Recitation 14

Spring 2021

Question 1

Suppose there are three stocks in the market - A, B, and C - with the following properties:

	Expected Return	Standard Deviation
A	$\bar{r}_A = 5\%$	$\sigma_A = 10\%$
B	$\bar{r}_B = 20\%$	$\sigma_B = 40\%$
C	$\bar{r}_C = 20\%$	$\sigma_C = 40\%$

The return on stock C is not correlated with A and B. Correlation of returns between A and B is $\rho_{A,B} = -0.5$.

(a) Suppose that an investor allocates 40% of her wealth to stock A, 60% to stock C, and nothing to stock B. What is the standard deviation and expected return of her portfolio?

(b) Suppose that an investor allocates 40% of her wealth to stock A, 60% to stock B, and nothing to stock C. What is the standard deviation and expected return of her portfolio?

(c) Suppose that an investor constructs an equally weighted portfolio out of stocks A, B, and C. What is the standard deviation and expected return of her portfolio?

Solutions:

(a) Portfolio: $w_A = 40\%$, $w_B = 0\%$, $w_C = 60\%$

Expected return:

$$\begin{aligned}\bar{r}_p &= w_A\bar{r}_A + w_B\bar{r}_B + w_C\bar{r}_C \\ &= 40\% \times 5\% + 60\% \times 20\% \\ &= 14\%\end{aligned}$$

Portfolio variance:

$$\sigma_p^2 = w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + w_C^2\sigma_C^2 + 2w_Aw_B\sigma_{A,B} + 2w_Aw_C\sigma_{A,C} + 2w_Bw_C\sigma_{B,C}$$

In this problem, $\sigma_{A,C} = 0$ and $\sigma_{B,C} = 0$. Also recall that

$$\sigma_{A,B} = \rho_{A,B} \times \sigma_A \times \sigma_B$$

This will be useful in parts b and c, where both w_A and w_B are non-zero.

Going back to the portfolio variance, we have:

$$\begin{aligned}\sigma_p^2 &= w_A^2\sigma_A^2 + w_C^2\sigma_C^2 \\ &= 0.4^2 0.1^2 + 0.6^2 .4^2 \\ &= .0592 \\ \sigma_p &= 24.33\%\end{aligned}$$

(b) Portfolio: $w_A = 40\%$, $w_B = 60\%$, $w_C = 0\%$

Expected return:

$$\begin{aligned}\bar{r}_p &= w_A\bar{r}_A + w_B\bar{r}_B + w_C\bar{r}_C \\ &= 40\% \times 5\% + 60\% \times 20\% \\ &= 14\%\end{aligned}$$

Portfolio variance:

$$\begin{aligned}\sigma_p^2 &= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + w_C^2\sigma_C^2 + 2w_Aw_B\sigma_{A,B} + 2w_Aw_C\sigma_{A,C} + 2w_Bw_C\sigma_{B,C} \\ &= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_{A,B} \\ &= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + 2w_Aw_B\sigma_A\sigma_B\rho_{A,B} \\ &= 0.4^2 0.1^2 + 0.6^2 0.4^2 + 2 \times 0.4 \times 0.6 \times 0.1 \times 0.4 \times (-0.5) \\ &= 0.0496 \\ \sigma_p &= 22.27\%\end{aligned}$$

(c) Portfolio: $w_A = 1/3$, $w_B = 1/3$, $w_C = 1/3$

Expected return:

$$\begin{aligned}\bar{r}_p &= w_A\bar{r}_A + w_B\bar{r}_B + w_C\bar{r}_C \\ &= 1/3 \times 5\% + 1/3 \times 20\% + 1/3 \times 20\% \\ &= 15\%\end{aligned}$$

Portfolio variance:

$$\begin{aligned}\sigma_p^2 &= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + w_C^2\sigma_C^2 + 2w_Aw_B\sigma_{A,B} + 2w_Aw_C\sigma_{A,C} + 2w_Bw_C\sigma_{B,C} \\ &= w_A^2\sigma_A^2 + w_B^2\sigma_B^2 + w_C^2\sigma_C^2 + 2w_Aw_B\sigma_A\sigma_B\rho_{A,B} \\ &= (1/3)^2 0.1^2 + (1/3)^2 0.4^2 + (1/3)^2 0.4^2 + 2 \times (1/3) \times (1/3) \times 0.1 \times 0.4 \times (-0.5) \\ &= 0.0322 \\ \sigma_p &= 17.95\%\end{aligned}$$

Altogether, we have:

Portfolio	Expected Return	Standard Deviation
a	$\bar{r}_p = 14\%$	$\sigma_p = 24.33\%$
b	$\bar{r}_p = 14\%$	$\sigma_p = 22.27\%$
c	$\bar{r}_p = 15\%$	$\sigma_p = 17.95\%$

Portfolio (a) has the same return as portfolio (b). However, because the correlation between stocks A and B is lower than that between stocks A and C, portfolio (b) has a lower volatility. The equally weighted portfolio (c) has higher expected return and lower volatility because it takes greater advantage of low correlation between the three stocks and, as a result, achieves a better risk-return trade-off.

Question 2

Suppose there are 2 stocks in the market, stocks A and C, with the following properties:

	Expected Return	Standard Deviation
A	$\bar{r}_A = 5\%$	$\sigma_A = 10\%$
C	$\bar{r}_C = 20\%$	$\sigma_C = 40\%$

The returns on stocks A and C are not correlated.

(a) Build a portfolio with an expected return of 10%

(b) Suppose that the maximum risk you can afford to take is 20%. Build a portfolio that achieves this target. What is the expected return of this portfolio?

(c) Suppose that there is an investor with low tolerance for risk. Construct a minimum variance portfolio (MPV) for this investor. What is the expected return of this minimum variance portfolio?

Solutions:

(a) Target $\bar{r}_p = 10\%$.

Suppose you invest x in stock A and $(1-x)$ in stock C. Then:

$$\begin{aligned} x\bar{r}_A + (1-x)\bar{r}_C &= 10\% \\ x \times 5\% + (1-x) \times 20\% &= 10\% \\ x &= 2/3 \end{aligned}$$

So we invest $2/3$ into stock A and $1/3$ into stock C.

(b) Target portfolio volatility of 20%.

Suppose you invest x in stock A and $(1-x)$ in stock C. Then:

$$\begin{aligned} x^2\sigma_A^2 + (1-x)^2\sigma_C^2 + x(1-x)\sigma_{A,C} &= 0.2^2 \\ 0.1^2x^2 + 0.4^2(1-x)^2 &= 0.2^2 \\ 0.17x^2 - 0.32x + 0.12 &= 0 \\ x &= 1.365, 0.517 \end{aligned}$$

We have two portfolio candidates:

1. 136.5% in A, -36.5% in C
2. 51.7% in A, 48.3% in C

Both achieve the 20% volatility target. To find which is better, we look at the expected returns:

1. $\bar{r}_1 = 1.365 \times 5\% - 0.365 \times 20\% = -0.48\%$
2. $\bar{r}_2 = 0.517 \times 5\% + 0.483 \times 20\% = 12.25\%$

So the optimal portfolio is 51.7% invested in A and 48.3% invested in C.

(c) Minimum variance portfolio.

Suppose you invest x in stock A and $(1-x)$ in stock C. Then we solve the following problem:

$$\min_x \{x^2 0.1^2 + (1-x)^2 0.4^2\}$$

Solving $\min_x\{0.17x^2 - 0.32x + 0.16\}$ yields $x = 94.12\%$.

So the minimum variance portfolio is 94.12% invested in A and 5.88% invested in C. The portfolio achieves:

- Standard deviation of $9.7\% = \sqrt{0.9412^2 0.1^2 + 0.0588^2 0.4^2}$
- Expected return of $5.88\% = 94.12\% \times 5\% + 5.88\% \times 20\%$

Question 3

Suppose there are 2 stocks in the market, stocks A and C, with the following properties:

	Expected Return	Standard Deviation
A	$\bar{r}_A = 5\%$	$\sigma_A = 10\%$
C	$\bar{r}_C = 20\%$	$\sigma_C = 40\%$

The returns on stocks A and C are not correlated.
There is a riskless asset with guaranteed return of 3%.
Suppose there are two investors, X and Z, who maximize risk-return tradeoff and have mean-variance preferences. Investor X can only invest in stocks A and C. Investor Z, in addition to being able to invest in stocks A and C, can also invest in the riskless bond.
Investor X targets standard deviation of 20% and investor Z targets standard deviation of 12%.

(a) What is the maximum Sharpe ratio investor X can achieve?

(b) What is the maximum Sharpe ratio investor Z can achieve?

(c) Describe the portfolio of investor Z. What is the expected return of this portfolio?

(d) If investor X had access to the risk-free asset, could her return have been improved? If yes, compute the expected return for a 20% target level of risk.

Solutions:

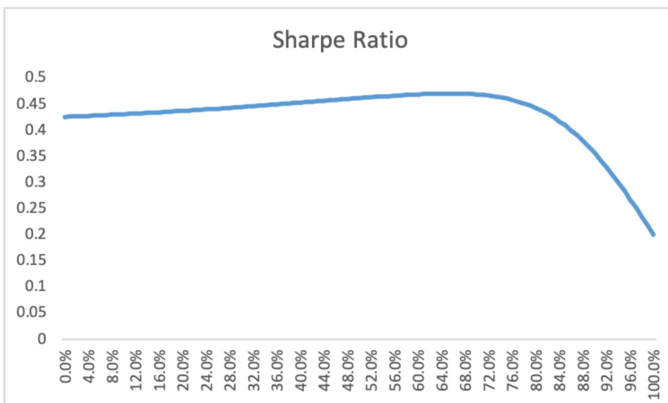
(a) Recall from question 2 part b that the expected return on X's portfolio is 12.25%. The Sharpe ratio of X's portfolio is:

$$SR = \frac{12.25\% - 3\%}{20\%} = 0.462$$

(b) We need to find the portfolio of risky assets, A and C, which has the highest Sharpe ratio. Suppose we invest x in A and $(1-x)$ in C. We solve:

$$\max_x \{ \frac{x\bar{r}_A + (1-x)\bar{r}_C - r_f}{\sqrt{x^2\sigma_A^2 + (1-x)^2\sigma_C^2}} \}$$

As a side note, we can visualize the behavior of the Sharpe ratio by plotting this ratio over risky portfolios (over different values of x). This plot is shown below, with the weight on stock A on the x axis and the Sharpe ratio on the y axis:



We can solve the Sharpe ratio maximization problem using Excel, as shown in the recitation video. We find that the tangency portfolio is the portfolio with 65.31% invested in A and 34.69% invested in C. The expected return of this portfolio is 10.20%, and the volatility is 15.34%. The Sharpe ratio is **0.47**.

(c) Since investor Z targets a 12% portfolio volatility, she needs to mix the risk-free asset with the tangency portfolio. Suppose she invests x in the tangency portfolio and $(1-x)$ in the risk-free asset. Then:

$$x^2\sigma_T^2 + (1-x)^2\sigma_f^2 = 0.12^2$$

$$x = \frac{0.12}{0.1534} = 78.24\%$$

So investor Z needs to invest 78.24% in the tangency portfolio and 21.76% in the risk-free asset to achieve the maximimum Sharpe ratio while targeting 12% volatility. Z's portfolio consists of:

- 21.76% risk-free asset
- 51.10% stock A (= 78.24% × 65.31%)
- 27.14% stock C (= 78.24% × 34.69%)

The expected return on Z's portfolio is:

$$21.76\% \times 3\% + 51.10\% \times 5\% + 27.14\% \times 20\% = 8.64\%$$

(d) Maximum return for investor X if she could invest in the risk-free asset.

Suppose investor X invests x into the tangency portfolio and $(1-x)$ into the risk-free asset. Then:

$$x^2\sigma_T^2 + (1-x)^2\sigma_f^2 = 0.2^2$$
$$x = \frac{0.2}{0.1534} = 130.40\%$$

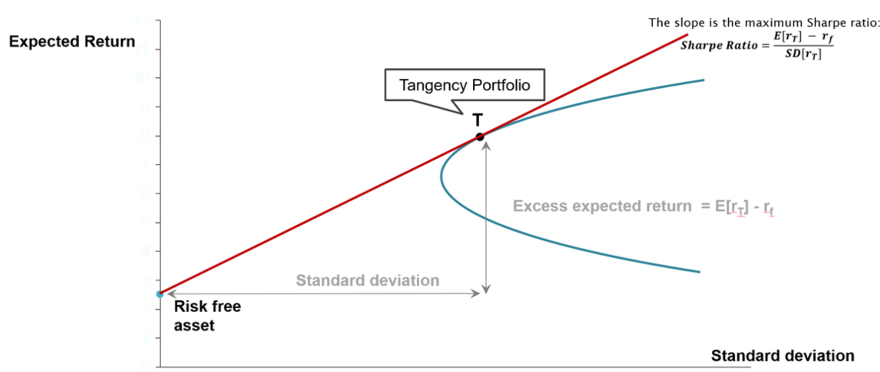
So investor X needs to invest 130.40% into the tangency portfolio and -30.40% into the risk-free asset. Recall that the tangency portfolio is 65.31% A and 34.69% C. So X's portfolio is:

– −30.40% risk-free asset
– 85.16% stock A (= 130.40% × 65.31%)
– 45.24% stock C (= 130.40% × 34.69%)

The expected return of X's portfolio is:

$$-30.40\% \times 3\% + 85.16\% \times 5\% + 45.24\% \times 20\% = 12.39\%$$

Without access to the risk-free asset, X's return was 12.25%. She is able to achieve an additional 0.14% return without increasing the risk of the portfolio. In the diagram below, she is able to achieve a return on the capital market line (red line) by including the risk-free asset in her portfolio. Otherwise, she is constrained to investing on the blue line shown.



Question 4

Suppose there are 10 stocks in the market. Each stock's expected returns follow this two factor model:

$$r_i = b_1 F_1 + b_{2,i} F_2 + \epsilon_i$$

The model parameters are defined as follows: $b_1 = 10$, $b_{2,i} = i$, F_1 has zero mean and standard deviation of 1%, F_2 has a mean of 1% and standard deviation of 1%, and ϵ_i has zero mean and standard deviation of 30%.
Assume that F_1, F_2, and ϵ_i are independent of each other. There is a risk-free asset with guaranteed return of 0.75%.

(a) Consider an equally-weighted portfolio. Compute its Sharpe ratio. Compute the RRR for each individual asset.

(b) Find the tangency portfolio. Compute its Sharpe ratio and compare it to the Sharpe ratio of the equally-weighted portfolio. Compute the RRR for each individual asset for the tangency portfolio.

Solutions:

We start by computing the expected return and variance of returns of stock i.
Expected return:

$$\begin{aligned} E[r_i] &= b_1 E[F_1] + b_{2,i} E[F_2] + E[\epsilon_i] \\ &= 10 \times 0 + i \times 1\% + 0 \\ &= i \times 1\% \end{aligned}$$

Variance:

$$\begin{aligned} V[r_i] &= V[b_1 F_1 + b_{2,i} F_2 + \epsilon_i] \\ &= b_1^2 V[F_1] + b_{2,i}^2 V[F_2] + V[\epsilon_i] \\ &= 10^2 \times 0.01^2 + i^2 \times 0.01^2 + 0.3^2 \end{aligned}$$

We also find the covariance between stocks i and j, $i \neq j$:

$$Cov(r_i) = Cov(b_1 F_1 + b_{2,i} F_2 + \epsilon_i, b_1 F_1 + b_{2,j} F_2 + \epsilon_j)$$

Since F_1, F_2, ϵ are independent and ϵ is uncorrelated across stocks:

$$\begin{aligned} Cov(r_i) &= V[b_1 F_1] + Cov(b_{2,i} F_2, b_{2,j} F_2) \\ &= b_1^2 V[F_1] + b_{2,i} b_{2,j} V[F_2] \\ &= 10 \times 0.01^2 + i \times j \times 0.01^2 \end{aligned}$$

We introduce the following matrix notation for this problem:

- w is the vector of portfolio weights
- $\bar{r}$ is the vector of expected returns
- $\bar{x}$ is the vector of expected excess returns

- Σ is the covariance matrix

The expected return on a portfolio can be computed as $w'\bar{r}$. The variance of the portfolio can be computed as $w'\Sigma w$.
The return-to-risk ratio (RRR) for asset i and portfolio p is:

$$RRR_{i,p} = \frac{r_i - r_f}{Cov(r_i, r_p)/\sigma_p}$$

Note that the $Cov(r_i, r_p)$ term is

$$Cov(r_i, r_p) = Cov(r_i, \sum_{j=1}^{I} w_j r_j) = \sum_{j=1}^{I} w_j Cov(r_i, r_j)$$

(a) We can compute the expected return and variance of the equally-weighted portfolio by using the above formulas in Excel, as shown in the video. This gives an expected return of 5.50%, variance of 0.02203, and standard deviation of 14.84%. This yields a Sharpe ratio of 0.320.

Now, we need to compute the RRR for the stocks in the portfolio. Note that, for the equally weighted portfolio:

$$Cov(r_i, r_p) = \sum_{j=1}^{I} w_j Cov(r_i, r_j) = \frac{1}{I} \sum_{j=1}^{I} Cov(r_i, r_j)$$

We can compute the covariance of each stock with the portfolio in Excel using MMULT, as shown in the recitation video. We then find the RRR for each stock i as $\frac{r_i - r_f}{Cov(r_i, r_p)/\sigma_p}$, which yields the following:

Stock	RRR
1	0.0190
2	0.0923
3	0.1617
4	0.2275
5	0.2900
6	0.3494
7	0.4059
8	0.4598
9	0.5112
10	0.5603

This shows that stock 1 contributes the least in terms of the risk-return trade-off, while stock 10 contributes the most. This relationship is monotonic - as the stock index increases, so does its contribution to the risk-return trade-off. This means that we can increase the portfolio's Sharpe ratio by putting a greater weight on stocks with higher indexes and a lower weight on stocks with lower indexes. The fact that the RRR is different for different assets means this portfolio is not efficient.

(b) The tangency portfolio weights are given by:

$$w_T = \frac{1}{\bar{x}'\Sigma^{-1}\iota}\Sigma^{-1}\bar{x}$$

We can solve this in Excel, giving us the following tangency portfolio weights:

$$w_T = \begin{bmatrix} -10.7\% \\ -6.1\% \\ -1.5\% \\ 3.1\% \\ 7.7\% \\ 12.3\% \\ 16.9\% \\ 21.5\% \\ 26.1\% \\ 30.7\% \end{bmatrix}$$

Note that the weight on each stock increases monotonically from stock 1 to stock 10.

We can compute the expected return and variance using the formulas given above. This yields an expected return of 9.3%, a variance of 0.0434, and a standard deviation of 20.8%. The resulting Sharpe ratio is 0.41. As expected, the tangency portfolio exhibits a much better risk-return trade-off than the equally-weighted portfolio.

We can compute RRR for each stock with the same method used ini part a. Note that the weights on each stock are no longer equal, so we need to use $Cov(r_i, r_p) = \sum_{j=1}^{I} w_j Cov(r_i, r_j)$.

The resulting RRRs are, as shown in the video, 0.4104 for every stock. The fact that the RRRs are equal across the assets and equal to the Sharpe ratio means that the portfolio allocation is optimal in terms of risk-return trade-off.