Special Relativity

We would like to understand how particles move in relativistic limit.

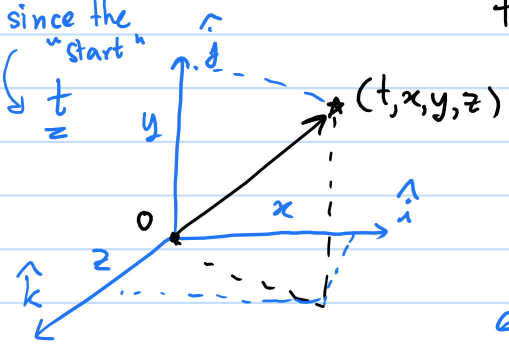
But first let us remind ourself about the Newtonian mechanics

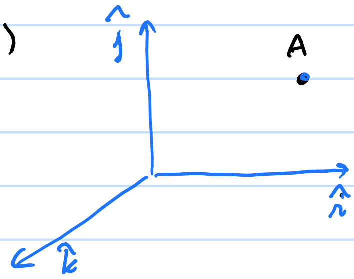
The Galilean transformation

☆ A reference frame = 1-1 correspondence between physical events and $\mathbb{R}^4$ space

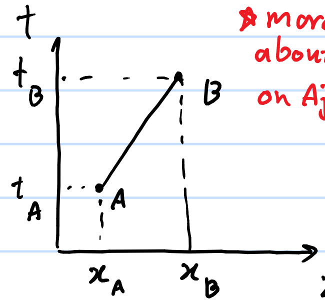
more about "vector" on Aj. Pamin's → (t, x, y, z) where each values represent how far the event from the reference point in space & time

time since the "start" $\underline{t}$





$\Rightarrow$



space-time diagram

* more about this on Aj. Ekapong

For example, a train leaving station A at time $t_A \Rightarrow (t_A, x_A, 0, 0)$

after that the train arrives at the station B at time $t_B \Rightarrow (t_B, x_B, 0, 0)$

the train speed in this ref frame $= (x_B - x_A)/(t_B - t_A)$

☆ Inertial frame = a ref frame which the motions of free particles is rectilinear : $\vec{r}(t) = \vec{u}t + \vec{r}_0$, $\vec{u}, \vec{r}_0$ are const.

* Remarks: 1) $\vec{r}$ is a 3d vector (x, y, z)

2) Free (inertial) particles = particles with no force (influence) acting on them

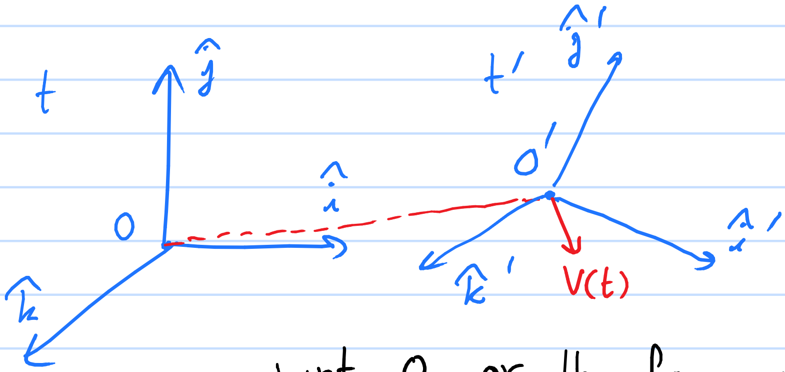
3) This is actually the first law of motion in Newtonian (writing in a fancy but simple way)

Galileon transformations

A transformation of space time = a change in reference frame

= a change in the reference point

(space & time)



For example, the ref frame O' (point O') could be moving wrt. O or the frame of axes are rotating wrt. O

☆ A change in ref frame ⇒ a change in the value of (t, x, y, z)

more on this "passive", "active" transformation in Aj. Pawin's

☆ Galileon transformation = a space-time transformation which leaves the following structures invariant

1). Time intervals of any two events $\Delta t = t_2 - t_1$

2). Spatial distances of any two events which happen at the same time (a set of simultaneous events)

$$\Delta S = |\vec{r}_2 - \vec{r}_1|$$

3). The rectilinear motion of free particles

$\vec{r}(t) = \vec{u}t + \vec{r}_0$ where $\vec{u}, \vec{r}_0$ are arbitrary const. vectors

(Here invariance means $\vec{u}, \vec{r}_0$ are different after transf. but the motion takes the same form)

It turns out that all Galilean transformation have the form (in coordinate transformation form)

$$t' = t + a \quad (a = \text{constant})$$

$$\vec{r}' = \tilde{R}\vec{r} - \vec{v}t + \vec{b}$$

(where $\tilde{R}$ is a rotation matrix $\tilde{R}^T\tilde{R} = \mathbb{1}$ and $\vec{v}, \vec{b} = $ const.)

Proof property 1 (time interval invariant) is easy to see

$$\Delta t' = t_2' - t_1' = t_2 + a - (t_1 + a) = t_2 - t_1 = \Delta t$$

Property 2 (spatial distance invariant) needs a little calculation

Let's consider the first part only $\vec{r}' = \tilde{R}\vec{r}$

$$\Delta s' = |\vec{r}_2' - \vec{r}_1'| = \sqrt{(\vec{r}_2' - \vec{r}_1') \cdot (\vec{r}_2' - \vec{r}_1')}$$

$$\Delta s'^2 = (\vec{r}_2' - \vec{r}_1') \cdot (\vec{r}_2' - \vec{r}_1') \equiv \Delta\vec{r}' \cdot \Delta\vec{r}'$$

☆ One useful thing about inner product of vector is the matrix representation of vectors : in Cartesian coordinate

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = \begin{pmatrix} A_x & A_y & A_z \end{pmatrix} \begin{pmatrix} B_x \\ B_y \\ B_z \end{pmatrix}$$

This means that we can write a vector in a column form (choosing & agreeing on using Cartesian coordinate)

$$\vec{A} = \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix} \Rightarrow |\vec{A}|^2 = \underbrace{\vec{A} \cdot \vec{A} = A^T A}_{\text{inner prod. in matrix rep.}} = (A_x \; A_y \; A_z) \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

This will become very useful representation later

Now it is easy to see that the spatial distance is invariant under the rotation part $\vec{r}' = R\vec{r} \Rightarrow \Delta\vec{r}' = \vec{r}_2' - \vec{r}_1' = \tilde{R}(\vec{r}_2 - \vec{r}_1) = \tilde{R}\Delta\vec{r}$

$$\Delta s'^2 = \Delta\vec{r}' \cdot \Delta\vec{r}' = \Delta r'^T \Delta r' = \Delta r^T R^T R \Delta r = \Delta r^T \Delta r = \Delta s^2$$

For the translation part of Galilean transformation, the simultaneity is crucial for the invariance since $\vec{r}' = \tilde{R}\vec{r} - \vec{v}t + \vec{b}$

$$\Delta\vec{r}' = \vec{r}_2'(t) - \vec{r}_1'(t) = (\tilde{R}\vec{r}_2 - \cancel{\vec{v}t} + \cancel{\vec{b}}) - (\tilde{R}\vec{r}_1 - \cancel{\vec{v}t} + \cancel{\vec{b}}) = \tilde{R}(\vec{r}_2 - \vec{r}_1) = \tilde{R}\Delta\vec{r}$$

Hence, $\Delta s'^2 = \Delta r'^T \Delta r' = \Delta r^T R^T R \Delta r = \Delta r^T \Delta r = \Delta s^2$

Property 3 If $\vec{r}(t) = \vec{u}t + \vec{r}_0$ then the transformation leads to

$$\vec{r}'(t) = \tilde{R}\vec{r}(t) - \vec{v}t + \vec{b} = \tilde{R}(\vec{u}t + \vec{r}_0) - \vec{v}t + \vec{b} = (\tilde{R}\vec{u} - \vec{v})t + \tilde{R}\vec{r}_0 + \vec{b}$$

which is a rectilinear motion (with velocity $\tilde{R}\vec{u} - \vec{v}$)

Exercise Can $\tilde{R}$ be nonconstant matrix? What is the physical interpretation?

Lorentz transformation

From the previous section, the velocity after Galilean transformation ($\tilde{R} = \mathbb{1}$) is simply additive, i.e. $\vec{u}' = \vec{u} - \vec{v}$

However, it has been demonstrated since 1888 that this velocity transformation does not quite work with the speed of light. (1888 was the year of Michealson Morley experiment). Eventually, the resolution came from Einstein (Principle of relativity in 1905) (More from Aj. Ekapong's)

Poincare and Lorentz transformation

In classical mechanics we assume that the space (set) of events (t, x, y, z) form a Galilean space (a space equipped with Galilean transformation). In Special Relativity (SR) the structure for invariance is different. Instead of seperate time interval & spatial distance, there is a single interval defined between pairs of events:

$$\Delta s^2 \equiv (c\Delta t)^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

- First short cut! There will be too many c's to write in the future and this is just a constant $\Rightarrow$ we will choose a system of units in such a way that $c = 1$. This is not so surprising as it seems since a constant can be different in different units, for example $c = 2.99 \times 10^8\ m/s = 1.08 \times 10^9\ km/hr$. However, a peculiar thing about this is that we don't write unit at all for c! $\Rightarrow$ In this new system of units, the length in space & the length in time share the same unit!

☆ Transformations which leave $\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$ invariant are called Poincaré transformation

In order to discuss the general form of Poincaré transf. we start by writing the interval Δs in matrix representation:

$$\Delta s^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2 \equiv \Delta r^T \tilde{G} \Delta r \quad \text{where}$$

$$r = \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}, \quad \Delta r = \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} \quad \text{and} \quad \tilde{G} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- It is easy to see that

$$(\Delta t \;\; \Delta x \;\; \Delta y \;\; \Delta z) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix} = \Delta s^2$$

☆☆ Note that I might use a different notation of Δs than the rest of lecturers in the school. I'm not changing, deal with it!

☆ Note that the spatial distance in Galilean space can be written in a similar structure $\Delta s^2 = \Delta r^T \Delta r = \Delta r^T \mathbb{1} \Delta r$

- The Poincaré transformation takes a general form as

$$r' = \tilde{\Lambda} r + a$$

where $\tilde{\Lambda}$ is 4×4 matrix, a is a constant column vector

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} + \begin{pmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

The subset of Poincaré transf. with no translation $(a = 0)$ is called Lorentz transformation (more from Aj. Patipan's)

Then we have

$$\Delta r' = r_2' - r_1' = \tilde{\Lambda} r_2 + a - (\tilde{\Lambda} r_1 + a)$$
$$= \tilde{\Lambda}(r_2 - r_1) = \tilde{\Lambda} \Delta r$$

☆ Therefore, the invariance of the interval Δs requires $\tilde{\Lambda}$ to have the following property:

$$\Delta s'^2 = \Delta r'^T \tilde{G} \Delta r' = \Delta r^T \tilde{\Lambda}^T \tilde{G} \tilde{\Lambda} \Delta r$$

$$\Delta s^2 = \Delta r^T \tilde{G} \Delta r$$

$$\therefore \quad \boxed{\tilde{\Lambda}^T \tilde{G} \tilde{\Lambda} = \tilde{G}}$$

The motivation for choosing the position of indices are
1). x' & x are object of the same type ⇒ both are up
2). The summation must happen with up-down $\Lambda^{\nu}{}_{\mu} x^{\mu}$

This is the property of Lorentz transformation

The second shortcut! (The Einstein's summation convention)

In the component form of vector/matrix operation we have

$$x' = \tilde{\Lambda} x \Rightarrow \begin{pmatrix} x'^0 \\ x'^1 \\ x'^2 \\ x'^3 \end{pmatrix} = \begin{pmatrix} \Lambda^0{}_0 & \Lambda^0{}_1 & \Lambda^0{}_2 & \Lambda^0{}_3 \\ \Lambda^1{}_0 & \Lambda^1{}_1 & \Lambda^1{}_2 & \Lambda^1{}_3 \\ \Lambda^2{}_0 & \Lambda^2{}_1 & \Lambda^2{}_2 & \Lambda^2{}_3 \\ \Lambda^3{}_0 & \Lambda^3{}_1 & \Lambda^3{}_2 & \Lambda^3{}_3 \end{pmatrix} \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}$$

there are 4 eq^{n}'s here $\begin{pmatrix} \vdots \end{pmatrix} = \begin{pmatrix} \vdots \end{pmatrix}$

where $x^0 \equiv t$, $x^1 = x$, $x^2 = y$, $x^3 = z$

Careful about the position
There is no $\cancel{\Lambda^0_0}$

Writing 4 equations explicitly as

$$x'^0 = \sum_{\mu=0}^{3} \Lambda^0{}_{\mu} x^{\mu}, \quad x'^1 = \sum_{\mu=0}^{3} \Lambda^1{}_{\mu} x^{\mu}, \quad x'^2 = \sum_{\mu=0}^{3} \Lambda^2{}_{\mu} x^{\mu}, \quad x'^3 = \sum_{\mu=0}^{3} \Lambda^3{}_{\mu} x^{\mu}$$

We can do better by writing generically as

$$x' = \tilde{\Lambda} x \Rightarrow x'^{\nu} = \sum_{\mu=0}^{3} \Lambda^{\nu}{}_{\mu} x^{\mu}$$

☆ The Einstein's convention is the realisation that every time the Σ appears, there are 2 repeating indices in the same side of the eq^{n}

$\therefore$ There is no need to write Σ at all!

$$\therefore \quad x' = \tilde{\Lambda} x \Rightarrow x'^{\nu} = \Lambda^{\nu}{}_{\mu} x^{\mu}$$

★ The rule is repeating indices = summation

$$x'^{\nu} = \Lambda^{\nu}{}_{\mu} x^{\mu}$$

"dummy" index
can be changed to anything

Free index
must appear on both sides of the eq$^{\underline{n}}$

The inverse Lorentz transformation can be written as

$$x' = \Lambda x \Rightarrow x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu}$$

a new free index

$$\Lambda^{-1}\times: \quad \Lambda^{-1}x' = \Lambda^{-1}\Lambda x \Rightarrow (\Lambda^{-1})^{\rho}{}_{\mu} x'^{\mu} = \underbrace{(\Lambda^{-1})^{\rho}{}_{\mu} \Lambda^{\mu}{}_{\nu}} x^{\nu}$$

dummy index
bec. of matrix mult.

$$(\Lambda^{-1}\Lambda)^{\rho}{}_{\nu} = (\mathbb{1})^{\rho}{}_{\nu}$$

$$\Lambda^{-1}x' = \mathbb{1}x = x \Rightarrow (\Lambda^{-1})^{\rho}{}_{\mu} x'^{\mu} = (\mathbb{1})^{\rho}{}_{\nu} x^{\nu} = \delta^{\rho}{}_{\nu} x^{\nu}$$

Since $\mathbb{1}$ is identity matrix, we define the Kronecker delta:

$$\delta^{\mu}{}_{\nu} = \begin{cases} 0 & \text{if } \mu \neq \nu \\ 1 & \text{if } \mu = \nu \end{cases}$$

$\therefore$ we get $\quad (\Lambda^{-1})^{\rho}{}_{\mu} x'^{\mu} = x^{\rho}$

Back to our space-time interval. We can also write it in component form with Einstein's convention

$$\Delta s^2 = \Delta x^T \tilde{G} \Delta x \Rightarrow \Delta s^2 = \Delta x^{\mu} (\tilde{G})_{\mu\nu} \Delta x^{\nu}$$

Now we define $(\tilde{G})_{\mu\nu} \equiv g_{\mu\nu}$ as a component form of "the metric tensor" where

$$g_{\mu\nu} = \begin{cases} 0 & \text{if } \mu \neq \nu \\ 1 & \text{if } \mu = \nu = 0 \\ -1 & \text{if } \mu = \nu > 0 \end{cases}$$

The position of indices are both down since x' & x are the same type + up-down summation

(more on another type of vectors from Aj. Parin's)

Exercise Derive the definition of Lorentz transformation in component form (Ans: $\tilde{G} = \Lambda^T \tilde{G} \Lambda \Rightarrow g_{\mu\nu} = \Lambda^\rho{}_\mu g_{\rho\sigma} \Lambda^\sigma{}_\nu$)

Exercise Check that $\det(\tilde{\Lambda}) = \pm 1$

Exercise Check that for any Lorentz transformation, we have either

$$\Lambda^0{}_0 \geq 1 \quad \text{or} \quad \Lambda^0{}_0 \leq -1$$

[Hint: work out $\begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \begin{pmatrix} \Lambda^0{}_0 & A^T \\ B & R^T \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \begin{pmatrix} \Lambda^0{}_0 & B^T \\ A & R \end{pmatrix}$]

Exercise What is the physical meaning of Lorentz transf. with $\Lambda^0{}_0 \geq 1$ and Lorentz transf. with $\Lambda^0{}_0 \leq -1$ [Hint: look at $\Delta t = \begin{pmatrix} \Delta t \\ 0 \\ 0 \\ 0 \end{pmatrix}$]

★ The set $\mathbb{R}^4$ of spacetime coordinates with this Poincaré-invariant interval structure is called Minkowski space (= the set of events)

- However Minkowski space $\neq$ vector space in Linear Algebra sense

Since the linear combination of 2 events does not necessary have Poincaré-invariant property. Let x^μ & y^μ be 2 events, $x^\mu + b y^\mu$ does not transform as Poincaré-transf.

$$x'^\mu + b y'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu + b(\Lambda^\mu{}_\nu y^\nu + a^\mu)$$

$$= \Lambda^\mu{}_\nu (x^\nu + b y^\nu) + (1+b) a^\mu$$ not invariant

★ But the difference between any two events do form a vector space

Since $x'^\mu - y'^\mu = \Lambda^\mu{}_\nu (x^\nu - y^\nu)$

- Minkowski space is an affine space

4-vectors

- Objects $x = x^\mu \hat{e}_\mu = x^0\hat{e}_0 + x^1\hat{e}_1 + x^2\hat{e}_2 + x^3\hat{e}_3$ are called 4-vectors
- The metric tensor has the inverse, $g^{\mu\nu}$ defined by
$$g^{\mu\nu} g_{\nu\rho} = \delta^\mu{}_\rho$$
- One can immediately check that the inverse has indentical elements
$$g^{\mu\nu} = \begin{cases} 0 & \text{if } \mu \neq \nu \\ 1 & \text{if } \mu = \nu = 0 \\ -1 & \text{if } \mu = \nu > 0 \end{cases}$$
- $g_{\mu\nu}$ & $g^{\mu\nu}$ will be used to "switch" between lower & upper index objects
For example, $u_\mu = g_{\mu\nu} u^\nu$, $u^\mu = g^{\mu\nu} u_\nu$
- For two 4-vectors $A = A^\mu \hat{e}_\mu$ and $B = B^\mu \hat{e}_\mu$, we can define their inner product to be the scalar
$$g(A, B) \equiv A^\mu B_\mu = A^\mu g_{\mu\nu} B^\nu = A_\nu B^\nu = A_\nu g^{\nu\mu} B_\mu$$
- We say that 2 vectors are orthogonal if $A^\mu B_\mu = 0$
- The magnitude of a 4vector is defined as $A^\mu A_\mu$
- A non-zero 4-vector A^μ is called
  - null if $A^\mu A_\mu = 0$
  - timelike if $A^\mu A_\mu > 0$
  - spacelike if $A^\mu A_\mu < 0$

☆ Again, my notation is different than the rest of lecturers. Deal with it!

- The set of all null 4-vectors is called the null cone
- The light cone at point p in Minkowski is the set of points in M that are connected to p by a null vector

Exercise Check that the "boost" with velocity v in the x-direction satisfies $\tilde{G} = \Lambda^T \tilde{G} \Lambda$

$$\Lambda = \begin{pmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \text{where } \gamma = \frac{1}{\sqrt{1-v^2}}$$

☆ This transformation is equivalent to changing the frame of reference to an inertial frame with speed v in x-direction

Exercise Verify that performing two Lorentz transformations with velocities v_1 and v_2 in x-direction successively is equivalent to a single Lorentz transformation with velocity

$$v = \frac{v_1 + v_2}{1 + v_1 v_2}$$

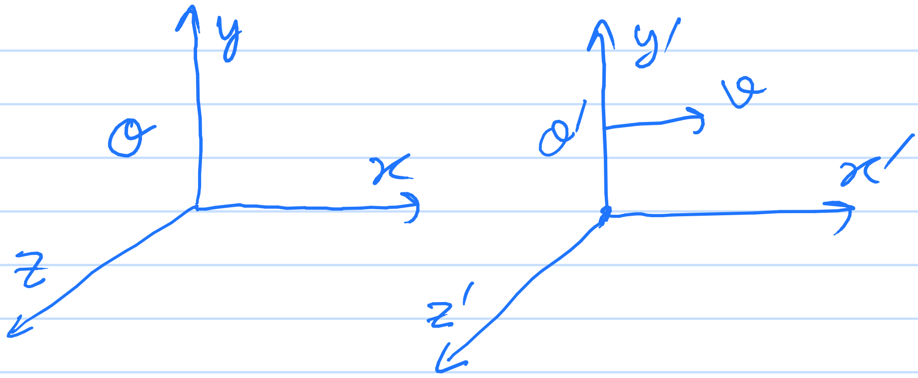
Relativity

- Consider 2 events $p = (t_1, x_1, y, z)$ and $q = (t_2, x_2, y, z)$ which happens at the same time $\Delta t = t_2 - t_1 = 0$. They are called simultaneous in an inertial frame $\mathcal{O}$.
- However the concept of simultaneity does not make sense in Special Relativity since a boost in x-direction is enough to break the simultaneousness of two events

$$\Delta x' = \gamma(\Delta x - v \Delta t), \quad \Delta t' = \gamma(\Delta t - v \Delta x)$$

For $\Delta t = 0$ in $\mathcal{O}$ frame, $\Delta t' = -\gamma v \Delta x \neq 0$ if $x_2 \neq x_1$

$\Rightarrow$ relativity of simultaneity



- Now consider a clock at rest in O' frame mak a successive ticks at (t_1', x', y', z') & (t_2', x', y', z'), time difference in O frame (let O' be a moving frame with v wrt. O) is given by the inverse Lorentz transf.

$$\Delta t = \gamma(\Delta t' + v\Delta x') = \gamma \Delta t' \quad (\Delta x' = 0 \text{ in } O')$$

$$\therefore \quad \Delta t = \frac{\Delta t'}{\sqrt{1-v^2}} \geq \Delta t' \quad \text{since } v < 1 \; (v < c)$$

☆ This is an effect known as time dilation

☆ " A moving clock appears to slow down "

- Now consider a rod of length $l = \Delta x$ at rest in O
- How do we measure a "moving" rod in O' frame?

⇒ We measure 2 simultaneous events at the end points (in O')

Consider the inverse Lorentz transf.

$$l = \Delta x = \gamma(\Delta x' + v\Delta t') = \gamma \Delta x'$$

$$\therefore \quad l' \equiv \Delta x' = \frac{1}{\gamma}\Delta x = \sqrt{1-v^2}\, l \leq l$$

- This is the effect known as length contraction
- " A rod is contracted when viewed by a moving observer "

- Let a particle have velocity $\vec{u} = (u_x, u_y, u_z)$ in O frame and $\vec{u}' = (u_x', u_y', u_z')$ in O' frame
- Let set $\quad u_x = \frac{dx}{dt}$, $u_x' = \frac{dx'}{dt'}$, $u_y = \frac{dy}{dt}$, etc...
- From Lorentz transf., it is easy to derive the relativistic transformation of velocities.

$$u_x' = \frac{dx'}{dt'} = \frac{d}{dt'}\Big(\gamma(x - vt)\Big) = \gamma \frac{d}{dt}(x - vt)\frac{dt}{dt'} = \gamma(u_x - v)\frac{dt}{dt'}$$

Consider $\frac{dt'}{dt} = \frac{d}{dt}\gamma(t - vx) = \gamma(1 - vu_x)$

Combine two eqn, we get $\quad u_x' = \dfrac{u_x - v}{1 - vu_x}$

Exercise show that

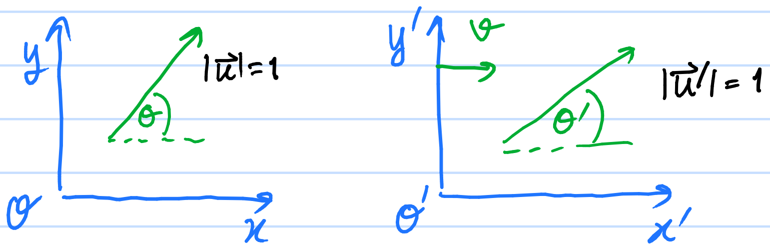
$$u_y' = \frac{u_y}{\gamma(1 - vu_x)}, \quad u_z' = \frac{u_z}{\gamma(1 - vu_x)}$$

Exercise Using the inverse Lorentz transf. to show that

$$u_x = \frac{u_x' + v}{1 + vu_x'}, \quad u_y = \frac{u_y'}{\gamma(1 + vu_x')}, \quad u_z = \frac{u_z'}{\gamma(1 + vu_x')}$$

Exercise Show that if the speed of the particle is $|\vec{u}| = 1$ the velocity in any inertial frame is also $|\vec{u}'| = 1$

Exercise

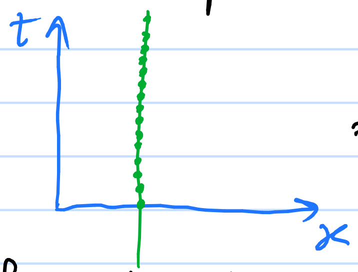


Show that the relativistic formula of light aberration

$$\sin\theta = \frac{\sqrt{1 - v^2}\,\sin\theta'}{1 + v\cos\theta'}$$

Particle dynamics

- In Minkowski space, a series of continuous events can form a "World-line".
  - We can parametrise the World-line using a parameter λ
  - For example, a particle at rest has a straight line as the World-line in space-time diagram



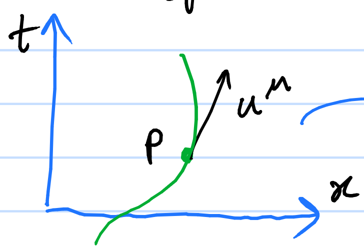
$\Rightarrow$ The "natural" parametrisation of this world-line is just "t"

- Let's define the tangent 4-vector to the curve at point p to be the 4-vector U given by

$$U = U^\mu e_\mu \quad \text{where} \quad U^\mu = \left.\frac{dx^\mu}{d\lambda}\right|_{\lambda=\lambda_0}$$

where p is at $x^\mu(\lambda_0)$

- Note that this 4-vector is independent of the basis choice $(\hat{e}_\mu)$
- The direction in space-time diagram = direction of world-line



U is tangent to the World-line

- The World-line of a physical object is assumed to stay inside the light-cone (null-cone) of every points in the World-line.
- This translate to the requirement on the tangent 4-vector to be timelike: $g(U(\lambda), U(\lambda)) = g_{\mu\nu} U^\mu(\lambda) U^\nu(\lambda) = g_{\mu\nu} \frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} > 0$

We have $\quad 0 < g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = \left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dx}{d\lambda}\right)^2 - \left(\frac{dy}{d\lambda}\right)^2 - \left(\frac{dz}{d\lambda}\right)^2$

$$0 < \left(\frac{dt}{d\lambda}\right)^2\left[1 - \left(\frac{dx}{dt}\right)^2 - \left(\frac{dy}{dt}\right)^2 - \left(\frac{dz}{dt}\right)^2\right]$$

$$0 < \left(\frac{dt}{d\lambda}\right)^2(1 - v^2) \Rightarrow v < 1$$

$\Rightarrow$ the velocity of physical objects is always less than 1

- Now let's talk about the choice of parametrisation λ

First consider two neighbouring events on the World-line $x(\lambda)$ and $x(\lambda + \Delta\lambda)$ where the difference is timelike vector $(\Delta x^\mu = x^\mu(\lambda+\Delta\lambda) - x^\mu)$: $\quad \Delta s^2 = g_{\mu\nu}\Delta x^\mu \Delta x^\nu > 0$

- There always exists an inertial frame of reference where $x^\mu(\lambda+\Delta\lambda)$ & x^μ are seperated only in time coordinate

$\Rightarrow$ the object stay at rest in this period $\Delta\lambda$

$\Rightarrow$ this is called "instantaneous rest frame" (i.r.f.) which can be varied from point to point.

$\therefore$ In this frame we can define Δs as the difference in time in i.r.f. (We can do this since Δs is invariant & $\Delta x = \Delta y = \Delta z = 0$ in i.r.f.) $\Rightarrow \quad \Delta\tau^2 \equiv \Delta s^2 = g_{\mu\nu}\Delta x^\mu \Delta x^\nu > 0$

Now, let's take the limit $\Delta\lambda \to 0$

$$\Delta\tau^2 = g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\times(\Delta\lambda)^2 = \left[\left(\frac{dt}{d\lambda}\right)^2 - \left(\frac{dx}{d\lambda}\right)^2 - \left(\frac{dy}{d\lambda}\right)^2 - \left(\frac{dz}{d\lambda}\right)^2\right]\Delta\lambda^2$$

$$= (1-v^2)\left(\frac{dt}{d\lambda}\right)^2\Delta\lambda^2 = \frac{1}{\gamma^2}\Delta t^2$$

- Therefore, we get the usual time dilation formula $\Delta\tau = \frac{1}{\gamma}\Delta t$

where $\Delta\tau$ is interpreted as the clock carried by the particle

- We can take this infinitesimal piece and integrate along the World-line

$$\tau_{pq} = \int_p^q d\tau = \int_{t_p}^{t_q} \frac{dt}{\gamma}$$

This could still vary along the path

- This is called the proper time from $p \to q$
- If we fix p and let q vary we can use this as a parameter λ

$$\lambda = \tau = \int_{t_p}^{t} \frac{dt}{\gamma}$$

- In this proper time parametrisation, the tangent 4-vector becomes the 4-velocity defined as

$$V^\mu = \frac{dx^\mu}{d\lambda} = \frac{dx^\mu}{d\tau} = \gamma\frac{dx^\mu}{dt} \Rightarrow V = \gamma\begin{pmatrix} \frac{dt}{dt} \\ \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt}\end{pmatrix} = \gamma\begin{pmatrix} 1 \\ \vec{v}\end{pmatrix}$$

- It is quicker to write vector component in 1 row as follow

$$V^\mu = \gamma(1, \vec{v})$$

Exercise Show that 4-velocity always has magnitude $= 1$

Exercise Show that the 4-acceleration can be written as

$$A^\mu \equiv \frac{dV^\mu}{d\tau} = \frac{d^2x^\mu}{d\tau^2} = \gamma\left(\frac{d\gamma}{dt}, \vec{v}\frac{d\gamma}{dt} + \gamma\frac{d\vec{v}}{dt}\right)$$

Exercise Show that 4-velocity is always $\perp$ to 4-acceleration

[Hint: Consider $\frac{d}{dt}(V^\mu V_\mu) = \frac{d}{dt}(1) = 0$]

Relativistic Particle Dynamics

We assume that each particle has a constant, m, attached to it, called "rest mass". This can be considered as the inertia mass in an i.r.f. of the particle satisfying $\vec{F} = m\vec{a}$ in that frame.

- The 4-momentum of the particle is defined as

$$p^\mu \equiv mV^\mu = m\gamma(1, \vec{v})$$
$$= (m\gamma, \gamma m\vec{v})$$

- We will define $p^\mu = (E, \vec{p})$, where E is energy, $\vec{p}$ = momentum

$$E = m\gamma = \frac{m}{\sqrt{1-v^2}}, \quad \vec{p} = \gamma m\vec{v} = \frac{m\vec{v}}{\sqrt{1-v^2}}$$

- We call them energy & momentum because in $v \ll 1$ limit

$$\vec{p} = \frac{m\vec{v}}{(1-v^2)^{1/2}} = m\vec{v}(1-v^2)^{-1/2} \simeq m\vec{v}\left(1 + \frac{1}{2}v^2 + O(v^4)\right) \simeq m\vec{v}$$

$$E = \frac{m}{(1-v^2)^{1/2}} = m(1-v^2)^{-1/2} \simeq m\left(1 + \frac{1}{2}v^2 + O(v^4)\right) \simeq m + \frac{1}{2}mv^2$$

(in the usual unit the last eqn reads $E = mc^2 + \frac{1}{2}mv^2$)

The energy contribution $E = m$ which is the energy of particle at rest

- The 4-momentum has magnitude m^2 since $p^\mu p_\mu = m^2 V^\mu V_\mu = m^2$
- Therefore, we have the dispersion relation of relativistic particle

$$m^2 = p^\mu p_\mu = (E, \vec{p})\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} E \\ \vec{p} \end{pmatrix} = E^2 - \vec{p}^2 = m^2 \quad \therefore \boxed{E^2 = \vec{p}^2 + m^2}$$

- Also $\frac{\vec{p}}{E} = \frac{\gamma m\vec{v}}{\gamma m} = \vec{v}$

- The above relation still holds even in the case $v \to 1$ as long as $m=0$. This is photon which satisfy

$$E = |\vec{p}| \quad \& \quad p^\mu = (E, E\hat{n}) \quad \& \quad p^\mu p_\mu = 0$$

where $\hat{n}$ is the direction of propagation

- We can define a 4-force as follow

$$F^\mu = \frac{dp^\mu}{d\tau} = mA^\mu$$

which is always orthogonal to V^μ

- Define the 3-force in the usual way as $\vec{f} = \frac{d\vec{p}}{dt}$
- We have $$F^\mu = \frac{dp^\mu}{d\tau} = \frac{d}{dt}(E, \vec{p}) \cdot \frac{dt}{d\tau}$$

$$F^\mu = \gamma\left(\frac{dE}{dt}, \vec{f}\right)$$

Exercise Show that the definition of power $\vec{f} \cdot \vec{v} = \frac{dE}{dt}$ can be derived from $F^\mu V_\mu = 0$