Assignment Question 5

1 Liouville's theorem and preservation of phase space volume

"All things on the Notes are written here."

I would like to add...

1.0.1 Alternative Proof

A fluid element volume for a Hamiltonian system in phase space is preserved in time. A volume element is

$$dV(t) = d^f q(t) d^f p(t),$$

the evolving system

$$t \rightarrow t + \delta(t)$$

$$dV(t + \delta t) = J(t + \delta t, t) dV(t),$$

where J is the Jacobian of the mapping. Now,

$$q_i(t + \delta t) = q_i + \dot{q}_i \delta t + O(\delta t^2)$$

$$p_i(t + \delta t) = p_i + \dot{p}_i \delta t + O(\delta t^2),$$

$$(q_i \equiv q_i(t) etc.,)$$

So,

$$J(t + \delta t, t) = \frac{\partial(\vec{q}(t + \delta t), \vec{p}(t + \delta t))}{\partial(\vec{q}(t) \vec{p}(t))}$$

$$= abs \begin{bmatrix} \frac{\partial q_i(t+\delta t)}{\partial q_j(t)} & \frac{\partial q_i(t+\delta t)}{\partial p_j(t)} \\ \frac{\partial p_i(t+\delta t)}{\partial q_j(t)} & \frac{\partial p_i(t+\delta t)}{\partial p_j(t)} \end{bmatrix}$$

$$= abs \begin{bmatrix} \delta_{ij} + \frac{\partial \dot{q}_i}{\partial q_j}\delta t & \frac{\partial \dot{q}_i}{\partial p_j}\delta t \\ \frac{\delta \dot{p}_i}{\delta q_j}\delta t & \delta_{ij} + \frac{\delta \dot{p}_i}{\partial p_j}\delta t \end{bmatrix} + O(\delta t^2)$$

But,

$$det(1 + \epsilon A) = 1 + \epsilon TrA + O(\epsilon^2)$$

as

$$det(1 + \epsilon A) = \epsilon_{i_1 ... i_N}(1 + \epsilon A)_{1 i_1} ... (1 + \epsilon A)_{1 N_N}$$

$$= \epsilon_{12...N} + \epsilon[\epsilon_{i_1 2...N} a_{1_{i1}} + \epsilon_{1 i_2 3...N} a_{2 i_2} + ...$$

$$= 1 + \epsilon TrA + O(\epsilon^2).$$

So,

$$J(t + \delta t, t) = 1 + \delta t \sum_i \left(\frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} \right) + O(\delta t^2)$$

$$= 1 + \delta t \sum_i \left(\frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(- \frac{\partial H}{\partial q_i}\right)\right) + O(\delta t^2)$$

$$= 1 + O(\delta t^2).$$

Consider a volume in phase space given by

$$V(t) = \int_{R(t)} dV(t),$$

hence,

$$V(t+\delta t) = \int_{R(t+\delta t)} dV(t+\delta t) = \int_{R(t)} J(t+\delta t) dV(t) = V(t) + O(\delta t^2)$$

So,

$$\frac{dV(t)}{dt} = 0 \rightarrow V(t) = const.,$$

"therefore" i.e. volumes in phase space do not change in time.
-Renz Dela Cruz