Week 17 Algebra, Vectors, and Matrices Reading Note 1

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Author:	SUKHJIT MANN
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Cornell Notes	**Topic:** **Algebra, Vectors, and Matrices**	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: July 24, 2020

Essential Question:

What are vectors and matrices?

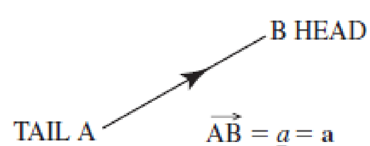
Questions/Cues:

- What is magnitude?
- What is a scalar?
- What is a vector?
- What is a unit vector?
- How do we multiply a vector by a scalar and what does it mean?
- How do we add two vectors together?
- How do we subtract two vectors?
- How do we use coordinates to describe vectors?
- What is the scalar product?

Notes

- Magnitude = size of a quantity, for example, 3 km or 10 amps
- Scalar = Quantities only described by their magnitude
  - Temperature, length, volume, and density are examples of scalars
- Vector = a quantity has both magnitude and direction, for example a force of 10 newtons acting vertically down
  - Often represented graphically by a straight line with an arrowhead to show the direction. Where the length of the line represents the magnitude and the direction of the line shows the direction of the vector

Figure 26.1
The line AB represents the vector **a**



- Vector from A to B can be written $\overrightarrow{AB}$, where B is called the head and A is called the tail of the vector
- The magnitude is written as $\left|\overrightarrow{AB}\right|$ or $|a|$
- Two vectors are equal if they have the same magnitude and same direction

- Unit vector = has a magnitude of 1, denoted by $\hat{a}$

It is possible to multiply a vector, say **a**, by a positive number (scalar), say k. The result is a new vector whose direction is the same as **a** and whose magnitude is k times that of the original vector **a**. So, for example, if **a** has magnitude 4, then the vector 3**a** has the same direction as **a** and magnitude of 12. The vector $\frac{1}{2}$**a** has magnitude 2. Figure 26.2 illustrates this.

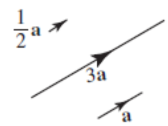


Figure 26.2
The vector 3**a** has the same direction as **a** and three times the magnitude. The vector $\frac{1}{2}$**a** has the same direction and half the magnitude

A vector may also be multiplied by a negative number. Again, if **a** has magnitude 4 then -3**a** has magnitude 12. The direction of -3**a** is opposite to that of **a**. Notice that the negative sign has the effect of reversing the direction. The vector $-$**a** has the same magnitude as that of **a** but the opposite direction. Figure 26.3 illustrates this.

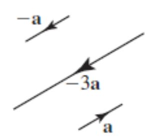


Figure 26.3
The vector -3**a** has the opposite direction to **a** and three times the magnitude. The vector $-$**a** has the same magnitude as **a** and the opposite direction

Recall that the magnitude of **a** is denoted by |**a**|. Note that this is a scalar quantity. Then a unit vector in the direction of **a** is given by

$$\text{unit vector in direction of } \mathbf{a} = \frac{1}{|\mathbf{a}|}\mathbf{a}$$

For example, given **a** has magnitude 4, then a unit vector in the direction of **a** is $\frac{1}{4}$**a**. If **b** has magnitude $\frac{1}{2}$ then a unit vector in the direction of **b** is 2**b**. Figure 26.4 illustrates some unit vectors.

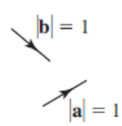


Figure 26.4
Unit vectors have a magnitude of 1

Adding vectors

If **a** and **b** are two vectors then we can form the vector **a** + **b**. We do this by moving **b**, still maintaining its length and direction, so that its tail coincides with the head of **a**. Figures 26.5(a) and (b) illustrate this. Then the vector **a** + **b** goes from the tail of **a** to the head of **b**. Figure 26.5 (c) illustrates this.

The method of adding vectors illustrated in Figure 26.5 is known as the **triangle law of addition**. The sum, **a** + **b**, is the same as **b** + **a**. The sum **a** + **b** is also known as the **resultant** of **a** and **b**.

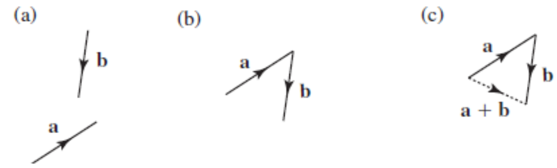


Figure 26.5
The tail of **b** is moved to join the head of **a**

Key point The resultant of **a** and **b** is the sum **a** + **b**.

WORKED EXAMPLE

26.1 Figure 26.6 illustrates a set of vectors. Use the triangle law of addition to find the resultant of (a) $\overrightarrow{AB}$ and $\overrightarrow{BC}$, (b) $\overrightarrow{BC}$ and $\overrightarrow{CD}$, (c) $\overrightarrow{AB}$ and $\overrightarrow{BD}$, (d) $\overrightarrow{AC}$ and $\overrightarrow{CD}$.

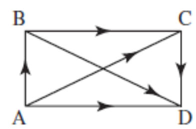


Figure 26.6
Figure for Worked Example 26.1

Solution

(a) $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$

(b) $\overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BD}$

(c) $\overrightarrow{AB} + \overrightarrow{BD} = \overrightarrow{AD}$

(d) $\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

The same principle used for adding two vectors can be applied to finding the sum of three or more vectors. Consider the vectors as shown in Figure 26.7.

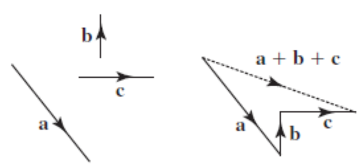


Figure 26.7
Addition of three vectors

By positioning the tail of **b** at the head of **a** and the tail of **c** at the head of **b**, the sum **a** + **b** + **c** can be found. It is a vector from the tail of **a** to the head of **c** as illustrated.

Subtraction of vectors

We wish to find the difference of two vectors, that is **a** − **b**. We do this by calculating **a** + (− **b**); that is, we add the vectors **a** and −**b**. Recall that −**b** is a vector in the opposite direction to **b** and with the same magnitude.

WORKED EXAMPLE

26.2 The vectors **a** and **b** are shown in Figure 26.8. On a diagram show (a) **a** − **b**, (b) **b** − **a**.

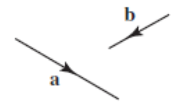


Figure 26.8 The vectors **a** and **b** for Worked Example 26.2

Solution

(a) We form the vector −**b**. This has the same magnitude (length) as **b** and the opposite direction. It is illustrated in Figure 26.9. This vector, −**b**, is now added to **a** by positioning the tail of −**b** at the head of **a** (see Figure 26.9). The sum **a** + (−**b**) is then determined using the triangle law of addition and is shown as a dotted line.

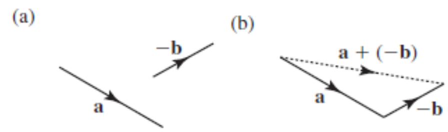


Figure 26.9 The vector **a** − **b** is found by adding **a** and (−**b**)

(b) To find **b** − **a** we calculate **b** + (−**a**). The vector −**a** has the same magnitude as **a** and the opposite direction. The tail of −**a** is positioned at the head of **b** and the resultant **b** + (−**a**) found using the triangle law of addition. Figure 26.10 illustrates this.

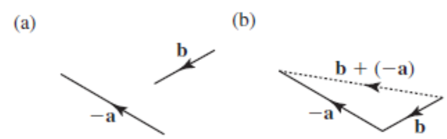


Figure 26.10 The vector **b** − **a** is found by adding **b** and (−**a**)

From Figures 26.8 and 26.9 we note that **b** − **a** = −(**a** − **b**).

We begin by introducing a **unit vector** in the x direction and a unit vector in the y direction. These are denoted by **i** and **j** respectively.

Key point

The unit vectors in the x and y directions are **i** and **j** respectively.

Consider Figure 26.12 which shows the x–y plane, the point P with coordinates (2, 4) and the vector $\overrightarrow{OP}$.

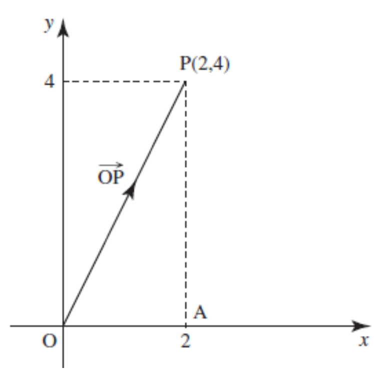


Figure 26.12 The vector $\overrightarrow{OP}$ is $2\mathbf{i} + 4\mathbf{j}$

From the triangle law of addition,

$$\overrightarrow{OP} = \overrightarrow{OA} + \overrightarrow{AP}$$

Now the magnitude of $\overrightarrow{OA}$ is 2 and the direction of $\overrightarrow{OA}$ is the positive x direction, so

$$\overrightarrow{OA} = 2\mathbf{i}$$

Similarly $\overrightarrow{AP}$ is in the y direction and has magnitude 4. Hence

$$\overrightarrow{AP} = 4\mathbf{j}$$

Then finally

$$\overrightarrow{OP} = 2\mathbf{i} + 4\mathbf{j}$$

The quantities 2 and 4 are the **Cartesian components** of the vector $\overrightarrow{OP}$. Note that $2\mathbf{i} + 4\mathbf{j}$ is any vector comprising 2 units in the x direction and 4 units in the y direction. Figure 26.13 illustrates three such vectors.

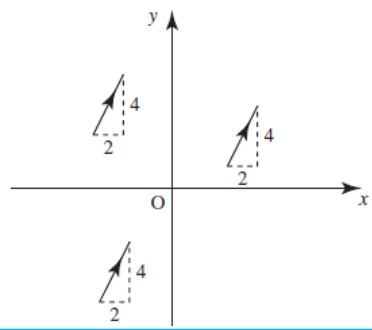


Figure 26.13 Each vector is $2\mathbf{i} + 4\mathbf{j}$ regardless of position

WORKED EXAMPLES

26.3 Sketch the vectors (a) $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$, (b) $\mathbf{b} = -2\mathbf{i} - 3\mathbf{j}$, (c) $\mathbf{c} = -\mathbf{i} + 3\mathbf{j}$, positioning the tail of each vector at the origin.

Solution Figure 26.14 illustrates the required vectors.

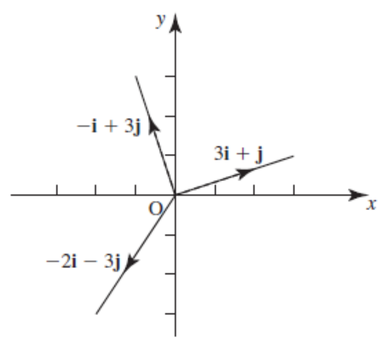


Figure 26.14 Vectors $\mathbf{a} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{b} = -2\mathbf{i} - 3\mathbf{j}$, $\mathbf{c} = -\mathbf{i} + 3\mathbf{j}$

26.4 Sketch the vector $\mathbf{v} = 4\mathbf{i} - \mathbf{j}$ with the point (1, 2) as the tail of the vector.

Solution Figure 26.15 illustrates the vector $\mathbf{v}$ with its tail at the point (1, 2).

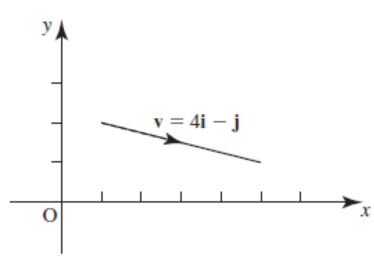


Figure 26.15 The vector $\mathbf{v} = 4\mathbf{i} - \mathbf{j}$ starts at the point (1, 2)

WORKED EXAMPLE

26.5 Given $\mathbf{a} = 7\mathbf{i} + 3\mathbf{j}$, $\mathbf{b} = \mathbf{i} - 2\mathbf{j}$ and $\mathbf{c} = -3\mathbf{i} + \mathbf{j}$, find (a) $\mathbf{a} + \mathbf{b}$, (b) $\mathbf{b} + \mathbf{c} + \mathbf{a}$, (c) $\mathbf{a} - \mathbf{b}$, (d) $\mathbf{c} - \mathbf{a}$.

Solution We separately add or subtract the x and y components:

(a) $\mathbf{a} + \mathbf{b} = 7\mathbf{i} + 3\mathbf{j} + \mathbf{i} - 2\mathbf{j}$

$= 8\mathbf{i} + 3\mathbf{j} - 2\mathbf{j}$ combining the $\mathbf{i}$ parts

$= 8\mathbf{i} + \mathbf{j}$ combining the $\mathbf{j}$ parts

(b) $\mathbf{b} + \mathbf{c} + \mathbf{a} = \mathbf{i} - 2\mathbf{j} + (-3\mathbf{i} + \mathbf{j}) + 7\mathbf{i} + 3\mathbf{j}$

$= 5\mathbf{i} + 2\mathbf{j}$

(c) $\mathbf{a} - \mathbf{b} = 7\mathbf{i} + 3\mathbf{j} - (\mathbf{i} - 2\mathbf{j})$

$= 7\mathbf{i} + 3\mathbf{j} - \mathbf{i} + 2\mathbf{j}$

$= 6\mathbf{i} + 5\mathbf{j}$

(d) $\mathbf{c} - \mathbf{a} = -3\mathbf{i} + \mathbf{j} - (7\mathbf{i} + 3\mathbf{j})$

$= -3\mathbf{i} + \mathbf{j} - 7\mathbf{i} - 3\mathbf{j}$

$= -10\mathbf{i} - 2\mathbf{j}$

WORKED EXAMPLE

26.6 Given $\mathbf{v} = 4\mathbf{i} + 3\mathbf{j}$ and $\mathbf{w} = -2\mathbf{i} + \mathbf{j}$, find (a) $2\mathbf{v}$, (b) $-3\mathbf{w}$, (c) $4\mathbf{v} + 5\mathbf{w}$.

Solution (a) $2\mathbf{v} = 2(4\mathbf{i} + 3\mathbf{j})$

$= 8\mathbf{i} + 6\mathbf{j}$

(b) $-3\mathbf{w} = -3(-2\mathbf{i} + \mathbf{j})$

$= 6\mathbf{i} - 3\mathbf{j}$

(c) $4\mathbf{v} + 5\mathbf{w} = 4(4\mathbf{i} + 3\mathbf{j}) + 5(-2\mathbf{i} + \mathbf{j})$

$= 16\mathbf{i} + 12\mathbf{j} - 10\mathbf{i} + 5\mathbf{j}$

$= 6\mathbf{i} + 17\mathbf{j}$

The magnitude of a vector was described as the ‘length’ of the vector. This idea can be made exact when using Cartesian coordinates. Figure 26.16 illustrates the vector $\mathbf{r}$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

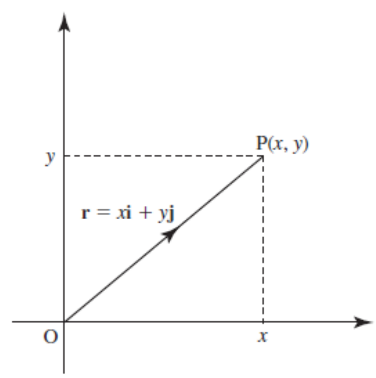


Figure 26.16 The vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$

The length (magnitude) of the vector is OP, that is $|\mathbf{r}| = \text{OP}$. Using Pythagoras’ theorem we have

$$\text{OP}^2 = x^2 + y^2$$

so

$$|\mathbf{r}| = \text{OP} = \sqrt{x^2 + y^2}$$

Key point If $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, then $|\mathbf{r}| = \sqrt{x^2 + y^2}$

WORKED EXAMPLE

26.7 Given the point A has coordinates (2, 7) and B has coordinates (4, 11) find

(a) the vector $\overrightarrow{AB}$

(b) the vector $\overrightarrow{BA}$

(c) the magnitude of $|\overrightarrow{AB}|$

Solution Figure 26.17 illustrates the situation.

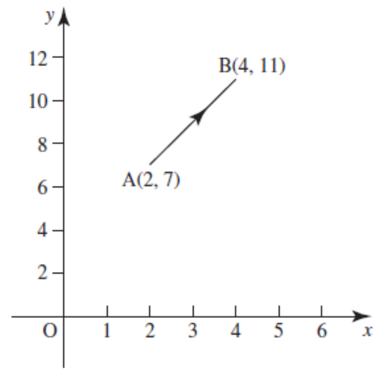


Figure 26.17 The vector $\overrightarrow{AB}$ for Worked Example 26.7

(a) Let $\overrightarrow{OA} = \mathbf{a} = 2\mathbf{i} + 7\mathbf{j}$ and $\overrightarrow{OB} = 4\mathbf{i} + 11\mathbf{j}$. From the triangle law of addition

$$\begin{aligned}
\overrightarrow{OA} + \overrightarrow{AB} &= \overrightarrow{OB} \\
\mathbf{a} + \overrightarrow{AB} &= \mathbf{b} \\
\overrightarrow{AB} &= \mathbf{b} - \mathbf{a} \\
&= 4\mathbf{i} + 11\mathbf{j} - (2\mathbf{i} + 7\mathbf{j}) \\
&= 2\mathbf{i} + 4\mathbf{j}
\end{aligned}$$

(b)
$$\begin{aligned}
\overrightarrow{BA} &= -\overrightarrow{AB} \\
&= -(2\mathbf{i} + 4\mathbf{j}) \\
&= -2\mathbf{i} - 4\mathbf{j}
\end{aligned}$$

(c) $\overrightarrow{AB} = 2\mathbf{i} + 4\mathbf{j}$

So

$$|\overrightarrow{AB}| = \sqrt{2^2 + 4^2} = \sqrt{20}$$

Key point

Given two vectors, **a** and **b**, their scalar product, denoted by $\mathbf{a} \cdot \mathbf{b}$, is given by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

where θ is the angle between **a** and **b**.

This product has applications in engineering. The scalar product is also known as the **dot product**.

WORKED EXAMPLE

26.8 Show that (a) $\mathbf{i}\cdot\mathbf{i} = 1$, (b) $\mathbf{i}\cdot\mathbf{j} = 0$.

Solution Recall that **i** and **j** are unit vectors in the x and y directions respectively.

(a) So

$$\begin{aligned}\mathbf{i}\cdot\mathbf{i} &= |\mathbf{i}||\mathbf{i}|\cos 0^\circ \quad \text{since the angle between } \mathbf{i} \text{ and itself is } 0^\circ\\ &= (1)(1)(1)\\ &= 1\end{aligned}$$

(b)
$$\begin{aligned}\mathbf{i}\cdot\mathbf{j} &= |\mathbf{i}||\mathbf{j}|\cos 90^\circ \quad \text{since the angle between } \mathbf{i} \text{ and } \mathbf{j} \text{ is } 90^\circ\\ &= (1)(1)(0)\\ &= 0\end{aligned}$$

Similarly it is easy to show that $\mathbf{j}\cdot\mathbf{j} = 1$, $\mathbf{j}\cdot\mathbf{i} = 0$.

Key point

$\mathbf{i}\cdot\mathbf{i} = 1 \qquad \mathbf{j}\cdot\mathbf{j} = 1 \qquad \mathbf{i}\cdot\mathbf{j} = 0 \qquad \mathbf{j}\cdot\mathbf{i} = 0$

We use the above Key point to find the scalar product of two vectors from their Cartesian form. This does not require the value of the angle between the two vectors.

WORKED EXAMPLE

26.9 Given $\mathbf{a} = 3\mathbf{i} + 2\mathbf{j}$ and $\mathbf{b} = 4\mathbf{i} - \mathbf{j}$, find $\mathbf{a}\cdot\mathbf{b}$.

Solution

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (3\mathbf{i} + 2\mathbf{j})\cdot(4\mathbf{i} - \mathbf{j})\\ &= 3\mathbf{i}\cdot(4\mathbf{i} - \mathbf{j}) + 2\mathbf{j}\cdot(4\mathbf{i} - \mathbf{j})\\ &= 12\mathbf{i}\cdot\mathbf{i} - 3\mathbf{i}\cdot\mathbf{j} + 8\mathbf{j}\cdot\mathbf{i} - 2\mathbf{j}\cdot\mathbf{j}\end{aligned}$$

Recalling $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = 1, \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{i} = 0$, we see that

$$\mathbf{a}\cdot\mathbf{b} = 12(1) - 3(0) + 8(0) - 2(1) = 10$$

We can use the technique of the above example to deduce the following general result:

Key point

If $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j}$, $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j}$ then

$$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2$$

That is, the **i** components are multiplied together, the **j** components are multiplied together and the sum of these products is the scalar product.

WORKED EXAMPLE

26.10 Given $\mathbf{a} = 4\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = 2\mathbf{i} + 3\mathbf{j}$

(a) Find the scalar product $\mathbf{a} \cdot \mathbf{b}$.

(b) Hence find the angle between $\mathbf{a}$ and $\mathbf{b}$.

Solution (a) We find the scalar product:

$$\begin{aligned}\mathbf{a} \cdot \mathbf{b} &= (4\mathbf{i} + \mathbf{j}) \cdot (2\mathbf{i} + 3\mathbf{j}) \\ &= 4(2) + 1(3) \\ &= 11\end{aligned}$$

(b) The scalar product is also given by

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$$

Now $|\mathbf{a}| = \sqrt{4^2 + 1^2} = \sqrt{17}$, $|\mathbf{b}| = \sqrt{2^2 + 3^2} = \sqrt{13}$. We have found the value of $\mathbf{a} \cdot \mathbf{b}$ in part (a). Hence using the values for $\mathbf{a} \cdot \mathbf{b}$, $|\mathbf{a}|$ and $|\mathbf{b}|$ we have

$$11 = \sqrt{17}\sqrt{13}\cos\theta$$

$$\cos\theta = \frac{11}{\sqrt{17}\sqrt{13}} = 0.7399$$

Using the inverse cosine function of a calculator we find that $\theta = 42.3°$.

Summary

In this week, we learned about what magnitude is, what a scalar and a vector quantity are, what a unit vector is, what scalar multiplication is, vector addition/subtraction, how to use coordinates to describe vector and finally what the scalar product of two vectors is.