Recitation 3

Problem 1

Consider a 2-year bond with a coupon rate of 4%, annual payments, and priced at \$98 per \$100 face value for an effective annual yield of 5.077%.

(a) Calculate the modified duration of this bond.

Solution: A bond's effective annual yield (EAY) is the simple interest rate earned on the bond over a year, taking into account all compounding opportunities. In order to use the convexity.xls calculator provided by Professor Lucas to find the bond's modified duration, we need to convert the EAY to a bond equivalent yield.

A bond equivalent yield is the single interest rate that equates the price of a bond to the present value of its future cash flows, assuming semiannual compounding. To convert an EAY to a bond equivalent yield, we can use the formula:

$$y_{ea} = (1 + y_{be}/2)^2 - 1, \tag{1}$$

where y_{ea} is the EAY and y_{be} is the bond equivalent yield.

Plugging in $y_{ea} = 5.077\%$ into Equation (1), we can solve for y_{be} to be 5.014%. Now, we can find the modified duration of the bond by inputting the following parameters into the convexity.xls calculator:

N	2	maturity in years		Convexity
Frequency	1	coupon payments per year (1 or 2)		
Coupon	4.00%	coupon rate		
Face	100	dollar face value of bond		Duration
Yield	5.014%	yield on a bond equivalent basis		
		Convexity	5.293494	
		Modified Duration	1.866401	
		Macauley Duration	1.961156	

So, the modified duration of the bond is 1.866 years.

(b) Use the modified duration calculated in Part (a) to estimate the new price of the bond if the bond's effective annual yield increases to 6.5%. Do you expect the actual price to be higher or lower than your estimate here?

Solution: Recall that we can approximate the dollar price change in a bond for a given change in its yield using the expression:

$$dP = -D_M \times P \times dy, \tag{2}$$

where P is the bond's price, D_M is its modified duration, and dy is the change in yield. Plugging in $D_M = 1.866$ from Part (a), $dy = 6.5 - 5.077 = 1.423\%$, and $P = \$98$ into Equation (2), we see that $dP = -\$2.60$. Thus, the new price of the bond after the yield increase is \$95.40.

Do we expect the actual change in price to be higher or lower than the one obtained using the approximation in Equation (2)? Recall that we can improve our approximation by including convexity—i.e., the second derivative of a bond's price with respect to its yield. The approximation that includes both duration and convexity can be written as:

$$dP = -D_M P(dy) + \frac{1}{2} C_0 P(dy)^2, \tag{3}$$

where C_0 is the convexity of the bond. Since a long position in a non-callable bond like the one in this example has positive convexity, the actual change in price will be *lower* than the one obtained using the approximation in Equation (2) that only includes duration.

More generally, positive convexity means that duration will underestimate the price increase resulting from a decline in the yield, and overestimate the price decrease from an increase in the yield.

(c) Imagine that you want to immunize a liability that comes due in 3.5 years. If you want to invest in an immunizing portfolio consisting of both the 2-year coupon bond in Part (a) and one of the following bonds, which one would you choose? (Assume coupons are paid annually.)

1. a 3.5-year, 6.25% coupon bond with an effective annual yield of 5.5%;
2. a 10-year, 3.9% coupon bond with an effective annual yield of 4.9%;
3. a 3.5-year, zero coupon bond with an effective annual yield 4.65%.

Solution: Immunization involves matching the durations of assets and liabilities in order to minimize interest rate risk. Which bond should we choose to combine with the 2-year bond in order to immunize our liability with a duration of 3.5 years?

Using the convexity.xls calculator, we can find the modified durations of each of the three bonds to be: (i) 3.24 years; (ii) 8 years; and (iii) 3.34 years.

The 3.5-year coupon bond is not suitable because it has a duration of less than 3.5 years, so any combination with the 2-year coupon bond—which has a duration of 1.866 years—would have too short a duration to immunize our liability.

The 3.5-year zero coupon bond is feasible, but almost 100% of our immunizing portfolio would have to be invested in this bond since the (Macaulay) duration of our assets and liabilities matches exactly.

Thus, a combination of the 2-year and 10-year bonds would be best. For suitable portfolio shares, it will have a higher yield and higher convexity than the 3.5-year zero coupon bond. (Intuitively, its payoffs are more like a "barbell" than the "bullet" payoffs of the 3.5-year zero coupon bond.)

Problem 2

You are managing a pension fund that has a single-payment liability in the amount of \$25 million coming due four years from today. You wish to immunize this liability by making investments today that will have a guaranteed terminal value of \$25 million in four years. Two investments are available:

1. 2-year, zero coupon notes;
2. 5-year, zero coupon notes.

(a) If today's yield curve is flat at 10%, how much must be invested in each security in order to immunize this liability? (Assume annual compounding.)

Solution: Immunization requires equalizing both the present values and durations of assets and liabilities.

Assume that the 2-year and 5-year zero-coupon notes have face values of $\$1,000$. Since the yield curve is flat at 10%, the price of the 2-year zero-coupon note is $1000/(1.1)^2 = \$826.45$, and the price of the 5-year zero-coupon note is $1000/(1.1)^5 = \$620.92$.

Since the liability involves a single payment of \$25 million in four years, its present value is \$25 million$/(1.1)^4 = \$17,075,336$. The Macaulay duration of a zero-coupon bond is equal to its time to maturity, so the modified duration of the 2-year zero-coupon note is $2/1.1 = 1.82$ years, the 5-year zero-coupon note is $5/1.1 = 4.54$ years, and the liability is $4/1.1 = 3.64$ years.

Let n_2 be the number of 2-year zero-coupon notes required to immunize the liability, and n_5 be the number of 5-year zero-coupon notes. Equating the present values of assets and liabilities, we have:

$$826.45 \times n_2 + 620.92 \times n_5 = 17,075,336.$$

Equating the value-weighted average durations of assets and liabilities implies that:

$$1.82 \times n_2 \left(\frac{826.45}{17,075,336} \right) + 4.54 \times n_5 \left(\frac{620.92}{17,075,336} \right) = 3.64.$$

Solving this system of two equations for n_2 and n_5, we find that we should invest in $n_2 = 6,836$ of the 2-year zero-coupon notes and $n_5 = 18,401$ of the 5-year zero-coupon notes in order to immunize the liability. In dollar amounts, we should invest in $6,836 \times 826.45 = \$5.65$ million of the 2-year zero-coupon notes and $18,401 \times 620.92 = \$11.42$ million of the 5-year zero-coupon notes.

(b) If interest rates at all maturities rise from 10% to 11%, would you consider yourself better or worse off than if interest rates had remained at 10%? Explain your choice with a calculation, and also explain intuitively why this is the case.

Solution: Since the cash flows of the immunizing portfolio are more "spread out" like a barbell than the single "bullet" liability that comes due in four years, the convexity of the immunizing portfolio is likely greater than that of the liability. Thus, if interest rates rise from 10% to 11%, we will be better off, as the value of the immunizing portfolio will fall less than that of the liability.

To verify this intuition, we can calculate the present values of the liability and the immunizing portfolio under the assumption that interest rates are 11%. The present value of the liability coming due in four years decreases from \$17,075,336 to \$25 million$/(1.11)^4 = \$16,468,274$. The price of the 2-year zero-coupon note is now $1000/(1.11)^2 = \$811.62$, and the price of the 5-year zero-coupon note is now $1000/(1.11)^5 = \$593.45$. The present value of the immunizing portfolio is:

$$811.62 \times 6,836 + 593.45 \times 18,401 = \$16,468,308$$

which is (slightly) greater than that of the liability.

In practice, once there is a change in interest rates, our portfolio shares have to be adjusted in order to re-immunize the liability. Such adjustments can involve non-trivial transaction costs.

Problem 3

You are a manager of a hedge fund that specializes in the fixed income market. The fund's portfolio currently consists of three positions (all yields are bond equivalent yields):

1. short position in $\$150,000$ face value of a 7-year 2.8% coupon bond, semiannual payments, priced to yield 3.95%;
2. long position in $\$350,000$ face value of a 3-year 2.25% coupon bond, semiannual payments, priced to yield 2.75%;
3. long position in $\$425,000$ face value of a 14-year 4.6% coupon bond, semiannual payments, priced to yield 4.125%.

You decide to delta-hedge the interest rate exposure of the fund's portfolio using a 3-year interest rate swap. The swap specifies a semi-annual exchange of 6-month LIBOR for a fixed payment 3.5%, paid semiannually (i.e., fixed payment of 1.75% every 6 months). The current 6-month LIBOR rate is 1.4%.

(a) Would you be a fixed or floating rate payor in the swap? Explain briefly.

Solution: Since on net you are receiving fixed payments from the portfolio—in other words, you are "long duration"—in order to delta-hedge the fund's portfolio you want to be a fixed rate payor in the swap.

(b) You calculate the dollar duration of the portfolio to be \$4,743,387. What is the notional principal of the swap that delta-hedges the portfolio?

Solution: Recall that an interest rate swap is equivalent to a long position in a fixed rate bond and a short position in a floating rate bond (or vice versa), with both positions initially priced at par.

Using the convexity.xls calculator, we can calculate the modified duration of the fixed leg of the swap with a semiannual fixed payment of 3.5% and 3 years to maturity to be 2.8245 years.

The modified duration of the floating leg of the swap is given by $0.5/(1 + 0.014/2) = 0.4965$ years, since there are 6 months until the next reset and the current 6-month LIBOR rate is 1.4%.

To find the notional principal of the swap, we equate the hedge ratio of the portfolio with that of the swap:

$$\$4,743,387 = S \times (2.8245 - 0.4965)$$

where S is the notional principal of the swap (in dollars). Solving for S, we find that the notional principal of the swap is \$2,037,560.