Week 9 Graph Sketching & Kinematics Reading Note

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Author:	SUKHJIT MANN

Cornell Notes	**Topic:** Graph Sketching & Kinematics Reading	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: May 17, 2020

Essential Question:

What is a function and what are its applications to kinematics (simple motion)?

Questions/Cues:

- What are axes?
- What is an interval?
- What is the notation used to concisely describe intervals?
- What are the different types of intervals?
- What is domain and range?

Notes

- Axes = horizontal & vertical lines that divide the x-y plane in four quadrants. These axes intersect at a point O called the origin. The horizontal axis is used to represent the independent variable, commonly x and the vertical axis is used to represent the dependent variable, commonly y.
    - A scale is drawn on both axes in such a way that at the origin x = 0 and y = 0. Positive x values lie to the right and negative x values to the left. Positive y values are above the origin, negative y values are below the origin.
        - It's not essential that the scales on both axes are the same
    - Each point in the plane corresponds to a specific value of x and y. We usually call the x value the x coordinate and call the y value the y coordinate. To refer to a specific point we give both coordinates in brackets in form (x, y), always giving the x coordinate first

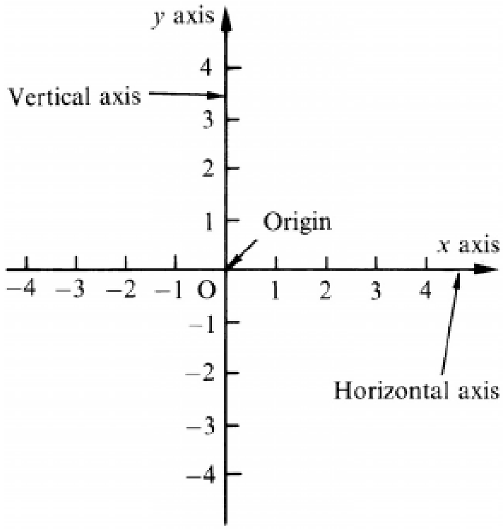


Figure 17.1
The x–y plane

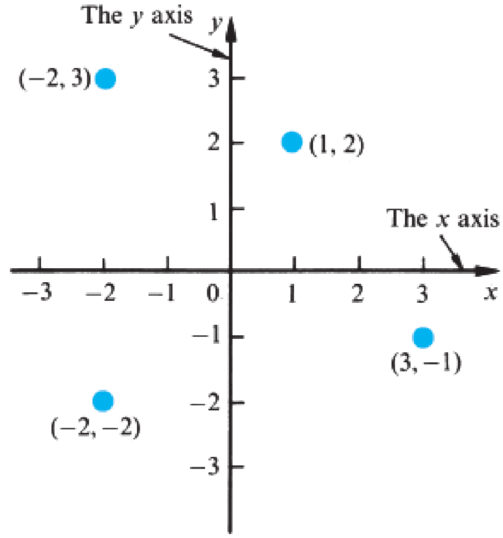


Figure 17.2
Several points in the x–y plane

WORKED EXAMPLE

17.1 Draw the x–y plane and on it mark the points whose coordinates are $(1, 2)$, $(3, -1)$, $(-2, 3)$, $(-2, -2)$.

Solution The plane is drawn in Figure 17.2, and the given points are indicated. Note that the first coordinate in each bracket is the x coordinate.

- Interval = part of an axis
- Greater than = $>$ means greater than. ex. 6 > 5. If $x_1 > x_2$ then x_1 is to the right of x_2 on the x axis.
  - The symbol $\geq$ means 'greater than or equal to', ie. $10 \geq 8$ and $8 \geq 8$ are both true.
- Less than = $<$ means less than. ie. 4 < 5. If $x_1 < x_2$ then x_1 is to the left of x_2 on the x axis
  - The symbol $\leq$ means 'less than or equal to'. ie. $6 \leq 9 \text{ and } 6 \leq 6$

** The symbols $>$, $\geq$, $<$, $\leq$ are known as inequalities

- Adding and subtracting the same quantity from both sides of an inequality leaves the inequality unchanged. For any quantity k if:

$$x > y$$

then

$$x + k > y + k \text{ and } x - k > y - k.$$

Similarly, if $x < y$ then

$$x + k < y + k \text{ and } x - k < y - k.$$

Multiplying and dividing an inequality needs greater care. If an inequality is multiplied or divided by a **positive** quantity then the inequality remains unchanged. If $x > y$ and k is positive, then

$$kx > ky \text{ and } \frac{x}{k} > \frac{y}{k}$$

Similarly, if $x < y$ and k is positive then

$$kx < ky \text{ and } \frac{x}{k} < \frac{y}{k}$$

If the inequality is multiplied or divided by a **negative** quantity then the inequality is reversed, that is, 'greater than' must be replaced by 'less than' and vice versa. If $x > y$ and k is negative, then

$$kx < ky \text{ and } \frac{x}{k} < \frac{y}{k}$$

Note that the 'greater than' has been replaced by a 'less than'. Similarly, if $x < y$ and k is negative then

$$kx > ky \text{ and } \frac{x}{k} > \frac{y}{k}$$

Note that the 'less than' has been replaced by 'greater than'.

Intervals

$\mathbb{R}$ is called the set of real numbers.

We often need to represent intervals on the number line. To help us do this we introduce the set $\mathbb{R}$. $\mathbb{R}$ is the symbol we use to denote all numbers from minus infinity to plus infinity. All numbers, including integers, fractions and decimals, belong to $\mathbb{R}$. To show that a number, x, is in this set, we write $x \in \mathbb{R}$ where $\in$ means 'belongs to'.

There are three different kinds of interval:

(a) *The closed interval* An interval that includes its end-points is called a **closed interval**. All the numbers from 1 to 3, including both 1 and 3, comprise a closed interval, and this is denoted using square brackets, [1, 3]. Any number in this closed interval must be greater than or equal to 1, and also less than or equal to 3. Thus if x is any number in the interval, then $x \geqslant 1$ and also $x \leqslant 3$. We write this compactly as $1 \leqslant x \leqslant 3$. Finally we need to show explicitly that x is any number on the x axis, and not just an integer for example. So we write $x \in \mathbb{R}$. Hence the interval $[1, 3]$ can be expressed as

$$\{x : x \in \mathbb{R},\ 1 \leqslant x \leqslant 3\}$$

This means the set contains all the numbers x with $x \geqslant 1$ and $x \leqslant 3$.

(b) *The open interval* Any interval that does not include its end-points is called an **open interval**. For example, all the numbers from 1 to 3, but excluding 1 and 3, comprise an open interval. Such an interval is denoted using round brackets, (1, 3). The interval may be written using set notation as

$$\{x : x \in \mathbb{R},\ 1 < x < 3\}$$

We say that x is **strictly greater** than 1, and **strictly less** than 3, so that the values of 1 and 3 are excluded from the interval.

(c) *The semi-open or semi-closed interval* An interval may be open at one end and closed at the other. Such an interval is called **semi-open** or, as some authors say, **semi-closed**. The interval $(1, 3]$ is a semi-open interval. The square bracket next to the 3 shows that 3 is included in the interval; the round bracket next to the 1 shows that 1 is not included in the interval. Using set notation we would write

$$\{x : x \in \mathbb{R},\ 1 < x \leqslant 3\}$$

When marking intervals on the x axis there is a notation to show whether or not the end-point is included. We use ● to show that an end-point is included (i.e. closed) whereas ○ is used to denote an end-point that is not included (i.e. open).

WORKED EXAMPLES

17.2 Describe the interval $[-3, 4]$ using set notation and illustrate it on the x axis.

Solution The interval $[-3, 4]$ is given in set notation by

$$\{x: x \in \mathbb{R},\ -3 \leqslant x \leqslant 4\}$$

Figure 17.3 illustrates the interval.

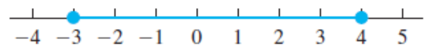


Figure 17.3
The interval $[-3, 4]$

17.3 Describe the interval $(1, 4)$ using set notation and illustrate it on the x axis.

Solution The interval $(1, 4)$ is expressed as

$$\{x: x \in \mathbb{R},\ 1 < x < 4\}$$

Figure 17.4 illustrates the interval on the x axis.

Figure 17.4
The interval $(1, 4)$

WORKED EXAMPLE

17.4 Plot a graph of $y = 2x - 1$ for $-3 \leqslant x \leqslant 3$.

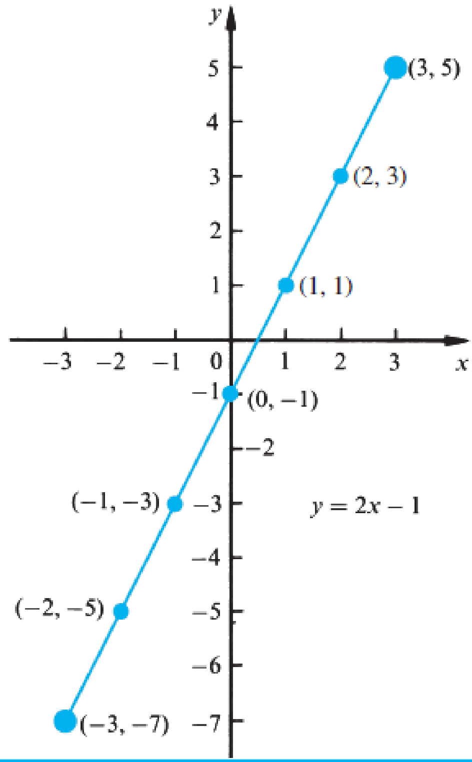
Solution We first calculate the value of y for several values of x. A table of x values and corresponding y values is drawn up as shown in Table 17.1.

Table 17.1
Values of x and y when $y = 2x - 1$

x	−3	−2	−1	0	1	2	3
y	−7	−5	−3	−1	1	3	5

The independent variable x varies from -3 to 3; the dependent variable y varies from -7 to 5. The x axis must accommodate values from -3 to 3; the y axis must accommodate values from -7 to 5. Each pair of values of x and y is represented by a unique point in the x–y plane, with coordinates (x, y). Each pair of values in the table is plotted as a point and then the points are joined to form the graph (Figure 17.5). By joining the points we see that, in this example, the graph is a straight line. We were asked to plot the graph for $-3 \leqslant x \leqslant 3$. Therefore, we have indicated each end-point of the graph by a ● to show that it is included. This follows the convention used for labelling closed intervals given in §17.2.

Figure 17.5
A graph of $y = 2x - 1$



WORKED EXAMPLE

17.5 (a) Plot a graph of $y = x^2 - 2x + 2$ for $-2 \leqslant x \leqslant 3$.

(b) Use your graph to determine which of the following points lie on the graph: (1, 2), (0, 2), and (1, 1).

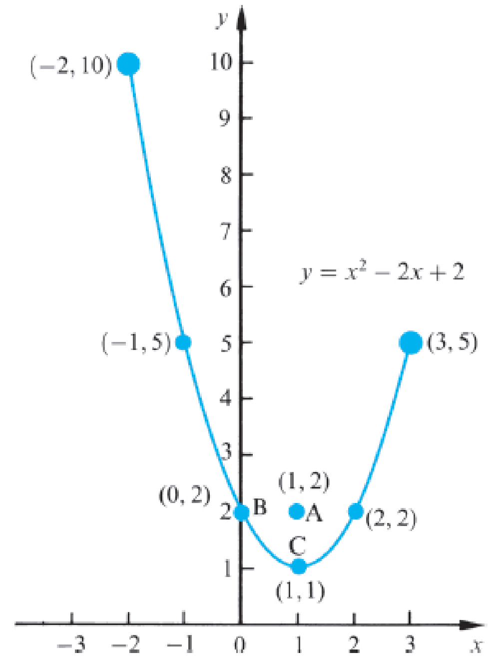
Solution (a) Table 17.2 gives values of x and the corresponding values of y. The independent variable is x and so the x axis is horizontal. The calculated points are plotted and joined. Figure 17.6 shows the graph. In this example the graph is not a straight line but rather a curve.

(b) The point (1, 2) has an x coordinate of 1 and a y coordinate of 2. The point is labelled by A on Figure 17.6. Points (0, 2) and (1, 1) are plotted and labelled by B and C. From the figure we can see that (0, 2) and (1, 1) lie on the graph.

Table 17.2
Values of x and y when $y = x^2 - 2x + 2$

x	−2	−1	0	1	2	3
y	10	5	2	1	2	5

Figure 17.6
A graph of
$y = x^2 - 2x + 2$



- Domain = the set of values we allow the independent variable to take is called the domain of the function. If the domain is not actually specified in any particular case, it's taken to be the largest set possible
- Range = The set of values we allow the dependent variable to take is called the range/output of the function

WORKED EXAMPLES

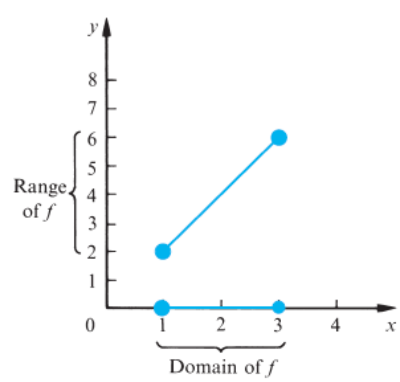
17.6 The function f is given by $y = f(x) = 2x$, for $1 \leqslant x \leqslant 3$.

(a) State the independent variable.
(b) State the dependent variable.
(c) State the domain of the function.
(d) Plot a graph of the function.
(e) State the range of the function.

Solution

(a) The independent variable is x.
(b) The dependent variable is y.
(c) Since we are given $1 \leqslant x \leqslant 3$ the domain of the function is the interval [1, 3], that is all values from 1 to 3 inclusive.
(d) The graph of $y = f(x) = 2x$ is shown in Figure 17.7.

Figure 17.7
Graph of $y = f(x) = 2x$



(e) The range is the set of values taken by the output, y. From the graph we see that as x varies from 1 to 3 then y varies from 2 to 6. Hence the range of the function is [2, 6].

17.7 Consider the function $y = f(t) = 4t$.

(a) State which is the independent variable and which is the dependent variable.

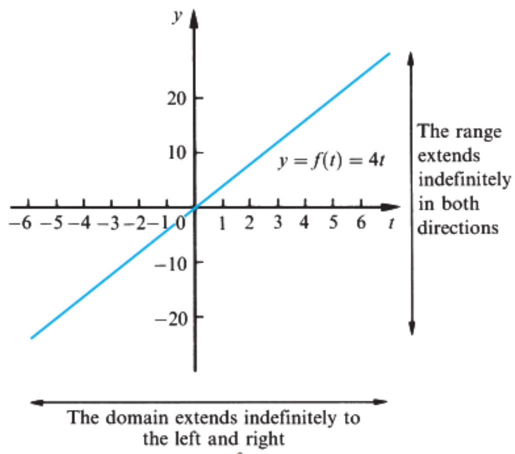
(b) Plot a graph of the function and give its domain and range.

Solution

(a) The independent variable is t and the dependent variable is y.

(b) No domain is specified so it is taken to be the largest set possible. The domain is $\mathbb{R}$, the set of all (real) numbers. However, it would be impractical to draw a graph whose domain was the whole extent of the t axis, and so a selected portion is shown in Figure 17.8. Similarly, only a restricted portion of the range can be drawn, although in this example the range is also $\mathbb{R}$.

Figure 17.8
A graph of $f(t) = 4t$



17.8 Consider the function $y = f(x) = x^2 - 1$: (a) state the independent variable; (b) state the dependent variable; (c) state the domain; (d) plot a graph of f and determine the range.

Solution

(a) The independent variable is x.

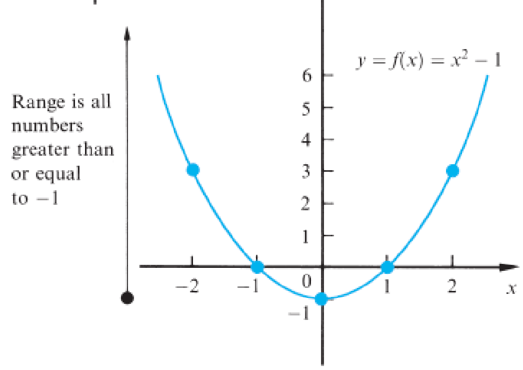
(b) The dependent variable is y.

(c) No domain is specified so it is taken to be the largest set possible. The domain is $\mathbb{R}$.

(d) A table of values and graph of $f(x) = x^2 - 1$ are shown in Figure 17.9. From the graph we see that the smallest value of y is -1, which occurs when $x = 0$. Whether x increases or decreases from 0, the value of y increases. The range is thus all values greater than or equal to -1.

Figure 17.9
A graph of $f(x) = x^2 - 1$

x	-2	-1	0	1	2
y	3	0	-1	0	3



We can write the range using the set notation introduced in §17.2 as

$$\text{range} = \{y : y \in \mathbb{R},\ y \geqslant -1\}$$

17.9 Given the function $y = 1/x$, state the domain and range.

Solution The domain is the largest set possible. All values of x are permissible except $x = 0$ since $\frac{1}{0}$ is not defined. It is never possible to divide by 0. Thus the domain comprises all real numbers except 0. A table of values and graph are shown in Figure 17.10. As x varies, y can take on any value except 0. In this example we see that the range is thus the same as the domain. We note that the graph is split at $x = 0$. For small, positive values of x, y is large and positive. The graph approaches the y axis for very small, positive values of x. For small, negative values of x, y is large and negative. Again, the graph approaches the y axis for very small, negative values of x. We say that the y axis is an **asymptote**. In general, if the graph of a function approaches a straight line we call that line an asymptote.

x	−3	−2	−1	0	1	2	3
y	−1/3	−1/2	−1	− (not defined)	1	1/2	1/3

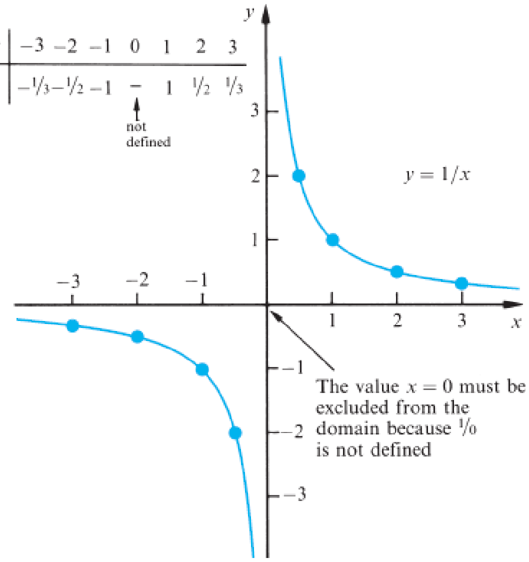


Figure 17.10
A graph of $y = 1/x$

WORKED EXAMPLE

17.10 Consider again the piecewise function in Worked Example 16.10:

$$y(x) = \begin{cases} x^2 + 1 & -1 \leqslant x \leqslant 2 \\ 3x & 2 < x \leqslant 6 \\ 2x + 1 & x > 6 \end{cases}$$

(a) Plot a graph of the function.

(b) State the domain of the function.

(c) State the range of the function.

Solution (a) Figure 17.11 shows a graph of $y(x)$.

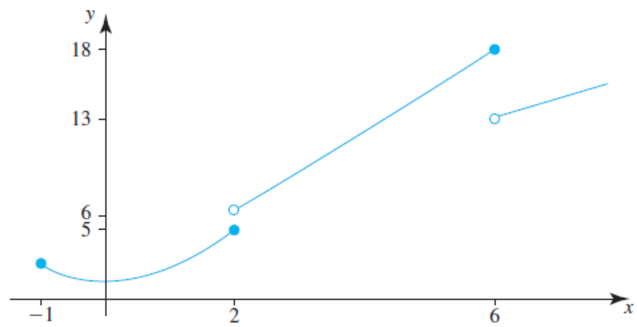


Figure 17.11 A graph of $y(x)$ for Worked Example 17.10

The different rules that make up the definition of y determine the three different pieces of the graph. Note the use of ● and ○ to denote the closed and open end-points.

(b) The domain is $[-1, \infty)$, that is

$$\text{domain} = \{x \in \mathbb{R}, x \geqslant -1\}$$

(c) The smallest value of y is $y = 1$, which occurs when $x = 0$. From the graph we see that the range is $[1, 5]$ and $(6, \infty)$, that is

$$\text{range} = \{y \in \mathbb{R}, 1 \leqslant y \leqslant 5 \text{ and } y > 6\}$$

WORKED EXAMPLES

17.11 Find graphically all solutions in the interval $[-3, 3]$ of the equation $x^2 + x - 3 = 0$.

Solution We draw a graph of $y(x) = x^2 + x - 3$. Table 17.3 gives x and y values and Figure 17.12 shows a graph of the function. We have plotted $y = x^2 + x - 3$ and wish to solve $0 = x^2 + x - 3$. Thus we read from the graph the coordinates of the points at which $y = 0$. Such points must be on the x axis. From the graph the points are $(1.3, 0)$ and $(-2.3, 0)$. So the solutions of

Table 17.3
Values of x and y when $y = x^2 + x - 3$

x	-3	-2	-1	0	1	2	3
y	3	-1	-3	-3	-1	3	9

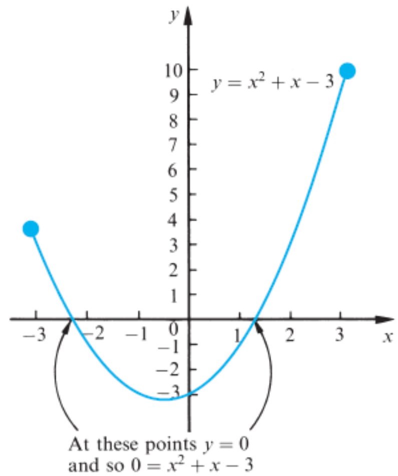


Figure 17.12
A graph of $y = x^2 + x - 3$

$x^2 + x - 3 = 0$ are $x = 1.3$ and $x = -2.3$. Note that these are approximate solutions, being dependent upon the accuracy of the graph and the x and y scales used. Increased accuracy can be achieved by drawing an enlargement of the graph around the values $x = 1.3$ and $x = -2.3$. So, for example, we could draw $y = x^2 + x - 3$ for $1.2 \leqslant x \leqslant 1.5$ and $y = x^2 + x - 3$ for $-2.4 \leqslant x \leqslant -2.2$.

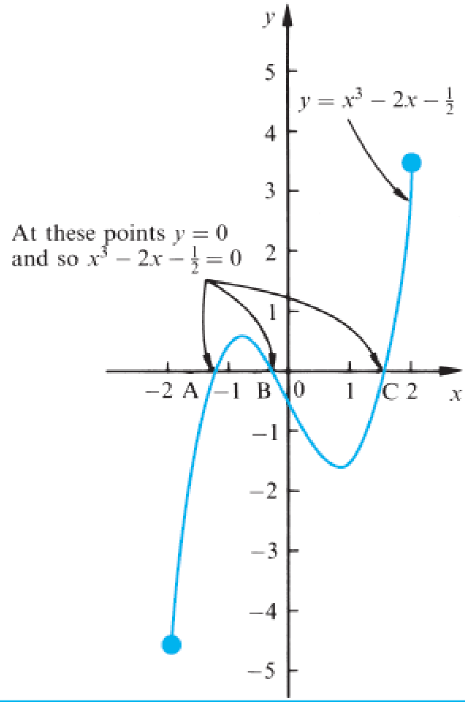
17.12 Find solutions in the interval $[-2, 2]$ of the equation $x^3 = 2x + \frac{1}{2}$ using a graphical method.

Solution The problem of solving $x^3 = 2x + \frac{1}{2}$ is identical to that of solving $x^3 - 2x - \frac{1}{2} = 0$. So we plot a graph of $y = x^3 - 2x - \frac{1}{2}$ for $-2 \leqslant x \leqslant 2$ and then locate the points where the curve cuts the x axis, that is, where the y coordinate is 0. Table 17.4 gives x and y values and Figure 17.13 shows a graph of the function. We now consider points on the graph where the y coordinate is zero. These points are marked A, B and C. Their x coordinates are -1.27, -0.26 and 1.53. Hence the solutions of $x^3 = 2x + \frac{1}{2}$ are approximately $x = -1.27$, $x = -0.26$ and $x = 1.53$.

Table 17.4
Values of x and y when $y = x^3 - 2x - \frac{1}{2}$

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
y	-4.5	-0.875	0.5	0.375	-0.5	-1.375	-1.5	-0.125	3.5

Figure 17.13
A graph of
$y = x^3 - 2x - \frac{1}{2}$



WORKED EXAMPLES

17.13 Solve graphically

$$3x - y = 9 \tag{17.1}$$

$$x + 2y = -4 \tag{17.2}$$

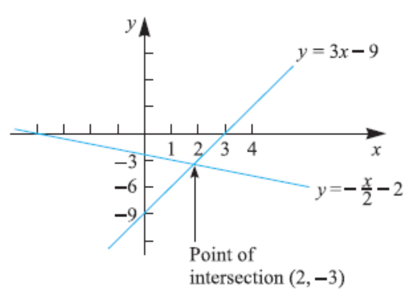
Solution Both equations are rearranged so that y is the subject. This gives

$$y = 3x - 9 \tag{17.3}$$

$$y = -\frac{x}{2} - 2 \tag{17.4}$$

Since Equations 17.3 and 17.4 are simple rearrangements of Equations 17.1 and 17.2 then they have the same solution. Equations 17.3 and 17.4 are drawn; Figure 17.14 illustrates this.

Figure 17.14
Points of intersection give the solution to simultaneous equations



We seek values of x and y that fit both equations. For a point to lie on both graphs, the point must be at the intersection of the graphs. In other words, solutions to simultaneous equations are given by the points of intersection. Reading from the graph, the point of intersection is at $x = 2$, $y = -3$. Hence $x = 2$, $y = -3$ is a solution of the given simultaneous equations.

17.14 Solve graphically

$$4x - y = 0$$
$$3x + y = 7$$

Solution We write the equations with y as subject:

$$y = 4x$$
$$y = -3x + 7$$

These are now plotted as shown in Figure 17.15.

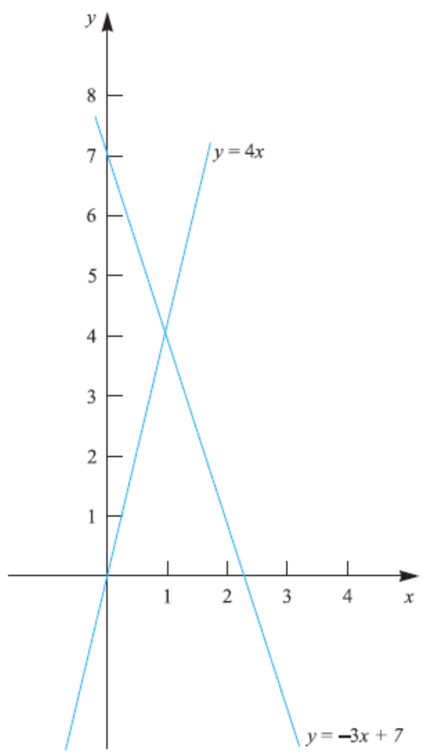


Figure 17.15 The graphs intersect at $x = 1$, $y = 4$

The point of intersection is $x = 1$, $y = 4$ and so this is the solution of the given simultaneous equations.

17.15 Solve graphically

$$x^2 - y = -1$$
$$2x - y = -3$$

Solution Writing the equations with y as subject gives

$$y = x^2 + 1$$

$$y = 2x + 3$$

These are plotted as shown in Figure 17.16.

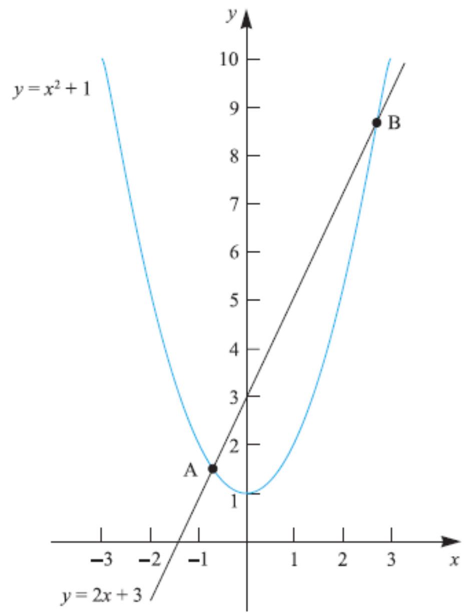


Figure 17.16
The graphs have two points of intersection and hence there are two solutions

There are two points of intersection, A and B. From the graph it is difficult to extract an accurate estimate. However, by using a graphics calculator or package, greater accuracy can be achieved. The coordinates of A are $x = -0.73$, $y = 1.54$; the coordinates of B are $x = 2.73$, $y = 8.46$. Hence there are two solutions to the given equations: $x = -0.73$, $y = 1.54$ and $x = 2.73$, $y = 8.46$.

17.16 (a) On the same axes draw graphs of $y = x^3$ and $y = 2x + \frac{1}{2}$ for $-2 \leqslant x \leqslant 2$.

(b) Note the x coordinates of the points where the two graphs intersect. By referring to Worked Example 17.12 what do you conclude? Can you explain your findings?

Solution (a) Table 17.5 gives x values and the corresponding values of x^3 and $2x + \frac{1}{2}$. Figure 17.17 shows a graph of $y = x^3$ together with a graph of $y = 2x + \frac{1}{2}$.

(b) The two graphs intersect at A, B and C. The x coordinates of these points are -1.27, -0.26 and 1.53. We note from Worked Example 17.12 that these values are the solutions of $x^3 = 2x + \frac{1}{2}$. We explain this as

Table 17.5
Values of x and y when $y = x^3$ and $y = 2x + \frac{1}{2}$

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
x^3	-8	-3.375	-1	-0.125	0	0.125	1	3.375	8
$2x + \frac{1}{2}$	-3.5	-2.5	-1.5	-0.5	0.5	1.5	2.5	3.5	4.5

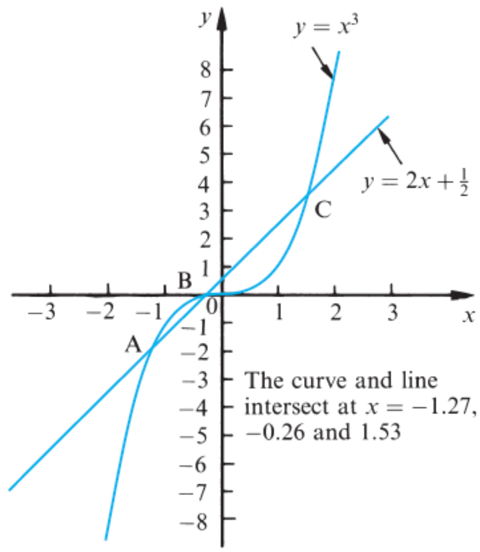


Figure 17.17
Graphs of $y = x^3$ and $y = 2x + \frac{1}{2}$

follows. Where $y = x^3$ and $y = 2x + \frac{1}{2}$ intersect, their y values are identical and so at these points

$$x^3 = 2x + \frac{1}{2}$$

By reading the x coordinates at these points of intersection we are finding those x values for which $x^3 = 2x + \frac{1}{2}$.

Summary

In this week, we learned about intervals, inequalities, solving a system of equations graphically and finding solutions to a function graphically within a given interval.