

#### Week 5 – Black-Scholes-Merton and the Greeks

MIT Sloan School of Management



#### **Topic outline**



- Calibrating and implementing multistep binomial trees
- The Black-Scholes-Merton model
- The Greeks
- Financial engineering application: Dynamic delta hedging of portfolio protection products

#### **Multi-step trees**



- Multi-step trees make it easy to numerically price derivatives under realistic assumptions
  about the distribution of future prices for the underlying security
- Stock returns are often modeled as having a constant expected return μ and constant volatility σ
  - That is, over a small interval of time *h*:

$$E\left[\frac{S_{t+h}}{S_t}\right] = e^{\mu \times h}; \quad \text{and} \quad E\left[\left(\frac{S_{t+h}}{S_t} - e^{\mu}h\right)^2\right] = \sigma^2 \times h$$

- Consider an option with maturity T
- We chop off the time interval [0, T] into n little intervals

$$\left[0,\frac{T}{n},\frac{2T}{n},\frac{3T}{n},\cdots,\frac{(n-1)T}{n},T\right]$$

• Each interval is of size h = T/n. We denote the steps i = 0, 1, ..., n  $\implies i \times h = \text{calendar time on the tree}$ 

#### **Multi-step trees**



Assume that the stock process follows:

$$S_{i+1} = \begin{cases} S_i \times u & \text{with probability} \quad q \\ S_i \times d & \text{with probability} \quad 1 - q \end{cases}$$
 (1)

Notice: *q* is the actual probability!

- We now choose u, d and q to approximate the expected return  $\mu$  and variance  $\sigma^2$  on the tree
- The following conditions must be satisfied:

1. Expected Return: 
$$E\left(\frac{S_{i+1}}{S_i}\right) = e^{\mu \times h} \Longrightarrow q \times u + (1-q) \times d = e^{\mu \times h}$$

2. Variance: 
$$E\left(\left(\frac{S_{i+1}}{S_i} - e^{\mu \times h}\right)^2\right) = \sigma^2 \times h \Longrightarrow$$

$$q\left(u-e^{\mu\times h}\right)^{2}+\left(1-q\right)\left(d-e^{\mu\times h}\right)^{2}=\sigma^{2}\times h$$

- Two equations, three parameters 

  we have one degree of freedom left. The following assumption gives symmetry to the problem.
  - 3. Percentage increase in stock = percentage decrease in stock

$$u = 1/d$$

# **Multi-step trees**



• These three equations in three unknowns give the solution:

$$u = e^{\sigma \times \sqrt{h}}$$
;  $d = 1/u$ ; and  $q = \frac{e^{\mu \times h} - d}{u - d}$ .

- We can then use these parameters and the procedure (1) to construct our tree relatively easily.
- Next figure shows an example from BinomialTree.xls
   The spreadsheet is available with this week' course materials

#### Multi-step trees: Example



#### BINOMIAL TREE MODEL

	Stock Assu	ımption	(	Option Ass	umption	1	ree		F	Risk Neutra	l Prob
	mu	0.1		T	0.1		n	10		q*	0.495834
	sigma	0.3		K	100		h	0.01			
	r div	0.02	(	Call or Put	1 (	=1 for call	u	1.030455		e Binomial	3.787
	div	0					d	0.970446	Delta	a Binomia	0.527
	S0	100					q	0.509173			
Carlon Control	020		1.272.27			04044	72722	0.0000000	121221		102.00
time ==>	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
i==>	0	1	2	3	4	5	6	7	8	9	10
0	100.000	103.045	106.184	109.417	112.750	116.183	119.722	123.368	127.125	130.996	134.986
1		97.045	100.000	103.045	106.184	109.417	112.750	116.183	119.722	123.368	127.125
2			94.176	97.045	100.000	103.045	106.184	109.417	112.750	116.183	119.722
3				91.393	94.176	97.045	100.000	103.045	106.184	109.417	112.750
4					88.692	91.393	94.176	97.045	100.000	103.045	106.184
5						86.071	88.692	91.393	94.176	97.045	100.000
6							83.527	86.071	88.692	91.393	94.176
7								81.058	83.527	86.071	88.692
8									78.663	81.058	83.527
9										76.338	78.663
10											74.082

- ① Time index: i = 0, 1, ..., n. Node index: j = 0, 1, ..., n
- Because we start from up-left corner
  - an *up* movement in S between i and i + 1 correspond to same index j;
  - a down movement in S between i and i + 1 corresponds to an increase in the index j.

## Multi-step trees: derivative pricing



How do we price a derivative security on this binomial tree?

- Risk neutral pricing provides an immediate answer
  - Compute the risk neutral probability:

$$q^* = \frac{e^{r \times h} - d}{u - d}$$

Since u, d, h are the same for each step,  $q^*$  remains constant.

Move backward on the tree one step at a time:

For European options, 
$$V_{i,j} = e^{-r \times h} \times E^* [V_{i+1} | (i,j)]$$

where i, j is the (time, node) state on the tree.

# Multi-step trees: derivative pricing of European call



 For instance, for call options, start from the end of the tree with the final condition

$$c_{n,j} = \max(S_{n,j} - K, 0)$$
 for  $j = 0, 1, ..., n$ 

Then move backward applying

$$c_{i,j} = e^{-r \times h} \times (q^* c_{i+1,j} + (1 - q^*) c_{i+1,j+1})$$

# **Multi-step trees: Example**



#### BINOMIAL TREE MODEL

	Stock Assu	mption		Option Ass	umption	1	Tree		F	Risk Neutra	l Prob
	mu	0.1		T	0.1		n	10		q*	0.495834
	sigma	0.3		K	100		h	0.01			
	ar.	0.02	(	Call or Put	1 (	=1 for call	u	1.030455		e Binomial	3.787
	div	0					d	0.970446	Delta	a Binomial	0.527
	S0	100					q	0.509173			
		0.04	0.00	0.00		0.05		0.07	0.00	0.00	
time ==>	0	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
i==>	0	1	2	3	4	5	6	/	8	9	10
O	100.000	103.045	106.184	109.417	112.750	116.183	119.722	123.368	127.125	130.996	134.986
1		97.045	100.000	103.045	106.184	109.417	112.750	116.183	119.722	123.368	127.125
2	2		94.176	97.045	100.000	103.045	106.184	109.417	112.750	116.183	119.722
3	3			91.393	94.176	97.045	100.000	103.045	106.184	109.417	112.750
4	ļ.				88.692	91.393	94.176	97.045	100.000	103.045	106.184
5	5					86.071	88.692	91.393	94.176	97.045	100.000
6	5						83.527	86.071	88.692	91.393	94.176
7	'							81.058	83.527	86.071	88.692
8	3								78.663	81.058	83.527
9										76.338	78.663
10	)										74.082

#### Option Price

time ==>	0.000	0.010	0.020	0.030	0.040	0.050	0.060	0.070	0.080	0.090	0.100
j==>	0.0	1.0	2.0	3.0	4.0	5.0	6.0	7.0	8.0	9.0	10.0
0	3.787	5.381	7.440	9.982	12.965	16.283	19.802	23.428	27.165	31.016	34.986
1		2.221	3.358	4.943	7.053	9.707	12.830	16.243	19.762	23.388	27.125
2			1.103	1.801	2.871	4.445	6.640	9.477	12.790	16.203	19.722
3				0.418	0.749	1.323	2.289	3.851	6.224	9.437	12.750
4					0.092	0.185	0.373	0.753	1.520	3.065	6.184
5						0.000	0.000	0.000	0.000	0.000	0.000
6							0.000	0.000	0.000	0.000	0.000
7								0.000	0.000	0.000	0.000
8									0.000	0.000	0.000
9										0.000	0.000
10											0.000

## Multi-step trees: Example with 250 steps instead of 10



Stock Assumption   mu   0.1   sigma   0.3   K   100   10.0004   N   100   100.0004   N   100.0004
mu sigma   0.1
time ==>         0 0.02 0 100         Call or Put 0 0 0.0012         1 (=1 for call 0.006018 0.094018 0.99
time ==> 0 0.0004 0.0008 0.0012 0.0016 0.002 0.0024 0.0028 0.0032 0.0036 0.004 0.002 0.0016 0.002 0.0024 0.0028 0.0032 0.0036 0.004 0.002 0.0028 0.0032 0.0036 0.004 0.002 0.0028 0.0032 0.0036 0.004 0.002 0.0036 0.004 0.0038 0.0038 0.0038 0.0038 0.004
time ==> 0 0.0004 0.0008 0.0012 0.0016 0.002 0.0024 0.0028 0.0032 0.0036 0.004
i==>         0         1         2         3         4         5         6         7         8         9         10           0         100.000         100.602         101.207         101.816         102.429         103.045         103.666         104.289         104.917         105.548         106.184           1         99.402         100.000         100.602         101.207         101.816         102.429         103.045         103.666         104.289         104.917           2         98.807         99.402         100.000         100.602         101.207         101.816         102.429         103.045         103.666         104.289         104.917           3         98.807         99.402         100.000         100.602         101.207         101.816         102.429         103.045         103.666           3         98.216         98.807         99.402         100.000         100.602         101.207         101.816         102.429           4         97.629         98.216         98.807         99.402         100.000         100.602         101.207         101.816         102.429           97.629         98.216         98.807         99.629         98.216
0 100.000 100.602 101.207 101.816 102.429 103.045 103.666 104.289 104.917 105.548 106.184 106.
1 99.402 100.000 100.602 101.207 101.816 102.429 103.045 103.666 104.289 104.917 98.807 99.402 100.000 100.602 101.207 101.816 102.429 103.045 103.666 104.289 103.066 104.289 103.066 104.289 103.066 104.289 103.066 103.666 104.289 103.066 103.066 103.066 104.289 103.066
2 98.807 99.402 100.000 100.602 101.207 101.816 102.429 103.045 103.666 98.216 98.807 99.402 100.000 100.602 101.207 101.816 102.429 97.629 98.216 98.807 99.402 100.000 100.602 101.207 97.045 97.629 98.216 98.807 99.402 100.000 100.602 101.207 97.045 97.629 98.216 98.807 99.402 100.000 96.464 97.045 97.629 98.216 98.807 99.402 100.000 96.464 97.045 97.629 95.313 95.887 96.464 94.743 95.313 94.176
98.216 98.807 99.402 100.000 101.207 101.816 102.429 97.629 98.216 98.807 99.402 100.000 100.602 101.207 97.045 97.629 98.216 98.807 99.402 100.000 100.602 101.207 97.045 97.629 98.216 98.807 99.402 100.000 96.464 97.045 97.629 98.216 98.807 95.887 96.464 97.045 97.629 95.313 95.887 96.464 94.743 95.313 94.176
9 94.743 95.313 94.176  time ==> 0.000 0.000 0.001 0.001 0.002 0.002 0.002 0.003 0.003 0.004 0.004
9 94.743 95.313 94.176  time ==> 0.000 0.000 0.001 0.001 0.002 0.002 0.002 0.003 0.003 0.004 0.004
9 94.743 95.313 94.176  time ==> 0.000 0.000 0.001 0.001 0.002 0.002 0.002 0.003 0.003 0.004 0.004
9 94.743 95.313 94.176  time ==> 0.000 0.000 0.001 0.001 0.002 0.002 0.002 0.003 0.003 0.004 0.004
time ==> 0.000 0.000 0.001 0.001 0.002 0.002 0.002 0.003 0.003 0.004 0.004
time ==> 0.000 0.000 0.001 0.001 0.002 0.002 0.002 0.003 0.003 0.004 0.004
0 3.876 4.193 4.527 4.879 5.247 5.633 6.036 6.457 6.894 7.348 7.818
1 3.561 3.861 4.177 4.511 4.863 5.231 5.618 6.021 6.442 6.879
2 3.262 3.545 3.845 4.161 4.495 4.847 5.216 5.602 6.005 3 2.980 3.246 3.529 3.828 4.145 4.479 4.831 5.200
4 2.715 2.964 3.230 3.513 3.812 4.129 4.463
5 2.466 2.699 2.949 3.214 3.497 3.796
6 2.233 2.451 2.684 2.933 3.198
6 2.233 2.451 2.684 2.933 3.198 7 2.016 2.218 2.436 2.668
8 1.814 2.002 2.204 9 1.628 1.801
9 1.628 1.801
10 1.455

## Multi-step trees: Probability distribution of $S_T$



- What is the probability distribution of the stock price at maturity  $S_T$ ?
- Without going into details, consider an n-step tree.
- There are n + 1 nodes at maturity, j = 0, 1, ..., n.
  - The top node j = 0 can only be reached with n ups u and 0 downs d
  - Node j = 1 can only be reached with n 1 ups u and 1 down d

:

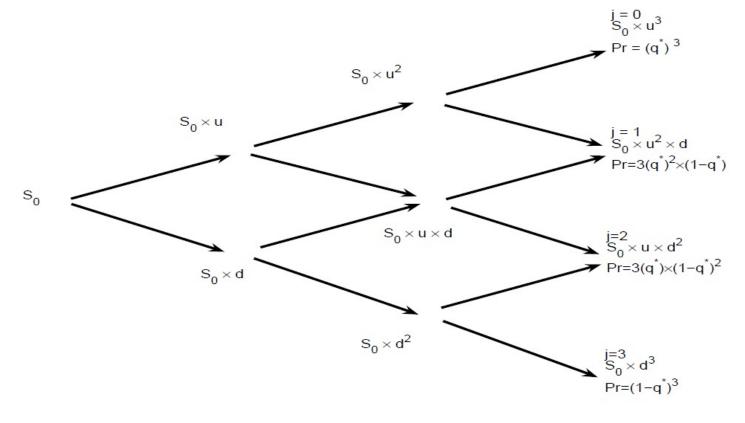
- Node j = n can only be reached with 0 ups u and n downs d
- If each up u has probability q\*, what is the probability of reaching exactly node j?

$$\Pr(\mathsf{node} = j \; \mathsf{at} \; T = n \times h) = \left(\frac{n!}{j!(n-j)!}\right) \times (q^*)^{n-j} \times (1-q^*)^j$$

where  $n! = 1 \times 2 \times 3 \times ... \times n$  (note: 0!=1)

#### For instance, if n = 3





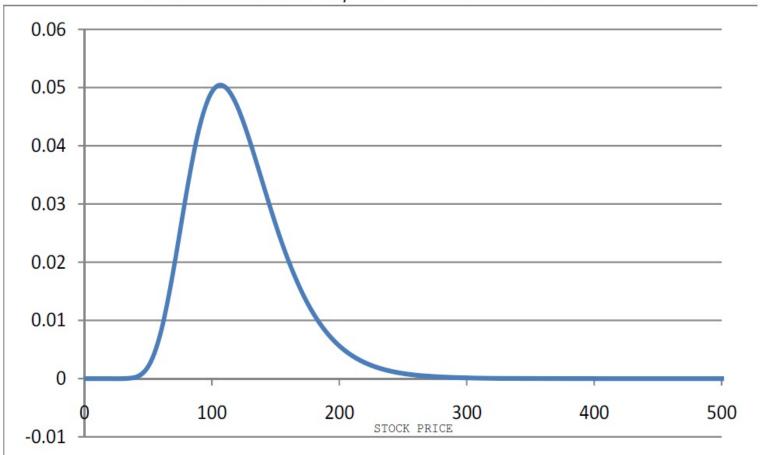
$$\Pr(j = 0 \text{ at } T = n \times h) = \left(\frac{3!}{0! \times 3!}\right) \times (q^*)^3 \times (1 - q^*)^0 = (q^*)^3$$

$$\Pr(j = 1 \text{ at } T = n \times h) = \left(\frac{3 \times 2}{2 \times 1}\right) \times (q^*)^2 \times (1 - q^*) = 3(q^*)^2 (1 - q^*)$$



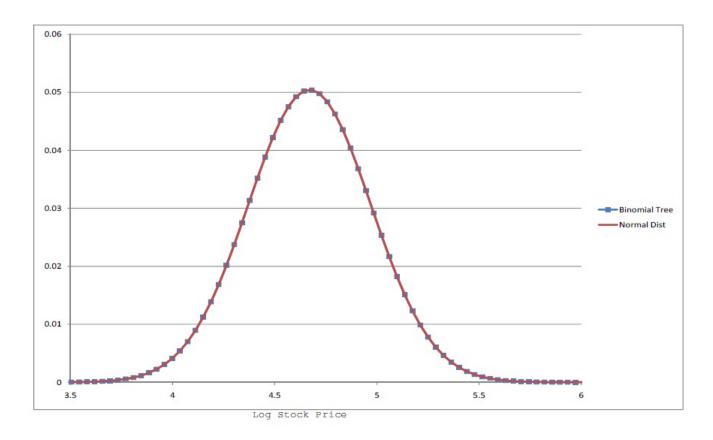
As *n* increases to infinity, the distribution at  $T = n \times h$  of the stock price  $S_T$  converges to a log-normal distribution.

Probabilities of  $S_T$  for n = 250 and T = 1





#### Or, plotting the probability density with respect to $log(S_T)$ , we obtain



Also plotted is the normal density with mean  $\mu_T = log(S_0) + (r - \varsigma - \frac{\sigma^2}{2})T$  and standard deviation  $\sigma_T = \sigma \sqrt{T}$ 

## Multi-step trees and the Black-Scholes-Merton (BSM) model



- It is evident that the probability distribution of  $log(S_T)$  implied by the binomial tree is the same as the normal distribution.
- The option price from the binomial tree is given by

$$c = E^* \left[ e^{-rT} \max(S_T - K, 0) \right] = \sum_{j=1}^n e^{-rT} \max(S_{n,j} - K, 0) Q_{n,j}^*$$

where  $Q_{n,j}^*$  is the risk neutral probability of  $S_{n,j}$ .

- Using the approximation that  $S_T$  is log-normally distributed, denote  $f(S_T) = \text{Log normal distribution with mean } \mu_T, \text{standard deviation } \sigma_T$
- Then, as *n* increases, the call option is given by:

$$c = \int_0^\infty e^{-rT} \max(S_T - K, 0) f(S_T) dS_T$$

This equation is mathematically equivalent to the Black-Scholes-Merton formula, but unlike that model can't be used directly to find option prices.

## **Assumptions for BSM Model**



- (1) Financial markets are frictionless:
  - No taxes or transactions costs
  - Assets are perfectly divisible
  - No restrictions on short sales
- (2) The interest rates for risk-free borrowing and lending are the same and constant. The annualized continuously compounded risk-free interest rate is *r*
- (3) The stock pays no dividends over the life of the option
- (4) Stock prices conform to the log-normal model:
  - Stock prices follow a continuous path
  - The mean and variance of the log return are constant
  - The log return over any period is independent of the log return over any other period
  - The log return is normally distributed





Black and Scholes (1973) and Merton (1974) show that under the assumptions, one can replicate the payoff on a European call option by dynamically rebalancing a portfolio of the stock and a risk-free security.

By no arbitrage, the value of the European call must equal the initial cost of the replicating portfolio The replicating portfolio has the properties:

- The stock position is always positive
- The fraction of stock in the replicating portfolio is the **hedge ratio** or the delta:

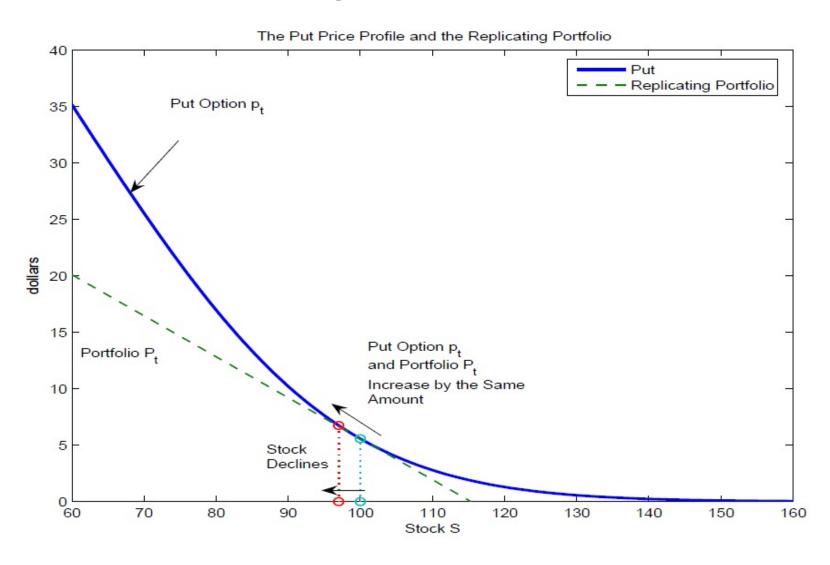
$$\Delta_c = \frac{\partial C}{\partial S}$$

- Since the  $\Delta$  of the call changes constantly, the fraction of the stock in the replicating portfolio is constantly adjust to replicate the payoff of the call.
- Similarly for a put option the hedge ratio is:

$$\Delta_p = \frac{\partial P}{\partial S}$$

## Graphical intuition behind dynamic replication





#### Black-Scholes-Merton formula



- The inputs into the original BSM formula are:
  - *S* = current stock price
  - *K* = exercise price
  - T t = time to expiration
  - *r* = continuously compounded and annualized risk-free interest rate
  - σ = annualized standard deviation of log returns (annualized "volatility")
- The most commonly used version of the formula also incorporates the dividend yield.
  - We'll look at this and other generalizations of BSM next week

#### **Basic BSM formula**



Black, Scholes and Merton show that the value of a European call option on a non-dividend paying stock is:

$$C(S, K, T - t, r, \sigma) = S \mathcal{N}(d_1) - K e^{-r(T - t)} \mathcal{N}(d_2)$$

where  $\mathcal{N}(\cdot)$  is the *cumulative density function* of a *standard normal* random variable, and:

$$d_{1} = \frac{\ln\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^{2}\right)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_{2} = d_{1} - \sigma\sqrt{T - t} = \frac{\ln\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^{2}\right)(T - t)}{\sigma\sqrt{T - t}}$$

Using put-call-parity, the value of a European style put option on the same stock with the same strike price is:

$$P(S, K, T - t, r, \sigma) = K e^{-r(T - t)} \mathcal{N}(-d_2) - S \mathcal{N}(-d_1)$$

**Important:** The expected return  $\mu$  is *not* an input to the formula. Of course, we would expect this from our discussion of the binomial model.

#### **Basic BSM formula: Example**



- Consider an at-the-money option: Current stock price S=100, strike price K=100, (continuously compounded) interest rate r=5%, maturity T=1, return volatility  $\sigma=30\%$
- We then have

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r + \sigma^2/2)T}{\sigma\sqrt{T}} = \frac{\ln\left(\frac{100}{100}\right) + (.05 + (0.30)^2/2) \times 1}{0.30\sqrt{1}} = 0.3167$$

$$d_2 = d_1 - \sigma\sqrt{T} = 0.3167 - .3\sqrt{1} = 0.0167$$

- Therefore  $\mathcal{N}(d_1) = 0.62425$  and  $\mathcal{N}(d_2) = 0.50665$
- The value of the call option is

$$c_0 = S\mathcal{N}(d_1) - Ke^{-rT}\mathcal{N}(d_2) = 100 \times 0.62425 - 100 \times e^{.05 \times 1} \times 0.50665 = 14.2312$$

The value of a put option can be computed from these data by recalling that

$$\mathcal{N}(-d_1) = 1 - \mathcal{N}(d_1) = 0.37575; \quad \mathcal{N}(-d_2) = 1 - \mathcal{N}(d_2) = 0.49335$$

so that

$$p_0 = Ke^{-rT}\mathcal{N}(-d_2) - S\mathcal{N}(-d_1) = -100 \times 0.37575 + 100 \times e^{.05 \times 1} \times 0.49335 = 9.3542$$

# **BSM** for dividend-paying stocks



The basic BSM formula prices European put and call options on an underlying security that pays no dividends and whose price is log-normally distributed

When there is a known dividend or dividend yield the formula can be easily adjusted:

#### 1. Options with known dividend

- Define  $S^* = S PV(D)$ where PV(D) = Present Value of Dividends before Expiration
- Use Black-Scholes Formula with S\* instead of S

For more on these variations see Hull Chapter 17

#### 2. Options with known dividend yield $\delta$

• Define  $S^* = S \times e^{-\delta \times T}$  and use

$$c = Se^{-\delta T} \mathcal{N}(d_1) - Ke^{-rT} \mathcal{N}(d_2); \quad p = Ke^{-rT} \mathcal{N}(-d_2) - Se^{-\delta T} \mathcal{N}(-d_1)$$

$$d_1 = \frac{\ln(S/K) + (r - \delta + \sigma^2/2)T}{\sigma\sqrt{T}}; \quad d_2 = d_1 - \sigma\sqrt{T}$$

#### **BSM** formula: Interpretation



#### The Black-Scholes formula expresses the option as a portfolio of stocks and bonds:

•  $\mathcal{N}(d_1)$  is the fraction of the share we hold in the replicating portfolio today (at t). It is equivalent to  $\Delta$  in the binomial model. In fact, for a call we can show that:

$$\Delta_c = \frac{\partial C}{\partial S} = \mathcal{N}(d_1) > 0$$

Similarly, for a put we can show that:

See Hull problem 15.17 for proofs

$$\Delta_{P} = \frac{\partial P}{\partial S} = -\mathcal{N}(-d_1) < 0$$

- For a call,  $Ke^{-r(T-t)}\mathcal{N}(d_2)$  is the amount of initial borrowing in the replicating portfolio
- Therefore, the value of the call is again the cost of the replicating portfolio:

$$C = \Delta_c \times S - B = \mathcal{N}(d_1)S - Ke^{-r(T-t)}\mathcal{N}(d_2)$$

# **Deriving the BSM formula**



- There are at least three different ways to derive the BSM formula:
  - (1) As  $n = \infty$ , the log normal binomial tree model for stock prices converges to the BSM price
  - (2) Black-Scholes partial differential equation (PDE)
  - (3) Risk-neutral pricing
    - All are helpful for developing intuition.
    - You can read about them in Hull or other books.
    - In the interest of time we won't go through those derivations here.



- Risk managers and options traders need to know the sensitivity of their options holdings to various factors
- Those sensitivities are known as "The Greeks." We'll discuss:
  - 1. Delta
  - 2. Gamma
  - 3. Theta
  - 4. Rho
  - 5. vega
- For options that can be priced using BSM, they often take a simple form



- Terminology:
  - 1. **Delta**: Sensitivity of option to changes in underlying

$$\Delta = \frac{d \text{ Option Price}}{d S} = \left\{ egin{array}{ll} \mathcal{N}(d_1) & \text{for Calls} \\ -\mathcal{N}(-d_1) & \text{for Puts} \end{array} \right.$$

 Gamma: Sensitivity of Delta (Δ) to changes in the underlying. For both calls and puts:

$$\Gamma = \frac{d \Delta}{d S} = \frac{N'(d_1)}{S\sigma\sqrt{T}}$$
 with  $N'(x) = \frac{e^{\frac{-x^2}{2}}}{\sqrt{2\pi}}$ 

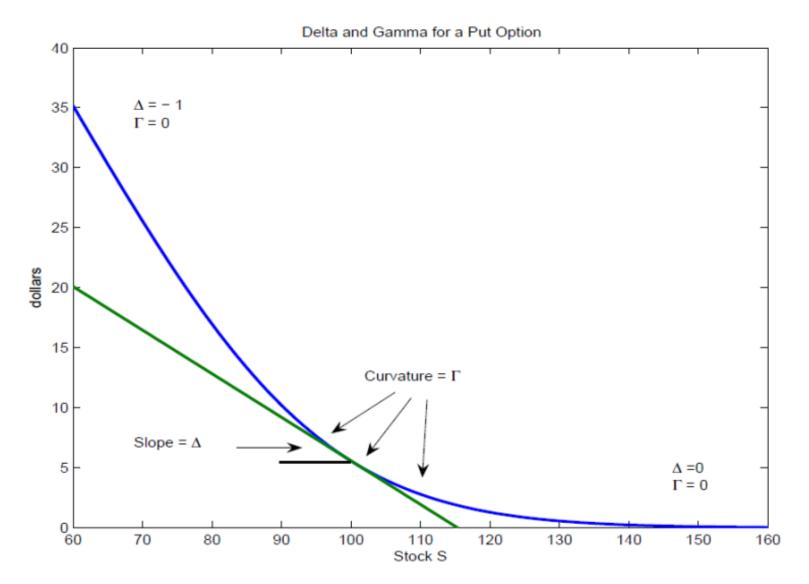
 $\Longrightarrow \Gamma = \text{curvature of option price with respect to stock } S$ 

Note: Expressions with *T* instead of *T-t* assume *t*=0

Q: Try to derive  $\Delta$  and  $\Gamma$  via the Black-Scholes formula

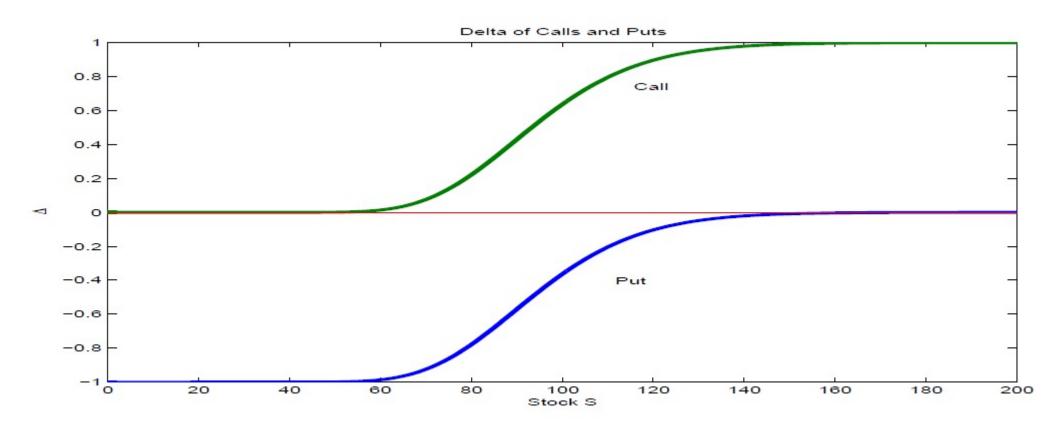






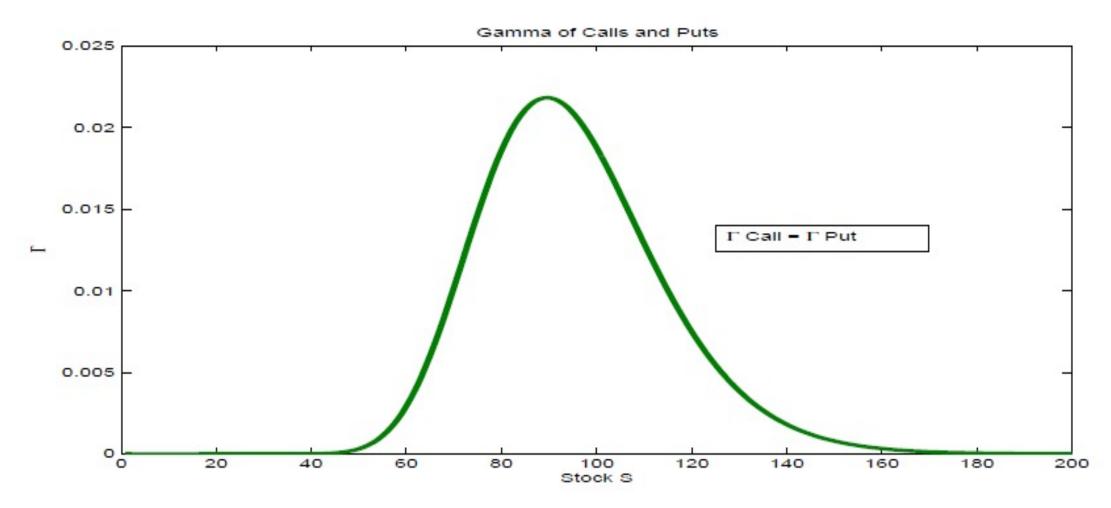


$$K = 100$$



<u>Note</u>: This holds all other parameters (K, T, r,  $\sigma$ ) constant



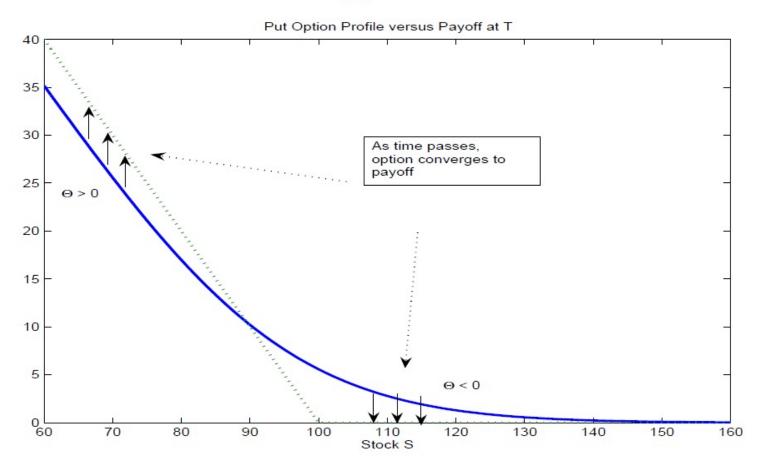


Note: This holds all other parameters  $(K, T, r, \sigma)$  constant



3. **Theta**: Sensitivity of option to passage of time *t* 

$$\Theta = \frac{d \text{ Option Price}}{d t} = \text{Long ugly formula}$$



Note: Remember we are holding *T* constant

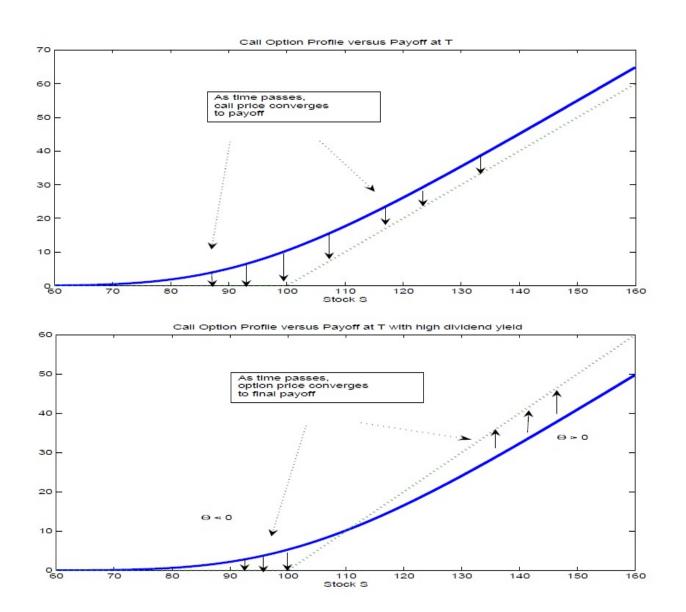


- For a put option,  $\Theta > 0$  for low S and  $\Theta < 0$  for high S. Why?
  - For S high, payoff is zero, but put price is positive
     As time passes (but keeping S fixed), the put price must decline
  - For S low, why is  $\Theta > 0$ ?
  - Simple example:
    - If the firm goes bankrupt  $\Longrightarrow S = 0$
    - Then, the put payoff at T is K
    - Value at time t < T is  $p = e^{-r(T-t)}K$ , which increases with t



- What about call options?
  - For non-dividend paying stock,  $\Theta < 0$  for call options
  - If S is very high, option holder will receive S K at maturity (with high probability)
  - This is the payoff of a long forward, so present value at t < T is  $c = S Ke^{-rT} > S K$ 
    - Intuition: for calls, we will pay K, whose present value today is smaller than K, pushing up the price of the call
  - What if S is low?
  - If the stock pays (a lot of) dividends, then Θ > 0 for high S, as the option holder misses the dividend payout during the life of the option (compared to a stock holder)







In BSM model, the volatility and risk-free interest rate are constant Still, it is interesting to ask how a change in  $\sigma$  or r change the value of the option

4. **Rho**: Change in option price due to a change in interest rate *r* 

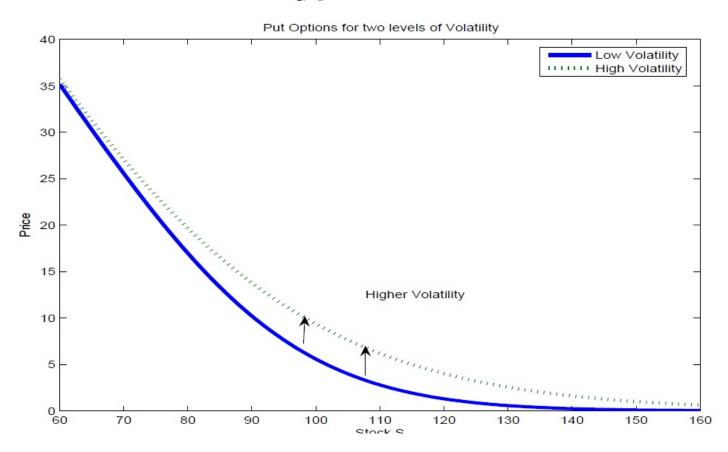
$$\rho = \frac{d \text{ Option Price}}{d r} = \left\{ \begin{array}{ll} \textit{KTe}^{-rT} \textit{N}(\textit{d}_2) > 0 & \text{for Calls} \\ -\textit{KTe}^{-rT} \textit{N}(-\textit{d}_2) < 0 & \text{for Puts} \end{array} \right.$$

<u>Intuition:</u> It depends on whether the option holder will pay K (call) or receive K (put). The PV of K declines as r increases, making the payment made smaller for the long call and payment received smaller for the long put.



5. **Vega**: Change in option price due to a change in volatility  $\sigma$ 

$$u = \frac{d \text{ Option Price}}{d \sigma} = S\sqrt{T}N'(d_1) > 0$$



# Using options for financial engineering



- Investors have a taste for securities where their principal is protected but they share in the upside if the market does well
- Investment banks often offer such securities, and hedge the short position with options or dynamic trading strategies
  - See examples of MITTS and SUN securities with class materials
- The example we will study is a "Capital Protected Note" loosely based on a security issued in 2008 by Morgan Stanley

## Using options for financial engineering: Example



It's Feb 22, 2008 and you sold a Capital Protected Note with:

- Maturity February 20, 2015
- Issue price \$10
- Principal \$10
- Interest 0%
- Principal protection 100%
- Payoff at maturity = principal + Supplemental Redemption Amount (SRA) if positive

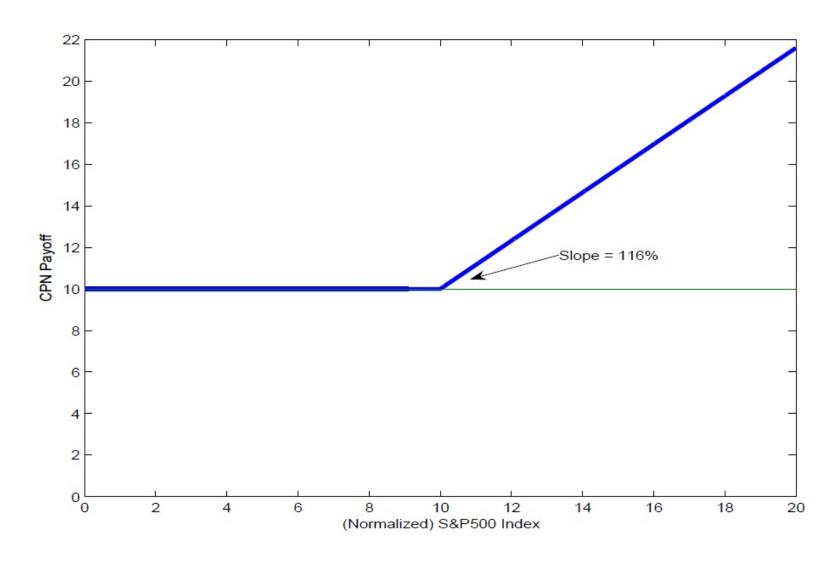
$$SRA = \$10 \times 116\% \times \frac{\text{Final Index Value} - \text{Initial Index Value}}{\text{Initial Index Value}}$$

Index is S&P 500 normalized to have Initial Index Value = \$10

You want to protect your short position against increases in the stock price index How to do this? Delta-hedge by setting up a replicating portfolio

# **Payoff on Capital Protected Note**





#### **Payoff on Capital Protected Note**



The payoff on the note can be decomposed into:

- (1) A zero coupon bond with principal \$10 and maturity T = 7
- (2) 1.16 at-the-money call options on the normalized S&P 500 with maturity T = 7
  - The reference index is normalized so that  $S_0 = \beta \times S\&P 500 = $10$
  - On 2/28/08, S&P  $500 = 1353.1 => \beta = 10/1353.1$
- Other data on 2/28/08
  - Interest rate *r* = 3.23% (continuously compounded)
  - Dividend yield on S&P 500  $\delta$  = 2%
  - Forecast of market volatility over the 7 years  $\sigma = 15\%$

#### The value of the security using BSM for dividend-paying stock is:

$$e^{-rT}$$
\$10+1.16× $Call$ (10,10, $r$ , $\delta$ , $\sigma$ , $T$ )  
=\$7.9764+1.16×\$1.7  
=\$9.9483

- A little less than \$10
- Investors give up interest on principal in exchange for a call option

#### Delta hedging at t=0



- You have a short position in the Capital Protected Note
- Hedge with an offsetting long position =>
  - Buy a zero coupon bond for \$7.9764 to hedge the bond component
  - Buy 1.16 units of the replicating portfolio for the embedded call option
- Setting up the replicating portfolio for each call:
  - We can calculate the call  $\Delta = N(d_1) = 0.5747$
  - Then the bond position = Call(10, 10, r,  $\delta$ ,  $\sigma$ , T)  $\Delta$  x  $S_0$  = 1.7 .05747 x 10 = -4.047
  - In sum, for each call option, invest 0.5747 x \$ 10 = \$5.747 in the S&P 500 and borrow \$4.047
- Value of replicating portfolio = \$5.747 \$4.047 = 1.7
- Multiply both positions by 1.16 to scale up to the replicating portfolio for the Capital Protected Bond

#### **Dynamic delta hedging**



Theoretically we need to frequently rebalance the portfolio as the  $\Delta$  changes.

- It will change with the stock price
- It will also change the passage of time, and any changes in r and σ

Question: Which Greeks tells us about these?

- With regard to stock prices:
  - If S<sub>t</sub> increases then Δ increases
  - If S<sub>t</sub> decreases then Δ decreases
- Recalculate Δ and new value of call.
- Adjust holdings of stocks and bonds in replicating portfolio to match new option value

Q: Why do dynamic replication here instead of initially just buying call options on the S&P 500 at the CBOE?

#### How well does dynamic replication work in practice?



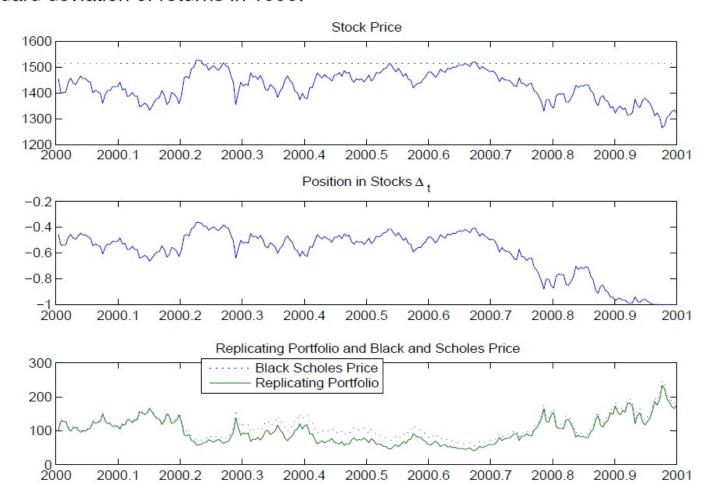
- Depends on a number of factors that include
  - Frequency of rebalancing
  - Stability of parameters
  - Accuracy of volatility estimate
  - Whether jumps in stock prices
- Well enough that may financial institutions are willing to offer products like Capital Protected Notes

#### How well does dynamic replication work in practice?



#### A put option example:

Let's look at Jan – Dec 2000. We replicate S&P500 option with T=1 year.  $\sigma=$  standard deviation of returns in 1999.



#### **Portfolio Insurance**



In 1981 UC-Berkeley Profs Hayne Leland and Mark Rubinstein joined forces with John O'Brien and formed Leland O'Brien and Rubinstein Associates, Incorporated (LOR).

<u>Business Idea</u>: Use the latest techniques from derivative security pricing and dynamic replication to offer investors insurance on their portfolios.

 For example, a pension plan fully invested in equity could purchase insurance to insure the portfolio against losses of value

LOR did not directly sell insurance. Rather they advised clients on the dynamic asset allocation that would insure against a drop in value of the portfolio.

Large potential demand from both mutual funds and pension plans

Business started slow but took off in 1984-86

By 1987 an estimated \$100 billion in assets were covered by portfolio insurance products

Total market cap of NYSE/Amex/Nasdaq of 2.2 trillion

#### **Portfolio insurance**



Various forms of portfolio insurance were developed.

• E.g. using out-of-the money put options where the investor could suffer a limited loss which is like a "deductible" that lowers the premium

The most significant development in this period was "perpetual" insurance

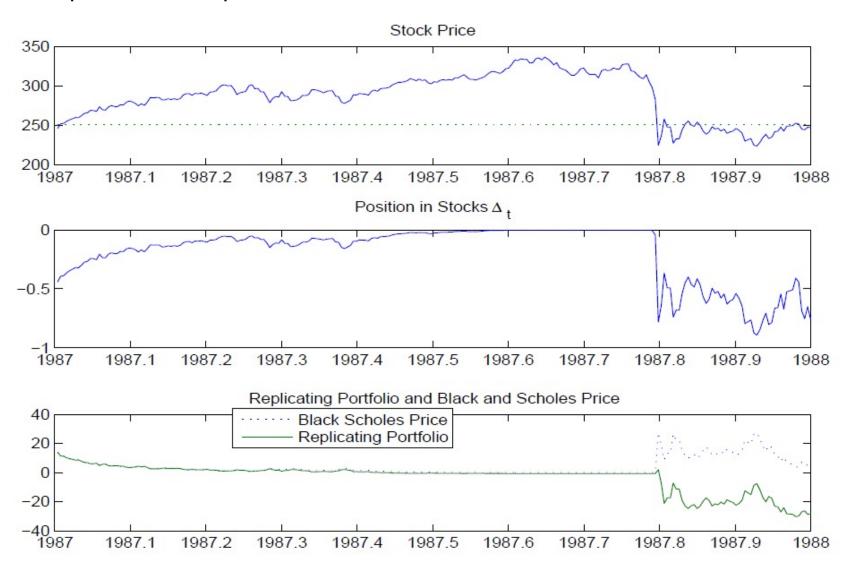
- In a standard fixed term contract, the investor was covered only for some period of time (typically, 3 years)
- Most pension funds have long term liabilities, limiting the value of term insurance
- Perpetual insurance allowed the investor (insured) to decide to exercise its option at any time in the future

But then came the 1987 Crash ...

#### How well does dynamic replication work in practice?



This example is for call options around the time of the 1987 market crash.







- This example illustrates in practice what we learned in theory: delta-hedging strategies only work well when stock price movement are fairly smooth
  - With large jumps in stock prices, it is impossible to rebalance fast enough to eliminate all risk
- Next week we will modify the hedging strategy to incorporate gamma hedging.
  - We'll see that as in fixed income strategies, incorporating this additional protection leads to a more robust hedge