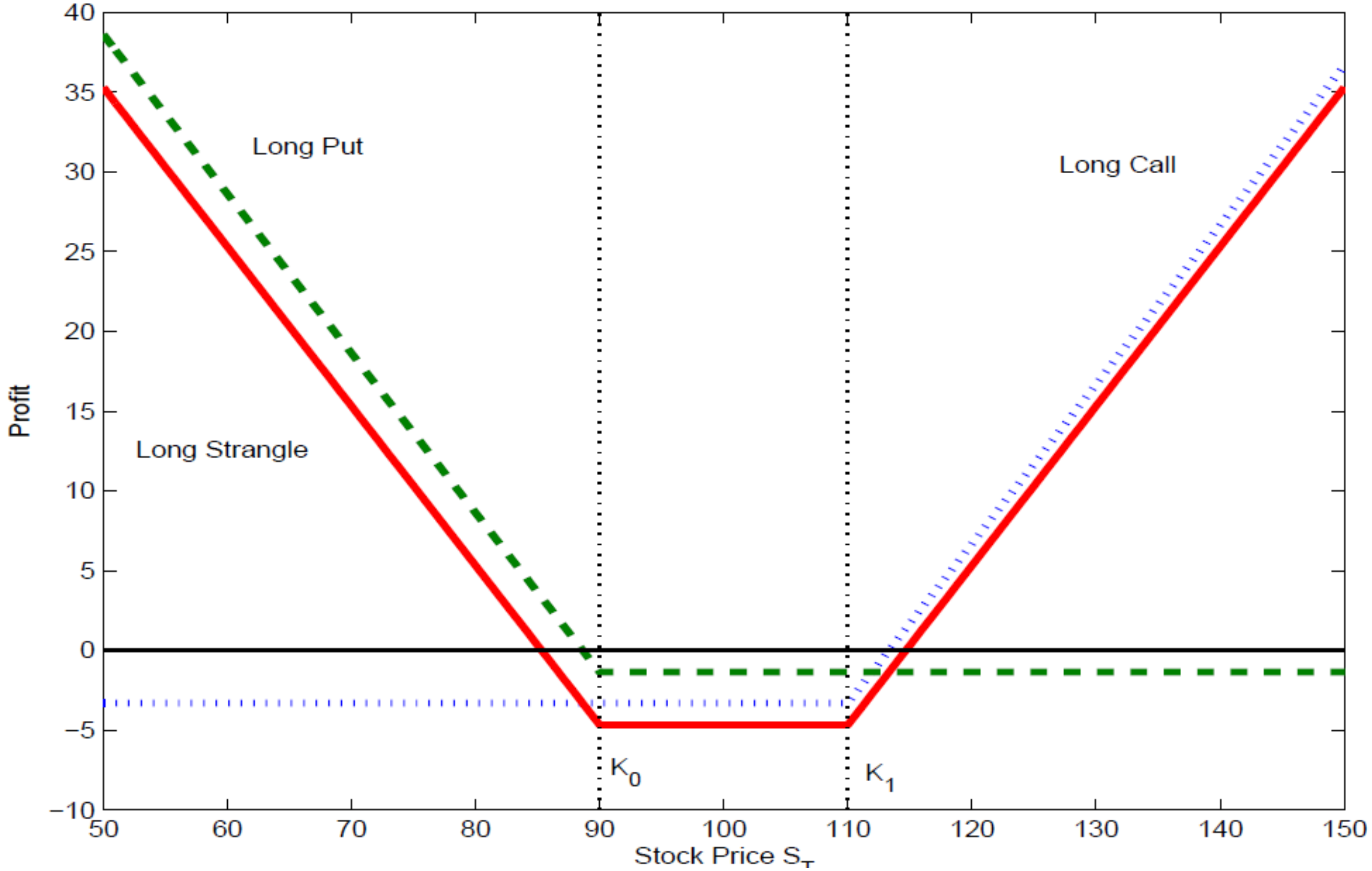
Options strategies: Straddle

Straddle. Bet on high volatility. Long a call and a put with the same strike



Options strategies: Strangle

Strangle. Bet on large movements. Long a put with strike K_0 and a call with strike $K_1 > K_0$



Pricing options on binomial trees and risk-neutral pricing

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How do we price an asset?

- NPV rule: Value of an asset = net present value of future cash flows, discounted at the opportunity cost of capital
- Biggest challenge for directly discounting future cash flows on options is difficulty of identifying the cost of capital
  - Technically that's because the implicit leverage in an option position is constantly changing over time, and the amount of leverage affects the discount rate
- A no-arbitrage approach, which can be implemented with binomial pricing, avoids the need to explicitly identify the relevant cost of capital
  - It is implicit in the value of the underlying asset in the options contract
- Binomial trees incorporate the **six main factors affecting the price of a stock option**:
  - (1) current stock price, S_0; (2) strike price, K; (3) time to expiration, T; (4) volatility of the stock price, σ; (5) risk-free interest rate, r; (6) expected dividends

One step binomial trees

Today is t = 0 and you are evaluating a stock

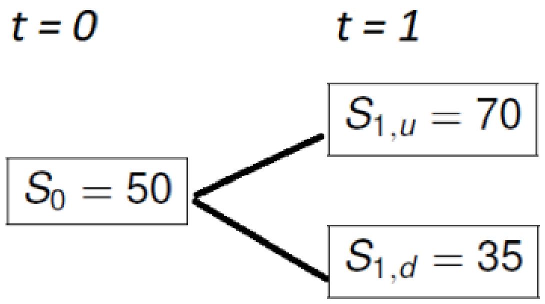
You approximate the distribution of the stock's price at time 1 with a 2-point distribution

- $S_{1,u}$ = 70 with probability q = 0.7
- $S_{1,d}$ = 35 with probability $1\text{-}q$ = 0.3

If the market expects a return R = 19% (simple interest basis), then the price today is

$$S_0 = \frac{E\left[S_1\right]}{1+R} = \frac{q \times S_{1,u} + (1-q) \times S_{1,d}}{1+R} = \frac{.7 \times 70 + .3 \times 35}{1.19} = 50$$

This model of the evolution of the stock price can be represented in a binomial tree:



Expected return and volatility implied by binomial tree

- Expected (Gross) Return:

$$
\begin{aligned}
E\left(\frac{S_1}{S_0}\right) &= q \times \left(\frac{S_{1,u}}{S_0}\right) + (1-q) \times \left(\frac{S_{1,d}}{S_0}\right) \\
&= 0.7 \times \left(\frac{70}{50}\right) + 0.3 \times \left(\frac{35}{50}\right) \\
&= 1.19
\end{aligned}
$$

- Variance. From the definition of variance:

$$
\begin{aligned}
E\left\{\left[\frac{S_1}{S_0} - E\left(\frac{S_1}{S_0}\right)\right]^2\right\} &= q \times \left(\frac{S_{1,u}}{S_0} - 1.19\right)^2 + (1-q) \times \left(\frac{S_{1,d}}{S_0} - 1.19\right)^2 \\
&= 0.7 \times \left(\frac{70}{50} - 1.19\right)^2 + 0.3 \times \left(\frac{35}{50} - 1.19\right)^2 \\
&= 0.1029
\end{aligned}
$$

The standard deviation (volatility) is $\sqrt{.1029} = 0.3207$

Option prices on a binomial tree

Can an option on the stock be valued using the NPV rule directly?

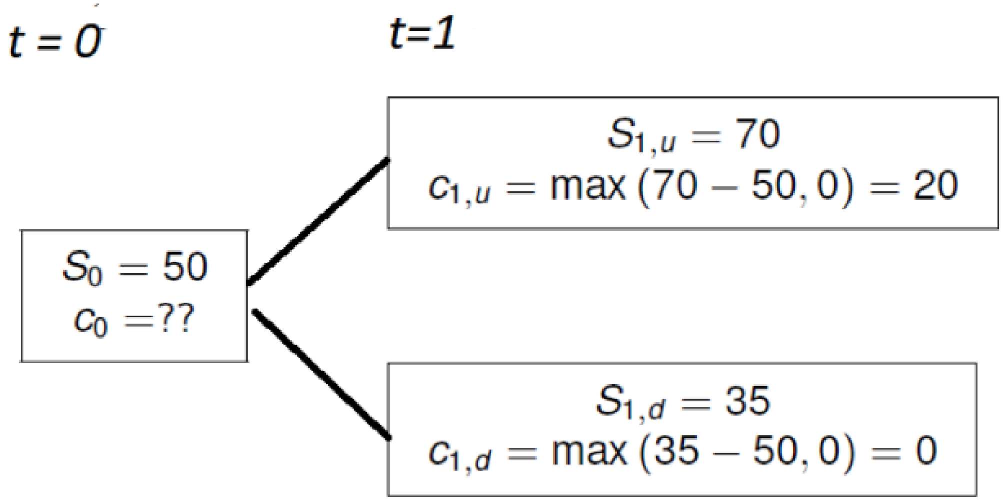
How can we price a call option with maturity $T = 1$ and strike $K = 50$?

Step 1: figure out the cash flows from the call at each node of tree

$$\text{In the Up Node} = c_{1,u} = \max(S_{1,u} - K, 0) = \max(70 - 50, 0) = 20$$

$$\text{In the Down Node} = c_{1,d} = \max(S_{1,d} - K, 0) = \max(35 - 50, 0) = 0$$

On the tree:



What should be the cost of capital for the call option?

We don't know yet, but it will turn out to be different than the cost of capital for the stock.

Pricing option with a "replicating portfolio"

Say r = .11 (continuously compounded risk-free rate)

Consider a portfolio that has the stock and risk-free bonds, with

- Position Δ = 0.5714 in stocks, for a dollar value $\Delta \times S_0$ = 28.5714
- Position of B_0 = -17.9167 in bonds (negative => short bonds) (note: $17.9167 \cdot e^{.11} = 20$)
- Value of portfolio today is V_0 = 28.5714 – 17.9167 = 10.6547

What is the value of the portfolio at time 1?

$$\text{In the Up Node} = V_{1,u} = \Delta \times S_{1,u} + B_0 \times e^r = 0.5714 \times 70 - 20 = 20$$
$$\text{In the Down Node} = V_{1,d} = \Delta \times S_{1,d} + B_0 \times e^r = 0.5714 \times 35 - 20 = 0$$

This is identical to the payoff on the call option!

No arbitrage =>

$$c_0 = V_0 = \Delta \times S_0 + B_0 = 10.6547$$

If this condition fails, "buy low, sell high"…

How do we find the replicating portfolio in general?

We want to match the payoff on the portfolio, π, with the payoff on the option on each node

That requires:

$$\pi_{1,u} = \Delta S_{1,u} + B_0 e^r = c_{1,u}$$

$$\pi_{1,d} = \Delta S_{1,d} + B_0 e^r = c_{1,d}$$

which implies:

$$B_0 e^r = c_{1,u} - \Delta S_{1,u} = c_{1,d} - \Delta S_{1,d}$$

- One equation in one unknown (Δ):

$$\Delta = \frac{c_{1,u} - c_{1,d}}{S_{1,u} - S_{1,d}}$$

- Interpretation: $\Delta =$ sensitivity of call price to changes in the stock price

How do we find the replicating portfolio in general?

Once we know Δ we can solve for B_0:

$$\pi_{1,u} = \Delta S_{1,u} + B_0 e^{r} = c_{1,u} \Rightarrow$$

$$B_0 = e^{-r}(c_{1,u} - \Delta S_{1,u})$$

This logic applies to put options too.

Example revisited

t = 0 *t=1*

$$S_0 = 50 \qquad c_0 = ??$$

$$S_{1,u} = 70 \qquad c_{1,u} = \max(70 - 50, 0) = 20$$

$$S_{1,d} = 35 \qquad c_{1,d} = \max(35 - 50, 0) = 0$$

$$\Longrightarrow \Delta = \frac{c_{1,u} - c_{1,d}}{S_{1,u} - S_{1,d}} = \frac{20 - 0}{70 - 35} = 0.5714$$

- The portfolio long the call and short Δ stocks is valued at time $i = 1$

$$\Pi_{1,u} = c_{1,u} - \Delta \times S_{1,u} = 20 - 0.5714 \times 70 = -20$$

$$\Pi_{1,d} = c_{1,d} - \Delta \times S_{1,d} = 0 - 0.5714 \times 35 = -20$$

- The bond position is then

$$B_0 = e^{-11\%} \times (-20) = -17.9167$$

Summing up

In order to price any derivative of the stock with payoff $V_{1,u}$ and $V_{1,d}$ on the tree, the general procedure is to:

1. Define delta to invest in stocks: $\Delta = \frac{V_{1,u} - V_{1,d}}{S_{1,u} - S_{1,d}}$
2. Compute amount of bonds: $B_0 = e^{-rT} \times (V_{1,u} - \Delta \times S_{1,u})$
3. Compute value of security: $V_0 = \Delta \times S_0 + B_0$

Where did the probabilities go?

- The pricing formula does not include the true probability of moving up or down (q, 1-q)
- Does this imply that those probabilities do not affect option prices?
- Answer is yes and no…

1. Given S_0, $S_{1,u}$ and $S_{1,d}$, options' payoffs can be replicated without reference to probabilities
$\Longrightarrow$ No impact of q on prices.

2. However, q determines the expected future stock price. Given a discount (e.g. 19%), S_0 is determined as the PV of future stock price.
$\Longrightarrow$ The current value of S_0 already depends on q!

3. Since option values depend on S_0, the probability q *does* impact the value of options

An important implication of no direct dependence on probabilities is that we can use the trick of "risk neutral" pricing

Risk neutral pricing

- The above procedure of finding a replicating portfolio is somewhat cumbersome
- There is an alternative procedure that is easier to work with called risk neutral pricing
- Since *given* S_0, $S_{1,u}$ and $S_{1,d}$ the probability q does not impact the price of the option, we can choose a fake probability q^* that simplifies the computations

With risk neutral pricing, we choose q* so that all risky assets earn the risk-free rate

- Find q^* such that

$$q^* \times S_{1,u} \times e^{-r \times T} + (1 - q^*) \times S_{1,d} \times e^{-r \times T} = S_0 \quad \text{(notice it is } r\text{, not } R\text{)}$$

$$q^* = \frac{S_0 \times e^{r \times T} - S_{1,d}}{S_{1,u} - S_{1,d}}$$

In other words,

$$S_0 = E^* \left(e^{-r \times T} S_1 \right)$$

Risk neutral pricing

We can likewise price any derivative simply by calculating the expected payoff using *q** and discounting by the risk-free rate

(1) $$\boxed{\textbf{Price of any derivative security} = E^{*}\left[e^{-rT}\ \textbf{Derivative Payoff}\right]}$$

The "*" on *E**[.} denotes that we used the risk-neutral probability *q**

Example revisited

Let's see if it works for the previous example

$$\text{Risk Neutral Probability: } q^* = \frac{50 \times e^{.11} - 35}{70 - 35} = 0.5947$$

Call Price:

$$c_0 = e^{-r \times T} \times E^*[c_1] = e^{-.11} \times [q^* \times 20 + (1 - q^*) \times 0] = 10.6547$$

Risk neutral pricing: A recipe

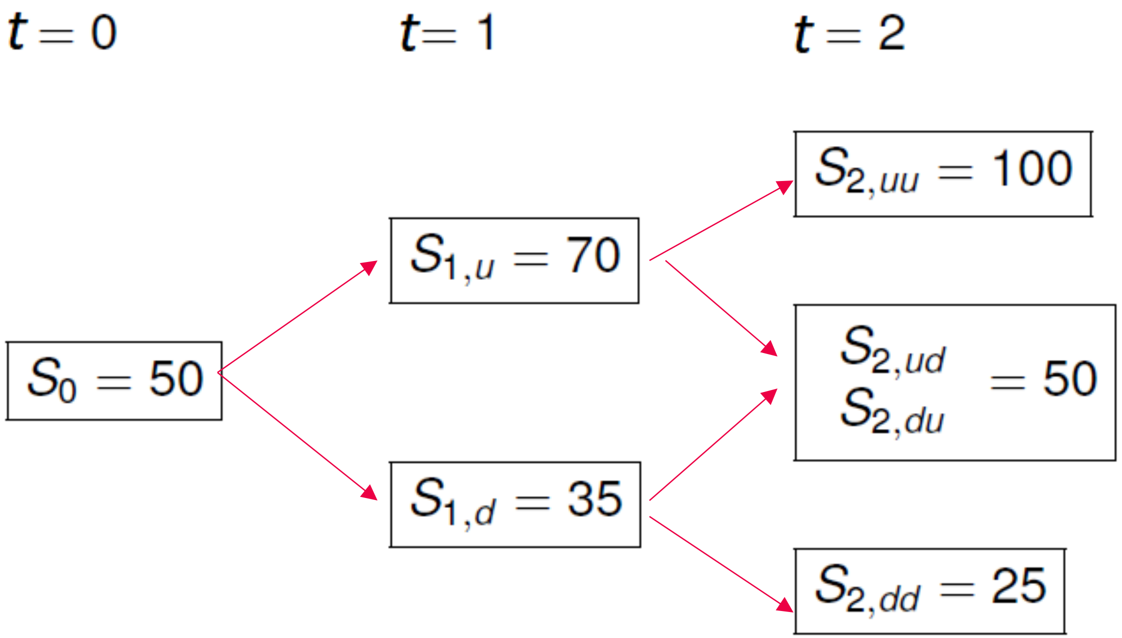
- The recipe to use risk neutral pricing to price derivatives securities with payoff at time T is
  - Assume (counterfactually) that present values can be calculated as if everyone is risk neutral
  - Based on model and current price of underlying asset, compute risk-neutral probabilities
  - Price any derivative security based on that underlying asset as

$$\text{Price of Derivative Security} = E^{*}\left[e^{-r \times T} \times (\text{Payoff at } T)\right]$$

- This works for any model of asset price evolution, not just the binomial tree model
  - It is a more general implication of no-arbitrage
  - We will use it again in Monte Carlo pricing

Two-step binomial trees

- Given the above methodology, we can now see it at work in a slightly more complicated example.
- Consider the two step binomial tree



- We want to price an option with maturity $T = 2$ and strike price $K = 50$.

Risk neutral pricing on 2-step tree

Move backward on the tree:

t=0

$\mathbf{S_0} = \mathbf{50}$

$q_0^* = 0.5947$

$c_0 = e^{-r} q_0^* \times c_{1,u}$

$= 13.4294$

t=1

$\mathbf{S_{1,u}} = \mathbf{70}$

$q_{1,u}^* = \frac{70e^r - 50}{100 - 50} = 0.5628$

$c_{1,u} = e^{-r} q_{1,u}^* c_{2,uu} = 25.2083$

$\mathbf{S_{1,d}} = \mathbf{35}$

$c_{1,d} = e^{-r} \times E^*[c_2] = 0$

t=2

$\mathbf{S_{2,uu}} = \mathbf{100}$

$c_{2,uu} = 50$

$\mathbf{S_{2,ud}} = \mathbf{S_{2,du}} = \mathbf{50}$

$c_{2,ud} = c_{2,du} = 0$

$\mathbf{S_{2,dd}} = \mathbf{25}$

$c_{2,dd} = 0$

American options

- An American Option is identical to a European Option, but it can be exercised *anytime* before expiration
- Examples of American options
  1. Individual stock options
  2. Some index options, such as options on S&P100 (CBOE)
  3. Some widely traded OTC interest rate options
     - American Interest Rate Swaptions (options to enter into an interest rate swap) are particularly popular
  4. American options embedded in other contracts
     - Callable and Puttable bonds; Convertible Bonds; Mortgages
  5. Real options
     - Real investments have optionality (e.g. option to invest in a new project, option to close down a plant etc.)
- In all these cases, not only decide **IF** we want to exercise the option, but also **WHEN**

American call options: Early exercise

An American Call on a non dividend paying stock is always worth “more alive than dead”

- Intuition: If we exercise early, we lose both
  1. the time value of money on the strike
  2. the value of the right to exercise the option in the future

=> better to sell it if you don’t want it than to early-exercise it

- But what about a stock that pays dividends?
  - If wait to exercise you could miss dividends
  - If a dividend is paid the stock price drops
  - Early exercise avoids this, at the cost of (1) and (2) above
  - Conclusion is that sometimes it is worth early exercising and sometimes it isn’t, you have to analyze the particular situation

American put options: Early exercise

If we exercise a Put early:

(Good) We receive K today instead of in the future $\Longrightarrow$ gain the interest on K

(Bad) We lose any dividends paid between now and maturity

(Bad) We receive K today for a stock that may be worth more than K at T

- Even if $S < K$ today, it may well be that $S_T > K$, in which case we will not want to exercise
- S today reflects all of the possible possibilities, including high S_T
- By waiting, we reserve the right of **not exercising** if $S_T > K$

Still, even if there are no dividends, put options may be worth exercising early

- For example, if the company goes bankrupt, then $S = 0$. Exercising immediately is then optimal, as you receive K today. It cannot get any better than this, and waiting leads to losing the interest

American options: pricing and optimal exercise

- How do we price American options? How do we compute the optimal exercise policy?
  - E.g., On May 11 you are long a Sept. put option on the S&P 100 index with strike price $K = 740$. The S&P 100 on May 11 stands at 689.83
  - The option is currently in the money by 7%
  - Do you exercise immediately or keep the option alive?
- General procedure is to compare the value of immediate exercise with the value of the option if it is kept alive. Exercise if and only if:

K – S > Discounted value of future distribution of payoffs if wait

- Binomial trees are an excellent tool to calculate the value of waiting

Valuing an American option on a 2-step tree

- $S_0 = 100,\ K = 100,\ T = 1,\ r = 2\%,\ \sigma = 30\%$
  $\Longrightarrow u = e^{\sigma\sqrt{T/2}} = 1.2363$ and $q^* = 0.4707$
- Consider a put option with $K = 100$. The European option is given by

$i = 0$

$S_0 = 100$
$p_0^E = e^{-\frac{r}{2}} E^*[p_1]$
$= e^{-\frac{r}{2}}$ 1-q^* $p_{1,d}$
$= 9.495$

$i = 1$

$S_{1,u} = 123.631$
$p_{1,u}^E = e^{-\frac{r}{2}} E^*[p_2 | S_{1,u}] = 0$

1-

$S_{1,d} = 80.886$
$p_{1,d}^E = e^{-\frac{r}{2}} E^*[p_2 | S_{1,d}]$
$= e^{-\frac{r}{2}}$ 1-q^* $p_{2,dd} = 18.119$

$i = 2$

$S_{2,uu} = 152.847$
$p_{2,uu}^E = 0$

$S_{2,ud} = S_{2,du} = 100$
$p_{2,ud}^E = p_{2,du}^E = 0$

$S_{2,dd} = 65.425$
$p_{2,dd}^E = 34.575$

Valuing an American option on a 2-step tree

Now let's consider the same situation but with an American put option

- At expiration (i =2) payoff is the same at each node for a European or American option
- At i =1 at each node u and d, compare the payoff if exercise with the PV of the expected payoff if wait
  - At node (1, u) it is OTM => no exercise => $p_{1,u}^{A} = 0$
  - At node (1, d)
    - If exercise immediately get K – S = 100 – 80.886 = 19.114
    - If wait value is equal to European counterpart

$$= e^{-r/2} E^* [p_2^A \mid S_{1,d}] = 18.119$$

  - It is better to exercise immediately and get 19.114 than to wait which is only worth 18.119

- Finally, the value of the option at i =0 is:

$$p_0^A = \max\left(K - S_0, e^{-r/2} E^* \left[p_1^A\right]\right) = e^{-r/2} \left(1 - q^*\right) p_{1,d}^A = 10.017$$

Valuing an American option on a 2-step tree

$i = 0$

$i = 1$

$$S_{1,u} = 123.631$$
$$p^A_{1,u} = \max\left(K - S_{1,u}, e^{-r/2}E^*[p^A_2 | S_{1,u}]\right)$$
$$= 0$$

$$S_0 = 100$$
$$p^A_0 = \max\left(K - S_0, e^{-r/2}E^*[p^A_1]\right)$$
$$= e^{-r/2}(1\text{-}q^*)\, p^A_{1,d} = 10.017$$

$$S_{1,d} = 80.886$$
$$p^A_{1,d} = \max\left(K - S_{1,d}, e^{-r/2}E^*[p^A_2 | S_{1,d}]\right)$$
$$= \max(19.114, 18.119) = 19.114$$

The American option is worth more than the otherwise similar European option in this case because early exercise is optimal at i=1 on the down node.

Valuing options on multi-step trees

- The tree methodology can be easily extended to many steps
- Remember that the pair (i, j) represents time $i = 0, 1, 2..., n$ and node $j = 1, 2, ..., n$
- With European style derivatives, we solve for prices $V_{i,j}$ using the rule

$$V^E_{i,j} = e^{-r\times h} \times E^* \left[V^E_{i+1} | (i, j) \right]$$

where $h = T/n$ is the time interval between steps

- With American style derivatives, we solve for prices $V_{i,j}$ using the rule

$$\mathbf{V^A_{i,j}} = \max \left\{ \mathbf{g_{i,j}} \, , \ \ \mathbf{e^{-r\times h}} \times \mathbf{E^*} \left[\mathbf{V^A_{i+1}} | (\mathbf{i}, \mathbf{j}) \right] \right\}$$

where $g_{i,j}$ is the payoff from the American derivative if exercise occurs in node (i, j)

- For instance, for the case of put options, we have

$$p^A_{i,j} = \max \left\{ K - S_{i,j} \, , \ \ e^{-r\times h} \times \left(q^* \times p^A_{i+1,j} + (1 - q^*) \times p^A_{i+1,j+1} \right) \right\}$$

Final thoughts

- The same solution could have been found using a replicating portfolio approach to value the option at each node and comparing that value to the intrinsic value.
- Important to remember about risk-neutral pricing:
  - It is a convenient pricing device, and it DOES NOT imply that market participants are risk neutral!
    - Market participants *are* risk averse in our setting.
    - We can account for risk aversion in two different ways:
      1. **by adding a risk premium to the opportunity cost of capital**
      2. **by "distorting" the probabilities towards the bad states**
    - With these distorted probabilities, we can *pretend* market participants are risk neutral and discount future payoffs with the riskfree rate.
- Adding more steps to the tree gives a better approximation to the true distribution of stock prices.
  - We'll see next time that the Black-Scholes-Merton model is the limit of the binomial model as step size goes to zero.