15.415x Foundations of Modern Finance

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Lecture 14: Portfolio Theory

Key concepts

- Introduction: portfolio choice with mean-variance preferences
- Portfolios with two assets
- Portfolio frontier with multiple risky assets
- Portfolio choice with a safe asset
- Analytics of the portfolio frontier
- Properties of the tangency portfolio
- Non-mean-variance objective

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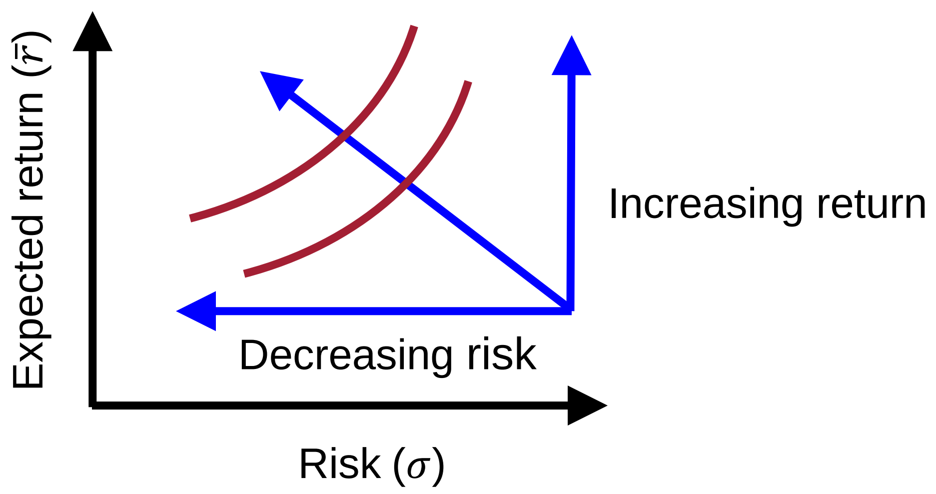
Mean-variance preferences

- How to choose a portfolio: Maximize expected utility.
- Special case – assume investors care only about the first two moments: average return and return variance/volatility (risk).
  - Minimize risk for a given expected return, Or:
  - Maximize expected return for a given risk.



Investor's problem

- Among all the portfolios with a target level of expected return, find the one with the lowest variance.

- Formally, we need to solve the following problem:

$$(P): \quad \underset{\{w_1,\ldots,w_n\}}{\text{Minimize}} \quad \sigma_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij}$$

$$\text{subject to:} \quad (1) \quad \textstyle\sum_{i=1}^{N} w_i = 1$$

$$(2) \quad \textstyle\sum_{i=1}^{N} w_i \bar{r}_i = \bar{r}_p$$

- Of all the portfolios [constraint (1)] with an expected return of $\bar{r}_p$ [constraint (2)], find the one that has the lowest variance.

Empirical examples

- We use ETFs to construct portfolios.
- Each ETF in our examples represents an exposure to a distinct asset class.

ETF	Description	Inception date
SPY	US Equity: S&P 500	Jan 22, 1993
AGG	Aggregate bonds	Sep 22, 2003
HYG	High yield bonds	Apr 04, 2007
IAU	Gold	Jan 21, 2005
IYR	US Equity: Real estate sector	Jun 12, 2000

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Two assets, long only

$$\bar{r}_p = w\bar{r}_1 + (1-w)\,\bar{r}_2$$

$$\sigma_p^2 = w^2\sigma_1^2 + (1-w)^2\sigma_2^2 + 2w(1-w)\sigma_{12}$$

- SPY and AGG (2008/01-2018/12).

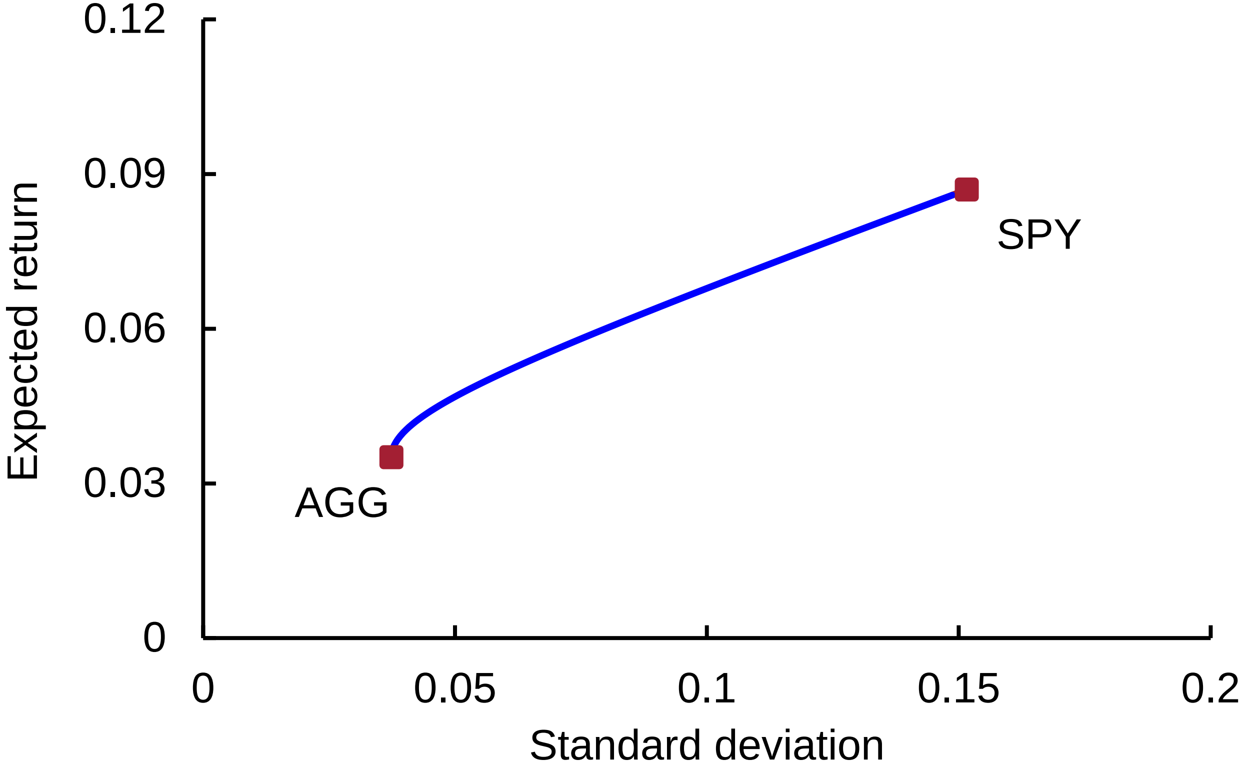
Mean Returns	
SPY	AGG
8.70%	3.26%

Covariances		
	SPY	AGG
SPY	0.0230	0.0004
AGG	0.0004	0.0014

- Portfolio return and risk (unique solution):

Weight in SPY	0.0%	10.0%	20.0%	30.0%	40.0%	50.0%	60.0%	70.0%	80.0%	90.0%	100.0%
Mean	3.26%	3.80%	4.35%	4.89%	5.44%	5.98%	6.52%	7.07%	7.61%	8.16%	8.70%
St Dev	3.80%	3.83%	4.43%	5.41%	6.61%	7.93%	9.32%	10.75%	12.20%	13.68%	15.16%

Two assets, long only



Two assets, with short sales allowed

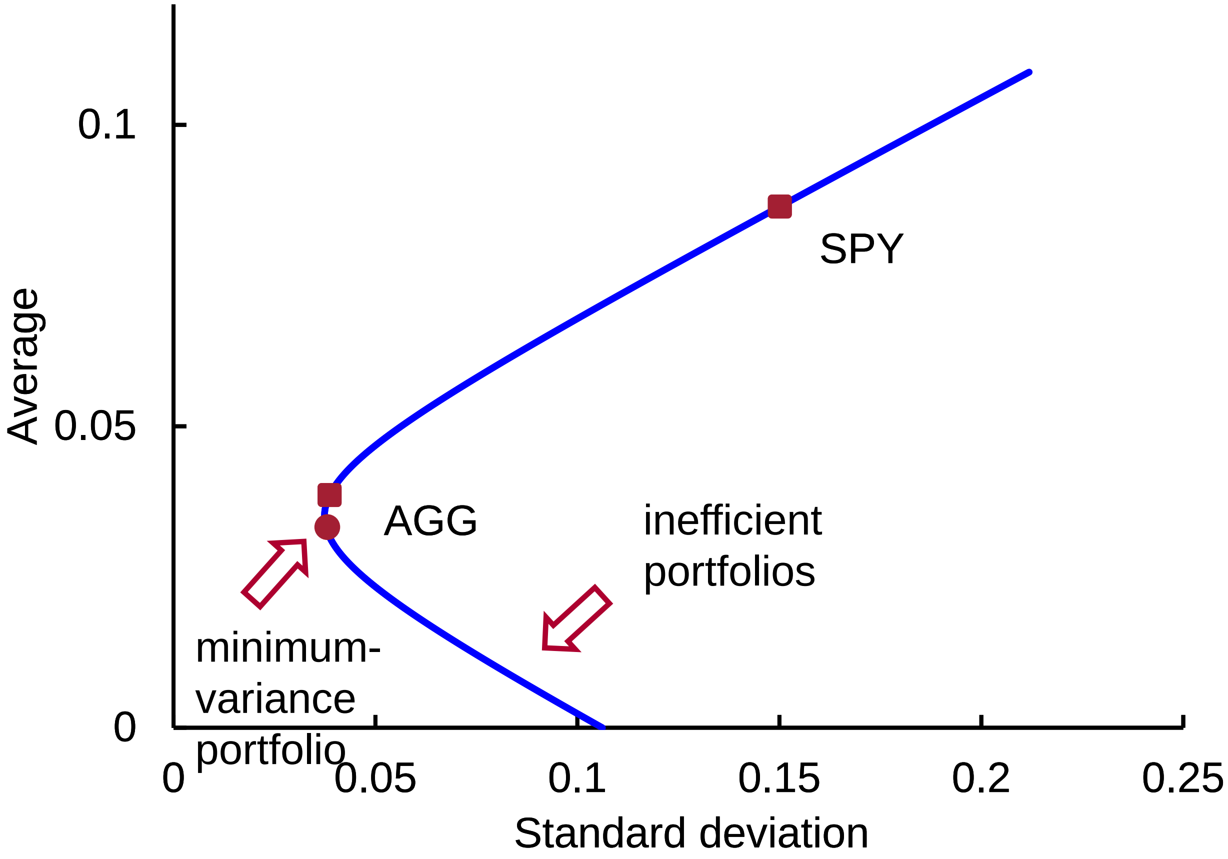
- When short sales are allowed, portfolio weights are unrestricted.

Mean Returns	
SPY	AGG
8.70%	3.26%

Covariances		
	SPY	AGG
SPY	0.0230	0.0004
AGG	0.0004	0.0014

Weight in SPY	-60.0%	-40.0%	-20.0%	0.0%	20.0%	40.0%	60.0%	80.0%	100.0%	120.0%	140.0%
Mean	-0.00%	1.09%	2.17%	3.26%	4.35%	5.44%	6.52%	7.61%	8.70%	9.79%	10.87%
St Dev	10.62%	7.81%	5.32%	3.80%	4.43%	6.61%	9.32%	12.20%	15.16%	18.16%	21.19%

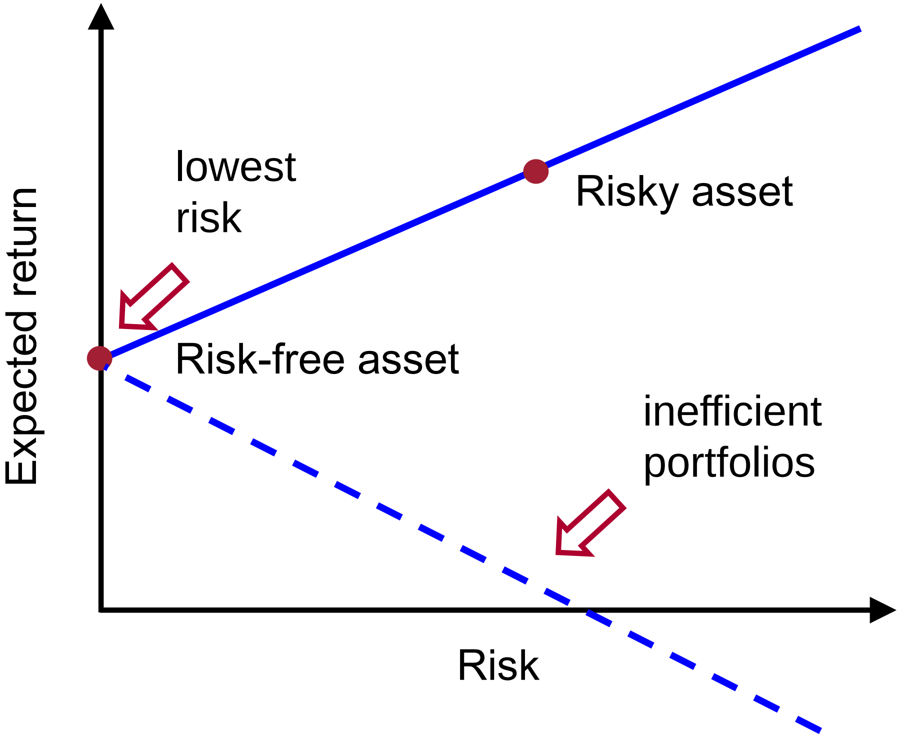
Two assets, with short sales allowed



Two assets: safe and risky

$$\bar{r}_p = w\bar{r}_1 + (1-w)r_F$$

$$\sigma_p = w\sigma_1$$



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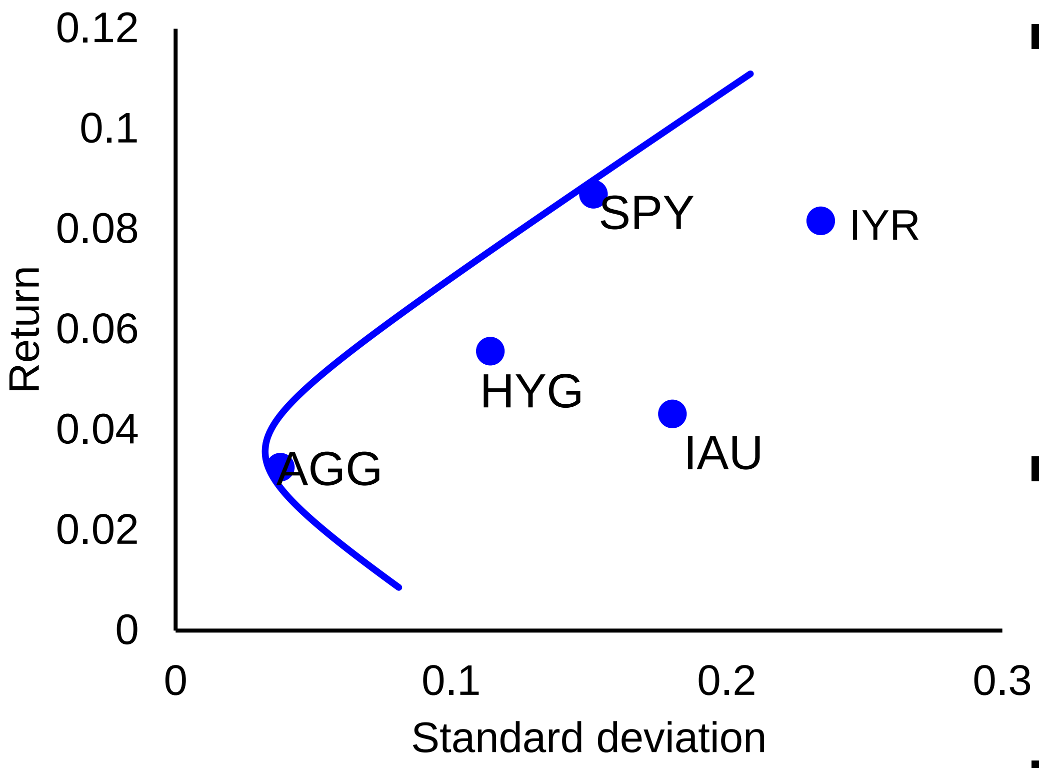
Multiple assets

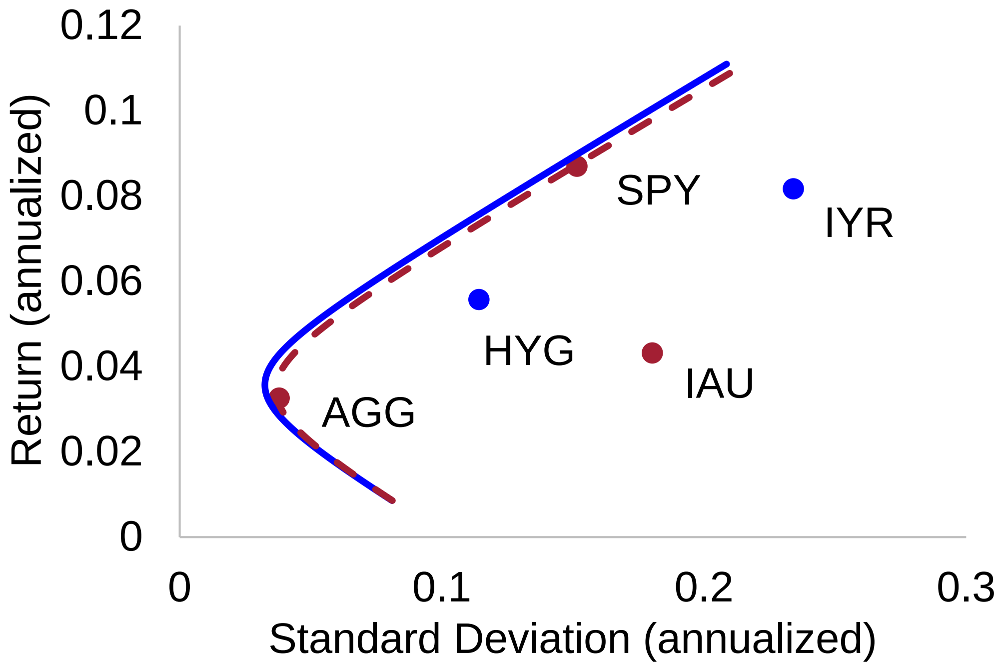
Optimal portfolios using in-sample return moments
(annualized returns)



- Given an expected return, the portfolio that minimizes risk (measured by SD or variance) is a mean-variance optimal portfolio.
- The locus of all frontier portfolios in the mean-SD plane is called portfolio frontier.
- The upper part of the portfolio frontier gives the efficient frontier portfolios.

Multiple assets

Add 3-asset frontier (dashed line): AGG, SPY, and IAU



- When more assets are included, the portfolio frontier improves, i.e., moves toward upper-left: higher mean returns and lower risk.
- Intuition: Since one can always choose to ignore the new assets, including them cannot make one worse off.

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Portfolio frontier with a safe asset

- Observation: A portfolio of risk-free and risky assets can be viewed as a portfolio of two portfolios:
  - the risk-free asset, and
  - a portfolio of only risky assets.
- Example: consider a portfolio with $40 invested in the risk-free asset and $30 each in the Equity Index and LT Bonds:
  - $w_0 = 40\%$ in the risk-free asset,
  - $w_1 = 30\%$ in Equities,
  - $w_2 = 30\%$ in LT Bonds.

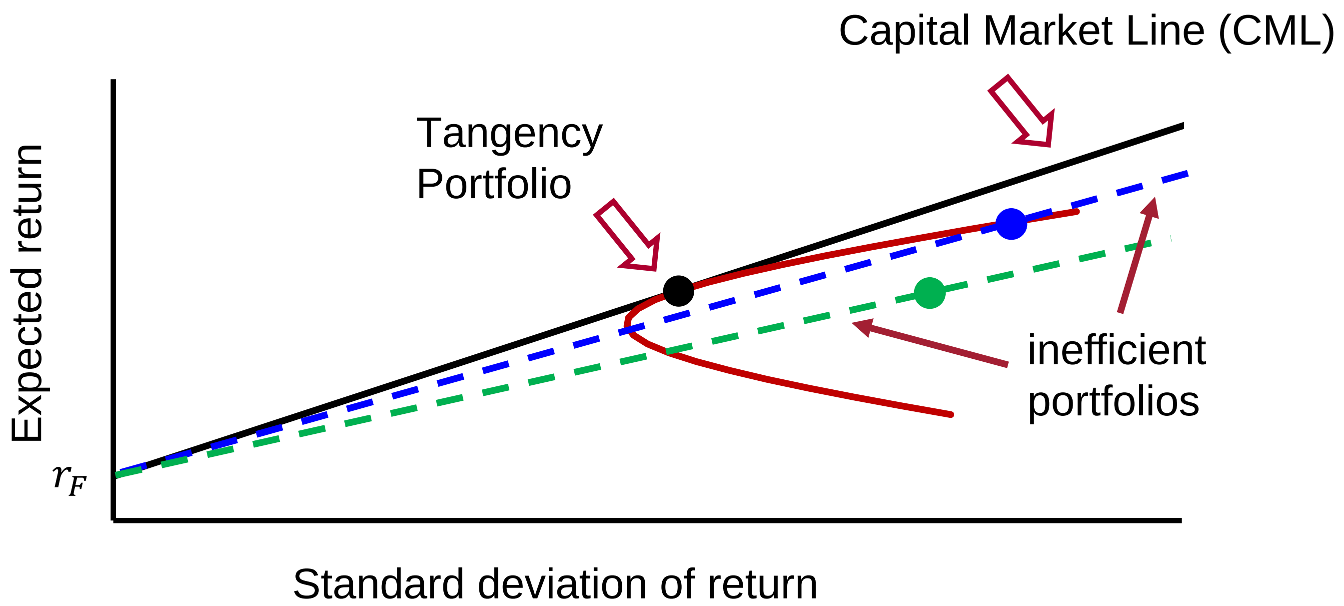
Portfolio frontier with a safe asset

- We can also view the portfolio as follows:
  - $1 - x = 40\%$ in the risk-free asset,
  - $x = 60\%$ in a portfolio of only risky assets, which has:
    - 50% in Equities,
    - 50% in LT Bonds.
- Consider a portfolio p with:
  - x invested in a risky portfolio q, and
  - $1 - x$ invested in the risk-free asset.

$$\bar{r}_p = (1 - x) r_F + x \bar{r}_q$$

$$\sigma_p^2 = x^2 \sigma_q^2 \text{ (use } \mathrm{Var}(r_F) = 0\text{)}$$

Portfolio frontier with a safe asset



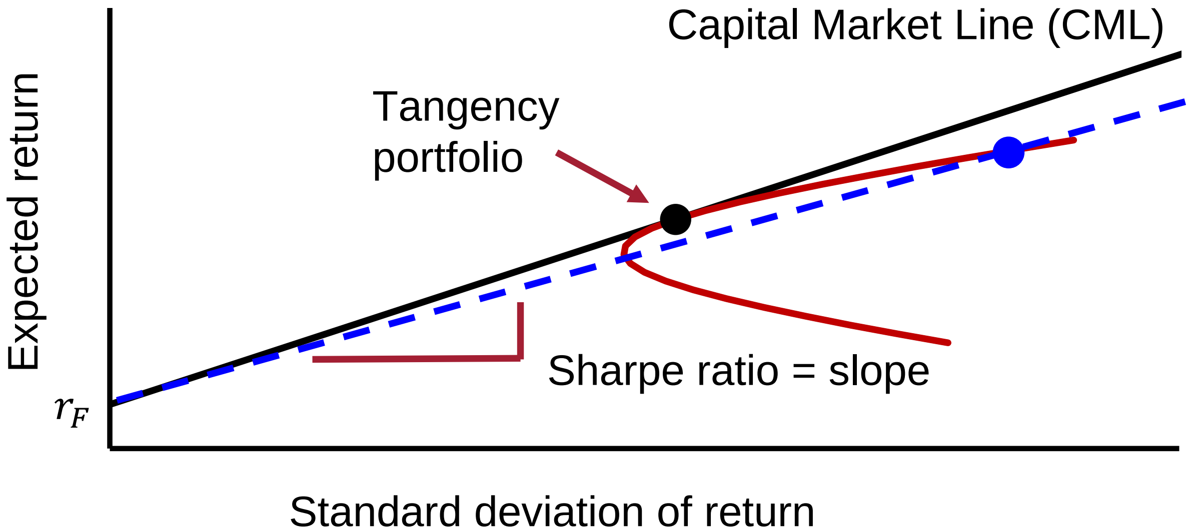
- With a risk-free asset, frontier portfolios are combinations of:
  - the risk-free asset,
  - the tangency portfolio (consisting of only risky assets).
- The frontier is also called the Capital Market Line (CML).

Sharpe ratio

- Sharpe ratio is a measure of a portfolio's risk-return trade-off, equal to the portfolio's risk premium divided by its volatility:

$$\text{Sharpe Ratio} \equiv \frac{\bar{r}_p - r_F}{\sigma_p} \quad \text{(higher is better !)}$$

- The tangency portfolio has the highest possible Sharpe ratio of all portfolios. So do all the portfolios on the CML.



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The tangency portfolio

- Use the optimality condition to characterize the Tangency portfolio explicitly.
  - N risky assets, $i = 1, 2, \dots, N$,
  - Expected return vector $\bar{r}$,
  - Expected excess return vector $\bar{x}$,
  - Covariance matrix Σ (positive definite),
  - $(N \times 1)$ vector of 1's ι,
  - $(N \times 1)$ vector of portfolio weights w of risky assets.

Then,

  - $\bar{x} = \bar{r} - r_F\, \iota$,
  - The weight in the risk-free asset is $1 - w'\iota$,
  - The expected excess return on portfolio w is $w'\bar{x}$.

The tangency portfolio

- Solve for the efficient portfolio:

$$\boxed{w'\Sigma\, w = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \Sigma_{ij}}$$

$$\boxed{w'\bar{x} = \sum_{i=1}^{N} w_i \bar{x}_i}$$

$$\min \quad w'\Sigma\, w \qquad \text{(portfolio variance)}$$

$$\text{s.t.} \quad w'\bar{x} = m \qquad \text{(portfolio expected excess return)}$$

- Solve the optimization portfolio using the method of Lagrange multipliers:

$$L = w'\Sigma w + 2\lambda\,(m - w'\bar{x}\,)$$

- The first order condition (FOC) $\partial L/\partial w = 0$:

$$2\Sigma w\; - 2\lambda\bar{x} = 0 \;\Rightarrow\; \text{Solution: } w_T = \lambda\, \Sigma^{-1}\, \bar{x}$$

- Tangency portfolio weights on risky assets sum up to one: $w_T'\,\iota = 1$. Find

$$\lambda = \frac{1}{\bar{x}'\Sigma^{-1}\iota}$$

$$w_T = \frac{1}{\bar{x}'\Sigma^{-1}\iota}\,\Sigma^{-1}\bar{x}$$

Example: optimal portfolio

- Consider an asset allocation problem with three asset classes: T-Bills, Stock Market Index, and a Hedge Fund Index.

- Returns on the Stock Market Index (SMI) and Hedge Fund Index (HFI) are uncorrelated.

- What fraction of the Tangency portfolio should be allocated to the Hedge Fund Index?

- What are the Sharpe ratios for the SMI, HFI, and the Tangency portfolio?

	Expected return	Standard deviation
T-Bills	3%	0%
Stock Market Index	9%	20%
Hedge Fund Index	5%	10%

Example: optimal portfolio

- Use the general portfolio solution

$$w_T = \lambda\, \Sigma^{-1} \bar{x}$$

- The covariance matrix of returns is diagonal, easy to invert:

$$w_{SMI} = \lambda \frac{(9\% - 3\%)}{0.20^2} = \lambda \times 1.5$$

$$w_{HFI} = \lambda \frac{(5\% - 3\%)}{0.10^2} = \lambda \times 2.0$$

$$w_{SMI} = \frac{1.5}{1.5 + 2.0} = 43\%$$

$$w_{HFI} = \frac{2.0}{1.5 + 2.0} = 57\%$$

  - Over 50% allocation into the HFI. How does HFI compare to SMI on a stand-alone basis?

Example: optimal portfolio

- Recall that the Sharpe ratio equals expected excess return over volatility:

$$SR_{SMI} = \frac{9\% - 3\%}{20\%} = 30\% \quad vs. \quad SR_{HFI} = \frac{5\% - 3\%}{10\%} = 20\%$$

- HFI has a lower Sharpe ratio yet receives a higher allocation.
  - HFI is an excellent diversifier: it is uncorrelated with SMI.
  - HFI has much lower volatility than SMI.
- Sharpe ratio of the Tangency portfolio:

vs. $SR_{SMI} = 30\%$
and $SR_{HFI} = 20\%$

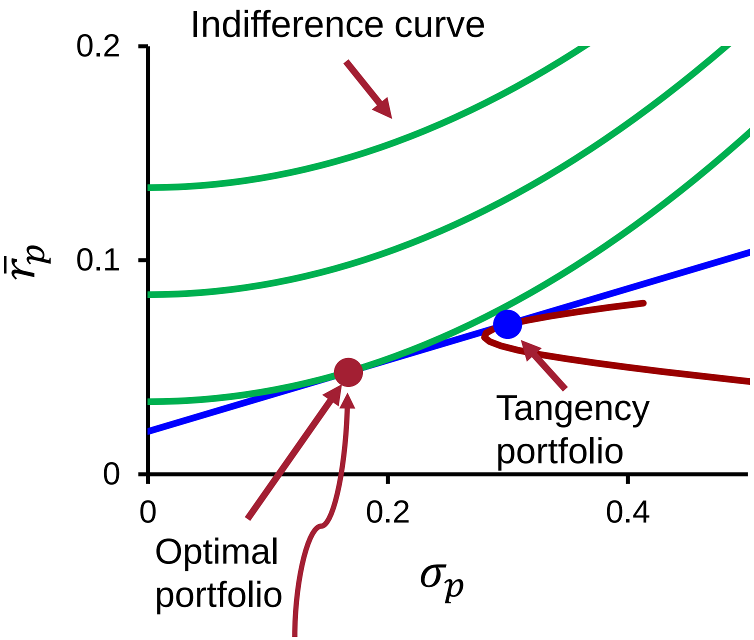
$$SR_T = \frac{0.43 \times 6\% + 0.57 \times 2\%}{\sqrt{0.43^2 \times 0.2^2 + 0.57^2 \times 0.10^2}} = 36\%$$

- Achieve higher Sharpe ratio thanks to diversification.

Risk aversion and portfolio choice

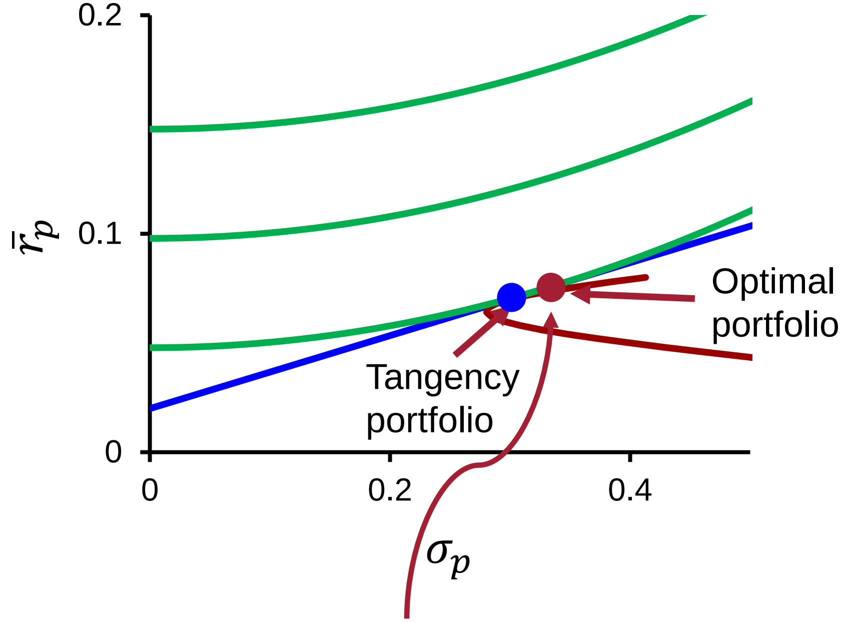
- All mean-variance investors choose the tangency portfolio of stocks.
- Individual risk aversion determines allocation to the risky assets.

High risk aversion



invest in the risk-free asset and the tangency portfolio

Low risk aversion



borrow at the risk-free rate and buy the tangency portfolio

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Portfolio and individual assets

- Contribution of an asset to a portfolio:
  - In the presence of a risk-free asset, portfolio's return is:

$$\tilde{r}_p = \left(1 - \sum_{i=1}^{N} w_i\right) r_F + \sum_{i=1}^{N} w_i \tilde{r}_i = r_F + \sum_{i=1}^{N} w_i (\tilde{r}_i - r_F)$$

- Each individual asset contributes to the portfolio on two dimensions:
  - Expected return;
  - Risk measured by return volatility (SD).
- We consider these two aspects separately.

Individual contribution to expected return

- Expected portfolio return is:

$$\bar{r}_p = r_F + \sum_{i=1}^{N} w_i \left(\bar{r}_i - r_F\right)$$

- Describe the marginal contribution of risky asset i to the expected portfolio return.
  - "Marginal contribution of x to A" means the incremental change of A when x changes by a small amount.
- Marginal contribution is the partial derivative: change in the portfolio properties per unit change in the weight on asset i, holding all other risky asset weights fixed (if we change w_i by δ, the weight on the risk-free asset must change by $-\delta$):

$$\frac{\partial \bar{r}_p}{\partial w_i} = \bar{r}_i - r_F \quad (\text{risk premium of asset } i)$$

Individual contribution to return volatility

- The variance of portfolio return is given by

$$\sigma_p^2 = \sum_{i=1}^{N}\sum_{j=1}^{N} w_i w_j \sigma_{ij} = w_i^2 \sigma_i^2 + 2\sum_{j\neq i}^{N} w_i w_j \sigma_{ij} + [\text{Terms not including } w_i]$$

- The marginal contribution of asset i to portfolio variance is $2\mathrm{Cov}(\tilde{r}_i, \tilde{r}_p)$:

$$\frac{\partial\, \sigma_p^2}{\partial w_i} = 2 w_i \sigma_i^2 + 2\sum_{j\neq i}^{N} w_j \sigma_{ij} = 2\sum_{j=1}^{N} w_j \sigma_{ij} = 2\mathrm{Cov}\left(\tilde{r}_i, \sum_{j=1}^{N} w_j \tilde{r}_j\right) = 2\mathrm{Cov}(\tilde{r}_i, \tilde{r}_p)$$

- The marginal contribution of asset i to portfolio standard deviation is

$$\frac{\partial \sigma_p}{\partial w_i} = \frac{\partial \left(\sigma_p^2\right)^{\frac{1}{2}}}{\partial w_i} = \frac{\mathrm{Cov}(\tilde{r}_i, \tilde{r}_p)}{\sigma_p} = \frac{\sigma_{ip}}{\sigma_p}$$

Optimality of the tangency portfolio

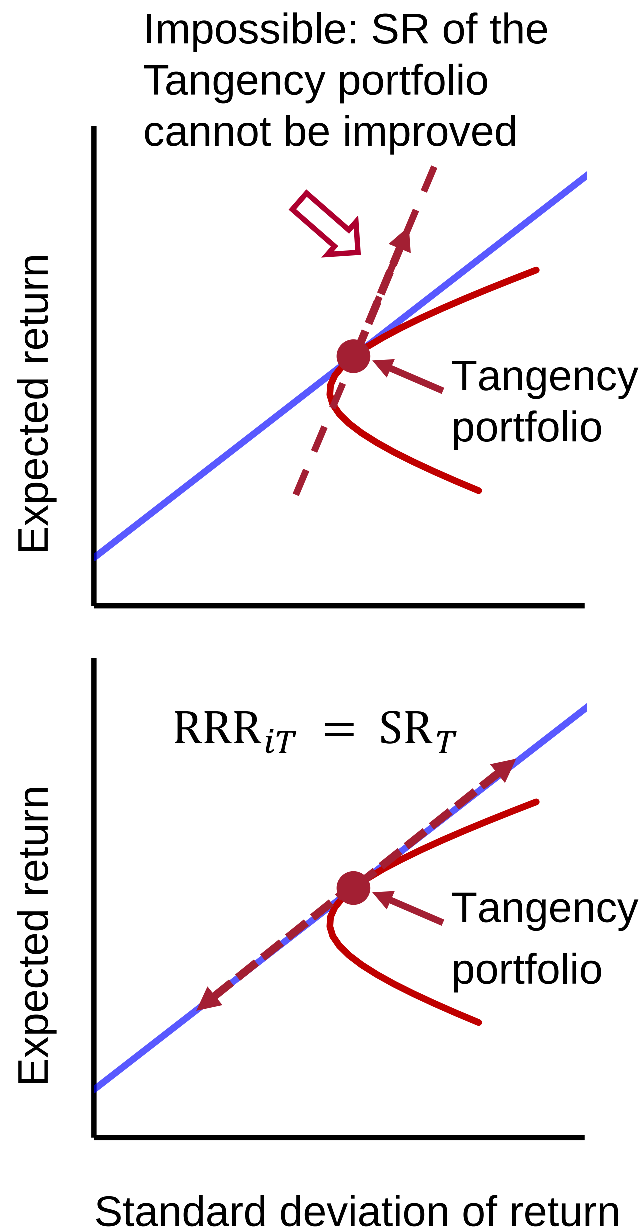
- Summarize the marginal contribution of risky asset i to portfolio p by its marginal return-to-risk ratio (RRR):

$$\mathrm{RRR}_{ip} = \frac{\left(\dfrac{\partial \bar{r}_p}{\partial w_i}\right)}{\left(\dfrac{\partial \sigma_p}{\partial w_i}\right)} = \frac{\bar{r}_i - r_F}{(\sigma_{ip}/\sigma_p)}$$

- Claim: For the tangency portfolio (T), which optimal, the return-to-risk ratio of all risl assets must be the same, and equal to tl Sharpe ratio of the tangency portfolio:

$$\mathrm{RRR}_{iT} = \frac{\bar{r}_i - r_F}{(\sigma_{iT}/\sigma_T)} = \frac{\bar{r}_T - r_F}{\sigma_T} = \mathrm{SR}_T$$

- Intuition: The SR of a frontier portfolio cann be improved.



Optimality of the tangency portfolio: a supplement

- Claim: For the tangency portfolio (T),

$$\mathrm{RRR}_{iT} = \frac{\bar{r}_i - r_F}{(\sigma_{iT}/\sigma_T)} = \frac{\bar{r}_T - r_F}{\sigma_T} = \mathrm{SR}_T$$

- Derive this result algebraically.
- Tangency portfolio composition is given by $w_T = \lambda\, \Sigma^{-1}\, \bar{x}$.
- Let e_i be the i'th basis vector, $e_i = (0, \ldots, \underbrace{1}_{i}, \ldots 0)'$. Then,

$$\sigma_{iT} = e_i' \Sigma w_T = e_i' \Sigma (\lambda\, \Sigma^{-1}\, \bar{x}) = \lambda e_i' \bar{x} = \lambda \bar{x}_i$$

$$\sigma_T^2 = w_T{}' \Sigma w_T = w_T{}' \Sigma (\lambda\, \Sigma^{-1}\, \bar{x}) = \lambda w_T{}' \bar{x} = \lambda \bar{x}_T$$

And therefore

$$\frac{\bar{x}_i}{\sigma_{iT}} = \frac{\bar{x}_T}{\sigma_T^2} \Rightarrow \text{this is the same as } \frac{\bar{r}_i - r_F}{(\sigma_{iT}/\sigma_T)} = \frac{\bar{r}_T - r_F}{\sigma_T}$$

Regression interpretation

- Optimality of the tangency portfolio is mathematically equivalent to the intercept α_i in the following regression being equal to zero:

$$\tilde{r}_i - r_F = \alpha_i + \beta_i(\tilde{r}_T - r_F) + \tilde{\varepsilon}_i$$

$$\beta_i = \frac{\sigma_{iT}}{\sigma_T^2}$$

$$RRR_{iT} = \frac{\bar{r}_i - r_F}{\beta_i \sigma_T} = \frac{\bar{r}_T - r_F}{\sigma_T} \Rightarrow \alpha_i = 0$$

- If α_i is non-zero, can improve on the Sharpe ratio of the tangency portfolio – the portfolio is not optimal.

- Can test for optimality in the data using regression methods.

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Beyond the static mean-variance theory

- In practice, portfolio choice problems are often dynamic:
  - Household managing investments to support several consumption objectives over the life cycle.
- Even in static problems, the objective may not be captured by the first two moments.
  - Maximize expected return with a penalty for portfolio value falling below a given threshold.
- Can capture various objectives within the expected utility framework, e.g.

$$\max \mathrm{E}\left[U\left(V_p\right)\right], \qquad U(v) = \begin{cases} v, & v \geq \underline{v} \\ v + a(v - \underline{v}), & v < \underline{v} \end{cases}, \qquad a > 0$$

Marginal contribution of an asset to portfolio

- Consider the objective

$$\max \mathrm{E}\left[U\left(1+\tilde{r}_p\right)\right],\ \ \tilde{r}_p = r_F + \sum_{i=1}^{N} w_i(\tilde{r}_i - r_F)$$

- The first-order optimality condition is

$$\mathrm{E}\left[U'\left(1+\tilde{r}_p\right) \times (\tilde{r}_i - r_F)\right] = 0$$

$$\mathrm{E}\left[U'\left(1+\tilde{r}_p\right)\right] \times (\bar{r}_i - r_F) + \mathrm{Cov}\left(U'\left(1+\tilde{r}_p\right), \tilde{r}_i\right) = 0$$

- Marginal contribution to portfolio risk is captured by $\mathrm{Cov}\left(U'\left(1+\tilde{r}_p\right), \tilde{r}_i\right)$: this places emphasis on nonlinear measures of return co-movement, e.g., $\mathrm{Cov}\left(\tilde{r}_p^2, \tilde{r}_i\right)$.

- Different investors no longer hold the same risky portfolio, the shape of the objective affects the optimal portfolio composition.

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