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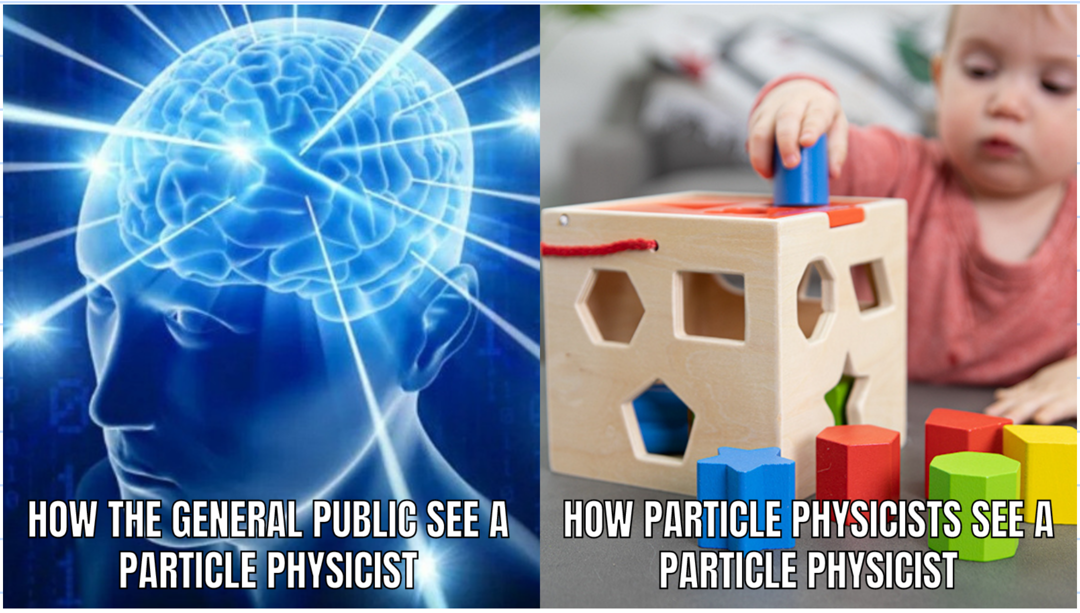
# SHEAP 2023

- This lecture is a pedagogical introduction to neutrino physics.
- The main references are
  - arXiv:0305245, TASI 2002 lecture on neutrinos, Yuval Grossman
  - arXiv: 0104147, Neutrino mass, mixing, and oscillation, Boris Kayser
- Let me start with my version of internet meme:




- We generally see the standard model (and neutrino physics) as nothing more than an attempt to put a square peg in a square hole! and I really hope this will be your point of view after this lecture.
- Before getting into why neutrino is massless in the standard model, let's familiarize ourself to the rules for the game we play, i.e., studying the shape of the holes.

## 1) Lorentz invariance

- The first and most important rule is that physical laws must be universal for any inertial frame of references.
- For this to work with Special Relativity which requires Lorentz transformation between reference frames, the Lagrangian must be invariant under Lorentz transformation

$$\mathcal{L} \longrightarrow \mathcal{L}' = \mathcal{L} ,$$

so that the equation of motions are universal for any inertial reference frame.

- The immediate consequence of this rule is that fields which are basic ingredients for cooking up the Lagrangian must transform in order to keep $\mathcal{L}$ invariant.
- The type of transformation is also the way to diffentiate types of fields (particles)

### 1.1) Scalar field (boson spin 0)

$$\phi \rightarrow \phi' = \phi$$

which is invariant under Lorentz transformation

### 1.2) Fermion field (fermion spin $1/2$)

$$\psi \rightarrow \psi' = e^{\frac{1}{2}\theta_{\mu\nu} S^{\mu\nu}} \psi$$

where $S^{\mu\nu}$ are generators (boost + rotation) of Lorentz transformation and $\theta_{\mu\nu}$ contains boost factors / rotation angles

- However, this type of field (Dirac fermion as appear in the Dirac equation) is not a fundamental ingredient we use in the Standard Model
- It turns out that $S^{\mu\nu}$ which is $4\times4$ matrix for each value of $\mu,\nu$ can be reduced further into 2 matrix blocks ($2\times2$ matrix).
- This also means that Dirac fermion can be decomposed into

$$\psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} , \quad \bar{\psi} \equiv \psi^\dagger \gamma^0 = (\psi_L^\dagger \ \psi_R^\dagger) \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} = (\psi_R^\dagger \ \psi_L^\dagger)$$

where $\psi_L, \psi_R$ are $2\times1$ complex vectors corresponding to left handed field & right handed field respectively.

- $\psi_L, \psi_R$ are treated as 2 independent fields which transform differently under boost, i.e.,

$$\psi_{L,R} \to \psi'_{L,R} = e^{i\vec{\theta}\cdot\vec{\sigma}/2} \psi_{L,R} \quad \text{under rotation}$$

$$\psi_{L,R} \to \psi'_{L,R} = e^{\pm\vec{\beta}\cdot\vec{\sigma}/2} \psi_{L,R} \quad \text{under boost}$$

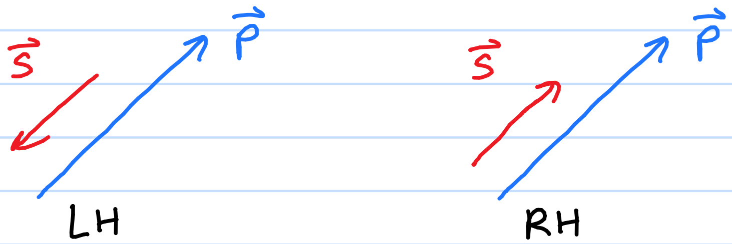
- The fact that they are indepedent manifests in the high energy (massless) limit since Dirac equation can be decoupled into

$$(i\not{\partial} - \overset{0}{\cancel{m}})\psi = 0 \Rightarrow \begin{cases} i\bar{\sigma}^\mu \partial_\mu \psi_L = 0 , & \bar{\sigma}^\mu \equiv (\mathbb{1}, -\sigma^i) \\ i\sigma^\mu \partial_\mu \psi_R = 0 , & \sigma^\mu \equiv (\mathbb{1}, \sigma^i) \end{cases}$$

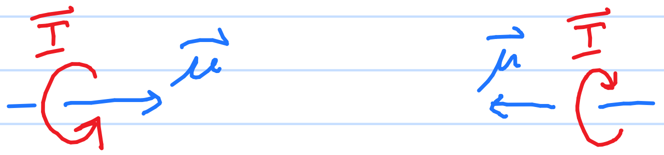
- They are called Weyl equations and it can be shown that the solutions $(\psi_L, \psi_R)$ are eigenvectors of the helicity operator:

$$h \equiv \frac{1}{2}\frac{\vec{\sigma}\cdot\vec{p}}{|\vec{p}\,|} \Rightarrow h\psi_{L,R} = \mp\frac{1}{2}\psi_{L,R}$$

- This means that massless right handed fields have spin in the direction of momentum, whereas left handed fields have the opposite




- Another interesting point is that left handed field & right handed field are related by the act of flipping the electric charge (negative $\longleftrightarrow$ positive), i.e., Charge conjugation.
- Classically, you can think of it as a flip in magnetic dipole moment due to the opposite charge (spin $\propto$ magnetic moment)




- In Quantum Field Theory, this relation stems from the Dirac equation with gauge field (photon) interaction:

$$(i\not{D} - m)\psi \equiv \left[i\gamma^\mu(\partial_\mu - ieA_\mu) - m\right]\psi = 0$$

- It can be shown that the transformed field defined as

$$\psi_c = \gamma^2 \psi^* \quad \text{where} \quad \gamma^2 = \begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix},$$

satisfies the same equation with the opposite charge:

$$\left[ i\gamma^\mu (\partial_\mu + ieA_\mu) - m \right] \psi_c = 0$$

- In terms of LH, RH, we can write

$$\psi_c = \begin{pmatrix} \sigma^2 \psi_R^* \\ -\sigma^2 \psi_L^* \end{pmatrix} = \begin{pmatrix} \psi_{c,L} \\ \psi_{c,R} \end{pmatrix}$$

which means that a charge conjugate of right handed field transforms as a left handed field and vice versa.

(We can check this explicitly, but I will leave as an exercise)

### 1.3) Gauge field (Boson spin 1)

$$A^\mu \rightarrow A'^\mu = \Lambda^\mu{}_\nu A^\nu$$

which transforms as a vector in Lorentz transformation.

- The whole structure of gauge fields is rather special and it leads us to the next rule

## 2). Gauge Invariance

- Gauge symmetry is the principle behind interactions of all elementary particles (fundamental forces of nature: electromagnetic, weak nuclear, strong nuclear forces)

- For example, the $U(1)$ gauge symmetry is responsible for EM (Electromagnetic) interaction.
- Similar to Lorentz transformation, the Lagrangian must be invariant under gauge transformation. (Note that this statement can be violated after the spontaneous symmetry breaking of the electroweak interaction)
- Also, different fields transform differently under gauge symmetry. The transformation rule for each particle gives us interactions between particles.
- Let's use EM gauge transformation as an example:

  Fermion with "charge" $Q$ : $\psi \rightarrow \psi' = e^{iQ\alpha(x)}\psi$

  Photon : $A^{\mu} \rightarrow A^{\mu\prime} = A^{\mu} + \frac{1}{e}\partial^{\mu}\alpha(x)$ "gauge"

- It is straightforward to check that the Lagrangian,

$$\mathcal{L} = \bar{\Psi}(i\not{D} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu},$$

  where $D_{\mu}\psi = (\partial_{\mu} - iQeA_{\mu})\psi$, $F_{\mu\nu} \equiv \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu}$ is invariant, i.e.,

$$\mathcal{L} \rightarrow \mathcal{L}' = \bar{\Psi}'(i\not{D}' - m)\psi' - \frac{1}{4}F'_{\mu\nu}F^{\mu\nu\prime} = \mathcal{L}$$

  (left as an exercise as usual)

- Using this structure, we know that electron has $U(1)_{EM}$ charge $Q = 1$. For $u$ quark, its $U(1)_{EM}$ charge $Q = 2/3$.

- We can clearly use these numbers to tell the interaction properties of particles. However, in the Standard Model $U(1)_{EM}$ is not one of the starting points. $U(1)_{EM}$ is just a left over symmetry from the spontaneous symmetry breaking of $SU(2)_L \times U(1)_Y$.
- $U(1)_Y$ is called "U(1) hypercharge", $SU(2)_L$ is called "SU(2) left" meaning that SU(2) transformation is only for left handed particles in the Standard Model
- Since we have to talk about handedness, let's rewrite U(1)

$$\mathcal{L} = \bar{\Psi}(i\not{D} - m)\Psi = \bar{\Psi}(i\gamma^\mu\partial_\mu - iQe\gamma^\mu A_\mu - m)\Psi$$

$$= (\psi_R^\dagger \ \psi_L^\dagger)\left[i\begin{pmatrix} 0 & \sigma^\mu\partial_\mu \\ \bar{\sigma}^\mu\partial_\mu & 0 \end{pmatrix} - i\begin{pmatrix} 0 & Qe\sigma^\mu A_\mu \\ Qe\bar{\sigma}^\mu A_\mu & 0 \end{pmatrix} - \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix}\right]\begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix}$$

$$= i\psi_R^\dagger\sigma^\mu\partial_\mu\psi_R + i\psi_L^\dagger\bar{\sigma}^\mu\partial_\mu\psi_L - iQe\psi_R^\dagger\sigma^\mu\psi_R A_\mu$$

$$-iQe\psi_L^\dagger\bar{\sigma}^\mu\psi_L A_\mu - m\psi_R^\dagger\psi_L - m\psi_L^\dagger\psi_R$$

- Note that apart from the mass terms, gauge interactions and kinetic terms are seperated nicely for $\psi_L, \psi_R$
- Let's use this for $SU(2)_L$ gauge transformation. However, the gauge fields for $SU(2)_L$ form a matrix structure:

$$\tilde{W}_\mu = \begin{pmatrix} W_\mu^3 & \sqrt{2}W_\mu^+ \\ \sqrt{2}W_\mu^- & W_\mu^3 \end{pmatrix}$$

- This means that fields that are gauged (have "charge") under $SU(2)_L$ must form a column vector (typically called doublet).

$$\tilde{L} = \begin{pmatrix} \psi_L^u \\ \psi_L^d \end{pmatrix}$$

- We now can construct the Lagrangian for $\tilde{L}$

$$\begin{aligned}
\mathcal{L} &= i\tilde{L}^\dagger \bar{\sigma}^\mu \partial_\mu \tilde{L} + \frac{ig}{2} \tilde{L}^\dagger \bar{\sigma}^\mu \tilde{W}_\mu \tilde{L} \\
&= i(\psi_L^{u\dagger} \ \psi_L^{d\dagger}) \begin{pmatrix} \bar{\sigma}^\mu \partial_\mu \psi_L^u \\ \bar{\sigma}^\mu \partial_\mu \psi_L^d \end{pmatrix} + \frac{ig}{2} (\psi_L^{u\dagger} \ \psi_L^{d\dagger}) \begin{pmatrix} \bar{\sigma}^\mu W_\mu^3 & \sqrt{2}\bar{\sigma}^\mu W_\mu^+ \\ \sqrt{2}\bar{\sigma}^\mu W_\mu^- & \bar{\sigma}^\mu W_\mu^3 \end{pmatrix} \begin{pmatrix} \psi_L^u \\ \psi_L^d \end{pmatrix} \\
&= i\psi_L^{u\dagger} \bar{\sigma}^\mu \partial_\mu \psi_L^u + i\psi_L^{d\dagger} \bar{\sigma}^\mu \partial_\mu \psi_L^d + \frac{ig}{2} \psi_L^{u\dagger} \bar{\sigma}^\mu \psi_L^u W_\mu^3 \\
&\quad + \frac{ig}{2} \psi_L^{d\dagger} \bar{\sigma}^\mu \psi_L^d W_\mu^3 + \frac{ig}{\sqrt{2}} \psi_L^{u\dagger} \bar{\sigma}^\mu \psi_L^d W_\mu^+ + \frac{ig}{\sqrt{2}} \psi_L^{d\dagger} \bar{\sigma}^\mu \psi_L^u W_\mu^-
\end{aligned}$$

- The gauge transformation in this case is a bit more complicated

Doublet: $\tilde{L} \to \tilde{L}' = \tilde{U}(x)\tilde{L}$, where $\tilde{U}$ is a unitary $2\times 2$ matrix

Gauge fields: $\tilde{W} \to \tilde{W}' = \tilde{U}(x)\tilde{W}\tilde{U}^\dagger(x) + \frac{2i}{g}(\partial_\mu \tilde{U}(x))\tilde{U}^\dagger(x)$

(The fact that the Lagrangian above is invariant is left as an optional exercise)

- For the right handed field, it is not charged therefore we write

$$\mathcal{L} = i\psi_R^{u\dagger} \sigma^\mu \partial_\mu \psi_R^u + i\psi_R^{d\dagger} \sigma^\mu \partial_\mu \psi_R^d \qquad \text{(no gauge)}$$

- Now we have enough ingredients to construct the Standard Model and argue why neutrino is massless

# The Standard Model

- The SM fermions are
  - Leptons left handed doublet for each generation

$$\tilde{L}^1_L = \begin{pmatrix} \nu^1_L \\ e^1_L \end{pmatrix}, \quad \tilde{L}^2_L = \begin{pmatrix} \nu^2_L \\ e^2_L \end{pmatrix}, \quad \tilde{L}^3_L = \begin{pmatrix} \nu^3_L \\ e^3_L \end{pmatrix}$$

where $\nu^i_L$, $e^i_L$ are LH neutrinos and electrons respectively under $SU(2)_L$ transform as doublet ($\mathbf{2}$), under $U(1)_Y$ transform with charge $Q_Y = -1/2$ and under $SU(3)_c$ (strong nuclear force) transform as invariant ($\mathbf{1}$). We normally write

$$\tilde{L}^i_L\,(\mathbf{1}, \mathbf{2}, -1/2)$$

- Electron right handed singlet for each generation

$$e^1_R, \; e^2_R, \; e^3_R$$

The charges are given by $e^i_R(\mathbf{1}, \mathbf{1}, -1)$

- Quarks left handed doublet for each generation

$$\tilde{Q}^1_L = \begin{pmatrix} u^1_L \\ d^1_L \end{pmatrix}, \quad \tilde{Q}^2_L = \begin{pmatrix} u^2_L \\ d^2_L \end{pmatrix}, \quad \tilde{Q}^3_L = \begin{pmatrix} u^3_L \\ d^3_L \end{pmatrix}$$

The charges are given by $\tilde{Q}^i_L(\mathbf{3}, \mathbf{2}, 1/6)$

($\mathbf{3}$ means triplet under $SU(3)$)

- Up-type quark right handed singlet for each generation

$$u_R^1,\ u_R^2,\ u_R^3$$

The charges are given by $u_R^i(\mathbf{3},\mathbf{1},2/3)$

- Down-type quark right handed singlet for each generation

$$d_R^1,\ d_R^2,\ d_R^3$$

The charges are given by $d_R^i(\mathbf{3},\mathbf{1},-1/3)$

- The SM bosons are

  - $U(1)_Y$ gauge boson: $B_\mu$
  - $SU(2)_L$ gauge bosons: $W_\mu^3,\ W_\mu^\pm$

    $\Rightarrow$ After spontaneous symmetry breaking

    Photon (massless): $A_\mu = \sin\theta_W W_\mu^3 + \cos\theta_W B_\mu$

    Massive weak bosons: $Z_\mu = \cos\theta_W W_\mu^3 - \sin\theta_W B_\mu,\ W_\mu^\pm$
  - $SU(3)_c$ gauge bosons: $G_\mu^a$, $a=1,\dots,8$
  - Higgs scalar doublet

$$\Phi = \begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix},$$ $\phi_1, \phi_2$ are complex scalar

The charges are given by $\Phi(\mathbf{1},\mathbf{2},1/2)$

- Let's work out quark mass first.

  The first term that is invariant under Lorentz transformation + Gauge transformation = Yukawa term

$$\mathcal{L} = -y_d {Q_L^1}^\dagger \Phi d_R^1 - y_d^* {d_R^1}^\dagger \Phi^\dagger Q_L^1$$

Hermitian conjugate of the first term $\Rightarrow$ h.c.

- This is Lorentz invariant because

  (LH fermion)$^\dagger \cdot$(RH fermion). Scalar, (RH fermion)$^\dagger\cdot$(LH fermion). Scalar

  are invariant.

- For gauge invariant, we have

$SU(3)_c$ : $\mathbf{3}^\dagger \cdot \mathbf{1} \cdot \mathbf{3}$ $\Rightarrow$ invariant

$SU(2)_L$ : $\mathbf{2}^\dagger \cdot \mathbf{2} \cdot \mathbf{1}$ $\Rightarrow$ invariant

$U(1)_Y$ : $-1/6 + 1/2 - 1/3 = 0$ $\Rightarrow$ invariant

- This gives mass for down quark when the spontaneous symmetry breaking, i.e., Higgs gets non-zero vacuum expectation value (vev.)

$$\Phi = \begin{pmatrix} 0 \\ v \end{pmatrix} \quad \text{at the minimum}$$

- The Lagrangian becomes

$$\mathcal{L} = -y_d v {d_L^1}^\dagger d_R^1 - y_d^* v {d_R^1}^\dagger d_L^1 = -m_d \left( {d_L^1}^\dagger d_R^1 + {d_R^1}^\dagger d_L^1 \right)$$

- The down quark mass $m_d = y_d v$

- However, if we think about 3 generations, we have

$$\mathcal{L} = -\sum_{i,j} y_d^{ij} {Q_L^i}^\dagger \Phi d_R^j$$

- However, if we think about 3 generations, we have

$$\mathcal{L} = -\sum_{i,j} y_d^{ij} Q_L^{i\dagger} \Phi d_R^j + h.c.$$

- This suggests that quark mass term should be written as a mass matrix which is not necessarily diagonal

$$\mathcal{L} = -\sum_{i,j} M_d^{ij} d_L^{i\dagger} d_R^j \Rightarrow M_d^{ij} = y_d^{ij} \cdot v$$

- We will come back to this point when we discuss neutrino oscillation

- For up quark, we can check that

$$\mathcal{L} = -y_u Q_L^{1\dagger} i\sigma^2 \Phi^* u_R^1 + h.c.$$

is gauge & Lorentz invariant.

- Everything else apart from SU(2) part is left as an exercise.

- For SU(2), it is straightforward to check that

$$\sigma_i^T \sigma_2 = -\sigma_2 \sigma_i \quad , \; i = 1, 2, 3$$

- Using the above identity, we can show that under SU(2) trans.

$$U^\dagger \sigma_2 U^* \cong (\hat{\mathbb{1}} - i\sum_i \theta_i \sigma_i^\dagger) \sigma_2 (\hat{\mathbb{1}} + i\sum \theta_i \sigma_i^*) \quad \Big) \; \sigma_i^* = (\sigma_i^\dagger)^T$$

$$= \sigma_2 + i\sum_i \theta_i (-\sigma_i \sigma_2 + \sigma_2 \sigma_i^T) = \sigma_2 \qquad = \sigma_i^T$$

- This means that $\bar{\Phi} \equiv i\sigma_2 \Phi^*$ transforms as $\Phi$ in SU(2)

$$\bar{\Phi} \rightarrow \bar{\Phi}' = i\sigma_2 U^* \Phi^* = U(i\sigma_2 \Phi^*) = U\bar{\Phi}$$

which results in the invariance of the Lagrangian above

- When Higgs gets non-zero vev $\Phi = \begin{pmatrix} 0 \\ v \end{pmatrix}$, the up quark mass is $m_u = Y_u \cdot v$

$$\mathcal{L} = -Y_u v u_L^{1\dagger} u_R^1 - Y_u^* v u_R^{1\dagger} u_L^1 = -m_u (u_L^{1\dagger} u_R^1 + u_R^{1\dagger} u_L^1)$$

- Similar to the 3 generations of down quarks, gauge symmetries allow us to write non-diagonal mass terms:

$$\mathcal{L} = -\sum_{i,j} Y_u^{ij} Q_L^{i\dagger} i\sigma^2 \Phi^* u_R^j + h.c.$$

which give the mass matrix $M_u^{ij} = Y_u^{ij} \cdot v$

- Let's consider electron mass next. The Lagrangian allowed by symmetries is similar to the down quark's

$$\mathcal{L} = -Y_e L_L^{1\dagger} \Phi e_R^1 + h.c.$$

- As usual, the invariance of this term is left as an exercise
- Considering 3 generations, we get

$$\mathcal{L} = -\sum_{i,j} Y_e^{ij} L_L^{i\dagger} \Phi e_R^j + h.c.$$

- The mass matrix is not necessarily diagonal

$$M_e^{ij} = Y_e^{ij} \cdot v$$

- Note that the masses for down quarks, up quarks, and electrons are in the form of the Dirac mass:

$$(\text{left handed})^\dagger (\text{right handed}) + h.c.$$

- We are now ready to discuss neutrino masses.

# Neutrino Masses

- Our first obstacle to construct the mass for neutrino is that we don't have the right handed neutrino fields in the Standard Model!

- Can we try to get around by construct the mass term using only the left handed fields?

- This sounds reasonable. To see this, let's go back to the charge conjugate field $\psi_c = \begin{pmatrix} \sigma^2\psi_R^* \\ -\sigma^2\psi_L^* \end{pmatrix} = \begin{pmatrix} \psi_{c,L} \\ \psi_{c,R} \end{pmatrix}$

- Since neutrino has no electric charge, what if the charge conjugate of neutrino is neutrino itself?

$$\psi_c = \eta\psi_c \quad , \quad \eta = e^{i\theta} = \text{a complex phase}$$

$$\begin{pmatrix} \sigma^2\psi_R^* \\ -\sigma^2\psi_L^* \end{pmatrix} = \begin{pmatrix} \eta\psi_L \\ \eta\psi_R \end{pmatrix} \Rightarrow \psi_R = -\eta^*\sigma^2\psi_L^* \overset{\text{my choice}}{=} -i\sigma^2\psi_L^*$$

- If this is true, we can write neutrino as

$$\therefore \quad \psi = \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} = \begin{pmatrix} \psi_L \\ -i\sigma^2\psi_L^* \end{pmatrix}$$

this field is called Majorana fermion.

- With this we can write the mass term as the Majorana mass

$$\mathcal{L} = -m\psi_R^\dagger\psi_L + h.c. \Rightarrow -M(-i\sigma^2\psi_L^*)^\dagger\psi_L + h.c.$$

$$\mathcal{L} = -iM\psi_L^T\sigma_2\psi_L + h.c.$$

- Note that any phase transformation is not a symmetry of the Majorana mass term (except $U(1)_{EM}$ because it has zero charge)
- Therefore, any global symmetry is broken by this term.
- For example, global $U(1)$ lepton number is clearly not a symmetry

$$\begin{pmatrix} \nu \\ e \end{pmatrix} \to \begin{pmatrix} \nu' \\ e' \end{pmatrix} = e^{i\alpha} \begin{pmatrix} \nu' \\ e' \end{pmatrix} , \quad \alpha = \text{constant phase}$$

- The Lagrangian with $U(1)_L$ symmetry has a conservation of the lepton number (the total number is always conserved)
- Experimentally, we seems to have this conservation so far...
- Of course, we can never be 100% with experiments especially with higher energy experiments. Something could be waiting at the next corner.
- This also agrees with the fact the simplest Majorana mass term from Higgs spontaneous symmetry breaking is nonrenormalizable:

$$\mathcal{L} = \frac{C}{\Lambda_{LN}} \left( L_L^{1\dagger} i\sigma^2 \underline{\Phi}^* \right)^* L_L^{1\dagger} i\sigma^2 \underline{\Phi}^*$$

- As usual, the Lorentz / Gauge invariant are left as an exercise
- $C$ = coupling, $\Lambda_{LN}$ = dimension full constant (in the unit of energy) which is equal to the energy scale that the $U(1)_L$ is broken.

- The word "nonrenormalizable" means that we are not allowed to use perturbation at arbitrary high energy (a lot of untamed infinities would appear.) It also means that there must be a new physics (new field/symmetry) at around $\sim \Lambda_{LN}$ energy scale.
- This means that constructing Majorana mass for neutrino begs us to go beyond the Standard Model (BSM).
- Let's come back to the starting point:
  - No right handed neutrino field to construct Dirac mass
  - Majorana mass using left handed field breaks lepton number and requires us to go beyond SM
- We therefore have to conclude that

  $\boxed{\text{" Neutrinos are massless in the Standard Model "}}$

- However, we have a lot of evidences that neutrinos have mass but they are extremely small. ⇒ Let's go beyond the SM!

## Neutrino masses beyond the SM

- The simplest extension is probably what have been in your mind since the start of the discussion of the neutrino masses:
- Just introduce right handed neutrino and let it join SM?

- We can work out its charge by requiring that

$$\mathcal{L} = -y_\nu L_L^{1\dagger} i\sigma^2 \Phi^* \nu_R^1 + h.c.,$$

the mass term is invariant. ( Exercise )

- The result is $\nu_R(\mathbf{1}, \mathbf{1}, 0)$ which means that $\nu_R$ has no transformation under SM gauge groups
- However, this new field alone offers no good explanation why neutrino masses are much smaller than other particles ( $m_\nu/m_e < 10^{-6}$ ),
- Since they comes from the same type of coupling (Yukawa) and the same mechanism ( Higgs mechanism ), we expect

$$m_e = y_e v \sim m_\nu = y_\nu v$$

- This means that the right handed neutrino is not enough
- The canonical way of suppressing neutrino mass is to introduce new heavy fields such that at low energy ( electroweak scale ) there are nonrenormalizable terms coming from integrating out heavy fields.
- One of those terms can be Majorana mass for right handed field

$$\mathcal{L} = -\frac{1}{2} M_N \nu_R^T (i\sigma^2) \nu_R + h.c.$$

where $M_N$ is expected to be the energy scale of heavy fields

- The presence of Majorana term implies lepton number violation at higher energy scale
- Combining with the Dirac mass term, one can show that

$$\mathcal{L} = \begin{pmatrix} \nu_L^T & \nu_{R,c}^T \end{pmatrix} \begin{pmatrix} 0 & y_\nu v \\ y_\nu v & M_N \end{pmatrix} \begin{pmatrix} \nu_L \\ \nu_{R,c} \end{pmatrix} = \tilde{N}^T \tilde{M} \tilde{N}$$

- Note that $\nu_L$ and $\nu_{R,c}$ can be mixed since they have the same set of quantum numbers (including helicity)
- Diagonalization of this matrix gives 2 eigenvalues

$$(m_\nu)_{R,L} = \frac{M_N \pm \sqrt{M_N^2 - 4y_\nu^2 v^2}}{2}$$

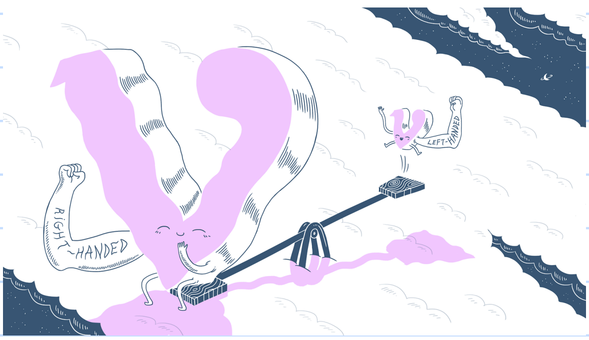
(This is left as an exercise but I can give a hint:

$$\mathrm{Tr}(\tilde{M}) = \sum \text{eigenvalues} \quad , \quad \mathrm{Det}(\tilde{M}) = \prod \text{eigenvalues} \;)$$

- In the limit that $v \ll M_N$, we can approximate

$$m_{\nu R} \cong M_N \quad , \quad m_{\nu L} \cong \frac{y_\nu^2 v^2}{M_N}$$

- This mechanism is called Seesaw mechanism from the fact that increasing $M_N$ makes $m_{\nu R}$ heavier and $m_{\nu L}$ lighter at the same time



- Using the seesaw mechanism alone, the choice of adding new heavy fields is practically endless.
- What we normally do is trying to fit this neutrino mass in the scheme of UV completion theories such as Grand Unified Theory, Supersymmetry, String/M theory, etc.
- The structure of seesaw mass matrix can also be extended to include more fields, more scales, which is popular among string inspired models. For example, the mass for $(\nu_L \;\; \nu_{R,c} \;\; S)$ can be of the form
$$\begin{pmatrix} 0 & m_D & 0 \\ m_D & 0 & M \\ 0 & M & M_S \end{pmatrix}$$
- This is called inverse seesaw or double seesaw mechanism
- Everything up to now is usually categorized broadly as Type I seesaw where right handed neutrino & heavy fields are introduced
- We can also build neutrino mass using triplet scalar field $\Delta(\mathbf{1}, \mathbf{3}, 1)$ (triplet Higgs) such that the mass term
$$\mathcal{L} = \lambda_N L_L'^{\dagger} \Delta L_L'$$
can be constructed and give mass by vev of $\Delta$
- Any seesaw model involving triplet Higgs ($\Delta$) is categorized as Type II seesaw

- Note that no BSM model has been confirmed by neutrino experiment so far.
- Therefore, the work on the neutrino phenomelogy of the BSM is still ongoing and very important for searching for new physics
- Although we don't have evidence for the specific BSM at the moment, we do have a strong evidence for massive neutrino and hence pointing toward new physics.
- The fact that neutrinos have masses manifests in the phenomenon called neutrino mixing and oscillation

## Neutrino Mixing

- Before we discuss neutrino mixing, we first discuss the up-type and down-type sector.
- There are 2 things we need to discuss, mass terms and gauge interactions.

$$\mathcal{L} = -\sum_{i,j} M_d^{ij} d_L^{i\dagger} d_R^j - \sum_{i,j} M_u^{ij} u_L^{i\dagger} u_R^j + h.c.$$

$$+ \sum_i \frac{ig}{\sqrt{2}} u_L^{i\dagger} \bar{\sigma}^\mu d_L^i W_\mu^+ + h.c.$$

- We define quarks (up, down, charm, strange, top, bottom) from the states with definite mass.

- We therefore have to rotate up-like quarks & down-like quarks

$$u_L^i = \sum_{a=\{u,c,t\}} \tilde{U}_L^{ia} u_L'^a \quad , \quad u_R^j = \sum_{a=\{u,c,t\}} \tilde{U}_R^{ja} u_R'^a$$

$$d_L^i = \sum_{b=\{d,s,b\}} \tilde{D}_L^{ib} d_L'^b \quad , \quad d_R^j = \sum_{b=\{d,s,b\}} \tilde{D}_L^{jb} d_R'^b$$

where $\tilde{U}_L, \tilde{U}_R, \tilde{D}_L, \tilde{D}_R$ are unitary matrices chosen in such a way that the mass terms are diagonal

$$\mathcal{L} = -\sum_{i,j} M_d^{ij} d_L^{i+} d_R^j - \sum_{i,j} M_u^{ij} u_L^{i+} u_R^j$$

$$= -\sum_a \sum_b d_L'^{a+} \Big( \sum_{i,j} D_L^{ai*} M^{ij} D_R^{jb} \Big) d_R'^b$$

$$- \sum_a \sum_b u_L'^{a+} \Big( \sum_{i,j} U_L^{ai*} M_u^{ij} U_R^{jb} \Big) u_R'^b$$

$$\sum_{i,j} D_L^{ai*} M_d^{ij} D_R^{jb} = M_d^a \delta^{ab} \quad , \quad \sum_{i,j} U_L^{ai*} M_u^{ij} U_R^{jb} = M_u^a \delta^{ab}$$

$$\mathcal{L} = -M_d^d d_L'^{d+} d_R'^d - M_d^s d_L'^{s+} d_R'^s - M_d^b d_L'^{b+} d_R'^b$$
$$- M_u^u u_L'^{u+} u_R'^u - M_u^c u_L'^{c+} u_R'^c - M_u^t u_L'^{t+} u_R'^t$$

where $\{ d'^d_{L,R}, d'^s_{L,R}, d'^b_{L,R} \} \equiv \{ d_{L,R}, s_{L,R}, b_{L,R} \}$ and

$\{ u'^u_{L,R}, u'^c_{L,R}, u'^t_{L,R} \} \equiv \{ u_{L,R}, c_{L,R}, t_{L,R} \}$ after rotations

are defined as physical quarks field with definite mass

- However, rotating quark fields results in a nondiagonal gauge interactions :

$$\mathcal{L} = \sum_i \frac{ig}{\sqrt{2}} u_L^{i\dagger} \bar{\sigma}^\mu d_L^i W_\mu^+ + h.c.$$

$$= \sum_{a,b} \sum_i \frac{ig}{\sqrt{2}} u_L'^{a\dagger} \tilde{U}^{ai*} \bar{\sigma}^\mu \tilde{D}^{ib} d_L'^{b} W_\mu^+ + h.c.$$

$$= \sum_{a,b} \frac{ig}{\sqrt{2}} u_L'^{a\dagger} \underbrace{\left( \sum_i \tilde{U}^{ai*} \tilde{D}^{ib} \right)}_{\equiv \tilde{V}_{CKM}^{ab}} \bar{\sigma}^\mu d_L'^{b} W_\mu^+ + h.c.$$

$$= \frac{ig}{\sqrt{2}} \left( u_L^\dagger \;\; c_L^\dagger \;\; t_L^\dagger \right) \begin{pmatrix} V_{CKM}^{ud} & V_{CKM}^{us} & V_{CKM}^{ub} \\ V_{CKM}^{cd} & V_{CKM}^{cs} & V_{CKM}^{cb} \\ V_{CKM}^{td} & V_{CKM}^{ts} & V_{CKM}^{tb} \end{pmatrix} \begin{pmatrix} \bar{\sigma}^\mu d_L \\ \bar{\sigma}^\mu s_L \\ \bar{\sigma}^\mu b_L \end{pmatrix} W_\mu^+$$

where $\tilde{V}_{CKM} = \tilde{U}^\dagger \tilde{D}$ is called Cabibbo-Kobayashi-Maskawa matrix which is nondiagonal.

- As a result, the mixing has been transferred to the gauge interaction sector due to the definition of quarks with definite mass.

- For lepton sector (electron, neutrino), the structure is almost identical

$$\mathcal{L} = -\sum_{i,j} M_e^{ij} e_L^{i\dagger} e_R^j - \sum_{i,j} M_\nu^{ij} \nu_L^{i\dagger} \nu_R^j + h.c.$$

$$+ \sum_i \frac{ig}{\sqrt{2}} \nu_L^{i\dagger} \bar{\sigma}^\mu e_L^i W_\mu^+ + h.c.$$

- We can rotate electrons and neutrinos to the mass eigenstates

$$\nu_L^i = \sum_{k=1,2,3} \tilde{U}_L^{ik} \nu_L'^k \,, \quad \nu_R^j = \sum_{k=1,2,3} \tilde{U}_R^{jk} \nu_R'^k$$

$$e_L^i = \sum_{a=\{e,\mu,\tau\}} \tilde{D}_L^{ia} e_L'^a \,, \quad e_R^j = \sum_{a=\{e,\mu,\tau\}} \tilde{D}_L^{ja} e_R'^a$$

- Note that I didn't label the neutrino states with definite mass with $\{\nu_e, \nu_\mu, \nu_\tau\}$ just yet.

- The rotations make mass terms diagonalized:

$$\mathcal{L} = -M_\nu^1 \nu_L'^{1\dagger} \nu_R'^1 - M_\nu^2 \nu_L'^{2\dagger} \nu_R'^2 - M_\nu^3 \nu_L'^{3\dagger} \nu_R'^3$$
$$- M_e e_L'^{e\dagger} e_R'^e - M_\mu e_L'^{\mu\dagger} e_R'^\mu - M_\tau e_L'^{\tau\dagger} e_R'^\tau$$

- The gauge interaction is also similar

$$\mathcal{L} = \sum_i \frac{ig}{\sqrt{2}} \nu_L^{i\dagger} \bar{\sigma}^\mu e_L^i W_\mu^+ + h.c.$$

$$= \sum_{k,a} \frac{ig}{\sqrt{2}} \nu_L'^{k\dagger} \underbrace{\left(\sum_i \tilde{U}^{ki*} \tilde{D}^{ia}\right)}_{\equiv \tilde{V}^{ka}_{PMNS}} \bar{\sigma}^\mu e_L'^a W_\mu^+ + h.c.$$

$$= \frac{ig}{\sqrt{2}} \left(\nu_L^{1\dagger}\ \nu_L^{2\dagger}\ \nu_L^{3\dagger}\right) \begin{pmatrix} V^{1e}_{PMNS} & V^{1\mu}_{PMNS} & V^{1\tau}_{PMNS} \\ V^{2e}_{PMNS} & V^{2\mu}_{PMNS} & V^{2\tau}_{PMNS} \\ V^{3e}_{PMNS} & V^{3\mu}_{PMNS} & V^{3\tau}_{PMNS} \end{pmatrix} \begin{pmatrix} \bar{\sigma}^\mu e_L \\ \bar{\sigma}^\mu \mu_L \\ \bar{\sigma}^\mu \tau_L \end{pmatrix} W_\mu^+$$

Where $\tilde{V}^{ka}_{PMNS} = \tilde{U}^\dagger \tilde{D}$ is called Pontecarvo-Maki-Nagakawa-Saki matrix which is equivalent to $\tilde{V}_{CKM}$ in the quark sect

- However, because when neutrino was first postulated we don't really see them.

- Therefore, it makes sense to define them by the type of lepton associated in the process.

- For example, the $\beta$-decay process: $n \to d + e^- + \bar{\nu}$ involves $W^-$ boson


$\Rightarrow$ We define $\bar{\nu}$ as electron type $\equiv \bar{\nu}_e$

- This means that we define $\{\nu_e, \nu_\mu, \nu_\tau\}$ as the fields couple with $\{e, \mu, \tau\}$ in the gauge interaction terms:

$$\mathcal{L} = \frac{ig}{\sqrt{2}} \underbrace{\left(\nu_L^{1\dagger}\ \nu_L^{2\dagger}\ \nu_L^{3\dagger}\right) \begin{pmatrix} V_{PMNS}^{1e} & V_{PMNS}^{1\mu} & V_{PMNS}^{1\tau} \\ V_{PMNS}^{2e} & V_{PMNS}^{2\mu} & V_{PMNS}^{2\tau} \\ V_{PMNS}^{3e} & V_{PMNS}^{3\mu} & V_{PMNS}^{3\tau} \end{pmatrix}}_{\equiv \left(\nu_{L,e}^\dagger\ \nu_{L,\mu}^\dagger\ \nu_{L,\tau}^\dagger\right)} \begin{pmatrix} \bar{\sigma}^\mu e_L \\ \bar{\sigma}^\mu \mu_L \\ \bar{\sigma}^\mu \tau_L \end{pmatrix} W_\mu^+$$

$$= \frac{ig}{\sqrt{2}} \left(\nu_{L,e}^\dagger\ \nu_{L,\mu}^\dagger\ \nu_{L,\tau}^\dagger\right) \begin{pmatrix} \bar{\sigma}^\mu e_L \\ \bar{\sigma}^\mu \mu_L \\ \bar{\sigma}^\mu \tau_L \end{pmatrix} W_\mu^+$$

- Due to this definition, the mass terms for $\{\nu_e, \nu_\mu, \nu_\tau\}$ becomes nondiagonal again

$$\mathcal{L} = -M_\nu^1 \nu_L^{1\dagger} \nu_R^1 - M_\nu^2 \nu_L^{2\dagger} \nu_R^2 - M_\nu^3 \nu_L^{3\dagger} \nu_R^3$$

$$\mathcal{L} = -\left(\nu_L^{1\dagger}\ \nu_L^{2\dagger}\ \nu_L^{3\dagger}\right)\begin{pmatrix} M_\nu^1 & 0 & 0 \\ 0 & M_\nu^2 & 0 \\ 0 & 0 & M_\nu^3 \end{pmatrix}\begin{pmatrix} \nu_R^1 \\ \nu_R^2 \\ \nu_R^3 \end{pmatrix}$$

$$= -\left(\nu_{L,e}^{\dagger}\ \nu_{L,\mu}^{\dagger}\ \nu_{L,\tau}^{\dagger}\right)\underbrace{\begin{pmatrix} V_{PMNS}^{1e*} & V_{PMNS}^{2e*} & V_{PMNS}^{3e*} \\ V_{PMNS}^{1\mu*} & V_{PMNS}^{2\mu*} & V_{PMNS}^{3\mu*} \\ V_{PMNS}^{1\tau*} & V_{PMNS}^{2\tau*} & V_{PMNS}^{3\tau*} \end{pmatrix}\begin{pmatrix} M_\nu^1 & 0 & 0 \\ 0 & M_\nu^2 & 0 \\ 0 & 0 & M_\nu^3 \end{pmatrix}}_{\text{nondiagonal}}\begin{pmatrix} \nu_R^1 \\ \nu_R^2 \\ \nu_R^3 \end{pmatrix}$$

- This nondiagonal mass matrix is the origin of the effect known as neutrino flavour oscillation

## Neutrino Oscillation

- Let's start by writing down the PMNS matrix

$$\begin{pmatrix} \nu_{L,e} \\ \nu_{L,\mu} \\ \nu_{L,\tau} \end{pmatrix} = \begin{pmatrix} V_{PMNS}^{e1} & V_{PMNS}^{e2} & V_{PMNS}^{e3} \\ V_{PMNS}^{\mu 1} & V_{PMNS}^{\mu 2} & V_{PMNS}^{\mu 3} \\ V_{PMNS}^{\tau 1} & V_{PMNS}^{\tau 2} & V_{PMNS}^{\tau 3} \end{pmatrix}\begin{pmatrix} \nu_L^1 \\ \nu_L^2 \\ \nu_L^3 \end{pmatrix}$$

- This looks horrible to deal with! Let's work in a special case where there are only $\nu_e$ & $\nu_\mu$ neutrinos (and $\nu_L^1, \nu_L^2$)
- $V_{PMNS}$ becomes a simple rotation in 2D

$$\begin{pmatrix} \nu_{L,e} \\ \nu_{L,\mu} \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta_{12} & \sin\theta_{12} \\ -\sin\theta_{12} & \cos\theta_{12} \end{pmatrix}}_{V}\begin{pmatrix} \nu_L^1 \\ \nu_L^2 \end{pmatrix}, \quad \begin{pmatrix} \nu_L^1 \\ \nu_L^2 \end{pmatrix} = \underbrace{\begin{pmatrix} \cos\theta_{12} & -\sin\theta_{12} \\ \sin\theta_{12} & \cos\theta_{12} \end{pmatrix}}_{V^\dagger}\begin{pmatrix} \nu_{L,e} \\ \nu_{L,\mu} \end{pmatrix}$$

- The question we want to answer is this:

  If we create only $\nu_e$ from $e^+$ processes, what is the chance that we can detect $\nu_\mu$ at the distance $L$ from the source?

- Surprisingly, the chance is non zero because the states that evolve with time by the phase factor (stationary states) are $(\nu_L^1, \nu_L^2)$.

- We can say this because they have definite masses and therefore have definite energy.

- Of course, this statement comes from Schrödinger equation which is not the right equation for relativistic particles.

- However, we can show the same dynamical equation using Dirac equation (Ask me how to this later if you are interested)

- For a moment, consider Schrodinger equation:

$$i\frac{\partial}{\partial t}|\psi\rangle = \hat{H}|\psi\rangle \qquad (\text{no } \hbar \text{ in this lecture})$$

- The eigenstates are

$$\hat{H}|\nu_L^1\rangle = \sqrt{p^2+(M_\nu^1)^2}\,|\nu_L^1\rangle = p\sqrt{1+\left(\frac{M_\nu^1}{p}\right)^2}\,|\nu_L^1\rangle$$

$$\cong \left(E + \frac{(M_\nu^1)^2}{2E}\right)|\nu_L^1\rangle$$

$$\& \quad \hat{H}|\nu_L^2\rangle \cong \left(E + \frac{(M_\nu^2)^2}{2E}\right)|\nu_L^2\rangle$$

$M_\nu^1, M_\nu^2 \ll p$

$p \sim E$

- Solving Schrodinger equation above gives

$$|\nu_L^1(t)\rangle = e^{-i\left(E-\frac{(M_\nu^1)^2}{2E}\right)t}|\nu_L^1\rangle, \quad |\nu_L^2(t)\rangle = e^{-i\left(E-\frac{(M_\nu^2)^2}{2E}\right)t}|\nu_L^2\rangle$$

- Write the rotation in bracket notation

$$|\nu_{L,e}\rangle = \cos\theta_{12}|\nu_L^1\rangle + \sin\theta_{12}|\nu_L^2\rangle$$

$$|\nu_{L,\mu}\rangle = -\sin\theta_{12}|\nu_L^1\rangle + \cos\theta_{12}|\nu_L^2\rangle$$

- Suppose we start with $|\nu_{L,e}\rangle$ at time $t=0$
- At time $t$, $|\nu_{L,e}\rangle$ evolves as

$$\begin{aligned}|\nu_{L,e}(t)\rangle &= e^{-i\hat{H}t}|\nu_{L,e}\rangle \\ &= \cos\theta_{12}\, e^{-i\left(E-\frac{(M_\nu^1)^2}{2E}\right)t}|\nu_L^1\rangle + \sin\theta_{12}\, e^{-i\left(E-\frac{(M_\nu^2)^2}{2E}\right)t}|\nu_L^2\rangle\end{aligned}$$

- Finding the probability of detecting $\nu_{L,\mu}$ can be written as

$$P(\nu_e\to\nu_\mu) = |\langle\nu_{L,\mu}|\nu_{L,e}(t)\rangle|^2 = \langle\nu_{L,\mu}|\nu_{L,e}(t)\rangle^*\langle\nu_{L,\mu}|\nu_{L,e}(t)\rangle$$

$$\begin{aligned}\langle\nu_{L,\mu}|\nu_{L,e}(t)\rangle &= \left(\langle\nu_L^1|(-\sin\theta_{12}) + \langle\nu_L^2|(\cos\theta_{12})\right)\cdot \\ &\quad \left(\cos\theta_{12}\, e^{-i\left(E-\frac{(M_\nu^1)^2}{2E}\right)t}|\nu_L^1\rangle + \sin\theta_{12}\, e^{-i\left(E-\frac{(M_\nu^2)^2}{2E}\right)t}|\nu_L^2\rangle\right) \\ &= \left(-\sin\theta_{12}\cos\theta_{12}\, e^{-i\left(E-\frac{(M_\nu^1)^2}{2E}\right)t} + \cos\theta_{12}\sin\theta_{12}\, e^{-i\left(E-\frac{(M_\nu^2)^2}{2E}\right)t}\right) \\ &= \frac{\sin(2\theta_{12})}{2}\left(e^{-i\left(E-\frac{(M_\nu^1)^2}{2E}\right)t} - e^{-i\left(E-\frac{(M_\nu^2)^2}{2E}\right)t}\right)\end{aligned}$$

$\sin(2\theta) = 2\sin\theta\cos\theta$

$$\therefore P(\nu_e\to\nu_\mu) = \frac{1}{4}\sin^2(2\theta_{12})\left(e^{i\left(E-\frac{(M_\nu^1)^2}{2E}\right)t} - e^{i\left(E-\frac{(M_\nu^2)^2}{2E}\right)t}\right) \times \left(e^{-i\left(E-\frac{(M_\nu^1)^2}{2E}\right)t} - e^{-i\left(E-\frac{(M_\nu^2)^2}{2E}\right)t}\right)$$

$$P(\nu_e \to \nu_\mu) = \frac{1}{4}\sin^2(2\theta_{12})\left( e^{i\left(\cancel{E} - \frac{(M_\nu^1)^2}{2E}\right)t} - e^{i\left(\cancel{E} - \frac{(M_\nu^2)^2}{2E}\right)t} \right)$$

$$\times \left( e^{-i\left(\cancel{E} - \frac{(M_\nu^1)^2}{2E}\right)t} - e^{-i\left(\cancel{E} - \frac{(M_\nu^2)^2}{2E}\right)t} \right)$$

$$= \frac{1}{4}\sin^2(2\theta_{12})\left[ 2 - 2\left( e^{i\left(-\frac{(M_\nu^1)^2}{2E} + \frac{(M_\nu^2)^2}{2E}\right)t} + e^{i\left(\frac{(M_\nu^1)^2}{2E} - \frac{(M_\nu^2)^2}{2E}\right)t} \right) \right]$$

$$= \frac{1}{2}\sin^2(2\theta_{12})\left[ 1 - \cos\left(\left(\frac{(M_\nu^2)^2}{2E} - \frac{(M_\nu^1)^2}{2E}\right)t\right) \right]$$

$$\Delta M_{21}^2 = (M_\nu^2)^2 - (M_\nu^1)^2$$

$$= \sin^2(2\theta_{12})\left( \frac{1 - \cos\left(\frac{\Delta M_{21}^2 t}{2E}\right)}{2} \right)$$

$$\frac{1-\cos(2\theta)}{2} = \sin^2\theta$$

$$= \sin^2(2\theta_{12})\sin^2\left(\frac{\Delta M_{21}^2 t}{4E}\right)$$

- We can assume that neutrino travels with speed of light

  $v \sim c = 1$ ( in natural unit, we don't write $c$ )

- At time $t$, the distance $= L = t$

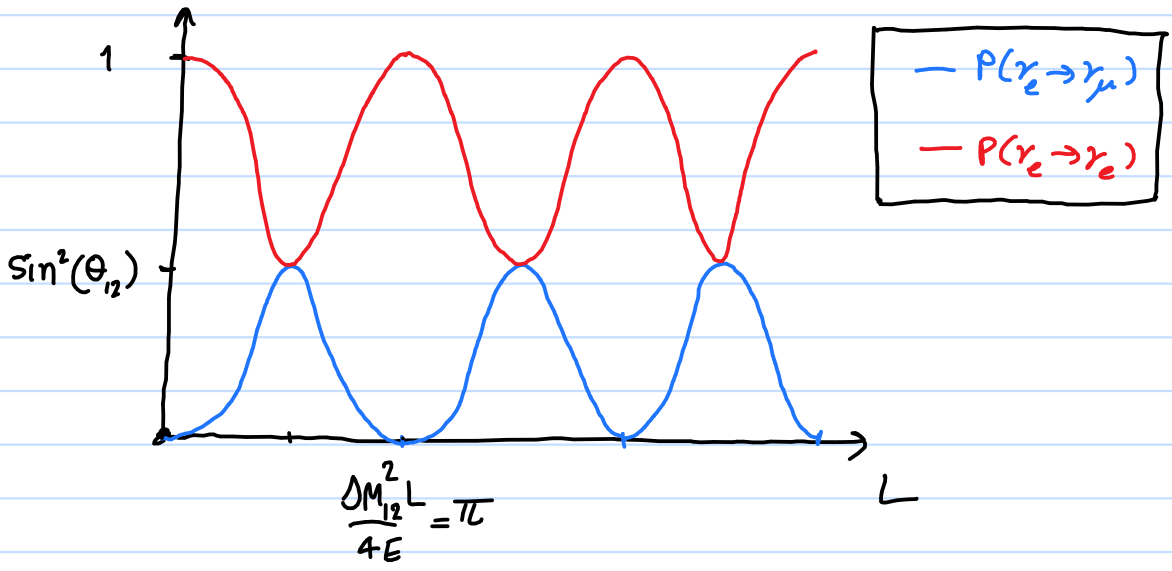
- The probability of detecting $\nu_{L,\mu}$ at a distance $L$ when the source producing only $\nu_e$ is

$$P(\nu_e \to \nu_\mu) = \sin^2(2\theta_{12})\sin^2\left(\frac{\Delta M_{21}^2 L}{4E}\right)$$

- Since the total probability $= 1$, we can write the disappearing probability

$$P(\nu_e \to \nu_e) = 1 - \sin^2(2\theta_{12})\sin^2\left(\frac{\Delta M_{21}^2 L}{4E}\right)$$

• This looks like an oscillation with distance:




• The oscillation length $\Rightarrow$ $L_{osc} = \frac{4\pi E}{\Delta M_{12}^2}$

• Although the SM contains 3 species of neutrino, the analysis for 2 species is still a very useful approximation

• Let's briefly review 3 species case, we can write

$$\begin{pmatrix} \nu_{L,e} \\ \nu_{L,\mu} \\ \nu_{L,\tau} \end{pmatrix} = \begin{pmatrix} V_{PMNS}^{e1} & V_{PMNS}^{e2} & V_{PMNS}^{e3} \\ V_{PMNS}^{\mu 1} & V_{PMNS}^{\mu 2} & V_{PMNS}^{\mu 3} \\ V_{PMNS}^{\tau 1} & V_{PMNS}^{\tau 2} & V_{PMNS}^{\tau 3} \end{pmatrix} \begin{pmatrix} \nu_L^1 \\ \nu_L^2 \\ \nu_L^3 \end{pmatrix}$$

$$\Rightarrow \quad |\nu_{L,a}\rangle = \sum_{k=1}^{3} V_{PMNS}^{ak} |\nu_L^k\rangle \quad , \quad a = \{e, \mu, \tau\}$$

• Repeat the same calculation (a bit more work $\rightarrow$ left as an exercise)

we get

$$P(\nu_{L,a} \to \nu_{L,b}) = \delta_{ab} - 4 \sum_{i<j} \mathrm{Re}\left[V^{*\alpha i} V^{\beta i} V^{\alpha j} V^{*\beta j}\right] \sin^2\left(\frac{\Delta M_{ji}^2 L}{4E}\right)$$
$$+ 2 \sum_{i<j} \mathrm{Im}\left[V^{*\alpha i} V^{\beta i} V^{\alpha j} V^{*\beta j}\right] \sin\left(\frac{\Delta M_{ji}^2 L}{4E}\right)$$

where $\Delta M_{ji}^2 \equiv (M_\nu^j)^2 - (M_\nu^i)^2$

- It turns out that the universe shows some mercy on us:

  Experimentally, the values for mass differences are

$$\Delta M_{21}^2 \equiv \Delta M_{sol}^2 \ll |\Delta M_{31}^2| \simeq |\Delta M_{32}^2| \equiv \Delta M_{atm}^2$$

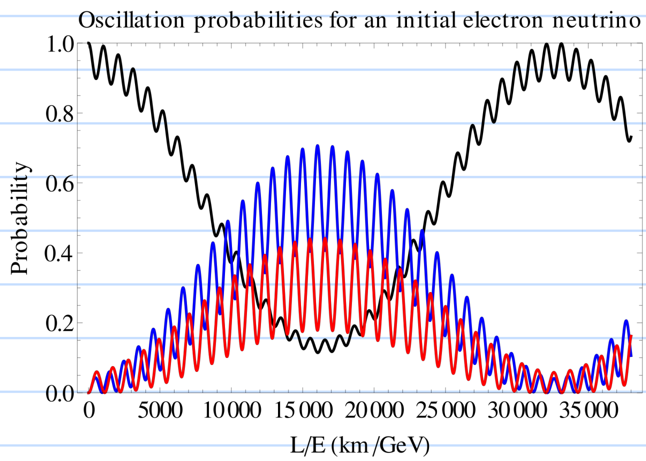
  where $\Delta M_{sol}^2$, $\Delta M_{atm}^2$ are defined as mass differences relevant to solar neutrino & atmospheric neutrino respectively.

- This means the oscillation lengths are seperated nicely

$$L_{sol} = \frac{4\pi E}{\Delta M_{sol}^2} \gg L_{atm} = \frac{4\pi E}{\Delta M_{atm}^2}$$

- In both limits ($L \sim L_{sol}$ & $L \sim L_{atm}$), the neutrino oscillation can be approximately described by 2 species oscillation

- For example, the below graph demonstrate this point perfectly




- The $L/E \sim 15{,}000$ the oscillation with the longer oscillation appears
- For $L/E \sim 1{,}000$ the oscillation with the shorter oscillation dominates
- If you want to know more how to probe neutrino oscillation

  I recommend [hep-ex] 1710.00715