Recitation 7

Problem 1

A financial firm is offering a synthetic zero-coupon putable convertible bond on the ABC Company. It works in the following way:

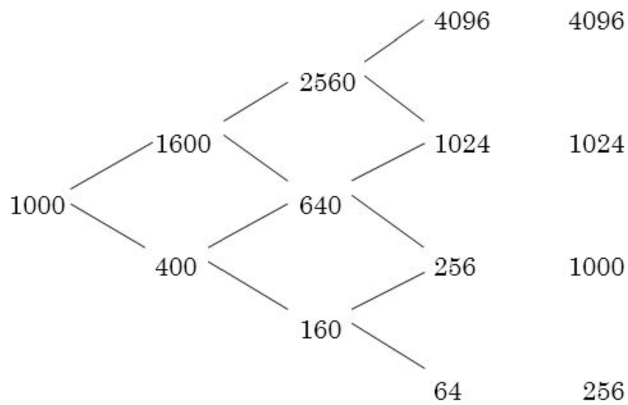
1. The bond has a promised principal payment of $\$1,000$ three periods from now.
2. At any time after the first period, the holder of the bond can either exchange it for one share of ABC stock or give it back to the securities firm in return for a payment of $\$600$.
3. The securities firm has set aside four shares of ABC stock to serve as collateral for each synthetic convertible.
4. The firm can at any time fulfill its entire obligation, and be required to make no other payments, by giving to the holder of the putable convertible full ownership of the four shares of stock.

Over each period, the total rate of return on holding ABC stock will be either 100% or -50%. At the end of each period, the stock will pay one fifth of its pre-dividend value as a cash dividend to the investor who held the stock during that period.

The remaining 80% of the pre-dividend value will then be the ex-dividend stock price at which one can purchase stock to hold over the following period or sell stock received from conversion or default. In other words, over each period the ex-dividend stock price will either increase by 60% or decrease by 60%.

By reinvesting dividends, an investor owning one share at the beginning of a period will own 1.25 shares at the end of the period. The current ex-dividend stock price is $\$1,000$, and the interest rate is 10% per period. The quoted price of the synthetic putable convertible is $\$1,000$. If the financial firm fully hedges its position, what will be its realized profit on each bond sold?

Solution: We can use a binomial tree risk-neutral pricing model to solve this problem. The possible paths for ABC stock over the next three periods are shown in the following diagram:



As always, we work backward through the binomial tree.

1. If the stock price on the maturity date is either \$4,096 or \$1,024, bondholders will choose to convert the bond to a stock.

2. If the stock price on the maturity date is \$256, bondholders will neither convert the bond to a stock nor exercise the put option, and the financial firm will pay them the principal amount of \$1,000.

3. If the stock price is \$64, the financial firm will settle its obligation by giving the bondholders the four shares of stock held as collateral.

The column to the right of the binomial tree diagram above reports the final payoffs to the bondholders at each node.

The value of the putable convertible at the other nodes on the tree can be found by applying the fundamental valuation equation to obtain the value if the bond is held for one more period. This value must then be compared with the value generated by immediate conversion or put and by the immediate forfeiture of the collateral. The final value assigned at each node on the tree must reflect the optimal use by both parties of the choices available to them.

Specifically, let $V(S, n)$ be the value of the bond when the stock price is S and there are n periods until maturity. We can solve for the risk-neutral probability of an "up" move, q^*, as:

$$1000 = (2000 \times q^* + 500 \times (1 - q^*))/1.1,$$

which yields $q^* = 0.4$.

Solving for $V(S, n)$ when $n = 1$, we have that:

$$\begin{aligned} V(2560, 1) &= \min\{4(2560), \max[2560, 600, (0.4(4096) + 0.6(1024))/1.1]\} \\ &= 2560 \end{aligned}$$

$$\begin{aligned} V(640, 1) &= \min\{4(640), \max[640, 600, (0.4(1024) + 0.6(1000))/1.1]\} \\ &= 917.8181 \end{aligned}$$

$$\begin{aligned} V(160, 1) &= \min\{4(160), \max[160, 600, (0.4(1000) + 0.6(256))/1.1]\} \\ &= 600 \end{aligned}$$

Thus, the bond is converted to a stock at the node of the tree where $S = 2560$, the put is exercised when $S = 160$, and the bond is neither converted nor put when $S = 640$.

Similarly, solving for $V(S, n)$ when $n = 2$, we have that:

$$\begin{aligned} V(1600, 2) &= \min\{4(1600), \max[1600, 600, (0.4(2560) + 0.6(917.8181))/1.1]\} \\ &= 1600 \end{aligned}$$

$$\begin{aligned} V(400, 2) &= \min\{4(400), \max[400, 600, (0.4(917.8181) + 0.6(600))/1.1]\} \\ &= 661.0248 \end{aligned}$$

Thus, the bond is converted to a stock at the node of the tree where $S = 1600$, and the bond is neither converted nor put when $S = 400$.

Finally, we can solve for the value of the bond at the initial node of the tree when the stock price $S = 1000$:

$$\begin{aligned} V(1000) &= [0.4(1600) + 0.6(661.0248)]/1.1 \\ &= 942.3771 \end{aligned}$$

In other words, if the financial firm fully hedges its position, its realized profit on each bond sold will be $1000 - 942.3771 = \$57.6229$.

Problem 2

Estimate the value of a new 6-month European-style (arithmetic) average price call option on a non-dividend-paying stock. The initial stock price is \$30, the strike price is \$30, the risk-free rate is 5%, and the stock price volatility is 30%.

(a) Perform the estimation by writing a program to run a Monte Carlo simulation to estimate the price of the option. Use 5,000 stochastic paths for the risk-neutral representation of the evolution of stock prices assuming lognormality, drawing innovations from a normal distribution, and with a time step h equal to 1 month.

Solution: Before performing the Monte Carlo simulation, let's see if we can derive a closed-form expression for the value of the average price call option. Since this material is more advanced, feel free to skip to the Monte Carlo simulation if desired.

The option's payoff depends on the arithmetic average of the price of the underlying stock during the life of the option. In particular, the payoff is $\max(0, S_{avg} - K)$, where S_{avg} is the average price of the stock.

Under the assumption that S_{avg} is lognormally distributed, the average price call can be valued using a similar formula to the one we've used to price a regular European call. Suppose M_1 and M_2 are the first two moments of S_{avg}. The value of the average price call is given by Black's model:

$$\begin{aligned} c &= e^{-rT}\left[F_0 \mathcal{N}(d_1) - K \mathcal{N}(d_2)\right] \\ d_1 &= \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma_F \sqrt{T}}; d_2 = d_1 - \sigma_F \sqrt{T} \end{aligned} \tag{1}$$

with $F_0 = M_1$ and $\sigma_F^2 = \frac{1}{T}\ln\left(\frac{M_2}{M_1^2}\right)$.

Assuming that the average is calculated continuously, then

$$M_1 = \frac{e^{(r-q)T}-1}{(r-q)T}S_0,$$

$$M_2 = \frac{2e^{\left[2(r-q)+\sigma^2\right]T}S_0^2}{\left(r-q+\sigma^2\right)\left(2r-2q+\sigma^2\right)T^2} + \frac{2S_0^2}{(r-q)T^2}\left(\frac{1}{2(r-q)+\sigma^2} - \frac{e^{(r-q)T}}{r-q+\sigma^2}\right).$$

Plugging in $r = 5\%$, $q = 0$, $\sigma = 30\%$, $T = 0.5$, $S_0 = 30$, and $K = 30$ to the expressions for M_1 and M_2 above, we get that $M_1 = 30.378$, $M_2 = 936.9$, and $\sigma_F^2 = \frac{1}{T}\ln\left(\frac{M_2}{M_1^2}\right) = 17.41$.

Finally, from Equation (1), we can calculate the value of the average price call as:

$$c = e^{-0.05\times 0.5}[30.378\mathcal{N}(0.163) - 30\mathcal{N}(0.04)] = \$1.64.$$

Using Monte Carlo, we can simulate the lognormal stock price process under the risk-neutral representation using the following algorithm:

$$S_{t+h} = S_t \times e^{\left(r-\frac{\sigma^2}{2}\right)h+\sigma\sqrt{h}\epsilon_t}$$

where the time step $h = \frac{1}{12}$ and $\epsilon_t \sim$ i.i.d. $N(0,1)$.

After generating 5,000 paths for the stock price, we can calculate the payoff on each path i as:

$$V_i = \max\left\{\text{avg}\left(S_1^i,\ldots,S_6^i\right) - K, 0\right\}.$$

Finally, the value of the average price call is given by $c = e^{-0.05\times 0.5}\frac{1}{5000}\sum_i V_i$. For $h = \frac{1}{12}$, $c = \$1.89$. This is close to the analytical result of \$1.64 we derived above! See the example R code below.

```
# Q2. Use Monte Carlo without trees to price average price call options

# key: construct risk-neutral representation of the evolution of stock
#      prices assuming log-normality

# (b) let h be one month
N = 5000
h = 1/12
Tm = 0.5   # 6 months
S0 = 30
sig = 0.3
r = 0.05
K = 30
v = rep.int(0,N)

set.seed(1234)
for (i in 1:N) {
  eps <-  rnorm(6,0,1)
  t_ <- (1:6)*h
  eps_ <- cumsum(eps)
  S_path <-  S0*exp( (r-0.5*sig^2)*t_ + sig*sqrt(h)*eps_ )
  v[i]<- max(mean(S_path) - K,0)
}

Voption <-  mean(v)*exp(-0.05*0.5) #[1] 1.885778
```

(b) Repeat this exercise but now set the time step h equal to 1 week (treating 6 months as 26 weeks).

Solution: Example R code is provided below. Notice that, with the smaller time step, the simulated value of the average price call of \$1.66 is nearly identical to the analytical result of \$1.64.

```
# (c) let h be one week

N = 5000
h = 1/52
Tm = 0.5   # 6 months
S0 = 30
sig = 0.3
r = 0.05
K = 30
v = rep.int(0,N)
set.seed(1234)
for (i in 1:N) {
  eps <-  rnorm(26,0,1)
  t_ <- (1:26)*h
  eps_ <- cumsum(eps)
  S_path <-  S0*exp( (r-0.5*sig^2)*t_ + sig*sqrt(h)*eps_ )
  v[i]<- max(mean(S_path) - K,0)
}

Voption <-  mean(v)*exp(-0.05*0.5) #[1] 1.659565
```

(c) Using the same Monte Carlo simulation of stock prices as in Part (b), what is the price of a knock-in call option with a strike price of \$30 and a barrier of \$35?

Solution: The knock-in call option comes into existence when the stock price reaches \$35 before expiration. Using the same algorithm to simulate the lognormal stock price process as in Part (b) with $h = \frac{1}{52}$, we can simulate the payoff of the knock-in call on a particular path as $\max(S_T - K, 0)$ if the stock price reaches \$35, and 0 otherwise.

Using the example R code below, the price of the knock-in call is estimated to be \$2.65.

```
B = 35 # the barrier

# Still, let h be one week, and characterize the lognormal stock price process
N = 5000
h = 1/52
Tm = 0.5   # 6 months
S0 = 30
sig = 0.3
r = 0.05
K = 30
v = rep.int(0,N) # storing option payoffs
set.seed(1234)
for (i in 1:N) {
  eps <-  rnorm(26,0,1)
  t_ <- (1:26)*h
  eps_ <- cumsum(eps)
  S_path <-  S0*exp( (r-0.5*sig^2)*t_ + sig*sqrt(h)*eps_ )
  if(max(S_path)>=B)
    v[i]<- max(S_path[26] - K,0)
  else v[i] <- 0

}

Voption <-  mean(v)*exp(-0.05*0.5) #[1] 2.651798
```

Problem 3

Explain why a regular European call option is the sum of a down-and-out European call option and a down-and-in European call option. Is the same true for American call options?

Solution: Recall that a **down-and-out** call option is a type of knock-out option. Specifically, it is a regular call option that ceases to exist if the underlying stock price reaches a certain barrier level H. The barrier level is *below* the initial stock price.

The corresponding knock-in option is a **down-and-in** call option. This is a regular call option that comes into existence only if the stock price reaches the barrier level H.

We have two cases to consider:

1. If the barrier level H is reached, then the down-and-out call option is worth nothing, while the down-and-in call option has the same value as a regular option.

2. If the barrier level H is not reached, then the down-and-in call option is worth nothing, while the down-and-out call option has the same value as a regular option.

Thus, the payoffs from a portfolio consisting of a down-and-out call option plus a down-and-in call option are identical to those of a regular European call option, regardless of whether the barrier level H is reached. A similar argument *cannot* be used for American options; generally, the possibility of early exercise results in different optimal exercise times for down-and-out and down-and-in American calls.