Week 15 Limits and differentiation Lecture note

Notebook:	Computational Mathematics
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Cornell Notes	**Topic:** **Limits and differentiation**	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: July 23, 2020

Essential Question:

What are limits and derivatives and how do they relate to the notion of continuity of a function?

Questions/Cues:

- What is the definition of a limit for a sequence?
- What is the definition of a limit and continuity for a function?
- What is the slope of a straight line?
- What is the definition of the derivative?

Notes

Definition of Limit for a sequence

	n=0	1	2	3	4	5	6	7	8	.......100.....
Examples $a_n=n/(n+1)$	0,	1/2,	2/3,	3/4,	4/5,	5/6,	6/7,	7/8,	8/9	.......100/101....

In general: $\lim_{n\to\infty} a_n=L$ if $\forall\ \varepsilon>0\ \exists\ N$: for $n>N$ $|a_n - L|<\varepsilon$

$\lim_{n\to\infty} a_n=L=1$ if $\varepsilon=1/5$ let $N=4$ for $n>N$ $\to 4/5<a_n<1 \to |a_n-1|<\varepsilon=\frac{1}{5}$

if $\varepsilon=1/9$ let $N=8$ for $n>N$ $\to 8/9<a_n<1 \to |a_n-1|<\varepsilon=\frac{1}{9}$

$\to$ if $\varepsilon=1/k$ let $N=k-1$ for $n>N$ $\to \frac{k-1}{k}<a_n<1 \to |a_n-1|<\varepsilon=\frac{1}{k}$

If limit exists finite, the sequence is <u>convergent</u>

$a_n=n/(n+1)$ converges to 1

Definition of Limit for a sequence

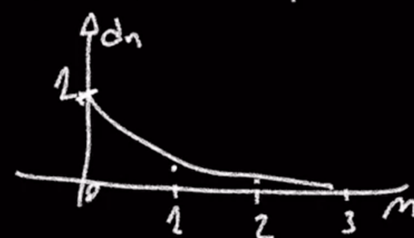
If limit doesn't exist the sequence is said to be <u>divergent</u>.
Examples:

	n=0	1	2	3	4	5	
1) $a_n=1\times 3^n$	1,	3,	9,	27,	81,	243....	diverges to ∞

	n=0	1	2	3	4	5	6	7....	
2) $a_n=\sin(\pi\, n/2)=$	0,	1,	0,	-1,	0,	1,	0,	-1…	does not converge

1) $\lim\limits_{n\to\infty} a_n = \dfrac{1}{(n+1)^3} = 0$

n	$(n+1)^3$	a_n
0	1	1
1	8	1/8
2	27	1/27
3	64	1/64
⋮	⋮	⋮



2) $\lim\limits_{n\to\infty} a_n = -2 + (-1)^n$

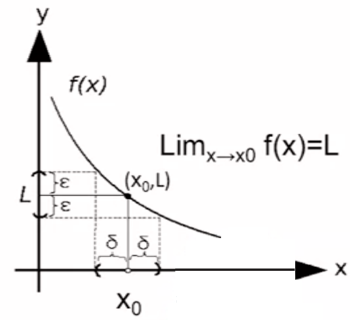
n	a_n
0	-1
1	-3
2	-1
3	-3
4	-1
5	-3
⋮	⋮

3) $$\lim_{n\to\infty} \frac{n^2+2n+5}{3n^2+2} = \frac{\cancel{n^2}\left(1+\underbrace{\frac{2}{n}}_{\to 0}+\underbrace{\frac{5}{n^2}}_{\to 0}\right)}{\cancel{n^2}\left(3+\underbrace{\frac{2}{n^2}}_{\to 0}\right)} = \boxed{\frac{1}{3}}$$

Definitions of limit and continuity for a function

$\mathrm{Lim}_{x \to x0} f(x)=L$

if $\forall\ \varepsilon>0\ \ \exists\ \delta>0$: for $|x-x_0|<\delta\ \rightarrow |f(x)-L|<\varepsilon$



If limit exists finite and coincides with the value of the function in x_0 , i.e. if

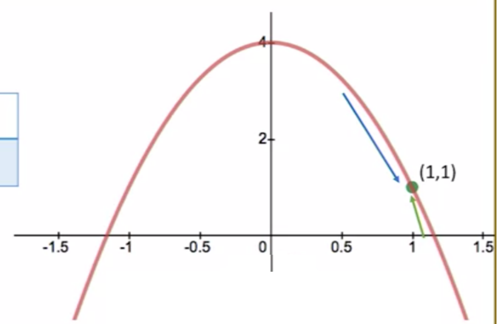
$f(x_0)=\lim_{x \to x0} f(x)=L$

→ the function is said to be <u>continuous</u> in x_0

Example

$f(x)=4-3x^2$ calculate $\lim_{x\to1} f(x)$

x=0.5	0.9	0.99	0.999	1	1.001	1.01	1.1
f(x)=3.25	1.57	1.06	1.006	?	0.993	0.94	0.37



Left and right limits:

left limit : $\lim_{x\to x_0^-} f(x)$ (blu arrow/numbers)

right limit: $\lim_{x\to x_0^+} f(x)$ (green arrow/numbers)

Limit exists if and only if $\lim_{x\to x_0^-} f(x)=\lim_{x\to x_0^+} f(x)=L$

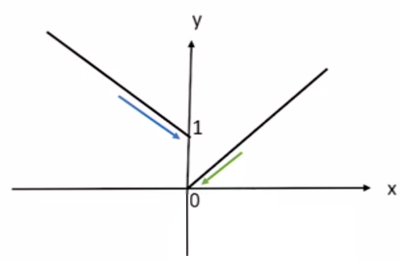
in our case $\lim_{x\to 1^-} f(x)=\lim_{x\to 1^+} f(x)=1=\lim_{x\to1} f(x)$

$f(1)=4-3=1$ → function is <u>continuous in x=1</u>

Discontinuous functions

Example $y=f(x)=\begin{cases} x & x\geq 0 \\ 1-x & x<0 \end{cases}$ $\lim_{x\to0} f(x)$?

x=-1	-0.5	-0.1	-0.01	0	0.01	0.1	0.5
f(x)=2	1.5	1.1	1.01	?	0.01	0.1	0.5

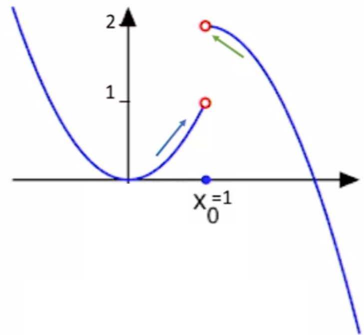


$\lim_{x\to 0^-} f(x)=1$ $\lim_{x\to 0^+} f(x)=0$

Left and right limits are different
→ limit does not exist, → <u>f not continuous in x=0</u>

Discontinuous functions

y=f(x) = $\begin{cases} x^2 & x < 1 \\ 1 + 2x - x^2 & x \geq 1 \end{cases}$ $\qquad \lim_{x\to 1} f(x)$?



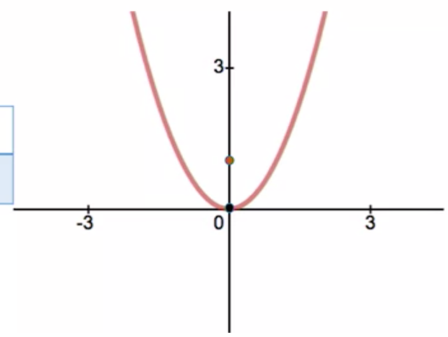
x=0	0.5	0.9	0.99	1	1.01	1.1	1.5
f(x)=0	0.25	0.81	0.98	?	1.99	1.9	1.75

$\lim_{x\to 1^-} f(x)=1 \qquad \lim_{x\to 1^+} f(x)=2$

Left and right limits are different
→ limit does not exist, → f not continuous in $x_0=1$

Discontinuous functions

y=f(x) = $\begin{cases} x^2 & x < 0 \\ 1 & x = 0 \\ x^2 & x > 0 \end{cases}$ $\qquad \lim_{x\to 0} f(x)$?

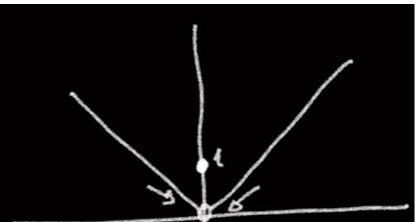


x=-1	-0.5	-0.1	-0.01	0	0.01	0.1	0.5
f(x)=1	-0.25	-0.01	0.0001	?	0.0001	0.01	0.25

$\lim_{x\to 0^-} f(x)=0= \lim_{x\to 0^+} f(x) \neq f(0)=1$

Left and right limits are equal
→ limit exists but different from f(0)
→ f not continuous in $x_0=0$

1) $f(x) = \begin{cases} |x| & x \neq 0 \\ \cancel{1}\ 0 & x = 0 \end{cases}$

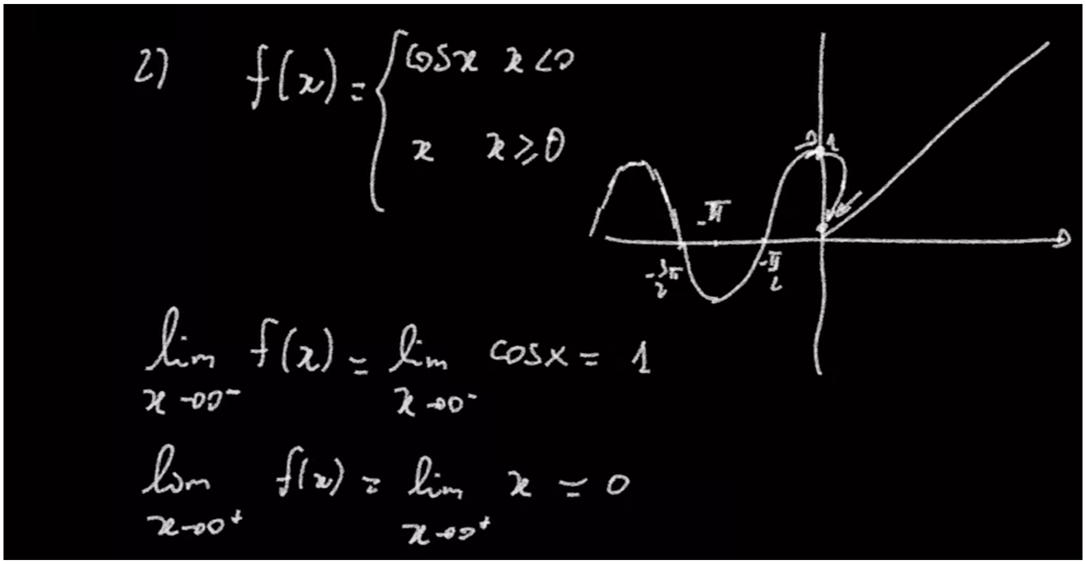


$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} -x = 0$

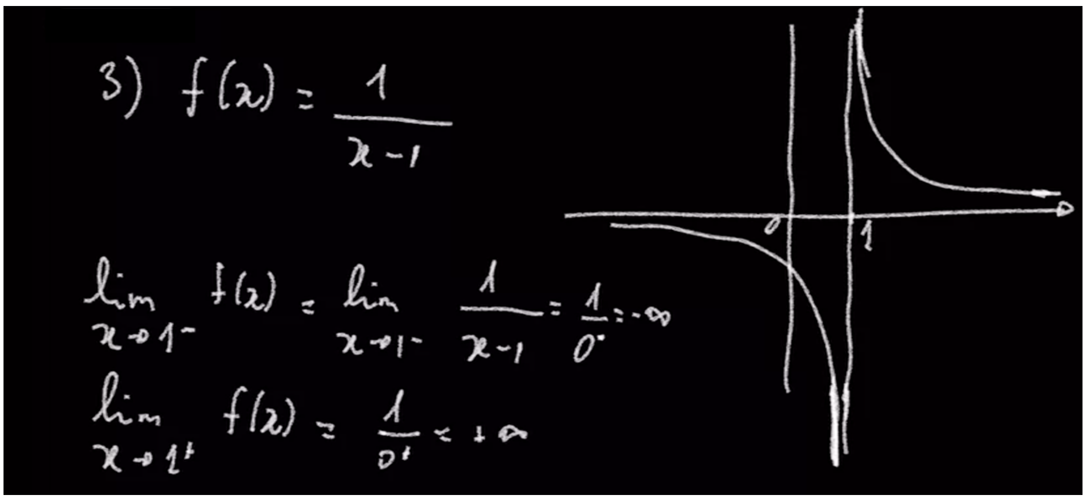
$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0$ $\Rightarrow \lim_{x\to 0} f(x) = ?$

$\boxed{f(0) = 1}$

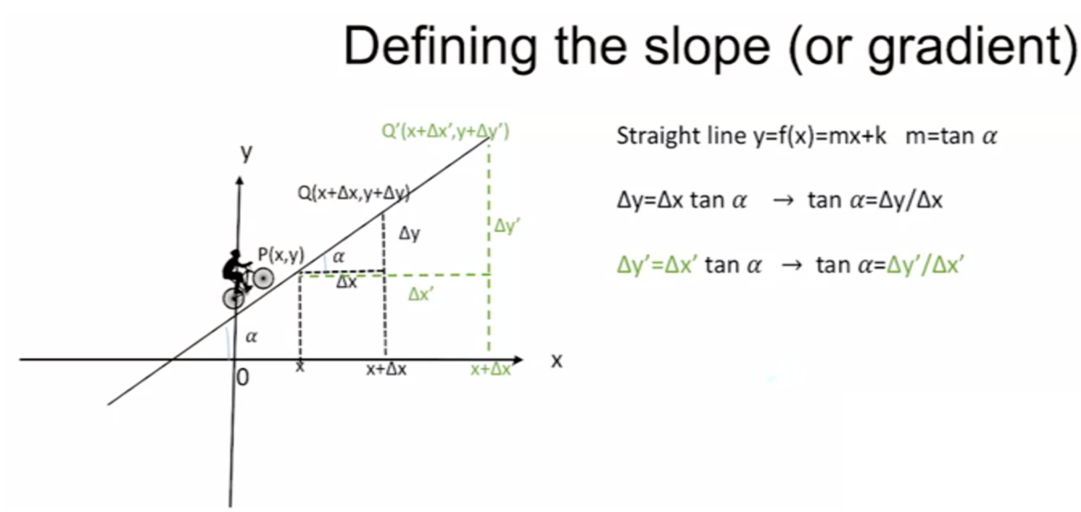
- This is called a removable discontinuity



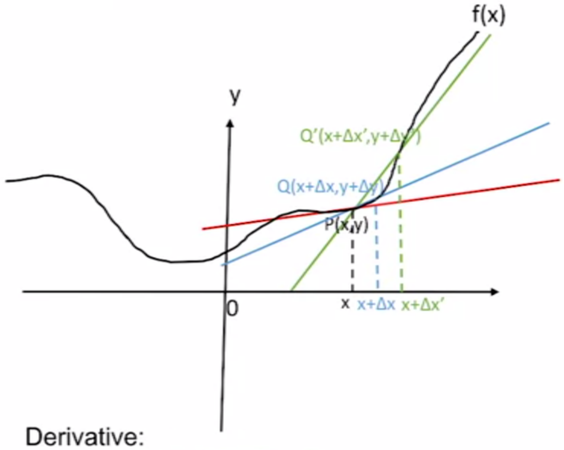
- This is a jump discontinuity



- This is called an essential singularity or infinite discontinuity



Definition of derivative



P(x,y) Q'(x+ Δx', y+ Δy')

y=f(x) y+ Δy'=f(x+ Δx') →Δy'=f(x+ Δx')-f(x)

Δy'=Δx' tan α' → tan α'=Δy'/Δx'

for Q(x+ Δx, y+ Δy) →Δy=f(x+ Δx)-f(x)

Δy=Δx tan α → tan α=Δy/Δx

Slope tangent in P (red line) →

$$\lim_{\Delta x\to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

Derivative:

$$f'(x) = \frac{df}{dx} = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}$$

$$y = f(x) = x$$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \frac{\cancel{x}+h-\cancel{x}}{h} = 1$$

$$f'(x) = \frac{d(x)}{dx} = 1$$

$$f(x) = \frac{1}{x} \qquad f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

$$f(x+h) = \frac{1}{x+h} \qquad \frac{f(x+h)-f(x)}{h} = \frac{\frac{1}{x+h}-\frac{1}{x}}{h}$$

$$\frac{\frac{x-(x+h)}{(x+h)x}}{h} = \frac{-\cancel{h}}{\cancel{h}x(x+h)} = -\frac{1}{x(x+h)}$$

$$\lim_{h\to 0} -\frac{1}{x(x+h)} = -\frac{1}{x^2} \Rightarrow f'(x) = -\frac{1}{x^2}$$

$$y = f(x) = x^n$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \frac{(x+h)^n - x^n}{h}$$

$$(x+h)^n = \left(x^n + n x^{n-1} \cdot h + C_n x^{n-2} \cdot h^2 + C x^{n-3} h^3 + \dots + h^n\right)$$

$$= \frac{\cancel{x^n} + n x^{n-1} \cdot h + \underbrace{C h^2}_{\to 0} - \cancel{x^n}}{h}$$

$$= \frac{n x^{n-1} \cancel{h}}{\cancel{h}} = n x^{n-1} \Rightarrow \frac{d x^\alpha}{dx} = \alpha x^{\alpha - 1}$$

$$f(x) = e^x$$

$$f'(x) = \lim_{h \to 0} \frac{e^{x+h} - e^x}{h} =$$

$$= \lim_{h \to 0} \frac{e^x (e^h - 1)}{h} = e^x \underbrace{\lim_{h \to 0} \frac{e^h - 1}{h}}_{= 1}$$

$$f'(x) = e^x$$

$$e^{\alpha x} \Rightarrow f'(x) = \alpha e^{\alpha x}$$

$$f(x) = \log x$$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\log(x+h) - \log(x)}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{\log\left(\frac{x+h}{x}\right)}{h} = \lim_{h \to 0} \frac{\log\left(1 + \frac{h}{x}\right)}{\frac{h}{x}x}$$

$$q = \frac{h}{x} \qquad \lim_{q \to 0} \frac{1}{x} \frac{\log(1+q)}{q} = \frac{1}{x} \underbrace{\lim_{q \to 0} \frac{\log(1+q)}{q}}_{=1}$$

$$\Rightarrow f'(x) = \frac{1}{x}$$

$$f(x) = \sin x$$

$$f'(x) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

$$\sin(x+h) = \sin x \cos h + \cos x \sin h$$

$$\lim_{h \to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h}$$

$$\lim_{h \to 0} \sin x \frac{(\cos h - 1)}{h} + \cos x \frac{\sin h}{h} =$$

$$= \cancel{\sin x \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{=0}} + \cos x \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{=1}$$

$$f'(x) = \cos x \qquad \to f(x) = \cos x \quad f'(x) = -\sin x$$

Summary

In this week, we learned about limit of a sequence, the limit/continuity of a function, discontinuous functions, the slope of a straight, the derivative and the derivative from first principles of some common functions.