15.415x Foundations of Modern Finance

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Lecture 6: Risk and Return

1. Introduction

Measurement and management of risk

- Measurement and management of risk is at the core of finance:
  - An investor saving for retirement: riskier strategy with more upside vs safer one, with less downside?
  - A hedge fund: how much capital to allocate to various trading strategies?
  - An insurance company: how to manage payout risk?
  - A sovereign wealth fund: how to structure financial investments given the composition of the country's economy?

Fundamental concepts and tools

- Need a systematic framework to making decisions under uncertainty – Expected Utility Theory.

- Develop analytical tools for quantifying risk, and for dealing with portfolios of investments – portfolio analytics.

- Key concept: Diversification -- the only “free lunch” in financial markets!

2. Expected Utility

Decisions under uncertainty

- Decisions under uncertainty boil down are choices among random payoffs:

$$x = \$1{,}000 \text{ or } \$0, \qquad 50/50 \text{ odds}$$

$$y = \$600 \text{ or } \$200, \qquad 70/30 \text{ odds}$$

- How should we model such choices?
  - Naïve approach: compute expected payoff, choose the higher one.
  - What about randomness? $50/50$ gamble $\$1{,}000$ / $\$0$ vs $\$500$ for sure?

“Rational” vs “Behavioral” approaches

- Two approaches: “rational” and “behavioral.”
- Rational approach is prescriptive: a model of choice with internal consistency and basic desired properties.
- Behavioral approach is descriptive: empirically-motivated model of observed individual behavior, captures behavioral biases and inconsistencies.
- We want to make investment decisions consistently.
- Focus on the rational model.

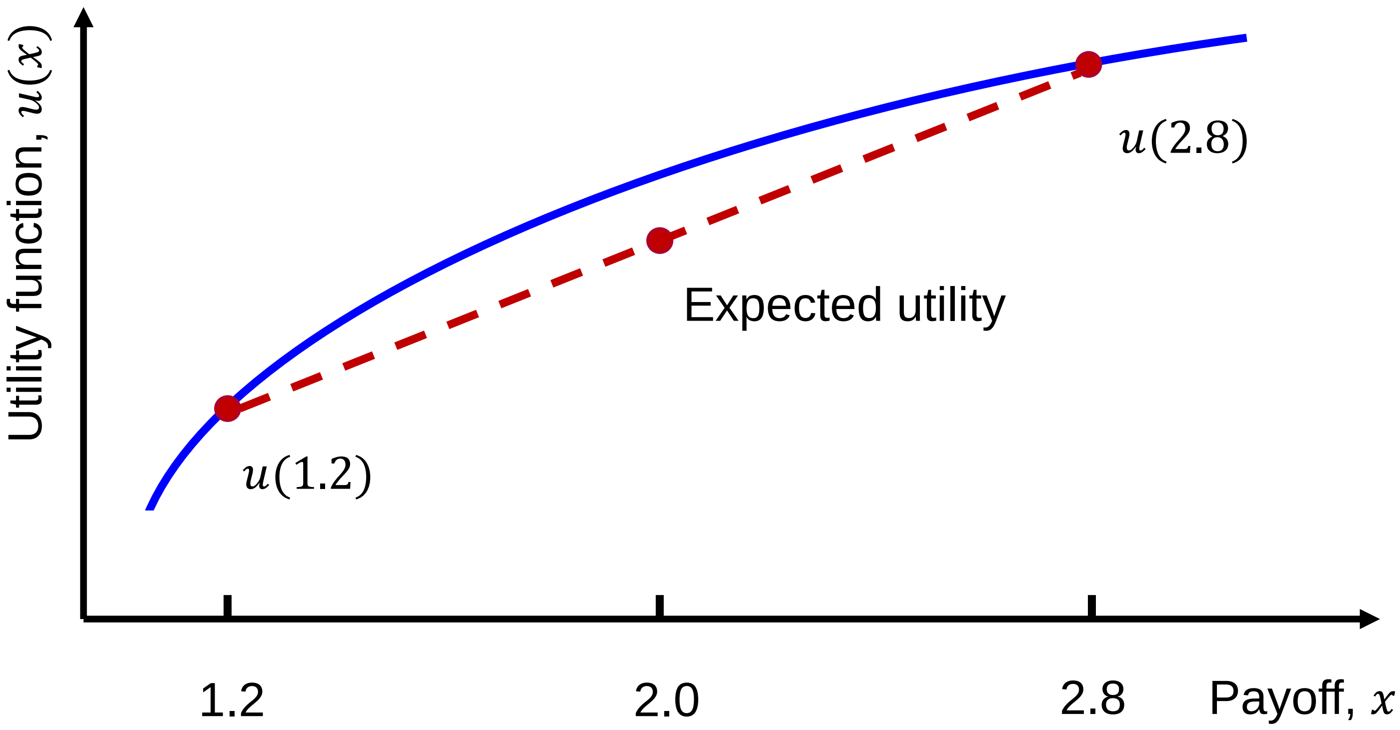
Basic assumptions

- Preferences are over outcomes only (cash flows).
- Our model abstracts from the mechanism by which cash flows are generated.

Expected utility theory

- Expected utility theory is the leading model of consistent decision making under uncertainty.
- Investor evaluates each gamble not by its expected payoff, but by its expected utility.
- A utility is a nonlinear transformation of the payoff: a $\$1,000$ payoff may not be twice as valuable as $\$500$.

An illustration



- Payoffs are transformed nonlinearly, with function $u(x)$.
- Use payoff probabilities as weights, linearly.

Choice among risky payoffs

- Utility function $u(\cdot)$ transforms payoffs.

- Investor then compares payoffs based on their expected utility:

$$x \text{ preferred to } y \quad \Leftrightarrow \quad \mathrm{E}[u(x)] > \mathrm{E}[u(y)]$$

- Select among investments consistently.

- For example, choices are transitive: if X is preferred to Y, and Y to Z -- then X is preferred to Z.

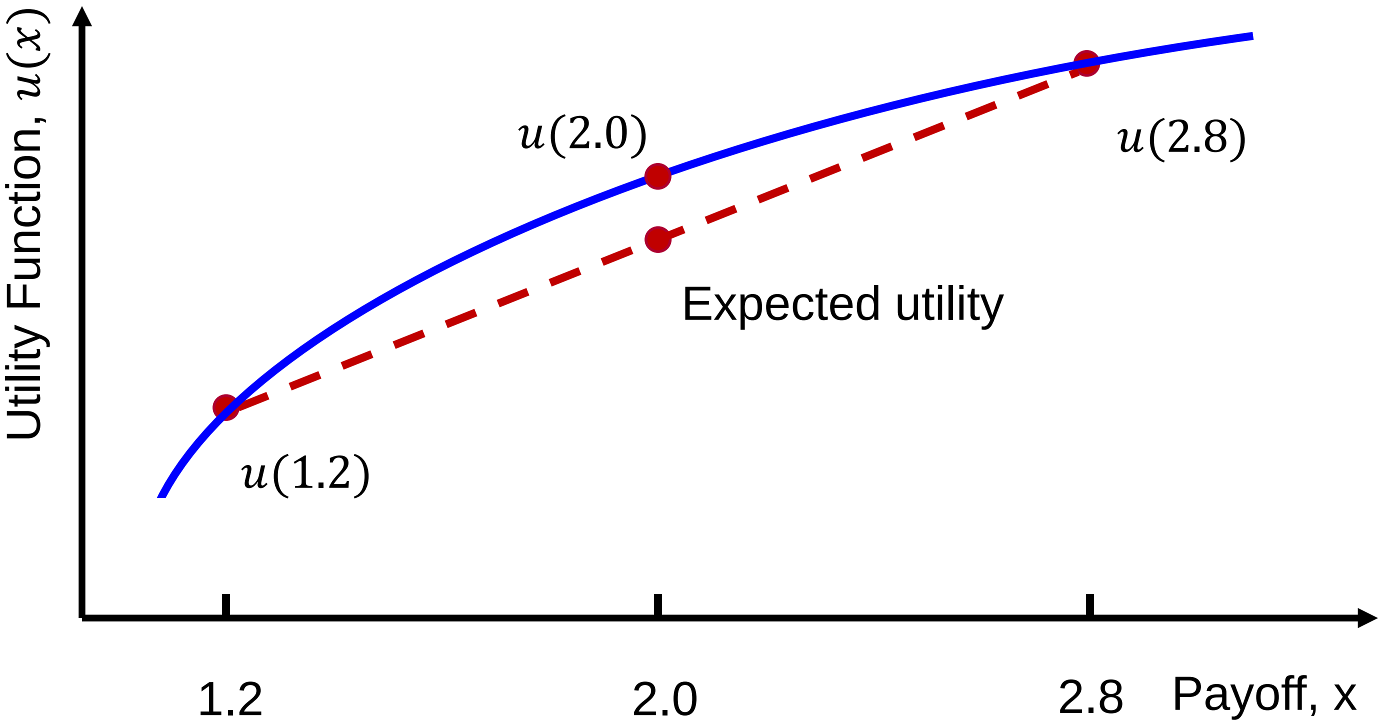
3. Expected Utility: Risk Aversion

Basic properties

- Prefer more to less: $x + \epsilon \succeq x$ for all $x, \epsilon \geq 0$.
  - Implies utility function is non-decreasing: $u'(x) \geq 0$.
- Aversion to risk: $u(\mathrm{E}[x]) \geq \mathrm{E}[u(x)]$.
  - Implies $u(\cdot)$ is concave: $u''(x) \leq 0$.

Risk aversion: a graphical illustration

- Increasing, concave utility: $u(\mathrm{E}[x]) \geq \mathrm{E}[u(x)]$ ($\mathrm{E}[x] = 2.0$).



Risk premium

- Consider an investment with random return x: investor starts with W and ends with $W(1+x)$.
- Expected return is zero: $\mathrm{E}[x]=0$, variance $\sigma_x^2=\mathrm{E}[x^2]$.
- Risk-averse investor would prefer a zero riskless return:

$$\mathrm{E}[u\big(W(1+x)\big)] \le u(W)$$

- Define the risk premium π, such that indifferent between a random return x and losing a fraction π of wealth for sure:

$$\mathrm{E}\big[u\big(W(1+x)\big)\big] = u\big(W(1-\pi)\big)$$

Derive π: use Taylor expansion

- Want to determine risk premium π based on

$$\mathrm{E}\big[u\big(W(1+x)\big)\big] = u\big(W(1-\pi)\big)$$

- Assume x is close to zero: $x \in (-\epsilon, \epsilon),\ \epsilon \ll 1.$

- Then risk π is also small in magnitude.

- Use Taylor expansion to simplify the problem: expand both sides of the equation around 0.

Derivation

$$\underbrace{\mathrm{E}\left[u\left(W(1+x)\right)\right]}_{\substack{\text{expected utility} \\ \text{of the risky payoff}}} = \mathrm{E}\left[u(W) + u'(W)Wx + 0.5u''(W)W^2x^2 + \cdots\right]$$

$$= u(W) + u'(W)W\,\underbrace{\mathrm{E}[x]}_{=0} + 0.5u''(W)W^2\sigma_x^2 + \cdots$$

$$= \underbrace{u\left(W(1-\pi)\right)}_{\substack{\text{utility of the} \\ \text{risk−free payoff}}} = u(W) - u'(W)W\pi + \cdots$$

Risk premium and risk aversion

- We find

$$u(W) + 0.5u''(W)W^2\sigma_x^2 = u(W) - \ u'(W)W\pi$$

- Risk premium π is given by

$$\pi = -\frac{1}{2}\frac{Wu''(W)}{u'(w)}\sigma_x^2$$

- Risk premium is a product of the relative risk aversion coefficient, $RRA(W)$:

$$RRA(W) = -\frac{Wu''(W)}{u'(w)}$$

and a measure of return risk – return variance σ_x^2.

Examples of utility functions

- Linear utility

$$u(W) = a + bW,\, b > 0$$

  - $RRA(W) = 0$ (agent is risk-neutral).

- Power utility

$$u(W) = \begin{cases} \dfrac{1}{1-\gamma} W^{1-\gamma}, & \gamma > 0, \neq 1 \\ \ln(W), & \gamma = 1 \end{cases}$$

  - $RRA(W) = \gamma$ (constant relative risk aversion).

4. Mean-Variance Preferences

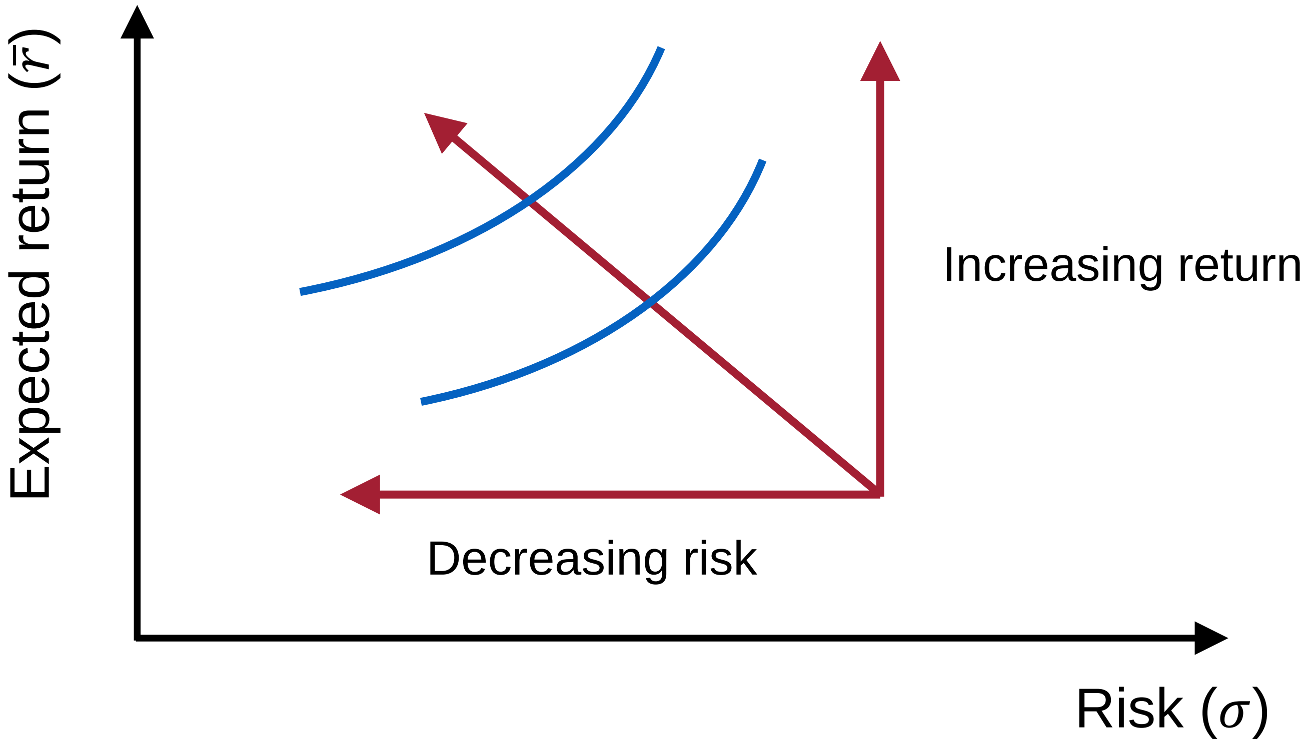
Ranking investments

- Assume all returns have a Normal (Gaussian) distribution: $\tilde{r}_i \sim N\left(\bar{r}_i, \sigma_i^2\right)$.
- Investors rank returns based on their expected utility:

$$\mathrm{E}[u(\tilde{r}_i)] = F(\bar{r}_i, \sigma_i^2)$$

- Investor prefers higher mean return $\bar{r}$.
- Investor dislikes higher variance of return σ^2.
- Mean-variance preferences.
- Variance (or standard deviation) is the only measure risk.

Investor preferences over risk and return



5. Asset Returns: Basic Statistics and Historical Data

Notation

- P_0 — initial price.
- $\tilde{P}_1$ — price at the end of the period (uncertain random variable).
- $\tilde{D}_1$ — dividend at the end of period.
- Return on an asset over a single period is random:

$$\tilde{r}_1 = \frac{\tilde{D}_1 + \tilde{P}_1 - P_0}{P_0} = \frac{\tilde{D}_1 + \tilde{P}_1}{P_0} - 1$$

- Expected return:

$$\mathrm{E}[\tilde{r}_1] = \frac{\mathrm{E}\left[\tilde{D}_1\right] + \mathrm{E}[\tilde{P}_1]}{P_0} - 1$$

- Excess return:

$$\tilde{r}_1^e = \tilde{r}_1 - r_f$$

Basic statistics

- Basic statistics: mean, variance, and standard deviation (volatility).
- Moments not known -- estimate from historical data.

	Return moments	Common sample estimators
Mean	$\bar{r} = \mathrm{E}[\tilde{r}]$	$\hat{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$
Variance	$\sigma^2 = \mathrm{E}[(\tilde{r} - \bar{r})^2]$	$\hat{\sigma}^2 = \frac{1}{T-1}\sum_{t=1}^{T}(r_t - \hat{r})^2$
Standard deviation	$\sigma = \sqrt{\sigma^2}$	$\hat{\sigma} = \sqrt{\hat{\sigma}^2}$

Riskier assets on average earn higher returns

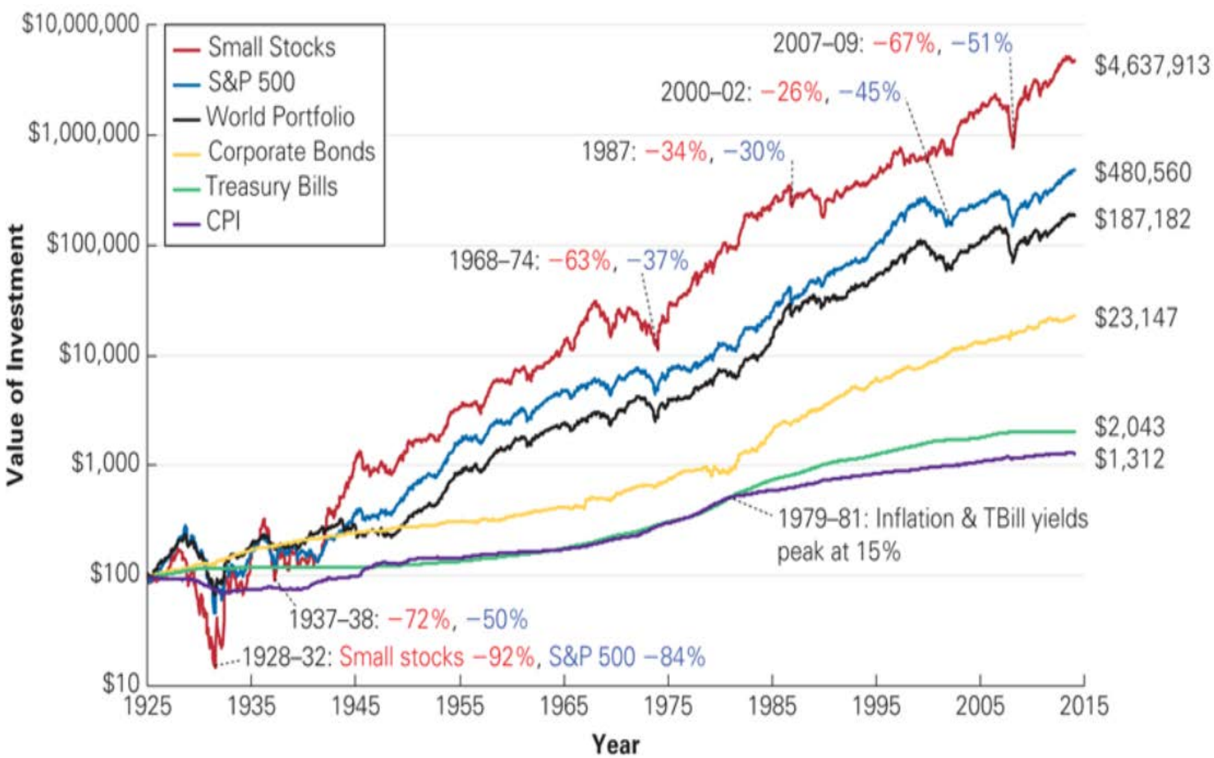
Average annual total returns from 1926 to 2018 (nominal)

Asset	Mean (%)	SD (%)
T-bills	3.4	3.1
Long term T-bonds	5.9	9.8
Long term corp. bonds	6.3	8.4
Large stocks	11.9	19.8
Small stocks	16.2	31.6
Inflation	3.0	4.0

Source: Stocks, bonds, bills and inflation, 2019 Year Book, Ibbotson Associates, Chicago, 2019.

Long-term returns

- Value at the end of 2015 of $100 invested at the end of 1925 in various asset classes.
- Why would any investor buy bonds? It is all about the risk-return tradeoff.



Source: Stocks, bonds, bills and inflation, 2016 Year Book, Ibbotson Associates, Chicago, 2016.

6. Other Dimensions of Risk

Risk is more than variance

- Risk has many dimensions: it is not just variance.
- Skewness: is the distribution symmetric? Negative vs positive outcomes.
- Derivatives may exhibit high skewness in returns (positive or negative).
- Kurtosis: does the distribution have fat tails?
- High kurtosis is common in financial markets: asset returns often have non-Normal distribution.
- Presence of tail risk implies that return risk is hard to estimate.

Example: Exchange-Traded Funds (ETFs)

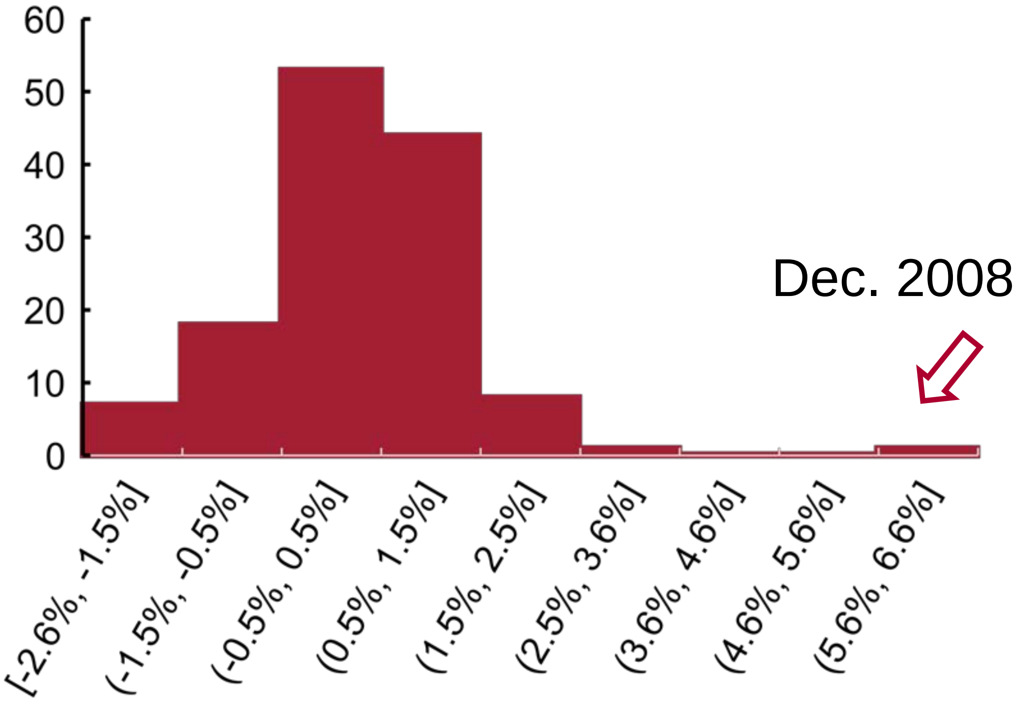
- Monthly returns (exchange-traded funds, CRSP US Stock Database):
- SPY: S&P 500 ETF;
- AGG: aggregate bond ETF.

Jan 2008 – Dec 2018

	SPY	AGG
Mean	0.68%	0.29%
Standard deviation	4.34%	1.10%

Heavy tails in returns

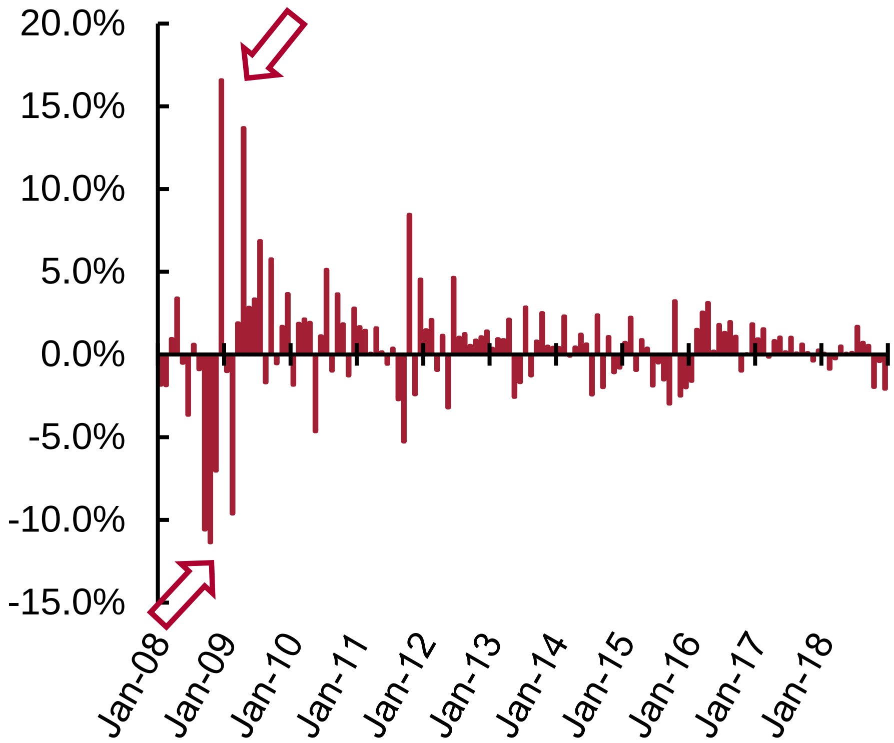
- Large returns more common than under normal distribution.
    - Monthly returns on AGG (aggregate bond ETF).
- Return on Dec. 2008 is 5.8 standard deviations above the sample mean.
- Under the normal distribution, this should happen once in 10 Million years.



Return volatility changes over time

- Properties of returns change over time.
- Extremely high volatility in 2008/2009.
- Return volatility tends to rise during economic distress.
- 5 months within a single year with returns exceeding 3 standard deviations (3-σ events).
- It is important to model time-variation in return volatility.

Monthly returns on HYG (high yield corporate bond ETF)



7. Joint Distribution of Returns

Correlation

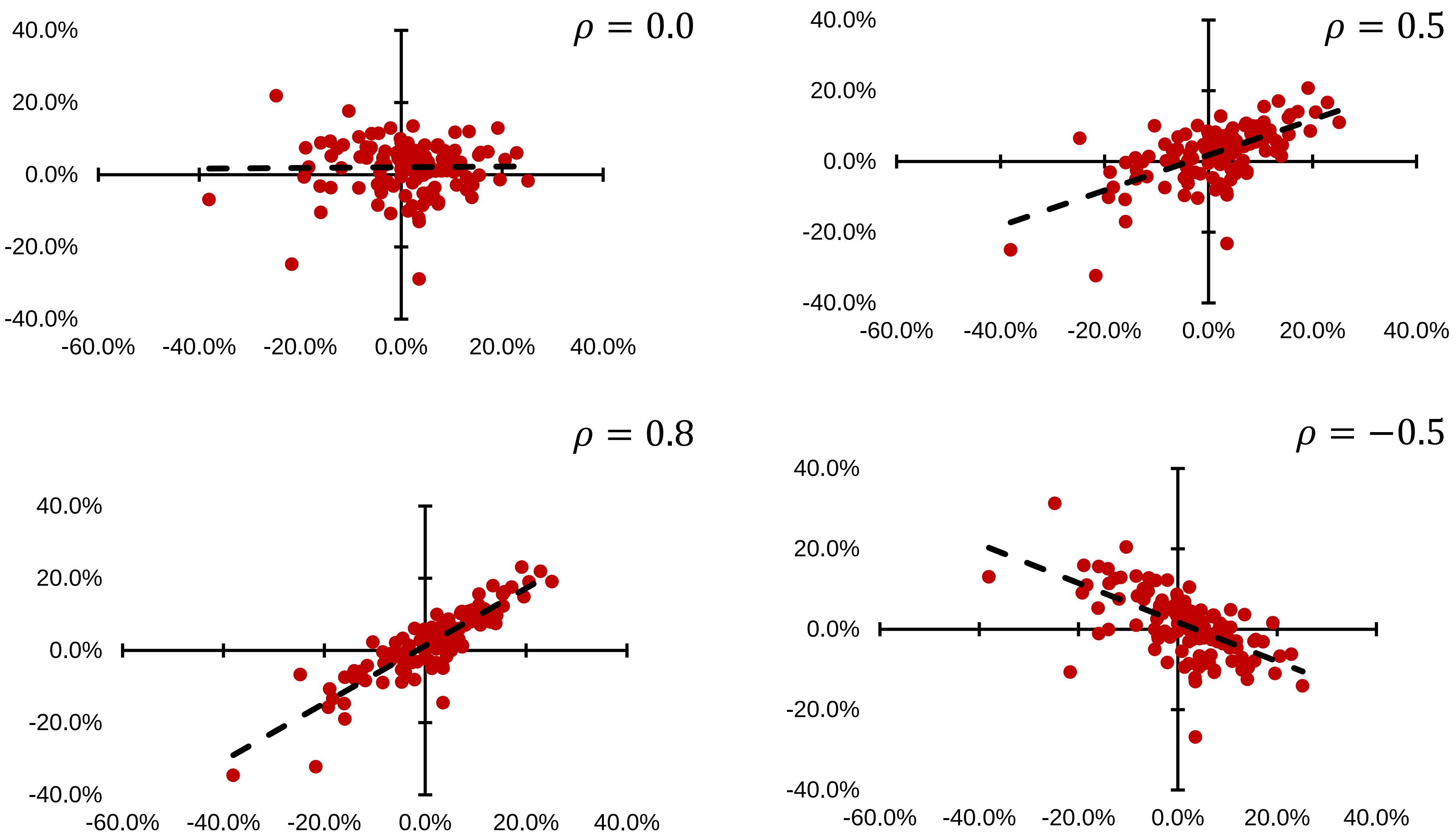
- Correlation: How closely do two variables move together?

$$\mathrm{Cov}(\tilde{r}_i, \tilde{r}_j) = \mathrm{E}\left[(\tilde{r}_i - \bar{r}_i)\left(\tilde{r}_j - \bar{r}_j\right)\right] = \sigma_{ij} \quad \text{[Covariance]}$$

$$\mathrm{Corr}(\tilde{r}_i, \tilde{r}_j) = \frac{\mathrm{E}\left[(\tilde{r}_i - \bar{r}_i)\left(\tilde{r}_j - \bar{r}_j\right)\right]}{\sigma_i \sigma_j} = \rho_{ij} \quad \text{[Correlation]}$$

$$\beta_{ij} = \frac{\sigma_{ij}}{\sigma_j^2} \quad \text{[Beta]}$$

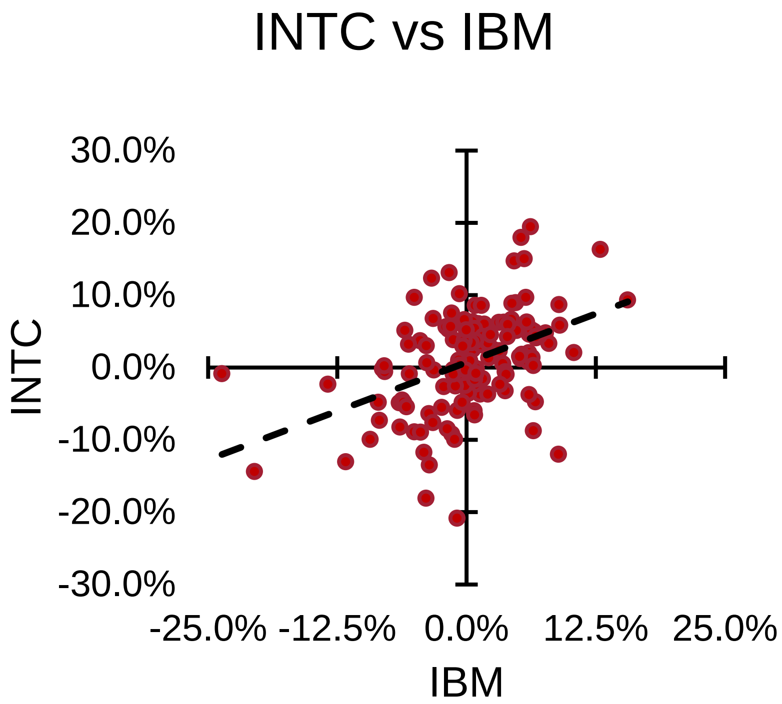
Correlation between two random variables

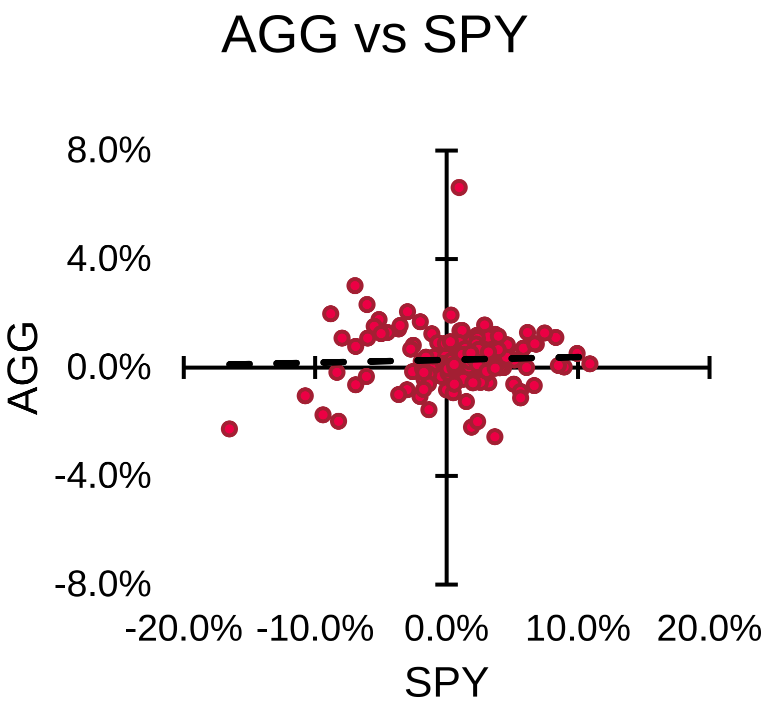


(Slope of the scattered plot gives the beta)

Empirical example of return correlation

- Monthly return correlation, Jan. 2008—Dec. 2018.





- Two technology stocks are positively correlated;
- Stock and bond markets are almost uncorrelated.

Historical correlations across assets

Annual Nominal Returns (1926-2018)						
	Bills	Long-term Treasury bonds	Long-term Corporate bonds	Large stocks	Small stocks	Inflation
T-bills	1.00	0.18	0.16	-0.02	-0.08	0.42
Long-term Treasury bonds		1.00	0.89	0.00	-0.10	-0.13
Long-term corporate bonds			1.00	0.16	0.06	-0.14
Large stocks				1.00	0.79	0.00
Small stocks					1.00	0.05
Inflation						1.00

Source: Stocks, bonds, bills and inflation, 2019 Year Book, Ibbotson Associates, Chicago, 2019.

Historical serial correlations

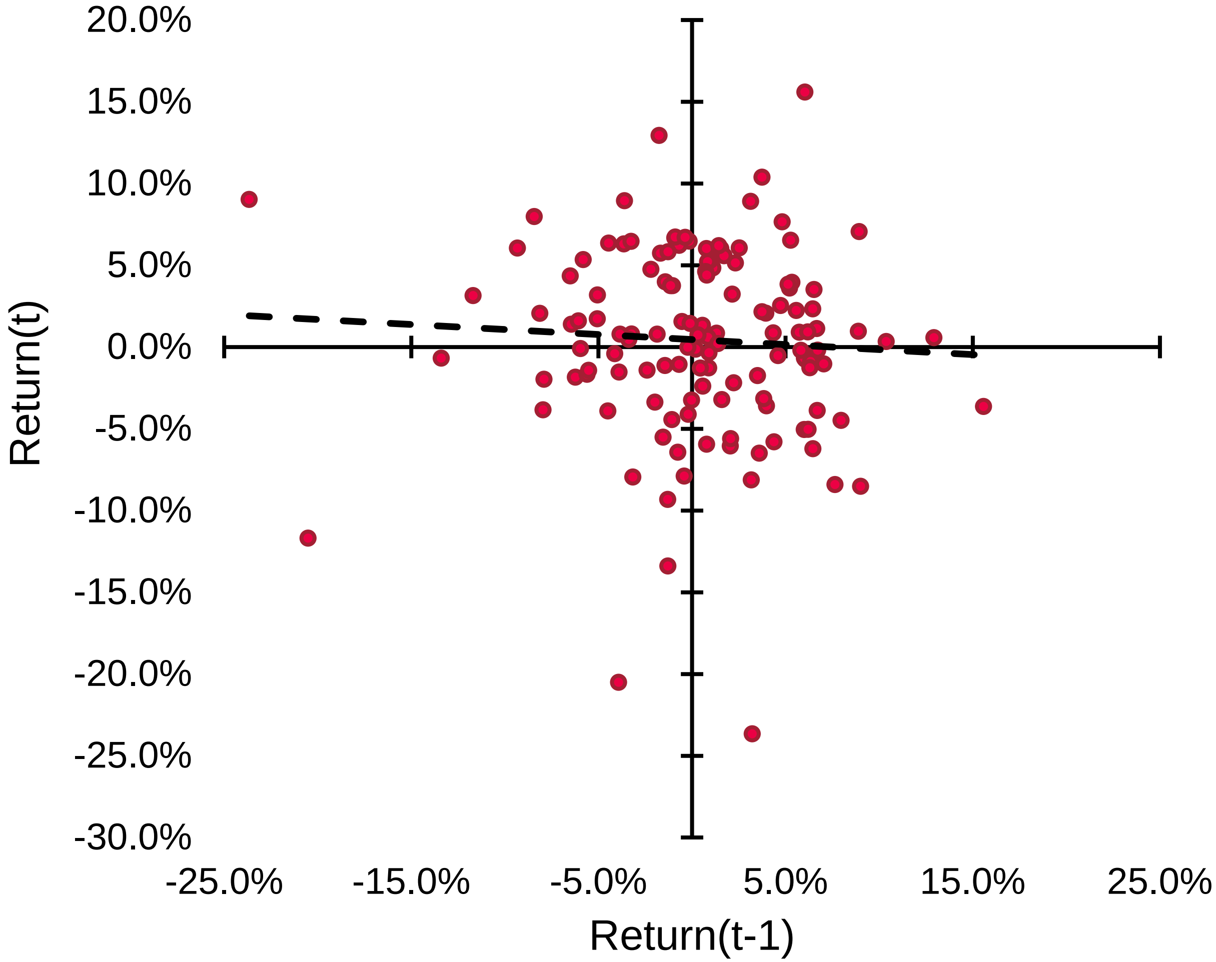
- Returns on risky assets are almost serially uncorrelated.
- High autocorrelation would imply that recent past returns help forecast future returns -- would be easy to profit from this by trading.

Serial Correlations of Annual Asset Returns (1926-2018)		
Asset	Serial Correlation	
	Nominal return	Real return
T-bills	0.92	0.67
Long-term Treasury Bonds	-0.15	-0.06
Long-term corporate Bonds	0.03	0.14
Large stocks	0.01	0.01
Small stocks	0.06	0.03

Source: Stocks, bonds, bills and inflation, 2019 Year Book, Ibbotson Associates, Chicago, 2019.

Historical serial correlations

- Monthly returns on IBM against last-month returns, Jan. 2008 – Dec. 2018.



8. Portfolio: definitions, margin, and leverage

Definitions

- Portfolio is a collection of n assets.
- Composition: N_i shares of each asset i, share price P_i.
- Portfolio value equals the sum of values of individual positions:

$$\text{Portfolio Value } V = N_1 P_1 + N_2 P_2 + \cdots + N_n P_n = \sum_{i=1}^{n} N_i P_i$$

Portfolio composition

- Portfolio composition can also be described by its asset weights:

$$w_i = \frac{N_i P_i}{N_1 P_1 + N_2 P_2 + \cdots + N_n P_n} = \frac{N_i P_i}{V}$$

- A typical portfolio has $V > 0$.

- When $V = 0$ (zero net investment), we call this an arbitrage portfolio.

- If $V > 0$, $w_1 + w_2 + \cdots w_n = 1$.

Example: portfolio composition

- Your investment account has $100,000.
- There are 3 positions:

1) 200 shares of stock A;
2) 1,000 shares of stock B;
3) 750 shares of stock C.

Example: portfolio composition

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	$50	$10,000	10%
B	1,000	$60	$60,000	60%
C	750	$40	$30,000	30%
Total			$100,000	100%

- Asset A: $200 \times \$50 = \$10{,}000$.
- Weight on A: $\$10{,}000/\$100{,}000 = 10\%$.
- Weights sum up to 100%: $10\% + 60\% + 30\% = 100\%$.

Add leverage

- Your broker informs you that you only need to keep $50,000 in your investment account to support the same portfolio.
- You can buy the same stocks on margin, using leverage.
- Withdraw $50,000 to use for other purposes, leave $50,000 in the account.

Portfolio composition with leverage

Asset	Shares	Price/Share	Dollar Investment	Portfolio Weight
A	200	$50	$10,000	20%
B	1,000	$60	$60,000	120%
C	750	$40	$30,000	60%
Riskless Bond	–50,000	$1	–$50,000	–100%
Total			$50,000	100%

- New position: riskless bond, -$\$50{,}000$.
- Total portfolio value is $\$50{,}000$ compared to $\$100{,}000$ without leverage.
- Weights of risky assets double; all weights still sum up to 100%.

Example: mortgage and leverage

- Purchase a home for $\$500{,}000$.
- Pay 20% down + mortgage for 80%.

Asset	Shares	Price per Share	Dollar Investment	Portfolio Weight
Home	1	$500,000	$500,000	500%
Mortgage	-1	$400,000	–$400,000	–400%
Total			$100,000	100%

- Leverage ratio = $\frac{\text{asset value}}{\text{net investment}}$ = $500\ K\ /\ 100\ K\ =\ 5.$

Leverage magnifies gains and losses

- Suppose house value declines by 15%.
- Cash buyer loses 15%; a buyer with a mortgage loses 75%.
  - New house value: $\$500{,}000 \times (1 - 0.15) = \$425{,}000$
  - Mortgage value is unchanged: $-\$400{,}000$ portfolio position.
  - New value of the portfolio (house value that belongs to the owner): $\$25{,}000$.
  - This is a 75% decline from initial investment of $\$100{,}000$:

$$\begin{aligned}&\text{Levered investment decline } (75\%)\\ &= \text{Leverage ratio } (5) \times \text{Original investment decline } (15\%)\end{aligned}$$

9. Portfolio: Risk and Return

Advantages of forming portfolios

- Why not pick the best asset instead of forming a portfolio?
- Don’t know which stock is best.
- Diversification -- reduce unnecessary risks.
- Enhance performance by focusing bets (hedging).
- Portfolios can customize and manage risk/reward trade-offs.

How to pick the "best" portfolio?

- What does "best" mean?
- What properties of a portfolio do we care about?
- Risk and reward:
  - Higher expected returns are preferred;
  - Higher risks are not desirable.

Portfolio properties

- Properties of a portfolio are determined by the returns of its assets and their weight in the portfolio.

- Start with expected return on the portfolio: it depends on expected returns on individual assets and their portfolio weights.

- Expected returns on portfolio assets

Asset	**1**	**2**	$\cdots$	n
Mean Return	$\bar{r}_1$	$\bar{r}_2$	$\ldots$	$\bar{r}_n$

- Expected return on the portfolio is a weighted average of expected returns on individual asset:

$$\bar{r}_p = E[r_p] = w_1\,\bar{r}_1 + w_2\,\bar{r}_2 + \cdots + w_n\,\bar{r}_n = \sum_{i=1}^{n} w_i\,\bar{r}_i$$

Expected return on the portfolio

- Expected returns on portfolio assets

Asset	1	2	$\cdots$	n
Mean Return	$\bar{r}_1$	$\bar{r}_2$	$\ldots$	$\bar{r}_n$

- Expected return on the portfolio is a weighted average of expected returns on individual asset:

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Variance of portfolio return

- Variance of portfolio returns depends on the entire covariance matrix of individual asset returns.
  - Derive the general expression below.

	$\mathbf{1}$	$\mathbf{2}$	$\cdots$	$\boldsymbol{n}$
$\mathbf{1}$	σ_1^2	σ_{12}	$\cdots$	σ_{1n}
$\mathbf{2}$	σ_{21}	σ_2^2	$\cdots$	σ_{2n}
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\boldsymbol{n}$	σ_{n1}	σ_{n2}	$\cdots$	σ_n^2

- Diagonal elements are individual return variances: $\sigma_{nn} = \sigma_n^2$.
- Off-diagonal elements capture pair-wise co-movement of asset returns.

Example: a portfolio with two assets

- Monthly stock returns, Jan. 2008 – Dec. 2018.
- SPY (equity ETF).
- AGG (bond ETF).

Sample mean	
SPY	AGG
0.68%	0.90%

Sample covariance matrix		
	SPY	AGG
SPY	0.00188	0.00002
AGG	0.00002	0.00012

- More intuitive: $\sigma_1 = 4.34\%$, $\sigma_2 = 1.1\%$, $\rho_{12} = 0.04$.

Example: portfolio return with two assets

- Portfolio return is a weighted average of individual returns:

$$\tilde{r}_p = w_1\,\tilde{r}_1 + w_2\,\tilde{r}_2$$

	SPY	AGG
Investment	\$600	\$400
Weight	0.6	0.4

- $r_{SPY} = 2\%$; $r_{AGG} = -1\%$ over the next month.

- Portfolio return:

$$r_p = \frac{(600)(2\%) + (400)(-1\%)}{1{,}000} = (0.6)(2\%) + (0.4)(-1\%) = 0.8\%$$

Portfolio mean and variance, two assets

- Expected portfolio return:

$$\bar{r}_p = w_1 \bar{r}_1 + w_2 \bar{r}_2$$

- Unexpected portfolio return:

$$\tilde{r}_p - \bar{r}_p = w_1(\tilde{r}_1 - \bar{r}_1) + w_2(\tilde{r}_2 - \bar{r}_2)$$

- The variance of the portfolio return:

	1	**2**
1	$w_1^2\sigma_1^2$	$w_1 w_2 \sigma_{12}$
2	$w_1 w_2 \sigma_{12}$	$w_2^2\sigma_2^2$

$$\sigma_p^2 = w_1^2\sigma_1^2 + w_2^2\sigma_2^2 + 2w_1 w_2 \sigma_{12}$$

Example: portfolio return with two assets

Sample mean	
SPY	AGG
0.68%	0.90%

- Equally weighted portfolio: $w_1 = w_2 = 0.5$.
- Mean of the portfolio return (use sample means to estimate expected asset returns):

$$\bar{r}_p = (0.5)(0.68\%) + (0.5)(0.90\%) = 0.79\%$$

- Expected return on a portfolio is a weighted average of expected returns on individual assets.

Example: portfolio variance with two assets

Covariance matrix		
	SPY	AGG
SPY	0.00188	0.00002
AGG	0.00002	0.00012

$$\sigma_p^2 = (0.5)^2(0.00188) + (0.5)^2(0.00012) + (2)(0.5)^2(0.00002) = 0.00051$$

- Portfolio volatility is not a weighted average of individual asset volatilities:

$$\sigma_p = 2.26\% < \text{ Weighted Average}$$

General expressions for portfolio mean and variance

- We now consider a portfolio of n assets.
- Portfolio weights are $\{w_1, w_2, \dots, w_n\}$, $\sum_i w_i = 1$
- Portfolio return is a weighted average of individual asset returns:

$$\tilde{r}_p = w_1\,\tilde{r}_1 + w_2\,\tilde{r}_2 + \cdots + w_n\,\tilde{r}_n = \sum_{i=1}^{n} w_i\,\tilde{r}_i$$

General expressions for portfolio mean and variance

- Expected return on the portfolio:

$$\bar{r}_p = E\left[r_p\right] = w_1\,\bar{r}_1 + w_2\,\bar{r}_2 + \cdots + w_n\,\bar{r}_n = \sum_{i=1}^{n} w_i\,\bar{r}_i$$

- Variance of the return on the portfolio = weighted sum of all the variances and covariances of its assets:

$$\sigma_p^2 = Var\left[\tilde{r}_p\right] = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij}\,, \qquad \sigma_{ii} = \sigma_i^2$$

10. Systematic and idiosyncratic risks

Example: diversification with a two-asset portfolio

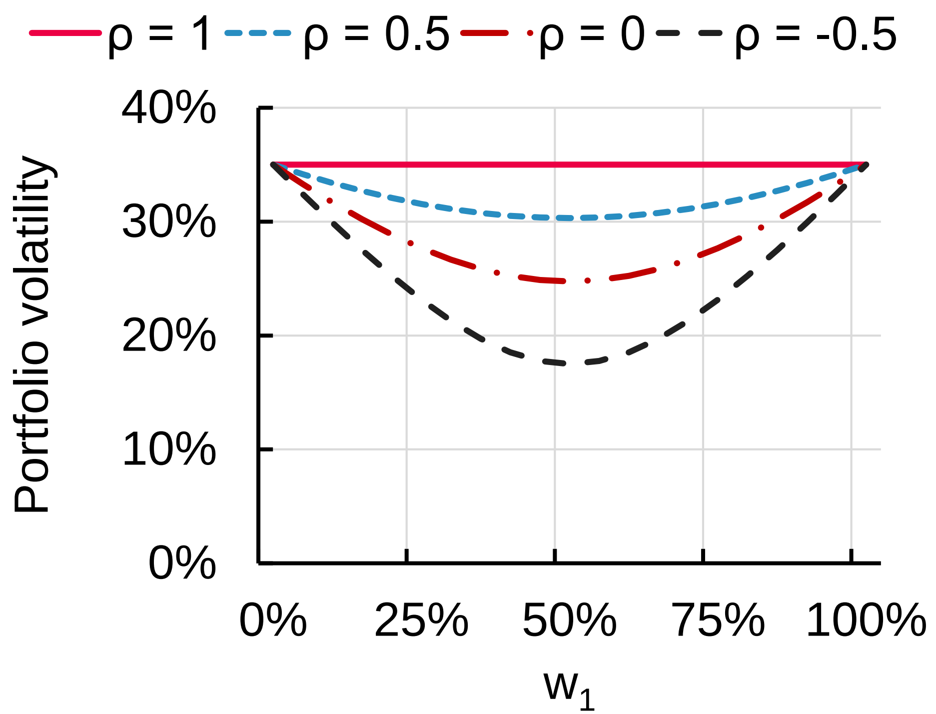
- Diversification reduces risk.
- Two assets: INTC and IBM.
- Compare their return volatility to an equally-weighted (50/50) portfolio.

	IBM	INTC	Portfolio
Volatility	5.69%	6.96%	5.38%

- Portfolio is less volatile than either of the two stocks individually!

Diversification and correlation

- Consider two assets with the same volatility, 35%.
- Portfolio with weight w in asset 1 and $1 - w$ in asset 2.
- Vary correlation ρ between the two assets: $\rho = 1, 0.5, 0, -0.5$.
- Volatility of the portfolio return is less than the volatility of each individual asset return.



Certain risks cannot be diversified away

- Diversification is effective up to a certain limit – risk cannot be fully eliminated through diversification.
- Remaining risk is known as non-diversifiable (also called market risk, systematic risk, common risk).
- Risk comes in two kinds:
  - Diversifiable risks;
  - Non-diversifiable risks.
- Sources of non-diversifiable risks include:
  - Business cycle;
  - Inflation;
  - Liquidity.

What determines limits of diversification?

- Consider an equally-weighted portfolio of n assets.
- Portfolio variance is the sum of all the terms in the matrix on the right:

	$\mathbf{1}$	$\cdots$	n
$\mathbf{1}$	$w_1^2\sigma_1^2$	$\cdots$	$w_1 w_n \sigma_{1n}$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$\boldsymbol{n}$	$w_n w_1 \sigma_{n1}$	$\cdots$	$w_n^2\sigma_n^2$

- A typical variance term: $\left(\frac{1}{n}\right)^2 \sigma_{ii}$ -- total number of variance terms is n.
- A typical covariance term: $\left(\frac{1}{n}\right)^2 \sigma_{ij},\ (i \neq j)$ -- total number of covariance terms is $n^2 - n$.
  - With 100 assets, 100 diagonal elements, and 9,900 off-diagonal.

Decompose portfolio variance

- Add all the terms:

$$\sigma_p^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j \sigma_{ij} = \sum_{i=1}^{n}\left(\frac{1}{n}\right)^2 \sigma_{ii} + \sum_{i=1}^{n}\sum_{j\neq i}^{n}\left(\frac{1}{n}\right)^2 \sigma_{ij}$$

$$= \left(\frac{1}{n}\right)\left(\frac{1}{n}\sum_{i=1}^{n}\sigma_i^2\right) + \left(\frac{n^2-n}{n^2}\right)\left(\frac{1}{n^2-n}\sum_{i=1}^{n}\sum_{j\neq i}^{n}\sigma_{ij}\right)$$

$$= \left(\frac{1}{n}\right)(\text{average variance}) + \left(1-\frac{1}{n}\right)(\text{average covariance})$$

- As n becomes very large:
  - Contribution of variance terms goes to zero.
  - Contribution of covariance terms goes to “average covariance.”

Portfolio variance and return correlation

- The average US stock has a monthly standard deviation of 10% and the average correlation between stocks is 40%.

- If you invest the same amount in each stock, what is variance of the portfolio?

$$Cov[R_i, R_j] = \rho_{ij}\sigma_i\sigma_j = 0.40 \times 0.10 \times 0.10 = 0.004$$

$$Var[R_p] = \frac{1}{n}0.10^2 + \frac{n-1}{n}0.004 \approx 0.004 \quad \text{if } n \text{ is large}$$

$$\sigma_p \approx \sqrt{0.004} = 6.3\%$$

Return correlation and limits of diversification