Week 7 Angles, Triangles, Trigonometry Reading Note

Notebook:	Computational Mathematics
Created:	2020-04-21 2:48 PM
Updated:	2020-05-14 2:45 PM
Author:	SUKHJIT MANN

Cornell Notes	Angles, triangles, trigonometry	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: May 12, 2020

Essential Question:

What are angles and what is trigonometry and how are these related to the study of triangles?

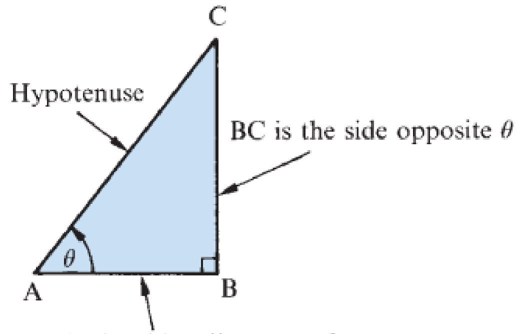
Questions/Cues:

- What is a right angle?
- How are a side and an angle referred to in a triangle?
- What is the hypotenuse?
- What are three basic trigonometric ratios?
- Given one or more of the trigonometric ratio, how do we find theta or how do we denote the inverse of a trig ratio?
- What are Quadrants?
- How do we calculate angles outside the range of 0° to 360°?
- What are the graphs of $y = \sin\theta, y = \cos\theta, \text{and } y = \tan\theta$ like?
- What is an identity?
- What is the most crucial identity in all of trigonometry?
- How do we solve equations involving trigonometric ratios?

Notes

- Right angle = A 90 degree angle, denoted a "square-box" or ∟
- Side (triangle) = A side is made my joining two points together and the resultant side is simply referred as the two uppercase letter that make up the side. ie. AB = the side joining points A and B
- Angle (triangle) = The angle at a particular point is made by joining two sides together. For example, in the figure below, the angle at A, made by the sides AB and AC, is written $\sphericalangle BAC$, $\sphericalangle A$ or simply as A. In the figure below, this angle is labelled as θ

Figure 22.1
A right-angled triangle ABC



- Hypotenuse = The side opposite the right angle

$$\sin\theta = \frac{\text{side opposite to }\theta}{\text{hypotenuse}} = \frac{BC}{AC}$$

$$\cos\theta = \frac{\text{side adjacent to }\theta}{\text{hypotenuse}} = \frac{AB}{AC}$$

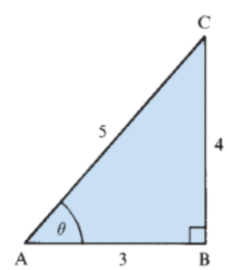
$$\tan\theta = \frac{\text{side opposite to }\theta}{\text{side adjacent to }\theta} = \frac{BC}{AB}$$

  - All trig ratios are defined as the ratio of two lengths and so they themselves are with no units
  - Since the hypotenuse is always the longest side of a triangle, $\sin\theta$ and $\cos\theta$ can never be greater than 1

WORKED EXAMPLES

22.1 Calculate $\sin\theta$, $\cos\theta$ and $\tan\theta$ for $\triangle ABC$ as shown in Figure 22.2.

Figure 22.2
$\triangle ABC$ for Worked Example 22.1



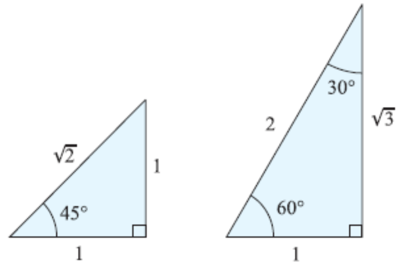
Solution

$$\sin\theta = \frac{BC}{AC} = \frac{4}{5} = 0.8$$

$$\cos\theta = \frac{AB}{AC} = \frac{3}{5} = 0.6$$

$$\tan\theta = \frac{BC}{AB} = \frac{4}{3} = 1.3333$$

Figure 22.3



22.2 Using the triangles shown in Figure 22.3 write down expressions for

(a) $\sin 45^\circ$, $\cos 45^\circ$, $\tan 45^\circ$

(b) $\sin 60^\circ$, $\cos 60^\circ$, $\tan 60^\circ$

(c) $\sin 30^\circ$, $\cos 30^\circ$, $\tan 30^\circ$

Solution

(a) $\sin 45^\circ = \frac{1}{\sqrt{2}}$, $\cos 45^\circ = \frac{1}{\sqrt{2}}$, $\tan 45^\circ = \frac{1}{1} = 1$

(b) $\sin 60^\circ = \frac{\sqrt{3}}{2}$, $\cos 60^\circ = \frac{1}{2}$, $\tan 60^\circ = \frac{\sqrt{3}}{1} = \sqrt{3}$

(c) $\sin 30^\circ = \frac{1}{2}$, $\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\tan 30^\circ = \frac{1}{\sqrt{3}}$

Notice that we have left our solutions in their exact forms using square roots if necessary rather than given their decimal approximations. These, so-called, **surd forms** are frequently used.

Note that the triangles in the above example are also referred to as special triangles, with one being a 45, 45, 90 triangle and the other being a 30, 60, 90 triangle

****Also an angle in degrees has the symbol ° Otherwise assume that the angle is measured in radians.**

WORKED EXAMPLES

22.3 Use a scientific calculator to evaluate

(a) $\sin 65^\circ$ (b) $\cos 17^\circ$ (c) $\tan 50^\circ$ (d) $\sin 1$ (e) $\cos 1.5$
(f) $\tan 0.5$

Solution

For (a), (b) and (c) ensure your calculator is set to DEGREES, and not to RADIANS or GRADS.

(a) $\sin 65^\circ = 0.9063$ (b) $\cos 17^\circ = 0.9563$ (c) $\tan 50^\circ = 1.1918$

Now, change the MODE of your calculator in order to work in radians. Check that

(d) $\sin 1 = 0.8415$ (e) $\cos 1.5 = 0.0707$ (f) $\tan 0.5 = 0.5463$

22.4 Prove

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

Solution From Figure 22.1, we have

$$\sin\theta = \frac{BC}{AC} \qquad \cos\theta = \frac{AB}{AC}$$

and so

$$\frac{\sin\theta}{\cos\theta} = \frac{BC/AC}{AB/AC} = \frac{BC}{AC} \times \frac{AC}{AB} = \frac{BC}{AB}$$

But

$$\tan\theta = \frac{BC}{AB}$$

and so

$$\frac{\sin\theta}{\cos\theta} = \tan\theta$$

- Finding angle given trig ratios = If $\sin\theta = x$, we write $\theta = \sin^{-1}x$
    - This is the same as saying that θ is the angle whose sine is x
    - The superscript -1 does not denote a power. It's a notation for the inverse of the trigonometrical ratios

WORKED EXAMPLES

22.5 Use a scientific calculator to evaluate

(a) $\sin^{-1} 0.5$ (b) $\cos^{-1} 0.3$ (c) $\tan^{-1} 2$

Solution (a) Using the $\sin^{-1}$ button we have

$$\sin^{-1} 0.5 = 30°$$

This is another way of saying that $\sin 30° = 0.5$.

(b) Using a scientific calculator we see

$$\cos^{-1} 0.3 = 72.5°$$

that is,

$$\cos 72.5° = 0.3$$

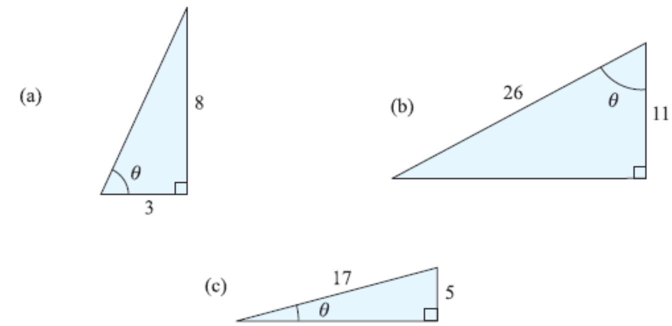
(c) Using a calculator we see $\tan^{-1} 2 = 63.4°$ and hence $\tan 63.4° = 2$.

22.6 Calculate the angles θ in Figures 22.4(a), (b) and (c).

Solution (a) In Figure 22.4(a) we are given the length of the side opposite θ and the length of the adjacent side. Hence we can make use of the tangent ratio. We write

$$\tan\theta = \frac{8}{3} = 2.6667$$

Figure 22.4



Therefore θ is the angle whose tangent is 2.6667, written

$$\theta = \tan^{-1} 2.6667$$

Using a calculator we can find the angle θ in either degrees or radians. If your calculator is set to degree mode, verify that $\theta = 69.44°$. Check also that you can obtain the equivalent answer in radians, that is 1.212.

(b) In Figure 22.4(b) we are given the length of the side adjacent to θ and the length of the hypotenuse. Hence we use the cosine ratio:

$$\cos\theta = \frac{11}{26} = 0.4231$$

Therefore

$$\theta = \cos^{-1} 0.4231$$

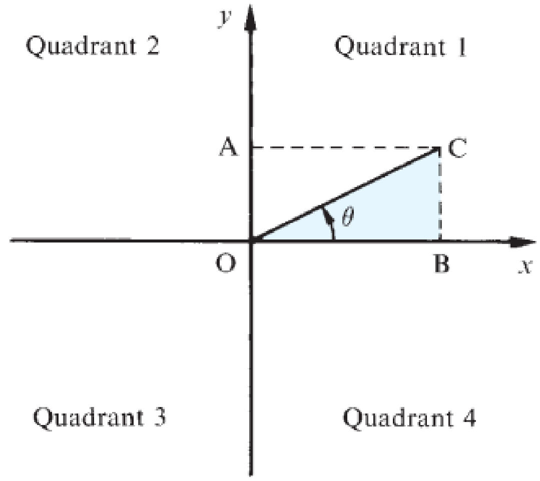
Using a calculator check that $\theta = 64.97°$.

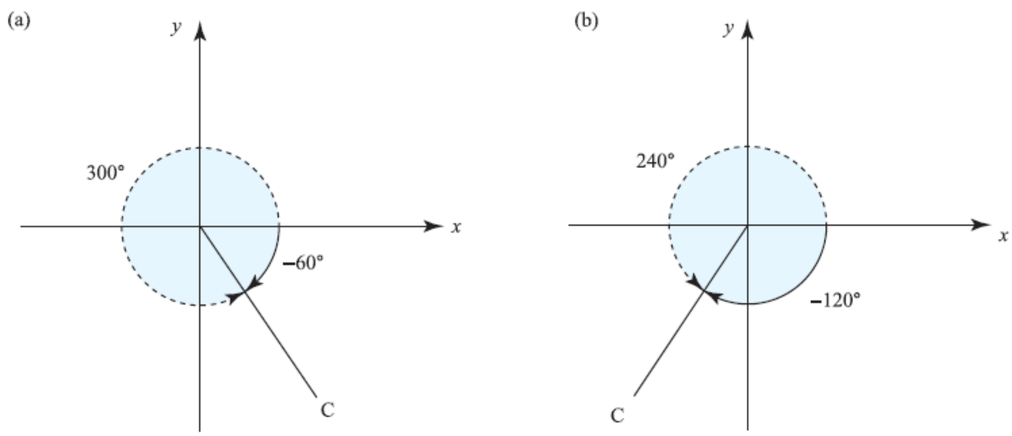
(c) In Figure 22.4(c) $\sin\theta = \frac{5}{17}$. Hence

$$\theta = \sin^{-1}\frac{5}{17} = 17.10°$$

- Quadrants (xy plane) = the xy plane can be divided into in four quadrants as shown in the figure below. The origin is O. The arm formed by OC can rotate into any of the quadrants. We measure the anticlockwise angle from the positive x axis to the arm and call this angle θ
  - When arm is in quadrant 1, $0° \le \theta \le 90°$
  - When arm is in quadrant 2, $90° \le \theta \le 180°$
  - When arm is in quadrant 3, $180° \le \theta \le 270°$
  - When arm is in quadrant 4, $270° \le \theta \le 360°$
    - On occasion, angles are measured in a clockwise direction from the positive x axis. In this case, these are conventionally taken to be negative. For example, angles of $-60°$ and $-120°$ correspond to same position as $300°$ and $240°$ each respectively as shown below

Figure 23.1
The plane is divided into four quadrants





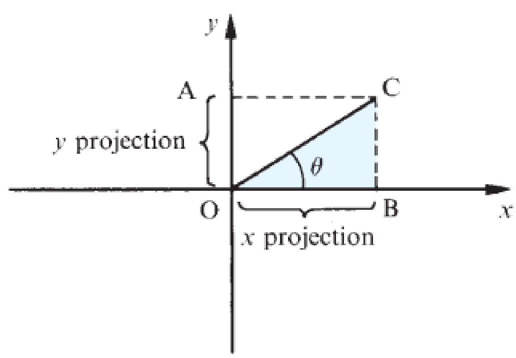
The negative angle −60°

The negative angle −120°

Figure 23.2
The negative angles of $-60°$ and $-120°$

- x-y projections = length of the side of the triangle formed by the arm OC. OB is the x-projection and OA is the y-projection. OA and OB can be negative depending on the position of the arm OC, but the length of the arm is always positive

Figure 23.3
The x projection of OC is OB; the y projection of OC is OA



$$\sin\theta = \frac{y \text{ projection of arm OC}}{\text{OC}} = \frac{\text{OA}}{\text{OC}}$$

$$\cos\theta = \frac{x \text{ projection of arm OC}}{\text{OC}} = \frac{\text{OB}}{\text{OC}}$$

$$\tan\theta = \frac{y \text{ projection of arm OC}}{x \text{ projection of arm OC}} = \frac{\text{OA}}{\text{OB}}$$

Note that the extended definition is in terms of projections and so θ is not limited to a maximum value of $90°$.

Projections in the first quadrant

Consider the arm OC in the first quadrant, as shown in Figure 23.3. From the right-angled triangle OCB we have

$$\sin\theta = \frac{\text{BC}}{\text{OC}} \qquad \cos\theta = \frac{\text{OB}}{\text{OC}} \qquad \tan\theta = \frac{\text{BC}}{\text{OB}}$$

Alternatively, using the extended definition we could also write

$$\sin\theta = \frac{\text{OA}}{\text{OC}} \qquad \cos\theta = \frac{\text{OB}}{\text{OC}} \qquad \tan\theta = \frac{\text{OA}}{\text{OB}}$$

Noting that OA = BC, we see that the two definitions are in agreement when $0° \leqslant \theta \leqslant 90°$.

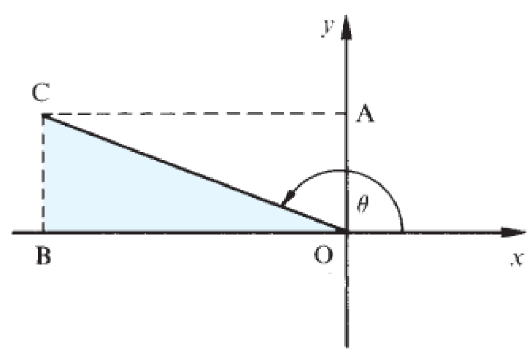


Figure 23.4
When OC is in the second quadrant the x projection is negative

Projections in the second quadrant

We now consider the arm in the second quadrant, as shown in Figure 23.4. The x projection of OC is onto the negative part of the x axis; the y projection of OC is onto the positive part of the y axis. Hence $\sin\theta$ is positive, whereas $\cos\theta$ and $\tan\theta$ are negative.

Projections in the third quadrant

When the arm is in the third quadrant, as shown in Figure 23.5, the x and y projections are both negative. Hence for $180° < \theta < 270°$, $\sin\theta$ and $\cos\theta$ are negative and $\tan\theta$ is positive.

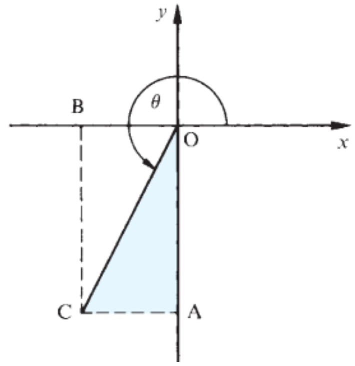


Figure 23.5
Both x and y projections are negative

Projections in the fourth quadrant

Finally the arm OC is rotated into the fourth quadrant, as shown in Figure 23.6. The x projection is positive, the y projection is negative, and so $\sin\theta$ and $\tan\theta$ are negative and $\cos\theta$ is positive.

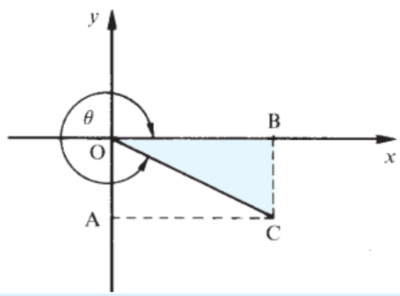


Figure 23.6
The x projection is positive and the y projection is negative

	Quadrant 1	Quadrant 2	Quadrant 3	Quadrant 4
$\sin\theta$	positive	positive	negative	negative
$\cos\theta$	positive	negative	negative	positive
$\tan\theta$	positive	negative	positive	negative

- Angles (outside the degree range) = This is done by adding or subtracting $360°$ from an angle. By adding/subtracting $360°$ is equivalent to rotating the arm through a complete revolution. This leaves the position unchanged and thus leaves the trigonometrical ration unaltered. This is mathematically stated by the following:

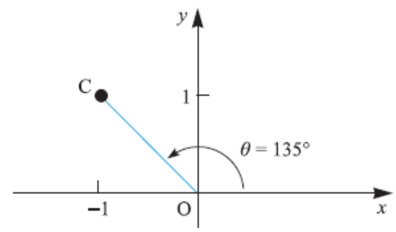
$$\sin\theta = \sin(\theta + 360°) = \sin(\theta - 360°)$$

$$\cos\theta = \cos(\theta + 360°) = \cos(\theta - 360°)$$

$$\tan\theta = \tan(\theta + 360°) = \tan(\theta - 360°)$$

23.1 Find $\tan\theta$ in Figure 23.7.

Figure 23.7



Solution Angle θ is in the second quadrant. Hence its tangent will be negative. We can find its value from

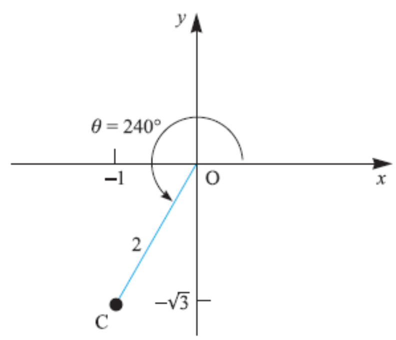
$$\tan\theta = \frac{y \text{ projection}}{x \text{ projection}}$$

$$= \frac{1}{-1}$$

$$= -1$$

23.2 Find $\sin\theta$ in Figure 23.8.

Figure 23.8



Solution Angle θ is in the third quadrant. Hence its sine will be negative. We can find its value from

$$\sin\theta = \frac{y \text{ projection}}{\text{OC}}$$

$$= \frac{-\sqrt{3}}{2}$$

$$= -\frac{\sqrt{3}}{2}$$

23.3 Find $\cos\theta$ in Figure 23.9.

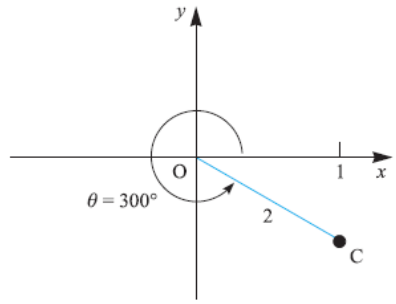


Figure 23.9

Solution Angle θ is in the fourth quadrant. Hence its cosine will be positive. We can find its value from

$$\cos\theta = \frac{x \text{ projection}}{OC}$$

$$= \frac{1}{2}$$

Table 23.1

θ	0°	30°	60°	90°	120°	150°	180°
$\sin\theta$	0	0.5	0.8660	1	0.8660	0.5	0
θ	210°	240°	270°	300°	330°	360°	
$\sin\theta$	−0.5	−0.8660	−1	−0.8660	−0.5	0	

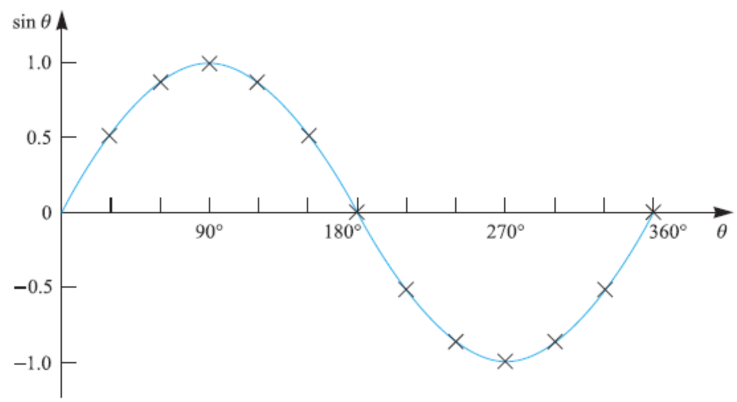


Figure 23.11
The function $y = \sin\theta$

- If we extend values of θ to below $0°$ and above $360°$, the values of $\sin\theta$ are simply repeated. The pattern established by graph of $\sin\theta$ is repeated every $360°$

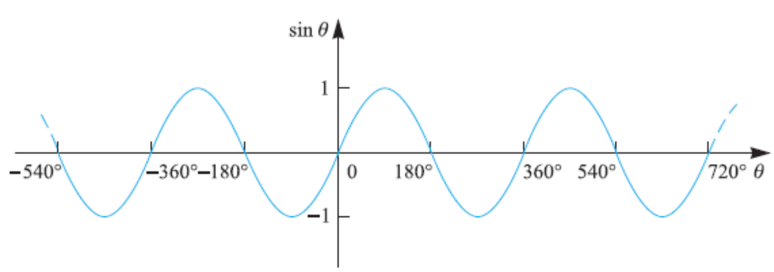


Figure 23.12
The values of $\sin\theta$ repeat every 360°

Table 23.2

θ	0°	30°	60°	90°	120°	150°	180°
$\cos\theta$	1	0.8660	0.5	0	−0.5	−0.8660	−1
θ	210°	240°	270°	300°	330°	360°	
$\cos\theta$	−0.8660	−0.5	0	0.5	0.8660	1	

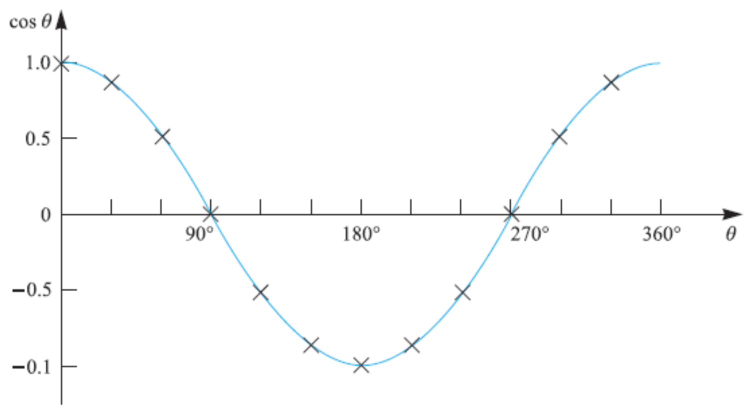


Figure 23.13
The function $y = \cos\theta$

- If we extend values of θ to below 0° and above 360°, the values of $\cos\theta$ are simply repeated. The pattern established by the graph of $\cos\theta$ is repeated every 360°

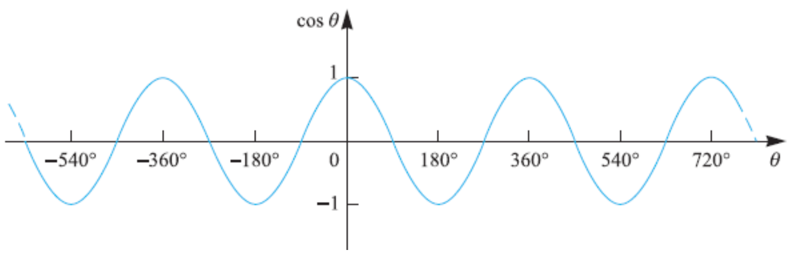


Figure 23.14
The values of $\cos\theta$ repeat every 360°

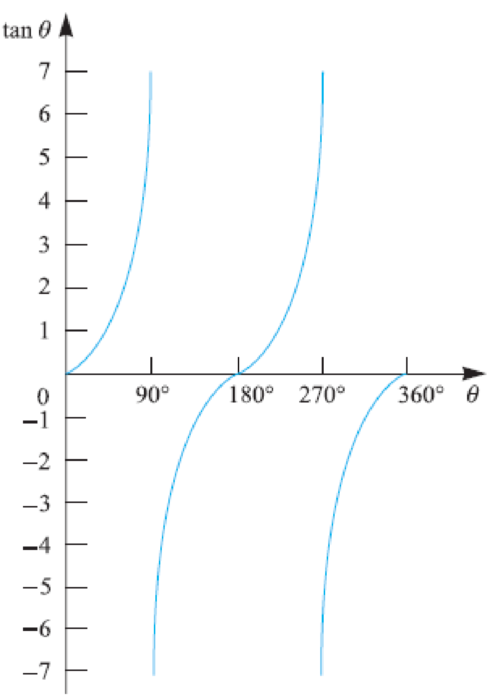
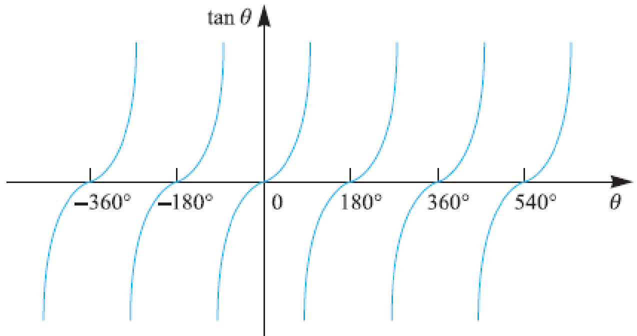


Figure 23.15
The function $y = \tan\theta$

- The pattern established by $\tan\theta$ is repeated every 180°. The values of $\tan\theta$ extend from minus infinity to plus infinity

Figure 23.16
The values of $\tan\theta$ repeat every $180°$



WORKED EXAMPLES

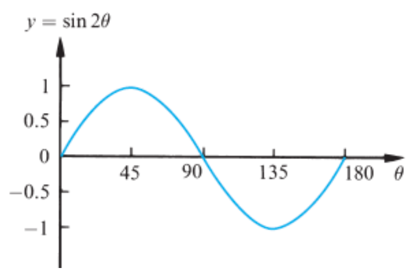
23.4 Draw $y = \sin 2\theta$ for $0° \leqslant \theta \leqslant 180°$.

Solution Values of θ and $\sin 2\theta$ are given in Table 23.3. A graph of $y = \sin 2\theta$ is shown in Figure 23.17.

Table 23.3
Values of θ and corresponding values of $\sin 2\theta$

θ	0	15	30	45	60	75	90	105
2θ	0	30	60	90	120	150	180	210
$\sin 2\theta$	0	0.5000	0.8660	1	0.8660	0.5000	0	−0.5000
θ		120	135	150	165	180		
2θ		240	270	300	330	360		
$\sin 2\theta$		−0.8660	−1	−0.8660	−0.5000	0		

Figure 23.17
A graph of $y = \sin 2\theta$



23.5 Draw a graph of $y = \cos(\theta + 30°)$ for $0° \leqslant \theta \leqslant 360°$.

Solution A table of values is drawn up in Table 23.4. Figure 23.18 shows a graph of $y = \cos(\theta + 30°)$.

Table 23.4
Values of θ and corresponding values of $\cos(\theta + 30°)$

θ	0	30	60	90	120	150	180	210
$\theta + 30$	30	60	90	120	150	180	210	240
$\cos(\theta + 30)$	0.8660	0.5000	0	−0.5000	−0.8660	−1	−0.8660	−0.5000

θ	240	270	300	330	360
$\theta + 30$	270	300	330	360	390
$\cos(\theta + 30)$	0	0.5000	0.8660	1	0.8660

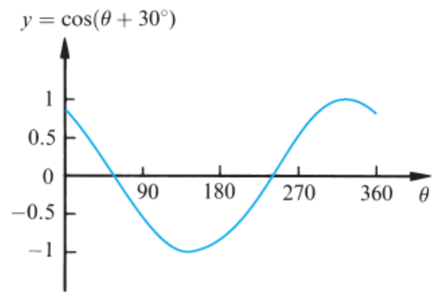


Figure 23.18
Graph of $y = \cos(\theta + 30°)$

- Identity = looks like an equation, crucial difference is that the left-hand side and right-hand side of an identity are equal for all values of the variable involved. On the other hand, an equation contains one or more unknown quantities, which must be found before the left-hand and right-hand sides are equal
  - $\frac{\sin\theta}{\cos\theta} = \tan\theta$ is an identity since $\frac{\sin\theta}{\cos\theta}$ and $\tan\theta$ have the same value whatever the value of θ
- The fundamental identity of trigonometry = $\sin^2 A + \cos^2 A = 1$
  - the symbols/variables A and B stand for any angle we choose
  - ****Note that** $(\cos A)^2$ is usually written as $\cos^2 A$ and $(\sin A)^2$ is written as $\sin^2 A$
  - The fundamental identity shows that sin A and cos A are closely related. Knowing sin A the identity can be used to calculate cos A and vice versa
  - By manipulating this identity we are able to derive a list of common trigonometric identities

Table 24.1
Common trigonometrical identities

$$\frac{\sin A}{\cos A} = \tan A$$

$$\sin(A+B) = \sin A \cos B + \sin B \cos A$$

$$\sin(A-B) = \sin A \cos B - \sin B \cos A$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos(A+B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A-B) = \cos A \cos B + \sin A \sin B$$

$$\cos 2A = (\cos A)^2 - (\sin A)^2 = \cos^2 A - \sin^2 A$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2 \sin\left(\frac{A-B}{2}\right) \cos\left(\frac{A+B}{2}\right)$$

$$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2 \sin\left(\frac{A-B}{2}\right) \sin\left(\frac{A+B}{2}\right)$$

$$\sin A = -\sin(-A)$$

$$\cos A = \cos(-A)$$

$$\tan A = -\tan(-A)$$

$$\sin \theta = \sin(180^\circ - \theta)$$

$$\cos \theta = -\cos(180^\circ - \theta)$$

$$\tan \theta = -\tan(180^\circ - \theta)$$

$$\sin \theta = -\sin(\theta - 180^\circ)$$

$$\cos \theta = -\cos(\theta - 180^\circ)$$

$$\tan \theta = \tan(\theta - 180^\circ)$$

$$\sin \theta = -\sin(360^\circ - \theta)$$

$$\cos \theta = \cos(360^\circ - \theta)$$

$$\tan \theta = -\tan(360^\circ - \theta)$$

WORKED EXAMPLE

24.1 (a) Show that $\sin\theta = \sin(180^\circ - \theta)$.

(b) Show that $\cos\theta = -\cos(180^\circ - \theta)$.

(c) From parts (a) and (b) deduce that $\tan\theta = -\tan(180^\circ - \theta)$.

Solution (a) Consider $\sin(180^\circ - \theta)$. Because the right-hand side is the sine of the difference of two angles we use the identity for $\sin(A - B)$:

$$\sin(A - B) = \sin A \cos B - \sin B \cos A$$

Letting $A = 180^\circ$ and $B = \theta$ we obtain

$$\begin{aligned}\sin(180^\circ - \theta) &= \sin 180^\circ \cos\theta - \sin\theta \cos 180^\circ \\ &= 0.\cos\theta - \sin\theta(-1) \\ &= \sin\theta\end{aligned}$$

Hence $\sin\theta = \sin(180^\circ - \theta)$.

(b) Consider $\cos(180^\circ - \theta)$. Here we are dealing with the cosine of the difference of two angles. We use the identity

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

Letting $A = 180^\circ$ and $B = \theta$ we obtain

$$\begin{aligned}\cos(180^\circ - \theta) &= \cos 180^\circ \cos\theta + \sin 180^\circ \sin\theta \\ &= -1(\cos\theta) + 0(\sin\theta) \\ &= -\cos\theta\end{aligned}$$

Hence $\cos\theta = -\cos(180^\circ - \theta)$.

(c) We note from Table 24.1 that $\tan\theta = \frac{\sin\theta}{\cos\theta}$ and so, using the results of parts (a) and (b),

$$\begin{aligned}\tan\theta &= \frac{\sin\theta}{\cos\theta} \\ &= \frac{\sin(180^\circ - \theta)}{-\cos(180^\circ - \theta)} \\ &= -\tan(180^\circ - \theta)\end{aligned}$$

WORKED EXAMPLE

24.2 Use a calculator to evaluate the following trigonometrical ratios and hence verify the identities obtained in Worked Example 24.1.

(a) $\sin 30^\circ$ and $\sin 150^\circ$

(b) $\cos 70^\circ$ and $\cos 110^\circ$

(c) $\tan 30^\circ$ and $\tan 150^\circ$

Solution

(a) Using a calculator verify that $\sin 30^\circ = 0.5$. Similarly, $\sin 150^\circ = 0.5$. We see that

$$\sin 30^\circ = \sin 150^\circ = \sin(180^\circ - 30^\circ)$$

This verifies the first of the identities for the case when $\theta = 30^\circ$.

(b) $\cos 70^\circ = 0.3420$ and $\cos 110^\circ = -0.3420$. Hence

$$\cos 70^\circ = -\cos 110^\circ = -\cos(180^\circ - 70^\circ)$$

This verifies the second of the identities for the case when $\theta = 70^\circ$.

(c) $\tan 30^\circ = 0.5774$ and $\tan 150^\circ = -0.5774$. So

$$\tan 30^\circ = -\tan 150^\circ = -\tan(180^\circ - 30^\circ)$$

This verifies the third of the identities for the case when $\theta = 30^\circ$.

WORKED EXAMPLES

24.3 Evaluate the following trigonometrical ratios using a calculator and hence verify the results in the previous Key point.

(a) $\sin 30^\circ$ and $\sin 210^\circ$

(b) $\cos 40^\circ$ and $\cos 220^\circ$

(c) $\tan 50^\circ$ and $\tan 230^\circ$

Solution

(a) $\sin 30^\circ = 0.5$ and $\sin 210^\circ = -0.5$. Note that $\sin 210^\circ = -\sin 30^\circ$.

(b) $\cos 40^\circ = 0.7660$ and $\cos 220^\circ = -0.7660$. Note that $\cos 220^\circ = -\cos 40^\circ$.

(c) $\tan 50^\circ = 1.1918$ and $\tan 230^\circ = 1.1918$. Note that $\tan 50^\circ = \tan 230^\circ$.

24.4 Evaluate the following trigonometrical ratios using a calculator and hence verify the results in the previous Key point.

(a) $\sin 70^\circ$ and $\sin 290^\circ$

(b) $\cos 40^\circ$ and $\cos 320^\circ$

(c) $\tan 20^\circ$ and $\tan 340^\circ$

Solution

(a) $\sin 70^\circ = 0.9397$, $\sin 290^\circ = -0.9397$ and so $\sin 290^\circ = -\sin 70^\circ$.

(b) $\cos 40^\circ = 0.7660$, $\cos 320^\circ = 0.7660$ and so $\cos 40^\circ = \cos 320^\circ$.

(c) $\tan 20^\circ = 0.3640$, $\tan 340^\circ = -0.3640$ and so $\tan 340^\circ = -\tan 20^\circ$.

WORKED EXAMPLES

24.5 Simplify

$$1 - \sin A \cos A \tan A$$

Solution

$$1 - \sin A \cos A \tan A = 1 - \sin A \cos A \left(\frac{\sin A}{\cos A}\right)$$
$$= 1 - \sin^2 A$$
$$= \cos^2 A \qquad \text{since} \cos^2 A + \sin^2 A = 1$$

24.6 Show that

$$\cos^4 A - \sin^4 A = \cos 2A$$

Solution We note that

$$\cos^4 A - \sin^4 A = (\cos^2 A + \sin^2 A)(\cos^2 A - \sin^2 A)$$
$$= 1(\cos^2 A - \sin^2 A) \qquad \text{using} \cos^2 A + \sin^2 A = 1$$
$$= \cos 2A \qquad \text{using Table 24.1}$$

24.7 Show that

$$\frac{\sin 6\theta - \sin 4\theta}{\sin \theta} = 2 \cos 5\theta$$

Solution We note the identity

$$\sin A - \sin B = 2 \sin\left(\frac{A - B}{2}\right) \cos\left(\frac{A + B}{2}\right)$$

Letting $A = 6\theta,\ B = 4\theta$ we obtain

$$\sin 6\theta - \sin 4\theta = 2 \sin \theta \cos 5\theta$$

and so

$$\frac{\sin 6\theta - \sin 4\theta}{\sin \theta} = 2 \cos 5\theta$$

- Trigonometric Equations = make use of the inverse trigonometric functions $\sin^{-1}x, \cos^{-1}x, \tan^{-1}x$
  - For example, since $\sin 30^\circ = 0.5$ we write $30^\circ = \sin^{-1}0.5$ and say 30° is the angle who sine is 0.5. It is also true that $\sin 150^\circ = 0.5$ and $150^\circ = \sin^{-1}0.5$. So the inverse sine, cosine, or tangent of a number can yield more than one answer
    - Your calculator will only give one value, the other values must be deduced from knowledge of the trig functions $y = \sin\theta, y = \cos^{-1}\theta, \text{and } \tan\theta$
    - Make sure to pay attention to the range that is stated in the question to know how many values are required for your answer

24.8 Find all values of $\sin^{-1} 0.4$ in the range $0°$ to $360°$.

Solution Let $\theta = \sin^{-1} 0.4$, that is $\sin\theta = 0.4$. Since $\sin\theta$ is positive then one value is in the first quadrant and another value is in the second quadrant. Using a scientific calculator we see

$$\theta = \sin^{-1} 0.4 = 23.6°$$

The formula $\sin\theta = \sin(180° - \theta)$ was derived on page 294

This is the value in the first quadrant. The value in the second quadrant is found using $\sin\theta = \sin(180° - \theta)$. Hence the value in the second quadrant is $180° - 23.6° = 156.4°$.

24.9 Find all values of θ in the range $0°$ to $360°$ such that

(a) $\theta = \cos^{-1}(-0.5)$ (b) $\theta = \tan^{-1} 1$

Solution (a) We have $\theta = \cos^{-1}(-0.5)$, that is $\cos\theta = -0.5$. As $\cos\theta$ is negative then θ must be in the second and third quadrants. Using a calculator we find $\cos^{-1}(-0.5) = 120°$. This is the value of θ in the second quadrant. We now seek the value of θ in the third quadrant.

We have, from the Key point on page 295,

$$\cos(\theta - 180°) = -\cos\theta$$

We are given $\cos\theta = -0.5$. So

$$\cos(\theta - 180°) = 0.5$$

Now, since θ is in the third quadrant, $\theta - 180°$ must be an acute angle whose cosine equals 0.5. That is,

$$\theta - 180° = 60°$$
$$\theta = 240°$$

This is the value of θ in the third quadrant. The required solutions are thus $\theta = 120°, 240°$.

(b) We are given $\theta = \tan^{-1} 1$, that is $\tan\theta = 1$. Since $\tan\theta$ is positive, there is a value of θ in the first and third quadrants. Using a calculator we find $\tan^{-1} 1 = 45°$. This is the value of θ in the first quadrant.

We have, from the Key point on page 295,

$$\tan(\theta - 180°) = \tan\theta$$

We are given $\tan\theta = 1$. So

$$\tan(\theta - 180°) = 1$$

Now $\theta - 180°$ must be an acute angle with tangent equal to 1. That is,

$$\theta - 180° = 45°$$
$$\theta = 225°$$

The required values of θ are $45°$ and $225°$.

24.10 Find all values of θ in the range $0°$ to $360°$ such that $\theta = \sin^{-1}(-0.5)$.

Solution We know $\sin\theta = -0.5$ and so θ must be in the third and fourth quadrants. Using a calculator we find

$$\sin^{-1}(-0.5) = -30°$$

We require solutions in the range $0°$ to $360°$. Recall that adding $360°$ to an angle leaves the values of the trigonometrical ratios unaltered. So

$$\sin^{-1}(-0.5) = -30° + 360° = 330°$$

We have found the value of θ in the fourth quadrant.

We now seek the value in the third quadrant. We have, from the Key point on page 295,

$$\sin(\theta - 180°) = -\sin\theta$$

We are given $\sin\theta = -0.5$. Therefore

$$\sin(\theta - 180°) = 0.5$$

Now $\theta - 180°$ must be an acute angle with sine equal to 0.5. That is,

$$\theta - 180° = 30°$$
$$\theta = 210°$$

The required values of θ are $210°$ and $330°$.

24.11 Find all values of θ in the range $0°$ to $360°$ such that $\sin 2\theta = 0.5$.

Solution We make the substitution $z = 2\theta$. As θ varies from $0°$ to $360°$ then z varies from $0°$ to $720°$. Hence the problem as given is equivalent to finding all values of z in the range $0°$ to $720°$ such that $\sin z = 0.5$.

$$\sin z = 0.5$$
$$z = \sin^{-1} 0.5$$
$$= 30°$$

Also

$$\sin(180° - 30°) = \sin 30°$$
$$\sin 150° = \sin 30° = 0.5$$
$$\sin^{-1} 0.5 = 150°$$

The values of z in the range $0°$ to $360°$ are $30°$ and $150°$. Recalling that adding $360°$ to an angle leaves its trigonometrical ratios unaltered, we see that $30° + 360° = 390°$ and $150° + 360° = 510°$ are values of z in the range $360°$ to $720°$. The solutions for z are thus

$$z = 30°, 150°, 390°, 510°$$

and hence the required values of θ are given by

$$\theta = \frac{z}{2} = 15°, 75°, 195°, 255°$$

24.12 Solve $\cos 3\theta = -0.5$ for $0° \leqslant \theta \leqslant 360°$.

Solution We substitute $z = 3\theta$. As θ varies from $0°$ to $360°$ then z varies from $0°$ to $1080°$. Hence the problem is to find values of z in the range $0°$ to $1080°$ such that $\cos z = -0.5$.

We begin by finding values of z in the range $0°$ to $360°$. Using Worked Example 24.9(a) we see that

$$z = 120°, 240°$$

To find values of z in the range $360°$ to $720°$ we add $360°$ to each of these solutions. This gives

$$z = 120° + 360° = 480° \qquad \text{and} \qquad z = 240° + 360° = 600°$$

To find values of z in the range $720°$ to $1080°$ we add $360°$ to these solutions. This gives

$$z = 840°, 960°$$

Hence

$$z = 120°, 240°, 480°, 600°, 840°, 960°$$

Finally, using $\theta = \frac{z}{3}$ we obtain values of θ:

$$\theta = \frac{z}{3} = 40°, 80°, 160°, 200°, 280°, 320°$$

24.13 Solve

$$\tan(2\theta + 20°) = 0.3 \qquad 0° \leqslant \theta \leqslant 360°$$

Solution We substitute $z = 2\theta + 20°$. As θ varies from $0°$ to $360°$ then z varies from $20°$ to $740°$.

First we solve

$$\tan z = 0.3 \qquad 0° \leqslant z \leqslant 360°$$

This leads to $z = 16.7°, 196.7°$. Values of z in the range $360°$ to $720°$ are

$$\begin{aligned} z &= 16.7° + 360°, 196.7° + 360° \\ &= 376.7°, 556.7° \end{aligned}$$

By adding a further $360°$ values of z in the range $720°$ to $1080°$ are found. These are

$$z = 736.7°, 916.7°$$

Hence values of z in the range $0°$ to $1080°$ are

$$z = 16.7°, 196.7°, 376.7°, 556.7°, 736.7°, 916.7°$$

Values of z in the range $20°$ to $740°$ are thus

$$z = 196.7°, 376.7°, 556.7°, 736.7°$$

The values of θ in the range $0°$ to $360°$ are found using $\theta = (z - 20°)/2$:

$$\theta = \frac{z - 20°}{2} = 88.35°, 178.35°, 268.35°, 358.35°$$

Summary

In this week, we learned the basic trigonometric ratios that govern the study of triangles. Also we examined special triangles and corresponding trig ratios that are generated through these triangles. We also looked at the extended definition of the three trig ratio, the inverse trig functions, the fundamental trig identity, the various derived trig identities and lastly trigonometric equations.