Week 13 Exponential and Logarithmic functions Reading Note

Notebook:	Computational Mathematics		
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Cornell Notes	**Topic:** **Exponential and Logarithmic functions**	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: July 20, 2020

Essential Question:

What are the exponential and logarithmic functions?

Questions/Cues:

- What is an exponential expression?
- What are laws of indices/exponents used to simplify exponential expressions?
- What is the exponential function and its properties?
- What is the function e^{-x} and its properties?
- What are logarithms?
- How do you calculate logarithms to bases other than 10 or e?
- What are the laws of logarithms?
- How do you solve equations involving logarithms?
- What is the logarithmic function and its properties?

Notes

- Exponential expression = of the form a^x, where the number a is the base and is the power or index
  - One of the most commonly used values for the base is the non-terminating number 2.71828..., denoted by the letter e
    - e^x is found throughout modelling problems in nature, i.e. population growth, spread of bacteria and radioactive decay

$$e^a e^b = e^{a+b}$$

$$\frac{e^a}{e^b} = e^{a-b}$$

$$e^0 = 1$$

$$(e^a)^b = e^{ab}$$

WORKED EXAMPLES

19.2 Simplify the following exponential expressions:

(a) e^2e^4 (b) $\frac{e^4}{e^3}$ (c) $(e^2)^{2.5}$

Solution

(a) $e^2e^4 = e^{2+4} = e^6$

(b) $\frac{e^4}{e^3} = e^{4-3} = e^1 = e$

(c) $(e^2)^{2.5} = e^{2\times 2.5} = e^5$

19.3 Simplify the following exponential expressions:

(a) e^xe^{3x} (b) $\frac{e^{4t}}{e^{3t}}$ (c) $(e^t)^4$

Solution

(a) $e^xe^{3x} = e^{x+3x} = e^{4x}$

(b) $\frac{e^{4t}}{e^{3t}} = e^{4t-3t} = e^t$

(c) $(e^t)^4 = e^{4t}$

19.4 Simplify

(a) $\sqrt{e^{6t}}$ (b) $e^{3y}(1+e^y) - e^{4y}$ (c) $(2e^t)^2(3e^{-t})$

Solution

(a) $\sqrt{e^{6t}} = (e^{6t})^{\frac{1}{2}} = e^{6t/2} = e^{3t}$

$\sqrt{a} = a^{\frac{1}{2}}$; that is, a square root sign is equivalent to the power $\frac{1}{2}$.

(b) $e^{3y}(1+e^y) - e^{4y} = e^{3y} + e^{3y}e^y - e^{4y}$

$$= e^{3y} + e^{4y} - e^{4y}$$

$$= e^{3y}$$

(c) $(2e^t)^2(3e^{-t}) = (2e^t)(2e^t)(3e^{-t})$

$$= 2\cdot 2\cdot 3\cdot e^te^te^{-t}$$

$$= 12e^{t+t-t}$$

$$= 12e^t$$

19.5 (a) Verify that

$$(e^t+1)^2 = e^{2t} + 2e^t + 1$$

(b) Hence simplify $\sqrt{e^{2t} + 2e^t + 1}$.

Solution

(a) $(e^t+1)^2 = (e^t+1)(e^t+1)$

$$= e^te^t + e^t + e^t + 1$$

$$= e^{2t} + 2e^t + 1$$

(b) $\sqrt{e^{2t} + 2e^t + 1} = \sqrt{(e^t+1)^2} = e^t + 1$

- Exponential Function = has the form $y = e^x$ and following properties:

(a) The exponential function is never negative.

(b) When $x = 0$, the function value is 1.

(c) As x increases, then e^x increases. This is known as **exponential growth**.

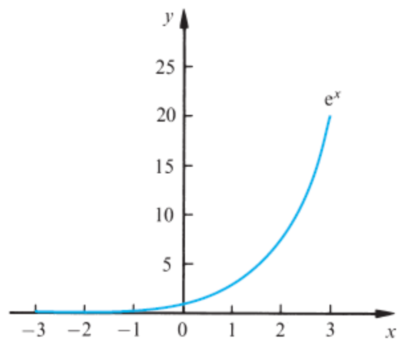
(d) By choosing x large enough, the value of e^x can be made larger than any given number. We say e^x increases without bound as x increases.

(e) As x becomes large and negative, e^x gets nearer and nearer to 0.

Table 19.1

x	-3	-2	-1	0	1	2	3
e^x	0.0498	0.1353	0.3679	1	2.7183	7.3891	20.086

Figure 19.1
A graph of $y = e^x$



WORKED EXAMPLE

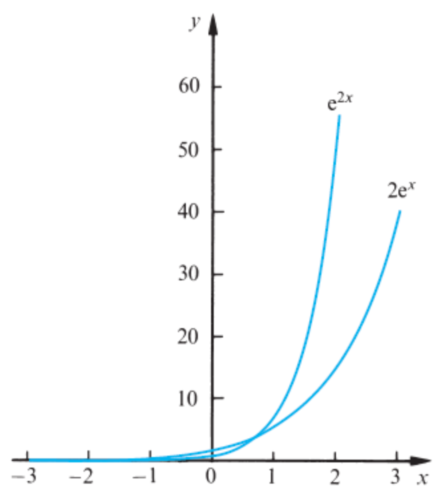
19.6 Sketch a graph of $y = 2e^x$ and $y = e^{2x}$ for $-3 \leqslant x \leqslant 3$.

Solution Table 19.2 gives values of $2e^x$ and e^{2x}, and the graphs are shown in Figure 19.2.

Table 19.2

x	-3	-2	-1	0	1	2	3
$2e^x$	0.0996	0.2707	0.7358	2	5.4366	14.7781	40.1711
e^{2x}	0.0025	0.0183	0.1353	1	7.3891	54.5982	403.429

Figure 19.2
Graphs of $y = e^{2x}$ and $y = 2e^x$



Note that when $x > 0$, the graph of $y = e^{2x}$ rises more rapidly than that of $y = 2e^x$.

We now turn our attention to an associated function: $y = e^{-x}$. Values are listed in Table 19.3 and the function is illustrated in Figure 19.3. From the table and the graph we note the following properties of $y = e^{-x}$:

(a) The function is never negative.

(b) When $x = 0$, the function has a value of 1.

(c) As x increases, then e^{-x} decreases, getting nearer to 0. This is known as **exponential decay**.

Table 19.3

x	-3	-2	-1	0	1	2	3
$-x$	3	2	1	0	-1	-2	-3
e^{-x}	20.086	7.3891	2.7183	1	0.3679	0.1353	0.0498

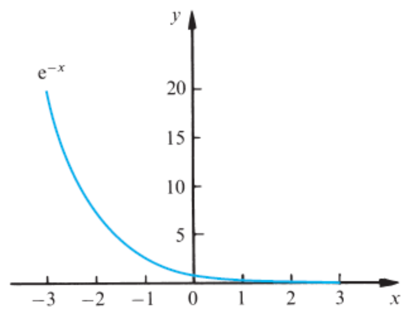


Figure 19.3
Graph of $y = e^{-x}$

WORKED EXAMPLE

19.7

(a) Plot $y = e^{x/2}$ and $y = 2e^{-x}$ for $-1 \leqslant x \leqslant 1$.

(b) Hence solve the equation

$$e^{x/2} - 2e^{-x} = 0$$

Solution

(a) A table of values is drawn up for $e^{x/2}$ and $2e^{-x}$ so that the graphs can be drawn. Table 19.4 lists the values and Figure 19.4 shows the graphs of the functions.

Table 19.4

x	-1	-0.75	-0.5	-0.25	0	0.25	0.5	0.75	1
$e^{x/2}$	0.607	0.687	0.779	0.882	1	1.133	1.284	1.455	1.649
$2e^{-x}$	5.437	4.234	3.297	2.568	2	1.558	1.213	0.945	0.736

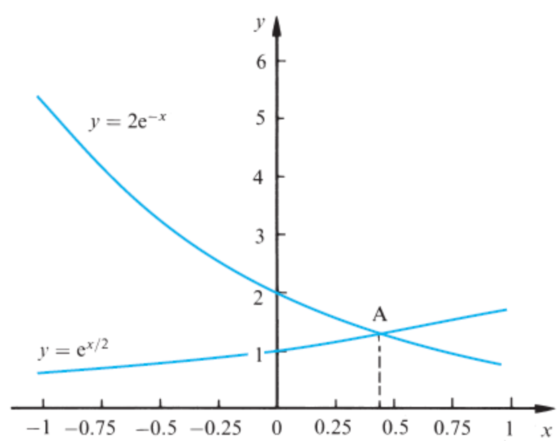


Figure 19.4
The graphs of $y = e^{x/2}$ and $y = 2e^{-x}$ intersect at A

(b) We note that the graphs intersect; the point of intersection is A. The equation

$$e^{x/2} - 2e^{-x} = 0$$

is equivalent to

$$e^{x/2} = 2e^{-x}$$

The graphs of $y = e^{x/2}$ and $y = 2e^{-x}$ intersect at A, where $x = 0.44$. Hence $x = 0.44$ is an approximate solution of $e^{x/2} - 2e^{-x} = 0$. The exact answer can be shown to be 0.46 (2 d.p.).

Given an equation such as $125 = 5^3$, we call 5 the base and 3 the power or index. We can use **logarithms** to write the equation in another form. The logarithm form is

$$\log_5 125 = 3$$

This is read as 'logarithm to the base 5 of 125 is 3'. In general if

$$y = a^x$$

then

$$\log_a y = x$$

Key point

$y = a^x$ and $\log_a y = x$ are equivalent.

'The logarithm to the base a of y is x' is equivalent to saying that 'y is a to the power x'. The word logarithm is usually shortened to just 'log'.

WORKED EXAMPLE

20.1 Write down the logarithmic form of the following:

(a) $16 = 4^2$ (b) $8 = 2^3$ (c) $25 = 5^2$

Solution

(a) $16 = 4^2$ may be written as

$$2 = \log_4 16$$

that is, 2 is the logarithm to the base 4 of 16.

(b) $8 = 2^3$ may be expressed as

$$3 = \log_2 8$$

which is read as '3 is the logarithm to the base 2 of 8'.

(c) $25 = 5^2$ may be expressed as

$$2 = \log_5 25$$

that is, 2 is the logarithm to the base 5 of 25.

Given an equation such as $16 = 4^2$ then 'taking logs' to base 4 will result in the logarithmic form $\log_4 16 = 2$.

WORKED EXAMPLE

20.2 Write the exponential form of the following:

(a) $\log_2 16 = 4$ (b) $\log_3 27 = 3$ (c) $\log_5 125 = 3$ (d) $\log_{10} 100 = 2$

Solution

(a) Here the base is 2 and so we may write $16 = 2^4$.
(b) The base is 3 and so $27 = 3^3$.
(c) The base is 5 and so $125 = 5^3$.
(d) The base is 10 and so $100 = 10^2$.

Recall that e is the constant 2.71828..., which occurs frequently in natural phenomena.

Although the base of a logarithm can be any positive number other than 1, the commonly used bases are 10 and e. Logarithms to base 10 are often denoted by 'log' or '$\log_{10}$'; logarithms to base e are denoted by 'ln' or '$\log_e$' and referred to as **natural logarithms**. Most scientific calculators possess 'log' and 'ln' buttons, which are used to evaluate logarithms to base 10 and base e.

20.4 Given $10^{0.6990} = 5$ evaluate the following:

(a) $10^{1.6990}$ (b) $\log 5$ (c) $\log 500$

Solution

(a) $10^{1.6990} = 10^1.10^{0.6990} = 10(5) = 50$

(b) We are given $10^{0.6990} = 5$ and by taking logs to the base 10 we obtain $0.6990 = \log 5$.

(c) From $10^{0.6990} = 5$ we can see that

$$100(5) = 100(10^{0.6990})$$

and so

$$500 = 10^2.10^{0.6990} = 10^{2.6990}$$

Taking logs to the base 10 gives

$$\log 500 = 2.6990$$

Key point

$$\log_a X = \frac{\log_{10} X}{\log_{10} a} \qquad \log_a X = \frac{\ln X}{\ln a}$$

WORKED EXAMPLE

20.5 Evaluate (a) $\log_6 19$, (b) $\log_7 29$.

Solution

(a) We use the formula

$$\log_a X = \frac{\log_{10} X}{\log_{10} a}$$

Comparing $\log_6 19$ with $\log_a X$ we see $X = 19$ and $a = 6$. So

$$\log_6 19 = \frac{\log_{10} 19}{\log_{10} 6} = \frac{1.2788}{0.7782} = 1.6433$$

We could have equally well used the formula

$$\log_a X = \frac{\ln X}{\ln a}$$

With this formula we obtain

$$\log_6 19 = \frac{\ln 19}{\ln 6} = \frac{2.9444}{1.7918} = 1.6433$$

(b) Comparing $\log_a X$ with $\log_7 29$ we see $X = 29$ and $a = 7$. Hence

$$\log_7 29 = \frac{\log_{10} 29}{\log_{10} 7} = \frac{1.4624}{0.8451} = 1.7304$$

Alternatively we use

$$\log_7 29 = \frac{\ln 29}{\ln 7} = \frac{3.3673}{1.9459} = 1.7304$$

By considering the formula

$$\log_a X = \frac{\log_{10} X}{\log_{10} a}$$

with $X = a$ we obtain

$$\log_a a = \frac{\log_{10} a}{\log_{10} a} = 1$$

Key point

$$\log_a a = 1$$

This same result could be derived by writing the logarithm form of $a = a^1$.

Key point

$$\log A + \log B = \log AB$$

This law holds true for any base. However, in any one calculation all bases must be the same.

WORKED EXAMPLES

20.7 Simplify to a single log term

(a) $\log 9 + \log x$
(b) $\log t + \log 4t$
(c) $\log 3x^2 + \log 2x$

Solution

(a) $\log 9 + \log x = \log 9x$
(b) $\log t + \log 4t = \log(t.4t) = \log 4t^2$
(c) $\log 3x^2 + \log 2x = \log(3x^2.2x) = \log 6x^3$

20.8 Simplify

(a) $\log 7 + \log 3 + \log 2$
(b) $\log 3x + \log x + \log 4x$

Solution

(a) We know $\log 7 + \log 3 = \log(7 \times 3) = \log 21$, and so

$$\begin{aligned}\log 7 + \log 3 + \log 2 &= \log 21 + \log 2\\ &= \log(21 \times 2) = \log 42\end{aligned}$$

(b) We have

$$\log 3x + \log x = \log(3x.x) = \log 3x^2$$

and so

$$\begin{aligned}\log 3x + \log x + \log 4x &= \log 3x^2 + \log 4x\\ &= \log(3x^2.4x) = \log 12x^3\end{aligned}$$

Key point

$$\log A - \log B = \log\left(\frac{A}{B}\right)$$

20.10 Use the second law of logarithms to simplify the following to a single log term:

(a) $\log 20 - \log 10$ (b) $\log 500 - \log 75$ (c) $\log 4x^3 - \log 2x$
(d) $\log 5y^3 - \log y$

Solution (a) Using the second law of logarithms we have

$$\log 20 - \log 10 = \log\left(\frac{20}{10}\right) = \log 2$$

(b) $$\log 500 - \log 75 = \log\left(\frac{500}{75}\right) = \log\left(\frac{20}{3}\right)$$

(c) $$\log 4x^3 - \log 2x = \log\left(\frac{4x^3}{2x}\right) = \log 2x^2$$

(d) $$\log 5y^3 - \log y = \log\left(\frac{5y^3}{y}\right) = \log 5y^2$$

20.11 Simplify

(a) $\log 20 + \log 3 - \log 6$
(b) $\log 18 - \log 24 + \log 2$

Solution (a) Using the first law of logarithms we see that

$$\log 20 + \log 3 = \log 60$$

and so

$$\log 20 + \log 3 - \log 6 = \log 60 - \log 6$$

Using the second law of logarithms we see that

$$\log 60 - \log 6 = \log\left(\frac{60}{6}\right) = \log 10$$

Hence

$$\log 20 + \log 3 - \log 6 = \log 10$$

(b) $$\begin{aligned}\log 18 - \log 24 + \log 2 &= \log\left(\frac{18}{24}\right) + \log 2 \\ &= \log\left(\frac{3}{4}\right) + \log 2 \\ &= \log\left(\frac{3}{4} \times 2\right) \\ &= \log 1.5\end{aligned}$$

20.12 Simplify

(a) $\log 2 + \log 3x - \log 2x$
(b) $\log 5y^2 + \log 4y - \log 10y^2$

Solution (a)
$$\log 2 + \log 3x - \log 2x = \log(2 \times 3x) - \log 2x$$
$$= \log 6x - \log 2x$$
$$= \log\left(\frac{6x}{2x}\right) = \log 3$$

(b)
$$\log 5y^2 + \log 4y - \log 10y^2 = \log(5y^2.4y) - \log 10y^2$$
$$= \log 20y^3 - \log 10y^2$$
$$= \log\left(\frac{20y^3}{10y^2}\right) = \log 2y$$

We consider a special case of the second law. Consider $\log A - \log A$. This is clearly 0. However, using the second law we may write

$$\log A - \log A = \log\left(\frac{A}{A}\right) = \log 1$$

Thus

Key point

$$\log 1 = 0$$

In any base, the logarithm of 1 equals 0.

Key point

$$n \log A = \log A^n$$

This law applies if n is integer, fractional, positive or negative.

WORKED EXAMPLES

20.14 Write the following as a single logarithmic expression:

(a) $3\log 2$ (b) $2\log 3$ (c) $4\log 3$

Solution

(a) $3\log 2 = \log 2^3 = \log 8$

(b) $2\log 3 = \log 3^2 = \log 9$

(c) $4\log 3 = \log 3^4 = \log 81$

20.15 Write as a single log term

(a) $\dfrac{1}{2}\log 16$ (b) $-\log 4$ (c) $-2\log 2$ (d) $-\dfrac{1}{2}\log 0.5$

Solution

(a) $\dfrac{1}{2}\log 16 = \log 16^{\frac{1}{2}} = \log\sqrt{16} = \log 4$

(b) $-\log 4 = -1.\log 4 = \log 4^{-1} = \log\left(\dfrac{1}{4}\right) = \log 0.25$

(c) $-2\log 2 = \log 2^{-2} = \log\left(\dfrac{1}{2^2}\right) = \log\left(\dfrac{1}{4}\right) = \log 0.25$

(d) $-\dfrac{1}{2}\log 0.5 = -\dfrac{1}{2}\log\left(\dfrac{1}{2}\right) = \log\left(\dfrac{1}{2}\right)^{-\frac{1}{2}} = \log 2^{\frac{1}{2}} = \log\sqrt{2}$

20.16 Simplify

(a) $3\log x - \log x^2$
(b) $3\log t^3 - 4\log t^2$
(c) $\log Y - 3\log 2Y + 2\log 4Y$

Solution

(a)
$$\begin{aligned}3\log x - \log x^2 &= \log x^3 - \log x^2\\ &= \log\left(\frac{x^3}{x^2}\right)\\ &= \log x\end{aligned}$$

(b)
$$\begin{aligned}3\log t^3 - 4\log t^2 &= \log(t^3)^3 - \log(t^2)^4\\ &= \log t^9 - \log t^8\\ &= \log\left(\frac{t^9}{t^8}\right)\\ &= \log t\end{aligned}$$

(c)
$$\begin{aligned}\log Y - 3\log 2Y + 2\log 4Y &= \log Y - \log(2Y)^3 + \log(4Y)^2\\ &= \log Y - \log 8Y^3 + \log 16Y^2\\ &= \log\left(\frac{Y.16Y^2}{8Y^3}\right)\\ &= \log 2\end{aligned}$$

20.17 Simplify

(a) $2\log 3x - \dfrac{1}{2}\log 16x^2$

(b) $\dfrac{3}{2}\log 4x^2 - \log\left(\dfrac{1}{x}\right)$

(c) $2\log\left(\dfrac{2}{x^2}\right) - 3\log\left(\dfrac{2}{x}\right)$

Solution

(a)
$$\begin{aligned}2\log 3x - \frac{1}{2}\log 16x^2 &= \log(3x)^2 - \log(16x^2)^{\frac{1}{2}}\\ &= \log 9x^2 - \log 4x\\ &= \log\left(\frac{9x^2}{4x}\right)\\ &= \log\left(\frac{9x}{4}\right)\end{aligned}$$

(b)
$$\begin{aligned}\frac{3}{2}\log 4x^2 - \log\left(\frac{1}{x}\right) &= \log(4x^2)^{\frac{3}{2}} - \log(x^{-1})\\ &= \log 8x^3 + \log x\\ &= \log 8x^4\end{aligned}$$

(c)
$$\begin{aligned}2\log\left(\frac{2}{x^2}\right) - 3\log\left(\frac{2}{x}\right) &= \log\left(\frac{2}{x^2}\right)^2 - \log\left(\frac{2}{x}\right)^3\\ &= \log\left(\frac{4}{x^4}\right) - \log\left(\frac{8}{x^3}\right)\\ &= \log\left(\frac{4/x^4}{8/x^3}\right)\\ &= \log\left(\frac{1}{2x}\right)\end{aligned}$$

20.18 Solve the following equations:

(a) $10^x = 59$ (b) $10^x = 0.37$ (c) $e^x = 100$ (d) $e^x = 0.5$

Solution

(a) $10^x = 59$
Taking logs to base 10 gives

$$x = \log 59 = 1.7709$$

(b) $10^x = 0.37$
Taking logs to base 10 gives

$$x = \log 0.37 = -0.4318$$

(c) $e^x = 100$
Taking logs to base e gives

$$x = \ln 100 = 4.6052$$

(d) $e^x = 0.5$
Taking logs to base e we have

$$x = \ln 0.5 = -0.6931$$

20.19 Solve the following equations:

(a) $\log x = 1.76$ (b) $\ln x = -0.5$ (c) $\log(3x) = 0.76$

(d) $\ln\left(\frac{x}{2}\right) = 2.6$ (e) $\log(2x - 4) = 1.1$ (f) $\ln(7 - 3x) = 1.75$

Solution

We note that if $\log_a X = n$, then $X = a^n$.

(a) $\log x = 1.76$ and so $x = 10^{1.76}$. Using the 'x^y' button of a scientific calculator we find

$$x = 10^{1.76} = 57.5440$$

(b) $\ln x = -0.5$

$$x = e^{-0.5} = 0.6065$$

(c) $\log 3x = 0.76$

$$3x = 10^{0.76}$$

$$x = \frac{10^{0.76}}{3} = 1.9181$$

(d) $\ln\left(\dfrac{x}{2}\right) = 2.6$

$$\frac{x}{2} = e^{2.6}$$

$$x = 2e^{2.6} = 26.9275$$

(e) $\log(2x - 4) = 1.1$

$$2x - 4 = 10^{1.1}$$

$$2x = 10^{1.1} + 4$$

$$x = \frac{10^{1.1} + 4}{2} = 8.2946$$

(f) $\ln(7 - 3x) = 1.75$

$$7 - 3x = e^{1.75}$$

$$3x = 7 - e^{1.75}$$

$$x = \frac{7 - e^{1.75}}{3} = 0.4151$$

20.20 Solve the following equations:

(a) $e^{3+x}.e^{x} = 1000$ (b) $\ln\left(\dfrac{x}{3} + 1\right) + \ln\left(\dfrac{1}{x}\right) = -1$

(c) $3e^{x-1} = 75$ (d) $\log(x + 2) + \log(x - 2) = 1.3$

Solution (a) Using the laws of logarithms we have

$$e^{3+x}.e^{x} = e^{3+2x}$$

Hence

$$e^{3+2x} = 1000$$

$$3 + 2x = \ln 1000$$

$$2x = \ln(1000) - 3$$

$$x = \frac{\ln(1000) - 3}{2} = 1.9539$$

(b) Using the laws of logarithms we may write

$$\ln\left(\frac{x}{3}+1\right)+\ln\left(\frac{1}{x}\right)=\ln\left(\left(\frac{x}{3}+1\right)\frac{1}{x}\right)=\ln\left(\frac{1}{3}+\frac{1}{x}\right)$$

So

$$\ln\left(\frac{1}{3}+\frac{1}{x}\right)=-1$$

$$\frac{1}{3}+\frac{1}{x}=e^{-1}$$

$$\frac{1}{x}=e^{-1}-\frac{1}{3}$$

$$\frac{1}{x}=0.0345$$

$$x=28.947$$

(c) $3e^{x-1}=75$

$$e^{x-1}=25$$

$$x-1=\ln 25$$

$$x=\ln 25+1=4.2189$$

(d) Using the first law we may write

$$\log(x+2)+\log(x-2)=\log(x+2)(x-2)=\log(x^2-4)$$

Hence

$$\log(x^2-4)=1.3$$

$$x^2-4=10^{1.3}$$

$$x^2=10^{1.3}+4$$

$$x=\sqrt{10^{1.3}+4}=4.8941$$

- Logarithmic Function= has the form $y=\log x$ or $y=\ln x$ and has the following properties:

(a) As x increases, the values of $\log x$ and $\ln x$ increase.
(b) $\log 1=\ln 1=0$.
(c) As x approaches 0 the values of $\log x$ and $\ln x$ increase negatively.
(d) When $x<1$, the values of $\log x$ and $\ln x$ are negative.
(e) $\log x$ and $\ln x$ are not defined when $x \leqslant 0$.

Table 20.1

x	0.01	0.1	0.5	1	2	5	10	100
$\log x$	-2	-1	-0.30	0	0.30	0.70	1	2
$\ln x$	-4.61	-2.30	-0.69	0	0.69	1.61	2.30	4.61

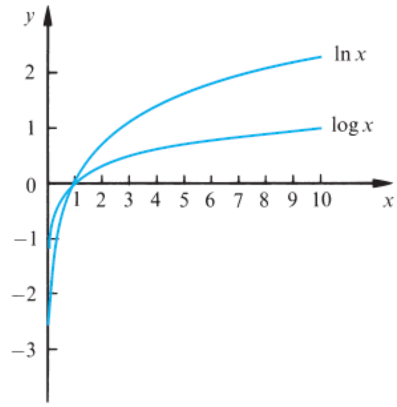


Figure 20.1
Graphs of $y = \log x$ and $y = \ln x$

Summary

In this week, we learned about what exponential expressions and logarithms are, how to calculate a base other than 10 or e, the laws of logarithms, how to solve equations involving logarithms and exponential expressions and finally what the exponential and logarithmic functions and their respective properties.