Unit-1, Electrostatic Fields-I

An electrostatic field is produced by a static charge (charge at rest or time-invariant) distribution. There are two fundamental laws in free space or vaccum. They are ① Coulomb's Law to study electrostatic fields. ② Gauss's Law. Both of these Laws are based on experimental studies.

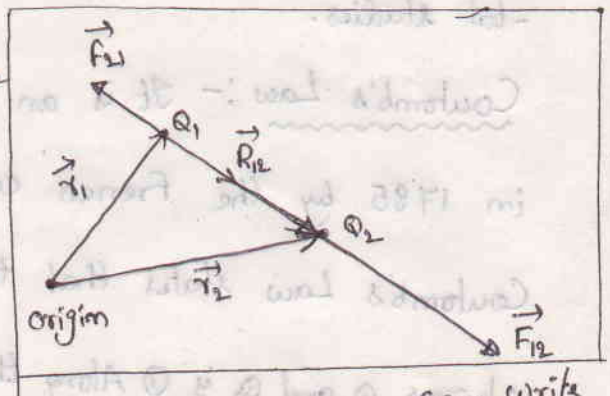
Coulomb's Law :- It is an experimental Law formulated in 1785 by the French colonel, Charles Augustin de Coulomb.

Coulomb's Law states that the force $\vec{F}$ between two point charges Q_1 and Q_2 is ① Along the line joining them. ② Directly proportional to the product $Q_1 Q_2$ of the charges ③ Inversely proportional to the square of the distance R b/w. them.

Mathematically, $F = \frac{K Q_1 Q_2}{R^2}$. where K is the proportionality constant. The charges Q_1 and Q_2 are in Coulombs (C), the distance, R is in meters (m), and the force F is in newtons (N) so that $K = \frac{1}{4\pi\varepsilon_0}$, The Constant ε_0 is known as the permittivity of the freespace (in farads per meter) and has the value $\varepsilon_0 = 8.854\times10^{-12} \simeq \frac{10^{-9}}{36\pi}$ F/m or $K = \frac{1}{4\pi\varepsilon_0} \simeq 9\times10^{9}$ m/F.

i.e. $F = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}$. If point charges Q_1 and Q_2 are located at points having position vectors $\vec{r_1}$ and $\vec{r_2}$, then the force, $\vec{F}_{12}$ on Q_2 due to Q_1 as shown in figure is given by

$$\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^2}\vec{a}_{R_{12}}$$



where: $\vec{R}_{12} = \vec{r}_2 - \vec{r}_1$

$R = |\vec{R}_{12}|$

$\vec{a}_{R_{12}} = \frac{\vec{R}_{12}}{R}$

Substituting all these in the above equation we can write

$$\boxed{\vec{F}_{12} = \frac{Q_1 Q_2}{4\pi\varepsilon_0 R^3}\vec{R}_{12} \quad \text{or} \quad \vec{F}_{12} = \frac{Q_1 Q_2(\vec{r}_2 - \vec{r}_1)}{4\pi\varepsilon_0 |\vec{r}_2 - \vec{r}_1|^3}}$$

From the figure, the force $\vec{F}_{21}$ on Q_1 due to Q_2 is given by

$\vec{F}_{21} = |\vec{F}_{12}|\vec{a}_{R_{21}} = |\vec{F}_{12}|(-\vec{a}_{R_{12}})$ or $\boxed{\vec{F}_{21} = -\vec{F}_{12}}$ since $\boxed{\vec{a}_{R_{21}} = -\vec{a}_{R_{12}}}$

Conditions to apply Coulomb's Law :-

1. Like charges (Charges of the same sign) repel each other while unlike charges attract.

2. The distance R between the charged bodies Q_1 and Q_2 must be large compared with the linear dimensions of Q_1 and Q_2. That is Q_1 and Q_2 must be point charges.

3. Q_1 and Q_2 must be static (at rest)

4. The signs of Q_1 and Q_2 must be taken into account.

Application of Coulomb's Law for more than 2 point charges :-

If we have more than 2 point charges, we can use the principle of Superposition to determine the force on a particular charge. The principle states that if there are N no. of. charges $Q_1, Q_2, Q_3, \ldots Q_N$ located respectively at points with position vectors $\vec{r}_1, \vec{r}_2, \vec{r}_3, \ldots \vec{r}_N$, the resultant force $\vec{F}$ on charge Q located at point $\vec{r}$ is the vector sum of the forces exerted on Q by each of the charges $Q_1, Q_2, Q_3, \ldots Q_N$. Hence

$$\vec{F} = \frac{QQ_1(\vec{r}-\vec{r}_1)}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_1|^3} + \frac{QQ_2(\vec{r}-\vec{r}_2)}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_2|^3} + \cdots + \frac{QQ_N(\vec{r}-\vec{r}_N)}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_N|^3}.$$

or

$$\boxed{\vec{F} = \frac{Q}{4\pi\varepsilon_0} \sum_{K=1}^{N} \frac{Q_K(\vec{r}-\vec{r}_K)}{|\vec{r}-\vec{r}_K|^3}}$$

Electric-field Intensity ($\vec{E}$) :- The electric field intensity or electric field strength or simply electric field. $\vec{E}$ is the force per unit charge, when placed in the electric field. Thus $\boxed{\vec{E} = \frac{\vec{F}}{Q}}$. The electric field intensity $\vec{E}$ is obviously in the direction of the force $\vec{F}$ and is measured in newtons/coulomb or Volts/meter. The electric field intensity at point $\vec{r}$ due to a point charge located at $\vec{r}'$ is given by

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 R^2}\vec{a}_R = \frac{Q(\vec{r}-\vec{r}')}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|^3}.$$

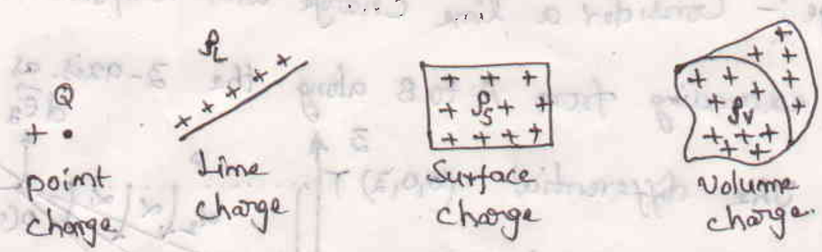
For N point charges $Q_1, Q_2, \ldots, Q_N$ located at $\vec{r}_1, \vec{r}_2, \ldots, \vec{r}_N$ the electric field intensity at point $\vec{r}$ is given by.

$$\vec{E} = \frac{Q_1(\vec{r}-\vec{r}_1)}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_1|^3} + \frac{Q_2(\vec{r}-\vec{r}_2)}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_2|^3} + \cdots + \frac{Q_N(\vec{r}-\vec{r}_N)}{4\pi\varepsilon_0 |\vec{r}-\vec{r}_N|^3}$$

or $\boxed{\vec{E} = \frac{1}{4\pi\epsilon_0} \sum_{K=1}^{N} \frac{Q_K(\vec{r}-\vec{r}_K)}{|\vec{r}-\vec{r}_K|^3}}$

Electric-fields due to Continous charge distributions :-

It is also posible to have Continous charge distributions in nature along a line, on a surface or in a Volume as shown in below figure.



In the figure the line charge density, Surface charge density and volume charge density are denoted as ρ_L (in C/m), ρ_S (in C/m^2) and ρ_V (in C/m^3) respectively.

The differential charge element, dQ and the total charge Q due to these charge distributions are given by.

$dQ = \rho_L d\vec{l}$ or $Q = \int_L \rho_L dl$ .... line charge

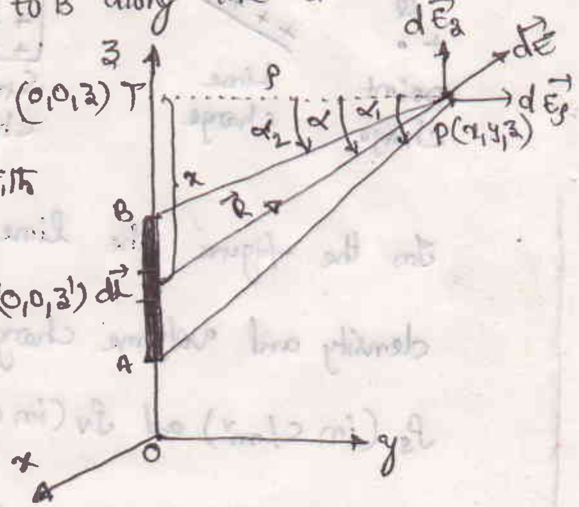
$dQ = \rho_S d\vec{s}$ or $Q = \int_S \rho_S d\vec{s}$ ..... Surface charge

$dQ = \rho_V dv$ or $Q = \int \rho_V dv$ ..... Volume charge and the electric field intensities due to each of these distributions are given by $\vec{E} = \int_L \frac{\rho_L \, d\vec{l}}{4\pi\varepsilon_0 R^2} \vec{a_R}$ ..... line charge

$$\vec{E} = \int_S \frac{\rho_s \, d\vec{s}}{4\pi\varepsilon_0 R^2} \vec{a_R} \text{ .... Surface charge}$$

$$\vec{E} = \int_V \frac{\rho_v \, dv}{4\pi\varepsilon_0 R^2} \vec{a_R} \text{ .... Volume charge.}$$

<u>Line charge</u> :- Consider a line charge with uniform charge density ρ_L extending from A to B along the z-axis as shown in figure. The differential charge element dQ associated with element $d\vec{l} = d\vec{z}$ of the line is $dQ = \rho_L d\vec{l} = \rho_L d\vec{z}$ and hence the total charge Q is



$$Q = \int_{z_A}^{z_B} \rho_L \, dz.$$

The electric field intensity $\vec{E}$ at any arbitrary point $P(x, y, z)$ can be found using the equation

$$\vec{E} = \int_L \frac{\rho_L \, dl}{4\pi\varepsilon_0 R^2} \vec{a_R}.$$

For this problem, the destination point is at $P(x,y,z)$ and the source is at a point (x', y', z'). From the figure,

$$dl = dz'$$

$$\vec{R} = (x,y,z) - (0,0,z') = x\,\vec{a}_x + y\,\vec{a}_y + (z-z')\,\vec{a}_z$$

or

$$\vec{R} = \rho\,\vec{a}_\rho + (z-z')\,\vec{a}_z$$

$$R^2 = |\vec{R}|^2 = x^2+y^2+(z-z')^2 = \rho^2 + (z-z')^2$$

$$\frac{\vec{a}_R}{R^2} = \frac{\vec{R}}{|\vec{R}|^3} = \boxed{\frac{\rho\,\vec{a}_\rho + (z-z')\,\vec{a}_z}{\left[\rho^2+(z-z')^2\right]^{3/2}}}$$

Substituting all these terms, we get

$$\vec{E} = \frac{\rho_L}{4\pi\varepsilon_0}\int \frac{\rho\,\vec{a}_\rho + (z-z')\,\vec{a}_z}{\left[\rho^2+(z-z')^2\right]^{3/2}}\,dz'$$

To evaluate this, it is convenient that we define α, α_1 and α_2 as shown in figure.

From fig. $\cos\alpha = \frac{\rho}{R}$

$R\cos\alpha = \rho$ or $R = \rho\sec\alpha$

$\sin\alpha = \frac{x}{R}$

$x = R\sin\alpha = \rho\sec\alpha\sin\alpha$

$= \rho\tan\alpha$

$$|\vec{R}| = \sqrt{\rho^2+(z-z')^2} = \rho\sec\alpha$$

$$z' = OT - \rho\tan\alpha,$$

$$dz' = -\rho\sec^2\alpha\,d\alpha$$

$$\therefore\quad \vec{E} = \frac{-\rho_L}{4\pi\varepsilon_0}\int_{\alpha_1}^{\alpha_2} \frac{\rho\sec^2\alpha\left[\cos\alpha\,\vec{a}_\rho + \sin\alpha\,\vec{a}_z\right]}{\rho^2\sec^2\alpha}\,d\alpha$$

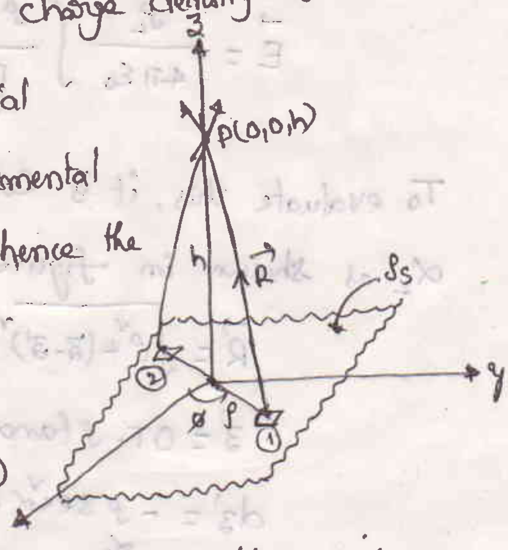
$$\vec{E} = -\frac{\rho_L}{4\pi\epsilon_0 \rho}\int_{\alpha_1}^{\alpha_2} + \left[\cos\alpha\, \vec{a_\rho} + \sin\alpha\, \vec{a_z}\right] d\alpha$$

$$\vec{E} = \frac{\rho_L}{4\pi\epsilon_0 \rho}\left[-(\sin\alpha_2 - \sin\alpha_1)\vec{a_\rho} + (\cos\alpha_2 - \cos\alpha_1)\vec{a_z}\right]$$

As a special case, for an infinite line charge, point B is at $(0,0,\infty)$ and A at $(0,0,-\infty)$ so that $\alpha_1 = \frac{\pi}{2}$ and $\alpha_2 = \frac{-\pi}{2}$. the z-component vanishes. then $\vec{E}$ becomes.

$$\boxed{\vec{E} = \frac{\rho_L}{2\pi\epsilon_0 \rho}\vec{a_\rho}}$$ - for infinite line charge.

<u>Surface charge</u>:- Consider an infinite sheet of charge in the xy plane with uniform charge density ρ_s. C/m^2 as shown in the figure. The differential charge associated with an elemental area ds is, $dQ = \rho_s\, ds$ and hence the total charge Q is $Q = \int_s \rho_s\, ds$.



The $\vec{E}$ field at point $(0,0,h)$ by the elemental surface ① shown in figure is

$$d\vec{E} = \frac{dQ}{4\pi\epsilon_0 R^2}\vec{a_R}.$$

From the figure, $\vec{R} = \rho(-\vec{a_\rho}) + h\vec{a_z}$

$$R = |\vec{R}| = [\rho^2 + h^2]^{1/2}$$

$$\vec{a_R} = \frac{\vec{R}}{R}, \quad dQ = \rho_s\, ds = \rho_s\, \rho\, d\rho\, d\phi$$

Substituting all These in the above equation, we get

$$d\vec{E} = \frac{\rho_s\, \rho\, d\phi\, d\rho\, [-\rho\vec{a_\rho} + h\vec{a_z}]}{4\pi\varepsilon_0\, [\rho^2 + h^2]^{3/2}}$$

Due to Symmetry of the charge distribution, for every element ①, there is corresponding element ② whose contribution along $\vec{a_\rho}$ Cancels that of element ①. so that $\vec{E}$ has only z-Component.

$$\text{Therefore, } \vec{E} = \int d\vec{E_z} = \frac{\rho_s}{4\pi\varepsilon_0} \int_{\phi=0}^{2\pi} \int_{\rho=0}^{\infty} \frac{h\,\rho\, d\rho\, d\phi}{[\rho^2 + h^2]^{3/2}}\, \vec{a_z}$$

$$\vec{E} = \frac{\rho_s h}{4\pi\varepsilon_0}\, 2\pi \int_0^{\infty} [\rho^2 + h^2]^{-3/2}\, \frac{1}{2}\, d(\rho^2)\, \vec{a_z}$$

$$\vec{E} = \frac{\rho_s h}{2\varepsilon_0} \left\{ -[\rho^2 + h^2]^{-1/2} \right\}_0^{\infty} \vec{a_z}$$

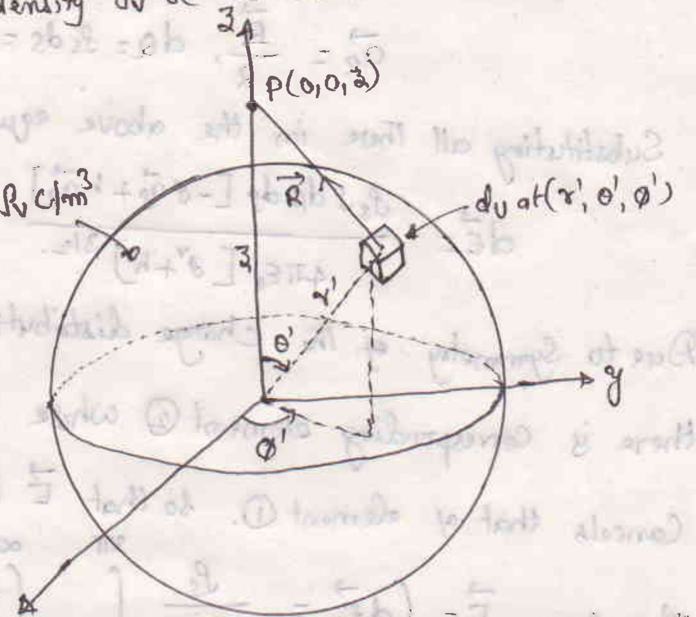
$$\vec{E} = \frac{\rho_s}{2\varepsilon_0}\, \vec{a_z}$$

that is $\vec{E}$ has only z-Component if the charge is in the xy-plane.

In general for an infinite sheet of charge $\boxed{\vec{E} = \frac{\rho_s}{2\varepsilon_0}\vec{a_n}}$

where $\vec{a_n}$ is the unit vector normal to the surface.

<u>Volume charge</u> :- Let the volume charge distribution with uniform charge density ρ_v be as shown in the figure.



The differential charge, dQ associated with ~~uniform charge density~~ ~~ρ_v c/m³ be as~~ the elemental volume dv is $dQ = \rho_v dv$. and hence the total charge in a sphere of radius a is

$$Q = \int \rho_v \, dv = \rho_v \int dv = \rho_v \frac{4\pi a^3}{3}.$$

The electric field $d\vec{E}$ at $P(0,0,z)$ due to the elementary volume charge is $d\vec{E} = \frac{\rho_v \, dv}{4\pi\varepsilon_0 R^2} \vec{a_R}$. Due to the symmetry of charge distribution, the contributions to $\vec{E_x}$ and $\vec{E_y}$ become zero. and there will be only $\vec{E_z}$ component. and it is given by.

$$\vec{E_z} = \vec{E} \cdot \vec{a_z} = \int dE \cos\alpha = \frac{\rho_v}{4\pi\varepsilon_0} \int \frac{dv \cos\alpha}{R^2}$$

From the figure, $dv = r'^2 \sin\theta' \, dr' \, d\theta' \, d\phi'$

Applying the Cosine rule to the fig. we have

$$R^2 = z^2 + r'^2 - 2zr'\cos\theta'$$

$$r'^2 = z^2 + R^2 - 2zR\cos\alpha.$$

Expressing $\cos\theta'$, $\cos\alpha$ and $\sin\theta' d\theta'$ in termes of R and r', that is

$$\cos\alpha = \frac{z^2 + R^2 - r'^2}{2zR}$$

$$\cos\theta' = \frac{z^2 + r'^2 - R^2}{2zr'}$$

Differentiating with respect to θ', we get

$$\sin\theta' \, d\theta' = \frac{R\,dR}{zr'}$$

Substituting all these into the above equation, we get

$$\vec{E}_z = \frac{\rho_v}{4\pi\varepsilon_0} \int_{\phi'=0}^{2\pi} d\phi' \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} r'^2 \frac{R\,dR}{zr'} \, dr' \, \frac{z^2 + R^2 - r'^2}{2zR} \, \frac{1}{R^2}$$

$$= \frac{\rho_v 2\pi}{8\pi\varepsilon_0 z^2} \int_{r'=0}^{a} \int_{R=z-r'}^{z+r'} r'\left[1 + \frac{z^2 - r'^2}{R^2}\right] dR \, dr'$$

$$= \frac{\rho_v \pi}{4\pi\varepsilon_0 z^2} \int_0^a 4r'^2 \, dr' = \boxed{\frac{1}{4\pi\varepsilon_0} \frac{1}{z^2}\left(\frac{4}{3}\pi a^3 \rho_v\right)}$$

$$\vec{E}_z = \boxed{\frac{Q}{4\pi\varepsilon_0 z^2}} \vec{a}_z \quad \text{or in general} \quad \boxed{\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2} \vec{a}_r}$$

which is identical to the electric field at the same point

 due to a point charge, Q located at the origin of sphere.

Electrix flux Density ($\vec{D}$) :- The force $\vec{F}$ and electric field $\vec{E}$ are dependent on the medium in which charge is placed. Suppose a new vector field $\vec{D}$ independent of medium is defined by $\vec{D} = \varepsilon_0 \vec{E}$, we define electric flux, ψ interms of $\vec{D}$ as.

$$\psi = \int \vec{D} \cdot d\vec{S}.$$

All the formulas derived ~~for~~ $\vec{E}$ from Coulomb's Law can be used in Calculating $\vec{D}$, except that we have to multiply those formulas by ε_0. For example for an infinite sheet of charge, $\vec{D} = \frac{\rho_s}{2} \vec{a}_n$ and for volume charge distribution $\vec{D} = \int_v \frac{\rho_v \, dv}{4\pi R^2} \vec{a}_R$ so that $\vec{D}$ is independent of the medium.

Gauss's Law :- Gauss's Law states that the total electric flux, ψ through any closed surface is equal to the total charge enclosed by that surface. Thus $\psi = Q_{encl.}$

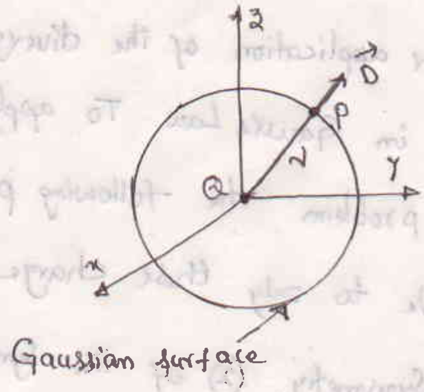
That is $\psi = \oint d\psi = \oint_S \vec{D} \cdot d\vec{S}$ = Total Charge enclosed $Q = \int \rho_v \, dv$.

or $\boxed{Q = \oint_S \vec{D} \cdot d\vec{S} = \int_v \rho_v \, dv.}$ Applying Divergence theorem to the middle term, we get $\oint_S \vec{D} \cdot d\vec{S} = \int_v \nabla \cdot \vec{D} \, dv$. Comparing the two volume integrals we can write $\boxed{\rho_v = \nabla \cdot \vec{D}}$ which is called Maxwell's 1st equation for Electrostatic fields which states the volume charge density is the same as the divergence of the electric flux density.

<u>procedure to apply Gauss's Law</u>: Gauss's Law is an alternative statement of Coulomb's Law. proper application of the divergence theorem to Coulomb's Law results in Gauss's Law. To apply Gauss's Law to any Electrostatic problem the following procedure is used. (1). Gauss's Law is applicable to only those charge distributions which are having Symmetry. (2) If the Symmetry exists then choose a mathematical surface called Gaussian Surface. (3) Gaussian Surface will be chosen such that the $\vec{D}$ field is every where normal or tangential to the Gaussian Surface. (4) The tangential $\vec{D}$ Component becomes zero and there exists only normal Component of $\vec{D}$. for this normal Component of $\vec{D}$ we can apply Gauss's Law to solve the Electrostatic problems at very fast level or at very quick.

<u>Applications of Gauss's Law</u>:- (1) point charge:- Suppose a point charge, Q is located at the origin. To determine $\vec{D}$ at any point P, it is easy to choose a spherical surface

Containing P will satisfy Symmetry Conditions. That is a spherical surface centered at the origin is the Gaussian surface in this case as shown in figure.



Since $\vec{D}$ is every where normal to the Gaussian surface, that is $\vec{D} = D_r \vec{a_r}$.

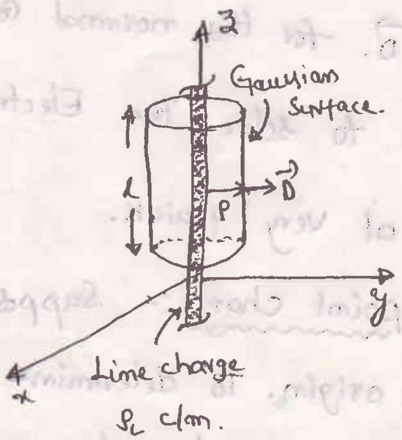
Applying Gauss's Law, we can write

$$Q = \oint_s \vec{D}\cdot d\vec{s} = D_r \oint_s d\vec{s} = D_r 4\pi r^2$$

where $\oint_s ds = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} r^2 \sin\theta \, d\theta \, d\phi = 4\pi r^2$ is the surface area of the Gaussian surface. Thus

$$\boxed{\vec{D} = \frac{Q}{4\pi r^2}\vec{a_r}}$$ as expected.

② Infinite Line charge :- Assume that the infinite line charge of uniform charge ρ_L C/m lies along the z-axis. To determine $\vec{D}$ at a point P, we choose a cylindrical surface containing P to satisfy symmetry condition as shown in figure. The $\vec{D}$ is normal to the rectangular face of the cylinder along it's length, l. and $\vec{D}$ is tangential to the Top and bottom surfaces hence $\vec{D} = 0$ for Top and bottom surfaces of Gaussian surface.

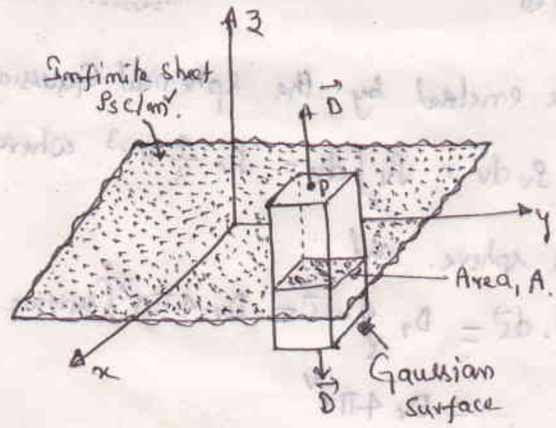


So $\vec{D} = D_\rho \vec{a_\rho}$. If we apply Gauss's Law to an arbitrary length, l of the line, $\rho_L l = Q = \oint_s \vec{D} \cdot \vec{ds} = D_\rho \oint_s \vec{ds} = D_\rho 2\pi \rho l$.

where $\vec{ds} = 2\pi\rho l$ is the surface area of the cylinder of radius ρ and length l. from the above equation $D_\rho = \frac{\rho_L l}{2\pi \rho l}$ or

$$\boxed{\vec{D} = \frac{\rho_L}{2\pi\rho} \vec{a_\rho}}$$ as expected.

③ <u>Infinite Sheet of Charge</u>:- Assume that the infinite sheet of uniform charge ρ_s C/m² is lying on the $z=0$ plane. To determine $\vec{D}$ at a point P, we choose a rectangular box or Cube as the Gaussian surface. as shown in figure.



$\vec{D}$ is normal to the sheet, $\vec{D} = D_z \vec{a_z}$. Applying Gauss's Law

$$\rho_s \int_s \vec{ds} = Q = \oint_s \vec{D} \cdot \vec{ds}$$

$$= D_z \left[\int_{Top} \vec{ds} + \int_{Bottom} \vec{ds} \right]$$

Note that $\vec{D} \cdot \vec{ds}$ evaluated on the sides of the box is zero because $\vec{D}$ is tangential in x and y directions. If the Top and Bottom area of the box each has area, A then $\rho_s A = D_z (A + A)$

$$D_z = \frac{\rho_s}{2} \vec{a_z}$$ or in general $$\boxed{\vec{D} = \frac{\rho_s}{2} \vec{a_n}}$$ as expected.

(14) Uniformly charged Sphere:- Consider a sphere of radius a with uniform charge ρ_v C/m³. To determine $\vec{D}$ everywhere, we construct Gaussian surfaces for cases $r<a$ and $r>a$ seperately. Since the charge has spherical symmetry, it is obvious that a spherical surface is an appropriate Gaussian surface, as shown in figures.

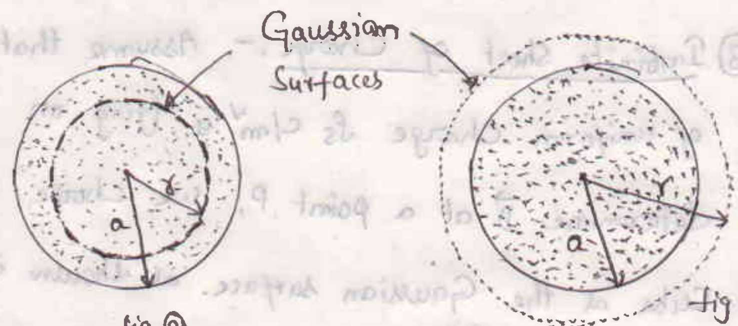


fig (a) fig (b).

For $r<a$, the total charge enclosed by the spherical Gaussian surface of radius r is $Q_{encl} = \int_v \rho_v \, dv = \rho_v \int_v dv = \rho_v \frac{4}{3}\pi r^3$ where $\frac{4}{3}\pi r^3$ is the ~~surface area~~ volume of the sphere. and

$$\psi = \oint_s \vec{D}\cdot d\vec{s} = D_r \oint_s d\vec{s} = D_r \cdot 4\pi r^2 \text{ (surface area)}.$$

$$= D_r \, 4\pi r^2$$

Hence, $\psi = Q_{encl}$. $\quad D_r 4\pi r^2 = \frac{4\pi r^3}{3}\rho_v$

$$\text{or } \boxed{\vec{D} = \frac{r}{3}\rho_v \vec{a}_r} \quad 0<r\le a$$

For $r \ge a$, The charge enclosed by the Gaussian surface is

$$Q_{encl} = \int_v \rho_v \, dv = \rho_v \int dv = \rho_v \frac{4}{3}\pi a^3$$

and $\psi = \oint_S \vec{D} \cdot d\vec{S} = D_r 4\pi r^2$

$\therefore D_r 4\pi r^2 = \frac{4}{3}\pi a^3 \rho_v$

or $D_r = \frac{a^3}{3r^2}\rho_v$ or $\boxed{\vec{D} = \frac{a^3}{3r^2}\rho_v \vec{a_r}}$ $r \geq a$.

Finally for uniformly charged sphere $\vec{D} = \begin{cases} \frac{r}{3}\rho_v \vec{a_r} & 0 < r \leq a \\ \frac{a^3}{3r^2}\rho_v \vec{a_r} & r \geq a. \end{cases}$

<u>Electric potential</u> :- The electric field intensity, $\vec{E}$ due to a charge distribution can be obtained from Coulomb's Law in general or from Gauss's Law when symmetry exists. Another way of obtaining $\vec{E}$ is from the electric scalar potential, V which is to be defined.

Suppose we wish to move a point charge, Q from point A to point B in an electric field, $\vec{E}$. Then from Coulomb's Law the force on Q is $\vec{F} = Q\vec{E}$ so that the workdone in ~~moving Q from A to B is~~ $W = -Q\int_A^B \vec{E} \cdot d\vec{l}$ in displacing the charge by $d\vec{l}$ is $dW = -\vec{F} \cdot d\vec{l} = -Q\vec{E} \cdot d\vec{l}$. The negative sign indicates that the work is being done by an external agent.

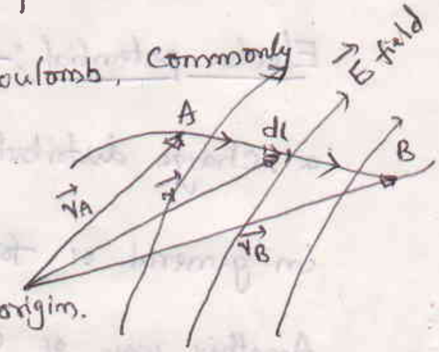
Thus, the total workdone in moving Q from A to B is

$$W = -Q\int_A^B \vec{E} \cdot d\vec{l}$$

Dividing W by Q gives the potential energy per unit charge. This can be denoted as V_{AB} known as the potential difference between points A and B. Thus

$$\boxed{V_{AB} = \frac{W}{Q} = -\int_A^B \vec{E}.d\vec{l}}$$

The potential difference V_{AB} is independent of the path taken and is measured in joules per coulomb, commonly referred to as volts (V).



As an example, if the $\vec{E}$ field is due to a point charge, Q located at the origin, then $\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a_r}$

$$\therefore V_{AB} = -\int_{r_A}^{r_B} \frac{Q}{4\pi\varepsilon_0 r^2}\vec{a_r} \cdot dr\,\vec{a_r} \qquad [\because d\vec{l} = dr\,\vec{a_r}]$$

$$= \frac{Q}{4\pi\varepsilon_0}\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

or $V_{AB} = V_B - V_A$ where V_B and V_A are the potentials at B and A respectively. If we move charge from infinity then the potential at infinity will be taken as zero. i.e. if $V_A = 0$ as $\vec{r}_A \to \infty$ and $\vec{r}_B \to \vec{r}$.

$\therefore$ Due to a point charge, Q located at the origin is

$\boxed{V = \frac{Q}{4\pi\varepsilon_0 r}}$, In other words, by assuming zero potential at infinity, the potential at a distance, r from the point charge is the work done per unit charge by an external agent, that is $\boxed{V = -\int_{\infty}^{r} \vec{E} \cdot d\vec{l}}$. If the point charge Q is not located at origin but at a point whose position vector is $\vec{r'}$ then the potential at a distance $\vec{r}$ is given by

$$\boxed{V(\vec{r}) = \frac{Q}{4\pi\varepsilon_0 |\vec{r} - \vec{r'}|}}$$

For point charges $Q_1, Q_2, \ldots Q_N$ located at points with position vectors $\vec{r_1}, \vec{r_2}, \ldots \vec{r_N}$, the potential at $\vec{r}$ is

$$V(\vec{r}) = \frac{Q_1}{4\pi\varepsilon_0 |\vec{r} - \vec{r_1}|} + \frac{Q_2}{4\pi\varepsilon_0 |\vec{r} - \vec{r_2}|} + \cdots + \frac{Q_N}{4\pi\varepsilon_0 |\vec{r} - \vec{r_N}|} \quad \text{or.}$$

$$\boxed{V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \sum_{K=1}^{N} \frac{Q_K}{|\vec{r} - \vec{r_K}|}}.$$

Similarly for continuous charge distributions, V is given by.

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_L \frac{\rho_L(\vec{r'})\, dl'}{|\vec{r} - \vec{r'}|} \text{ for line charge.}$$

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_S \frac{\rho_S(\vec{r'})\, dS'}{|\vec{r} - \vec{r'}|} \text{ for surface charge } \& \ V(\vec{r}) = \frac{1}{4\pi\varepsilon_0} \int_V \frac{\rho_V(\vec{r'})\, dv}{|\vec{r} - \vec{r'}|} \text{ for volume charge.}$$

Relationship between $\vec{E}$ and V :- The potential difference between points A and B is independent of path taken. Hence

$$V_{BA} = -V_{AB}$$

that is $V_{BA} + V_{AB} = \oint_L \vec{E} \cdot d\vec{l} = 0$ or $\boxed{\oint_L \vec{E} \cdot d\vec{l} = 0}$

This shows that the potential for closed loop is zero. Since work done is zero for closed path or loop. Applying Stoke's theorem to above equation we can write

$$\oint_L \vec{E} \cdot d\vec{l} = \int_S (\nabla \times \vec{E}) \cdot d\vec{s} = 0 \quad \text{or} \quad \boxed{\nabla \times \vec{E} = 0}$$ is

referred to as Maxwell's 2nd equation for electrostatic fields. which says that the electricfield $\vec{E}$ is Conservative or irrotational.

from the way we defined potential, $V = -\int \vec{E} \cdot d\vec{l}$, it follows that

$$dV = -\vec{E} \cdot d\vec{l} = -E_x dx - E_y dy - E_z dz$$

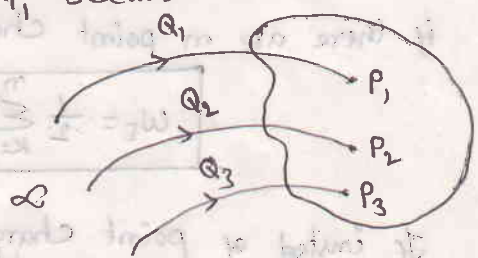
But we know that $dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$.

Comparing the two expressions for dV, we can write

$$E_x = \frac{-\partial V}{\partial x}, \quad E_y = \frac{-\partial V}{\partial y} \text{ and } E_z = \frac{-\partial V}{\partial z}$$

or $\boxed{\vec{E} = -\nabla V}$ i.e. electric field intensity is the gradient of V.

Energy Density in Electrostatic fields :- To determine the energy present in an assembly of charges, we must first determine the amount of work necessary to assemble them. Suppose we wish to position three point charges Q_1, Q_2 and Q_3 in an initially empty space as shown in figure. No work is required to transfer Q_1 from infinity to P_1 because the space is initially charge free and there is no electric field. The workdone in transferring Q_2 from ∞ to P_2 is equal to the product of Q_2 and the potential V_{21} at P_2 due to Q_1. Similarly the workdone in positioning Q_3 at P_3 is equal to $Q_3(V_{31}+V_{32})$. where V_{32} and V_{31} are the potentials at P_3 due to Q_2 and Q_1 respectively. Hence the total workdone in positioning 3 charges is



$$W_E = W_1 + W_2 + W_3$$
$$= 0 + Q_2 V_{21} + Q_3 (V_{31} + V_{32}).$$

If the charges were positioned in reverse order

$$W_E = W_3 + W_2 + W_1$$
$$= 0 + Q_2 V_{23} + Q_1 (V_{12} + V_{13})$$

where V_{23} is the potential at P_2 due to Q_3. V_{12} and V_{13} respectively the potentials at P_1 due to Q_2 and Q_3.

Unit-2, Electrostatics-II (Electric Fields in Material Space).

Introduction:- Just as Electric fields can exist in free space, they can exist in material media also. All the materials available in nature may be classified interms of their Conductivity, σ in mhos/permeter ($\mho$/m) or Siemens/meter (S/m) as Conductors and non conductors or technically as metals and insulators (or Dielectrics). A material with high Conductivity ($\sigma >> 1$) is referred to as a metal where as one with low Conductivity ($\sigma << 1$) is referred to as an insulator. The Conductivity of any metal means that the no. of charge Carriers available. Dielectric materials have few electrons available for conduction in Contrast to metals, which have more no. of electrons.

Convection and Conduction Currents:- The two fundamental quantities required for the study of Electric fields are Electric Voltage and Current. To see how Electric field behaves in a Conductor or dielectric, it is appropriate to Consider electric Current. Electric current is generally Caused by the motion of electric charges. The current in Amperes through a given area is the electric charge passing through the area per unit time.

That is $\vec{I} = \frac{dQ}{dt}$. Thus in a current of one ampere, charge is being transferred at a rate of one coulomb per second.

We now introduce the concept of current density, $\vec{J}$. If current ΔI flows through a surface ΔS, the current density is $\vec{J} = \frac{\Delta I}{\Delta S}$ or $\Delta I = \vec{J}\Delta S$. Then the total current through a surface, S is $\boxed{\vec{I} = \int_S \vec{J} \cdot d\vec{S}}$. Depending on how $\vec{I}$ is produced, there are three different kinds of current densities. They are Convection, Conduction and Displacement current densities. The displacement current exists in Magneto statics.

Convection current, as distinct from conduction current, does not involve conductors and consequently doesnot satisfy Ohm's Law. It occurs in freespace or vaccum. The convection current density, $\vec{J}$ is the current through a unit area at that point. In freespace, $\vec{J}$ can be expressed as $\boxed{\vec{J} = \rho_v \vec{u}}$, where ρ_v is the charge density and $\vec{u}$ is the velocity of charge carriers. In similar way the current density, $\vec{J}$ in conductors is given by $\boxed{\vec{J} = \sigma \vec{E}}$.

Resistance and Capacitance:- All the materials having Conductivity, σ can also have the resistivity in contrast to conductivity. That is the resistance of any metal will the opposing nature for the flow of charges. From ohm's Law, we can express the resistance, R in ohms can as

$$\boxed{R = \frac{V}{I} = \frac{\int_L \vec{E}\cdot d\vec{l}}{\oint_S \vec{J}\cdot d\vec{s}} = \frac{\int_L \vec{E}\cdot d\vec{l}}{\oint_S \sigma\vec{E}\cdot d\vec{s}}}$$

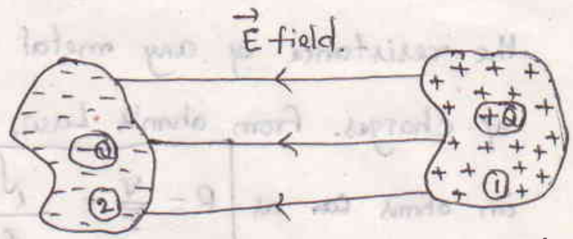
using this equation, the Resistance, R (or conductance, $G = \frac{1}{R}$) of a given conducting material can be found by following these steps:

1. Choose a suitable coordinate system.
2. Assume, V as the potential difference b/w conductor materials.
3. Determine $\vec{E}$ from $\vec{E} = -\nabla V$ and $\vec{I}$ from $\vec{I} = \oint_S \sigma\vec{E}\cdot d\vec{s}$.
4. Finally obtain R as $\frac{V}{I}$.

Alternatively, it is possible to assume current I, finding the corresponding potential difference V and determine R from $R = \frac{V}{I}$.

Basically, to have a capacitor we must have two (or more) conductors carrying equal but opposite charges. The conductors are sometimes referred to as the plates of the capacitor. For example consider a two-conductor

Capacitor as shown in the figure. The conductors are maintained at a potential difference, V. and $\vec{E}$ is the electric field existing between the conductors.



We define the capacitance, C of the capacitor as the ratio of magnitude of the charge on one of the conductors to the potential difference between them, that is

$$\boxed{C = \frac{Q}{V} = \frac{\oint_S \vec{D}\cdot d\vec{S}}{-\int_l \vec{E}\cdot d\vec{l}} = \frac{\varepsilon \oint_S \vec{E}\cdot d\vec{S}}{-\int_l \vec{E}\cdot d\vec{l}}}$$ (in Farads).

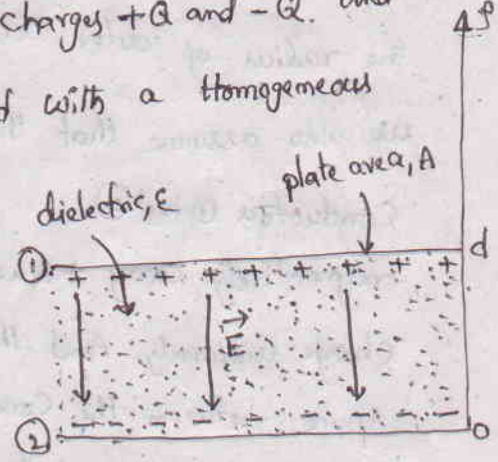
The capacitance of two conductors can be found by following these steps: (1) Choose a suitable co-ordinate system.

(2). Let the two conducting plates carry charges $+Q$ and $-Q$ respectively.

(3) Determine $\vec{E}$ from either Coulomb's Law or Gauss's Law and find V from $V = -\int_l \vec{E}\cdot d\vec{l}$.

(4) Finally obtain C from $C = Q/V$.

parallel-plate Capacitor:- Consider the arrangement of parallel plate Capacitor as shown in figure. Assume that each of the plates has, A and they are seperated by a distance, d. and plates ① and ②, respectively Carrying charges +Q and −Q. and also the space between the plates is filled with a Homogeneous dielectric with permittivity, ε.



For this arrangement of two Conductors or parallel plates, we can write

$$Q = \oint_S \vec{D} \cdot d\vec{s} = \oint_S \epsilon \vec{E} \cdot d\vec{s}$$

$$= \varepsilon \oint_S \vec{E} \cdot d\vec{s}$$

$$= \varepsilon E_\rho \oint_S d\vec{s}$$

$$= \varepsilon E_\rho A.$$

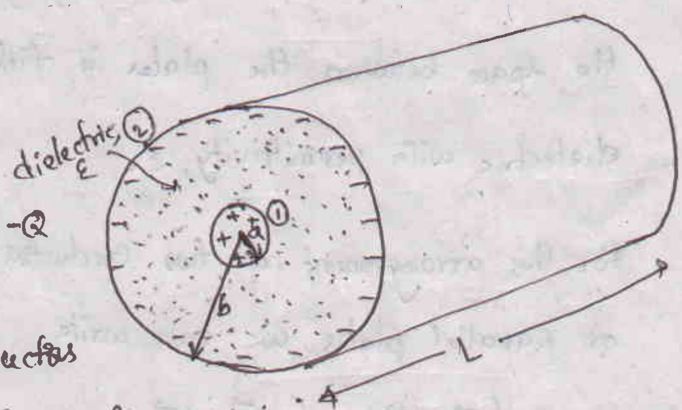
$$E_\rho = \frac{Q}{\varepsilon A} \text{ or ingeneral } \vec{E} = \frac{Q}{\varepsilon A} \vec{a_\rho} \text{ and}$$

$$V = -\int_1^2 \vec{E} \cdot d\vec{L} = -\int_d^0 \frac{Q}{\varepsilon A} \vec{a_\rho} \cdot d\rho \, \vec{a_\rho} = \frac{-Q}{\varepsilon A} \int_d^0 d\rho = \frac{-Q}{\varepsilon A} [\rho]_d^0$$

$$V = 0 - \left(\frac{-Qd}{\varepsilon A} \right) = \frac{Qd}{\varepsilon A}.$$

we Know that, Capacitance $\boxed{C = \frac{Q}{V} = \frac{\varepsilon A}{d}}$

Coaxial or Cylindrical Capacitor :- This is essentially a coaxial Cable or Coaxial Cylindrical Capacitor. Consider the length of the Conductors or cylinder is L. The radius of inner conductor is a and the radius of outer Conductor is b ($b>a$) as shown in the figure.



We also assume that the Conductors ① and ② respectively carry $+Q$ and $-Q$ Charge uniformly. And the space between the Conductors is filled with a dielectric medium of pirmittivity, ε.

Applying Gauss's Law to an arbitary Gaussian cylindrical surface of radius ρ ($a<\rho<b$), we obtain

$$Q = \varepsilon \oint_s \vec{E}\cdot d\vec{s} = \varepsilon E_\rho \oint_s d\vec{s} = \varepsilon E_\rho \, 2\pi\rho L \quad \text{or} \quad E_\rho = \frac{Q}{2\pi\varepsilon\rho L}$$

or in General $\vec{E} = \dfrac{Q}{2\pi\varepsilon\rho L}\vec{a}_\rho$ and

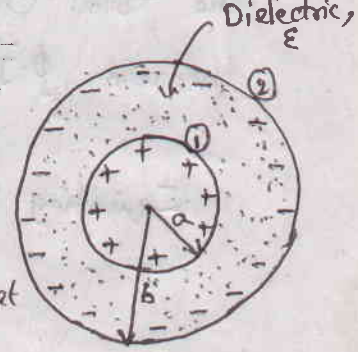
$$V = -\int_2^1 \vec{E}\cdot d\vec{l} = -\int_b^a \frac{Q}{2\pi\varepsilon\rho L}\cdot d\rho\,\vec{a}_\rho = -\frac{Q}{2\pi\varepsilon L}\int_b^a \frac{1}{\rho}\,d\rho$$

$$V = \frac{-Q}{2\pi\varepsilon L}\left[\log\rho\right]_b^a = \frac{-Q}{2\pi\varepsilon L}\left[\log a - \log b\right]$$

$$V = \frac{Q}{2\pi\varepsilon L}(\log b - \log a) = \frac{Q}{2\pi\varepsilon L}\log\left(\frac{b}{a}\right).$$

we Know that $\boxed{C = \frac{Q}{V} = \frac{2\pi\varepsilon L}{\log\left(\frac{b}{a}\right)}}$

Spherical Capacitor:- This is the case of two Concentric spherical Conductors. Consider the inner radius, a and outer radius b. (b>a). Seperated by dielectric medium with permittivity, ε as shown in the figure. we assume charges +Q and -Q on the inner and outer spheres respectively.



Applying Gauss's Law to an arbitary Gaussian spherical surface of radius ρ $(a < \rho < b)$, we get

$$Q = \varepsilon \oint_s \vec{E}\cdot d\vec{s} = \varepsilon E_\rho \oint_s d\vec{s} = \varepsilon E_\rho (4\pi\rho^2)$$

or $E_\rho = \frac{Q}{4\pi\varepsilon\rho^2}$ or in General $\vec{E} = \frac{Q}{4\pi\varepsilon\rho^2}\vec{a_\rho}$ and

$$V = -\int_2^1 \vec{E}\cdot d\vec{l} = -\int_b^a \frac{Q}{4\pi\varepsilon\rho^2}\vec{a_\rho}\cdot d\rho\,\vec{a_\rho} = \frac{-Q}{4\pi\varepsilon}\int_b^a \frac{1}{\rho^2}d\rho.$$

$$V = \frac{-Q}{4\pi\varepsilon}\left[\frac{-1}{\rho}\right]_b^a = \frac{+Q}{4\pi\varepsilon a} - \frac{+Q}{4\pi\varepsilon b} = \frac{Q}{4\pi\varepsilon}\left(\frac{1}{a} - \frac{1}{b}\right)$$

we Know that, Capacitance, $\boxed{C = \frac{Q}{V} = \frac{4\pi\varepsilon}{\left(\frac{1}{a} - \frac{1}{b}\right)}}$

Continuity of Current equation & Relaxation Time :- Due to the principle of Conservation of charge, the time rate of decrease of charge with in a given volume must be equal to the net outward current flow through the closed surface & Volume. That is $I_{out} = \oint_S \vec{J} \cdot d\vec{S} = \frac{-dQ_{in}}{dt}$. where Q_{in} is the total charge enclosed by the surface. Applying Divergence theorem $\oint_S \vec{J} \cdot d\vec{S} = \int_V \nabla \cdot \vec{J} \, dv$. But $\frac{-dQ_{in}}{dt} = \frac{-d}{dt}\int_V \rho_v dv = -\int_V \frac{\partial \rho_v}{\partial t} dv$

Equating the two volume integrals $\int_V \nabla \cdot \vec{J} dv = -\int_V \frac{\partial \rho_v}{\partial t} dv$.

or $\boxed{\nabla \cdot \vec{J} = \frac{-\partial \rho_v}{\partial t}}$ which is called the Continuity of Current equation.

Relaxation time :- It is the time taken by the charges placed inside the material to drop or reduce by an amount $e^{-1} = 36.8\%$ of its initial value. The relaxation time, T_r for any material can be equivalent to the ratio of the medium permittivity to the conductivity i.e.

$$\boxed{T_r = \frac{\varepsilon}{\sigma}}.$$

Dielectric Constant and Strength :- We can write for any medium of permittivity, ε, $\vec{D} = \varepsilon \vec{E}$ where $\varepsilon = \varepsilon_0 \varepsilon_r$ or $\varepsilon_r = \frac{\varepsilon}{\varepsilon_0}$. ε_0 is the permittivity of free space, and ε_r is called the Dielectric Constant or relative permittivity. Therefore the Dielectric Constant (or relative permittivity) ε_r is the ratio of the permittivity of the Dielectric to that of free space or Vaccum. The dielectric strength is the maximum electric field that a dielectric can withstand.

Linear, Isotopic and Homogeneous Dielectrics :- A material is said to be linear if $\vec{D}$ varies linearly with $\vec{E}$ and non linear otherwise. Materials for which $\vec{D}$ and $\vec{E}$ are in the same direction are said to be isotropic. non isotropic otherwise. Materials for which ε (or σ) does not vary in the region being considered and is therefore the same at all points are said to be Homogeneous. otherwise inhomogeneous or non-Homogeneous.

Boundary Conditions :- If the field exists in a region consists of two different media, the Conditions that the field must satisfy at the interface seperating the media are called Boundary Conditions