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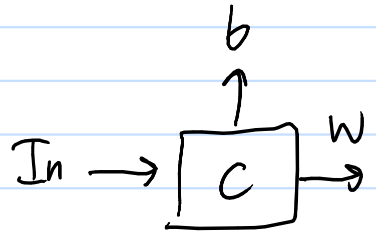
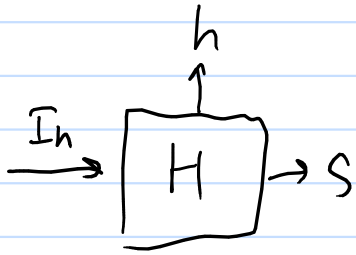
QUANTUM MECHANICS

LECTURE NOTES

Quantum Mechanics

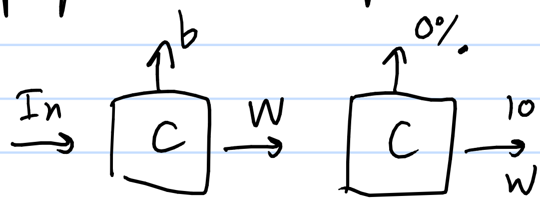
- We start the course on Quantum Mechanics (QM) by considering the following experiments.
  - Lets focus on properties of electrons: color & hardness
  - The only observable colours are black & white
  - " ——————— " hardness are hard & soft
  - It is possible to build a box that measures colours or hardness

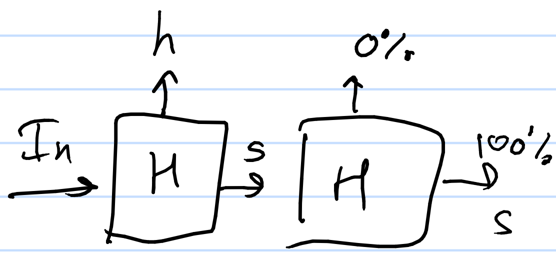




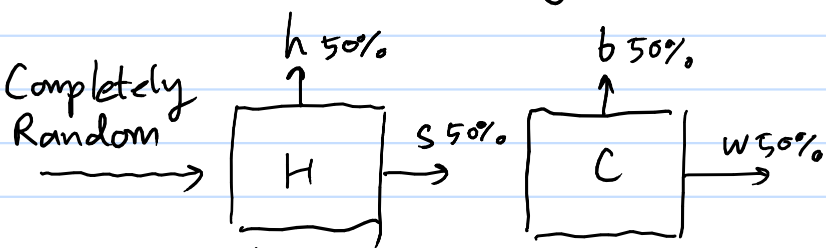
[* In reality this is just spin & magnetic field experiments]

- The properties are repeatable

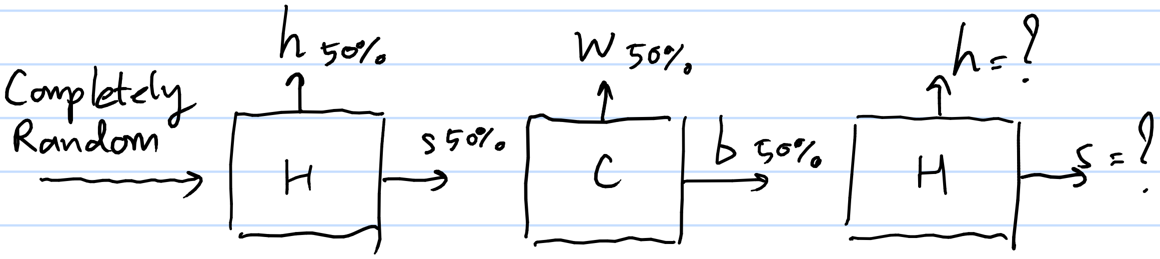




- Now the question is colour & hardness correlated?
- The experiments show the following



- They are not correlated!
- Now consider the following experiment



• From previous experiment electrons going into $\boxed{C}$ must be soft ⇒ All electrons going into $\boxed{H}$ must be soft

• The logic has lead us to believe that the final output must be 100% S & 0% H

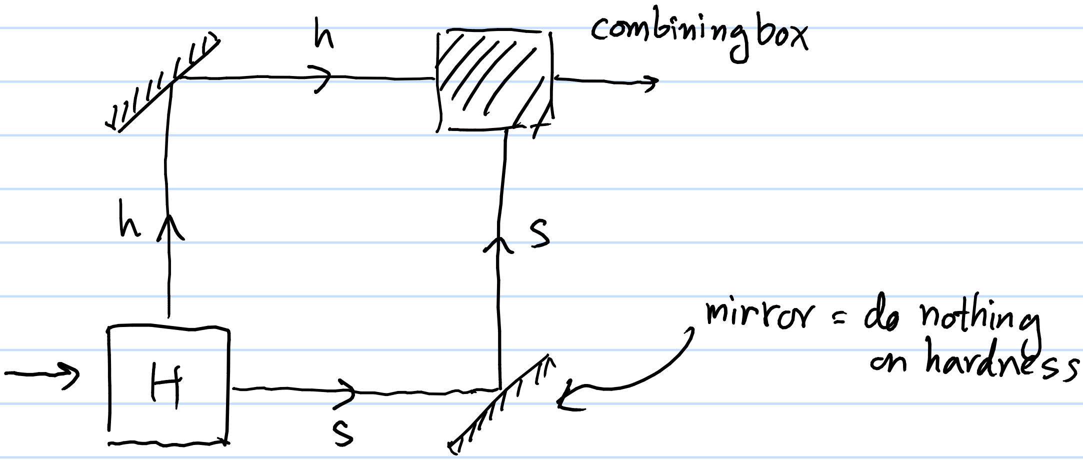
• However, this is completely wrong, the experiment shows that the final output is 50% S & 50% H !

☆ It has been tested extensively that this is not just our ignorance in building a box !

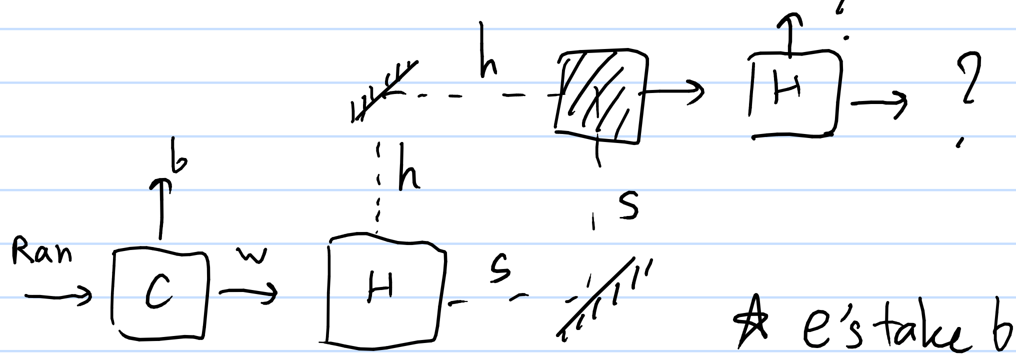
☆ The immediate consequence is that the simultaneous measurement of hardness and color is fundamentally forbidden! ⇒ Uncertainty principle

⇒ Measurable physical quantities can be "incompatible"

☆ Let's go a bit deeper, consider the following setup



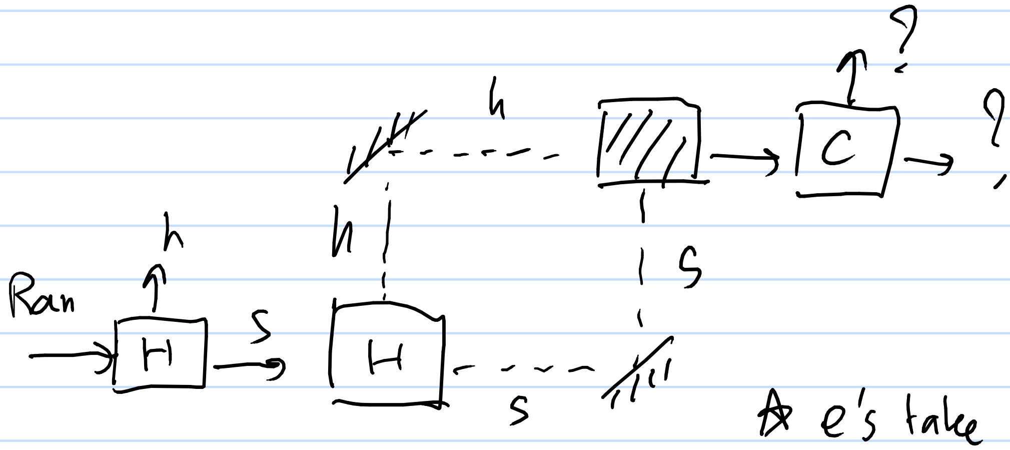
✩ Let's first feed in white electrons into this device



h : 50%

S : 50%

✩ e's take both path

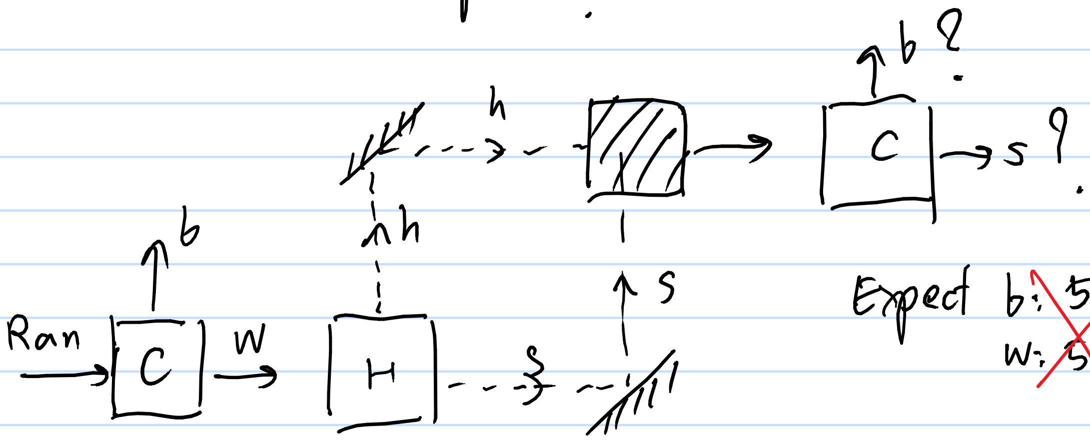


b : 50%

W : 50%

✩ e's take the lower path

Now this one is a surprise!

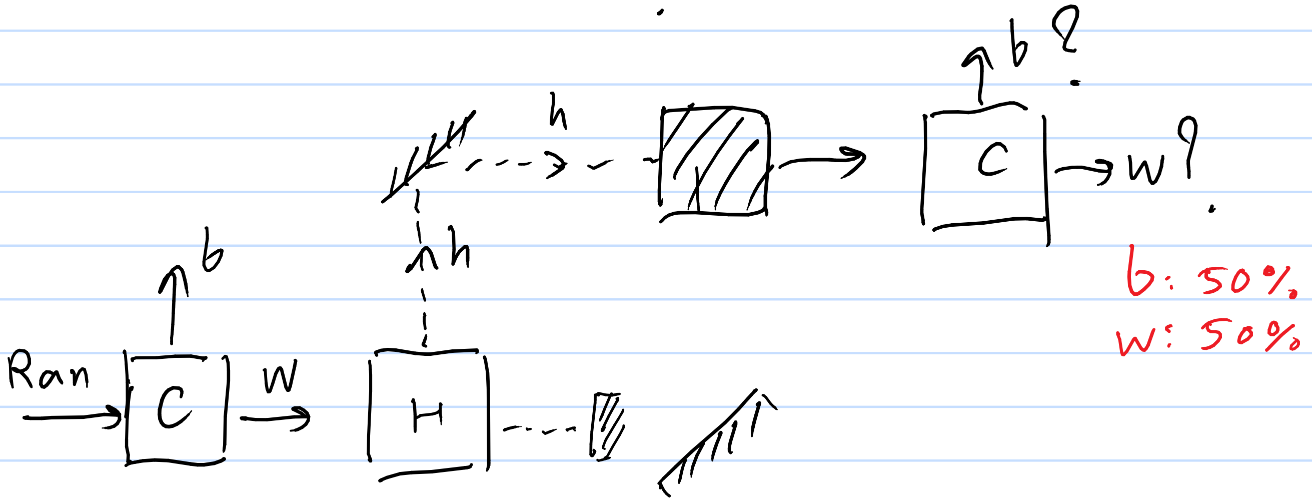


Expect ~~b: 50%~~
~~W: 50%~~ ?

From previous experiment white electrons after $\boxed{H} \xrightarrow{s}$ will be b:50% w:50%. The same for $\boxed{H} \xrightarrow{h}$ electrons will be 50% b 50% W. => The combine box will gives b:w = 50%/50%

No, the experiment turns out to be 100% white!

✩ Now, consider a wall that blocks elecctron



☆ We might expect 100% white? ⇒ Wrong! the result is 50%/50%

☆ The same is true if we block the h path

☆ The same is true if we fire 1 e^- at a time!

We are in trouble!

- e^- cannot take h path? ⇒ No, it would've been 50/50
- " ——— " s path? ⇒ " —— " ———.
- Can it take both paths? ⇒ No, we can fire 1 e^- at a time and look for where e^- is ⇒ we see 1 e^- at 1 path
- Can it take neither? ⇒ No, if we block both we see nothing!

- This doesn't make any sense!
- We cannot think of electron as an object anymore
- Electrons have a different nature (unlike what we used to)

⇒ The new way of moving ⇒ superposition

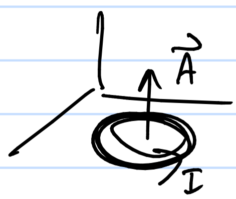
⇒ e^- is in superposition of h path & s path!

Stern-Gerlach Experiment

- Frankfurt 1922
- Pauli suggested that there are at least 2 d.o.f. of e^-
- Kröning suggested that electron "spins" around itself
- But Pauli says that the speed of the spin $> c \Rightarrow$ non-sense
- Ehrenfest had the same idea and published it $\Rightarrow$
- The idea of sending particles through magnetic field and measure the deflecting angle has been known for quite some time.
- Stern & Gerlach tried measuring orbital angular momentum of Silver atoms $\Rightarrow$ but it has no orbital angular momentum

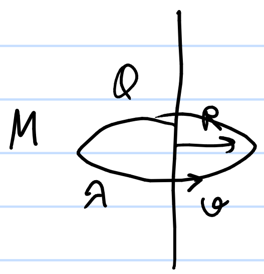
$\Rightarrow$ They see something else entirely! $\Rightarrow$ and the result is weird.

- We are going to extract QM lesson from it!
- We don't see spin directly we see magnetic moment.
- Classically we have



$\vec{\mu} = I\vec{A}$

- The unit: $[\mu] = \frac{\text{Joules}}{\text{Tesla}}$ $\left([\mu B] = [E]\right)$

- 

A ring of charges spinning

$$I = \lambda v = \frac{Q}{2\pi R} v$$

$$\mu = IA = \frac{Qv}{2\pi R} \cdot \pi R^2 = \frac{QvR}{2}$$

- We have $\mu = \frac{Q v R}{2}$
- Now consider the angular momentum $L = M v R$

$$\therefore \quad \mu = \frac{1}{2}\frac{Q}{M}(M v R) = \frac{Q}{2M} L$$

$$\boxed{\mu = \frac{1}{2}\frac{Q}{M} L}$$

universal / indep. on the velocity or shape.

- Is this also true for electrons? $\left(\mu = \frac{e}{2m_e} S = \frac{e\hbar}{2m_e}\left(\frac{S}{\hbar}\right)\right)$

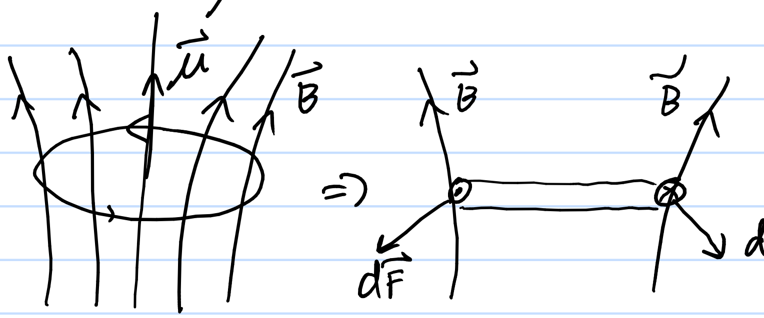
This is just a number

- We can parametrize this with a factor

$$\boxed{\mu = \text{"}g\text{"}\left(\frac{e\hbar}{2m_e}\right)\left(\frac{S}{\hbar}\right)}$$

$\hbar$ = Planck const. $\cong 1.054 \times 10^{-34}$ Js which has the same unit as angular moment-um

- "g"-factor can be calculated $\Rightarrow$ $g_e = 2$
- Dirac's equation gives this value $\Rightarrow$ agrees with the experiments.
- $\mu_B \equiv \frac{e\hbar}{2m_e}$ = Bohr magnetron $= 9.3 \times 10^{-24}$ J/T
- $g \neq 1 \Rightarrow$ electron is not spinning around classically!
- We write $\vec{\mu}_e = -2\mu_B \frac{\vec{S}}{\hbar}$ (−ve sign is due to the negative charge)
- Consider μ in $\vec{B}$



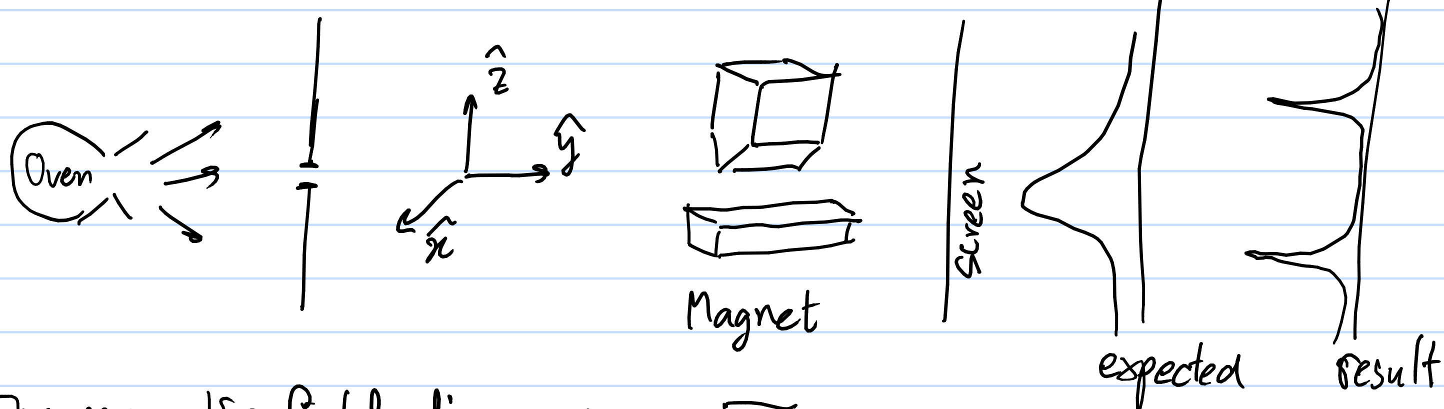
$$\vec{F} = \vec{\nabla}(\vec{\mu}\cdot\vec{B})$$

Directional Derivative

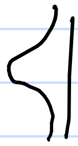
$\Rightarrow$ Force is in the direction that $\vec{\mu}\cdot\vec{B}$ increase fastest!

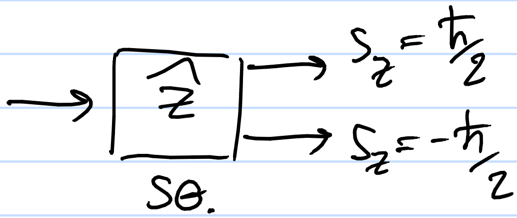
- Silver Atoms $\Rightarrow 47\,e^- / 46\,e^-$ fill $n=1,2,3,4$
- The last one is in the spherical symmetric shell (5s) which has zero angular momentum

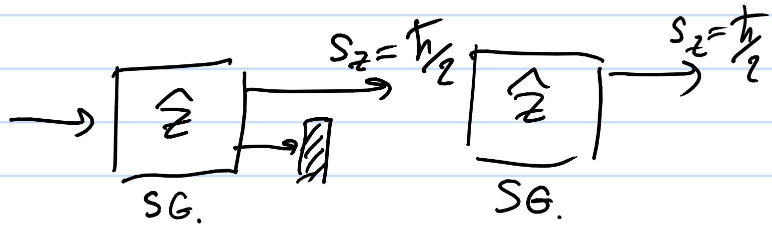
$\Rightarrow$ Throwing Ag in $\approx$ Throwing free e^- (magnetic moment)

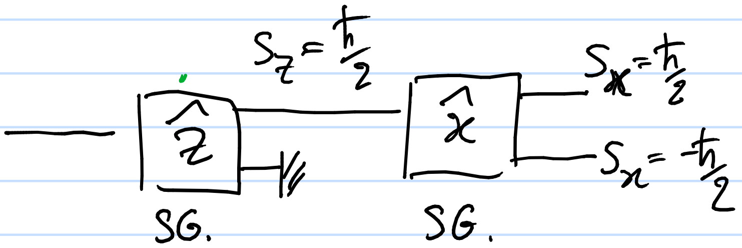


- The magnetic field diverges

- $\vec{F} = \vec{\nabla}(\vec{\mu}\cdot\vec{B}) \approx \vec{\nabla}(\mu_z B_z) = \mu_z \vec{\nabla} B_z = \mu_z \left(\frac{\partial B_z}{\partial z}\right)\hat{z}$
- Since $|F| \propto \mu_z$ and the distribution is Boltzman
- We expect  profile but we get  !
- The spin is measured! $\mu = -2\mu_B\left(\frac{S}{\hbar}\right)$
- The split $\Rightarrow$ $\boxed{\left(\frac{S}{\hbar}\right) = \pm\frac{1}{2}}$
- The interpretation $\Rightarrow$ The superposition of electrons being in the state $|\uparrow\rangle$ & $|\downarrow\rangle$
- Let's put things in a box and repeat the set up last time



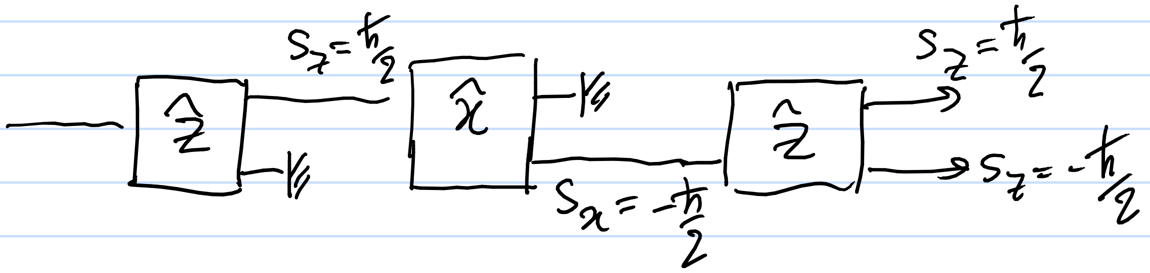




$\langle x,+|z,+\rangle \neq 0$

$\langle x,-|z,+\rangle \neq 0$

- The $\hat{x}$ box is the Stern-Gerlach magnetic with 90°rot.



- The S_z, S_x are incompatible (uncertainty)!
- Let's set up the machinery!
- We say that electrons can be described using 2 Complex vectors

  $|z,+\rangle$ with $S_z = \frac{\hbar}{2}$ & $|z,-\rangle$ with $S_z = -\hbar/2$
- Mathematically, this is the idea of operator & eigenvalues
- Suppose there is $\hat{S}_z$ operator (a linear map from a vector space onto itself).
- Then we write the "measurement" as

$$\hat{S}_z|z,+\rangle = \frac{\hbar}{2}|z,+\rangle \quad , \quad \hat{S}_z|z,-\rangle = -\frac{\hbar}{2}|z,-\rangle$$

- We then say $|z,\pm\rangle$ are eigenvectors of $\hat{S}_z$ ($\pm\frac{\hbar}{2}$ = eigenvalues)

- All possible spin state of an electron is a superposition of 2 basis

$$|\psi\rangle = C_1 \overbrace{|z,+\rangle}^{\text{First basis state}} + C_2 \overbrace{|z,-\rangle}^{\text{second basis state}}$$

We call this "Ket"

where $C_1, C_2 \in \mathbb{C}$

- This is similar to $\vec{v} = v_x \hat{x} + v_y \hat{y}$ (2D real vector space)

☆ The matrix representation

- is a way to represent a vector!
- The usual way of thinking about 3D vector is a 3 tuple (x, y, z) ⇒ We can do the same here
- We replace

$$|z,+\rangle \equiv |1\rangle \longleftrightarrow \begin{pmatrix}1\\0\end{pmatrix} \quad \& \quad |z,-\rangle \equiv |2\rangle \longleftrightarrow \begin{pmatrix}0\\1\end{pmatrix}$$

- This means $|\psi\rangle = C_1|z,+\rangle + C_2|z,-\rangle = C_1\begin{pmatrix}1\\0\end{pmatrix} + C_2\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}C_1\\C_2\end{pmatrix}$

- Now consider the "dot" product by defining "bra" basis

$$|1\rangle \leftrightarrow \begin{pmatrix}1\\0\end{pmatrix}, \quad \langle 1| \longleftrightarrow \begin{pmatrix}1 & 0\end{pmatrix}$$

$$|2\rangle \leftrightarrow \begin{pmatrix}0\\1\end{pmatrix}, \quad \langle 2| \longleftrightarrow \begin{pmatrix}0 & 1\end{pmatrix}$$

- Then the bra-ket product is naturally a matrix multiplication!

$$\langle 1|1\rangle = \langle 2|2\rangle = \begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0 & 1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 1$$

$$\& \ \langle 1|2\rangle = \langle 2|1\rangle = 0$$

Similar to

$\hat{x}\cdot\hat{x} = \hat{y}\cdot\hat{y} = 1$

$\hat{x}\cdot\hat{y} = 0$

- The general bra is written as

$$|\alpha\rangle = \alpha_1|1\rangle + \alpha_2|2\rangle \longleftrightarrow \begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix}$$
$$|\beta\rangle = \beta_1|1\rangle + \beta_2|2\rangle \longleftrightarrow \begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix}$$

$$\langle\alpha| = \alpha_1^*\langle 1| + \alpha_2^*\langle 2| \longleftrightarrow (\alpha_1^* \ \alpha_2^*)$$
$$\langle\beta| = \beta_1^*\langle 1| + \beta_2^*\langle 2| \longleftrightarrow (\beta_1^* \ \beta_2^*)$$

- Now the dot product (bra-ket) is defined as

$$\langle\alpha|\beta\rangle \equiv (\alpha_1^* \ \alpha_2^*)\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} = \alpha_1^*\beta_1 + \alpha_2^*\beta_2$$

- ☆ Note that the whole idea of complex conjugate is to ensure that the length of any vector is real and positive (we will use this as a probability later)

$$\langle\alpha|\alpha\rangle = (\alpha_1^* \ \alpha_2^*)\begin{pmatrix}\alpha_1\\ \alpha_2\end{pmatrix} = \alpha_1^*\alpha_1 + \alpha_2^*\alpha_2 = |\alpha_1|^2 + |\alpha_2|^2 \geq 0$$

- Now consider the operator

<u>Claim</u>
$$\hat{S}_z = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

- It is easy to check that

$$\hat{S}_z|1\rangle = \frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix}1\\ 0\end{pmatrix} = \frac{\hbar}{2}|1\rangle$$

- We are almost ready to explain SG. using our new tools!
- Except that we need $\hat{S}_x$ & $|x,+\rangle, |x,-\rangle$ (also $\hat{S}_y$)

- We are looking for matrices which has similar properties as $\hat{S}_z$

☆ Notice that $\hat{S}_z$ is Hermitian* $\left(\left(\hat{S}_z^T\right)^* \equiv \hat{S}_z^\dagger = \hat{S}_z \right)$

This is corresponding to the "physical" operator.

- The most general matrix is

$$S = \begin{pmatrix} 2c & a-ib \\ a+ib & 2d \end{pmatrix}, \quad a,b,c,d \in \mathbb{R}$$

- Now we don't want $\hat{S}_x, \hat{S}_y$ to do nothing!

$\Rightarrow$ we remove the identity $\mathbb{1}_{2\times2}$

$$\hat{S} - (c+d)\mathbb{1}_{2\times2} = \begin{pmatrix} c-d & a-ib \\ a+ib & d-c \end{pmatrix} = \hat{S}'$$

$\Rightarrow$ We also don't want to get involved

$\Rightarrow$ We remove the part $\hat{S}_z$

$$\hat{S}' - \frac{(c-d)}{(\hbar/2)}\hat{S}_z = \begin{pmatrix} 0 & a-ib \\ a+ib & 0 \end{pmatrix}$$

$$= a\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Pauli matrices

We define

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

☆ Remark 1: The set of any 2×2 Hermitian matrices form a real "vector space" with $\{\mathbb{1}, \sigma_x, \sigma_y, \sigma_z\}$ as a basis

☆ Remark 2: The choices of $\hat{S}_x, \hat{S}_y$ from $\hat{S}'$ is not unique! But we will choose the convenient notation

$$\hat{S}_x = \frac{\hbar}{2}\sigma_x, \quad \hat{S}_y = \frac{\hbar}{2}\sigma_y, \quad \text{and } \hat{S}_z = \frac{\hbar}{2}\sigma_z$$

- Why $\frac{\hbar}{2}$? $\Rightarrow$ We need the spin $\frac{\hbar}{2}$ back if we start SG with $\hat{S}_x$ instead of $\hat{S}_z$
- Now, what kind of state has spin "up" and "down" with respect to $\hat{S}_x$?

$\Rightarrow$ Consider the $\hat{S}_z$ case, we have

$$\hat{S}_z|z,+\rangle = \frac{\hbar}{2}|z,+\rangle \ , \quad \hat{S}_z|z,-\rangle = -\frac{\hbar}{2}|z,-\rangle$$

They are eigenvectors!

- We want eigenvectors of $\hat{S}_x$ & $\hat{S}_y$
- After a calculation we have

$\hat{S}_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$: There are 2 eigenvalues

$\langle x,+|x,+\rangle = 1$
$\langle x,-|x,-\rangle = 1$
$\langle x,+|x,-\rangle = 0$
$\langle x,-|x,+\rangle = 0$
$\vdots$

$$\frac{\hbar}{2} : |x,+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } -\frac{\hbar}{2} : |x,-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$\hat{S}_y : \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$: There are 2 eigenvalues

$$\frac{\hbar}{2} : |y,+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \text{ and } -\frac{\hbar}{2} : |y,-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

☆ We need $1/\sqrt{2}$ factor since we want the length for the basis vector to be 1

Homework : There are famous relations among spins operators

Check the following: $\hat{S}_x\hat{S}_y - \hat{S}_y\hat{S}_x = i\hbar\hat{S}_z$

$$\hat{S}_y\hat{S}_z - \hat{S}_z\hat{S}_y = i\hbar\hat{S}_z \ \& \ \hat{S}_z\hat{S}_x - \hat{S}_x\hat{S}_z = i\hbar\hat{S}_y$$

- We can write $|x,+\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|z,+\rangle + |z,-\rangle\right)$

  and $|x,-\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left(|z,+\rangle - |z,-\rangle\right)$

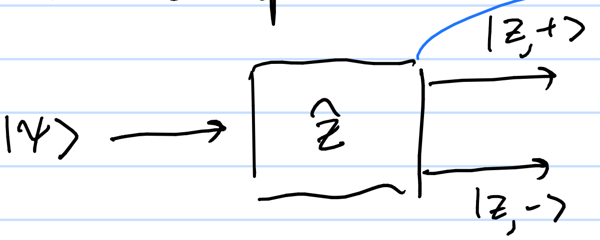
- Inversely, we have

$$|z,+\rangle = \frac{1}{\sqrt{2}}\left(|x,+\rangle + |x,-\rangle\right)$$

$$|z,-\rangle = \frac{1}{\sqrt{2}}\left(|x,+\rangle - |x,-\rangle\right)$$

- This is promising. We built a mathematical framework that could explain SG experiment

- Consider the set up



What this box does is split the beam into 2 beams which are the eigenvectors of $\hat{S}_z$

- The probability of $|\psi\rangle$ that coming out at $|z,+\rangle$ is $|\langle z,+|\psi\rangle|^2$

- For example, if $|\psi\rangle$ is completely random (thermal equilibrium). $|\psi\rangle$ should have equal parts

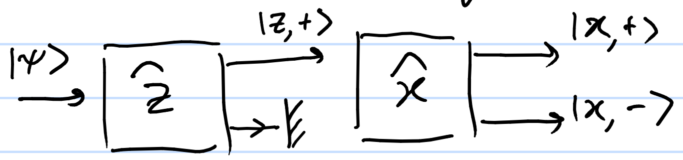
$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|z,+\rangle + |z,-\rangle\right)$$

$\therefore$ After $\boxed{\hat{z}}$, the probability of e^- coming out with spin up is

$$|\langle z,+|\psi\rangle|^2 = \left|\frac{1}{\sqrt{2}}\overbrace{\cancel{\langle z,+|z,+\rangle}}^{1} + \frac{1}{\sqrt{2}}\overbrace{\cancel{\langle z,+|z,-\rangle}}^{0}\right|^2$$

$$= \frac{1}{2} \Rightarrow \text{This is why we have } 50:50$$

- Now, consider the sequencial SG devices:



- What is the probability of particle with $S_x = \frac{\hbar}{2}$?

$\Rightarrow$ If we think of $|z,+\rangle$ going in $\boxed{\hat{x}}$ as 100%.

The $|z,+\rangle$ must have a "1" infront of it

- The question is then equivalent to finding $|\langle x,+|z,+\rangle|^2$

$$\langle x,+|z,+\rangle = \left[\left(\frac{1}{\sqrt{2}}\right)^*\langle z,+| + \left(\frac{1}{\sqrt{2}}\right)^*\langle z,-|\right]|z,+\rangle$$

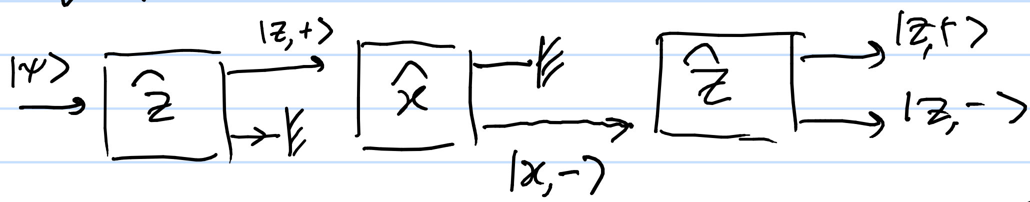
$$= \frac{1}{\sqrt{2}}\left(\overset{1}{\cancel{\langle z,+|z,+\rangle}} + \overset{0}{\cancel{\langle z,-|z,+\rangle}}\right) = \frac{1}{\sqrt{2}}$$

Alternatively $\langle x,+|z,+\rangle = \langle x,+|\left(\frac{1}{\sqrt{2}}|x,+\rangle + \frac{1}{\sqrt{2}}|x,-\rangle\right)$

$$= \frac{1}{\sqrt{2}}\left(\overset{1}{\cancel{\langle x,+|x,+\rangle}} + \overset{0}{\cancel{\langle x,+|x,-\rangle}}\right) = \frac{1}{\sqrt{2}}$$

$\therefore\ |\langle x,+|z,+\rangle|^2 = \frac{1}{2}$ the prob is still 50:50

- What about

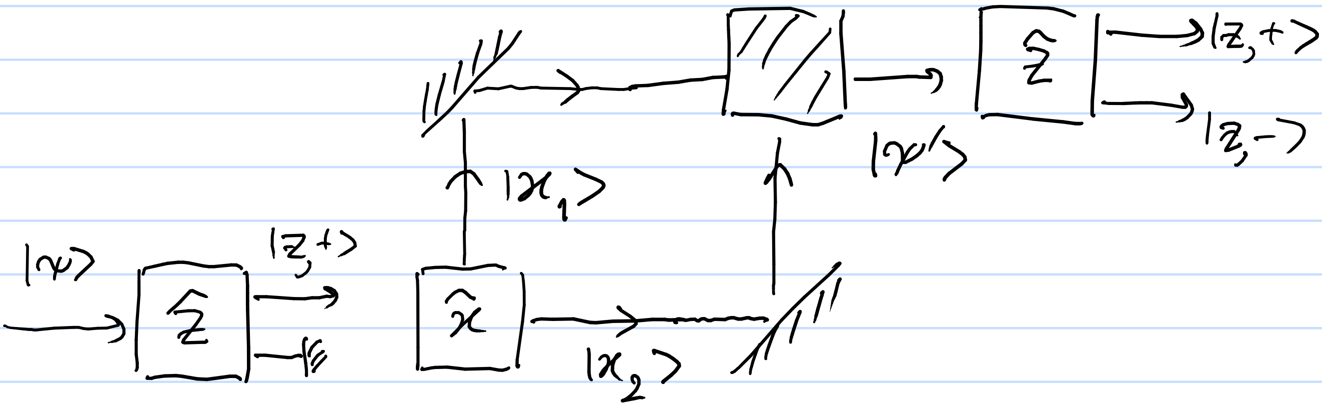


- What is the probability of $|z,-\rangle$ at the very end?
- We can think about $|x,-\rangle$ as 100% (coeff = 1)

$$\langle z,-|x,-\rangle = \langle z,-|\left[\frac{1}{\sqrt{2}}|z,+\rangle - \frac{1}{\sqrt{2}}|z,-\rangle\right] = \frac{-1}{\sqrt{2}}$$

$\therefore\ |\langle z,-|x,-\rangle|^2 = \frac{1}{2}$ the prob is still 50:50

- Consider the tricky one:



- Thinking $|z,+\rangle$ as 100% we have

$$|z,+\rangle = \frac{1}{\sqrt{2}}\left(|x,+\rangle + |x,-\rangle\right)$$

$$\therefore \quad |x_1\rangle = \frac{1}{\sqrt{2}}|x,+\rangle \quad \& \quad |x_2\rangle = \frac{1}{\sqrt{2}}|x,-\rangle$$

- At the combining box, we have

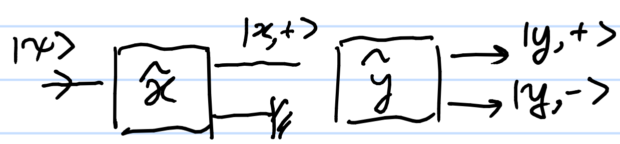
$$|\psi'\rangle = |x_1\rangle + |x_2\rangle = \frac{1}{\sqrt{2}}|x,+\rangle + \frac{1}{\sqrt{2}}|x,-\rangle$$

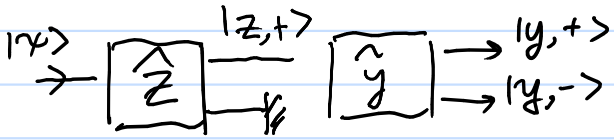
$$= |z,+\rangle$$

$$\therefore \quad \langle z,+|\psi'\rangle = 1 \quad \& \quad \langle z,-|\psi'\rangle = 0$$

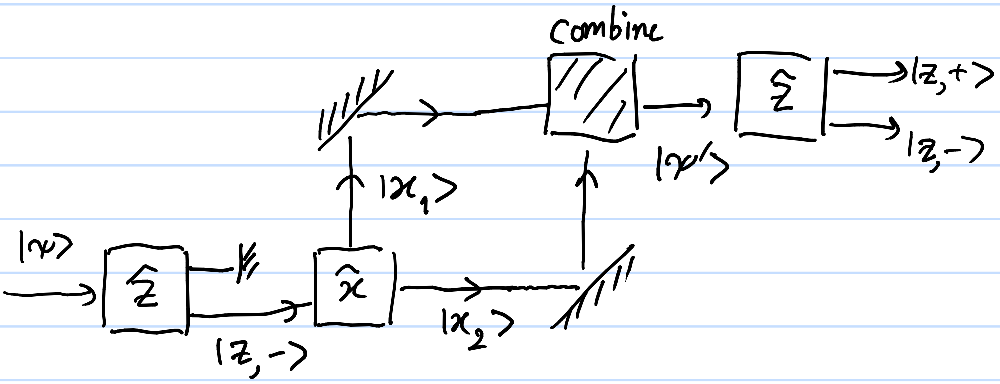
Homework

Work out

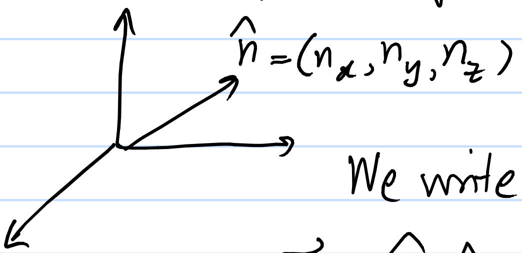
① 

② 

③



General Spin (spin a generic direction)



$$\vec{n} = n_x \hat{e}_x + n_y \hat{e}_y + n_z \hat{e}_z$$

We write $\vec{S} = (\hat{S}_x, \hat{S}_y, \hat{S}_z)$

$$\Rightarrow \quad \vec{S} = \hat{S}_x \hat{e}_x + \hat{S}_y \hat{e}_y + \hat{S}_z \hat{e}_z \quad ??$$

This is strange / $\hat{e}$ has nothing to do with $\hat{S}$

We define the spin projected in $\hat{n}$ direction as

$$\hat{S}_n = \vec{n} \cdot \vec{S} = n_x \hat{S}_x + n_y \hat{S}_y + n_z \hat{S}_z$$

For the unit vector in polar coordinate

$$n_x = \sin\theta\cos\phi, \quad n_y = \sin\theta\sin\phi, \quad n_z = \cos\theta$$

$$\hat{S}_n = \frac{\hbar}{2}(n_x \sigma_1 + n_y \sigma_2 + n_z \sigma_3) = \frac{\hbar}{2}\begin{pmatrix} n_z & n_x - i n_y \\ n_x + i n_y & -n_z \end{pmatrix}$$

$$\hat{S}_n = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

Exercise ① Find the eigenvalues of this $\hat{S}_n$ (ans: $\pm\frac{\hbar}{2}$)

② Find the eigenvectors, starting by assuming $|n,\pm\rangle$: $\hat{S}_n|n,\pm\rangle = \pm\frac{\hbar}{2}|n,\pm\rangle$ and expand this in the basis of $|z,+\rangle, |z,-\rangle$: $|n,\pm\rangle = C_1|z,+\rangle + C_2|z,-\rangle$

that is $|n,\pm\rangle = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$

[ans: $|n,+\rangle = \cos\left(\frac{\theta}{2}\right)|z,+\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\phi}|z,-\rangle$

$|n,-\rangle = \sin\left(\frac{\theta}{2}\right)|z,+\rangle - \cos\left(\frac{\theta}{2}\right)e^{i\phi}|z,-\rangle$]

③ From ② take $\theta = 0$, can you recover $|z,\pm\rangle$? if not How do we fix it?

Homework 1

1.1) The Levi-Civita symbol is given by

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{, if } ijk \text{ is an even permutation of } (1,2,3) \\ -1 & \text{, if } \text{"} \quad \text{" odd "} \quad \text{"} \\ 0 & \text{, if } i=j \text{ or } j=k \text{ or } i=k \end{cases}$$

For example, $\varepsilon_{123} = 1$, $\varepsilon_{312} = 1$ since $(1,2,3) \to (2,1,3) \to (3,1,2)$ two swaps

$\varepsilon_{321} = -1$ since $(3,2,1) \to (1,2,3)$ one swap!

- Show explicitly the famous relation among spin is written as

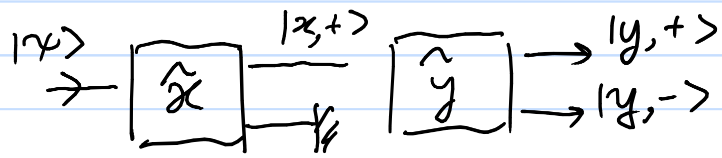
$$[S_i, S_j] = \sum_{k=1}^{3} i\hbar\, \varepsilon_{ijk} S_k$$

where $S_1 = S_x = \frac{\hbar}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $S_2 = S_y = \frac{\hbar}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $S_3 = S_z = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

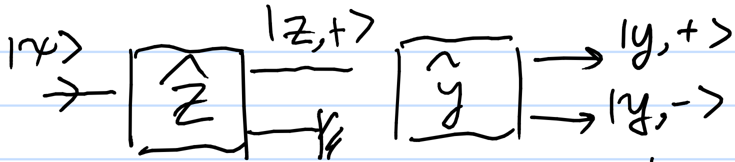
and $[A,B] \equiv AB - BA$ is called commutator of A & B

1.2) Find out the probability at the end of each experiments:

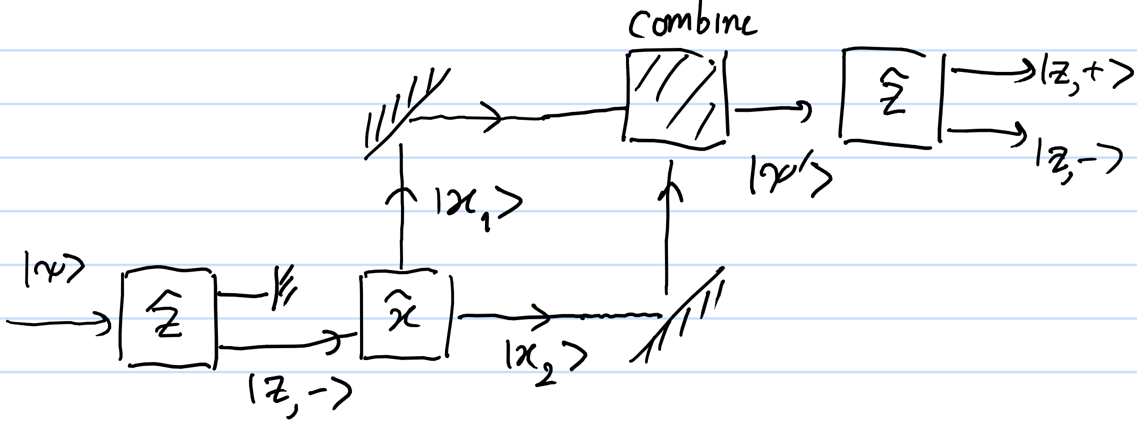
a)



b)



c)



1.3). Let's define anti-commutator as $\{A, B\} = AB + BA$

a). Show that

$$\{\hat{S}_i, \hat{S}_j\} = \frac{\hbar^2}{2}\delta_{ij}\mathbb{1}$$

b). Show that any matrix multiplication can be decomposed

as
$$XY = \frac{1}{2}\{X, Y\} + \frac{1}{2}[X, Y]$$

Then show that

$$\hat{S}_i\hat{S}_j = \frac{\hbar^2}{4}\delta_{ij}\mathbb{1} + \sum_k \frac{i\hbar}{2}\varepsilon_{ijk}\hat{S}_k$$

c). From previous exercise, show that if

$$\vec{n}\cdot\vec{S} \equiv n_x\hat{S}_x + n_y\hat{S}_y + n_z\hat{S}_z \text{ then}$$

i) $(\vec{a}\cdot\vec{S})(\vec{b}\cdot\vec{S}) = \frac{\hbar^2}{4}(\vec{a}\cdot\vec{b}) + \frac{i\hbar}{2}(\vec{a}\times\vec{b})\cdot\vec{S}$

ii) $(\vec{a}\cdot\vec{S})^2 = \frac{\hbar^2}{4}|\vec{a}|^2$

1.4) a) Find the eigenvalues of a spin operator in general direction

$$\hat{S}_n = \frac{\hbar}{2}\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

b) Find the eigenvectors, starting by assuming

$$|n, \pm\rangle : \hat{S}_n|n, \pm\rangle = \pm\frac{\hbar}{2}|n, \pm\rangle$$

and expand this in the basis of $|z, +\rangle, |z, -\rangle$

c) There will be an ambiguity when $\theta = 0$. How should we fix this?

1.5) We can define an exponential of a matrix using a power series:

$$e^{\hat{X}} = \sum_{n=0}^{\infty} \frac{1}{n!} \hat{X}^n = \mathbb{1} + \hat{X} + \frac{1}{2!}\hat{X}\hat{X} + \dots$$

Consider $\hat{M} = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$

Show that $e^{i\hat{M}\theta}$ takes the form

$$e^{i\hat{M}\theta} = A(\theta)\,\mathbb{1} + B(\theta)\,\hat{M}$$

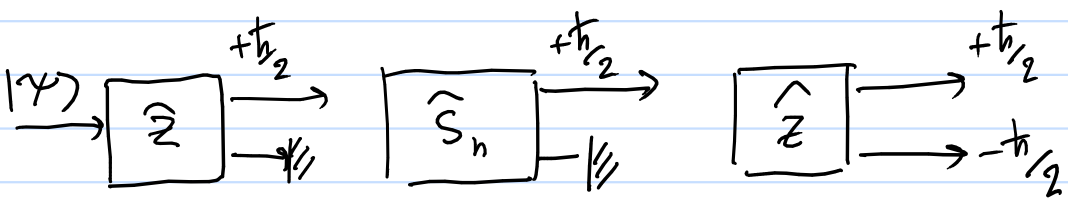
and determine the function $A(\theta)$, $B(\theta)$

1.6) A spin state is given by

$$(1+i)|z,+\rangle - (1+\sqrt{3}\,i)|z,-\rangle$$

What is the direction this spin is pointing to?

1.7) Consider the setup



where the $\hat{S}_n$ is Spin in the (x,z) plane $(\phi=0, \theta\neq 0)$

Find the probability of the final outcome as a function of θ and justify your answer for $\theta=0$, $\theta=\pi/2$ & $\theta=\pi$

Linear Algebra

- You are probably familiar with vectors.
- However, we would like to generalise this for our use.

Definition A vector space $\mathbb{V}$ is a collection of objects called vectors together with

- A definite rule for the vector sum $|v\rangle + |w\rangle$
- A definite rule for scalar multiplication $a|v\rangle$

with the following properties:

1) Closure: For any $|v\rangle$ & $|w\rangle$, $|v\rangle + |w\rangle \in \mathbb{V}$
2) Scalar multiplication is distributive & associative
$$a(|v\rangle + |w\rangle) = a|v\rangle + a|w\rangle$$
$$(a+b)|v\rangle = a|v\rangle + b|v\rangle, \quad a(b|v\rangle) = (ab)|v\rangle$$
3) Vector addition is commutative & associative
$$|v\rangle + |w\rangle = |w\rangle + |v\rangle, \quad |v\rangle + (|w\rangle + |z\rangle) = (|v\rangle + |w\rangle) + |z\rangle$$
4) There exists a null vector $|0\rangle$ obeying $|v\rangle + |0\rangle = |v\rangle$
5) For every vector $|v\rangle$ there exists an inverse vector denoted $|-v\rangle$, such that $|v\rangle + |-v\rangle = |0\rangle$
6) The multiplication of scalar 1 does nothing: $1|v\rangle = |v\rangle$

☆ If the set of scalar is $\mathbb{R} \Rightarrow$ real vector space

If the set of scalar is $\mathbb{C} \Rightarrow$ Complex vector space

Example of vector space

1) The set of $\mathbb{C}^n$ of all n-tuples $|x\rangle = (x_1, x_2, \dots, x_n)$ where $x_i \in \mathbb{C}$ is a vector space under

$$|x\rangle + |y\rangle = (x_1+y_1, x_2+y_2, \dots, x_n+y_n)$$

$$a|x\rangle = (ax_1, ax_2, \dots, ax_n)$$

※ we write in column most of the time.

2) The set of $\mathbb{C}^\infty$: the set of all sequences $|x\rangle = (x_1, x_2, \dots)$ with the usual addition & multipication rules : $|x\rangle + |y\rangle = (x_1+y_1, x_2+y_2, \dots)$ & $a|x\rangle = (ax_1, ax_2, \dots)$

3) The set of all $m \times n$ matrices over $\mathbb{C}$, denoted $M^{(m,n)}(\mathbb{C})$ is a vector space. Vectors are written as $\tilde{A} = [a_{ij}]$ where $i = 1, \dots, m$ and $j = 1, \dots, n$ and $a_{ij} \in \mathbb{C}$. Addition and multiplication are

$$\tilde{A} + \tilde{B} = [a_{ij} + b_{ij}], \quad c\tilde{A} = [ca_{ij}]$$

4) Real-valued functions on $\mathbb{R}^n$, denoted $\mathcal{F}(\mathbb{R}^n)$, form a vector space over $\mathbb{R}$. The vectors can be thought of as functions of n arguments,

$$f(\vec{x}) = f(x_1, x_2, \dots, x_n)$$

and vector addition $f+g$ and scalar multiplication are

$$(f+g)(\vec{x}) = f(\vec{x}) + g(\vec{x}), \quad (af)(\vec{x}) = af(\vec{x})$$

5) Let S be an arbitrary set, the set $\mathcal{F}(S, \mathbb{C})$ of all complex functions on S forms a complex vector space

☆ When the vectors can be uniquely identified by a finite number of scalars ($\mathbb{R}$ or $\mathbb{C}$), the vector space is said to be finite dimensional

☆ The number of independent components needed to specify an arbitrary vector is called the dimension of vector space.

☆ Example 1) & 3) are finite dimensional with dimension n and mn respectively

- On the other hand, it is not possible to specify the vectors in example 2), 4), and 5) by a finite number of scalars

  ⇒ These vector spaces are said to be infinite dimensional

Vector Space Homomorphism (linear map)

For vector spaces, V, W, a map $T: V \to W$ is called linear mapping (or vector space homomorphism) if

$$T(a|x\rangle + b|y\rangle) = aT(|x\rangle) + bT(|y\rangle)$$

If a linear map is one-to-one and onto, T is called a vector space isomorphism (V & W are called isomorphic)

Example Consider the set $P_n(x)$ of all real-valued polynomials of degree $\leq n$, $f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$.

Addition: $f(x) + g(x) = (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2 + \dots + (a_n + b_n)x^n$

Multiplication: $cf(x) = ca_0 + ca_1 x + ca_2 x^2 + \dots + ca_n x^n$

The set $P_n(x)$ is obviously a vector space. Now consider the map $S: P_n(x) \to \mathbb{R}^{n+1}$ defined by

$$S(a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n) = (a_0, a_1, a_2, \ldots, a_n)$$

It is then easy to check that S is isomorphism

$$S(f(x) + g(x)) = S(f(x)) + S(g(x)) \;\&\; S(af(x)) = aS(f(x))$$

☆ The set of all real polynomials $P(x)$:

$P(x) = P_0(x) \cup P_1(x) \cup P_2(x) \cup \ldots$ is also a vector space

☆ The linear map onto the same vector space is called linear operators

☆ The identity operator is denoted as $\mathbb{1}$ where

$$\mathbb{1}|x\rangle = |x\rangle \quad \text{for any } |x\rangle \in V$$

☆ We define the product of 2 operators as the composition map:

$$(ST)|x\rangle = S(T|x\rangle)$$

☆ The inverse operator of an operator T is written as T^{-1}:

$$T^{-1}T = TT^{-1} = \mathbb{1}$$

☆ The relation of inverse operator is then

$$(ST)^{-1} = T^{-1}S^{-1}$$

Example On the vector space $P(x)$ of all polynomials with real coefficients over a variable x

Let x be the operation of multiplying by x

and D be the operation of differentiation

$$x: f(x) \to x f(x) \quad \& \quad D: f(x) \to \frac{d f(x)}{dx}$$

These are linear operator since

$$x\left(a f(x) + b g(x)\right) = a x f(x) + b x g(x)$$

$$\& \quad \frac{d}{dx}\left(a f(x) + b g(x)\right) = a \frac{d f(x)}{dx} + b \frac{d}{dx} g(x)$$

Exercise: show that 2 operators x & D obey a very familiar relation

$$[D, x] \equiv Dx - xD = \mathbb{1}$$

Vector Subspace & Basis

☆ If A is any subset of a vector space V, the set of all linear combinations of elements of A:

$$L(A) = \left\{ \sum_{i=1}^{n} a_i |x_i\rangle \;\middle|\; a_i \in \mathbb{R} (\text{or } \mathbb{C}), |x_i\rangle \in A \right\}$$

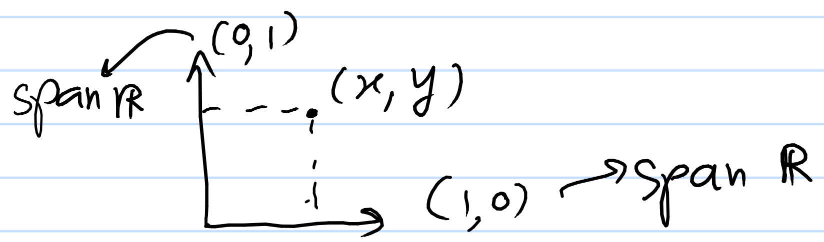
is called subspace spanned/generated by A

☆ A subspace is also a vector space

☆ A vector space V is said to be the direct sum of subspaces U & W if every vector $|v\rangle \in V$ has a unique decomposition $|v\rangle = |u_1\rangle + |w_1\rangle$ where $|u_1\rangle \in U$, $|w_1\rangle \in W$, and we write

$$V = U \oplus W$$

Example $\mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R}$



☆ A set of vectors A is said to be linearly independent if it has the following property

$$\sum_{i=1}^{n} a_i |x_i\rangle = 0 \Rightarrow a_j = 0, \forall j \qquad \left(A = \{|x_i\rangle\} \right)$$

Example $\mathbb{R}^2$: consider $|e_1\rangle = (1,0)$, $|e_2\rangle = (0,1)$

If $c_1|e_1\rangle + c_2|e_2\rangle = |0\rangle$

$$(c_1, c_2) = (0,0) \Rightarrow c_1 = c_2 = 0$$

$\{|e_1\rangle, |e_2\rangle\}$ is linealy independent.

☆ A basis of V is a list of vectors that spans the entire vector space V & is linearly independent

- Any finite dimensional vector space has basis
- The choice of basis is not unique!

☆ The dimension of a vecter space V is the number of vectors in a basis of V

Example $\mathbb{C}^n$: Consider the list of vectors

$$|e_1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; |e_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \; \dots, \; |e_n\rangle \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

They are clearly linearly independent & any vector $|v\rangle \in V$ can be written as $|v\rangle = \sum_{i=1}^{n} c_i |e_i\rangle$

The dimensionality $= n = \#$ of basis $= \#$ of parameters

Back to operators: Consider an operator T on vector space V

Null space of $T \equiv null(T) = \{ |v\rangle \in V \; | T|v\rangle = 0 \} \supseteq \{0\}$

$null(T)$ is always a subspace of V

Range of $T \equiv range(T) = T(V) = \{ T|v\rangle : v \in V \}$

$T(V)$ is always a subspace of V

Theorem $\quad dim(V) = dim(null(T)) + dim(T(V))$

Matrix Representation (of operators)

Let's choose a basis $\{|v_1\rangle, |v_2\rangle, \dots, |v_n\rangle\}$ s.t. any vector can be uniquely written as

$$|v\rangle = a_1|v_1\rangle + a_2|v_2\rangle + \dots + a_n|v_n\rangle$$

Let's apply an operator

$$T|v\rangle = a_1 T|v_1\rangle + a_2 T|v_2\rangle + \dots + a_n T|v_n\rangle$$

Since $T|v_1\rangle$ is also a vector, it can be uniquely written as

$$T|v_1\rangle = T_{11}|v_1\rangle + T_{21}|v_2\rangle + \dots + T_{n1}|v_n\rangle$$

where $T_{11}, T_{12}, \dots, T_{1n}$ are scalar

and $\quad T|v_2\rangle = T_{12}|v_1\rangle + T_{22}|v_2\rangle + \dots + T_{n2}|v_n\rangle$

and so on..., such that we can write as a sum.

$$T|v_i\rangle = \sum_{j=1}^{n} T_{ji}|v_j\rangle \quad , \; i = 1, \dots, n$$

We can also write

$$T|v\rangle = \sum_{i=1}^{n} a_i T|v_i\rangle = \sum_{i=1}^{n} a_i \sum_{j=1}^{n} T_{ji}|v_j\rangle$$

$$T|v\rangle = \sum_{j=1}^{n} \left(\sum_{i=1}^{n} a_i T_{ji}\right)|v_j\rangle$$

Since $T|v\rangle$ is also a vector, we have

$$T|v\rangle = |v'\rangle \equiv \sum_{j=1}^{n} b_j |v_j\rangle$$

We have
$$\boxed{b_j = \sum_{i=1}^{n} T_{ji} a_i}$$

where a_i are scalar in $|v\rangle = \sum_{i=1}^{n} a_i |v_i\rangle$

and b_j are scalar in $|v'\rangle = T|v\rangle = \sum_{j=1}^{n} b_j |v_j\rangle$

$\therefore$ The above equation can be written in matrix form

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} T_{11} & T_{12} & \cdots & T_{1n} \\ T_{21} & T_{22} & \cdots & T_{2n} \\ \vdots & & \ddots & \\ T_{n1} & & & T_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

which represents

$$|v'\rangle = T|v\rangle$$

☆ Note that the values of T_{ij} are depending on basis

The multiplication of operators can be represented:

$$|v'\rangle = ST|v\rangle \Rightarrow \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} S_{11} & \cdots & S_{1n} \\ & \ddots & \\ S_{n1} & \cdots & S_{nn} \end{pmatrix} \begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \\ T_{n1} & & T_{nn} \end{pmatrix} \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

☆ The $\mathbb{1}$ operator & $\hat{O}$ operator can be represented as

$$\mathbb{1} = \begin{pmatrix} 1 & & 0 \\ & 1 & \\ & & \ddots \\ 0 & & & 1 \end{pmatrix}, \quad \hat{O} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & \ddots & & \\ \vdots & & \ddots & \\ 0 & & & 0 \end{pmatrix}$$

Basis independent properties (of matrix rep.)

① $tr(T) = \sum_{i=1}^{n} T_{ii}$ (trace = sum of all diagonal components)

② $\det(T)$ = determinance of matrix T_{ij}

The reason why this is basis independent is the eigenvalues/vectors.

☆ U is a T-invariant subspace if

$$T(U) = \{ T|u\rangle \mid |u\rangle \in U \} \subseteq U$$

Example $\{0\}$, V ⇒ Trivial invariant subspace

1-Dimensional invariant subspace ⇒ $\mathcal{U} = \{ \alpha|u\rangle \mid \alpha \in \mathbb{C} \}$

If U is T-invariant, $T|u\rangle \in U \Rightarrow T|u\rangle = \lambda|u\rangle$

for some λ, $|u\rangle$ is called eigenvector and

λ is called eigenvalue.

☆ $|u\rangle$ must not be zero ⇒ uninteresting

☆ $\text{Spectrum}(T) = \{ \lambda \mid \lambda \text{ is an eigenvalues of } T \}$

If λ is eigenvalue, $Tu = \lambda u$ or we could write

$$(T - \lambda \mathbb{1}) u = 0$$

$\Rightarrow$ null$(T - \lambda \mathbb{1}) = \{$eigenvectors with eigenvalue $\lambda\}$

☆ Also $(T-\lambda\mathbb{1})$ is not invertible (no inverse)

Theorem Let T be a linear operator and $\lambda_1, \dots, \lambda_n$ are eigenvalues with corresponding eigenvectors $|u_1\rangle, \dots, |u_n\rangle$. The set $\{|u_1\rangle, \dots, |u_n\rangle\}$ is linearly independent.

☆ The fact that $(T-\lambda\mathbb{1})$ is not invertible is indp. of the basis. $\Rightarrow$ We can work as a matrix rep.

☆ A matrix with no inverse = singular matrix

$$\det(T - \lambda\mathbb{1}) \equiv f(\lambda) = 0$$

We call $f(\lambda)$ a characteristic polynomial

☆ If we know that $f(\lambda_1) = f(\lambda_2) = \dots f(\lambda_n) = 0$

we can write $f(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$

☆ $\lambda_1, \dots, \lambda_n$ are zeros of $f(\lambda)$ and are eigenvalues.

☆ Some of them can be equal $\Rightarrow$ degenerate eigenvalues.

Exercise Find the eigenvalues / eigenvectors of an operator on $\mathbb{C}^4$ written in orthogonal basis

$$M = \begin{pmatrix} -1 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & -1 & 2 & -1 \end{pmatrix}$$

Inner product

Inner product on a vector space is a map $V \otimes V \to \mathbb{C}$ (or $\mathbb{R}$)

This is similar to the "dot" product: Motivation

$$V = \mathbb{R}^n \Rightarrow \quad \vec{a} = (a_1, \dots, a_n) \ \& \ \vec{b} = (b_1, \dots, b_n)$$

$$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \dots + a_n b_n \in \mathbb{R}$$

Properties of dot product

1). $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

2). $\vec{a} \cdot \vec{a} \geq 0$ with $\vec{a} \cdot \vec{a} = 0 \Rightarrow \vec{a} = \vec{0}$

3) $\vec{a} \cdot (\beta_1 \vec{b}_1 + \beta_2 \vec{b}_2) = \beta_1 \vec{a} \cdot \vec{b}_1 + \beta_2 \vec{a} \cdot \vec{b}_2$

The length of the vector (norm of vector)

$$|\vec{a}| = \sqrt{\vec{a} \cdot \vec{a}}$$

☆ Note that the "dot" product is not unique

We can just $\vec{a} \cdot \vec{b} \equiv c_1 a_1 b_1 + c_2 a_2 b_2 + \dots + c_n a_n b_n$

☆ Schwarz inequality:

$$|\vec{a} \cdot \vec{b}| \leq |\vec{a}| |\vec{b}|$$

Complex spaces $\mathbb{C}^n$

$$|z\rangle \in \mathbb{C}^n, \quad |z\rangle = (z_1, \dots, z_n), \quad z_i \in \mathbb{C}$$

If we think the length of this vector, we need

$$|z| = \sqrt{|z_1|^2 + |z_2|^2 + \dots + |z_n|^2} = \sqrt{z_1 z_1^* + z_2 z_2^* + \dots + z_n z_n^*}$$

Inner product on a complex vector space

We write the "dot" of $|a\rangle$ & $|b\rangle$ as

$$\langle a|b\rangle \in \mathbb{C}, \quad |a\rangle, |b\rangle \in V$$

Axioms

1). $\langle a|b\rangle = \langle b|a\rangle^*$

2). $\langle a|a\rangle \geq 0$ only equal if $|a\rangle = |0\rangle$

3). $\langle a|\beta_1 b_1 + \beta_2 b_2\rangle = \beta_1\langle a|b_1\rangle + \beta_2\langle a|b_2\rangle$, $\beta_1, \beta_2 \in \mathbb{C}$

It is easy to see that

$$\langle \alpha_1 a_1 + \alpha_2 a_2 | \beta\rangle = \alpha_1^*\langle a_1|b\rangle + \alpha_2^*\langle a_2|b\rangle$$

Example

1). $V \in \mathbb{C}^n$, $\langle a|b\rangle = a_1^* b_1 + a_2^* b_2 + \dots + a_n^* b_n$

2). V = a set of all complex functions $f(x) \in \mathbb{C}$

where $x = [0, L]$

If $f, g \in V$, we can define the inner product

$$\langle f|g\rangle = \int_0^L f^*(x) g(x)\, dx$$

☆ If $|a\rangle, |b\rangle \in V$ and $\langle a|b\rangle = 0$, we say vectors $|a\rangle$ & $|b\rangle$ are orthogonal

☆ If we have a set $\{|e_1\rangle, |e_2\rangle, \dots, |e_n\rangle\}$ s.t. $\langle e_i|e_j\rangle = \delta_{ij}$ the set is orthonormal

δ_{ij} = Kronecker delta where

$$\delta_{ij} = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$$

☆ If $\{|e_i\rangle\}$ is a basis of V and $\{e_i\}$ are orthonormal

$$|a\rangle \in V, \quad |a\rangle = \sum_{i=1}^{n} a_i |e_i\rangle$$

The dot product with $|e_j\rangle$ is then

$$\langle e_j | a\rangle = \sum_{i=1}^{n} a_i \langle e_j | e_i\rangle = \sum_{i=1}^{n} a_i \delta_{ji} = a_j$$

$$\therefore \quad |a\rangle = \sum_{i=1}^{n} \left(\langle e_i | a\rangle\right) |e_i\rangle$$

Sometimes we choose to write

$$|a\rangle = \sum_{i=1}^{n} |e_i\rangle \langle e_i | a\rangle$$

☆ The norm $|a|^2 = \langle a | a\rangle \geq 0$

☆ Schwarz inequality $|\langle a|b\rangle| \leq |a||b|$

☆ Triangular inequality $|a+b| \leq |a| + |b|$

☆ Finite dim vector space with an inner product is called Hilbert space

☆ Infinite dim vector space with an inner product must be "complete" to be called Hilbert space

Ex $\quad p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_N x^N \Rightarrow \sum_{i=1}^{N} \frac{x^i}{i!} = e^N$

incomplete $\qquad\qquad$ complete

Gram-Schmidt Procedure

Suppose there is a set $\{|v_1\rangle, \dots, |v_n\rangle\}$ (linearly indep.)

We can construct the orthonormal basis $\{|e_1\rangle, |e_2\rangle, \dots, |e_n\rangle\}$

$$|e_j\rangle = \frac{|v_j\rangle - \sum_{i<j} \langle e_i | v_j \rangle |e_i\rangle}{\left| |v_j\rangle - \sum_{i<j} \langle e_i | v_j \rangle |e_i\rangle \right|}$$

☆ Let V be a vector space & U is a set of vectors $\in V$

We can define : $U^\perp$ (orthogonal compliment of U)

$$U^\perp = \{ |v\rangle \in V \mid \langle v | u \rangle = 0 \ \forall u \in U \}$$

$U^\perp$ is automatically subspace

Theorem If U is a subspace, then $V = U \oplus U^\perp$

Ex If $V = \mathbb{R}^3$ & $U = \mathbb{R}^2 \Rightarrow \mathbb{R}^3 = \mathbb{R}^2 \oplus \mathbb{R}$

☆ Since any vector $|v\rangle \in V$ can be uniquely decomposed as

$|v\rangle = |u\rangle + |w\rangle$, $|u\rangle \in U$, $|w\rangle \in U^\top$, we can define

the orthogonal projector P_U as : $P_U |v\rangle = |u\rangle$

☆ If the basis of U subspace is $\{|e_1\rangle, \dots, |e_k\rangle\}$

$$P_U |v\rangle = \langle e_1 | v \rangle |e_1\rangle + \dots + \langle e_k | v \rangle |e_k\rangle$$

☆ Projector : $P_U |u\rangle = |u\rangle \Rightarrow P_U P_U = P_U^2 = P_U$

Exercise: Show that in an orthonormal basis

$$\underbrace{\{|g_1\rangle, \dots, |g_n\rangle\}}_{V} = \{\underbrace{|e_1\rangle, \dots, |e_k\rangle}_{U}, \underbrace{|f_1\rangle, \dots, |f_{n-k}\rangle}_{U^\top}\}$$

cont'd the orthogonal projection is written by the matrix

$$P_U = \mathrm{diag}(\underbrace{1, 1, \dots, 1}_{k}, \underbrace{0, 0, \dots, 0}_{n-k})$$

Dirac's notation

- $|b\rangle = \text{ket} \in V$
- breaking the inner product up

$$\underset{\text{bracket}}{\langle a|b\rangle} \longleftrightarrow \underset{\text{bra}}{\langle a|} \quad \underset{\text{ket}}{|b\rangle}$$

- We can think of bras are maps from $V \to \mathbb{C}$
- Bra space is not V but a dual vector space V^*
- We can define a vector addition & scalar multiplication in V^*: $\langle w| = \alpha\langle \alpha| + \beta\langle b|$, $\langle a|, \langle b| \in V^*$ such that $\langle w|v\rangle = \alpha\langle \alpha|v\rangle + \beta\langle b|v\rangle$, $\forall v \in V$

$\Rightarrow V^*$ is a vector space.

☆ For any $|v\rangle \in V$, there is a unique $\langle v| \in V^*$

$$|v\rangle = \alpha_1|a_1\rangle + \alpha_2|a_2\rangle \xleftrightarrow[\text{dual}]{} \langle v| = \alpha_1^*\langle a_1| + \alpha_2^*\langle a_2|$$

☆ If we think about

$$|\alpha\rangle \Rightarrow \begin{pmatrix} \alpha_1 \\ \vdots \\ \vdots \\ \alpha_n \end{pmatrix} \qquad \langle \beta| \Rightarrow (\beta_1^* \; \beta_2^* \cdots \beta_n^*)$$

$$\therefore \langle \beta|\alpha\rangle = (\beta_1^* \; \beta_2^* \cdots \beta_n^*)\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \beta_1^*\alpha_1 + \beta_2^* a_2 + \dots + \beta_n^* a_n$$

Adjoint Operator

Consider a linear operator T acting on $|v\rangle$

$$|w\rangle = T|v\rangle \equiv |Tv\rangle$$

The dot product with $|u\rangle$ is then $\langle u|Tv\rangle$

The adjoint operator of T denoted as $T^\dagger$ is defined by the following property:

Let $T^\dagger|u\rangle \equiv |T^\dagger u\rangle$, then

$$\boxed{\langle T^\dagger u|v\rangle = \langle u|Tv\rangle}$$

☆ We can show immediately that $(TS)^\dagger = S^\dagger T^\dagger$

$$\langle u|TSv\rangle = \langle T^\dagger u|Sv\rangle = \langle S^\dagger T^\dagger u|v\rangle = \langle (TS)^\dagger u|v\rangle$$

☆ The adjoint of the adjoint is the original operator

$(S^\dagger)^\dagger = S$, since

$$\langle (S^\dagger)^\dagger u|v\rangle = \langle u|S^\dagger v\rangle = \langle S^\dagger v|u\rangle^* = \langle v|Su\rangle^* = \langle Su|v\rangle$$

Example For $|v\rangle \in \mathbb{C}^3$, an operator T acting on $|v\rangle$ is given by

$$T(v_1|e_1\rangle + v_2|e_2\rangle + v_3|e_3\rangle) = (2v_2 + iv_3)|e_1\rangle + (v_1 - iv_2)|e_2\rangle$$
$$+ (3iv_1 + v_2 + 7v_3)|e_3\rangle$$

Now, consider a vector $|u\rangle = u_1|e_1\rangle + u_2|e_2\rangle + u_3|e_3\rangle$

$$\langle u| = u_1^*\langle e_1| + u_2^*\langle e_2| + u_3^*\langle e_3|$$

$$\langle u|Tv\rangle = u_1^*(2v_2 + i v_3) + u_2^*(v_1 - i v_2) + u_3^*(3i v_1 + v_2 + 7v_3)$$

$$= (u_1^* + 3i u_3^*)v_1 + (2u_1^* - i u_2^* + u_3^*)v_2 + (i u_1^* + 7u_3^*)v_3$$

$$= \langle T^\dagger u | v\rangle$$

$$\therefore \; |T^\dagger u\rangle = T^\dagger|u\rangle = (u_1^* + 3i u_3^*)|e_1\rangle + (2u_1^* - i u_2^* + u_3^*)|e_2\rangle + (i u_1^* + 7u_3^*)|e_3\rangle$$

which gives the action of $T^\dagger$

☆ We can work in matrix representation now. Consider $T|v\rangle$

$$T\big(v_1|e_1\rangle + v_2|e_2\rangle + v_3|e_3\rangle\big) = v_1\big(T|e_1\rangle\big) + v_2\big(T|e_2\rangle\big) + v_3\big(T|e_3\rangle\big)$$

last page

$$= v_1\big(|e_2\rangle + 3i|e_3\rangle\big) + v_2\big(2|e_1\rangle - i|e_2\rangle + |e_3\rangle\big) + v_3\big(i|e_1\rangle + 7|e_3\rangle\big)$$

$\therefore$ $T_{11} = 0$, $T_{21} = 1$, $T_{31} = 3i$ and $T_{12} = 2$, $T_{22} = -i$, $T_{32} = 1$

& $T_{13} = i$, $T_{23} = 0$, $T_{33} = 7$

The matrix rep is

$$T = \begin{pmatrix} 0 & 2 & i \\ 1 & -i & 0 \\ 3i & 1 & 7 \end{pmatrix}$$

<u>Exercise</u> Find the matrix rep of $T^\dagger$ and conclude that in matrix rep

$$T^\dagger = (T^*)^T$$

☆☆ Note that this is true only if we use orthonormal basis

Hermitian Operator

- An operator T is said to be Hermitian if
$$T^\dagger = T$$

- Hermitian is very important in Physics since we measure real values & Hermitian operators have real eigenvalues!

☆ **Theorem** $T^\dagger = T \iff \langle v | Tv \rangle \in \mathbb{R}, \forall |v\rangle$

$\rightarrow$: $\langle v|Tv\rangle = \langle T^\dagger v | v\rangle = \langle Tv|v\rangle = \langle v|Tv\rangle^*$

$\leftarrow$: $\langle v|Tv\rangle = \langle Tv|v\rangle = \langle v|T^\dagger v\rangle$

$$\langle v|(T-T^\dagger)v\rangle = 0 \Rightarrow T = T^\dagger$$

☆ **Theorem** The eigenvalues of Hamiltonian operators are real

proof: Let $|v\rangle$ be a nonzero eigenvector of the Hermitian operator T with eigenvalue λ: $T|v\rangle = \lambda|v\rangle$, we have

$$\langle v|Tv\rangle = \langle v|\lambda v\rangle = \lambda\langle v|v\rangle$$

Since T is hermitian, we can write

$$\langle v|Tv\rangle = \langle Tv|v\rangle = \lambda^*\langle v|v\rangle$$

$$\therefore \quad (\lambda - \lambda^*)\langle v|v\rangle = 0$$

Since $|v\rangle$ is non-zero $\Rightarrow \lambda$ is real

☆ <u>Theorem</u> Different eigenvalues of a Hermitian operator correspond to orthogonal eigenvectors

Proof Let $|v_1\rangle$ & $|v_2\rangle$ be eigenvectors of T

$$T|v_1\rangle = \lambda_1 |v_1\rangle \quad \& \quad T|v_2\rangle = \lambda_2 |v_2\rangle$$

where λ_1 & λ_2 are different (and real)

Consider $\langle v_2 | T v_1 \rangle = \langle v_2 | \lambda_1 v_1 \rangle = \lambda_1 \langle v_2 | v_1 \rangle$

$\langle T v_2 | v_1 \rangle = \langle \lambda_2 v_2 | v_1 \rangle = \lambda_2^* \langle v_2 | v_1 \rangle$

Since $\langle v_2 | T v_1 \rangle = \langle T^\dagger v_2 | v_1 \rangle = \langle T v_2 | v_1 \rangle$

$$\Rightarrow \quad (\lambda_1 - \lambda_2) \langle v_2 | v_1 \rangle = 0$$

If we have $\lambda_1 \neq \lambda_2$, then $\langle v_2 | v_1 \rangle = 0$

<u>Unitary Operator</u>

An operator on a complex vector space is called Unitary if it does not change the length (magnitude) of the vector it acts on:

$$\big| U|v\rangle \big| = \big| |v\rangle \big| \quad , \quad \text{for all } |v\rangle \in V$$

or $\quad \langle Uv | Uv \rangle = \langle v | U^\dagger U v \rangle = \langle v | v \rangle$

$\therefore$ This is equal to

$$U^\dagger U = U U^\dagger = \mathbb{1}$$

Operator in bra-ket

- Notice that we can write $|a\rangle\langle b| = A$ and call this an operator, since it can act on a vector and becomes another vector:

$$A|u\rangle = |b\rangle\overbrace{\langle a|u\rangle}^{\text{inner product} \in \mathbb{C}} = (\langle a|u\rangle)|b\rangle$$

- We can act from the left as well

$$\langle v|A = \langle v|b\rangle\langle a|$$

- Now consider 2 vectors expanded in an orthonormal basis $\{|e_i\rangle\}$

$$|a\rangle = \sum_{i=1}^{n} a_i|e_i\rangle, \quad |b\rangle = \sum_{i=1}^{n} b_i|e_i\rangle$$

Assume that $|b\rangle$ is obtained by the operator Ω

$$\Omega|a\rangle = |b\rangle \longrightarrow \sum_{i=1}^{n} a_i(\Omega|e_i\rangle) = \sum_{i=1}^{n} b_i|e_i\rangle$$

Acting both sides with $\langle e_j|$

$$\sum_{i=1}^{n} a_i\langle e_j|\Omega|e_i\rangle = \sum_{i=1}^{n} b_i\overbrace{\langle e_j|e_i\rangle}^{\delta_{ij}} = b_j$$

Therefore, if we define

$$\boxed{\Omega_{ji} \equiv \langle e_j|\Omega|e_i\rangle} \Rightarrow \sum_{i=1}^{n} \Omega_{ji}a_i = b_j$$

we clearly see that this is a matrix rep

$$\boxed{\Omega_{ij} = \langle e_i|\Omega|e_j\rangle} = \begin{pmatrix} \Omega_{11} & \Omega_{12} & \cdots & \Omega_{1n} \\ \Omega_{21} & \Omega_{22} & & \\ \vdots & & \ddots & \\ \Omega_{n1} & & & \Omega_{nn} \end{pmatrix}$$

Therefore, the operator can be written as

$$\boxed{\Omega = \sum_{i=1}^{n}\sum_{j=1}^{n} |e_i\rangle \Omega_{ij} \langle e_j|}$$

Since we can act on $|a\rangle$ and get $|b\rangle$

$$\Omega|a\rangle = \sum_{i,j} |e_i\rangle \Omega_{ij} \langle e_j| \sum_k a_k |e_k\rangle$$

$$= \sum_{i,j,k} |e_i\rangle \Omega_{ij}\, a_k \delta_{jk}$$

$$= \sum_{i,j} |e_i\rangle \Omega_{ij}\, a_j = \sum_i |e_i\rangle b_i = |b\rangle$$

Projection Operator

Consider an orthonormal basis $\{|e_i\rangle\}$, choose an element $|m\rangle$ and form an operator P^m defined by

$$P^m = |e_m\rangle\langle e_m|$$

This operator maps any vector to a vector along $|m\rangle$

$$P^m|v\rangle = |e_m\rangle\langle e_m|v\rangle \propto |e_m\rangle$$

Now consider 2 basis vector $|e_m\rangle, |e_n\rangle$, the projection operator $P^{m,n} = |e_m\rangle\langle e_m| + |e_n\rangle\langle e_n|$

$$P^{m,n}|v\rangle = |e_m\rangle\langle e_m|v\rangle + |e_n\rangle\langle e_n|v\rangle$$

$P^{m,n}$ maps a vector to a subspace spanned by $\{|e_m\rangle, |e_n\rangle\}$

☆ We can write a matrix rep. by

$$\langle e_i|P^{m,n}|e_j\rangle = P^{m,n}_{ij} = \mathrm{diag}(0,0,\ldots,0,\overset{m}{1},0,\ldots,0,\overset{n}{1},0,\ldots 0)$$

• With the same principle, we can construct the projector to any subspace.

☆ If we include every basis vector, we get the identity

$$P \equiv |e_1\rangle\langle e_1| + |e_2\rangle\langle e_2| + \ldots + |e_n\rangle\langle e_n|$$

• The matrix rep. is simply $P = \mathrm{diag}(1,1,\ldots,1)$

☆ We write this and call the completeness relation

$$\boxed{\mathbb{1} = \sum_{i=1}^{n} |e_i\rangle\langle e_i|}$$

Adjoint Operator

• The adjoint of Ω is $\Omega^\dagger$ which is defined by

$$\langle \Omega^\dagger u|v\rangle = \langle u|\Omega v\rangle, \quad \text{Inner product on l.h.s.}$$

$$\langle v|\Omega^\dagger u\rangle^* = \langle u|\Omega v\rangle$$

$$\langle v|\Omega^\dagger|u\rangle = \langle u|\Omega|v\rangle^* \quad \text{c.c. on both side.}$$

• This is another definition of $\Omega^\dagger$

$$\boxed{\langle v|\Omega^\dagger|u\rangle = \langle u|\Omega|v\rangle^*}$$

• Since

$$\langle u|\Omega|v\rangle^* = \langle u|\Omega v\rangle^* = \langle \Omega v|u\rangle$$

- Taking ket out we have

$$\langle v|\Omega^\dagger = \langle \Omega v|$$

- This means that the dual correspondence is

$$\underset{\text{bra}}{\Omega|v\rangle} \longleftrightarrow \underset{\text{ket}}{\langle v|\Omega^\dagger}$$

<u>Exercise</u>

1). Show that $(\Omega_1\Omega_2)^\dagger = \Omega_2^\dagger\Omega_1^\dagger$

2). Show that if $\Omega = |a\rangle\langle b|$ then $\Omega^\dagger = |b\rangle\langle a|$

<u>Hermitian operator</u>

☆ An operator is hermitian if $\Omega^\dagger = \Omega$

" ————— " anti-hermitian if $\Omega^\dagger = -\Omega$

- Applying this in the Dirac notation, we get

$$\langle v|\Omega|u\rangle = \langle u|\Omega|v\rangle^*$$

- If we take $|u\rangle = |v\rangle$, we immediately see that $\langle v|\Omega|v\rangle$ is real if Ω is hermitian

- Sometimes we call $\langle v|\Omega|v\rangle$ the expectation value of an operator is state $|v\rangle$

<u>Exercise</u> For wavefunction $f(x) \in \mathbb{C}$ and

$$\langle f|g\rangle \equiv \int_{-\infty}^{\infty} f^*(x)\, g(x)\, dx$$

show that the operator $\Omega = \frac{\hbar}{i}\frac{d}{dx}$ is hermitian.

Non-denumerable basis

- Let's talk about position & momentum operaters
- We define $|x\rangle$ where the label is the value of the position
- This is an infinite vector space where the basis is

  $\{ |x\rangle \mid \forall x \in \mathbb{R} \}$

- Notice that scalar multiplication is not equal to changing

the position, i.e., $a|x\rangle \neq |ax\rangle$

- Also $|x_1 + x_2\rangle \neq |x_1\rangle + |x_2\rangle$
- The inner product is defined as

$$\langle x|y\rangle = \delta(x-y) = \text{Dirac delta function}$$

- The Dirac delta function has 2 properties:

1) $\displaystyle\int_{-\infty}^{\infty} \delta(x-x_0)\,dx = 1$

2) $\displaystyle\int_{-\infty}^{\infty} f(x)\delta(x-x_0)\,dx = f(x_0)$

- The norm of a position state is infinite! $\Rightarrow$ not physical

state $\langle x|x\rangle = \delta(0) = \infty$

- The completeness relation : $\mathbb{1} = \int dx\, |x\rangle\langle x|$
- This is consistent to the inner product since

$$|y\rangle = \int dx |x\rangle\langle x|y\rangle = \int dx |x\rangle\, \delta(x-y) = |y\rangle$$

- The position operator $\hat{x}$ is defined by its action

$$\hat{x}|x\rangle = x|x\rangle$$

- The $\hat{x}$ operator is hermitian:

$$\langle x_1|\hat{x}^\dagger|x_2\rangle = \langle x_2|\hat{x}|x_1\rangle^* = \left(x_1\,\delta(x_2-x_1)\right)^* = x_1\,\delta(x_2-x_1)$$

$$= x_2\,\delta(x_1-x_2) = \langle x_1|\hat{x}|x_2\rangle \qquad \hat{x}^\dagger = \hat{x}$$

- Given the state $|\psi\rangle$ of a particle, we define the position-stated wave function $\psi(x)$ by

$$\psi(x) \equiv \langle x|\psi\rangle \in \mathbb{C}$$

- We can write any state as a superposition of position eigenstates:

$$|\psi\rangle = \mathbb{1}|\psi\rangle = \int dx\,|x\rangle\langle x|\psi\rangle = \int dx\,|x\rangle\psi(x)$$

- We can interpret $\psi(x)$ as the component of $|\psi\rangle$ along the state $|x\rangle \Rightarrow \psi(x) =$ wave function in x-space
- The normalisation condition is

$$\langle\psi|\psi\rangle = 1 \Rightarrow \quad 1 = \int dx\,\langle\psi|x\rangle\langle x|\psi\rangle = \int dx\,\psi^*(x)\psi(x)$$

- The inner product can be written as

$$\langle\phi|\psi\rangle = \int dx\,\langle\phi|x\rangle\,\langle x|\psi\rangle = \int dx\,\phi^*(x)\psi(x)$$

- The matrix element is written as

$$\langle\phi|\hat{x}|\psi\rangle = \langle\phi|\hat{x}\,\mathbb{1}|\psi\rangle = \int dx\,\langle\phi|\hat{x}|x\rangle\langle x|\psi\rangle$$

$$= \int dx\,\langle\phi|x\rangle\,x\,\langle x|\psi\rangle = \int dx\,\phi^*(x)\,x\,\psi(x)$$

- Now, let's introduce the momentum states:

$$\text{Basis} = \{\,|p\rangle \mid \forall p \in \mathbb{R}\,\}, \quad \langle p|p'\rangle = \delta(p-p')$$

$$\mathbb{1} = \int dp\,|p\rangle\langle p| \quad \& \quad \hat{p}|p\rangle = p|p\rangle$$

Translation operator

- We define a translation operator by dx as

$$T(dx)|x\rangle = |x+dx\rangle$$

- Consider a state $|\psi\rangle$

$$T(dx)|\psi\rangle = T(dx)\int dx\, |x\rangle\langle x|\psi\rangle$$
$$= \int dx\, |x+dx\rangle\langle x|\psi\rangle$$
$$= \int dx'\, |x'\rangle\langle x'-dx|\psi\rangle$$

$$\langle x|\psi\rangle = \psi(x) \Rightarrow \langle x|T(dx)|\psi\rangle = \psi(x-dx)$$

- We know that

$$\langle \psi|T^{\dagger}(dx)T(dx)|\psi\rangle = \langle\psi|\psi\rangle \Rightarrow T^{\dagger}(dx)T(dx) = \mathbb{1}$$

$$T(-dx) = T^{-1}(dx), \quad \lim_{dx\to 0} T(dx) = \mathbb{1}$$

and $\quad T(dx_1)T(dx_2) = T(dx_1+dx_2)$

- Our guess would be

$$T(dx) = \mathbb{1} - i\underset{\text{operator}}{K}\,dx, \quad \text{where } K^{\dagger}=K$$

Exercise Show that this form satisfies all requirements

- Note that

$$\hat{x}\,T(dx)|x\rangle = \hat{x}|x+dx\rangle = (x+dx)|x+dx\rangle$$

$$T(dx)\hat{x}|x\rangle = T(dx)x|x\rangle = x|x+dx\rangle$$

$$\therefore [\hat{x}, T(dx)]|x\rangle = dx\,|x+dx\rangle$$

$$\therefore \boxed{[\hat{x}, T(dx)] = dx}$$

or $[\hat{x}, \mathbb{1} - iK\,dx] = \hat{x}(\mathbb{1} - iK\,dx) - (\mathbb{1} - iK\,dx)\hat{x}$

$$= -i(\hat{x}K - K\hat{x})\,dx = dx$$

$$\therefore \quad [\hat{x}, K] = i$$

This looks familiar! How about $K = \dfrac{\hat{p}}{\hbar}$?

$$[\hat{x}, \hat{p}] = i\hbar$$

☆ This means that momentum operator is the generator for space translation!

Momentum Operator in the Position Basis

· Consider momentum operator acting on $|\psi\rangle$

$$\left(\cancel{\mathbb{1}} - i\frac{\hat{p}\Delta x}{\hbar}\right)|\psi\rangle = T(\Delta x)\int dx\, |x\rangle\langle x|\psi\rangle$$

$$= \int dx\, |x+\Delta x\rangle \psi(x) = \int dx'\, |x'\rangle \psi(x'-\Delta x)$$

$$= \int dx'\, |x'\rangle \left(\psi(x') + (-\Delta x)\frac{\partial \psi(x')}{\partial x'}\right)$$

$$= \int dx'\, |x'\rangle \left(\langle x'|\psi\rangle - \Delta x \frac{\partial}{\partial x'}\langle x'|\psi\rangle\right)$$

$$= \cancel{|\psi\rangle} - \Delta x \int dx'\, |x'\rangle \frac{\partial}{\partial x'}\langle x'|\psi\rangle$$

$$\therefore \quad p|\psi\rangle = \int dx'\, |x'\rangle \left(-i\hbar \frac{\partial}{\partial x'}\langle x'|\psi\rangle\right)$$

$$\langle x|p|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\psi\rangle$$

$$\langle \phi|p|\psi\rangle = \int dx'\, \langle\phi|x'\rangle\left(-i\hbar\frac{\partial}{\partial x}\langle x'|\psi\rangle\right) = \int dx\, \phi^*(x)\left(-i\hbar\frac{\partial}{\partial x}\psi(x)\right)$$

Momentum-Space Wave Function

- We can define the momentum-space wave function

$$\langle p|\psi\rangle \equiv \psi(p)$$

- We will consider the transformation function $\langle x|p\rangle$

consider $$\langle x|\hat{p}|\psi\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|\psi\rangle$$

$$\langle x|\hat{p}|p\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|p\rangle$$

Since $\hat{p}|p\rangle = p|p\rangle$ we can write

$$p\langle x|p\rangle = -i\hbar\frac{\partial}{\partial x}\langle x|p\rangle$$

$\therefore$ The solution is

$$\langle x|p\rangle = Ne^{\frac{ipx}{\hbar}}$$

- We can find the normalisation factor by

$$\langle x|x'\rangle = \delta(x-x') = \int dp\,\langle x|p\rangle\langle p|x'\rangle$$

$$= |N|^2\int dp\, e^{ip(x-x')/\hbar}$$

Since the integral representation of the Dirac delta is

$$\delta(x-x') = \frac{1}{2\pi}\int dk\, e^{ik(x-x')}$$

$\therefore$ The normalisation is

$$|N|^2 = 1/(2\pi\hbar)$$

$\therefore$ The transformation function is

$$\boxed{\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}}$$

Uncertainties

- Let's talk about physics! We will say that if an electron (or other particles/stuff) describing by $|\psi\rangle$, a physical quantities associated with Hermitian operator H of $|\psi\rangle$ can be measured by

$$\langle H\rangle \equiv \langle\psi|H|\psi\rangle$$

- Clearly, if $|\psi\rangle$ is an eigenvector of H, this quantities is simply

$$\langle H\rangle = \langle\psi|H|\psi\rangle = \lambda\langle\psi|\psi\rangle = \lambda$$

- We don't have uncertainty here!
- What happen if $|\psi\rangle$ is a linear combination of eigenstates?

$$\begin{aligned}\langle H\rangle = \langle\psi|H|\psi\rangle &= \left(\langle h_1|a_1^* + \langle h_2|a_2^*\right)H\left(a_1|h_1\rangle + a_2|h_2\rangle\right)\\ &= a_1^*a_1\langle h_1|H|h_1\rangle + a_1^*a_2\langle h_1|H|h_2\rangle + a_2^*a_1\langle h_2|H|h_1\rangle\\ &\quad + a_2^*a_2\langle h_2|H|h_2\rangle\\ &= a_1^*a_1\lambda_1\overset{1}{\cancel{\langle h_1|h_1\rangle}} + a_1^*a_2\lambda_2\overset{0}{\cancel{\langle h_1|h_2\rangle}} + a_2^*a_1\lambda_1\overset{0}{\cancel{\langle h_2|h_1\rangle}}\\ &\quad + a_2^*a_2\lambda_2\overset{1}{\cancel{\langle h_2|h_2\rangle}}\\ &= |a_1|^2\lambda_1 + |a_2|^2\lambda_2\end{aligned}$$

- This is an average of λ_1 & λ_2 $\Rightarrow$ There is an uncertainty!
- We need a way to quantify this uncertainty.

- We define the uncertainty $\Delta H(\psi)$ of the Hermitian operator H on state $|\psi\rangle$

$$\Delta H(\psi) \equiv \left| (H - \langle H \rangle \mathbb{1}) |\psi\rangle \right|$$

- This is real & positive. When $\Delta H(\psi) = 0$, then $|\psi\rangle$ is an eigenvalue

$$\Delta H(\psi) = 0 \Rightarrow (H - \langle H \rangle \mathbb{1})|\psi\rangle = |0\rangle \rightarrow H|\psi\rangle = \langle H \rangle |\psi\rangle$$

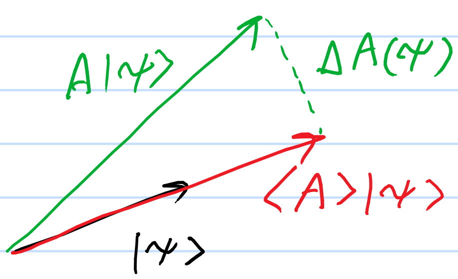
which means $H|\psi\rangle = \lambda|\psi\rangle$

- Computationally, it is easier to write

$$\begin{aligned}
(\Delta H(\psi))^2 &= \langle \psi | (H - \langle H \rangle \mathbb{1})^\dagger (H - \langle H \rangle \mathbb{1}) |\psi\rangle \\
&= \langle \psi | (H - \langle H \rangle \mathbb{1})(H - \langle H \rangle \mathbb{1}) |\psi\rangle \\
&= \langle \psi | (H^2 - 2H\langle H \rangle + \langle H \rangle^2 \mathbb{1}) |\psi\rangle \\
&= \underbrace{\langle \psi | H^2 |\psi\rangle}_{\equiv \langle H^2 \rangle} - 2\langle H \rangle \underbrace{\langle \psi | H | \psi \rangle}_{\langle H \rangle} + \langle H \rangle^2 \overset{1}{\cancel{\langle \psi | \psi \rangle}} \\
&= \langle H^2 \rangle - 2\langle H \rangle^2 + \langle H \rangle^2 \\
&= \langle H^2 \rangle - \langle H \rangle^2
\end{aligned}$$

Since $(\Delta H(\psi))^2 \geq 0 \Rightarrow \langle H^2 \rangle \geq \langle H \rangle^2$

☆ There is a useful geometrical picture.



The projector $P_\psi = |\psi\rangle\langle\psi|$ brings $A|\psi\rangle$ to $\langle A \rangle |\psi\rangle$:

$$P_\psi A|\psi\rangle = |\psi\rangle\langle\psi|A|\psi\rangle = \langle A \rangle |\psi\rangle$$

The Uncertainty Principle

Let's define $|f\rangle \equiv (A - \langle A \rangle \mathbb{1})|\psi\rangle$

$$|g\rangle \equiv (B - \langle B \rangle \mathbb{1})|\psi\rangle$$

- we can see that $\langle f|f\rangle = \Delta A(\psi)$, $\langle g|g\rangle = \Delta B(\psi)$

☆ From Schwarz inequality, we immediately have

$$\langle f|f\rangle\langle g|g\rangle \geq |\langle f|g\rangle|^2$$

Therefore we have

$$(\Delta A)^2(\Delta B)^2 \geq |\langle f|g\rangle|^2 = (\mathrm{Re}\langle f|g\rangle)^2 + (\mathrm{Im}\langle f|g\rangle)^2$$

Now consider the inner product

$$\langle f|g\rangle = \langle\psi|(A - \langle A\rangle\mathbb{1})(B - \langle B\rangle\mathbb{1})|\psi\rangle$$

$$= \langle\psi|\, AB - \langle A\rangle B - A\langle B\rangle + \langle A\rangle\langle B\rangle \,|\psi\rangle$$

$$= \langle\psi|AB|\psi\rangle - \langle A\rangle\langle B\rangle - \cancel{\langle A\rangle\langle B\rangle} + \cancel{\langle A\rangle\langle B\rangle}$$

$$= \langle\psi|AB|\psi\rangle - \langle A\rangle\langle B\rangle$$

Similarly, $\langle g|f\rangle = \langle\psi|BA|\psi\rangle - \langle B\rangle\langle A\rangle$

since, $\langle f|g\rangle = \langle g|f\rangle^*$, we can write

$$\mathrm{Im}(\langle f|g\rangle) = \frac{1}{2i}\left(\langle f|g\rangle - \langle f|g\rangle^*\right) = \frac{1}{2i}\left(\langle f|g\rangle - \langle g|f\rangle\right)$$

$$= \frac{1}{2i}\left(\langle\psi|AB|\psi\rangle - \langle\psi|BA|\psi\rangle\right) = \langle\psi|\frac{1}{2i}[A,B]|\psi\rangle$$

The real part is a little more messy, we have

$$\mathrm{Re}(\langle f|g\rangle) = \frac{1}{2}\left(\langle f|g\rangle + \langle f|g\rangle^*\right) = \frac{1}{2}\left(\langle f|g\rangle + \langle g|f\rangle\right)$$

$$Re(\langle f|g\rangle) = \frac{1}{2}\left(\langle\psi|AB|\psi\rangle - \langle A\rangle\langle B\rangle + \langle\psi|BA|\psi\rangle - \langle A\rangle\langle B\rangle\right)$$

$$= \frac{1}{2}\langle\psi|\{A,B\}|\psi\rangle - \langle A\rangle\langle B\rangle$$

$\therefore$ We have

$$(\Delta A)^2(\Delta B)^2 \geq \left(\langle\psi|\frac{1}{2i}[A,B]|\psi\rangle\right)^2 + \left(\langle\psi|\frac{1}{2}\{A,B\}|\psi\rangle - \langle A\rangle\langle B\rangle\right)^2$$

This is an inequality, so we can simply drop the last term

$$(\Delta A)^2(\Delta B)^2 \geq \left(\langle\psi|\frac{1}{2i}[A,B]|\psi\rangle\right)^2$$

$$\boxed{\Delta A\,\Delta B \geq \left|\langle\psi|\frac{1}{2i}[A,B]|\psi\rangle\right|}$$

This is the form that we are familiar with.

From $[\hat{x},\hat{p}] = i\hbar\mathbb{1}$, the uncertainty is simply

$$\Delta x\,\Delta p \geq \left|\langle\psi|\frac{1}{2i}(i\hbar\mathbb{1})|\psi\rangle\right|$$

$$\Delta x\,\Delta p \geq \frac{\hbar}{2}$$

Transformation Operator

- As we saw in the spin example earlier, the 2D complex vector space ($|\psi\rangle$) is spanned by $\{|z,+\rangle, |z,-\rangle\}$ or equally $\{|x,+\rangle, |x,-\rangle\}$

- We are interested in changing an orthonormal basis to another orthonormal basis $\{|e_i\rangle\} \to \{|f_i\rangle\}$

☆ We can construct the transformation operator relating 2 orthonormal basis by

$$\boxed{U = \sum_{k=1}^{n} |f_k\rangle\langle e_k|}$$

- It's obvious to see

$$U|e_i\rangle = \sum_{k=1}^{n} |f_k\rangle\langle e_k|e_i\rangle = \sum_{k=1}^{n} |f_k\rangle \delta_{ki} = |f_i\rangle$$

- This transformation matrix is unitary*

- Since the adjoint of U is written as

$$U^{+} = \sum_{k=1}^{n} |e_k\rangle\langle f_k|$$

- This is because $|x\rangle = U|y\rangle = \sum_{k=1}^{n} \langle f_k|y\rangle |e_k\rangle$ then

$$\langle x| = \sum_{k=1}^{n} \langle e_k| (\langle f_k|y\rangle)^{*} = \sum_{k=1} \langle e_k| \langle y|f_k\rangle$$

$$= \langle y| \sum_{k=1}^{n} |f_k\rangle\langle e_k| \equiv \langle y| U^{+}$$

- It is now easy to check that

$$UU^{+} = \left(\sum_{k} |f_k\rangle\langle e_k|\right)\left(\sum_{j} |e_j\rangle\langle f_j|\right) = \sum_{k,j} |f_k\rangle \underbrace{\langle e_k|e_j\rangle}_{\delta_{kj}} \langle f_j|$$

$$= \sum_{k=1}^{n} |f_k\rangle\langle f_k| \equiv \mathbb{1}$$

$$\& \quad U^{+}U = \left(\sum_{j} |e_j\rangle\langle f_j|\right)\left(\sum_{k} |f_k\rangle\langle e_k|\right) = \sum_{k,j} |e_j\rangle \underbrace{\langle f_j|f_k\rangle}_{\delta_{jk}} \langle e_k|$$

$$= \sum_{k=1}^{n} |e_k\rangle\langle e_k| = \mathbb{1}$$

- The matrix rep. of U in $\{|e_i\rangle\}$ basis can be written as

$$U_{ij} \equiv \langle e_i | U | e_j \rangle = \sum_k \langle e_i | f_k \rangle \underbrace{\langle e_k | e_j \rangle}_{\delta_{kj}}$$

$$= \langle e_i | f_j \rangle$$

- Also $U^\dagger_{ij} = \langle e_i | U^\dagger | e_j \rangle = \sum_k \langle e_i | e_k \rangle \langle f_k | e_j \rangle = \langle f_i | e_j \rangle$

The transformation rule for a vector can be found

The vector $|\alpha\rangle$ in $\{|e_i\rangle\}$ basis and $\{|f_i\rangle\}$ basis are

$$|\alpha\rangle = \sum_i |e_i\rangle\langle e_i|\alpha\rangle$$

$$|\alpha\rangle = \sum_i |f_i\rangle\langle f_i|\alpha\rangle$$

Consider only the components (column vector)

$$\langle f_j | \alpha \rangle = \sum_i \langle f_j | e_i \rangle \langle e_i | \alpha \rangle = \sum_i U^\dagger_{ji} \langle e_i | \alpha \rangle$$

which is $\begin{pmatrix} \alpha_1' \\ \alpha_2' \\ \vdots \\ \alpha_n' \end{pmatrix} = \begin{pmatrix} U^\dagger_{12} & \cdots & U^\dagger_{1n} \\ \vdots & \ddots & \\ U^\dagger_{n1} & & U^\dagger_{nn} \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} \Rightarrow \boxed{\alpha' = U^\dagger \alpha}$

- For an operator $\hat{X}$ the matrix rep.'s are related by

$$\langle f_i | \hat{X} | f_j \rangle = \sum_{k,m} \langle f_i | e_k \rangle \langle e_k | \hat{X} | e_m \rangle \langle e_m | f_j \rangle$$

$$= \sum_{k,m} U^\dagger_{ik} \langle e_k | \hat{X} | e_m \rangle U_{mj}$$

$\therefore$ Operator in different basis are related by

$$\boxed{\hat{X}' = U^\dagger \hat{X} U}$$

$\Rightarrow$ " Similarity transformation "

Diagonalisation

- When we have an operator, it can be useful to find a basis for which the operators are represented by matrices that take a simple form : diagonal matrices = all non-diagonal elements are zero.

☆ If we can find such a basis, we say that the operator is diagonalisable

☆ If an operator T is diagonal in basis $\{|v_i\rangle\}$ (orthonormal) its matrix takes the form $\mathrm{diag}(\lambda_1, \lambda_2, \dots, \lambda_n)$:

$$\langle v_i | T | v_j \rangle = \lambda_j \delta_{ij}$$
$$= \lambda_j \langle v_i | v_j \rangle$$

$\therefore$ we have $T|v_j\rangle = \lambda_j |v_j\rangle$

This is eigenvalue/vector equation!

☆ Now what happen if we work in a basis $\{|w\rangle\}$ and the matrix rep of T is not diagonal?

$\Rightarrow$ We find a basis $\{|v_i\rangle\}$ and change a basis!

$\Rightarrow$ This is a process of "diagonalisation"

☆ Let, $\{|w_i\rangle\}$ & $\{|v_i\rangle\}$ are 2 orthonormal basis

we are looking for the transformation $|v_i\rangle = U|w_i\rangle$

- The matrix rep of U is $\langle w_i | v_j \rangle = \langle w_i | U | w_j \rangle = U_{ij}$

Suppose that $\{|v_i\rangle\}$ is the basis that T is diagonal

$$T|v_i\rangle = \lambda_i |v_i\rangle$$

Rewrite this a bit, we have

$$\sum_j T|w_j\rangle\langle w_j|v_i\rangle = \lambda_i |v_i\rangle$$

$\times \langle w_k|$

$$\sum_j \underbrace{\langle w_k|T|w_j\rangle}_{T \text{ in } |w\rangle \text{ basis}} \underbrace{\langle w_j|v_i\rangle}_{U_{ji}} = \lambda_i \underbrace{\langle w_k|v_i\rangle}_{U_{ki}}$$

$$\sum_j (T)_{kj} U_{ji} = \lambda_i U_{ki}$$

- Consider $i=1$ for example, we can write

$$\begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \\ T_{n1} & & T_{nn} \end{pmatrix} \begin{pmatrix} U_{11} \\ U_{21} \\ \vdots \\ U_{n1} \end{pmatrix} = \lambda_1 \begin{pmatrix} U_{11} \\ U_{21} \\ \vdots \\ U_{n1} \end{pmatrix}$$

This is the eigenvalue equation $\begin{pmatrix} U_{11} \\ \vdots \\ U_{n1} \end{pmatrix}$ is eigenvector with eigenvalue λ_1

- Consider $i=2$ we can write

$$\begin{pmatrix} T_{11} & \cdots & T_{1n} \\ \vdots & \ddots & \\ T_{n1} & & T_{nn} \end{pmatrix} \begin{pmatrix} U_{12} \\ U_{22} \\ \vdots \\ U_{n2} \end{pmatrix} = \lambda_2 \begin{pmatrix} U_{12} \\ U_{22} \\ \vdots \\ U_{n2} \end{pmatrix}$$

☆ This means that we can find U_{ij} from the eigenvector equation!

☆ The process of diagonalisation = finding eigenvector/values

- Then we construct the transformation matrix by stacking normalised eigenvectors as

$$\begin{array}{c} \text{n^{th} eigenvector} \\ \begin{pmatrix} U_{11} & U_{22} & \cdots & U_{1n} \\ U_{21} & U_{22} & & U_{2n} \\ \vdots & \vdots & & \vdots \\ U_{n1} & U_{n2} & & U_{nn} \end{pmatrix} \\ \text{1st eigenvector} \quad \text{2nd eigenvector} \end{array}$$

- Then the similarity transformation relates T' & T

$$T^D_{ij} \equiv \langle v_i | T | v_j \rangle = \langle w_i | U^\dagger T U | w_j \rangle$$

$$= \sum_{k,m} \langle w_i | U^\dagger | w_k \rangle \underbrace{\langle w_k | T | w_m \rangle}_{T_{km}} \langle w_m | U | w_j \rangle$$

$$T^D_{ij} = \sum_{k,m} U^\dagger_{ik} T_{km} U_{mj}$$

$$T^D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} = U^\dagger T U$$

It follows immediately that

$$T|v_i\rangle = \lambda_i |v_i\rangle$$

$$TU^\dagger U |v_i\rangle = \lambda_i |v_i\rangle$$

$$UTU^\dagger |w_i\rangle = \lambda_i |w_i\rangle$$

T & $UTU^\dagger$ have exactly the same eigenvalues!

Compatible Operator

Operators A & B are called compatible when A & B commute:

$$[A, B] \equiv AB - BA = 0$$

and called incompatible when

$$[A, B] = AB - BA \neq 0$$

For example, $\hat{x}, \hat{p}$ are incompatible, $\hat{S}_x, \hat{S}_z$ are incomp

Exercise We know that $[S_i, S_j] = \sum_k i\varepsilon_{ijk}\hbar S_k$.

Check that $\hat{S}^2 \equiv \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2$ is compatible with S_x, S_y, S_z, i.e.,

$$[\hat{S}^2, \hat{S}_i] = 0$$

(Hermitian operators)

☆ Suppose that A and B are compatible observables, and the eigenvalues of A are nondegenerate (different). Then it is easy to show that $\langle a_i | B | a_j \rangle$ are all diagonal. ($|a_i\rangle$ are eigenvectors of A)

- Consider

$$\langle a_i | [A, B] | a_j \rangle = \langle a_i | AB - BA | a_j \rangle$$
$$= (a_i - a_j) \langle a_i | B | a_j \rangle$$
$$= 0$$

Therefore, if $i \neq j$ then $a_i \neq a_j$ (non degenerate)

then $\langle a_i | B | a_j \rangle = 0$

This means that $\langle a_i | B | a_j \rangle$ is diagonal

$$\langle a_i | B | a_j \rangle = \delta_{ij} \langle a_i | B | a_i \rangle$$

This implies directly that $|a_i\rangle$ is eigenvector of B

$$B|a_i\rangle = b_i |a_i\rangle$$

Since $\langle a_i | B | a_j \rangle = b_j \langle a_i | a_j \rangle = b_j \delta_{ij}$

$$\therefore \quad b_i = \langle a_i | B | a_i \rangle$$

b_i is eigenvalue corresponding to $|a_i\rangle$

☆ Compatible hermitian operators have simultaneous eigenvectors

- Even though we proved it in nondegenerated case the statement holds in degenerated case as well.
- Then the common eigenvectors can be found by finding a proper linear combination of degenerated eigenvectors.
- The example of angular momentum will be discussed later on in this course

A simultaneous eigenvectors of A & B are $|a_i, b_j\rangle$

$$A|a_i, b_j\rangle = a_i |a_i, b_j\rangle$$

$$B|a_i, b_j\rangle = b_j |a_i, b_j\rangle$$

where $\langle a_i, b_j | a_k, b_m \rangle = \delta_{ik} \delta_{jm}$

& $\mathbb{1} = \sum_{i,j} |a_i, b_j\rangle\langle a_i, b_j|$

Homework 2

2.1) Suppose that A, B and C are linear operators

a). Prove that

$$[A, BC] = [A, B]C + B[A, C]$$

b). Prove the Jacobi identity

$$\big[[A, B], C\big] + \big[[B, C], A\big] + \big[[C, A], B\big] = 0$$

c). Using the famous commutation $[\hat{x}, \hat{p}] = i\hbar$ show that

$$[\hat{x}^n, \hat{p}] = i\hbar n \hat{x}^{n-1}$$

d). For any function $f(\hat{x})$ that can be expanded in a power series in $\hat{x}$, show that

$$[f(\hat{x}), \hat{p}] = i\hbar f'(\hat{x})$$

2.2) Suppose that A & B are two linear operators that do not commute, $[A, B] \neq 0$

a). Let t be a variable, show that

$$\frac{d}{dt} e^{t(A+B)} = (A+B) e^{t(A+B)} = e^{t(A+B)} (A+B)$$

b). Suppose that $[A, B] = c\mathbb{1}$ where c is a complex number

Prove that $$e^A B e^{-A} = B + c$$

[Hint: Define $F(t) = e^{tA} B e^{-tA}$, find $F(0)$ and then $\frac{dF}{dt}$

Integrate the differential equation and take $F(1)$]

c) Let a be a real number and $\hat{p}$ be the momentum operator.

Show that the translation operator

$$\hat{T}(a) = e^{-\frac{ia\hat{p}}{\hbar}}$$

translates the position operator by similarity transformation:

$$\hat{T}^{+}(a)\,\hat{x}\,\hat{T}(a) = \hat{x} + a$$

2.3) Prove that for any two operators A & B, we have

$$e^{A}Be^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \ldots$$

[Hint: define $F(t) = e^{tA}Be^{-tA}$, then calculate $F'(0)$ $F''(0)$, $F'''(0)$ then combine them in Taylor expansions]

2.4) Consider two operators A & B such that $[A,B] = c\mathbb{1}$, where c is a complex number. We will prove the following

$$e^{(A+B)} = e^{B}e^{A}e^{c/2} = e^{A}e^{B}e^{c/2}$$

a) For this purpose consider the function

$$G(t) = e^{t(A+B)}e^{-tA}$$

Use the previous identities (2.1-2.3) to show that

$$G^{-1}(t)\frac{d}{dt}G(t) = B + ct$$

b). Note that this is equivalent to $\frac{d}{dt}G(t) = G(t)(B+ct)$

Verify that the solution is

$$G(t) = G(0)e^{tB}e^{\frac{1}{2}ct^{2}}$$

Then consider $G(1)$ to prove the first identity.

2.5) Consider a 3D vector space with orthonormal basis $|1\rangle, |2\rangle, |3\rangle$. Using complex constants a & b, we define kets

$$|\psi\rangle = a|1\rangle - b|2\rangle + a|3\rangle \quad \& \quad |\phi\rangle = b|1\rangle + a|2\rangle$$

a) Calculate $\langle\phi|\psi\rangle$ and $\langle\psi|\phi\rangle$, and check that $\langle\phi|\psi\rangle = \langle\psi|\phi\rangle^*$

b) Let $A = |\phi\rangle\langle\psi|$, find 3×3 matrix that represents A in $\{|1\rangle, |2\rangle, |3\rangle\}$ basis

c) Give a simple argument to show that A has a zero eigenvalue [Hint: consider $\det(U^\dagger A U) = \det(A)$ if U is unitary]

d) Let $Q = |\psi\rangle\langle\psi| + |\phi\rangle\langle\phi|$, is Q hermitian?

2.6) a) Show that the following matrices are unitary

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}, \quad \frac{1}{2}\begin{pmatrix} 1+i & 1-i \\ 1-i & 1+i \end{pmatrix}$$

b) The trace of a matrix is defined to be the sum of its diagonal matrix elements

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} A_{ii}$$

Show that

b.I) $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$

b.II) $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB)$

b.III) $\mathrm{Tr}(U^\dagger X U) = \mathrm{Tr}(X)$ (similarity transf.)

2.7) Let $R = |1\rangle\langle 1| + |3\rangle\langle 2| - |2\rangle\langle 3|$ where $\{|1\rangle, |2\rangle, |3\rangle\}$ are orthonormal basis. Find eigenvalues eigenvectors

2.8) Find eigenvalues / eigenvectors following matrices

$$\Omega = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad R = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

2.9) Considering the commutator of

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 2 \end{pmatrix}$$

to show that they can be simultaneously diagonalised

☆Then find the eigenvectors common to both operators and verify that under a unitary transformation to the basis, both A, B are diagonal

2.10) From the properties of matrix, $\det(U^\dagger X U) = \det(X)$ & $\mathrm{Tr}(U^\dagger X U) = \mathrm{Tr}(X)$, we can conclude that

$$\det(X) = \prod_{i=1}^{n} \lambda_i \quad \text{(product of all eigenvalues)}$$

$$\mathrm{Tr}(X) = \sum_{i=1}^{n} \lambda_i \quad \text{(sum of " ")}$$

In this problem we will check this with explicit calculation

Consider

$$T = \begin{pmatrix} 2 & i & 1 \\ -i & 2 & i \\ 1 & -i & 2 \end{pmatrix}$$

a). Calculate $\det(T)$ & $Tr(T)$

b). Find eigenvalues and check that they are consistent with det & Tr

c). Find eigenvectors and construct unitary matrix that diagonalise T. (Be careful with the degenerate sector, you might need to construct 2 orthogonal vectors in that sector first)

2.11) Show that $e^{-x^2/2}$ is an eigenvector of the operator

$$\hat{Q} = \frac{d^2}{dx^2} - x^2$$

and find its eigenvalue

2.12) If two matrices commute $[A, B] = 0$, and apply similarity transformation to both of them $A' = U^{\dagger} A U$, $B' = U^{\dagger} B U$, show that the resulting matrices also commute

$$[A', B'] = 0$$

2.13) Prove that diagonal matrices alway commute.

(Then it follows from 2.12) that 2 simultaneously diagonalisable matrices must commute)

Quantum Dynamics

☆ The state space of quantum mechanics "the Hilbert space" is best thought as a space with time-independent basis vectors.

☆ In Schrödinger picture of the quantum dynamics the state that represents a quantum system depends on time. We can write

$$|\psi, t\rangle = \sum_i c_i(t) |u_i\rangle ,$$

where the $c_i(t)$ are time-dependent function and $|u_i\rangle$ are basis vectors.

Unitary time evolution

- We propose that for any quantum system, there is a unitary operator $\mathcal{U}(t,t_0)$ mapping from state at time t_0 to the state at time t:

$$|\psi,t\rangle = \mathcal{U}(t,t_0)|\psi,t_0\rangle , \quad \forall t, t_0$$

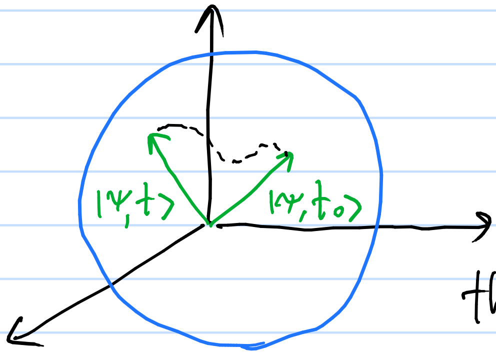
- Note that $\mathcal{U}$ can relate any pair of t_0 & t
- The operator $\mathcal{U}$ is unique.
- The unitary property means that

$$\mathcal{U}^\dagger(t,t_0)\mathcal{U}(t,t_0) = \mathcal{U}(t,t_0)\mathcal{U}^\dagger(t,t_0) = \mathbb{1}$$

- Unitarity implies that the norm is conserved

$$\langle\psi,t|\psi,t\rangle = \langle\psi,t_0|\mathcal{U}^\dagger(t,t_0)\mathcal{U}(t,t_0)|\psi,t_0\rangle = \langle\psi,t_0|\psi,t_0\rangle$$

The picture is that $\mathcal{U}$ operates as a rotation on a unit vector



Unit vector bec. $\langle\psi,t|\psi,t\rangle = 1$

$\Rightarrow$ $\mathcal{U}$ is called the time evolution operator

- For $t=t_0$, the unitary evolution operator becomes $\mathbb{1}$

$$|\psi,t_0\rangle = \mathcal{U}(t_0,t_0)|\psi,t_0\rangle \Rightarrow \mathcal{U}(t_0,t_0) = \mathbb{1}$$

- 2 successive evolutions = 1 evolution

$$|\psi,t_2\rangle = \mathcal{U}(t_2,t_1)|\psi,t_1\rangle = \mathcal{U}(t_2,t_1)\mathcal{U}(t_1,t_0)|\psi,t_0\rangle$$

$$|\psi,t_2\rangle \equiv \mathcal{U}(t_2,t_0)|\psi,t_0\rangle$$

$$\therefore \quad \mathcal{U}(t_2,t_0) = \mathcal{U}(t_2,t_1)\mathcal{U}(t_1,t_0)$$

- The inverse can be written by swapping $t \leftrightarrow t_0$

$$\mathcal{U}(t_0,t)\mathcal{U}(t,t_0) = \mathbb{1}$$

- We have

$$\mathcal{U}(t_0,t) = \mathcal{U}^{-1}(t,t_0) = \mathcal{U}^{\dagger}(t,t_0)$$

Schröding equation

- We can now derive the famous equation. This is in order to find a differential equation
- Take the time derivative, we get

$$\frac{\partial}{\partial t}|\psi,t\rangle = \left(\frac{\partial}{\partial t}\mathcal{U}(t,t_0)\right)|\psi,t_0\rangle$$

• Now we change R.H.S to $|\psi,t\rangle$:

$$\frac{\partial}{\partial t}|\psi,t\rangle = \left(\frac{\partial}{\partial t}\mathcal{U}(t,t_0)\right)\mathcal{U}(t_0,t)\,|\psi,t\rangle$$

$$= \left(\frac{\partial}{\partial t}\mathcal{U}(t,t_0)\right)\mathcal{U}^\dagger(t,t_0)\,|\psi,t\rangle$$

$$\frac{\partial}{\partial t}|\psi,t\rangle = \Lambda(t,t_0)\,|\psi,t\rangle$$

where we define $\quad \Lambda(t,t_0) \equiv \left(\frac{\partial \mathcal{U}(t,t_0)}{\partial t}\right)\mathcal{U}^\dagger(t,t_0)$

☆ $\Lambda(t,t_0)$ is anti-hermitian since

$$\mathcal{U}(t,t_0)\mathcal{U}^\dagger(t,t_0) = \mathbb{1}$$

$$\left(\frac{\partial \mathcal{U}(t,t_0)}{\partial t}\right)\mathcal{U}^\dagger(t,t_0) + \mathcal{U}(t,t_0)\frac{\partial}{\partial t}\mathcal{U}^\dagger(t,t_0) = 0$$

$$\Lambda(t,t_0) + \Lambda^\dagger(t,t_0) = 0$$

☆ $\Lambda(t,t_0)$ is independent of t_0. This should be the case since t_0 appears no where on the L.H.S

$$\Lambda(t,t_0) = \left(\frac{\partial}{\partial t}\mathcal{U}(t,t_0)\right)\mathcal{U}^\dagger(t,t_0)$$

$$= \left(\frac{\partial \mathcal{U}(t,t_0)}{\partial t}\right)\left(\mathcal{U}(t_0,t_1)\mathcal{U}^\dagger(t_0,t_1)\right)\mathcal{U}^\dagger(t,t_0)$$

$$= \frac{\partial}{\partial t}\left(\mathcal{U}(t,t_0)\mathcal{U}(t_0,t_1)\right)\mathcal{U}(t_1,t_0)\mathcal{U}(t_0,t)$$

$$= \frac{\partial}{\partial t}\left(\mathcal{U}(t,t_1)\right)\mathcal{U}(t_1,t)$$

$$= \frac{\partial}{\partial t}\left(\mathcal{U}(t,t_1)\right)\mathcal{U}^\dagger(t,t_1) = \Lambda(t,t_1)$$

- Since t_0, t_1 can be any value, we can conclude that $\Lambda(t,t_0) \equiv \Lambda(t)$ and the equation becomes

$$\frac{\partial}{\partial t}|\psi,t\rangle = \Lambda(t)|\psi,t\rangle$$

- We can define hermitian operator $\boxed{H(t) \equiv i\hbar\Lambda(t)}$

such that the equation above becomes "the Schrödinger equation"

$$\boxed{i\hbar\frac{\partial}{\partial t}|\psi,t\rangle = H(t)|\psi,t\rangle}$$

- This is our result! Unitary time evolution implies this equation!

☆ Note that $[\hbar] = J\cdot s \Rightarrow [H] = [\hbar][\Lambda] = J\cdot s \times \frac{1}{s} = J$

H has an energy unit ⇒ we call this Hamiltonian

- To see the meaning of the Hamiltonian operator more clearly, we consider a time derivative of the average of an observable Q (with respect to $|\psi\rangle$)

$$\frac{d}{dt}\langle Q\rangle = \frac{d}{dt}\left(\langle\psi|Q|\psi\rangle\right)$$

$$= \frac{d(\langle\psi|)}{dt}Q|\psi\rangle + \langle\psi|\overset{0}{\cancel{\frac{dQ}{dt}}}|\psi\rangle + \langle\psi|Q\frac{d}{dt}|\psi\rangle$$

$$= -\frac{1}{i\hbar}\langle\psi|HQ|\psi\rangle + \langle\psi|Q\cdot\frac{1}{i\hbar}H|\psi\rangle$$

$$= \frac{1}{i\hbar}\langle\psi|[Q,H]|\psi\rangle = \frac{1}{i\hbar}\langle[Q,H]\rangle$$

This is just a natural generalisation of Hamiltonian mechanics

In classical mechanics, we have Poisson brackets

$$\{A, B\}_{PB} \equiv \frac{\partial A}{\partial x}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial x}$$

Then it turns out that for any observable function $Q(x,p)$ its time derivative is given by

$$\frac{dQ}{dt} = \{Q, H\}_{PB}$$

- We can see their similarity by

$$i\hbar[A, B] \Longleftrightarrow \{A, B\}_{PB}$$

which is consistent with

$$[x, p] = i\hbar \Longleftrightarrow \{x, p\}_{PB} = 1$$

☆ We can also write the Schrödinger equation in a more familiar form (wave function).

- The wave function $\Psi(x,t)$ can be obtained from

$$\langle x|\Psi,t\rangle = \Psi(x,t)$$

- We have $\quad i\hbar\frac{\partial}{\partial t}\langle x|\Psi,t\rangle = \langle x|H(t)|\Psi,t\rangle$
- If we use $\quad H(t) = \frac{\hat{p}^2}{2m} + V(\hat{x})$

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\Psi(x,t) &= \frac{1}{2m}\langle x|\hat{p}^2|\Psi,t\rangle + \langle x|V(\hat{x})|\Psi,t\rangle \\ &= \frac{1}{2m}(-i\hbar)^2\frac{\partial^2}{\partial x^2}\Psi(x,t) + V(x)\Psi(x,t) \\ &= \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right)\Psi(x,t) \end{aligned}$$

Calculation of the unitary time evolution operator

☆ In QM problems, it's relatively easy to obtain the Hamiltonian, after all this is just classical analog

☆ The solution of this Hamiltonian is however much harder to obtain.

☆ We start by calculation of $\mathcal{U}$.

- From
$$H(t) = i\hbar\left(\frac{\partial}{\partial t}\mathcal{U}(t,t_0)\right)\mathcal{U}^{\dagger}(t,t_0)$$

- We can write
$$\frac{\partial \mathcal{U}(t,t_0)}{\partial t} = -\frac{i}{\hbar} H(t)\,\mathcal{U}(t,t_0)$$

- Since there is no confusion on $\partial/\partial t$ or d/dt, we get
$$\boxed{\frac{d\mathcal{U}(t)}{dt} = -\frac{i}{\hbar} H(t)\,\mathcal{U}(t)}$$

- Solving the differential equation for operators (matrices) is quite difficult. We will seperate in 3 cases:

Case I : H is time independent

- Then the equation becomes $\frac{d}{dt}\mathcal{U}(t) = -\frac{i}{\hbar}H\,\mathcal{U}(t) \equiv K\,\mathcal{U}(t)$

- The solution is
$$\mathcal{U}(t) = e^{tK}\,\mathcal{U}(0)$$

- We can check by take a derivative, since
$$e^{tK} = 1 + tK + \frac{1}{2!}(tK)^2 + \frac{1}{3!}(tK)^3 + \ldots = \sum_{n=0}^{\infty} \frac{1}{n!} t^n K^n$$

$$\frac{d}{dt} e^{tk} = k + tk^2 + \frac{t^2}{2!}k^3 + \dots = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} k^n = \sum_{n=0}^{\infty} \frac{t^n}{n!} k^{n+1}$$

$$= k\left(1 + tk + \frac{(tk)^2}{2!} + \dots\right)$$

$$= k e^{tk}$$

$$\therefore \frac{d\mathcal{U}(t)}{dt} = \frac{d}{dt}(e^{tk})\mathcal{U}(0) = k e^{tk}\mathcal{U}(0) = k\,\mathcal{U}(t)$$

The solution is $\mathcal{U}(t,t_0) = e^{-\frac{i}{\hbar}Ht}\,\mathcal{U}_0$

Since $\mathcal{U}(t_0,t_0) = \mathbb{1} \Rightarrow e^{\frac{i}{\hbar}Ht_0}\mathcal{U}_0 = \mathbb{1}$

$$\therefore \mathcal{U}_0 = e^{-\frac{i}{\hbar}Ht_0}$$

Therefore the full solution is

$$\boxed{\mathcal{U}(t,t_0) = e^{-\frac{i}{\hbar}H(t-t_0)}}$$

Case II: H is time dependent and $[H(t_1), H(t_2)] = 0$ for all t_1, t_2

We first claim that the time evolution operator is given by

$$\boxed{\mathcal{U}(t,t_0) = e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt' H(t')}}$$

- To show that this is the solution in this case, consider

$$R(t) \equiv -\frac{i}{\hbar}\int_{t_0}^{t} dt' H(t') \quad \text{such that} \quad \frac{dR}{dt} = R' = -\frac{i}{\hbar}H(t)$$

- Notice that $R(t)$ & $R'(t)$ commute in this case:

$$[R'(t), R(t)] = \left[-\frac{i}{\hbar}H(t), -\frac{i}{\hbar}\int_{t_0}^{t} dt' H(t')\right] = \left(-\frac{i}{\hbar}\right)^2 \int_{t_0}^{t} dt' [H(t), H(t')] = 0$$

• The solution is

$$\mathcal{U}(t) = e^{R(t)} = \mathbb{1} + R(t) + \frac{1}{2!}R(t)R(t) + \frac{1}{3!}R(t)R(t)R(t) + \dots$$

The time derivative is

$$\frac{d\mathcal{U}}{dt} = \frac{d}{dt}e^{R(t)} = R' + \frac{1}{2!}(R'R + RR') + \frac{1}{3!}(R'RR + RR'R + RRR') + \dots$$

since R commutes with R', we can write

$$\frac{d\mathcal{U}}{dt} = R' + \frac{1}{2}\cdot 2R'R + \frac{1}{3!}\cdot 3R'RR + \dots$$
$$= R'\left(\mathbb{1} + R + \frac{1}{2!}R^2 + \dots\right)$$
$$= R'e^R = \frac{-i}{\hbar}H(t)\mathcal{U} \quad \text{which is what we want.}$$

Case III H is time dependent & $[H(t_1), H(t_2)] \neq 0$ for all t_1, t_2

This is the most general solution and we might not need it for this course. We will simply show the solution:

$$\mathcal{U}(t,t_0) = T\left[e^{-\frac{i}{\hbar}\int_{t_0}^{t} dt' H(t')}\right] \equiv \mathbb{1} + \left(\frac{-i}{\hbar}\right)\int_{t_0}^{t} dt_1 H(t_1)$$
$$+ \left(\frac{-i}{\hbar}\right)^2 \int_{t_0}^{t} dt_1 H(t_1) \int_{t_0}^{t_1} dt_2 H(t_2)$$
$$+ \left(\frac{-i}{\hbar}\right)^3 \int_{t_0}^{t} dt_1 H(t_1) \int_{t_0}^{t_1} dt_2 H(t_2) \int_{t_0}^{t_2} dt_3 H(t_3)$$

where T is the time-ordering operator defined as above

• This formula is sometimes called Dyson formula which is useful in Quantum Field theory

Heisenberg Picture

- In previous section, we treat $|\psi, t\rangle$ as time dependent this approach (state as time dependent) is called Schrödinger picture

- We can transfer time dependency to operators

$\Rightarrow$ This approach is known as Heisenberg picture

Heisenberg operators

- Consider a Schrödinger operator $\hat{A}_S$. We consider a matrix rep. of $\hat{A}_S$, specifically an element between time dependent states $|\alpha, t\rangle$ and $|\beta, t\rangle$:

$$\langle \alpha, t | \hat{A}_S | \beta, t\rangle = \langle \alpha, 0 | \mathcal{U}^\dagger(t,0) \hat{A}_S \mathcal{U}(t,0) | \beta, 0\rangle$$

- We simply define the Heisenberg operator $\hat{A}_H(t)$ as the operator between the time equal zero states

$$\boxed{\hat{A}_H(t) \equiv \mathcal{U}^\dagger(t,0) \hat{A}_S \mathcal{U}(t,0)}$$

- It is obvious to see that at $t=0$

$$\hat{A}_H(0) = \hat{A}_S$$

- The identity operator is the same in both picture

$$\mathbb{1}_H = \mathcal{U}^\dagger(t,0)\, \mathbb{1}\, \mathcal{U}(t,0) = \mathbb{1}$$

- The product of Heisenberg operators is the Heisenberg operator of the product of associated Schrödinger operators

$$\hat{C}_H(t) = \hat{A}_H(t)\hat{B}_H(t) = \mathcal{U}^\dagger(t,0)\hat{A}_S\underbrace{\mathcal{U}(t,0)\mathcal{U}^\dagger(t,0)}_{\mathbb{1}}\hat{B}_S\mathcal{U}(t,0)$$

$$= \mathcal{U}^\dagger(t,0)\hat{A}_S\hat{B}_S\mathcal{U}(t,0)$$

$$\equiv \mathcal{U}^\dagger(t,0)\hat{C}_S\mathcal{U}(t,0)$$

- It follows that

$$[\hat{A}_S, \hat{B}_S] = \hat{C}_S \implies [\hat{A}_H, \hat{B}_H] = \hat{C}_H$$

Therefore, the uncertainty relation is similar in both pictures

$$[\hat{x}, \hat{p}] = i\hbar\mathbb{1} \implies [\hat{x}_H, \hat{p}_H] = i\hbar\mathbb{1}$$

☆ Previous properties imply that replacing operators $\hat{x}, \hat{p}$ inside the Schrödinger Hamiltonian

$$\hat{H}_S(\hat{x}, \hat{p}, t) \Longrightarrow \hat{H}_S(\hat{x}_H, \hat{p}_H, t)$$

is the Heisenberg Hamiltonian

$$\hat{H}_S(\hat{x}_H, \hat{p}_H, t) = \hat{H}_H(t)$$

☆ In the case II, where $[\hat{H}_S(t), \hat{H}_S(t')] = 0$ for all t, t'

we have

$$\mathcal{U}(t,0) = e^{-\frac{i}{\hbar}\int_0^t dt' \hat{H}(t')}$$

This means that

$$[\hat{H}_S(t), \mathcal{U}(t,0)] = [\hat{H}_S(t), \mathcal{U}^\dagger(t,0)] = 0$$

$$\therefore \quad \hat{H}_H(t) = \mathcal{U}^\dagger(t,0)\hat{H}_S\mathcal{U}(t,0) = \hat{H}_S(t)$$

Use previous result, we have

$$\hat{H}_S(\hat{x}_H(t), \hat{p}_H(t), t) = \hat{H}_S(\hat{x}, \hat{p}, t)$$

☆ If $[\hat{A}_s, \hat{H}_s(t)] = 0$ at all times, then $[\hat{A}_s, \mathcal{U}(t,0)] = 0$

The consequence is that

$$\hat{A}_H = \mathcal{U}^\dagger(t,0)\hat{A}_s\mathcal{U}(t,0) = \hat{A}_s$$

☆ It follows from the definition of the Heisenberg operator that

$$\langle \psi, t|\hat{A}_s|\psi, t\rangle = \langle \psi, 0|\hat{A}_H|\psi, 0\rangle$$

Heisenberg equation of motion

- The equation for Heisenberg turns out to be similar to the equations of motion for associated classical variables

- Consider the Schrödinger equation & its conjugate

$$i\hbar\frac{\partial}{\partial t}\mathcal{U}(t,t_0) = H_s(t)\mathcal{U}(t,t_0)$$

$$-i\hbar\frac{\partial}{\partial t}\mathcal{U}^\dagger(t,t_0) = \mathcal{U}^\dagger(t,t_0)H_s(t)$$

- Now we can calculate

$$i\hbar\frac{d}{dt}\hat{A}_H(t) = \overbrace{\left(i\hbar\frac{\partial}{\partial t}\mathcal{U}^\dagger(t,0)\right)}^{-\mathcal{U}^\dagger(t,t_0)H_s(t)}\hat{A}_s(t)\,\mathcal{U}(t,0)$$

$$+\ \mathcal{U}^\dagger(t,0)\left(i\hbar\frac{\partial}{\partial t}\hat{A}_s(t)\right)\mathcal{U}(t,0)$$

$$+\ \mathcal{U}^\dagger(t,0)\,\hat{A}_s(t)\overbrace{\left(i\hbar\frac{\partial}{\partial t}\mathcal{U}(t,0)\right)}^{H_s(t)\mathcal{U}(t,t_0)}$$

$$
\begin{aligned}
i\hbar \frac{d}{\partial t}\hat{A}_H(t) &= -\mathcal{U}^\dagger(t,0)\hat{H}_s(t)\hat{A}_s(t)\mathcal{U}(t,0) \\
&\quad + \mathcal{U}^\dagger(t,0)\, i\hbar \frac{\partial}{\partial t}\hat{A}_s(t)\,\mathcal{U}(t,0) \\
&\quad + \mathcal{U}^\dagger(t,0)\,\hat{A}_s(t) H_s(t)\,\mathcal{U}(t,0) \\
&= -\hat{H}_H(t)\hat{A}_H(t) + \hat{A}_H(t)\hat{H}_H(t) \\
&\quad + i\hbar\left(\frac{\partial \hat{A}_s(t)}{\partial t}\right)_H
\end{aligned}
$$

$$
\boxed{i\hbar \frac{\partial \hat{A}_H}{\partial t} = [\hat{A}_H(t), \hat{H}_H(t)] + i\hbar\left(\frac{\partial \hat{A}_s(t)}{\partial t}\right)_H}
$$

☆ This is called Heisenberg equation of motion

☆ Note that if $\hat{A}_s$ has no time dependence, the above equation becomes simpler

$$
i\hbar \frac{\partial \hat{A}_H}{\partial t} = [\hat{A}_H, \hat{H}_H]
$$

☆ Let A_s be an operator without time dependence

The time derivative of the expectation value is

$$
\begin{aligned}
i\hbar\frac{\partial}{\partial t}\langle\psi,t|\hat{A}_s|\psi,t\rangle &= i\hbar\frac{\partial}{\partial t}\langle\psi,0|\hat{A}_H(t)|\psi,0\rangle \\
&= \langle\psi,0|\, i\hbar\frac{\partial}{\partial t}\hat{A}_H(t)\,|\psi,0\rangle \\
&= \langle\psi,0|[\hat{A}_H,\hat{H}_H]|\psi,0\rangle \\
&= \langle\psi,t|[\hat{A}_s,\hat{H}_s]|\psi,t\rangle
\end{aligned}
$$

$$
\therefore\ i\hbar\frac{\partial}{\partial t}\langle\hat{A}_s\rangle = i\hbar\frac{\partial}{\partial t}\langle\hat{A}_H(t)\rangle = \langle[\hat{A}_H,\hat{H}_H]\rangle = \langle[\hat{A}_s,\hat{H}_s]\rangle
$$

☆ The operator $\hat{A}_s$ is said to be conserved if it commutes with the Hamiltonian:

$$[\hat{A}_s, \hat{H}_s] = 0 \Rightarrow \hat{A}_s \text{ is conserved.}$$

- This is obvious from previous relations

$$i\hbar \frac{d}{dt}\hat{A}_H = [\hat{A}_H, \hat{H}_H] = [\hat{A}_s, \hat{H}_s] = 0$$

- The Heisenberg operator is constant.
- The expectation value is also constant.

Quantum Harmonic Oscillator

- Harmonic Oscillator is the system containing a lot of details in physics
- In fact, we will learn a lot just like simple harmonic oscillation has taught us in classical mechanics
- We will approach this by an elegant operator method
- Just like the classical mechanics, we write the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{m\omega^2\hat{x}^2}{2}$$

- It might be more familiar if we write

$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$ and substitute $p = mv$, $\omega = \sqrt{\frac{k}{m}}$

- The operator $\hat{x}$ & $\hat{p}$ are Hermitian
- We would like to deal with dimension first.

since $[\hbar\omega] = J\cdot s\cdot \frac{1}{s} = J$, we will factor this out

$$\hat{H} = \hbar\omega\left(\frac{m\omega}{2\hbar}\hat{x}^2 + \frac{1}{2m\hbar\omega}\hat{p}^2\right)$$

- Let's define the scale $d_0^2 \equiv \hbar/m\omega$ which has a unit of distance $[d_0^2] = [\hbar]/[m][\omega] = J\cdot s\cdot\frac{1}{kg}\cdot s$

$[d_0^2] = kg\, m^2/s^2\cdot s\cdot 1/kg\cdot s = m^2$

- We can write

$$\hat{H} = \hbar\omega\left(\frac{1}{2d_0^2}\hat{x}^2 + \frac{1}{2d_0^2m^2\omega^2}\hat{p}^2\right)$$

$$\hat{H} = \hbar\omega\left(\frac{1}{2d_0^2}\right)\left(\hat{X}^2 + \frac{1}{m^2\omega^2}\hat{P}^2\right)$$

- It is extremely useful to define two non-Hermitian operators which factor the Hamiltonian nicely

$$\hat{a} = \frac{1}{\sqrt{2}d_0}\left(\hat{X} + \frac{i\hat{P}}{m\omega}\right), \quad \hat{a}^+ = \frac{1}{\sqrt{2}d_0}\left(\hat{X} - \frac{i\hat{P}}{m\omega}\right)$$

- Before we show factorisation, let's consider

$$[\hat{a}, \hat{a}^+] = \frac{1}{2d_0^2}\left[\hat{X} + \frac{i\hat{P}}{m\omega}, \hat{X} - \frac{i\hat{P}}{m\omega}\right]$$

$$= \frac{1}{2d_0^2}\left(\cancel{[\hat{X},\hat{X}]}^{\,0} + \frac{i}{m\omega}[\hat{P},\hat{X}] - \frac{i}{m\omega}[\hat{X},\hat{P}] + \frac{1}{m^2\omega^2}\cancel{[\hat{P},\hat{P}]}^{\,0}\right)$$

$$= \frac{-\cancel{2}i}{\cancel{2}d_0^2} \times \frac{1}{m\omega}[\hat{X},\hat{P}] = \frac{-i}{d_0^2 m\omega}(i\hbar) = 1$$

- It is easy to see that

$$\hat{a}^+\hat{a} = \frac{1}{2d_0^2}\left(\hat{X} - \frac{i\hat{P}}{m\omega}\right)\left(\hat{X} + \frac{i\hat{P}}{m\omega}\right)$$

$$= \frac{1}{2d_0^2}\left(\hat{X}\hat{X} - \frac{i\hat{P}\hat{X}}{m\omega} + \frac{i\hat{X}\hat{P}}{m\omega} + \frac{\hat{P}\hat{P}}{m^2\omega^2}\right)$$

$$= \frac{1}{2d_0^2}\left(\hat{X}^2 + \frac{1}{m^2\omega^2}\hat{P}\hat{P} + \frac{i[\hat{X},\hat{P}]}{m\omega}\right)$$

$$= \frac{1}{2d_0^2}\left(\hat{X}^2 + \frac{1}{m^2\omega^2}\hat{P}^2\right) + \frac{1}{2d_0^2}\frac{(i)(i\hbar)}{m\omega}$$

$$= \frac{\hat{H}}{\hbar\omega} - \frac{1}{2}$$

Therefore, we can write the Hamiltonian as

$$\hat{H} = \hbar\omega\left(\hat{a}^{+}\hat{a} + \frac{1}{2}\right)$$

where we define the annihilation & creation operator

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} + \frac{i\hat{p}}{m\omega}\right), \quad \hat{a}^{+} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{x} - \frac{i\hat{p}}{m\omega}\right)$$

where $$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a} + \hat{a}^{+}\right), \quad \hat{p} = i\sqrt{\frac{\hbar m\omega}{2}}\left(\hat{a}^{+} - \hat{a}\right)$$

- The reason for their name will become clear soon.
- Let's define the number operator

$$\hat{N} \equiv \hat{a}^{+}\hat{a}$$

$$\therefore \quad \hat{H} = \hbar\omega\left(\hat{N} + \frac{1}{2}\right) = \hbar\omega\left(\hat{N} + \frac{1}{2}\mathbb{1}\right)$$

- Since $\hat{H}$ is a linear function of $\hat{N}$, $\hat{N}$ and $\hat{H}$ can be simultaneously diagonalised.
- $\hat{H}$ & $\hat{N}$ share the same eigenvectors!
- Suppose $|n\rangle$ is an eigenvector of $\hat{N}$:

$$\hat{N}|n\rangle = n|n\rangle$$

- Therefore, we have

$$\hat{H}|n\rangle = \hbar\omega\left(n + \frac{1}{2}\right)|n\rangle$$

which means that the energy eigenvalues are

$$\hat{H}|n\rangle = E_n|n\rangle \Rightarrow E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

- The introduction of $\hat{a}, \hat{a}^+, \hat{N}$ has a deeper meaning
- Consider the commutator

$$[\hat{N}, \hat{a}] = [\hat{a}^+\hat{a}, \hat{a}] = \hat{a}^+\overset{0}{\cancel{[\hat{a}, \hat{a}]}} + \overset{-1}{\cancel{[\hat{a}^+, \hat{a}]}}\hat{a}$$
$$= -\hat{a}$$

$$[\hat{N}, \hat{a}^+] = [\hat{a}^+\hat{a}, \hat{a}^+] = \hat{a}^+\overset{1}{\cancel{[\hat{a}, \hat{a}^+]}} + \overset{0}{\cancel{[\hat{a}^+, \hat{a}^+]}}\hat{a}$$
$$= \hat{a}^+$$

- We then have

$$\hat{N}\hat{a}^+|n\rangle = \left([\hat{N}, \hat{a}^+] + \hat{a}^+\hat{N}\right)|n\rangle$$
$$= \left(\hat{a}^+ + \hat{a}^+\hat{N}\right)|n\rangle$$
$$= (n+1)\hat{a}^+|n\rangle$$

which implies that $\hat{a}^+|n\rangle$ is proportional to $|n+1\rangle$ (since it has eigenvalue $= n+1$)

$$\hat{a}^+|n\rangle = c|n+1\rangle$$

- Similarly

$$\hat{N}\hat{a}|n\rangle = \left([\hat{N}, \hat{a}] + \hat{a}\hat{N}\right)|n\rangle$$
$$= \left(-\hat{a} + \hat{a}\hat{N}\right)|n\rangle$$
$$= (n-1)\hat{a}|n\rangle$$

which implies that ($\hat{a}|n\rangle$ has eigenvalue $= n-1$)

$$\hat{a}|n\rangle = c'|n-1\rangle$$

- The coefficients c & c' can be evaluated by normalisation condition:

$$\langle n|\hat{a}^{\dagger}\hat{a}|n\rangle = (c')^2 \langle n-1|n-1\rangle = (c')^2$$
$$= \langle n|\hat{N}|n\rangle = n$$
$$\therefore \quad c' = \sqrt{n}$$
$$\Rightarrow \quad \hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

Similarly

$$\langle n|\hat{a}\hat{a}^{\dagger}|n\rangle = c^2\langle n+1|n+1\rangle = c^2$$
$$= \langle n|[\hat{a},\hat{a}^{\dagger}] + \hat{a}^{\dagger}\hat{a}|n\rangle$$
$$= \langle n|n\rangle + \langle n|\hat{N}|n\rangle$$
$$= n+1$$
$$\therefore \quad C = \sqrt{n+1}$$
$$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle$$

- Therefore, the picture is that all eigenvectors span the energy ladder.
- Starting from $|n\rangle$, we can construct the rest by applying $\hat{a}$, $\hat{a}^{\dagger}$ successively:

$$\vdots$$
$$E_n = \hbar\omega(n+\tfrac{5}{2}) \quad \text{——} \quad |n+2\rangle = \frac{1}{\sqrt{n+2}}\hat{a}^{\dagger}|n+1\rangle = \frac{1}{\sqrt{(n+2)(n+1)}}\hat{a}^{\dagger}\hat{a}^{\dagger}|n\rangle$$
$$E_n = \hbar\omega(n+\tfrac{3}{2}) \quad \text{——} \quad |n+1\rangle = \frac{1}{\sqrt{n+1}}\hat{a}^{\dagger}|n\rangle$$
$$E_n = \hbar\omega(n+\tfrac{1}{2}) \quad \text{——} \quad |n\rangle$$
$$E_n = \hbar\omega(n-\tfrac{1}{2}) \quad \text{——} \quad |n-1\rangle = \frac{1}{\sqrt{n}}\hat{a}|n\rangle$$
$$E_n = \hbar\omega(n-\tfrac{1}{2}) \quad \text{——} \quad |n-2\rangle = \frac{1}{\sqrt{n-1}}\hat{a}|n-1\rangle = \frac{1}{\sqrt{n(n-1)}}\hat{a}\hat{a}|n\rangle$$
$$\vdots$$

- Can we use $\hat{a}$ to go down to $-\infty$?
- Suppose we start with $|n\rangle$ where n is an integer.

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle$$

$$\hat{a}^2|n\rangle = \sqrt{n(n-1)}\,|n-2\rangle$$

$$\hat{a}^3|n\rangle = \sqrt{n(n-1)(n-2)}\,|n-3\rangle$$

$$\vdots$$

- There will be k where $n=k$

$$\hat{a}^{k+1}|n\rangle = \sqrt{n(n-1)\cdots(n-k)}\,|n-k-1\rangle = 0\,|n-k-1\rangle$$

$$= 0 \longrightarrow \text{This is "the zero" vector} \neq |n=0\rangle$$

- The sequence terminates here since $\hat{a}|0\rangle = 0$
- The case where n is not an integer, the property of the norm still forces us to consider only $n \geq 0$

$$n = \langle n|\hat{N}|n\rangle = \left(\langle n|\hat{a}^{+}\right)\left(\hat{a}|n\rangle\right) \geq 0$$

- Therefore the sequence must terminate before n becomes negative.
- Since the smallest number is $n=0$
- The ground state is $|0\rangle$ with $E_0 = \frac{1}{2}\hbar\omega$
- The rest of the spectrum can be found by

$$|1\rangle = \hat{a}^{+}|0\rangle, \quad |2\rangle = \frac{\hat{a}^{+}}{\sqrt{2}}|1\rangle = \frac{(\hat{a}^{+})^2}{\sqrt{2}}|0\rangle$$

$$|3\rangle = \frac{\hat{a}^{+}}{\sqrt{3}}|2\rangle = \frac{(\hat{a}^{+})^3}{\sqrt{3!}}|0\rangle, \ldots, |n\rangle = \frac{(\hat{a}^{+})^n}{\sqrt{n!}}|0\rangle$$

Using $\{|n\rangle\}$ as a basis, we can write matrix element of

$$\langle n'|\hat{a}|n\rangle = \sqrt{n}\langle n'|n-1\rangle = \sqrt{n}\,\delta_{n',n-1}$$

$$\langle n'|\hat{a}^{+}|n\rangle = \sqrt{n+1}\langle n'|n+1\rangle = \sqrt{n+1}\,\delta_{n',n+1}$$

- They are not diagonal since $[\hat{N},\hat{a}]\neq 0$, $[\hat{N},\hat{a}^{+}]\neq 0$
- We can use

$$\hat{x} = \frac{d_0}{\sqrt{2}}(\hat{a}+\hat{a}^{+}) = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}+\hat{a}^{+})$$

$$\hat{p} = \frac{i\hbar}{\sqrt{2}d_0}(\hat{a}^{+}-\hat{a}) = i\sqrt{\frac{\hbar m\omega}{2}}(\hat{a}^{+}-\hat{a})$$

$\therefore$ The matrix element of $\hat{x}$ & $\hat{p}$ in $\{|n\rangle\}$ basis is

$$\langle n'|\hat{x}|n\rangle = \sqrt{\frac{\hbar}{2m\omega}}\left(\sqrt{n}\,\delta_{n',n-1} + \sqrt{n+1}\,\delta_{n',n+1}\right)$$

$$\langle n'|\hat{p}|n\rangle = i\sqrt{\frac{\hbar m\omega}{2}}\left(\sqrt{n+1}\,\delta_{n',n+1} - \sqrt{n}\,\delta_{n',n-1}\right)$$

- Notice that $\hat{x},\hat{p}$ are not diagonal in this basis since

$$[\hat{N},\hat{x}]\neq 0 \;,\; [\hat{N},\hat{p}]\neq 0$$

- Let's find wave functions corresponding to eigenvectors $|n\rangle$ (sometimes we call eigenfunctions)
- Starting with $\hat{a}|0\rangle = 0$
- Inner product with $|x\rangle$ reads $\left(d_0 = \sqrt{\frac{\hbar}{m\omega}}\right)$

$$\langle x|\hat{a}|0\rangle = \frac{1}{\sqrt{2}d_0}\langle x|\hat{x}+\frac{i\hat{p}}{m\omega}|0\rangle = 0$$

- Since $\langle x|\hat{x}|\psi\rangle = x\langle x|\psi\rangle$ & $\langle x|\hat{p}|\psi\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\langle x|\psi\rangle$

- We then have

$$\left(x + \frac{\hbar}{m\omega}\frac{\partial}{\partial x}\right)\langle x|0\rangle = 0$$

- Let's define the wavefunction of the ground state as

$$\langle x|0\rangle = \psi_0(x)$$

- Hence, we have

$$\left(x + d_0^2\frac{\partial}{\partial x}\right)\psi_0(x) = 0 \qquad \left(d_0 = \sqrt{\frac{\hbar}{m\omega}}\right)$$

- We can solve this to obtain the Gaussian function

$$d_0^2\frac{\partial}{\partial x}\psi_0(x) = -x\psi_0(x)$$

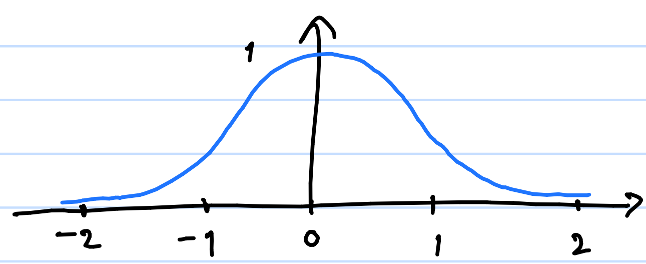
$$\frac{1}{\psi_0}\frac{\partial}{\partial x}\psi_0(x) = -\frac{x}{d_0^2} = \frac{\partial}{\partial x}\ln(\psi_0(x))$$

$$\therefore \quad \ln(\psi_0(x)) + C = -\frac{x^2}{2d_0^2}$$

$$\psi_0(x) = Ne^{-(x^2/2d_0^2)}$$

Digression : Gaussian function

- This is one of the most important functions in Science (frequently used in data analysis, statistics, Experimental physics, theoretical physics, etc.)
- The shape of $f(x) = e^{-x^2}$ is



• The total area of $f(x)$ is $\sqrt{\pi}$

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

• This can be shown easily by squaring an integral

$$I = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2} dy\right)$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx dy$$

• Changing $x, y \to r, \theta$ (polar coordinate)

$$I = \int_{r=0}^{\infty}\int_{\theta=0}^{2\pi} e^{-r^2} r dr d\theta$$

$$= 2\pi \int_0^{\infty} e^{-r^2} \frac{1}{2} dr^2 = \pi(-1)\int_0^{\infty} e^{-r^2} d(-r^2)$$

$$= \pi(-1)\left[e^{-r^2}\right]_0^{\infty} = \pi(-1)(0-1) = \pi$$

$$\therefore \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

Exercise show that

1) $$\int_{-\infty}^{\infty} e^{-a(x-b)^2} dx = \sqrt{\frac{\pi}{a}}$$

2) $$\int_{-\infty}^{\infty} x^2 e^{-ax^2} dx = -\frac{\partial}{\partial a}\left(\sqrt{\frac{\pi}{a}}\right) = \frac{1}{2a}\sqrt{\frac{\pi}{a}}$$

3) $$\int_{-\infty}^{\infty} x^{2n} e^{-ax^2} dx = (-1)^n \frac{\partial^n}{\partial a^n}\left(\sqrt{\frac{\pi}{a}}\right)$$

- Back to the ground state wave function

$$\psi_0(x) = N e^{-(x^2/2d_0^2)}$$

- The normalisation can be found by imposing

$$\langle 0|0\rangle = \int_{-\infty}^{\infty} dx \langle 0|x\rangle\langle x|0\rangle = \int_{-\infty}^{\infty} dx\, \psi_0(x)\psi_0^*(x) = 1$$

$$\therefore\quad 1 = |N|^2 \int_{-\infty}^{\infty} e^{-x^2/d_0^2}\, dx = |N|^2 d_0 \sqrt{\pi}$$

$$N = \frac{1}{\pi^{1/4}\sqrt{d_0}} = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4}$$

$$\therefore\quad \langle x|0\rangle = \psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} e^{-\left(\frac{m\omega x^2}{2\hbar}\right)}$$

Then the rest of eigenfunctions can be calculated from

$$\langle x|1\rangle = \langle x|a^\dagger|0\rangle = \langle x|\frac{1}{\sqrt{2}d_0}\left(\hat{X} - \frac{i\hat{P}}{m\omega}\right)|0\rangle$$

$$= \frac{1}{\sqrt{2}d_0}\left(x - d_0^2\frac{\partial}{\partial x}\right)\langle x|0\rangle$$

$$\langle x|2\rangle = \frac{1}{\sqrt{2}}\langle x|(a^\dagger)^2|0\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}d_0}\right)^2\left(x - d_0^2\frac{\partial}{\partial x}\right)^2\langle x|0\rangle$$

$$\vdots$$

$$\langle x|n\rangle = \left(\frac{1}{\pi^{1/4}\sqrt{2^n n!}}\right)\left(\frac{1}{d_0}\right)^{n+\frac{1}{2}}\left(x - d_0^2\frac{\partial}{\partial x}\right)^n e^{-(x^2/2d_0^2)}$$

Time Evolution of the harmonic oscillator

- We will work out the time development of quantum harmonic oscillation
- In Schrödinger's picture, the Hamiltonian is time independent

$$H_s = \frac{\hat{p}^2}{2m} + \frac{1}{2} m\omega^2 \hat{x}^2$$

- This falls into case I, such that $[H_s, H_s] = 0$
- Therefore the Heisenberg Hamiltonian is

$$H_H = \mathcal{U}^\dagger H_s \mathcal{U} = e^{\frac{i}{\hbar}H_s t} H_s e^{-\frac{i}{\hbar}H_s t} = H_s$$

- We can work out time evolution operator to determine the evolution of quantum state as usual

$$|\psi(t)\rangle = \mathcal{U}(t,0)|\psi_0\rangle = e^{-\frac{i}{\hbar}Ht}|\psi_0\rangle$$

$$= \sum_{n=0}^{\infty} e^{-\frac{i}{\hbar}Ht} |n\rangle\langle n|\psi_0\rangle$$

$H|n\rangle = E_n|n\rangle$

$$= \sum_{n=0}^{\infty} \langle n|\psi_0\rangle e^{-\frac{i}{\cancel{\hbar}}\cancel{\hbar}\omega(n+\frac{1}{2})t} |n\rangle$$

$$= \sum_{n=0}^{\infty} \langle n|\psi_0\rangle e^{-i[(n+\frac{1}{2})\omega]t} |n\rangle$$

- To complete the description, we will use Heisenberg picture of the operator

- Suppose that we replace $\hat{X}$ & $\hat{P}$ with $\hat{X}_H$ & $\hat{P}_H$, we get

$$H_H = H_s(\hat{X}_H, \hat{P}_H) = \frac{\hat{P}_H^2}{2m} + \frac{1}{2} m\omega^2 \hat{X}_H^2$$

- We know that $\hat{X}_H$ & $\hat{P}_H$ are time dependent, so we can calculate its time dependence from

$$\hat{X}_H = e^{\frac{i}{\hbar}Ht} \hat{X} e^{-\frac{i}{\hbar}Ht} \quad \& \quad \hat{P}_H = e^{\frac{i}{\hbar}Ht} \hat{P} e^{-\frac{i}{\hbar}Ht}$$

- However, this is too complicated so we will deal with this later

- Let's find an easy way! ⇒ Heisenberg equation of motion

$$i\hbar \frac{\partial}{\partial t}\hat{X}_H = [\hat{X}_H, \hat{H}_H] + i\hbar \left(\cancel{\frac{\partial \hat{X}_s}{\partial t}}\right)_H^{0}$$

$$i\hbar \frac{\partial}{\partial t}\hat{X}_H = [\hat{X}_H, \frac{\hat{P}_H^2}{2m} + \cancel{\frac{1}{2}m\omega^2 \hat{X}_H^2}^{0}]$$

$$= \frac{1}{2m}\left([\hat{X}_H, \hat{P}_H]\hat{P}_H + \hat{P}_H[\hat{X}_H, \hat{P}_H]\right)$$

$$[\hat{X}_H, \hat{P}_H] = [\hat{X}_s, \hat{P}_s] = i\hbar$$

$$= \frac{1}{2m}\left(2i\hbar \hat{P}_H\right) = \frac{2i\hbar \hat{P}_H}{2m}$$

$$\boxed{\frac{\partial}{\partial t}\hat{X}_H = \frac{\hat{P}_H}{m}} \quad \text{———} \ (*)$$

$$i\hbar \frac{\partial}{\partial t}\hat{P}_H = [\hat{P}_H, \hat{H}_H] + i\hbar \left(\cancel{\frac{\partial \hat{P}_s}{\partial t}}\right)_H^{0}$$

$$= [\hat{P}_H, \cancel{\frac{\hat{P}_H^2}{2m}}^{0} + \frac{1}{2}m\omega^2 \hat{X}_H^2]$$

$$= \frac{m\omega^2}{2}\left([\hat{P}_H, \hat{X}_H]\hat{X}_H + \hat{X}_H[\hat{P}_H, \hat{X}_H]\right)$$

$$i\hbar \frac{\partial \hat{P}_H}{\partial t} = \frac{m\omega^2}{2} \cdot 2(-i\hbar) X_H = -i\hbar m\omega^2 X_H$$

$$\boxed{\frac{\partial \hat{P}_H}{\partial t} = -m\omega^2 X_H} \quad \text{——} \quad (**)$$

- Combining $(*)$ & $(**)$, we have the classical equation of motion:

$$\frac{\partial \hat{P}_H}{\partial t} = m\frac{\partial^2 \hat{X}_H}{\partial t^2} = -m\omega^2 \hat{X}_H$$

$$\therefore \quad \frac{\partial^2 \hat{X}_H}{\partial t^2} = -\omega^2 \hat{X}_H$$

The solution is

$$\hat{X}_H = \hat{A}\cos\omega t + \hat{B}\sin\omega t$$

&

$$\hat{P}_H = m\frac{\partial \hat{X}_H}{\partial t} = -m\omega\hat{A}\sin\omega t + m\omega\hat{B}\cos\omega t$$

The Schrödinger operator can be obtained from

$$\hat{X}_H(t=0) = \hat{X}_S = \hat{A} \quad , \quad \hat{P}_H(t=0) = \hat{P}_S = m\omega\hat{B}$$

$\therefore$ The full solution is

$$\hat{X}_H = \hat{X}_S\cos\omega t + \frac{\hat{p}}{m\omega}\sin\omega t$$

$$\hat{P}_H = \hat{P}_S\cos\omega t - m\omega\hat{X}\sin\omega t$$

☆ This means that Heisenberg operators of $\hat{X}$ & $\hat{P}$ oscillate like the classical variables would

- With this form, we are able to calculate quantities like $\langle\Psi(t)|\hat{A}_S(\hat{X},\hat{P})|\Psi(t)\rangle = \langle\Psi(0)|\hat{A}(\hat{X}_H,\hat{P}_H)|\Psi(0)\rangle$

- To complete this discussion, we evaluate $\hat{X}_H, \hat{P}_H$ directly

- The Heisenberg operator $\hat{X}_H$ is

$$\hat{X}_H = e^{\frac{i}{\hbar}Ht}\hat{X}_s e^{-\frac{i}{\hbar}Ht}$$

- We can use the formula in previous homework:

$$e^{A}Be^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \dots$$

$$\therefore \quad \hat{X}_H = \hat{X} + \frac{it}{\hbar}[\hat{H},\hat{X}] + \frac{1}{2!}(it)^2[\hat{H},[\hat{H},\hat{X}]]$$

$$+ \frac{1}{3!}(it)^3[\hat{H},[\hat{H},[\hat{H},\hat{X}]]] + \dots$$

Since the commutator can be written as (in S-picture)

$$[\hat{H},\hat{X}] = [\frac{\hat{P}^2}{2m} + \cancel{\frac{1}{2}m\omega^2\hat{X}^2}, \hat{X}] = \frac{1}{2m}\left(\hat{P}[\hat{P},\hat{X}] + [\hat{P},\hat{X}]\hat{P}\right)$$

$$= -\frac{i\hbar}{m}\hat{P}$$

$$[\hat{H},\hat{P}] = [\cancel{\frac{\hat{P}^2}{2m}} + \frac{1}{2}m\omega^2\hat{X}^2, \hat{P}] = \frac{m\omega^2}{2}\left(\hat{X}[\hat{X},\hat{P}] + [\hat{X},\hat{P}]\hat{X}\right)$$

$$= i\hbar m\omega^2\hat{X}$$

$$\therefore \hat{X}_H = \hat{X} + \frac{it}{\hbar}\left(-\frac{i\hbar}{m}\hat{P}\right) + \frac{1}{2!}\left(\frac{it}{\hbar}\right)^2[\hat{H},\left(-\frac{i\hbar}{m}\hat{P}\right)]$$

$$+ \frac{1}{3!}\left(\frac{it}{\hbar}\right)^3[\hat{H},[\hat{H},\left(-\frac{i\hbar}{m}\hat{P}\right)]] + \dots$$

$$\hat{X}_H = \hat{X} + t\left(\frac{\hat{P}}{m}\right) + \frac{1}{2!}\left(\frac{-t^2}{\cancel{\hbar^2}}\right)\left(-\frac{i\cancel{\hbar}}{\cancel{m}}\right)i\cancel{\hbar}\cancel{m}\omega^2\hat{X}$$

$$+ \frac{1}{3!}\left(\frac{-it^3}{\hbar^3}\right)[\hat{H},\underbrace{\left(\frac{-i\hbar}{\cancel{m}}\right)i\hbar\cancel{m}\omega^2\hat{X}}] \quad \hbar^2\omega^2\left(-\frac{i\hbar}{m}\hat{P}\right)$$

$$= \hat{X} - \frac{1}{2!}(\omega t)^2\hat{X} + (\omega t)\frac{\hat{P}}{m\omega} - \frac{1}{3!}(\omega t)^3\frac{\hat{P}}{m\omega} + \dots$$

- The pattern repeats and forms as sin/cos

$$\hat{x}_H = \left(1 - \frac{1}{2!}(\omega t)^2 + \frac{1}{4!}(\omega t)^4 - \dots\right)\hat{x} + \left(\omega t - \frac{1}{3!}(\omega t)^3 + \frac{1}{5!}(\omega t)^5 - \dots\right)\frac{\hat{p}}{m\omega}$$

$$= \hat{x}\cos(\omega t) + \frac{\hat{p}}{m\omega}\sin(\omega t)$$

- We can also calculate directly for $\hat{P}_H$ and obtain

$$\hat{P}_H = e^{\frac{i}{\hbar}Ht}\,\hat{p}\,e^{-\frac{i}{\hbar}Ht} = \dots$$

$$= \hat{p}\cos(\omega t) - m\omega\hat{x}\sin(\omega t)$$

- Another important point is that if $[\hat{H}_s(t), \hat{H}_s(t')] = 0$ $\forall t, t'$, then

$$H_H(t) = H_s(\hat{x}_H, \hat{P}_H) = e^{\frac{i}{\hbar}H_s t} H_s e^{-\frac{i}{\hbar}H_s t}$$

$$= H_s(\hat{x}_s, \hat{P}_s)$$

- we can show this explicitly as follow:

$$H_H(t) = \frac{\hat{P}_H^2}{2m} + \frac{1}{2}m\omega^2\hat{x}_H^2$$

$$= \frac{1}{2m}\left(\hat{p}\cos\omega t - m\omega\hat{x}\sin\omega t\right)^2 + \frac{1}{2}m\omega^2\left(\hat{x}\cos\omega t + \frac{\hat{p}}{m\omega}\sin\omega t\right)^2$$

$$= \frac{\cos^2\omega t}{2m}\hat{p}^2 + \frac{m\omega^2\sin^2\omega t}{2}\hat{x}^2 - \cancel{\frac{\omega}{2}\sin\omega t\cos\omega t(\hat{x}\hat{p} + \hat{p}\hat{x})}$$

$$+ \frac{\sin^2\omega t}{2m}\hat{p}^2 + \frac{m\omega^2\cos^2\omega t}{2}\hat{x}^2 + \cancel{\frac{\omega}{2}\sin\omega t\cos\omega t(\hat{x}\hat{p} + \hat{p}\hat{x})}$$

$$= \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2 = H_s$$

- One last point we need to know is the fact that energy eigenstate is rather special in finding expectation value.

Time Dependence of Expectation Values

- Suppose that initially the state is an eigenstate of an observable $\hat{A}$ which commutes with Hamiltonian $[\hat{A},\hat{H}]=0$
- At later time (in S-picture), we have

$$|a,t\rangle = \mathcal{U}(t,0)|a\rangle = e^{-\frac{i}{\hbar}Ht}|a\rangle = e^{-\frac{i}{\hbar}E_a t}|a\rangle$$

- Consider another observable $\hat{B}$ (no need to commute with $\hat{A}$ or $\hat{H}$) the expectation value in this picture is

$$\begin{aligned}\langle\hat{B}\rangle &= \langle a|\mathcal{U}^\dagger(t,0)\hat{B}\,\mathcal{U}(t,0)|a\rangle \\ &= \langle a|e^{\frac{i}{\hbar}E_a t}\hat{B}\,e^{-\frac{i}{\hbar}E_a t}|a\rangle \\ &= \langle a|\hat{B}|a\rangle,\end{aligned}$$

which is constant in time

☆ The expectation value of any observable with respect to an energy eigenstate does not change in time

- Sometimes energy eigenstates are called stationary state
- The non-stationary state is a superposition of energy eigenstates :

$$|\alpha,0\rangle = \sum_a c_a|a\rangle$$

- The expectation is then

$$\begin{aligned}\langle\hat{B}\rangle &= \langle\alpha,t|\hat{B}|\alpha,t\rangle \\ &= \left(\sum_a c_a^*\langle a|e^{\frac{iE_a t}{\hbar}}\right)\hat{B}\left(\sum_{a'}c_{a'}e^{\frac{-iE_{a'}t}{\hbar}}|a'\rangle\right)\end{aligned}$$

$$\langle B \rangle = \sum_a \sum_{a'} C_a^* C_{a'} \langle a | \hat{B} | a' \rangle e^{\frac{i}{\hbar}(E_a - E_{a'})t}$$

which is clearly time dependent (each term oscillates with frequency:

$$\omega_{aa'} = \frac{(E_a - E_{a'})}{\hbar}$$

- Let's consider the time dependence in our QHM example
- Suppose that the initial state is

$$|\psi, 0\rangle = |n\rangle$$

- Let's calculate the time-dependent expectation value of $\hat{X}$

$$\langle \hat{X} \rangle = \langle \psi, t | \hat{X} | \psi, t \rangle = \langle n | e^{\frac{i}{\hbar}E_n t} \hat{X} e^{-\frac{i}{\hbar}E_n t} | n \rangle$$

$$= \langle n | \hat{X} | n \rangle = \langle n | \sqrt{\frac{\hbar}{2m\omega}} (\hat{a} + \hat{a}^+) | n \rangle$$

$$= \sqrt{\frac{\hbar}{2m\omega}} \langle n | \left(\sqrt{n} | n-1 \rangle + \sqrt{n+1} | n+1 \rangle \right) = 0$$

This is zero (also time independent)

- Do we get the same result in H-picture?

$$\langle \hat{X} \rangle = \langle \psi, 0 | \hat{X}_H | \psi, 0 \rangle = \langle n | \hat{X} \cos\omega t + \frac{\hat{P}}{m\omega} \sin\omega t \, | n \rangle$$

$$= \langle n | \hat{X} | n \rangle \cos\omega t + \langle n | \hat{P} | n \rangle \frac{\sin\omega t}{m\omega}$$

Since $\langle n | \hat{X} | n \rangle = 0$ and

$$\langle n | \hat{p} | n \rangle = \langle n | i \sqrt{\frac{\hbar m\omega}{2}} (\hat{a}^+ - \hat{a}) | n \rangle = 0$$

- This means that the energy eigenstates don't have the Classical behaviour of Harmonic oscillator. ($\langle\hat{X}\rangle$ should oscillate)

☆ We will study the specific combination such that $\langle\hat{X}\rangle$ Oscillates classically. $\Rightarrow$ coherent state

- Now, we can work out the Heisenberg operator $\hat{a}_H$

$$\hat{a}_H = e^{\frac{i}{\hbar}Ht}\hat{a}e^{-\frac{i}{\hbar}Ht} = e^{i\omega t(\hat{N}+\frac{1}{2})}\hat{a}e^{-i\omega t(\hat{N}+\frac{1}{2})}$$
$$= e^{i\omega t\hat{N}}\hat{a}e^{-i\omega t\hat{N}}$$

- We can evaluate this using a differential equation

$$\frac{d\hat{a}_H}{dt} = e^{i\omega t\hat{N}}(i\omega\hat{N})\hat{a}e^{-i\omega t\hat{N}} - e^{i\omega t\hat{N}}\hat{a}(i\omega\hat{N})e^{-i\omega t\hat{N}}$$
$$= i\omega e^{i\omega t\hat{N}}[\hat{N},\hat{a}]e^{-i\omega t\hat{N}}$$
$$= -i\omega e^{i\omega t\hat{N}}\hat{a}e^{-i\omega t\hat{N}} = -i\omega\hat{a}_H$$

- The solution is

$$\hat{a}_H(t) = e^{-i\omega t}\hat{a}$$
$$\& \quad \hat{a}^{+}_H(t) = e^{i\omega t}\hat{a}^{+}$$

- We can plug back to $\hat{X}_H$ & $\hat{P}_H$ and get
- The Heisenberg version is

$$\hat{X}_H = \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}_H + \hat{a}^{+}_H) = \sqrt{\frac{\hbar}{2m\omega}}\left(e^{-i\omega t}\hat{a} + e^{i\omega t}\hat{a}^{+}\right)$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\hat{a}(\cos\omega t - i\sin\omega t) + \hat{a}^{+}(\cos\omega t + i\sin\omega t)\right)$$
$$= \sqrt{\frac{\hbar}{2m\omega}}\left((\hat{a}+\hat{a}^{+})\cos\omega t + i(\hat{a}^{+}-\hat{a})\sin\omega t\right)$$

$$\hat{x}_H = \sqrt{\frac{\hbar}{2m\omega}}\left((\hat{a}+\hat{a}^{\dagger})\cos\omega t + i(\hat{a}^{\dagger}-\hat{a})\sin\omega t \right)$$

$$= \hat{x}\cos\omega t + \frac{\hat{p}}{m\omega}\sin\omega t$$

which is in agreement with previous result.

Homework 3

1) We have shown in the lecture that the expectation value of $\hat{X}$ & $\hat{P}$ in the energy eigenstate is zero

$$\langle \hat{X} \rangle = \langle n | \hat{X} | n \rangle = 0 \quad , \quad \langle \hat{P} \rangle = \langle n | \hat{P} | n \rangle = 0 .$$

1.1) Consider at $t=0$, the initial state is in $|\psi,0\rangle = |n\rangle$

Show that

$$\langle n | \hat{X}^2 | n \rangle = \frac{\hbar}{m\omega}\left(n+\frac{1}{2}\right) = d_0^2\left(n+\frac{1}{2}\right)$$

$$\langle n | \hat{P}^2 | n \rangle = m\hbar\omega\left(n+\frac{1}{2}\right) = \frac{\hbar^2}{d_0^2}\left(n+\frac{1}{2}\right)$$

$$\langle n | \hat{X}\hat{P} | n \rangle = \frac{\hbar}{2} \quad \& \quad \langle n | \hat{P}\hat{X} | n \rangle = -\frac{\hbar}{2}$$

1.2) Consider $t=0$, calculate the uncertainty ΔX & ΔP and show that

$$\Delta X \Delta P = \hbar\left(n+\frac{1}{2}\right)$$

[This satisfies the Heisenberg uncertainty principle and only the ground state $|n=0\rangle$ saturates the bound $\Delta x \Delta p = \hbar/2$]

1.3) Now consider at later time, using the Heisenberg picture to show that

$$\langle \psi,t | \hat{X}^2 | \psi,t \rangle = d_0^2\left(n+\frac{1}{2}\right), \quad \langle \psi,t | \hat{P}^2 | \psi,t \rangle = \frac{\hbar^2}{d_0^2}\left(n+\frac{1}{2}\right)$$

$(\Delta X \Delta P)(t) = \hbar\left(n+\frac{1}{2}\right)$ which are all time-independent.

2). Dynamics of 2-state system

Let $|1\rangle$ & $|2\rangle$ be eigenstates of an observable $\hat{A}$ with eigenvalue a_1 & a_2 respectively (non-degerated)

The Hamiltonian is given by

$$\hat{H} = h\,|1\rangle\langle 2| + h|2\rangle\langle 1|$$

where h is a real number

2.1) Find energy eigenvalues/eigenvectors

2.2) Suppose that at time $t=0$, $|\psi,0\rangle = |1\rangle$

Write down the state $|\psi,t\rangle$ for later time.

2.3) Suppose that at time $t=0$, $|\psi,0\rangle = |1\rangle$

Calculate the probability of finding the system in $|2\rangle$

3) Neutrino Oscillation: Neutrinos are neutral particles that comes in three flavours (ν_e, ν_μ, ν_τ). The observation of neutrino oscillations is a proof of their masses

Consider 2 neutrino species, namely the flavour eigenstates $|\nu_e\rangle$ & $|\nu_\mu\rangle$. Suppose that the Hamiltonian with eigenvectors $|\nu_1\rangle$ & $|\nu_2\rangle$ with eigenvalues $E_{1,2} = \sqrt{p^2c^2 + m_{1,2}^2c^4} \cong pc + \frac{m_{1,2}^2c^3}{2p}$

$$\hat{H}|\nu_1\rangle = \left(pc + \frac{m_1^2c^3}{2p}\right)|\nu_1\rangle \quad , \quad \hat{H}|\nu_2\rangle = \left(pc + \frac{m_2^2c^3}{2p}\right)|\nu_1\rangle$$

If the flavour eigenstates are related to the energy eigenstates by

$$|\nu_e\rangle = \cos\theta\,|\nu_1\rangle + \sin\theta\,|\nu_2\rangle$$

$$|\nu_\mu\rangle = -\sin\theta\,|\nu_1\rangle + \cos\theta\,|\nu_2\rangle$$

3.1) Calculate the matrix representation of the time evolution operator $\left(\mathcal{U}(t,0)\right)$ in

a) $\{|\nu_1\rangle, |\nu_2\rangle\}$ basis

b) $\{|\nu_e\rangle, |\nu_\mu\rangle\}$ basis

3.2) Suppose that the initial state is in electron neutrino eigenstate : $|\psi,0\rangle = |\nu_e\rangle$

Find the probability of finding muon neutrino eigenstate at later time, i.e.,

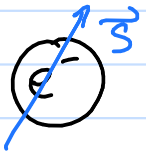
$$|\langle\nu_\mu|\psi,t\rangle|^2 = ?$$

Also find the probability of finding electron neutrino eigenstate at later time

$$|\langle\nu_e|\psi,t\rangle|^2 = ?$$

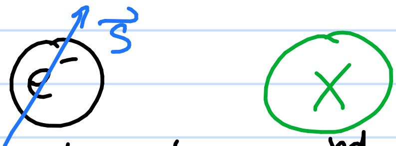
Multiparticle states

- In a real system, we often deal with more than 1 particle! Atoms, gas, solid state, ...
- The way we use to describe system ⇒ 1 particle



$\rightsquigarrow$ $|\psi\rangle \in \mathbb{V}$ → vector space for 1st particle

$\hat{T}$ = operator $\quad \hat{T} : \mathbb{V} \to \mathbb{V} \quad \left(\hat{U} : |\psi(t)\rangle = \hat{U}\,|\psi_0\rangle \right)$



1st particle $\qquad$ 2nd particle

$\rightsquigarrow$ $|\chi\rangle \in \mathbb{W}$ → vector space for 2nd particle

Since $\mathbb{V}$ is not $\mathbb{W}$ in general ($|\psi\rangle$ & $|\chi\rangle$ live in a different space)

$$\begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \qquad \begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix}$$

∴ We cannot simply $\cancel{|\psi\rangle + |\chi\rangle}$ ⇒ represents 2 particle system

⇒ We need a new way of combining states

★ we will propose "tensor product" $\boxed{g_{\mu\nu} \sim V_\mu \otimes W_\nu}$

$|\psi\rangle \otimes |\chi\rangle$ → A new vector in a $\underbrace{\text{new vector space}}_{\mathbb{V} \otimes \mathbb{W}}$

$$\therefore \quad |\psi\rangle \otimes |\chi\rangle \in \mathbb{V} \otimes \mathbb{W}$$

Properties of tensor product

1) $\boxed{(c|\psi\rangle)\otimes|\chi\rangle = |\psi\rangle\otimes(c|\chi\rangle) = c(|\psi\rangle\otimes|\chi\rangle)}$, $c\in\mathbb{C}$

2) $(|\psi_1\rangle + |\psi_2\rangle)\otimes|\chi\rangle = |\psi_1\rangle\otimes|\chi\rangle + |\psi_2\rangle\otimes|\chi\rangle$

or $|\psi\rangle\otimes(|\chi_1\rangle + |\chi_2\rangle) = |\psi\rangle\otimes|\chi_1\rangle + |\psi\rangle\otimes|\chi_2\rangle$

Let's consider $V\otimes W$

1) What is its $|0\rangle = 0(|\psi\rangle\otimes|\chi\rangle)$

$$= (\underbrace{0|\psi\rangle}_{|0\rangle})\otimes|\chi\rangle = |\psi\rangle\otimes(\underbrace{0|\chi\rangle}_{|0\rangle})$$

$|0\rangle\in V\otimes W = |0\rangle\otimes|\chi\rangle = |\psi\rangle\otimes|0\rangle$

2) $\underline{V\otimes W}$ is much larger V & W

There is a vector in $V\otimes W$ that cannot be written in terms of $|\psi\rangle\otimes|\chi\rangle$

$\Rightarrow$ We can use the linear combination of 2 vectors in $V\otimes W$

$\in$ $c_1(|\psi_1\rangle\otimes|\chi_1\rangle) + c_2(|\psi_2\rangle\otimes|\chi_2\rangle)$

$\neq |\psi'\rangle\otimes|\chi'\rangle$

3). What about its basis? $V\otimes W$

One easy way to define the basis of $V\otimes W$

$\Rightarrow$ Use basis of V & W

3). What about its basis? $V \otimes W$

One easy way to define the basis of $V \otimes W$

$\Rightarrow$ Use basis of V & W

Suppose we have $\{|e_i\rangle\}^{\nearrow n}$ is a basis of V

$$\left(|\psi\rangle = \sum_i c_i |e_i\rangle \right)$$

& we have $\{|f_i\rangle\}^{\nearrow m}$ is a basis of W

$$\left(|\chi\rangle = \sum_i d_i |f_i\rangle \right)$$

If we consider

$$|\psi\rangle \otimes |\chi\rangle = \left(\sum_i c_i |e_i\rangle\right) \otimes \left(\sum_j d_j |f_j\rangle\right)$$

properties of tensor product

$$= \sum_i \sum_j c_i d_j \underbrace{\left(|e_i\rangle \otimes |f_j\rangle\right)}_{\text{basis of } V \otimes W}$$

∴ We can use $\{|e_i\rangle \otimes |f_j\rangle\}$ as a basis

$\hookrightarrow n \times m$

<u>EX</u> A system of 2 spins (electrons)

1^{st} electron $\Rightarrow V$, $\{|z,+\rangle, |z,-\rangle\}$

2^{nd} electron $\Rightarrow W$, $\{|x,+\rangle, |x,-\rangle\}$ / $\{|z,+\rangle, |z,-\rangle\}$

We have a new basis of $V \otimes W$

$$\{|z,+\rangle \otimes |x,+\rangle, |z,+\rangle \otimes |x,-\rangle, |z,-\rangle \otimes |x,+\rangle, |z,-\rangle \otimes |x,-\rangle\}$$

∴ $|\Psi\rangle \in V \otimes W$ can be written as

$$|\Psi\rangle = a_1 |z,+\rangle \otimes |x,+\rangle + a_2 |z,+\rangle \otimes |x,-\rangle + a_3 |z,-\rangle \otimes |x,+\rangle + a_4 |z,-\rangle \otimes |x,-\rangle$$

Note that we can also choose $\{|z,\pm\rangle\}$ for $\mathbb{V}$

& $\{|z,\pm\rangle\}$ for $\mathbb{W}$

$\therefore$ A new basis $= \{ |z,+\rangle_1 \otimes |z,+\rangle_2 ,\ |z,+\rangle_1 \otimes |z,-\rangle_2 ,$

$|z,-\rangle_1 \otimes |z,+\rangle_2 ,\ |z,-\rangle_1 \otimes |z,-\rangle_2 \}$

We will from now on define

$|z,+\rangle \equiv |+\rangle \equiv |\uparrow\rangle$ (and keep in mind that $\hat{z}$ is the direction)

$|z,-\rangle \equiv |-\rangle \equiv |\downarrow\rangle$

Note that if $\{|e_i\rangle\}$ has n basis $\{|f_j\rangle\}$ has m basis

$\Rightarrow \{|e_i\rangle \otimes |f_j\rangle\}$ has $\underline{n \times m}$ basis $\Rightarrow \text{Dim}(\mathbb{V} \otimes \mathbb{W}) = n \times m$

$= \text{Dim}(\mathbb{V}) \times \text{Dim}(\mathbb{W})$

How do we define an operator on $\mathbb{V} \otimes \mathbb{W}$

We know for sure that an operator must map

$\mathbb{V} \otimes \mathbb{W} \to \mathbb{V} \otimes \mathbb{W}$

Suppose we have $\hat{T}$ such that it acts on $\mathbb{V}$

(it acts on 1st particle)

$\Rightarrow$ We can construct the tensor product of operators

$\hat{T} : \mathbb{V} \to \mathbb{V}$

Operator in $\mathbb{W}$

$\hat{T} \otimes \hat{\mathbb{1}} : \mathbb{V} \otimes \mathbb{W} \to \mathbb{V} \otimes \mathbb{W}$

acts on 1st particle

1st particle

do nothing

$\Rightarrow (\hat{T} \otimes \hat{\mathbb{1}}) \cdot (|\psi\rangle \otimes |\chi\rangle) = \hat{T}|\psi\rangle \otimes \hat{\mathbb{1}}|\chi\rangle$

A tensor product of operators.

$$(\hat{T}\otimes\hat{\mathbb{1}})\cdot(|\psi\rangle\otimes|\chi\rangle) = \hat{T}|\psi\rangle\otimes\hat{\mathbb{1}}|\chi\rangle$$

$\therefore$ If have $\hat{T}: \underline{\underline{V}} \to \underline{\underline{V}}$ & $\hat{S}: \underline{\underline{W}} \to \underline{\underline{W}}$

$$\boxed{(\hat{T}\otimes\hat{S})(|\psi\rangle\otimes|\chi\rangle) \equiv \hat{T}|\psi\rangle\otimes\hat{S}|\chi\rangle}$$

Suppose we have $\hat{T}$ & $\hat{S}$

we can write $(\hat{T}\otimes\hat{S}) = (\hat{T}\otimes\hat{\mathbb{1}})(\hat{\mathbb{1}}\otimes\hat{S})$

? check

$$(\hat{T}\otimes\hat{\mathbb{1}})(\hat{\mathbb{1}}\otimes\hat{S})(|\psi\rangle\otimes|\chi\rangle) = (\hat{T}\otimes\hat{\mathbb{1}})(\hat{\mathbb{1}}|\psi\rangle\otimes\hat{S}|\chi\rangle)$$

$$= \hat{T}|\psi\rangle\otimes\hat{\mathbb{1}}\hat{S}|\chi\rangle = \hat{T}|\psi\rangle\otimes\hat{S}|\chi\rangle$$

$\therefore$ We also have

$$\hat{T}\otimes\hat{S} = (\hat{\mathbb{1}}\otimes\hat{S})\cdot(\hat{T}\otimes\hat{\mathbb{1}})$$

we can check that this is true

$$\Rightarrow \quad (\hat{\mathbb{1}}\otimes\hat{S})(\hat{T}\otimes\hat{\mathbb{1}}) = (\hat{T}\otimes\hat{\mathbb{1}})(\hat{\mathbb{1}}\otimes\hat{S})$$

$$\boxed{[\hat{T}\otimes\hat{\mathbb{1}}, \hat{\mathbb{1}}\otimes\hat{S}] = 0}$$

From this relation, we can write the total Hamiltenian by combining the Hamiltenian of 2 systems

From this relation, we can write the total Hamiltonian by combining the Hamiltonian of 2 systems

$\hat{H}_1$ is the Hamiltonian of 1st particle.

$\hat{H}_2$ is ——————— . 2nd particle.

$$\cancel{\hat{H}_{Total} = \hat{H}_1 \otimes \hat{H}_2} \quad ? \qquad (1)$$

$$\boxed{\hat{H}_{Total} = \hat{H}_1 \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{H}_2} \quad ? \qquad (2)$$

Suppose $|E_1\rangle \otimes |E_2\rangle$ such that $H_1|E_1\rangle = E_1|E_1\rangle$

& $H_2|E_2\rangle = E_2|E_2\rangle$

(1) $\cancel{\hat{H}_{Total}} = (\hat{H}_1 \otimes \hat{H}_2)(|E_1\rangle \otimes |E_2\rangle) = (\hat{H}_1|E_1\rangle) \otimes (\hat{H}_2|E_2\rangle)$

$= E_1|E_1\rangle \otimes E_2|E_2\rangle = E_1E_2(|E_1\rangle \otimes |E_2\rangle)$

This is a product of $E_1E_2 \neq$ Total Energy!

$\therefore$ We should write

$$\hat{H}_{total} = (\hat{H}_1 \otimes \hat{\mathbb{1}}) + (\hat{\mathbb{1}} \otimes \hat{H}_2)$$

$$\hat{H}_{total}(|E_1\rangle \otimes |E_2\rangle) = \left((\hat{H}_1 \otimes \hat{\mathbb{1}}) + (\hat{\mathbb{1}} \otimes \hat{H}_2)\right)\left(|E_1\rangle \otimes |E_2\rangle\right)$$

$= (\hat{H}_1 \otimes \hat{\mathbb{1}})(|E_1\rangle \otimes |E_2\rangle) + (\hat{\mathbb{1}} \otimes \hat{H}_2)(|E_1\rangle \otimes |E_2\rangle)$

$= (\hat{H}_1|E_1\rangle \otimes |E_2\rangle) + (|E_1\rangle \otimes \hat{H}_2|E_2\rangle)$

$= E_1|E_1\rangle \otimes |E_2\rangle + |E_1\rangle \otimes E_2|E_2\rangle$

$= (E_1 + E_2)(|E_1\rangle \otimes |E_2\rangle) \Rightarrow$ total Energy!

$$\therefore \quad \boxed{\hat{H}_{Tot} = \hat{H}_1 \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{H}_2}$$

$\Rightarrow$ What about time evolution operator? ($\hat{H}_1$ & $\hat{H}_2$ are time independent)

$$\hat{U}_{Tot}(t,0) = e^{-\frac{i}{\hbar}\hat{H}_{Tot}t}$$

$$= \hat{\mathbb{1}} + \left(-\frac{it}{\hbar}\right)\hat{H}_{Tot} + \frac{1}{2!}\left(\frac{-it}{\hbar}\right)^2 \hat{H}_{Tot}^2 + \ldots$$

$$= (\hat{\mathbb{1}} \otimes \hat{\mathbb{1}}) + \left(\frac{-it}{\hbar}\right)\left(\hat{H}_1 \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{H}_2\right)$$

$$+ \frac{1}{2!}\left(\frac{-it}{\hbar}\right)^2 (\hat{H}_1 \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{H}_2)(\hat{H}_1 \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{H}_2)$$

$$(\hat{H}_1^2 \otimes \hat{\mathbb{1}} + \hat{H}_1 \otimes \hat{H}_2 + \hat{H}_1 \otimes \hat{H}_2 + \hat{\mathbb{1}} \otimes \hat{H}_2^2)$$

$$= \hat{\mathbb{1}} \otimes \hat{\mathbb{1}} + \left(\frac{-it}{\hbar}\right)(\hat{H}_1 \otimes \hat{\mathbb{1}}) + \frac{1}{2!}\left(\frac{-it}{\hbar}\right)^2 (\hat{H}_1^2 \otimes \hat{\mathbb{1}})$$

$$+ \left(\frac{-it}{\hbar}\right)(\hat{\mathbb{1}} \otimes \hat{H}_2) + \frac{1}{2!}\left(\frac{-it}{\hbar}\right)^2 (\hat{\mathbb{1}} \otimes \hat{H}_2^2) + \frac{1}{\cancel{2!}}\left(\frac{-it}{\hbar}\right)^2 \cdot \cancel{2}(\hat{H}_1 \otimes \hat{H}_2) + \ldots$$

We can write in terms of

$$= \left(\hat{\mathbb{1}} + \left(\frac{-it}{\hbar}\right)\hat{H}_1 + \frac{1}{2!}\left(\frac{-it}{\hbar}\right)^2 \hat{H}_1^2 + \ldots\right) \otimes \left(\hat{\mathbb{1}} + \left(\frac{-it}{\hbar}\right)\hat{H}_2 + \frac{1}{2!}\left(\frac{-it}{\hbar}\right)^2 \hat{H}_2^2 + \ldots\right)$$

$$= e^{-\frac{it}{\hbar}\hat{H}_1} \otimes e^{-\frac{it}{\hbar}\hat{H}_2}$$

$$\therefore \boxed{e^{-\frac{i}{\hbar}t\hat{H}_{tot}} = e^{-\frac{it}{\hbar}\hat{H}_1} \otimes e^{-\frac{it}{\hbar}\hat{H}_2}} \Rightarrow \boxed{\hat{U}_{Tot} = \hat{U}_1 \otimes \hat{U}_2}$$

Example on total Spin of 2 e^-

short handed for $\{|z,+\rangle, |z,-\rangle\}$

We want to describe $2e^-$ system

e^- 1st $\quad$ e^- 2nd

V_1 has a basis $\{|+\rangle, |-\rangle\}$

V_2 has a basis $\{|+\rangle, |-\rangle\}$

$\therefore$ $V_1 \otimes V_2$ has basis:

$$\{|+\rangle\otimes|+\rangle,\ |+\rangle\otimes|-\rangle,\ |-\rangle\otimes|+\rangle,\ |-\rangle\otimes|-\rangle\}$$

The most general vector (state) in $V_1\otimes V_2$

$$|\Psi\rangle = \alpha_1 |+\rangle\otimes|+\rangle + \alpha_2 |+\rangle\otimes|-\rangle + \alpha_3 |-\rangle\otimes|+\rangle + \alpha_4 |-\rangle\otimes|-\rangle$$

Now let's consider total spin in z-direction.

$$\hat{S}_z^{Tot} = \left(\hat{S}_z^{(1)}\otimes\mathbb{1}\right) + \left(\mathbb{1}\otimes\hat{S}_z^{(2)}\right)$$

Recall that $\hat{S}_z^{(1)}|+\rangle = +\frac{\hbar}{2}|+\rangle$ / $\hat{S}_z^{(2)}|+\rangle = +\frac{\hbar}{2}|+\rangle$

& $\hat{S}_z^{(1)}|-\rangle = -\frac{\hbar}{2}|-\rangle$ / $\hat{S}_z^{(2)}|-\rangle = -\frac{\hbar}{2}|-\rangle$

$$\underline{\underline{S_z^{Tot}}}\left(\underline{|+\rangle\otimes|+\rangle}\right) = \left[\left(\hat{S}_z^{(1)}\otimes\hat{\mathbb{1}}\right) + \left(\hat{\mathbb{1}}\otimes\hat{S}_z^{(2)}\right)\right]\left(|+\rangle\otimes|+\rangle\right)$$

$$= \left(\hat{S}_z^{(1)}\otimes\hat{\mathbb{1}}\right)\left(|+\rangle\otimes|+\rangle\right) + \left(\hat{\mathbb{1}}\otimes\hat{S}_z^{(2)}\right)\left(|+\rangle\otimes|+\rangle\right)$$

$$= \hat{S}_z^{(1)}|+\rangle\otimes|+\rangle + |+\rangle\otimes\hat{S}_z^{(2)}|+\rangle$$

$$= +\frac{\hbar}{2}|+\rangle\otimes|+\rangle + |+\rangle\otimes\frac{\hbar}{2}|+\rangle$$

$$= \left(\frac{\hbar}{2}+\frac{\hbar}{2}\right)\left(|+\rangle\otimes|+\rangle\right) = \hbar\,\underline{\left(|+\rangle\otimes|+\rangle\right)}$$

$\uparrow\uparrow \Rightarrow \uparrow\ \hbar$

$\frac{\hbar}{2}\ \frac{\hbar}{2}$

$$\hat{S}_z(|-\rangle\otimes|-\rangle) = \left(\hat{S}_z^{(1)}\otimes\hat{\mathbb{1}} + \hat{\mathbb{1}}\otimes\hat{S}_z^{(2)}\right)\left(|-\rangle\otimes|-\rangle\right)$$

$$= \hat{S}_z^{(1)}|-\rangle\otimes|-\rangle + |-\rangle\otimes\hat{S}_z^{(2)}|-\rangle$$

$$= -\frac{\hbar}{2}|-\rangle\otimes|-\rangle + |-\rangle\otimes\left(-\frac{\hbar}{2}\right)|-\rangle \qquad \underset{-\frac{\hbar}{2}\,-\frac{\hbar}{2}}{\downarrow\downarrow} \Rightarrow \underset{-\hbar}{\downarrow}$$

$$= \left(-\frac{\hbar}{2}-\frac{\hbar}{2}\right)\left(|-\rangle\otimes|-\rangle\right) = (-\hbar)\left(|-\rangle\otimes|-\rangle\right)$$

$$\hat{S}_z^{Tot}(|+\rangle\otimes|-\rangle) = \left(\hat{S}_z^{(1)}\otimes\hat{\mathbb{1}} + \hat{\mathbb{1}}\otimes\hat{S}_z^{(2)}\right)\left(|+\rangle\otimes|-\rangle\right)$$

$$= \left(\hat{S}_z^{(1)}|+\rangle\otimes|-\rangle + |+\rangle\otimes\hat{S}_z^{(2)}|-\rangle\right)$$

$$= \left(\frac{\hbar}{2}-\frac{\hbar}{2}\right)\left(|+\rangle\otimes|-\rangle\right) = 0 \qquad \underset{\frac{\hbar}{2}\,-\frac{\hbar}{2}}{\uparrow\downarrow} \Rightarrow 0$$

Similarly

$$\hat{S}_z^{Tot}(|-\rangle\otimes|+\rangle) = 0$$

$\therefore$ If we want to find the total spin in z direction of the general state

$$|\Psi\rangle = \alpha_1|+\rangle\otimes|+\rangle + \alpha_2|+\rangle\otimes|-\rangle + \alpha_3|-\rangle\otimes|+\rangle + \alpha_4|-\rangle\otimes|-\rangle$$

$$\hat{S}_z^{Tot}|\Psi\rangle = \alpha_1\overset{\hbar}{\hat{S}_z^{Tot}(|+\rangle\otimes|+\rangle)} + \alpha_2\overset{0}{\cancel{\hat{S}_z^{Tot}(|+\rangle\otimes|-\rangle)}}$$

$$+\alpha_3\overset{0}{\cancel{\hat{S}_z^{Tot}(|-\rangle\otimes|+\rangle)}} + \alpha_4\overset{-\hbar}{\hat{S}_z^{Tot}(|-\rangle\otimes|-\rangle)}$$

$$= \alpha_1\hbar(|+\rangle\otimes|+\rangle) - \alpha_4\hbar(|-\rangle\otimes|-\rangle)$$

If we want to find the state with no total spin in z-direction, $\hat{S}_z^{Tot}|\Psi\rangle = 0$

$$= \alpha_1 \hbar (|+\rangle \otimes |+\rangle) - \alpha_4 \hbar (|-\rangle \otimes |-\rangle)$$

If we want to find the state with no total spin in z-direction, $\hat{S}_z^{Tot} |\Psi\rangle = 0$

$$\Rightarrow \quad \alpha_1 \hbar (|+\rangle \otimes |+\rangle) - \alpha_4 \hbar (|-\rangle \otimes |-\rangle) = 0$$

$$\alpha_1 = \alpha_4 = 0$$

∴ We do not allow $|+\rangle \otimes |+\rangle$ & $|-\rangle \otimes |-\rangle$

$\Rightarrow$ The state with no total spin in z-direction is

$$|\Psi\rangle = \alpha_2 |+\rangle \otimes |-\rangle + \alpha_3 |-\rangle \otimes |+\rangle$$

How about spin in x-direction?

Consider 1 particle system

$$\underset{|z,+\rangle}{|+\rangle} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \underset{|z,-\rangle}{|-\rangle} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \hat{S}_x = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\therefore \hat{S}_x |+\rangle = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2} |-\rangle \quad - \#$$

$$\hat{S}_x |-\rangle = \frac{\hbar}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\hbar}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\hbar}{2} |+\rangle \quad - *$$

Now consider the total spin in x-direction of 2-e^- system

$$\boxed{\hat{S}_x^{Tot} = (\hat{S}_x^{(1)} \otimes \hat{\mathbb{1}}) + (\hat{\mathbb{1}} \otimes \hat{S}_x^{(2)})}$$

Exercise

Now consider the total spin in x-direction of 2-e^- system

$$\boxed{\hat{S}_x^{Tot} = \left(\hat{S}_x^{(1)} \otimes \hat{\mathbb{1}}\right) + \left(\hat{\mathbb{1}} \otimes \hat{S}_x^{(2)}\right)}$$

Exercise

Check
$$\hat{S}_x^{tot}\left(|+\rangle \otimes |+\rangle\right) = ?$$
$$\hat{S}_x^{tot}\left(|+\rangle \otimes |-\rangle\right) = ?$$
$$\hat{S}_x^{tot}\left(|-\rangle \otimes |+\rangle\right) = ?$$
$$\hat{S}_x^{Tot}\left(|-\rangle \otimes |-\rangle\right) = ?$$

$$\hat{S}_x^{tot}\left(|+\rangle \otimes |+\rangle\right) = \left(\left(\hat{S}_x^{(1)} \otimes \hat{\mathbb{1}}\right) + \left(\hat{\mathbb{1}} \otimes \hat{S}_x^{(2)}\right)\right)\left(|+\rangle \otimes |+\rangle\right)$$
$$= \underbrace{\hat{S}_x^{(1)}|+\rangle}\otimes|+\rangle + |+\rangle\otimes\underbrace{\hat{S}_x^{(2)}|+\rangle}$$
$$= \frac{\hbar}{2}|-\rangle\otimes|+\rangle + |+\rangle\otimes\frac{\hbar}{2}|-\rangle$$
$$= \frac{\hbar}{2}\left(|-\rangle\otimes|+\rangle + |+\rangle\otimes|-\rangle\right)$$

$$\hat{S}_x^{tot}\left(|-\rangle \otimes |-\rangle\right) = \dots = \frac{\hbar}{2}\left(|+\rangle\otimes|-\rangle + |-\rangle\otimes|+\rangle\right)$$

$$\hat{S}_x^{tot}\left(|+\rangle \otimes |-\rangle\right) = \left(\left(\hat{S}_x^{(1)} \otimes \hat{\mathbb{1}}\right) + \left(\hat{\mathbb{1}} \otimes \hat{S}_x^{(2)}\right)\right)\left(|+\rangle \otimes |-\rangle\right)$$
$$= \hat{S}_x^{(1)}|+\rangle\otimes|-\rangle + |+\rangle\otimes\hat{S}_x^{(2)}|-\rangle$$
$$= \frac{\hbar}{2}|-\rangle\otimes|-\rangle + |+\rangle\otimes\frac{\hbar}{2}|+\rangle$$
$$= \frac{\hbar}{2}\left(|-\rangle\otimes|-\rangle + |+\rangle\otimes|+\rangle\right)$$

$$\hat{S}_x^{tot}\left(|-\rangle \otimes |+\rangle\right) = \dots = \frac{\hbar}{2}\left(|+\rangle\otimes|+\rangle + |-\rangle\otimes|-\rangle\right)$$

Now let's consider a state that has no total spin in z-direction & no total spin in x-direction

The state with no total spin in z-direction is

$$|\Psi\rangle = \alpha_2 |+\rangle \otimes |-\rangle + \alpha_3 |-\rangle \otimes |+\rangle$$

If we want to find the state with no spin in x-dir

$$\Rightarrow \hat{S}_x^{tot} |\Psi\rangle = \alpha_2 \hat{S}_x^{tot}(|+\rangle \otimes |-\rangle) + \alpha_3 \hat{S}_x^{tot}(|-\rangle \otimes |+\rangle)$$

$$= \alpha_2 \frac{\hbar}{2}\left(|+\rangle \otimes |+\rangle + |-\rangle \otimes |-\rangle\right)$$
$$+ \alpha_3 \frac{\hbar}{2}\left(|-\rangle \otimes |-\rangle + |+\rangle \otimes |+\rangle\right)$$

$$= (\alpha_2 + \alpha_3)\frac{\hbar}{2}\left(|+\rangle \otimes |+\rangle\right) \quad = 0$$
$$+ (\alpha_2 + \alpha_3)\frac{\hbar}{2}\left(|-\rangle \otimes |-\rangle\right)$$

Since the basis are linearly independent

$$\hat{S}_x^{tot} |\Psi\rangle = 0 \quad \Rightarrow \quad \alpha_2 + \alpha_3 = 0 \quad \Rightarrow \quad \alpha_3 = -\alpha_2$$

$$\therefore |\Psi\rangle = \alpha_2 |+\rangle \otimes |-\rangle - \alpha_2 |-\rangle \otimes |+\rangle$$

$$\boxed{|\Psi\rangle = \alpha\left(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle\right)}$$

no spin in $\hat{z}$ & $\hat{x}$ direction $\Rightarrow$ has no total spin!

Exercise This state has no total spin in y-direction!

$$\hat{S}_y |+\rangle = ? \quad , \quad \hat{S}_y |-\rangle = ? \qquad \frac{\hbar}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$$

We need to normalise α ! $\Rightarrow$ inner product of $V_1 \otimes V_2$

Suppose that we have $\{|e_i\rangle\}$ of V_1 & $\{|f_i\rangle\}$ of V_2 } orthonormal basis

$\Rightarrow$ It makes sense to require that the tensor product of $\{|e_i\rangle\}$ & $\{|f_i\rangle\}$ are orthonormal (no need to normalise again)

$$\{ |e_i\rangle \otimes |f_i\rangle \} = \text{orthonormal basis}$$

Let's write it systematically.

$$\langle e_i | e_j \rangle = \delta_{ij} \quad , \quad \langle f_i | f_j \rangle = \delta_{ij}$$

$\therefore$ We require that

$$\boxed{\big(\underbrace{\langle e_i| \otimes \langle f_m|}_{\langle \psi_{im}|}\big) \cdot \big(\underbrace{|e_j\rangle \otimes |f_n\rangle}_{|\psi_{jn}\rangle}\big) = \underline{\delta_{ij}}\,\underline{\delta_{mn}}}$$

Ex $\quad |e_1\rangle \otimes |f_2\rangle \quad \begin{cases} (\langle e_1| \otimes \langle f_2|) \cdot (|e_1\rangle \otimes |f_2\rangle) = 1 \\ \text{the rest} = 0 \end{cases}$

This looks more natural if we have

$$(\langle e_i| \otimes \langle f_m|) \cdot (|e_j\rangle \otimes |f_n\rangle) = \underbrace{\langle e_i | e_j \rangle}_{\delta_{ij}\,\times} \times \underbrace{\langle f_m | f_n \rangle}_{\delta_{mn}}$$

With this rule it is obvious to have a rule of inner product of tensor product as

With this rule it is obvious to have a rule of inner product of tensor product as

$$\boxed{(\langle v'| \otimes \langle w'|)\cdot(|v\rangle \otimes |w\rangle) = \langle v'|v\rangle \cdot \langle w'|w\rangle}$$

Ex Let's apply this to

$$|\Psi\rangle = \alpha\left(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle\right)$$

$$\langle\Psi|\Psi\rangle = \underbrace{\left(\langle +| \otimes \langle -| - \langle -| \otimes \langle +|\right)\alpha^*}_{\langle\Psi|} \cdot$$

$$\underbrace{\alpha\left(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle\right)}_{|\Psi\rangle}$$

$$= |\alpha|^2\Big((\langle +| \otimes \langle -|)\cdot(|+\rangle \otimes |-\rangle)$$
$$- (\langle -| \otimes \langle +|)\cdot(|+\rangle \otimes |-\rangle)$$
$$- (\langle +| \otimes \langle -|)\cdot(|-\rangle \otimes |+\rangle)$$
$$+ (\langle -| \otimes \langle +|)\cdot(|-\rangle \otimes |+\rangle) \Big)$$

$$= |\alpha|^2\Big(\overset{1}{\langle +|+\rangle}\,\overset{1}{\langle -|-\rangle} - \underset{0}{\langle -|+\rangle}\,\underset{0}{\langle +|-\rangle}$$
$$- \overset{0}{\langle +|-\rangle}\,\overset{0}{\langle -|+\rangle} + \overset{1}{\langle -|-\rangle}\,\overset{1}{\langle +|+\rangle} \Big)$$

$$= |\alpha|^2 \cdot (1 + 1) = 2|\alpha|^2$$

If we want to normalise $|\Psi\rangle$

$$\langle\Psi|\Psi\rangle = 1 = 2|\alpha|^2 \Rightarrow \alpha = \frac{1}{\sqrt{2}} \cancel{e^{i\phi}}$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle\right)$$

Entanglement states

Let's go back to the point that $V \otimes W$ is quite large!

$\Rightarrow$ There is a state that cannot be written as a tensor product of 2 vectors, i.e., $|v\rangle \otimes |w\rangle$

Ex. A 2-e^- system

e^- (1st) $|v\rangle$ e^- (2nd) $|w\rangle$ $\Rightarrow$ Let's put them together

$$|\Psi\rangle = |v\rangle \otimes |w\rangle$$

$\Rightarrow$ We can think of each e^- seperately

$\Rightarrow$ We know for sure that in this system, 1^{st} e^- is in state $|v\rangle$ & 2^{nd} e^- is in state $|w\rangle$

? What about

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle\right)$$

We cannot tell exactly what is the state of 1^{st} e^-

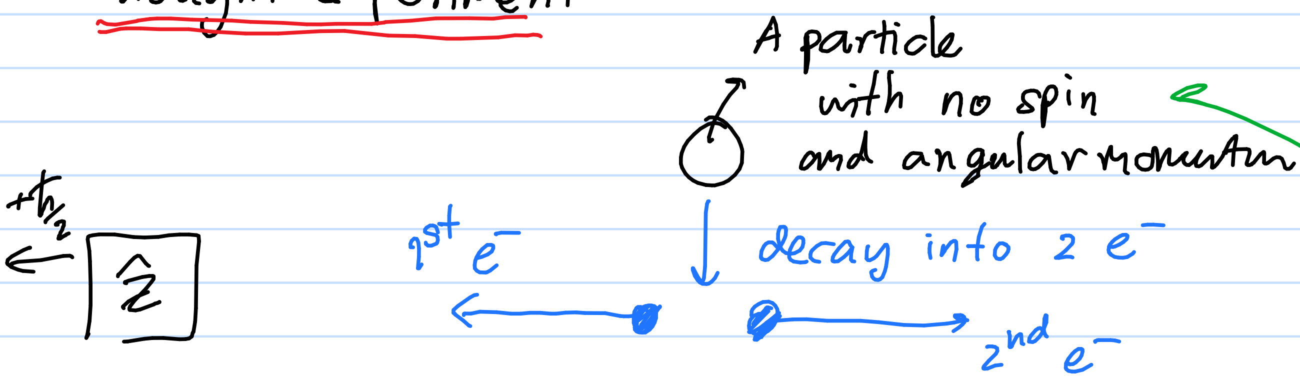
not $|+\rangle$, not $|-\rangle$, $\frac{1}{\sqrt{2}}(|+\rangle - |-\rangle)$

the same goes for 2^{nd} e^-

$\Rightarrow$ entangled state $\rightarrow$ obvious bec. we cannot tell the state of e^- exactly.

Also the state of 1^{st} e^- depends on the state of 2^{nd} e^-

Thought experiment



$\Rightarrow$ We cannot tell the spin of 1^{st} e^-

since spin conserves $\Rightarrow$ 1^{st} spin + 2^{nd} spin = 0

$\Rightarrow$ State of 1^{st} e^- depends state of 2^{nd} e^-

- If we measure the state of 1^{st} e^- to be $|+\rangle$

  $\Rightarrow$ then we can infer that the state of 2^{nd} e^- is $|-\rangle$
- Also if we have $|-\rangle$ for 1^{st} e^- then we know that 2^{nd} has $|+\rangle$

A more rigid way in order to identify entangled state!

A suppose that we have $V_1 : \{|e_1\rangle, |e_2\rangle\}$

& $V_2 : \{|f_1\rangle, |f_2\rangle\}$

$V_1 \otimes V_2 : \{ |e_1\rangle \otimes |f_1\rangle, |e_1\rangle \otimes |f_2\rangle, |e_2\rangle \otimes |f_1\rangle, |e_2\rangle \otimes |f_2\rangle \}$

$|\Psi\rangle \in V_1 \otimes V_2$

In general, we write

$$|\Psi\rangle = a_{11} |e_1\rangle \otimes |f_1\rangle + a_{12} |e_1\rangle \otimes |f_2\rangle + a_{21} |e_2\rangle \otimes |f_1\rangle + a_{22} |e_2\rangle \otimes |f_2\rangle$$

Note that we can write $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

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Note that we can write $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

Suppose that $|\Psi\rangle$ is not entangled (We can identify the state of each particle)

$$|\Psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle$$

$$|\Psi\rangle = (\alpha_1|e_1\rangle + \alpha_2|e_2\rangle) \otimes (\beta_1|f_1\rangle + \beta_2|f_2\rangle)$$

$$= \alpha_1\beta_1|e_1\rangle\otimes|f_1\rangle + \alpha_1\beta_2|e_1\rangle\otimes|f_2\rangle + \alpha_2\beta_1|e_2\rangle\otimes|f_1\rangle + \alpha_2\beta_2|e_2\rangle\otimes|f_2\rangle$$

A general state $|\Psi\rangle$ is not entangled if

$$a_{11} = \alpha_1\beta_1, \quad a_{12} = \alpha_1\beta_2, \quad a_{21} = \alpha_2\beta_1, \quad a_{22} = \alpha_2\beta_2$$

Let's consider

$$a_{11}a_{22} = \alpha_1\beta_1\alpha_2\beta_2, \quad a_{12}a_{21} = \alpha_1\beta_2\alpha_2\beta_1$$

$\therefore$ $a_{11}a_{22} - a_{12}a_{21} = 0$ if $\quad$ is true

$\therefore$ $a_{11}a_{22} - a_{12}a_{21} = 0$ if $|\Psi\rangle$ is not entangled!

$$\boxed{\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = 0 \text{ if } |\Psi\rangle \text{ is not entangled}}$$

Example

1). $V_1 = V_2 : \{ |+\rangle, |-\rangle \}$

$V_1 \otimes V_2 : \{ |+\rangle \otimes |+\rangle, |+\rangle \otimes |-\rangle, |-\rangle \otimes |+\rangle, |-\rangle \otimes |-\rangle \}$

$|\Psi\rangle = |+\rangle \otimes |-\rangle \Rightarrow$ not entangled

2). $|\Psi\rangle = \frac{1}{\sqrt{2}} |+\rangle \otimes |-\rangle - \frac{1}{\sqrt{2}} |-\rangle \otimes |-\rangle$

$= \frac{1}{\sqrt{2}} (|+\rangle - |-\rangle) \otimes |-\rangle$

We can tell their states $\Rightarrow$ not entangled.

$|\Psi\rangle = a_{11} |+\rangle \otimes |+\rangle + a_{12} |+\rangle \otimes |-\rangle + a_{21} |-\rangle \otimes |+\rangle + a_{22} |-\rangle \otimes |-\rangle$

$= 0 |+\rangle \otimes |+\rangle + \frac{1}{\sqrt{2}} |+\rangle \otimes |-\rangle + 0 |-\rangle \otimes |+\rangle - \frac{1}{\sqrt{2}} |-\rangle \otimes |-\rangle$

$\det \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = 0 \Rightarrow$ non-entangle

3). $|\Psi\rangle = \frac{1}{\sqrt{2}} (|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle)$

$|\Psi\rangle = \underset{0}{a_{11}} |+\rangle \otimes |+\rangle + \underset{\frac{1}{\sqrt{2}}}{a_{12}} |+\rangle \otimes |-\rangle + \underset{-\frac{1}{\sqrt{2}}}{a_{21}} |-\rangle \otimes |+\rangle + \underset{0}{a_{22}} |-\rangle \otimes |-\rangle$

$\det \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 \end{pmatrix} = 0 + \frac{1}{2} \neq 0 \Rightarrow$ entangle

4). $|\Psi\rangle = \frac{1}{\sqrt{2}} (|+\rangle \otimes |+\rangle - |-\rangle \otimes |-\rangle) \Rightarrow$ entangle ✓

$\det \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} = \frac{1}{2} \neq 0$

Entanglement as a communication tool

$\Rightarrow$ Quantum Information

In order to use the entanglement property, we start by finding a basis set of $V_1 \otimes V_2$ such that all of them are entangled.

$$V_1 \otimes V_2 : \{ |+\rangle \otimes |+\rangle,\ |+\rangle \otimes |-\rangle,\ |-\rangle \otimes |+\rangle,\ |-\rangle \otimes |-\rangle \}$$

non entangle

The Bell basis

$$|\Phi_0\rangle = \frac{1}{\sqrt{2}} (|+\rangle \otimes |+\rangle + |-\rangle \otimes |-\rangle) \qquad \det\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{pmatrix} \neq 0 \checkmark$$

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}} (|+\rangle \otimes |-\rangle + |-\rangle \otimes |+\rangle) \qquad \det\begin{pmatrix} 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \neq 0 \checkmark$$

$$|\Phi_2\rangle = \frac{i}{\sqrt{2}} (|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle) \qquad \det\begin{pmatrix} 0 & \frac{i}{\sqrt{2}} \\ -\frac{i}{\sqrt{2}} & 0 \end{pmatrix} \neq 0 \checkmark$$

$$|\Phi_3\rangle = \frac{1}{\sqrt{2}} (|+\rangle \otimes |+\rangle - |-\rangle \otimes |-\rangle) \qquad \det\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \neq 0 \checkmark$$

$\Rightarrow \{ |\Phi_0\rangle, |\Phi_1\rangle, |\Phi_2\rangle, |\Phi_3\rangle \}$ are orthonormal

Note that
$$|\Phi_3\rangle = (\mathbb{1} \otimes \hat{\sigma}_z) |\Phi_0\rangle$$
$$= (\mathbb{1} \otimes \hat{\sigma}_z) \left(\frac{1}{\sqrt{2}} |+\rangle \otimes |+\rangle + \frac{1}{\sqrt{2}} |-\rangle \otimes |-\rangle \right)$$
$$= \frac{1}{\sqrt{2}} |+\rangle \otimes \underline{\hat{\sigma}_z |+\rangle} + \frac{1}{\sqrt{2}} |-\rangle \otimes \underline{\hat{\sigma}_z |-\rangle}$$
$$= \frac{1}{\sqrt{2}} |+\rangle \otimes |+\rangle - \frac{1}{\sqrt{2}} |-\rangle \otimes |-\rangle$$

$$\hat{S}_z = \frac{\hbar}{2} \hat{\sigma}_z = \frac{\hbar}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\hat{\sigma}_z |+\rangle = |+\rangle$$

$$\hat{\sigma}_z |-\rangle = -|-\rangle$$

Also, $|\Phi_1\rangle = (\hat{\mathbb{1}} \otimes \hat{\sigma}_x)|\Phi_0\rangle$

$$= (\hat{\mathbb{1}} \otimes \hat{\sigma}_x)\left(\frac{1}{\sqrt{2}}|+\rangle\otimes|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\otimes|-\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|+\rangle\otimes\hat{\sigma}_x|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\otimes\hat{\sigma}_x|-\rangle$$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle\otimes|-\rangle + |-\rangle\otimes|+\rangle\right)$$

$$\boxed{\begin{aligned}\hat{\sigma}_x &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \therefore\ \hat{\sigma}_x|+\rangle &= |-\rangle \\ \hat{\sigma}_x|-\rangle &= |+\rangle\end{aligned}}$$

$$|\Phi_2\rangle = (\hat{\mathbb{1}} \otimes \hat{\sigma}_y)|\Phi_0\rangle$$

$$= (\hat{\mathbb{1}} \otimes \hat{\sigma}_y)\left(\frac{1}{\sqrt{2}}|+\rangle\otimes|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\otimes|-\rangle\right)$$

$$= \frac{1}{\sqrt{2}}|+\rangle\otimes\hat{\sigma}_y|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\otimes\hat{\sigma}_y|-\rangle$$

$$= \frac{1}{\sqrt{2}}i|+\rangle\otimes|-\rangle + \frac{1}{\sqrt{2}}(-i)|-\rangle\otimes|+\rangle$$

$$= \frac{i}{\sqrt{2}}\left(|+\rangle\otimes|-\rangle - |-\rangle\otimes|+\rangle\right)$$

$$\boxed{\begin{aligned}\hat{\sigma}_y &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ \hat{\sigma}_y|+\rangle &= i|-\rangle \\ \hat{\sigma}_y|-\rangle &= -i|+\rangle\end{aligned}}$$

$$\therefore \boxed{|\Phi_i\rangle = (\hat{\mathbb{1}} \otimes \hat{\sigma}_i)|\Phi_0\rangle} \quad \hat{\sigma}_1 = \hat{\sigma}_x,\ \hat{\sigma}_2 = \hat{\sigma}_y,\ \hat{\sigma}_3 = \hat{\sigma}_z$$

Exercise

Show that $\langle\Phi_i|\Phi_j\rangle = \delta_{ij}$

$|\Phi_i\rangle = (\hat{\mathbb{1}} \otimes \hat{\sigma}_i)|\Phi_0\rangle \Rightarrow \langle\Phi_i| = \langle\Phi_0|(\hat{\mathbb{1}} \otimes \underset{\hat{\sigma}_i}{\overset{\Downarrow}{\hat{\sigma}_i^\dagger}})$

Then use Pauli matrices to conclude that

$$\langle\Phi_i|\Phi_j\rangle = \delta_{ij}$$

From the old basis

$$\{ |+\rangle\otimes|+\rangle, |+\rangle\otimes|-\rangle, |-\rangle\otimes|+\rangle, |-\rangle\otimes|-\rangle \}$$

$$\Rightarrow \{ |\Phi_0\rangle, |\Phi_1\rangle, |\Phi_2\rangle, |\Phi_3\rangle \}$$

$$\left.\begin{aligned}
|\Phi_0\rangle &= \frac{1}{\sqrt{2}}\left(|+\rangle\otimes|+\rangle + |-\rangle\otimes|-\rangle \right) \\
|\Phi_1\rangle &= \frac{1}{\sqrt{2}}\left(|+\rangle\otimes|-\rangle + |-\rangle\otimes|+\rangle \right) \\
|\Phi_2\rangle &= \frac{i}{\sqrt{2}}\left(|+\rangle\otimes|-\rangle - |-\rangle\otimes|+\rangle \right) \\
|\Phi_3\rangle &= \frac{1}{\sqrt{2}}\left(|+\rangle\otimes|+\rangle - |-\rangle\otimes|-\rangle \right)
\end{aligned}\right\}$$

$$|\Phi_0\rangle + |\Phi_3\rangle = \frac{\overset{\sqrt{2}}{\cancel{2}}}{\cancel{\sqrt{2}}} |+\rangle\otimes|+\rangle + 0$$

$$|+\rangle\otimes|+\rangle = \frac{1}{\sqrt{2}}\left(|\Phi_0\rangle + |\Phi_3\rangle \right)$$

Similarly

$$|+\rangle\otimes|-\rangle = \frac{1}{\sqrt{2}}\left(|\Phi_1\rangle - i\,|\Phi_2\rangle \right)$$

$$|-\rangle\otimes|+\rangle = \frac{1}{\sqrt{2}}\left(|\Phi_1\rangle + i\,|\Phi_2\rangle \right)$$

$$|-\rangle\otimes|-\rangle = \frac{1}{\sqrt{2}}\left(|\Phi_0\rangle - |\Phi_3\rangle \right)$$

Measurement of states (with tensor product)

1). The full measurement of $\mathbb{V}_1 \otimes \mathbb{V}_2$

This is similar to $\{ |+\rangle, |-\rangle \}$

$$|\psi\rangle = \alpha_1 |+\rangle + \alpha_2 |-\rangle \Rightarrow P(|+\rangle) = |\langle + | \psi \rangle|^2$$

$$P(|-\rangle) = |\langle - | \psi \rangle|^2$$

1). The full measurement of $V_1 \otimes V_2$

☆ This is similar to $\{|+\rangle, |-\rangle\}$

$$|\psi\rangle = \alpha_1 |+\rangle + \alpha_2 |-\rangle \Rightarrow P(|+\rangle) = |\langle + | \psi \rangle|^2$$

$$P(|-\rangle) = |\langle - | \psi \rangle|^2 \Rightarrow$$

☆ We have a version of $V_1 \otimes V_2$

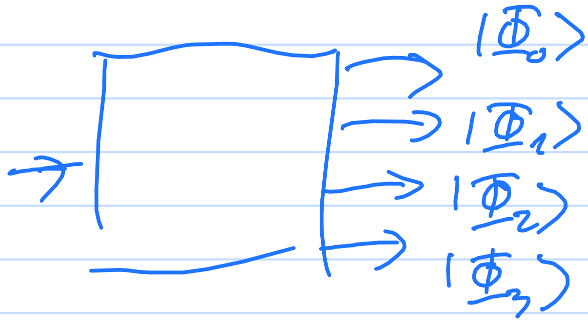
$$|\Psi\rangle = \alpha_0 |\Phi_0\rangle + \alpha_1 |\Phi_1\rangle + \alpha_2 |\Phi_2\rangle + \alpha_3 |\Phi_3\rangle$$

$$P(|\Phi_0\rangle) = |\langle \Phi_0 | \Psi \rangle|^2$$

$$P(|\Phi_1\rangle) = |\langle \Phi_1 | \Psi \rangle|^2$$

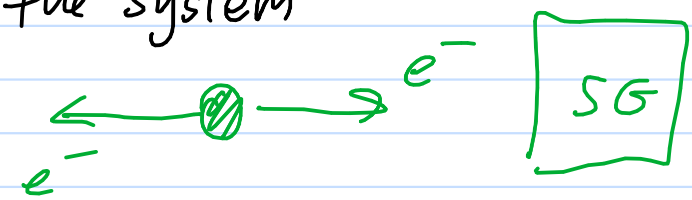
$$P(|\Phi_2\rangle) = |\langle \Phi_2 | \Psi \rangle|^2$$

$$P(|\Phi_3\rangle) = |\langle \Phi_3 | \Psi \rangle|^2$$



2). The partial measurement

We can also measure the state of one particle of the system



1) What is the prob of finding the state of the first particle?

2) After measurement, what is the state of the second particle?

Example

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle\right)$$

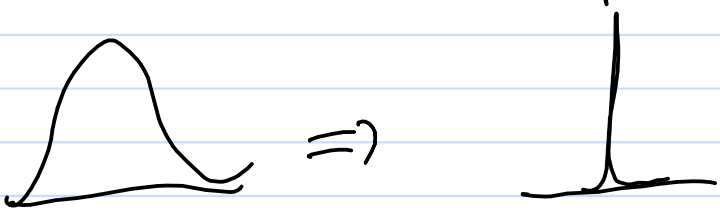
2   Prob

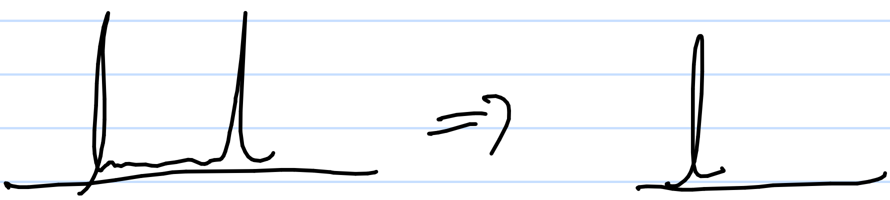
$$= |+\rangle \otimes \frac{1}{\sqrt{2}}|-\rangle + |-\rangle \otimes \left(-\frac{1}{\sqrt{2}}\right)|+\rangle)$$

The prob of finding the first particle with spin up $= \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} \Rightarrow$ After measurement $|+\rangle \otimes |-\rangle$

The prob of finding the first particle with spin down $= \left(\frac{1}{\sqrt{2}}\right)^2 = \frac{1}{2} \Rightarrow$ After measurement $|-\rangle \otimes |+\rangle$

$\Rightarrow$ Wave function collapse concept





EX $\quad |\Psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \otimes |-\rangle + |+\rangle \otimes |+\rangle\right)$

The prob of finding the first particle in $|+\rangle = 1$

$$|\Psi\rangle = |+\rangle \otimes \frac{1}{\sqrt{2}}\left(|-\rangle + |+\rangle\right)$$

The state after measurement $= |\Psi\rangle$

Partial measurement

Suppose we have $V_1 \otimes V_2$, we measure the first particle (V_1) with respect the basis $\{|e_i\rangle\}$.

Suppose that $|\Psi\rangle = \sum_i |e_i\rangle \otimes |v_i\rangle$

we can write $|\psi\rangle$ using normalisation of $|v_i\rangle$

$$|\Psi\rangle = \sum_i \sqrt{\langle v_i|v_i\rangle}\,|e_i\rangle \otimes \boxed{\frac{|v_i\rangle}{\sqrt{\langle v_i|v_i\rangle}}}$$

$|\vec{v}|$

unit vector

☆ The probability of finding 1^{st} particle in $|e_j\rangle$

$= \left(\sqrt{\langle v_j|v_j\rangle}\right)^2 = \langle v_j|v_j\rangle$

$$\boxed{\begin{array}{l} |\psi\rangle = \alpha_1|+\rangle + \alpha_2|-\rangle \\ P(|+\rangle) = \alpha_1^2 = |\langle +|\psi\rangle|^2 \end{array}}$$

☆ After measurement (and know that 1^{st} particle is in $|e_j\rangle$, the state of system is in

$$|e_j\rangle \otimes \frac{|v_j\rangle}{\sqrt{\langle v_j|v_j\rangle}}$$

Ex $\quad |\Psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \otimes |-\rangle - |-\rangle \otimes |+\rangle\right)$

We can write $|\Psi\rangle = |+\rangle \otimes \frac{1}{\sqrt{2}}|-\rangle + |-\rangle \otimes \left(\frac{-1}{\sqrt{2}}\right)|+\rangle$

$|\Psi\rangle = \sum_i |e_i\rangle \otimes |v_i\rangle$

Prob of find 1^{st} parnticle in $|+\rangle = \langle -|\frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}}|-\rangle = \frac{1}{2}$

" ——— " $\quad |-\rangle = \langle +|\left(\frac{-1}{\sqrt{2}}\right)\left(\frac{-1}{\sqrt{2}}\right)|+\rangle = \frac{1}{2}$

After measurement

If 1^{st} particle is $|+\rangle$, the system is in $|+\rangle \otimes |-\rangle$

" ——— " $|-\rangle$, " ——— " $|-\rangle \otimes |+\rangle$

Ex

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|+\rangle \otimes |-\rangle + |+\rangle \otimes |+\rangle\right)$$

$$|\Psi\rangle = |+\rangle \otimes \frac{1}{\sqrt{2}}\left(|-\rangle + |+\rangle\right)$$

$$|\Psi\rangle = \sum_i |e_i\rangle \otimes |v_i\rangle$$

The prob of finding 1^{st} particle in $|+\rangle$

$$= \frac{1}{\sqrt{2}}\left(\langle -| + \langle +|\right) \frac{1}{\sqrt{2}}\left(|-\rangle + |+\rangle\right)$$

$$= \frac{1}{2}\left(\overset{1}{\cancel{\langle -|-\rangle}} + \overset{1}{\cancel{\langle +|+\rangle}}\right) = 1$$

※ Note that $|\Psi\rangle$ must be normalised !

Remark We can change state $|\psi\rangle$ into $\sigma_i|\psi\rangle$

by apply magnetic field into a system

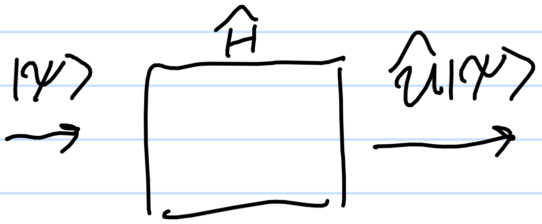
$\Rightarrow$ We can construct the Hamiltonian of a box

such that the time evolution $= \sigma_i$

$$|\psi\rangle \longrightarrow \boxed{\hat{H}} \longrightarrow \hat{U}|\psi\rangle$$

* We can choose

$$\hat{U} = e^{\underbrace{-i\frac{\pi}{2}(\hat{\mathbb{1}} - \hat{\sigma}_i)}_{\hat{H}t/\hbar}}$$

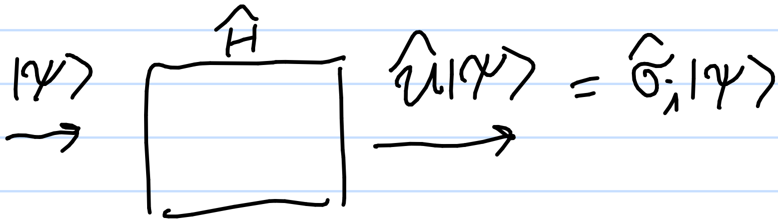


* We can choose

$$\hat{U} = e^{\underbrace{-i\frac{\pi}{2}(\hat{\mathbb{1}} - \hat{\sigma}_i)}_{\frac{\hat{H}t}{\hbar}}}$$

$$e^{\frac{i\pi}{2}\hat{\sigma}_i} = \hat{\mathbb{1}} - \hat{\sigma}_i \qquad \hat{U} = e^{-i\frac{\pi}{2}} e^{\frac{i\pi}{2}\hat{\sigma}_i} = \hat{\sigma}_i$$

⇒ We can construct the Hamiltonian of a box such that the time evolution = σ_i



Quantum Teleportation

⇒ Digital information is encoded in bit $\underbrace{\boxed{1}, \boxed{0}}$ ↗ classical bit

1 bit = 2 possible state

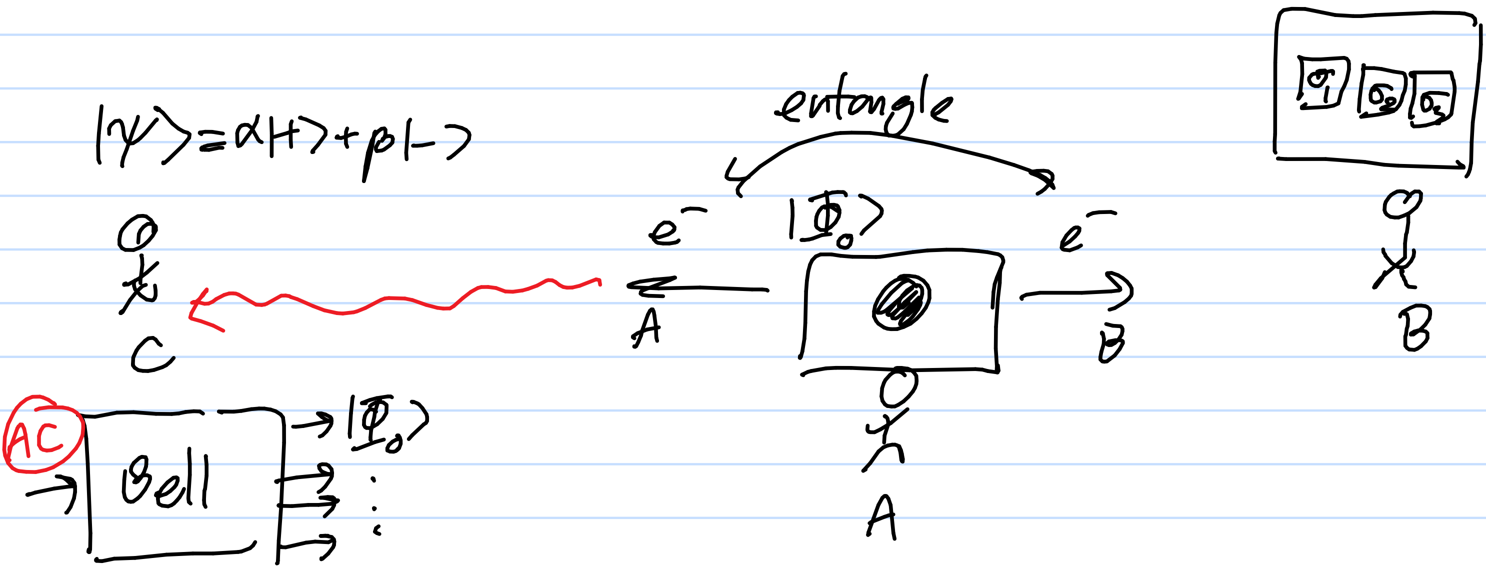
⇒ Quantum bit (qubit)

$$|\psi\rangle = \alpha|+\rangle + \beta|-\rangle \Rightarrow 1 \text{ bit} = 2 \text{ complex numbers (normalisation)}$$

⇒ Easier to perturb

⇒ We can use quantum property in order to "teleport" the qubit into other places

⇒ We donot use the measurement of value α, β & send this information classically



Let's do this mathematically.

Suppose $|\psi\rangle_C = \alpha|+\rangle_C + \beta|-\rangle_C$ is the qubit we want to send over

Suppose A send $|\Phi_0\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|+\rangle_A \otimes |+\rangle_B + |-\rangle_A \otimes |-\rangle_B\right)$

The state representing of 3 particles

$$|\Psi\rangle = |\Phi_0\rangle_{AB} \otimes |\psi\rangle_C$$

$$= \frac{1}{\sqrt{2}}\left(|+\rangle_A \otimes |+\rangle_B + |-\rangle_A \otimes |-\rangle_B\right) \otimes \left(\alpha|+\rangle_C + \beta|-\rangle_C\right)$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big[\alpha|+\rangle_A \otimes |+\rangle_B \otimes |+\rangle_C + \alpha|-\rangle_A \otimes |-\rangle_B \otimes |+\rangle_C$$
$$+ \beta|+\rangle_A \otimes |+\rangle_B \otimes |-\rangle_C + \beta|-\rangle_A \otimes |-\rangle_B \otimes |-\rangle_C\Big]$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big[\alpha|+\rangle_A\otimes|+\rangle_B\otimes|+\rangle_C + \alpha|-\rangle_A\otimes|-\rangle_B\otimes|+\rangle_C$$

$$+\beta|+\rangle_A\otimes|+\rangle_B\otimes|-\rangle_C + \beta|-\rangle_A\otimes|-\rangle_B\otimes|-\rangle_C\Big]$$

Since A & C particles can be measured using Bell basis measurement.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big[\alpha(\underline{|+\rangle_A\otimes|+\rangle_C})\otimes|+\rangle_B + \alpha(\underline{|-\rangle_A\otimes|+\rangle_C})\otimes|-\rangle_B$$

$$+\beta(\underline{|+\rangle_A\otimes|-\rangle_C})\otimes|+\rangle_B + \beta(\underline{|-\rangle_A\otimes|-\rangle_C})\otimes|-\rangle_B\Big]$$

$$|+\rangle\otimes|+\rangle = \tfrac{1}{\sqrt{2}}(|\Phi_0\rangle + |\Phi_3\rangle)$$

$$|+\rangle\otimes|-\rangle = \tfrac{1}{\sqrt{2}}(|\Phi_1\rangle - i|\Phi_2\rangle)$$

$$|-\rangle\otimes|+\rangle = \tfrac{1}{\sqrt{2}}(|\Phi_1\rangle + i|\Phi_2\rangle)$$

$$|-\rangle\otimes|-\rangle = \tfrac{1}{\sqrt{2}}(|\Phi_0\rangle - |\Phi_3\rangle)$$

$$|\Psi\rangle = \frac{1}{2}\Big[\alpha(|\Phi_0\rangle_{AC} + |\Phi_3\rangle_{AC})\otimes|+\rangle_B$$

$$+\alpha(|\Phi_1\rangle_{AC} + i|\Phi_2\rangle_{AC})\otimes|-\rangle_B$$

$$+\beta(|\Phi_1\rangle_{AC} - i|\Phi_2\rangle_{AC})\otimes|+\rangle_B$$

$$+\beta(|\Phi_0\rangle_{AC} - |\Phi_3\rangle_{AC})\otimes|-\rangle_B\Big]$$

$$|\Psi\rangle = \frac{1}{2}\Big[\alpha\left(|\Phi_0\rangle_{AC} + |\Phi_3\rangle_{AC}\right)\otimes|+\rangle_B$$
$$+\alpha\left(|\Phi_1\rangle_{AC} + i|\Phi_2\rangle_{AC}\right)\otimes|-\rangle_B$$
$$+\beta\left(|\Phi_1\rangle_{AC} - i|\Phi_2\rangle_{AC}\right)\otimes|+\rangle_B$$
$$+\beta\left(|\Phi_0\rangle_{AC} - |\Phi_3\rangle_{AC}\right)\otimes|-\rangle_B\Big]$$

$$|\Psi\rangle = \frac{1}{2}|\Phi_0\rangle_{AC}\otimes\left(\alpha|+\rangle_B + \beta|-\rangle_B\right)$$
$$+\frac{1}{2}|\Phi_1\rangle_{AC}\otimes\left(\alpha|-\rangle_B + \beta|+\rangle_B\right)$$
$$+\frac{1}{2}|\Phi_2\rangle_{AC}\otimes\left(i\alpha|-\rangle_B - i\beta|+\rangle_B\right)$$
$$+\frac{1}{2}|\Phi_3\rangle_{AC}\otimes\left(\alpha|+\rangle_B - \beta|-\rangle_B\right)$$

Suppose that C measure A&C particle

A&C $\rightarrow$ $\boxed{\text{Bell basis}}$ $\rightarrow$ $|\Phi_0\rangle$ $\qquad |\Psi\rangle_B = \alpha|+\rangle_B + \beta|-\rangle_B$

= original qubit

Quantum teleportation

What if C measure A&C particle and get

A&C $\rightarrow$ $\boxed{\text{Bell basis}}$ $\rightarrow$ $|\Phi_1\rangle$ $\qquad |\Psi\rangle_B = \alpha|-\rangle_B + \beta|+\rangle_B$

C then tells B that you should apply $\hat{\sigma}_1$

What if C measure A & C particle and get

$$A\&C \longrightarrow \boxed{\text{Bell basis}} \longrightarrow |\Phi_1\rangle \qquad |\psi\rangle_B = \alpha|-\rangle_B + \beta|+\rangle_B$$

C then tells B that you should apply $\hat{\sigma}_1$
(classical infor)

$$|\psi'_B\rangle = \hat{\sigma}_1|\psi_B\rangle$$
$$= \alpha\hat{\sigma}_1|-\rangle_B + \beta\hat{\sigma}_1|+\rangle_B$$
$$= \alpha|+\rangle_B + \beta|-\rangle_B$$

Original qubit !

What if C measure A&C particle and get

$$A\&C \longrightarrow \boxed{\text{Bell basis}} \longrightarrow |\Phi_2\rangle$$

C tells B to apply $\hat{\sigma}_2$

$$|\psi'_B\rangle = \hat{\sigma}_2|\psi_B\rangle$$
$$= \text{original qubit}$$

If C measure A&C particle and get

$$A\&C \longrightarrow \boxed{\text{Bell basis}} \longrightarrow |\Phi_3\rangle$$

C tells B to apply $\hat{\sigma}_3$

$$|\psi'_B\rangle = \hat{\sigma}_3|\psi_B\rangle$$
$$= \text{original qubit}$$

Homework 4

1) Use the properties of tensor product to work out the action of

$$\hat{S}_x^T \equiv \hat{S}_x \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_x$$

$$\hat{S}_y^T \equiv \hat{S}_y \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_y$$

$$\hat{S}_z^T \equiv \hat{S}_z \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_z$$

on Bell basis $\{|\Phi_0\rangle, |\Phi_1\rangle, |\Phi_2\rangle, |\Phi_3\rangle\}$

2) Let's work in $\{|z,\pm\rangle\}$ basis for matrix representation.

2.1) Find the length of $|A\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} i \\ 2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ i \end{pmatrix}$

2.2) Find the length of $|A'\rangle = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1+i \\ 2+i \end{pmatrix}$. Is it equal to the length of $|A\rangle$?

2.3) Consider the state $|A\rangle$ in 2.1). Is this an entangled state? (Please show this explicitly)

2.4) Consider $|A\rangle$. What are the probabilities of a measurement $\hat{S}_z$ on the first particle giving the values i) $\hbar/2$ ii) $-\frac{\hbar}{2}$? What are the states of the second particle after the partial measurement?

2.5) Consider $|A\rangle$. What are the probabilities of measuring $\hat{S}_z^T = \hat{S}_z \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_z$ and obtain values i) $\hbar$ ii) $-\hbar$

Angular Momentum

QM in 3D $\Rightarrow$ rotation $\rightarrow$ there is a bigger structure

↳ Hydrogen atom

QM in 1D $\hat{x}, \hat{p} \Rightarrow \hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x})$

we cannot tell x & p simultaneously

$[\hat{x}, \hat{p}] \neq 0 \Rightarrow \Delta x \Delta p \geq$ [illegible]

$[\hat{x}, \hat{p}] = i\hbar \Rightarrow \Delta x \Delta p \geq \frac{\hbar}{2}$

QM in 3D

Can we tell x, y simultaneously? Yes! no uncertainty!

$[\hat{x}, \hat{y}] = 0, \quad [\hat{x}, \hat{z}] = 0, \quad [\hat{y}, \hat{z}] = 0$ ✓

Can we tell p_x, p_y simultaneously? Yes!

$[\hat{p}_x, \hat{p}_y] = [\hat{p}_y, \hat{p}_z] = [\hat{p}_z, \hat{p}_x] = 0$ ✓

Can we tell x, p_y simultaneously? Yes

$[\hat{x}, \hat{p}_x] = i\hbar, \quad [\hat{y}, \hat{p}_y] = i\hbar, \quad [\hat{z}, \hat{p}_z] = i\hbar$ ✓

$\Rightarrow [\hat{x}, \hat{p}_y] = 0, \quad [\hat{x}, \hat{p}_z] = 0, \quad [\hat{y}, \hat{p}_x] = 0$ ✓

$[\hat{y}, \hat{p}_z] = 0, \quad [\hat{z}, \hat{p}_x] = 0, \quad [\hat{z}, \hat{p}_y] = 0$ ✓

We now use indices

$\hat{x}_1 = \hat{x}, \quad \hat{x}_2 = \hat{y}, \quad \hat{x}_3 = \hat{z}, \quad \hat{p}_1 = \hat{p}_x, \quad \hat{p}_2 = \hat{p}_y, \quad \hat{p}_3 = \hat{p}_z$

$$\Rightarrow \boxed{[\hat{x}_i, \hat{x}_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}}$$

Let's consider angular momentum in CM

$\vec{L} = \vec{r} \times \vec{p}$

$\vec{r} = x_1\hat{e}_1 + x_2\hat{e}_2 + x_3\hat{e}_3$

$\vec{p} = p_1\hat{e}_1 + p_2\hat{e}_2 + p_3\hat{e}_3$

$$\vec{L} = L_1\hat{e}_1 + L_2\hat{e}_2 + L_3\hat{e}_3 = \left|\begin{matrix} \hat{e}_1 & \hat{e}_2 & \hat{e}_3 \\ x_1 & x_2 & x_3 \\ p_1 & p_2 & p_3 \end{matrix}\right| \begin{matrix} \hat{e}_1 & \hat{e}_2 \\ x_1 & x_2 \\ p_1 & p_2 \end{matrix}$$

Unit vector in x_1, x_2, x_3 direction (Cartesian)

$$= \hat{e}_1(x_2p_3 - x_3p_2) + \hat{e}_2(x_3p_1 - p_3x_1) + \hat{e}_3(x_1p_2 - x_2p_1)$$

We kept order x, p

$$\Rightarrow \boxed{L_1 = x_2p_3 - x_3p_2}, \quad L_2 = x_3p_1 - p_3x_1, \quad L_3 = x_1p_2 - x_2p_1$$

We now use Levi-Civita symbol ε_{ijk}

$$\varepsilon_{123} = 1, \quad \varepsilon_{ijk} = -\varepsilon_{ikj}, \quad \varepsilon_{ijk} = -\varepsilon_{kji}, \quad \varepsilon_{ijk} = -\varepsilon_{jik}$$

antisymmetric

$$\varepsilon_{iik} = 0$$

$$\Rightarrow \quad L_1 = \underbrace{\varepsilon_{123}}_{(+1)} x_2p_3 + \underbrace{\varepsilon_{132}}_{(-1)} x_3p_2 + \cancel{\varepsilon_{111}x_1p_1}^{0} + \cancel{\varepsilon_{112}x_1p_2}^{0} + \cancel{\varepsilon_{121}x_2p_1}^{0} + \cancel{\varepsilon \ldots}^{0}$$

$$L_1 = \sum_{j,k} \varepsilon_{1jk}x_jp_k$$

Similarly $\quad L_2 = \sum_{j,k} \varepsilon_{2jk}x_jp_k, \quad L_3 = \sum_{j,k} \varepsilon_{3jk}x_jp_k$

$$\Rightarrow \quad \boxed{L_i = \sum_{j,k} \varepsilon_{ijk}x_jp_k}$$

$$\left[(\vec{A}\times\vec{B})_i = \sum_{jk} \varepsilon_{ijk}A_jB_k \right]$$

CM

Now let's consider QM angular momentum.

$$\hat{\vec{r}} = \underbrace{\hat{X}_1}_{\text{operator}} \underbrace{\vec{e}_1}_{\text{unit vector in 3D}} + \hat{X}_2\vec{e}_2 + \hat{X}_3\vec{e}_3 \qquad \hat{\vec{p}} = \hat{P}_1\vec{e}_1 + \hat{P}_2\vec{e}_2 + \hat{P}_3\vec{e}_3$$

$$\hat{\vec{L}} = \hat{L}_1\vec{e}_1 + \hat{L}_2\vec{e}_2 + \hat{L}_3\vec{e}_3$$

$$\therefore \quad \hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}} \Rightarrow \boxed{\hat{L}_i = \sum_{j,k} \varepsilon_{ijk} \hat{X}_j \hat{P}_k}$$

☆ Can we write $\hat{\vec{L}} = -\hat{\vec{p}} \times \hat{\vec{r}}$?

In CM $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$, is this in QM?

$\Rightarrow$ In general, the answer is no! but in our case we can switch!

General case: $\hat{\vec{A}} = \sum_i \hat{A}_i \vec{e}_i$, $\hat{\vec{B}} = \sum_i \hat{B}_i \vec{e}_i$

$$(\hat{\vec{A}} \times \hat{\vec{B}})_i = \sum_{j,k} \varepsilon_{ijk} \hat{A}_j \hat{B}_k = \sum_{j,k} \varepsilon_{ijk} \big(\underbrace{\hat{A}_j \hat{B}_k - \hat{B}_k \hat{A}_j}_{[\hat{A}_j, \hat{B}_k]} + \hat{B}_k \hat{A}_j\big)$$

$$= \sum_{j,k} \varepsilon_{ijk} [\hat{A}_j, \hat{B}_k] + \sum_{j,k} \underbrace{\varepsilon_{ijk}}_{= -\varepsilon_{ikj}} \hat{B}_k \hat{A}_j = -\sum_{j,k} \varepsilon_{ikj} \hat{B}_k \hat{A}_j = -(\hat{\vec{B}} \times \hat{\vec{A}})_i$$

$$\boxed{(\hat{\vec{A}} \times \hat{\vec{B}})_i = -(\hat{\vec{B}} \times \hat{\vec{A}})_i + \sum_{j,k} \varepsilon_{ijk} [\hat{A}_j, \hat{B}_k]}$$

$$\Rightarrow (\hat{\vec{r}} \times \hat{\vec{r}})_i = -(\hat{\vec{r}} \times \hat{\vec{r}})_i + \sum_{j,k} \varepsilon_{ijk} \cancel{[\hat{X}_j, \hat{X}_k]}^{\,0}$$

$$(\hat{\vec{r}} \times \hat{\vec{r}})_i = -(\hat{\vec{r}} \times \hat{\vec{r}})_i \Rightarrow \boxed{\hat{\vec{r}} \times \hat{\vec{r}} = 0}$$

Similar $\boxed{\hat{\vec{p}} \times \hat{\vec{p}} = 0}$ (same reasoning)

$$\Rightarrow (\hat{\vec{r}} \times \hat{\vec{p}})_i = -(\hat{\vec{p}} \times \hat{\vec{r}})_i + \sum_{j,k} \underbrace{\varepsilon_{ijk} \cancel{[\hat{X}_j, \hat{P}_k]}^{\,i\hbar\delta_{jk}}}_{=0} \quad j=k$$

$$\boxed{\hat{\vec{r}} \times \hat{\vec{p}} = -\hat{\vec{p}} \times \hat{\vec{r}}} \quad \checkmark$$

Is $\hat{L}$ Hermitian? (real eigenvalue)

$$\Rightarrow \quad \hat{L}_i = \sum_{j,k} \varepsilon_{ijk} \hat{X}_j \hat{p}_k$$

Hermitian

$\Downarrow$

$\hat{X}^+ = \hat{X}$

$\hat{p}^+ = \hat{p}$

$$\Rightarrow \quad \hat{L}_i^+ = \sum_{j,k} \varepsilon_{ijk} (\hat{X}_j \hat{p}_k)^+ = \sum_{j,k} \varepsilon_{ijk} \hat{p}_k^+ \hat{X}_j^+$$

$$= \sum_{j,k} \varepsilon_{ijk} \hat{p}_k \hat{X}_j = -\sum_{j,k} \varepsilon_{ikj} \hat{p}_k \hat{X}_j$$

$$= -(\hat{\vec{p}} \times \hat{\vec{r}})_i = (\hat{\vec{r}} \times \hat{\vec{p}})_i = \hat{L}_i$$

$\Uparrow$

Since $\hat{\vec{r}} \times \hat{\vec{p}} = -\hat{\vec{p}} \times \hat{\vec{r}}$

$$\therefore \quad \hat{L}_i^+ = \hat{L}_i \quad \text{Hermitian}$$

We will now want to deal with $\hat{L}^2 = \hat{L}\cdot\hat{L}$

$$\hat{L}^2 = \hat{L}_1^2 + \hat{L}_2^2 + \hat{L}_3^2 \qquad (\underline{\vec{e}_1, \vec{e}_2, \vec{e}_3 = \text{orthonormal}})$$

What about $\hat{r}\cdot\hat{L} = \sum_i \hat{x}_i \hat{L}_i$

$$= \sum_i \hat{x}_i \left(\sum_{j,k} \varepsilon_{ijk} \hat{x}_j \hat{p}_k\right)$$

$$= \sum_{i,j,k} \varepsilon_{ijk} \cancel{\hat{x}_i \hat{x}_j \hat{p}_k} \quad 0$$

$$[\hat{x}_i, \hat{x}_j] = 0$$

$$\underbrace{\varepsilon_{123}}_{(+1)} \hat{x}_1 \hat{x}_2 \hat{p}_3 + \underbrace{\varepsilon_{213}}_{(-1)} \underbrace{\hat{x}_2 \hat{x}_1}_{\hat{x}_1\hat{x}_2} \hat{p}_3$$

there is alway a term to cancel

$$\therefore \quad \boxed{\hat{r}\cdot\hat{L} = 0}$$

Similarly, $\boxed{\hat{p}\cdot\hat{L} = 0}$

Now consider $\hat{L}^2 = (\hat{r}\times\hat{p})^2$

We will show that

$$\boxed{(\hat{L})^2 = \hat{r}^2\hat{p}^2 - (\hat{r}\cdot\hat{p})^2 + i\hbar\, \hat{r}\cdot\hat{p}}$$

$$\hat{L}^2 = (\hat{r}\times\hat{p})^2 = \sum_i (\hat{r}\times\hat{p})_i (\hat{r}\times\hat{p})_i$$

$$= \sum_i \left(\sum_{j,k} \varepsilon_{ijk} \hat{x}_j \hat{p}_k\right)\left(\sum_{m,n} \varepsilon_{imn} \hat{x}_m \hat{p}_n\right)$$

$$\hat{L}^2 = \sum_{i,j,k,m,n} (\varepsilon_{ijk}\varepsilon_{imn}) \hat{x}_j \hat{p}_k \hat{x}_m \hat{p}_n$$

$$\hat{L}^2 = \sum_{i,j,k,m,n} (\varepsilon_{ijk}\varepsilon_{imn})\hat{x}_j\hat{p}_k\hat{x}_m\hat{p}_n$$

Property of $\varepsilon_{ijk} \rightsquigarrow \delta_{ij}$

1). $$\sum_i \varepsilon_{ijk}\varepsilon_{imn} = \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}$$

$j=2,\ k=3,\ m=2,\ n=3$

$$\underbrace{\varepsilon_{123}\varepsilon_{123}}_{1} + \cancel{\varepsilon_{223}}\varepsilon_{223} + \cancel{\dots} = 1\times 1 - 0\times 0 = 1$$

2). $$\sum_{i,j} \varepsilon_{ijk}\varepsilon_{ijm} = 2\delta_{km}$$

$$\therefore\ \hat{L}^2 = \sum_{j,k,m,n} (\delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km})\hat{x}_j\hat{p}_k\hat{x}_m\hat{p}_n$$

$$= \sum_{j,k,\cancel{m},\cancel{n}} \underbrace{\delta_{jm}}_{m=j}\underbrace{\delta_{kn}}_{n=k}\hat{x}_j\hat{p}_k\hat{x}_{\underset{j}{m}}\hat{p}_{\underset{k}{n}} - \sum_{j,k,\cancel{m},\cancel{n}} \underbrace{\delta_{jn}}_{n=j}\underbrace{\delta_{km}}_{m=k}\hat{x}_j\hat{p}_k\hat{x}_{\underset{k}{m}}\hat{p}_{\underset{j}{n}}$$

$$= \sum_{j,k} \hat{x}_j\hat{p}_k\hat{x}_j\hat{p}_k - \sum_{j,k}\hat{x}_j\hat{p}_k\hat{x}_k\hat{p}_j \qquad \hat{\vec{r}}\cdot\hat{\vec{p}} = \sum_j \hat{x}_j\hat{p}_j$$

$$= \sum_{j,k}\left(\hat{x}_j(\underbrace{\hat{p}_k\hat{x}_j - \hat{x}_j\hat{p}_k}_{[\hat{p}_k,\hat{x}_j] = -i\hbar\delta_{kj}} + \hat{x}_j\hat{p}_k)\hat{p}_k - \hat{x}_j(\underbrace{\hat{p}_k\hat{x}_k - \hat{x}_k\hat{p}_k}_{[\hat{p}_k,\hat{x}_k] = -i\hbar\delta_{kk}} + \hat{x}_k\hat{p}_k)\hat{p}_j\right)$$

$$= \sum_{j,k}\left(-i\hbar\underbrace{\delta_{kj}}_{k=j}\hat{x}_j\hat{p}_k + \hat{x}_j\hat{x}_j\hat{p}_k\hat{p}_k + i\hbar\underbrace{\delta_{kk}}_{\delta_{11}+\delta_{22}+\delta_{33}=3}\hat{x}_j\hat{p}_j - \hat{x}_j\hat{x}_k\hat{p}_k\hat{p}_j\right)$$

$$= \sum_j\underbrace{(-i\hbar\hat{x}_j\hat{p}_j + 3i\hbar\hat{x}_j\hat{p}_j)}_{2i\hbar\hat{x}_j\hat{p}_j} + \left(\sum_j\hat{x}_j\hat{x}_j\right)\left(\sum_k\hat{p}_k\hat{p}_k\right) - \sum_{j,k}\hat{x}_j\hat{x}_k\hat{p}_k\hat{p}_j$$

$$= \sum_j \underbrace{\left(-i\hbar \hat{x}_j \hat{p}_j + 3 i\hbar \hat{x}_j \hat{p}_j\right)}_{2i\hbar \hat{x}_j \hat{p}_j} + \left(\sum_j \hat{x}_j \hat{x}_j\right)\left(\sum_k \hat{p}_k \hat{p}_k\right) - \sum_{j,k} \hat{x}_j \underbrace{\hat{x}_k \hat{p}_k}\hat{p}_j$$

$$\hat{x}_k \hat{x}_j$$

$$= 2i\hbar\, \hat{\vec{r}}\cdot\hat{\vec{p}} + (\hat{\vec{r}}\cdot\hat{\vec{r}})(\hat{\vec{p}}\cdot\hat{\vec{p}}) - \sum_{j,k} \hat{x}_k \Big(\underbrace{\hat{x}_j \hat{p}_k - \hat{p}_k \hat{x}_j}_{[\hat{x}_j, \hat{p}_k] = i\hbar\delta_{jk}} + \hat{p}_k \hat{x}_j\Big)\hat{p}_j$$

$$= 2i\hbar\, \hat{\vec{r}}\cdot\hat{\vec{p}} + (\hat{\vec{r}}\cdot\hat{\vec{r}})(\hat{\vec{p}}\cdot\hat{\vec{p}}) - \underbrace{\sum_{j,k} i\hbar\, \underbrace{\delta_{jk}}_{k=j} \hat{x}_k \hat{p}_j}_{i\hbar\, \hat{\vec{r}}\cdot\hat{\vec{p}}} - \sum_{j,k} \hat{x}_k \hat{p}_k \hat{x}_j \hat{p}_j$$

$$\boxed{\hat{L}^2 = (\hat{\vec{r}}\cdot\hat{\vec{r}})(\hat{\vec{p}}\cdot\hat{\vec{p}}) - (\hat{\vec{r}}\cdot\hat{\vec{p}})(\hat{\vec{r}}\cdot\hat{\vec{p}}) + i\hbar\, \hat{\vec{r}}\cdot\hat{\vec{p}}}$$

Algebra of $\hat{\vec{L}}$

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$$

$$[\hat{A}\hat{B}, \hat{C}] = [\hat{A}, \hat{C}]\hat{B} + \hat{A}[\hat{B}, \hat{C}]$$

Consider $[\hat{L}_i, \hat{x}_j] = \left[\sum_{m,n} \varepsilon_{imn} \hat{x}_m \hat{p}_n, \hat{x}_j\right]$

$$= \sum_{m,n} \varepsilon_{imn} \overset{0}{\cancel{[\hat{x}_m, \hat{x}_j]}}\hat{p}_n + \varepsilon_{imn} \hat{x}_m \underbrace{[\hat{p}_n, \hat{x}_j]}_{-i\hbar\delta_{nj}}$$

$$= \sum_{m,n} \varepsilon_{imn} \hat{x}_m (-i\hbar) \underbrace{\delta_{nj}}_{n=j}$$

$$= -i\hbar \sum_m \underbrace{\varepsilon_{imj}}_{(-1)} \hat{x}_m$$

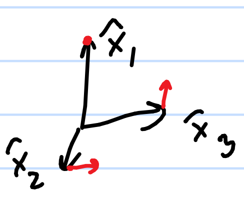
$$\boxed{[\hat{L}_i, \hat{x}_j] = +i\hbar \sum_m \varepsilon_{ijm} \hat{x}_m}$$

$$[\hat{L}_1, \hat{x}_1] = 0$$

$$[\hat{L}_1, \hat{x}_2] = +i\hbar \hat{x}_3$$

$$[\hat{L}_1, \hat{x}_3] = -i\hbar \hat{x}_2$$

↳ $\hat{x}_j$ is a vector under rotation



$\hat{L}_1$ = rotation generator around $\hat{x}_1$ axis

Now consider $[\hat{L}_i, \hat{p}_j] = \left[\sum_{m,n} \varepsilon_{imn} \hat{x}_m \hat{p}_n , \hat{p}_j\right]$

$$= \sum_{m,n} \Big(\underbrace{\varepsilon_{imn}}_{m \to j} \underbrace{[\hat{x}_m, \hat{p}_j]}_{i\hbar \underbrace{\delta_{mj}}_{m=j}} \hat{p}_n + \varepsilon_{imn} \hat{x}_m \cancelto{0}{[\hat{p}_n, \hat{p}_j]} \Big)$$

$$= i\hbar \sum_n \varepsilon_{ijn} \hat{p}_n$$

$$\boxed{[\hat{L}_i, \hat{p}_j] = i\hbar \sum_n \varepsilon_{ijn} \hat{p}_n}$$

$\hat{p}_j$ is a vector under rotation (similar to $\hat{x}_j$)

Now, let's do $[\hat{L}_i, \hat{L}_j] = \left[\hat{L}_i , \sum_{m,n} \varepsilon_{jmn} \hat{x}_m \hat{p}_n \right]$

$$= \sum_{m,n} \Big(\varepsilon_{jmn} \underbrace{[\hat{L}_i, \hat{x}_m]}_{i\hbar \sum_k \varepsilon_{imk} \hat{x}_k} \hat{p}_n + \varepsilon_{jmn} \hat{x}_m \underbrace{[\hat{L}_i, \hat{p}_n]}_{i\hbar \sum_k \varepsilon_{ink} \hat{p}_k} \Big)$$

$$= i\hbar \sum_{m,n,k} \Big((\varepsilon_{jmn} \varepsilon_{imk}) \hat{x}_k \hat{p}_n + (\varepsilon_{jmn} \varepsilon_{ink}) \hat{x}_m \hat{p}_k \Big)$$

$\varepsilon_{mjn} \varepsilon_{mik} \cdot (-1)^2$ $\qquad$ $\varepsilon_{nmj} \varepsilon_{nik} (-1)^2$

$(\delta_{ji} \delta_{nk} - \delta_{jk} \delta_{ni})$ $\qquad$ $(\delta_{mi} \delta_{jk} - \delta_{mk} \delta_{ji})$

$$= i\hbar \sum_{m,n,k} \Big(\delta_{ji} \underbrace{\delta_{nk}}_{n=k} \hat{x}_m \hat{p}_n - \delta_{jk} \underbrace{\delta_{ni}}_{n=i} \hat{x}_m \hat{p}_n + \underbrace{\delta_{mi}}_{m=i} \delta_{jk} \hat{x}_m \hat{p}_k - \underbrace{\delta_{mk}}_{m=k} \delta_{ji} \hat{x}_m \hat{p}_k \Big)$$

$$= i\hbar \Big(\sum_{m,k} \delta_{ji} \hat{x}_m \hat{p}_k - \sum_{m,k} \delta_{jk} \hat{x}_m \hat{p}_i + \sum_{n,k} \delta_{jk} \hat{x}_i \hat{p}_k - \sum_{n,k} \delta_{ji} \hat{x}_k \hat{p}_k \Big)$$

abort!

$$\hat{L}_1 = \sum_{j,k} \varepsilon_{1jk} \hat{X}_j \hat{P}_k = \hat{X}_2\hat{P}_3 - \hat{P}_3\hat{X}_2$$

$$\hat{L}_2 = \sum_{j,k} \varepsilon_{2jk} \hat{X}_j \hat{P}_k = \hat{X}_3\hat{P}_1 - \hat{X}_1\hat{P}_3$$

$$[\hat{L}_i, \hat{X}_j] = i\hbar \sum_k \varepsilon_{ijk} \hat{X}_k \ , \quad [\hat{L}_i, \hat{P}_j] = i\hbar \sum_k \varepsilon_{ijk} \hat{P}_k$$

$$[\hat{L}_1, \hat{X}_2] = i\hbar \hat{X}_3 \ , \quad [\hat{L}_1, \hat{X}_3] = -i\hbar \hat{X}_2$$

$$[\hat{L}_1, \hat{P}_2] = i\hbar \hat{P}_3 \ , \quad [\hat{L}_1, \hat{P}_3] = -i\hbar \hat{P}_2$$

$$\therefore \quad [\hat{L}_1, \hat{L}_2] = [\hat{L}_1, \hat{X}_3\hat{P}_1 - \hat{X}_1\hat{P}_3]$$

$$= [\hat{L}_1, \hat{X}_3\hat{P}_1] - [\hat{L}_1, \hat{X}_1\hat{P}_3]$$

$$= \hat{X}_3 \cancelto{0}{[\hat{L}_1, \hat{P}_1]} + \underbrace{[\hat{L}_1, \hat{X}_3]}_{-i\hbar\hat{X}_2}\hat{P}_1 - \cancelto{0}{[\hat{L}_1, \hat{X}_1]}\hat{P}_3 - \hat{X}_1 \underbrace{[\hat{L}_1, \hat{P}_3]}_{-i\hbar\hat{P}_2}$$

$$= -i\hbar \hat{X}_2\hat{P}_1 + i\hbar \hat{X}_1\hat{P}_2 = i\hbar \underbrace{(\hat{X}_1\hat{P}_2 - \hat{X}_2\hat{P}_1)}_{\hat{L}_3}$$

$$\therefore \quad [\hat{L}_1, \hat{L}_2] = i\hbar \hat{L}_3$$

Similarly $[\hat{L}_1, \hat{L}_3] = -i\hbar \hat{L}_2$

$$\boxed{[\hat{L}_i, \hat{L}_j] = i\hbar \sum_k \varepsilon_{ijk} \hat{L}_k}$$

$\hat{X}, \hat{P}$ $\hat{X}, \hat{P}$

$\hat{L}$ = vector under rotation

$$\bigstar \quad \boxed{[\hat{L}_i, \hat{L}_j] = i\hbar \sum_k \varepsilon_{ijk} \hat{L}_k}$$

Note $\hat{\sigma}_i = \{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ has a similar structure of algebra.

Let's consider the Hamiltonian of the form.

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\underbrace{|\hat{r}|^2}_{\text{magnitude of } \hat{r}})$$

$\Rightarrow$ Central Potential problem in CM

e.g. Gravity, Coulomb force $\quad \underbrace{\frac{Gm_1m_2}{r^2}}, \underbrace{\frac{kq_1q_2}{r^2}}$

(could be $-k|\vec{r}|$)

$\Rightarrow$ In CM, we can reduce the equation of motion in 3D into radial part (equation with r as a single variable)

(We use $L = $ const , $\vec{r} \times \vec{F} = 0 \Rightarrow \frac{d\vec{L}}{dt} = 0$)

What about QM?

In the form $\hat{H} = \frac{\hat{p}^2}{2m} + V(|\hat{r}|)$

? Is $\hat{L}_i$ conserved? $\Rightarrow$ Heisenberg picture

$$i\hbar \frac{d}{dt} \hat{L}_i^H = [\hat{L}_i^H, \hat{H}]$$

We can check $[\hat{L}_i, \hat{H}] = 0$

We can first check

$$[\hat{L}_i, \frac{\hat{p}^2}{2m}] = \frac{1}{2m} \sum_j [\hat{L}_i, \hat{P}_j \hat{P}_j]$$

$$= \frac{1}{2m} \sum_j \left(\underbrace{[\hat{L}_i, \hat{P}_j]}_{i\hbar \sum_k \varepsilon_{ijk} \hat{P}_k} \hat{P}_j + \hat{P}_j \underbrace{[\hat{L}_i, \hat{P}_j]}_{i\hbar \sum_k \varepsilon_{ijk} \hat{P}_k} \right)$$

$$= \frac{1}{2m}\sum_j \Big(\underbrace{[\hat{L}_i, \hat{p}_j]}_{i\hbar \sum_k \varepsilon_{ijk}\hat{p}_k} \hat{p}_j + \hat{p}_j \underbrace{[\hat{L}_i, \hat{p}_j]}_{i\hbar \sum_k \varepsilon_{ijk}\hat{p}_k} \Big)$$

$$= \frac{i\hbar}{2m}\sum_{j,k} \Big(\varepsilon_{ijk}\hat{p}_k\hat{p}_j + \varepsilon_{ijk}\hat{p}_j\hat{p}_k \Big) \rightarrow 0$$

$$\underset{(+1)}{\varepsilon_{123}}\hat{p}_2\hat{p}_3 + \ldots$$

$$+\underbrace{\varepsilon_{132}}_{-1}\underbrace{\hat{p}_3\hat{p}_2}_{\hat{p}_2\hat{p}_3} + \ldots$$

antisymmetric tensor contract with symmetric tensor $= 0$

$$\left[\hat{L}_i, \frac{\hat{p}^2}{2m}\right] = 0$$

Similarly we have $[\hat{L}_i, \hat{r}^2] = 0$

$$\therefore \quad \hat{H} = \frac{\hat{p}^2}{2m} + V(|\hat{r}|) \rightarrow V(\hat{r}^2)$$

$$\Rightarrow [\hat{L}_i, \hat{H}] = 0 \quad \Rightarrow \quad L = \text{conserved quantity.}$$

We fix the mistake last time.

$$[\hat{L}_i, \hat{L}_j] = i\hbar \sum_{m,n,k} \Big((\varepsilon_{jmn}\varepsilon_{imk})\hat{x}_{m}\hat{p}_n + (\varepsilon_{jmn}\varepsilon_{ink})\hat{x}_m\hat{p}_k \Big)$$

$$= i\hbar \sum_{m,n,k} \Big((\varepsilon_{jmn}\varepsilon_{imk})\hat{x}_k\hat{p}_n + (\varepsilon_{jmk}\varepsilon_{ikn})\hat{x}_m\hat{p}_n \Big)$$

$$= i\hbar \sum_{m,n,k} \Big(\varepsilon_{jmn}\varepsilon_{imk}\hat{x}_k\hat{p}_n + \varepsilon_{jkm}\varepsilon_{imn}\hat{x}_k\hat{p}_n \Big)$$

$$= i\hbar \sum_{m,n,k} \Big(\varepsilon_{jmn}\varepsilon_{imk} + \varepsilon_{jkm}\varepsilon_{imn} \Big)\hat{x}_k\hat{p}_n$$

$$= i\hbar \sum_{m,n,k} \left(\varepsilon_{jmn}\varepsilon_{imk} + \varepsilon_{jkm}\varepsilon_{imn} \right) \hat{x}_k \hat{p}_n$$

$$= i\hbar \sum_{n,k} \left(\sum_m \varepsilon_{mjn}\varepsilon_{mik} + \sum_m \varepsilon_{mkj}\varepsilon_{min} \right) \hat{x}_k \hat{p}_n$$

$$= i\hbar \sum_{n,k} \left(\cancel{\delta_{ji}\delta_{nk}} - \delta_{jk}\delta_{ni} + \delta_{ki}\delta_{jn} - \cancel{\delta_{kn}\delta_{ji}} \right) \hat{x}_k \hat{p}_n$$

$$= i\hbar \sum_{n,k} \underbrace{\left(\delta_{ki}\delta_{nj} - \delta_{kj}\delta_{ni} \right)}_{\sum_m \varepsilon_{mkn}\varepsilon_{mij}} \hat{x}_k \hat{p}_n$$

$$(\hat{\vec{r}} \times \hat{\vec{p}})_m = \hat{L}_m$$

$$\overbrace{\sum_{n,k} \varepsilon_{mkn} \hat{x}_k \hat{p}_n}$$

$$= i\hbar \sum_{m,n,k} \varepsilon_{mkn} \varepsilon_{mij} \hat{x}_k \hat{p}_n$$

$$= i\hbar \sum_m \varepsilon_{mij} \hat{L}_m = i\hbar \sum_m \varepsilon_{ijm} \hat{L}_m \quad \#$$

From last time we have $[\hat{L}_i, \hat{H}] = 0$

$\Rightarrow$ $\hat{L}_i$ & $\hat{H}$ share eigenvectors

$i = 3$ $(\hat{L}_z)$ $\Rightarrow$ $|\psi\rangle$ = eigenvectors of $\hat{L}_z$ & $\hat{H}$

Can we extend beyond $\hat{L}_z$ (include $\hat{L}_x$ & $\hat{L}_y$)?

Are $\hat{L}_x, \hat{L}_z$ sharing eigenvectors?

$$[\hat{L}_x, \hat{L}_z] \neq 0 = -i\hbar \hat{L}_y$$

$\Rightarrow$ We need to find a set of operators that share eigenvectors

$\Rightarrow$ We need to find a set of operators that share eigenvectors

$\{\hat{L}_z, \hat{H}\}$ can we include more?

$\Rightarrow$ $\cancel{\hat{L}_x}$, $\cancel{\hat{L}_y}$

$[\hat{L}_x, \hat{L}_z] \neq 0$ $\quad$ $[\hat{L}_x, \hat{L}_z] \neq 0$

But there is 1 more

$\hat{L}^2 \equiv \hat{\vec{L}}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$

Let's check this 1) $[\hat{L}_z, \hat{L}^2] = 0$?

2) $[\hat{H}, \hat{L}^2] = 0$?

1) $[\hat{L}_z, \hat{L}^2] = [\hat{L}_z, \hat{L}_x^2 + \hat{L}_y^2 + \cancelto{0}{\hat{L}_z^2}]$ $\rightarrow$ $[\hat{L}_z, \hat{L}_z] = 0$

$= [\hat{L}_z, \hat{L}_x^2] + [\hat{L}_z, \hat{L}_y^2]$

$\Sigma \varepsilon_i \ldots$

$[\hat{L}_x, \hat{L}_z] = -i\hbar \hat{L}_y$

$[\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y$

$[\hat{L}_y, \hat{L}_z] = i\hbar \hat{L}_x$

$[\hat{L}_z, \hat{L}_y] = -i\hbar \hat{L}_x$

$= [\hat{L}_z, \hat{L}_x]\hat{L}_x + \hat{L}_x[\hat{L}_z, \hat{L}_x]$

$+ [\hat{L}_z, \hat{L}_y]\hat{L}_y + \hat{L}_y[\hat{L}_z, \hat{L}_y]$

$= +\cancel{i\hbar \hat{L}_y \hat{L}_x} + i\hbar \cancel{\hat{L}_x \hat{L}_y}$ $= 0$

$+ -i\hbar \cancel{\hat{L}_x \hat{L}_y} - i\hbar \cancel{\hat{L}_y \hat{L}_x}$

$\therefore [\hat{L}_z, \hat{L}^2] = 0$

$\hat{L}_z \to \hat{L}_x \to \hat{L}_y \to \hat{L}_z$

$[\hat{L}_x, \hat{L}_y] = i\hbar \hat{L}_z$

Now 2). $[\hat{H}, \hat{L}^2] = [\hat{H}, \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2]$

$= \dots \cancel{[\hat{H}, \hat{L}_x]} + \dots \cancel{[\hat{H}, \hat{L}_y]}$

$+ \dots \cancel{[\hat{H}, \hat{L}_z]}$

bec. $[\hat{H}, \hat{L}_i] = 0$

$= 0$

$\therefore$ $\boxed{\{\hat{L}_z, \hat{H}, \hat{L}^2\} \text{ share eigenvectors!}}$

(This is for $\hat{H} = \frac{\hat{p}^2}{2m} + V(|\hat{r}|)$)

Note The fact that $\hat{L}_z, \hat{H}, \hat{L}^2$ share eigenvectors.

$\Rightarrow$ This is the reason why we have $\underbrace{\overset{\hat{H}}{n}, \overset{\hat{L}^2}{\ell}, \overset{\hat{L}_z}{m}}$ for

Hydrogen atom $\psi_{n\ell m}(r, \theta, \phi)$ quantum numbers

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Algebra of angular momentum

We will focus on shared eigenvectors of $\{\hat{L}_z, \hat{L}^2\}$

$\Rightarrow$ Since we know that we can apply

$$[\hat{L}_i, \hat{L}_j] = i\hbar \sum_k \varepsilon_{ijk} \hat{L}_k \Rightarrow [\hat{L}_z, \hat{L}^2] = 0$$

to the spin

$$[\hat{S}_i, \hat{S}_j] = \frac{i\hbar}{2} \sum_k \varepsilon_{ijk} \hat{S}_k \Rightarrow [\hat{S}_z, \hat{S}^2] = 0$$

$\Rightarrow$ We will combine $\hat{L}, \hat{S} \Rightarrow \hat{J}$ (could be both spin & angular momentum)

Theory of $\hat{J}_i$

1). $$[\hat{J}_i, \hat{J}_j] = i\hbar \sum_k \varepsilon_{ijk} \hat{J}_k$$

2). $$[\hat{J}_z, \hat{J}^2] = 0$$

The structure of $\hat{J}$ operators is very similar to the Harmonic oscillator.

Recap $\{\hat{x}, \hat{p}\} \Rightarrow \{\hat{a}, \hat{a}^+\}$, $\hat{N} = \hat{a}^+\hat{a} \propto \hat{H}$

$\hat{a}, \hat{a}^+, \hat{N}$ system

$$\boxed{[\hat{a}, \hat{a}^+] = 1 \quad , [\hat{N}, \hat{a}^+] = \hat{a}^+ , \quad [\hat{N}, \hat{a}] = -\hat{a}}$$

We can write $|n\rangle$ = eigenvector of $\hat{N}$

$$\hat{a}|n\rangle \propto |n-1\rangle \quad , \quad \hat{a}^+|n\rangle \propto |n+1\rangle$$

In our case, we can propose

(adjoint)
dagger

$$\hat{J}_+ = \hat{J}_x + i\hat{J}_y \quad , \quad \hat{J}_- = \hat{J}_x - i\hat{J}_y \Rightarrow \underbrace{\hat{J}_+}_{\text{plus}}{}^{+} = \underbrace{\hat{J}_-}_{\text{minus}}$$

What about we identify $\hat{J}_+ \Rightarrow \hat{a}^+$, $\hat{J}_- \Rightarrow \hat{a}$

Let's start with

$$\hat{J}_+\hat{J}_- = (\hat{J}_x + i\hat{J}_y)(\hat{J}_x - i\hat{J}_y) = \hat{J}_x^2 + i\hat{J}_y\hat{J}_x - i\hat{J}_x\hat{J}_y + \hat{J}_y^2$$

$$= \hat{J}_x^2 + \hat{J}_y^2 - i\underbrace{(\hat{J}_x\hat{J}_y - \hat{J}_y\hat{J}_x)}_{[\hat{J}_x, \hat{J}_y] = i\hbar\hat{J}_z}$$

$$\hat{J}_+\hat{J}_- = \hat{J}_x^2 + \hat{J}_y^2 + \hbar\hat{J}_z$$

$$\hat J_+\hat J_- = (\hat J_x + i\hat J_y)(\hat J_x - i\hat J_y) = \hat J_x^2 + i\hat J_y\hat J_x - i\hat J_x\hat J_y + \hat J_y^2$$

$$= \hat J_x^2 + \hat J_y^2 - i\underbrace{(\hat J_x\hat J_y - \hat J_y\hat J_x)}_{[\hat J_x,\hat J_y] = i\hbar\hat J_z}$$

$$\left.\begin{aligned} \hat J_+\hat J_- &= \hat J_x^2 + \hat J_y^2 + \hbar\hat J_z \\ \text{Also}\quad \hat J_-\hat J_+ &= \hat J_x^2 + \hat J_y^2 - \hbar\hat J_z \end{aligned}\right\} \quad (1)$$

$$\Rightarrow \quad [\hat J_+, \hat J_-] = \hat J_+\hat J_- - \hat J_-\hat J_+ = 2\hbar\hat J_z \quad \text{— } (2)$$

Consider (1)

$$\hat J_+\hat J_- = \underbrace{\hat J_x^2 + \hat J_y^2 + \hat J_z^2}_{} - \hat J_z^2 + \hbar\hat J_z$$

$$= \hat J^2 - \hat J_z^2 + \hbar\hat J_z$$

$$\left.\begin{aligned} \hat J^2 &= \hat J_+\hat J_- + \hat J_z^2 - \hbar\hat J_z \\ \text{or}\quad \hat J^2 &= \hat J_-\hat J_+ + \hat J_z^2 + \hbar\hat J_z \end{aligned}\right\} \quad \text{— } (3)$$

Next, $[\hat J_+, \hat J^2] = [\hat J_x + i\hat J_y, \hat J^2]$

$$= \cancelto{0}{[\hat J_x, \hat J^2]} + i\,\cancelto{0}{[\hat J_y, \hat J^2]} = 0$$

Also $\quad [\hat J_\pm, \hat J^2] = 0 \quad$ — (4)

Now, consider

$$[\hat J_z, \hat J_+] = [\hat J_z, \hat J_x + i\hat J_y] = \underbrace{[\hat J_z, \hat J_x]}_{+i\hbar\hat J_y} + i\underbrace{[\hat J_z, \hat J_y]}_{-i\hbar\hat J_x}$$

$$= i\hbar\hat J_y + \hbar\hat J_x = \hbar(\hat J_x + i\hat J_y)$$

$$\therefore \quad [\hat J_z, \hat J_+] = \hbar\hat J_+ \qquad \left([\hat N, \hat a^+] = \hat a^+\right)$$

$$[\hat{J}_z, \hat{J}_-] = [\hat{J}_z, \hat{J}_x - i\hat{J}_y] = \underbrace{[\hat{J}_z, \hat{J}_x]}_{i\hbar\hat{J}_y} - i\underbrace{[\hat{J}_z, \hat{J}_y]}_{-i\hbar\hat{J}_x}$$

$$= i\hbar\hat{J}_y - \hat{J}_x$$

$$= -\hbar(\hat{J}_x - i\hat{J}_y) = -\hbar\hat{J}_-$$

$\therefore$ We have $\boxed{[\hat{J}_z, \hat{J}_\pm] = \pm\hbar\hat{J}_\pm}$

(This is similar to $[\hat{N}, \hat{a}^\dagger] = +\hat{a}^\dagger$, $[\hat{N}, \hat{a}] = -\hat{a}$)

$\hat{N}|n\rangle = n|n\rangle \Rightarrow \hat{a}^\dagger|n\rangle \propto |n+1\rangle$, $\hat{a}|n\rangle \propto |n-1\rangle$

We can use $\hat{J}_\pm$ as ladder operators of $\hat{J}_z$

☆ Suppose that the eigenvectors of $\hat{J}_z$ is $|m\rangle$

$$\hat{J}_z|m\rangle = \hbar m|m\rangle \quad , \text{ m is a real number}$$

we can use $\hat{J}_\pm$ to raise $|m\rangle$

Consider $\hat{J}_+|m\rangle$, now lets apply $\hat{J}_z$

$$\hat{J}_z(\underbrace{\hat{J}_+|m\rangle}) = (\underbrace{\hat{J}_z\hat{J}_+ - \hat{J}_+\hat{J}_z}_{[\hat{J}_z, \hat{J}_+] = \hbar\hat{J}_+} + \hat{J}_+\hat{J}_z)|m\rangle$$

eigenvector of $\hat{J}_z$

$$= (\hbar\hat{J}_+ + \hat{J}_+\hat{J}_z)|m\rangle$$

$\hbar m|m\rangle$

$$= \hbar(\hat{J}_+|m\rangle) + \hat{J}_+(\hat{J}_z|m\rangle)$$

$$= \hbar(\hat{J}_+|m\rangle) + \hbar m\,\hat{J}_+|m\rangle$$

$$= (\hbar + \hbar m)(\hat{J}_+|m\rangle)$$

$$= \underbrace{\hbar(m+1)}_{\text{eigenvalue}}(\underbrace{\hat{J}_+|m\rangle}_{\text{eigenvectors of } \hat{J}_z})$$

$$\therefore \boxed{\hat{J}_+|m\rangle \propto |m+1\rangle}$$

Similarly, $\hat{J}_-|m\rangle \propto |m-1\rangle$

$\hat{J}_z$

$\vdots$

$|m+1\rangle, \propto \hat{J}_+|m\rangle$

$|m\rangle$

$|m-1\rangle \propto \hat{J}_-|m\rangle$

$\vdots$

But what happen to $\hat{J}^2$ $\Rightarrow$ since we know that $\{\hat{J}_z, \hat{J}^2\}$ share eigenvectors

$\Rightarrow$ Suppose that the shared eigenvectors are labeled by $|j, m\rangle$ such that

identify eigenvalue of $\hat{J}_z$

$$\hat{J}_z \underline{|j,m\rangle} = \hbar m \underline{|j,m\rangle}$$

identify eigenvalue of $\hat{J}^2$

$$\& \quad \hat{J}^2 \underline{|j,m\rangle} = \underbrace{\hbar^2 j(j+1)}\underline{|j,m\rangle}$$

note that we have an unusual convention $\Rightarrow$ $\hbar^2 j^2$

We already know that $\hat{J}_\pm|m\rangle \propto |m\pm1\rangle$

what about $\hat{J}_\pm|j\rangle = ?$

Let's check this

$$\hat{J}^2\hat{J}_+|j,m\rangle = (\underbrace{\hat{J}^2\hat{J}_+ - \hat{J}_+\hat{J}^2}_{[\hat{J}^2, \hat{J}_+] = 0} + \hat{J}_+\hat{J}^2)|j,m\rangle$$

$$= \hat{J}_+\hat{J}^2|j,m\rangle = \hat{J}_+(\hbar^2 j(j+1))|j,m\rangle$$

$$\hat{J}^2 \underbrace{\hat{J}_+|j,m\rangle} = (\underbrace{\hat{J}^2\hat{J}_+ - \hat{J}_+\hat{J}^2}_{[\hat{J}^2,\hat{J}_+]=0} + \hat{J}_+\hat{J}^2)|j,m\rangle$$

$$= \hat{J}_+\hat{J}^2|j,m\rangle = \hat{J}_+(\hbar^2 j(j+1))|j,m\rangle$$

$$= \underbrace{\hbar^2 j(j+1)}(\underbrace{\hat{J}_+|j,m\rangle})$$

eigenvalue ⇒ the same as $|j,m\rangle$

$$\therefore\ \hat{J}_+|j,m\rangle \propto |j,m+1\rangle$$

$$\Rightarrow\quad \boxed{\hat{J}_\pm|j,m\rangle \propto |j,m\pm1\rangle}\ *$$

Now let's find the normalization factor of $|j,m\rangle$

$$\langle j',m'|j,m\rangle = \delta_{jj'}\delta_{mm'}$$ orthonormal basis

From this we can find $C_\pm$ → coefficient

$$\hat{J}_\pm|j,m\rangle = C_\pm(j,m)|j,m\pm1\rangle$$ → normalised

Let's consider

$$\hat{J}_\pm = \hat{J}_x \pm i\hat{J}_y \Rightarrow \hat{J}_\pm^\dagger = \hat{J}_\mp$$

$$\|\hat{J}_\pm|j,m\rangle\|^2 = \langle j,m|\hat{J}_\pm^\dagger\hat{J}_\pm|j,m\rangle$$

$$= \langle j,m|\hat{J}_\mp\hat{J}_\pm|j,m\rangle$$

from

$$\left.\begin{aligned}\hat{J}^2 &= \hat{J}_+\hat{J}_- + \hat{J}_z^2 - \hbar\hat{J}_z\\ \hat{J}^2 &= \hat{J}_-\hat{J}_+ + \hat{J}_z^2 + \hbar\hat{J}_z\end{aligned}\right\} \Rightarrow \hat{J}_\mp\hat{J}_\pm = \hat{J}^2 - \hat{J}_z^2 \mp \hbar\hat{J}_z$$

$$\|\hat{J}_\pm|j,m\rangle\|^2 = \langle j,m|(\hat{J}^2 - \hat{J}_z^2 \mp \hbar\hat{J}_z)|j,m\rangle$$

$$\|\hat{J}_\pm|j,m\rangle\|^2 = \langle j,m|(\hat{J}^2-\hat{J}_z^2\mp\hbar\hat{J}_z)|j,m\rangle$$

$$\|C_\pm(j,m)|j,m\pm1\rangle\|^2 = \langle j,m\pm1|\,C_\pm^*(j,m)\,C_\pm(j,m)\,|j,m\pm1\rangle$$

$$= |C_\pm(j,m)|^2\,\underbrace{\langle j,m\pm1|j,m\pm1\rangle}_{1}$$

$$\therefore\quad |C_\pm(j,m)|^2 = \langle j,m|(\hat{J}^2-\hat{J}_z^2\mp\hbar\hat{J}_z)\underbrace{|j,m\rangle}_{\text{eigen vector}}$$

$$= \langle j,m|\left(\hbar^2 j(j+1)-(\hbar m)^2\mp\hbar\cdot\hbar m\right)|j,m\rangle$$

$$= \hbar^2\left(j(j+1)-m^2\underbrace{\mp}_{(-1)(\pm)} m\right)\underbrace{\langle j,m|j,m\rangle}_{1}$$

$$\therefore\quad \boxed{C_\pm(j,m) = \hbar\sqrt{j(j+1)-m(m\pm1)}}$$

$$\boxed{J_\pm|j,m\rangle = \hbar\sqrt{j(j+1)-m(m\pm1)}\;|j,m\pm1\rangle}$$

Recall that in QHO we have the lowest possible state $|0\rangle$

such that $\hat{a}\underbrace{|0\rangle}_{\text{vacuum state (ground state)}} = \underbrace{0}_{\text{zero vector}}$

Can we have such limit in this case?

Let's consider $\hat{J}_+|j,m\rangle = \hbar\sqrt{j(j+1)-m(m+1)}\;|j,m\pm1\rangle$

See next page for the real reason

~~if this term is negative then the whole coefficient is imaganary~~ $i=\sqrt{-1}$ $a+bi$

~~$\Rightarrow$ then the magnitude of $\hat{J}_+|j,m\rangle$ could be less than zero~~

$\Rightarrow$ In order to prevent this $j(j+1)-m(m+1)\geq 0$

The real reason why $j(j+1)-m(m+1)<0$

Consider

$$\hat{J}^2 = \hat{J}_x^2 + \hat{J}_y^2 + \hat{J}_z^2 = \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+) + \hat{J}_z^2$$

$$\therefore \hat{J}^2 - \hat{J}_z^2 = \frac{1}{2}(\hat{J}_+\hat{J}_- + \hat{J}_-\hat{J}_+)$$

$$\langle j,m|(\hat{J}^2 - \hat{J}_z^2)|j,m\rangle = \frac{1}{2}\left(\langle j,m|\hat{J}_-^\dagger\hat{J}_-|j,m\rangle + \langle j,m|\hat{J}_+^\dagger\hat{J}_+|j,m\rangle\right)$$

$$j(j+1)-m(m-1) + j(j+1)-m(m+1)$$

If this is negative

then $\hat{J}_z^2 > \hat{J}^2$ not possible

⇒ In order to prevent this $\underline{j(j+1)-m(m+1)\geq 0}$

This places a bound on m

Solve inequality ⇒ $j(j+1)\geq m(m+1)$

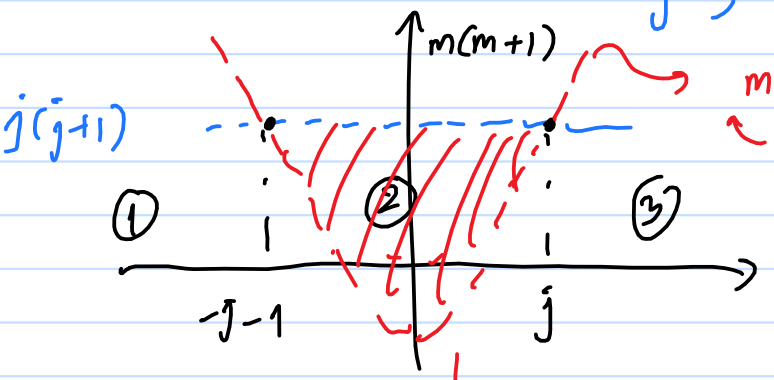
let's consider the saturation $j(j+1)=m(m+1)$

⇒ $m=j$, $m=-j-1$

$(-j-1)(-j-1+1)$
$=(-j-1)(-j)$
$=j(j+1)$

m^2+m ⇒ parabola

+



⇒ The range is

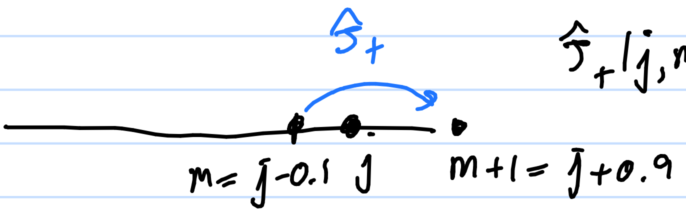
$m(m+1)\leq j(j+1)$

$-j-1\leq m\leq j$

A m is bounded by $-j-1\leq m\leq j$

m can be any value between these values

but if the maximum m differs from j by a fraction, ex $m=j-0.1$



$\hat{J}_+|j,m\rangle=\hbar\sqrt{\cancel{j(j+1)-m(m+1)}}\,|j,m+1\rangle$

imaginary
⇒ ill defined.

⇒ We can prevent this by choosing the maximum

$m=j$

$\hat{J}_+|j,j\rangle=\hbar\sqrt{\cancelto{0}{j(j+1)-j(j+1)}}\,|j,j+1\rangle = 0$

$$\hat{J}_+|j,j\rangle = \hbar\sqrt{\cancelto{0}{j(j+1)-j(j+1)}}\;|j,j+1\rangle = 0$$

$\therefore$ The state with highest value of m is $|j,j\rangle$

$-j-1 \le m \le j \Rightarrow$ the lowest of m is $|j,-j\cancel{-1}\rangle$

this is not true (yet)

Consider $\hat{J}_-|j,m\rangle = \hbar\sqrt{j(j+1)-m(m-1)}\;|j,m-1\rangle$

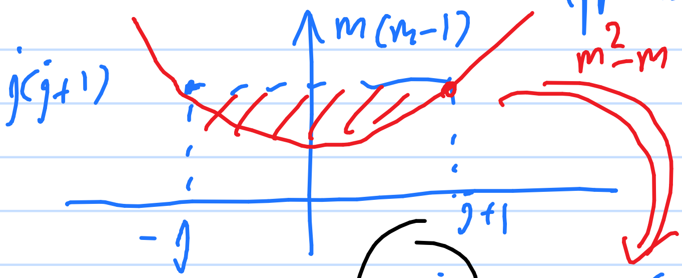
$\Rightarrow$ inside square root ≥ 0

$$\Rightarrow \quad j(j+1) \geq m(m-1)$$

The saturation happens when $m = j+1$, $m=-j$

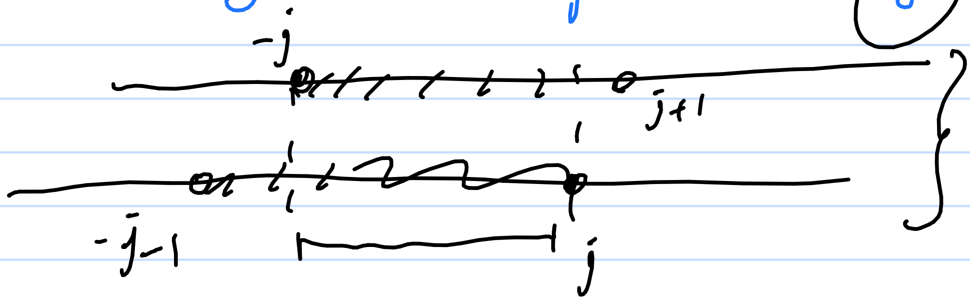
$m(m-1) = (j+1)(j+1-1) = j(j+1)$ ✓

$m(m-1) = -j(-j-1) = j(j+1)$ ✓



$\therefore$ $-j \le m \le j+1$

combining with $-j-1 \le m \le j$



$-j \le m \le j$

$\Rightarrow$ The lowest state is $|j,-j\rangle$

$$\hat{J}_-|j,-j\rangle = \hbar\sqrt{\cancelto{0}{j(j+1)-\underbrace{(-j)(-j-1)}_{j(j+1)}}}\;|j,-j-1\rangle = 0$$

$$\boxed{\text{The whole spectrum of eigenstates is } \{\,|j,-j\rangle,\ |j,-j+1\rangle,\ \ldots\ldots,\ |j,j-1\rangle,\ |j,j\rangle\,\}}$$

The whole spectrum of eigenstates is

$$\{ |j,-j\rangle, |j,-j+1\rangle, \ldots\ldots, |j,j-1\rangle, |j,j\rangle \}$$

Remark 1). These are eigenstates of $\hat{J}^2, \hat{J}_z$

$\Rightarrow$ they have the same eigenvalue for $\hat{J}^2$

$\Rightarrow$ degenaracy of $\hat{J}^2$

(of course they have different eigenvalue of $\hat{J}_z$)

2). Suppose that there are N state $\Rightarrow N = 0, 1, 2, 3, \ldots$

$$N = j - (-j) = 2j \Rightarrow j = N/2 = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \ldots$$

☆ The possible value of j can only be

i) integer $0, 1, 2, \ldots$

ii) half integer $\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{5}{2} \ldots$

Example 1). $j = 0 \Rightarrow \{ |0,0\rangle \} \Rightarrow$ singlet state

2). $j = 1/2 \Rightarrow \{ |\tfrac{1}{2}, -\tfrac{1}{2}\rangle, |\tfrac{1}{2}, \tfrac{1}{2}\rangle \} \Rightarrow$ doublet states

* This is the case of electron spin $\{ |\hat{z}, -\rangle, |\hat{z}, +\rangle \}$

$$\hat{S}_z |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \hbar(-\tfrac{1}{2}) |\tfrac{1}{2}, -\tfrac{1}{2}\rangle = -\frac{\hbar}{2} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$(\hat{J}_z$

$$\hat{S}_z |\hat{z}, -\rangle = -\frac{\hbar}{2} |\hat{z}, -\rangle$$

Similar, $\hat{S}_z |\tfrac{1}{2}, \tfrac{1}{2}\rangle = +\frac{\hbar}{2} |\tfrac{1}{2}, \tfrac{1}{2}\rangle$

$$\hat{S}_z |\hat{z}, +\rangle = +\frac{\hbar}{2} |\hat{z}, +\rangle$$

3). $j=1 \Rightarrow \{ |1,-1\rangle, |1,0\rangle, |1,1\rangle \} \Rightarrow$ triplet states

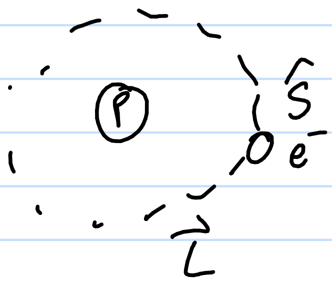
4). $j=\frac{3}{2} \Rightarrow \{ |\frac{3}{2},-\frac{3}{2}\rangle, |\frac{3}{2},-\frac{1}{2}\rangle, |\frac{3}{2},\frac{1}{2}\rangle, |\frac{3}{2},\frac{3}{2}\rangle \} \Rightarrow$ four states

5). $j=2 \Rightarrow$ . . . . $\Rightarrow$ 5 states

Addition of angular momentum

Since $\hat{L}$ & $\hat{S}$ are similar (under the same structure of math)

$\Rightarrow$ It is not so surprising to combine them



$\Rightarrow$ under the presence of magnetic field is not only $\hat{S}$ but $\hat{S}$ & $\hat{L}$

But this is not the direct addition

$\Rightarrow$ tensor product

$\rightsquigarrow$ multiple particle state $\quad |v\rangle \otimes |w\rangle$, $|v\rangle \in V_1$, $|w\rangle \in V_2$

Recap

$$(\hat{T} \otimes \hat{S}) \cdot (|v\rangle \otimes |w\rangle) = \hat{T}|v\rangle \otimes \hat{S}|w\rangle$$

$$(\hat{T}_2 \otimes \hat{S}_2)(\hat{T}_1 \otimes \hat{S}_1) = \hat{T}_2\hat{T}_1 \otimes \hat{S}_2\hat{S}_1$$

$$(|v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes |w\rangle) = (|v_1\rangle + |v_2\rangle) \otimes |w\rangle$$

$$\alpha(|v\rangle \otimes |w\rangle) = \alpha|v\rangle \otimes |w\rangle = |v\rangle \otimes \alpha|w\rangle$$

$$(\hat{T}_1 \otimes \hat{S}) + (\hat{T}_2 \otimes \hat{S}) = (\hat{T}_1 + \hat{T}_2) \otimes \hat{S}$$

How do we sum 2 $\hat{J}$'s ?

$\Rightarrow$ We can take a lesson from a system of 2 electrons.

How do we sum 2 $\hat{J}$'s?

$\Rightarrow$ We can take a lesson from a system of 2 electrons.

Suppose that we have 1st particle with spin operator

$$\hat{\vec{J}}^{(1)} = \hat{J}_x^{(1)}\vec{e}_x + \hat{J}_y^{(1)}\vec{e}_y + \hat{J}_z^{(1)}\vec{e}_z$$

that acts on the first particle only

& 2nd particle with

$$\hat{\vec{J}}^{(2)} = \hat{J}_x^{(2)}\vec{e}_x + \hat{J}_y^{(2)}\vec{e}_y + \hat{J}_z^{(2)}\vec{e}_z$$

that acts on the second particle only

$\therefore$ We can write the total spin operator as

$$\boxed{\hat{J}_i^{tot} = \underbrace{\hat{J}_i^{(1)} \otimes \hat{\mathbb{1}}} + \underbrace{\hat{\mathbb{1}} \otimes \hat{J}_i^{(2)}}}$$

☆ not $\cancel{\hat{J}_i^{(1)} \otimes \hat{J}_i^{(2)}}$

What is the algebra of $\hat{J}_i^{tot}$?

$$[\hat{J}_i^{tot}, \hat{J}_j^{tot}] = [(\hat{J}_i^{(1)} \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{J}_i^{(2)}), (\hat{J}_j^{(1)} \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{J}_j^{(2)})]$$

$$= [\hat{J}_i^{(1)} \otimes \hat{\mathbb{1}}, \hat{J}_j^{(1)} \otimes \hat{\mathbb{1}}] + [\hat{\mathbb{1}} \otimes \hat{J}_i^{(2)}, \hat{J}_j^{(1)} \otimes \hat{\mathbb{1}}]$$

$$+ [\hat{J}_i^{(1)} \otimes \hat{\mathbb{1}}, \hat{\mathbb{1}} \otimes \hat{J}_j^{(2)}] + [\hat{\mathbb{1}} \otimes \hat{J}_i^{(2)}, \hat{\mathbb{1}} \otimes \hat{J}_j^{(2)}]$$

0 0

Since $(\hat{\mathbb{1}} \otimes \hat{J}_i^{(2)})(\hat{J}_j^{(1)} \otimes \hat{\mathbb{1}}) = \hat{J}_j^{(1)} \otimes \hat{J}_i^{(2)}$

$(\hat{J}_j^{(1)} \otimes \hat{\mathbb{1}})(\hat{\mathbb{1}} \otimes \hat{J}_i^{(2)}) = \hat{J}_j^{(1)} \otimes \hat{J}_i^{(2)}$

$$[\hat{J}_i^{(1)} \otimes \hat{\mathbb{1}}, \hat{J}_j^{(1)} \otimes \hat{\mathbb{1}}] = ([\hat{J}_i^{(1)}, \hat{J}_j^{(1)}]) \otimes \hat{\mathbb{1}} = i\hbar \sum_k \varepsilon_{ijk} \hat{J}_k^{(1)} \otimes \hat{\mathbb{1}}$$

$$[\hat{\mathbb{1}} \otimes \hat{J}_i^{(2)}, \hat{\mathbb{1}} \otimes \hat{J}_j^{(2)}] = \hat{\mathbb{1}} \otimes ([\hat{J}_i^{(2)}, \hat{J}_j^{(2)}]) = \hat{\mathbb{1}} \otimes i\hbar \sum_k \varepsilon_{ijk} \hat{J}_k^{(2)}$$

$$[\hat{J}_i^{(1)} \otimes \hat{\mathbb{1}}, \hat{J}_j^{(1)} \otimes \hat{\mathbb{1}}] = \left([\hat{J}_i^{(1)}, \hat{J}_j^{(1)}]\right) \otimes \hat{\mathbb{1}} = i\hbar \sum_k \varepsilon_{ijk} \hat{J}_k^{(1)} \otimes \hat{\mathbb{1}}$$

$$[\hat{\mathbb{1}} \otimes \hat{J}_i^{(2)}, \hat{\mathbb{1}} \otimes \hat{J}_j^{(2)}] = \hat{\mathbb{1}} \otimes \left([\hat{J}_i^{(2)}, \hat{J}_j^{(2)}]\right) = \hat{\mathbb{1}} \otimes i\hbar \sum_k \varepsilon_{ijk} \hat{J}_k^{(2)}$$

$$\therefore [\hat{J}_i^{tot}, \hat{J}_j^{tot}] = i\hbar \sum_k \varepsilon_{ijk} \hat{J}_k^{(1)} \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes i\hbar \sum_k \varepsilon_{ijk} \hat{J}_k^{(2)}$$

$$= i\hbar \sum_k \varepsilon_{ijk} \left(\hat{J}_k^{(1)} \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{J}_k^{(2)}\right)$$

$$= i\hbar \sum_k \varepsilon_{ijk} \hat{J}_k^{tot}$$

$\Rightarrow$ We can conclude that $\hat{J}_i^{tot}$ still obeys the same structure $\Rightarrow$ This is the way to add angular momentum

$$\hat{J}_i^{tot} = \hat{J}_i^{(1)} \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{J}_i^{(2)}$$

Example Two e^-'s system

$\Rightarrow$ We have 4 basis vectors (as eigenstates of $\hat{S}_z$)

$$\hat{S}_z|z,\pm\rangle = \pm\frac{\hbar}{2}|z,\pm\rangle \Rightarrow \hat{J}_z|z,+\rangle = \hbar m|z,+\rangle \quad m = \tfrac{1}{2}$$

$$\hat{J}_z|z,-\rangle = \hbar m|z,-\rangle \quad m = -\tfrac{1}{2}$$

$\Rightarrow$ We know that $j = \frac{1}{2}$, $m = \frac{1}{2}$

$$|z,+\rangle = |\tfrac{1}{2}, \tfrac{1}{2}\rangle, \quad |z,-\rangle = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$\left.\begin{array}{l} |+\rangle \otimes |+\rangle \\ |z,+\rangle \otimes |z,+\rangle \end{array}\right\} \Rightarrow |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle$$

The basis of the system of 2 e^-'s

$$\{ |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle, |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle, |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle, |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle \}$$

The basis of the system of 2 e^-'s

$$\left\{ |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle,\ |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle,\ |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle,\ |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle \right\}$$

Let's consider $|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle$

omit for brevity.

What is the eigenvalue of $\hat{J}_z^{tot} = \hat{J}_z^{(1)} \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{J}_z^{(2)}$

$$\underline{\hat{J}_z^{tot}\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle\right)} = \left(\hat{J}_z \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{J}_z\right)\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$$

$$= \left(\hat{J}_z \otimes \hat{\mathbb{1}}\right)\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle\right) + \left(\hat{\mathbb{1}} \otimes \hat{J}_z\right)\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$$

$$= \underbrace{\hat{J}_z|\tfrac{1}{2},\tfrac{1}{2}\rangle}_{\hbar\cdot\frac{1}{2}|\frac{1}{2},\frac{1}{2}\rangle} \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes \underbrace{\hat{J}_z|\tfrac{1}{2},\tfrac{1}{2}\rangle}_{\hbar\frac{1}{2}|\frac{1}{2},\frac{1}{2}\rangle}$$

$$= \left(\hbar\tfrac{1}{2} + \hbar\tfrac{1}{2}\right)|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle = \underbrace{\hbar}_{m=1}\underline{\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle\right)}$$

$\Rightarrow$ The vector $|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle$ is eigenvector of $\hat{J}_z^{tot}$ with $m=1$ $\Rightarrow$ $|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle = |j, 1\rangle$

We can do this for the rest

$$\hat{J}_z^{tot}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle\right) = \ldots = \hbar\cdot 0\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$$

$$\hat{J}_z^{tot}\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle\right) = \ldots = \hbar\cdot 0\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$

$$\hat{J}_z^{tot}\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle\right) = \ldots = \hbar\cdot(-1)\left(|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle\right)$$

$\therefore$ we can summarise that

$\therefore$ we can summarise that

$$\boxed{\begin{array}{l} m=1; \quad |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle \\ m=0; \quad |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle, \quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle \\ m=-1; \quad |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle \end{array}}$$

$\Rightarrow$ Can we rearrange into the eigenstates of $\hat{S}_i^{tot}$?

$\Rightarrow$ Notice that the case $j=1$ (triplet states)

$\{ \underbrace{|1,1\rangle}_{m=1}, \underbrace{|1,0\rangle}_{m=0}, \underbrace{|1,-1\rangle}_{m=-1} \}$ $\Rightarrow$ We are missing another $m=0$ state

$\Rightarrow$ What about the case $j=0$ (singlet state)

$\{ |0,0\rangle \} \rightarrow m=0$

$\Rightarrow$ Sometimes we write $\boxed{2 \otimes 2 = 3 \oplus 1}$

Can we identify them ?

$|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle = |1,1\rangle$ (no other states with $m=1$)

$|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle = |1,-1\rangle$ (no other states with $m=-1$)

In $m=0$ case, there is an ambiguity! $\Rightarrow$ no obvious option!

However, we know that $|1,0\rangle$ relates to $|1,1\rangle$ & $|1,-1\rangle$

$\Rightarrow$ We can exploit this to find the correct combination

Now, we can use $\hat{J}_\pm$ to raise / lower the $\hat{J}_z$ eigenvalue!

Recall that $\boxed{\hat{J}_\pm |j,m\rangle = \hbar\sqrt{j(j+1) - m(m\pm1)}\,|j, m\pm1\rangle}$

$\Rightarrow$ This is true for the total $\hat{J}^{tot}_\pm = \hat{J}^{tot}_x \pm i\,\hat{J}^{tot}_y$

$$\boxed{\hat{J}^{tot}_\pm = \hat{J}_\pm \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{J}_\pm}$$

$$\hat{J}^{tot}_+ |1,1\rangle = 0 \quad , \quad \hat{J}^{tot}_- |1,1\rangle = \hbar\sqrt{1(1+1) - 1(1-1)}\,|1,0\rangle$$
$$= \sqrt{2}\hbar\,|1,0\rangle$$

On the other hand,

$$\hat{J}^{tot}_-\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle\right) = \left(\hat{J}_- \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{J}_-\right)\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$$
$$= \left(\hat{J}_- \otimes \hat{\mathbb{1}}\right)\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$$
$$+ \left(\hat{\mathbb{1}} \otimes \hat{J}_-\right)\left(|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle\right)$$
$$= \underbrace{\hat{J}_- |\tfrac{1}{2},\tfrac{1}{2}\rangle}_{\hbar\sqrt{\frac{1}{2}(\frac{1}{2}+1) - \frac{1}{2}(\frac{1}{2}-1)}\,|\frac{1}{2},\frac{1}{2}-1\rangle} \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle + |\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes \underbrace{\hat{J}_- |\tfrac{1}{2},\tfrac{1}{2}\rangle}_{=\hbar|\frac{1}{2},-\frac{1}{2}\rangle}$$

$$= \hbar\sqrt{\tfrac{1}{2}+\tfrac{1}{2}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \hbar\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\sqrt{2}\hbar\,|1,0\rangle = \hbar\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle + \hbar\,|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$

$$\therefore \boxed{|1,0\rangle = \frac{1}{\sqrt{2}}\,|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}\,|\tfrac{1}{2},\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle}$$

Clebsh-Gordan Coefficients

$1/2 \times 1/2$		1			
		+1			
+1/2	+1/2	1	1	0	
			0	0	
+1/2	−1/2		1/2	1/2	1
−1/2	+1/2		1/2	−1/2	−1
			−1/2	−1/2	1

$$|1,+1\rangle = \overset{1}{\sqrt{1}}\,|\tfrac{1}{2},+\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},+\tfrac{1}{2}\rangle$$

$$|1,0\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2},+\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle + \frac{1}{\sqrt{2}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},+\tfrac{1}{2}\rangle$$

Square root

$$|0,0\rangle = \frac{1}{\sqrt{2}}|\tfrac{1}{2},+\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle - \frac{1}{\sqrt{2}}|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},+\tfrac{1}{2}\rangle$$

Another point for $|0,0\rangle$ is that $\langle 1,0|0,0\rangle = 0$

(They stay orthogonal)

Another interesting calculation is the $(\hat{J}^{tot})^2$ operator.

We define

$$(\hat{J}^{tot})^2 = (\hat{\vec{J}}^{tot})^2 = \sum_i \hat{J}_i^{tot}\hat{J}_i^{tot}$$

$$= \sum_i (\hat{J}_i \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{J}_i)(\hat{J}_i \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{J}_i)$$

$$= \sum_i \Big(\underbrace{\hat{J}_i\hat{J}_i \otimes \hat{\mathbb{1}}}_{\hat{J}^2} + \underbrace{\hat{J}_i \otimes \hat{J}_i + \hat{J}_i \otimes \hat{J}_i}_{} + \underbrace{\hat{\mathbb{1}} \otimes \hat{J}_i\hat{J}_i}_{\hat{J}^2} \Big)$$

$$= \Big(\sum_i \hat{J}_i\hat{J}_i\Big) \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \Big(\sum_i \hat{J}_i\hat{J}_i\Big) + 2\sum_i \hat{J}_i \otimes \hat{J}_i$$

$$= \hat{J}^2 \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{J}^2 + 2\sum_i \hat{J}_i \otimes \hat{J}_i$$

Consider $\sum_i \hat{J}_i \otimes \hat{J}_i = \hat{J}_z \otimes \hat{J}_z + \hat{J}_x \otimes \hat{J}_x + \hat{J}_y \otimes \hat{J}_y$

$\hat{J}_+ , \hat{J}_-$

$$= \hat{J}^2 \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{J}^2 + 2 \sum_i \hat{J}_i \otimes \hat{J}_i$$

Consider $\sum_i \hat{J}_i \otimes \hat{J}_i = \hat{J}_z \otimes \hat{J}_z + \hat{J}_x \otimes \hat{J}_x + \hat{J}_y \otimes \hat{J}_y$

$\hat{J}_+ , \hat{J}_-$

$$\left.\begin{aligned} \hat{J}_+ &= \hat{J}_x + i \hat{J}_y \\ \hat{J}_- &= \hat{J}_x - i \hat{J}_y \end{aligned}\right\} \quad \hat{J}_x = \frac{\hat{J}_+ + \hat{J}_-}{2} \qquad \hat{J}_y = \frac{\hat{J}_+ - \hat{J}_-}{2i}$$

$$\hat{J}_x \otimes \hat{J}_x + \hat{J}_y \otimes \hat{J}_y = \left(\frac{\hat{J}_+ + \hat{J}_-}{2}\right) \otimes \left(\frac{\hat{J}_+ + \hat{J}_-}{2}\right) + \left(\frac{\hat{J}_+ - \hat{J}_-}{2i}\right) \otimes \left(\frac{\hat{J}_+ - \hat{J}_-}{2i}\right)$$

-1

$$= \frac{1}{4}\Big[\cancel{\hat{J}_+ \otimes \hat{J}_+} + \hat{J}_+ \otimes \hat{J}_- + \hat{J}_- \otimes \hat{J}_+ + \cancel{\hat{J}_- \otimes \hat{J}_-} - \left(\cancel{\hat{J}_+ \otimes \hat{J}_+} - \hat{J}_+ \otimes \hat{J}_- - \hat{J}_- \otimes \hat{J}_+ + \cancel{\hat{J}_- \otimes \hat{J}_-}\right)\Big]$$

$$= \frac{1}{2}\left(\hat{J}_+ \otimes \hat{J}_- + \hat{J}_- \otimes \hat{J}_+\right)$$

$$\Rightarrow \sum_i \hat{J}_i \otimes \hat{J}_i = \hat{J}_z \otimes \hat{J}_z + \frac{1}{2}\left(\hat{J}_+ \otimes \hat{J}_- + \hat{J}_- \otimes \hat{J}_+\right)$$

$$\Rightarrow \boxed{(\hat{J}^{tot})^2 = \hat{J}^2 \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{J}^2 + 2 \hat{J}_z \otimes \hat{J}_z + \hat{J}_+ \otimes \hat{J}_- + \hat{J}_- \otimes \hat{J}_+}$$

Check

$$|0,0\rangle = \frac{1}{\sqrt{2}} |\tfrac{1}{2}, +\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, -\tfrac{1}{2}\rangle - \frac{1}{\sqrt{2}} |\tfrac{1}{2}, -\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2}, +\tfrac{1}{2}\rangle$$

$$(\hat{J}^{tot})^2 |0,0\rangle = \hbar^2 (0)(0+1) |0,0\rangle = 0$$

$\hbar^2 j(j+1)$

$$\boxed{(\hat{J}^{tot})^2 = \hat{J}^2 \otimes \hat{\mathbb{1}} + \hat{\mathbb{1}} \otimes \hat{J}^2 + 2\hat{J}_z \otimes \hat{J}_z + \hat{J}_+ \otimes \hat{J}_- + \hat{J}_- \otimes \hat{J}_+}$$

Check

$$|0,0\rangle = \frac{1}{\sqrt{2}} |\tfrac{1}{2},+\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle - \frac{1}{\sqrt{2}} |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},+\tfrac{1}{2}\rangle$$

$$(\hat{J}^{tot})^2 |0,0\rangle = \hbar^2 (0)(0+1) |0,0\rangle = 0$$

$\hbar^2 j(j+1)$

$$= \frac{1}{\sqrt{2}} \hbar^2 \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}+1\right) |\tfrac{1}{2},+\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$
$$- \frac{1}{\sqrt{2}} \hbar^2 \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}+1\right) |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},+\tfrac{1}{2}\rangle \quad \hat{J}^2 \otimes \hat{\mathbb{1}}$$
$$+ \frac{1}{\sqrt{2}} \hbar^2 \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}+1\right) |\tfrac{1}{2},+\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$$
$$- \frac{1}{\sqrt{2}} \hbar^2 \left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}+1\right) |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},+\tfrac{1}{2}\rangle \quad \hat{\mathbb{1}} \otimes \hat{J}^2$$

+ .. try this yourself!

☆ Exam 9:00–1X.XX 29 November

HW also on the same day.

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{r}) \xrightarrow{\text{derivative}} \frac{\partial}{\partial r} \cdots + \hat{L}^2 \cdots$$

Homework 4

1). Consider a state in $j=1$ space

$$|\psi\rangle = \frac{i}{\sqrt{5}}|1,-1\rangle + \sqrt{\frac{2}{5}}|1,0\rangle - \sqrt{\frac{2}{5}}|1,1\rangle$$

1.1) Find $\langle \hat{J}_z \rangle$ & uncertainty $\sigma_{J_z}^2$

1.2) Find $\langle \hat{J}^2 \rangle$

1.3) Find $\langle \hat{J}_+ \rangle$, $\langle \hat{J}_- \rangle$

1.4) Find $\langle \hat{J}_- \hat{J}_+ \rangle$, $\langle \hat{J}_+ \hat{J}_- \rangle$

2). Show that for a general state in $j=1/2$ space

$$|\psi\rangle = a|\tfrac{1}{2},-\tfrac{1}{2}\rangle + b|\tfrac{1}{2},\tfrac{1}{2}\rangle$$

The expectation value of the total angular momentum is

$$\langle \hat{J}^2 \rangle = \langle \hat{J}_x^2 \rangle + \langle \hat{J}_y^2 \rangle + \langle \hat{J}_z^2 \rangle$$

3) Starting from the most general state:

$$|\psi\rangle = a|+\rangle \otimes |+\rangle + b|+\rangle \otimes |-\rangle + c|-\rangle \otimes |+\rangle + d|-\rangle \otimes |-\rangle$$

find the normalised state which has the following properties:

$$\hat{S}_x^{tot}|\psi\rangle = (\hat{S}_x \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_x)|\psi\rangle = 0$$

Homework 5

1). Investigate the following tensor products. Which ones are entangled?

i) $|\psi\rangle = |\tfrac{1}{2},+\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},+\tfrac{1}{2}\rangle + |\tfrac{1}{2},+\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$
$+ 2|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},+\tfrac{1}{2}\rangle + 2|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$

ii) $|\psi\rangle = |\tfrac{1}{2},+\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle + 3|\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$

iii) $|\psi\rangle = |\tfrac{1}{2},-\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},+\tfrac{1}{2}\rangle + 2|\tfrac{1}{2},+\tfrac{1}{2}\rangle \otimes |\tfrac{1}{2},-\tfrac{1}{2}\rangle$

2) Work out the Clebsch-Gordan coefficients in

1×1/2	3/2 +3/2	3/2 +1/2	1/2 +1/2	3/2 −1/2	1/2 −1/2	3/2 −3/2
+1 +1/2	1					
+1 −1/2		1/3	2/3			
0 +1/2		2/3	−1/3			
0 −1/2				2/3	1/3	
−1 +1/2				1/3	−2/3	
−1 −1/2						1

using creation/annihilation operators.

3) We use $\{|z,\pm\rangle\}$ basis for matrix representation in this problem

3.1). Find the length of $|A\rangle = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}i\\2\end{pmatrix} + \begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\i\end{pmatrix}$

3.2) Find the length of $|A'\rangle = \begin{pmatrix}2\\1\end{pmatrix} \otimes \begin{pmatrix}1+i\\2+i\end{pmatrix}$. Is it equal to the length of $|A\rangle$?

3.3) Consider the state $|A\rangle$ in 3.1). Is this an entangled state? (Please show this explicitly)

3.4) Consider $|A\rangle$. What are the probabilities of a measurement $\hat{S}_z$ on the first particle giving the values i) $\hbar/2$ ii) $-\hbar/2$?

What are the states of the second particle after the partial measurement?

3.5) Consider $|A\rangle$. What are the probabilities of measuring $\hat{S}_z^T = \hat{S}_z \otimes \mathbb{1} + \mathbb{1} \otimes \hat{S}_z$ and obtain values i) $\hbar$ ii) $-\hbar$