Week 18 Algebra, Vectors, and Matrices continued Reading Note 2

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Cornell Notes	**Topic:** **Algebra, Vectors, and Matrices continued**	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: July 29, 2020

Essential Question:

What are vectors and matrices?

Questions/Cues:

- What does the term linear equation mean?
- What does the term system of linear equations mean?
- What operations can be performed on a system of linear equations?
- What are the three possible solutions to a linear system with two unknowns
- What are the three possible solutions to a linear system comprising of three equations and three unknowns?
- What is an augmented matrix?
- What are elementary row operations?
- What is Gaussian Elimination with Back Substitution?
- What is Row-Echelon and Reduced Row-Echelon Form?
- What are the conditions for Reduced Row-Echelon Form?
- What is a Vector Space?
- What is a linear combination?

Notes

*What does the term **linear equation** mean?*

An equation is where two mathematical expressions are defined as being equal.

A linear equation is one where all the variables such as x, y, z have index (power) of 1 or 0 only, for example

$$x + 2y + z = 5$$

is a linear equation. The following are also linear equations:

$$x = 3;\ x + 2y = 5;\ 3x + y + z + w = -8$$

The following are *not* linear equations:

1. $x^2 - 1 = 0$
2. $x + y^4 + \sqrt{z} = 9$
3. $\sin(x) - y + z = 3$

Why not?

In equation (1) the index (power) of the variable x is 2, so this is actually a quadratic equation.
In equation (2) the index of y is 4 and z is 1/2. Remember, $\sqrt{z} = z^{1/2}$.
In equation (3) the variable x is an argument of the trigonometric function sine.

Note that if an equation contains an **argument** of trigonometric, exponential, logarithmic or hyperbolic functions then the equation is not linear.

A set of linear equations is called a **linear system**.

What does the term system of linear equations mean?

Generally a finite number of linear equations with a finite number of unknowns $x, y, z, w, \ldots$ is called a **system of linear equations** or just a **linear system**.

For example, the following is a linear system of three simultaneous equations with three unknowns x, y and z:

$$\begin{aligned} x + 2y - 3z &= 3 \\ 2x - y - z &= 11 \\ 3x + 2y + z &= -5 \end{aligned}$$

In general, a linear system of m equations in n unknowns $x_1, x_2, x_3, \ldots, x_n$ is written mathematically as

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \cdots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \cdots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \cdots & + & a_{mn}x_n & = & b_m \end{array} \qquad (*)$$

where the coefficients a_{ij} and b_j represent real numbers. The unknowns $x_1, x_2, \ldots, x_n$ are **placeholders** for real numbers.

Example 1.1

Solve the equations about the cost of ice creams and drinks by algebraic means

$$\begin{aligned} 2x + 2y &= 3 \qquad (1) \\ 2x + y &= 2.5 \qquad (2) \end{aligned}$$

Solution

How do we solve these linear simultaneous equations, (1) and (2)?

Let's think about the information contained in these equations. The x in the first line represents the cost of an ice cream, so must have the same value as the x in the second line. Similarly, the y in the first line that represents the cost of a drink must have the same value as the y in the second line.

It follows that we can combine the two equations to see if together they offer any useful information.

How?

In this case, we subtract equation (2) from equation (1):

$$\begin{array}{rl} 2x + 2y = 3 & \qquad (1) \\ -(2x + y = 2.5) & \qquad (2) \\ \hline 0 + y = 0.5 & \end{array}$$

Note that the unknown x is eliminated in the last line which leaves $y = 0.5$.

What else do we need to find?

The other unknown x.

How?

By substituting $y = 0.5$ into equation (1):

$$2x + 2\,(0.5) = 3 \quad \text{implies that} \quad 2x + 1 = 3 \quad \text{gives} \quad x = 1$$

Hence the cost of an ice cream is £1 because $x = 1$ and the cost of a drink is £0.50 because $y = 0.5$; this is the solution to the given simultaneous equations (1) and (2).

This is also the **point of intersection**, (1, 0.5), of the graphs in Fig. 1.1. The procedure outlined in Example 1.1 is called the method of **elimination**. The values $x = 1$ and $y = 0.5$ is the solution of equations (1) and (2). In general, values which satisfy the above linear system are called the **solution** or the **solution set** of the linear system. Here is another example.

Example 1.2

Solve

$$9x + 3y = 6 \qquad (1)$$
$$2x - 7y = 9 \qquad (2)$$

Solution

We need to find the values of x and y which satisfy both equations.

How?

Taking one equation from the other doesn't help us here, but we can multiply through either or both equations by a non-zero constant.

If we multiply equation (1) by 2 and (2) by 9 then in both cases the x coefficient becomes 18. Carrying out this operation we have

$$18x + 6y = 12 \quad \left[\text{multiplying equation (1) by } 2\right]$$
$$18x - 63y = 81 \quad \left[\text{multiplying equation (2) by } 9\right]$$

How do we eliminate x from these equations?

To eliminate the unknown x we subtract these equations:

$$\begin{array}{r} 18x + 6y = 12 \\ -(18x - 63y = 81) \\ \hline \end{array}$$

$$\begin{aligned} 0 + [6 - (-63)]\,y &= 12 - 81 \quad \left[\text{subtracting}\right] \\ 69y &= -69 \quad \text{which gives } y = -1 \end{aligned}$$

We have $y = -1$.

What else do we need to find?

The value of the placeholder x.

How?

By substituting $y = -1$ into the given equation $9x + 3y = 6$:

$$\begin{aligned} 9x + 3(-1) &= 6 \\ 9x - 3 &= 6 \\ 9x &= 9 \text{ which gives } x = 1 \end{aligned}$$

(continued...)

Hence our solution to the linear system of (1) and (2) is

$$x = 1 \text{ and } y = -1$$

We can check that this is the solution to the given system, (1) and (2), by substituting these values, $x = 1$ and $y = -1$, into the equations (1) and (2).

Note that we can carry out the following operations on a linear system of equations:

1. Interchange any pair of equations.
2. Multiply an equation by a non-zero constant.
3. Add or subtract one equation from another.

Example 1.3

Solve the linear system

$$\begin{aligned} x + 2y + 4z &= 7 \qquad (1)\\ 3x + 7y + 2z &= -11 \qquad (2)\\ 2x + 3y + 3z &= 1 \qquad (3) \end{aligned}$$

Solution

What are we trying to find?

The values of x, y and z that satisfy all three equations (1), (2) and (3).

How do we find the values of x, y and z?

By elimination. To eliminate one of these unknowns, we first need to make the coefficients of x (or y or z) equal.

Which one?

There are three choices but we select so that the arithmetic is made easier, in this case it is x. Multiply equation (1) by 2 and then subtract the bottom equation (3):

$$\begin{array}{rl} 2x + 4y + 8z = 14 & [\text{multiplying (1) by 2}] \\ -(2x + 3y + 3z = 1) & (3) \\ \hline 0 + y + 5z = 13 & [\text{subtracting}] \end{array}$$

Note that we have eliminated x and have the equation $y + 5z = 13$.
How can we determine the values of y and z from this equation?
We need another equation with only y and z.
How can we get this?
Multiply equation (1) by 3 and then subtract the second equation (2):

$$\begin{array}{rl} 3x + 6y + 12z = 21 & [\text{multiplying (1) by 3}] \\ \underline{-(3x + 7y + 2z = -11)} & (2) \\ 0 - y + 10z = 32 & [\text{subtracting}] \end{array}$$

Again there is no x and we have the equation $-y + 10z = 32$.
How can we find y and z?
We now solve the two simultaneous equations that we have obtained

$$\begin{aligned} y + 5z &= 13 \qquad (4) \\ -y + 10z &= 32 \qquad (5) \end{aligned}$$

We add equations (4) and (5), because $y + (-y) = 0$, which eliminates y.

$$0 + 15z = 45 \text{ gives } z = \frac{45}{15} = 3$$

Hence $z = 3$, but how do we find the other two unknowns x and y?
We first determine y by substituting $z = 3$ into equation (4) $y + 5z = 13$:

$$\begin{aligned} y + (5 \times 3) &= 13 \\ y + 15 &= 13 \qquad \text{which gives } y = -2 \end{aligned}$$

We have $y = -2$ and $z = 3$. We still need to find the value of last unknown x.
How do we find the value of x?
By substituting the values we have already found, $y = -2$ and $z = 3$, into the given equation $x + 2y + 4z = 7\ (1)$:

$$x + (2 \times -2) + (4 \times 3) = 7 \quad \text{gives} \quad x = -1$$

Hence the solution of the given three linear equations is $x = -1$, $y = -2$ and $z = 3$.
We can illustrate the given equations in a three-dimensional coordinate system as shown in Fig. 1.3.

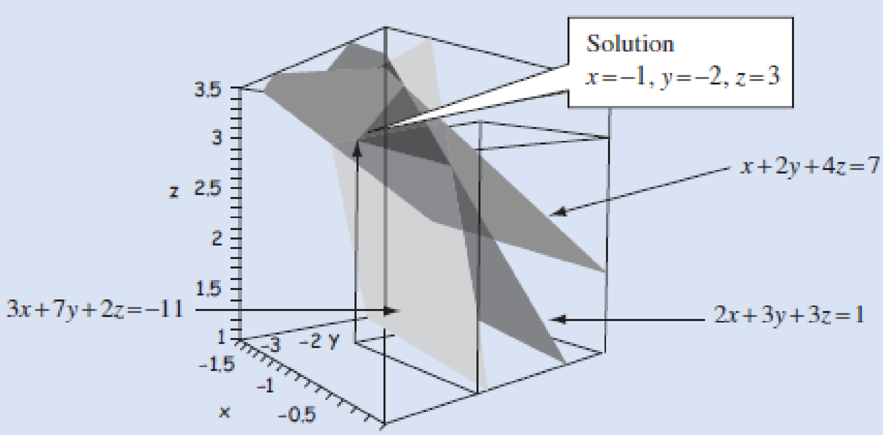


Figure 1.3

Example 1.4

Solve the linear system

$$\begin{aligned} 2x + 3y &= 6 \qquad (1) \\ 4x + 6y &= 9 \qquad (2) \end{aligned}$$

Solution

How do we solve these equations?

Multiply (1) by 2 and then subtract equation (2):

$$\begin{array}{rl} 4x + 6y = 12 & \text{[Multiplying (1) by 2]} \\ -(4x + 6y = 9) & (2) \\ \hline 0 + 0 = 3 & \end{array}$$

But how can we have $0 = 3$*?*

A plot of the graphs of the given equations is shown in Fig. 1.4.

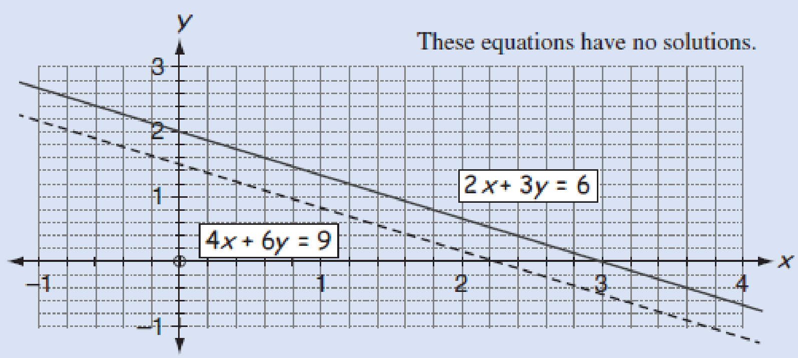


Figure 1.4

Can you see why there is no common solution to these equations?

The solution of the given equations would be the intersection of the lines shown in Fig. 1.4, but these lines are parallel so there is no intersection, therefore no solution.

By examining the given equations,

$$\begin{aligned} 2x + 3y &= 6 \qquad (1) \\ 4x + 6y &= 9 \qquad (2) \end{aligned}$$

can you see why there is no solution?

If you multiply the first equation (1) by 2 we have

$$4x + 6y = 12$$

This is a contradiction.

Why?

Because we have

$$\begin{aligned} 4x + 6y &= 12 \\ 4x + 6y &= 9 \qquad (2) \end{aligned}$$

that is, $4x + 6y$ equals both 9 and 12. This is clearly impossible. Hence the given linear system has no solution.

A system that has no solution is called **inconsistent.** If the linear system has at least one solution then we say the system is **consistent.**

Example 1.5

Graph the equations and determine the solution of this system:

$$2x + 3y = 6 \qquad (1)$$
$$4x + 6y = 12 \qquad (2)$$

Solution

The graph of the given equations is shown in Fig. 1.5.

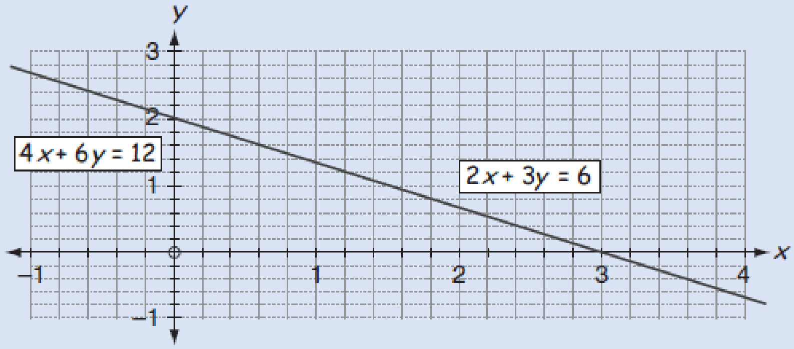


Figure 1.5

(continued...)

What do you notice?

Both the given equations produce exactly the same line; that is they coincide.

How many solutions do these equations have?

An infinite number of solutions, as you can see on the graph of Fig. 1.5. Any point on the line is a solution, and since there are an infinite number of points on the line we have an infinite number of solutions.

How can we write these solutions?

Let $x = a$ – where a is any real number – be a solution.

What then is y equal to?

Substituting $x = a$ into the given equation (1) yields

$$2a + 3y = 6 \qquad [2x + 3y = 6]$$
$$3y = 6 - 2a$$
$$y = \frac{6 - 2a}{3} = \frac{6}{3} - \frac{2}{3}a = 2 - \frac{2}{3}a$$

Hence if $x = a$ then $y = 2 - \frac{2}{3}a$.

The solution of the given linear system, (1) and (2), is $x = a$ and $y = 2 - 2a/3$ where a is any real number. You can check this by substituting various values of a. For example, if $a = 1$ then

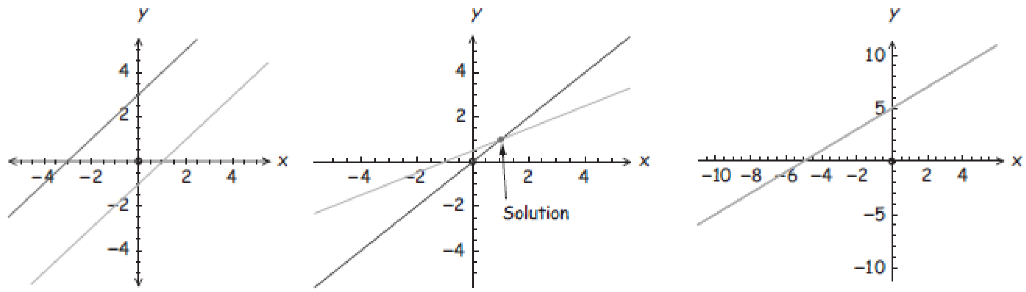
$$x = 1, \quad y = 2 - 2(1)/3 = 4/3$$

We can check that this answer is correct by substituting these values, $x = 1$ and $y = 4/3$, into equations (1) and (2):

$$\begin{aligned} 2\,(1) + 3\,(4/3) &= 2 + 4 = 6 \\ 4\,(1) + 6\,(4/3) &= 4 + 8 = 12 \end{aligned} \qquad \begin{bmatrix} 2x + 3y = 6 & (1) \\ 4x + 6y = 12 & (2) \end{bmatrix}$$

Hence our solution works. This solution $x = a$ and $y = 2 - 2a/3$ will satisfy the given equations for any real value of a.

The graphs in Fig. 1.6 represent the three possible solutions to a linear system with two unknowns.

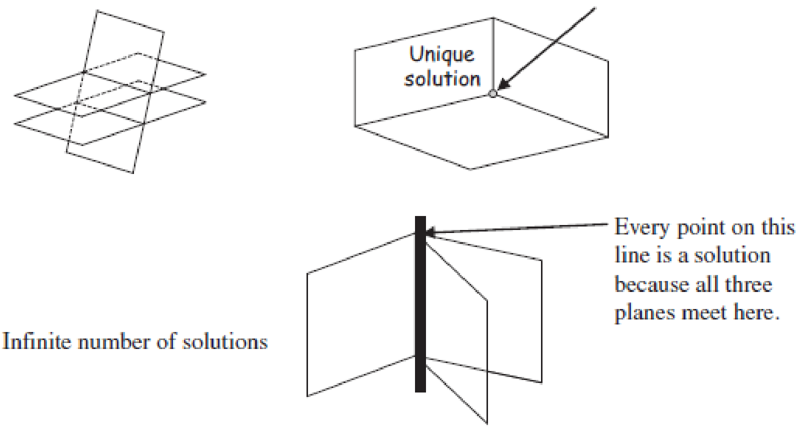


(a) No solution (b) Unique solution (c) Infinite number of solutions

Figure 1.6

The graphs in Fig. 1.7 illustrate solutions arising from three linear equations and three unknowns.

(a) No solution (b) Unique solution



(c) Infinite number of solutions

Figure 1.7

Fig. 1.7(a) shows three planes (equations) which have no point in common, hence no solution.

Fig. 1.7(b) shows the three planes (equations) with a unique point in common.

Fig. 1.7(c) shows three planes (equations) with a line in common. Every point on this line is a solution, which means we have an infinite number of solutions.

Suppose we have a general linear system of m equations with n unknowns labelled $x_1, x_2, x_3, \ldots$ and x_n given by:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

where the coefficients a_{ij}, b_i are real numbers and $x_1, x_2, x_3, \ldots$ and x_n are placeholders for real numbers that satisfy the equations. This general system can be stored in matrix form as

$$\left(\begin{array}{ccc|c} a_{11} & \ldots & a_{1n} & b_1 \\ \vdots & \ddots & \vdots & \\ a_{m1} & \cdots & a_{mn} & b_m \end{array}\right)$$

This is an **augmented matrix**, which is a matrix containing the coefficients of the unknowns x_1, x_2, $x_3, \ldots, x_n$ and the constant values on the right hand side of the equations. In everyday English language, augmented means 'to increase'. Augmenting a matrix means adding one or more columns to the original matrix. In this case, we have added the b's column to the matrix. These are divided by a vertical line as shown above.

Example 1.6

Consider Example 1.3 section 1.1, where we had to solve the following linear system:

$$\begin{aligned} x + 2y + 4z &= 7 \\ 3x + 7y + 2z &= -11 \\ 2x + 3y + 3z &= 1 \end{aligned}$$

Write the augmented matrix of this linear system.

Solution

An augmented matrix is simply a shorthand way of representing a linear system of equations. Rather than write x, y and z after each coefficient, recognize that the first column contains the coefficients of x, the second column the coefficients of y and so on. Placing the coefficients of x, y and z on the left hand side of the vertical line in the augmented matrix and the constant values 7, -11 and 1 on the right hand side we have

$$\left(\begin{array}{ccc|c} 1 & 2 & 4 & 7 \\ 3 & 7 & 2 & -11 \\ 2 & 3 & 3 & 1 \end{array}\right)$$

Because each row of the augmented matrix corresponds to one equation of the linear system, we can carry out analogous operations such as:

1. Multiply a row by a non-zero constant.
2. Add or subtract a multiple of one row to another.
3. Interchange rows.

We refer to these operations as **elementary row operations**.

Example 1.7

Solve the following linear system by using the Gaussian elimination procedure:

$$\begin{aligned} x - 3y + 5z &= -9 \\ 2x - y - 3z &= 19 \\ 3x + y + 4z &= -13 \end{aligned}$$

Solution

What is the augmented matrix in this case?

Let R_1, R_2 and R_3 represent rows 1, 2 and 3 respectively. We have

$$\begin{aligned} x - 3y + 5z &= -9 \\ 2x - y - 3z &= 19 \\ 3x + y + 4z &= -13 \end{aligned} \quad \text{and} \quad \begin{array}{l} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \end{array} \quad \begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 2 & -1 & -3 & 19 \\ 3 & 1 & 4 & -13 \end{array}\right)$$

Note that each row represents an equation.

How can we find the unknowns x, y and z?

The columns in the matrix represent the x, y and z coefficients respectively. If we can transform this augmented matrix into

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \\ \downarrow & \downarrow & \downarrow \end{array} \\ \text{Final row} & \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & A & B \end{array}\right) \end{array}$$ where A, B and * represents any real number

then we can find z.

How?

Look at the final row.

What does this represent?

$(0 \times x) + (0 \times y) + (A \times z) = B$

$$Az = B \text{ which gives } z = \frac{B}{A} \text{ provided } A \neq 0$$

Hence we have a value for $z = B/A$.

But how do we find the other two unknowns x and y?

Now we can use a method called **back substitution**. Examine the second row of the above matrix:

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \text{Second row} & \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & A & B \end{array}\right) \end{array}$$

By expanding the second row we get an equation in terms of y and z. From above we already know the value of $z = B/A$, so we can substitute $z = B/A$ and obtain y. Similarly from the first row we can find x by substituting the values of y and z.

We need to perform row operations on the augmented matrix to transform it from:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 2 & -1 & -3 & 19 \\ 3 & 1 & 4 & -13 \end{array}\right) \text{ to } \left(\begin{array}{ccc|c} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & A & B \end{array}\right)$$

We need to convert this augmented matrix to an equivalent matrix with zeros in the bottom left hand corner. That is 0 in place of 2, 3 and 1.

How do we get 0 in place of 2?

Remember, we can multiply an equation by a non-zero constant, and take one equation away from another. In terms of matrices, this means that we can multiply a row and take one row away from another because each row represents an equation.

To get 0 in place of 2 we multiple row 1, R_1, by 2 and subtract the result from row 2, R_2; that is, we carry out the row operation $R_2 - 2R_1$:

$$\begin{array}{c} R_1 \\ R_2^* = R_2 - 2R_1 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 2-2(1) & -1-(2\times(-3)) & -3-(2\times 5) & 19-(2\times(-9)) \\ 3 & 1 & 4 & -13 \end{array}\right)$$

We call the new middle row R_2^*. Completing the arithmetic, the middle row becomes

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 3 & 1 & 4 & -13 \end{array}\right)$$

Where else do we need a zero?

Need to get a 0 in place of 3 in the bottom row.

How?

We multiply the top row R_1 by 3 and subtract the result from the bottom row R_3; that is, we carry out the row operation, $R_3 - 3R_1$:

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3^* = R_3 - 3R_1 \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 3-3(1) & 1-(3\times(-3)) & 4-(3\times 5) & -13-(3\times(-9)) \end{array}\right)$$

We can call R_3^* the new bottom row of this matrix. Simplifying the arithmetic in the entries gives:

$$\begin{array}{c} R_1 \\ R_2^* \\ R_3^* \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 0 & 10 & -11 & 14 \end{array}\right)$$

Note that we now only need to convert the 10 into zero in the bottom row.

How do we get a zero in place of 10?

We can only make use of the bottom two rows, R_2^* and R_3^*.

Why?

Looking at the first column, it is clear that taking any multiple of R_1 away from R_3 will interfere with the zero that we have just worked to establish.

(continued...)

We execute $R_3^* - 2R_2^*$ because

$$10 - (2 \times 5) = 0 \quad \text{(gives a 0 in place of 10)}$$

Therefore we have

$$\begin{array}{r} R_1 \\ R_2^* \\ R_3^{**} = R_3^* - 2R_2^* \end{array} \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 0-(2\times 0) & 10-(2\times 5) & -11-[2\times(-13)] & 14-(2\times 37) \end{array} \right)$$

which simplifies to

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{r} R_1 \\ R_2^* \\ R_3^{**} \end{array} & \left(\begin{array}{ccc|c} 1 & -3 & 5 & -9 \\ 0 & 5 & -13 & 37 \\ 0 & 0 & 15 & -60 \end{array} \right) \end{array} \qquad (\dagger)$$

From the bottom row R_3^{**} we have

$$15z = -60 \text{ which gives } z = -\frac{60}{15} = -4$$

How do we find the other two unknowns x and y?
By expanding the middle row R_2^* of $(\dagger)$ we have:

$$5y - 13z = 37$$

We can find y by substituting $z = -4$ into this

$$5y - 13(-4) = 37 \qquad [\text{Substituting } z = -4]$$
$$5y + 52 = 37 \text{ which implies } 5y = -15 \text{ therefore } y = -\frac{15}{5} = -3$$

How can we find the last unknown x?
By expanding the first row R_1 of $(\dagger)$ we have:

$$x - 3y + 5z = -9$$

Substituting $y = -3$ and $z = -4$ into this:

$$x - (3 \times (-3)) + (5 \times (-4)) = -9$$
$$x + 9 - 20 = -9 \text{ which gives } x = 2$$

Hence our solution to the linear system is $x = 2, y = -3$ and $z = -4$.
Remember, each of the given equations can also be graphically represented by planes in a 3d coordinate system.
The solution is where all three planes (equations) meet. The equations are illustrated in Fig. 1.9.

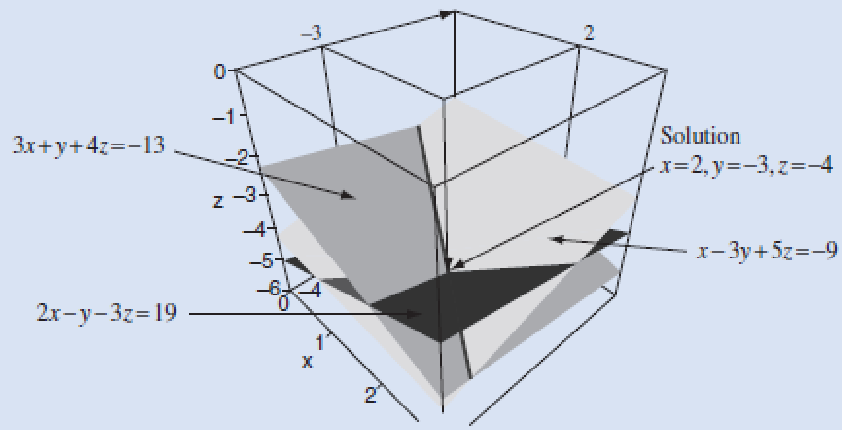


Figure 1.9

The above process is called **Gaussian elimination** with back substitution. The aim of Gaussian elimination is to produce a 'triangular' matrix with zeros in the bottom left corner of the matrix. This is achieved by the elementary row operations:

1. Multiply a row by a non-zero constant.
2. Add or subtract a multiple of one row from another.
3. Interchange rows.

We say two matrices are **row equivalent** if one matrix is derived from the other by using these three operations.

If augmented matrices of two linear systems are row equivalent then the two systems have the same solution set. You may like to check this for the above Example 1.7, to see that the solution $x = 2$, $y = -3$ and $z = -4$ satisfies the given equations.

In summary, the given linear system of equations is written in an augmented matrix, which is then transformed into a much simpler equivalent augmented matrix, which then allows us to use back substitution to find the solution of the linear system.

Example 1.8

Solve the linear system:

$$\begin{aligned} x + 3y + 2z &= 13 \\ 4x + 4y - 3z &= 3 \\ 5x + y + 2z &= 13 \end{aligned}$$

(continued...)

Solution

How can we find the unknowns x, y and z?

We use Gaussian elimination with back substitution.

The augmented matrix is:

$$\begin{array}{ll} \text{Row 1} & R_1 \\ \text{Row 2} & R_2 \\ \text{Row 3} & R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 4 & 4 & -3 & 3 \\ 5 & 1 & 2 & 13 \end{array}\right)$$

Need to convert the entries in this triangle to zeros.

Our aim is to convert this augmented matrix so that there are 0's in the bottom left hand corner, that is; the first 4 in the second row reduces to zero, and the 5 and 1 from the bottom row reduce to zero. Hence $4 \to 0, 5 \to 0$ and $1 \to 0$.

To get 0 in place of the first 4 in the middle row we multiply row 1, R_1, by 4 and take the result away from row 2, R_2, that is $R_2 - 4R_1$. To get 0 in place of 5 in the bottom row we multiply row 1, R_1, by 5 and take the result away from row 3, R_3, that is $R_3 - 5R_1$. Combining the two row operations, $R_2 - 4R_1$ and $R_3 - 5R_1$, we have

$$\begin{array}{r} R_1 \\ R_2^{\dagger} = R_2 - 4R_1 \\ R_3^{\dagger} = R_3 - 5R_1 \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 4-(4\times 1) & 4-(4\times 3) & -3-(4\times 2) & 3-(4\times 13) \\ 5-(5\times 1) & 1-(5\times 3) & 2-(5\times 2) & 13-(5\times 13) \end{array}\right)$$

We call the new row 2 and 3 – $R_2^{\dagger}$ and $R_3^{\dagger}$ respectively. This simplifies to:

$$\begin{array}{c} R_1 \\ R_2^{\dagger} \\ R_3^{\dagger} \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & -14 & -8 & -52 \end{array}\right)$$

We have nearly obtained the required matrix with zeros in the bottom left hand corner. We need a 0 in place of -14 in the new bottom row, $R_3^{\dagger}$. We can only use the second and third rows, $R_2^{\dagger}$ and $R_3^{\dagger}$.

Why?

Because if we use first row, R_1, we will get a non-zero number in place of the zero already established in $R_3^{\dagger}$.

How can we obtain a 0 in place of -14?

$$R_3^{\dagger} - \frac{14}{8}R_2^{\dagger} \text{ because } -14 - \left[\frac{14}{8} \times (-8)\right] = 0 \text{ (gives 0 in place of } -14\text{)}$$

Therefore

$$\begin{array}{r} R_1 \\ R_2^{\dagger} \\ R_3^{\dagger\dagger} = R_3^{\dagger} - \frac{14}{8}R_2^{\dagger} \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & -14-\left[\frac{14}{8}\times(-8)\right] & -8-\left[\frac{14}{8}\times(-11)\right] & -52-\left(\frac{14}{8}\times(-49)\right) \end{array}\right)$$

which simplifies to

$$\begin{array}{cc} & \begin{array}{ccc} x & y & z \end{array} \\ \begin{array}{c} R_1 \\ R_2^{\dagger} \\ R_3^{\dagger\dagger} \end{array} & \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & 0 & 45/4 & 135/4 \end{array}\right) \end{array} \qquad (*)$$

This time we have called the bottom row $R_3^{\dagger\dagger}$. From this row, $R_3^{\dagger\dagger}$, we have

$$\frac{45}{4}z = \frac{135}{4} \text{ which gives } z = \frac{135}{45} = 3$$

How can we find the unknown, y?
By expanding the second row $R_2^{\dagger}$ in (*) we have

$$-8y - 11z = -49$$

We know from above that $z = 3$, therefore, substituting $z = 3$ gives

$$\begin{aligned} -8y - 11\,(3) &= -49 \\ -8y - 33 &= -49 \\ -8y &= -49 + 33 = -16 \text{ which yields } y = (-16)/(-8) = 2 \end{aligned}$$

So far we have $y = 2$ and $z = 3$.
How can we find the last unknown, x?
By expanding the first row, R_1, of (*) we have

$$x + 3y + 2z = 13$$

Substituting our values already found, $y = 2$ and $z = 3$, we have

$$\begin{aligned} x + (3 \times 2) + (2 \times 3) &= 13 \\ x + 6 + 6 &= 13 \text{ which gives } x = 1 \end{aligned}$$

Hence $x = 1$, $y = 2$ and $z = 3$ is our solution. We can illustrate these equations as shown in Fig. 1.10.

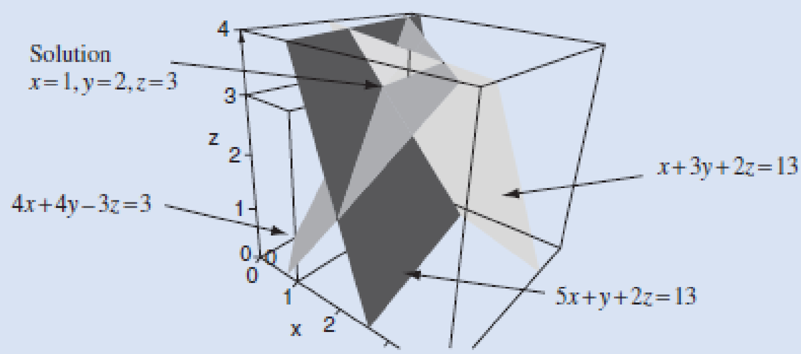


Figure 1.10

We can check that this solution is correct by substituting these $x = 1$, $y = 2$ and $z = 3$ into the given equations.

The Gaussian elimination process can be extended in the above example so that the first non-zero number in the bottom row of (*) is 1, that is

$$\begin{array}{c} R_1 \\ R_2^{\dagger} \\ R_3^{\dagger\dagger} \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & 0 & 45/4 & 135/4 \end{array}\right)$$

Convert this into 1

How do we convert 45/4 into 1?

Multiply the bottom row $R_3^{\dagger\dagger}$ by $\frac{4}{45}$.

$$\begin{array}{c} R_1 \\ R_2^{\dagger} \\ R_3' = \dfrac{4R_3^{\dagger\dagger}}{45} \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & 0 & \dfrac{4}{45}\left(\dfrac{45}{4}\right) & \dfrac{4}{45}\left(\dfrac{135}{4}\right) \end{array}\right)$$

which simplifies to

$$\begin{array}{c} R_1 \\ R_2^{\dagger} \\ R_3' \end{array} \left(\begin{array}{ccc|c} 1 & 3 & 2 & 13 \\ 0 & -8 & -11 & -49 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

The advantage of this is that we get the z value directly. From the bottom row, R_3', we have $z = 3$. We can extend these row operations further and obtain the following matrix:

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right) \qquad (*)$$

Why would we want to achieve this sort of augmented matrix?

Because we can read off the x, y and z values directly from this augmented matrix. The only problem is in doing the arithmetic, because achieving this sort of matrix can be a laborious process.

This augmented matrix (*) is said to be in reduced row echelon form.

A matrix is in **reduced row echelon form**, normally abbreviated to **rref**, if it satisfies all the following conditions:

1. If there are any rows containing only zero entries then they are located in the bottom part of the matrix.
2. If a row contains non-zero entries then the first non-zero entry is a 1. This 1 is called a **leading 1**.
3. The leading 1's of two consecutive non-zero rows go strictly from top left to bottom right of the matrix.
4. The only non-zero entry in a column containing a leading 1 is the leading 1.

If condition (4) is *not* satisfied then we say that the matrix is in **row echelon form** and drop the qualification 'reduced'. In some linear algebra literature the leading 1 condition is relaxed and it is enough to say that any non-zero number is the **leading coefficient**.

For example, the following are all in reduced row echelon form:

$$\begin{pmatrix} 0 & \boxed{1} & 0 & 8 & 0 \\ 0 & 0 & \boxed{1} & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} \boxed{1} & 0 & 0 & 5 \\ 0 & \boxed{1} & 0 & -6 \\ 0 & 0 & \boxed{1} & 9 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & \boxed{1} & 3 & 0 & 4 & 0 & 6 \\ 0 & 0 & 0 & \boxed{1} & 5 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The following matrices are *not* in reduced row echelon form:

$$\mathbf{A} = \begin{pmatrix} 0 & \boxed{1} & 5 & 0 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 & \boxed{1} \end{pmatrix}, \mathbf{B} = \begin{pmatrix} \boxed{1} & 0 & 0 \\ 0 & \boxed{1} & 0 \\ \boxed{1} & 0 & 0 \end{pmatrix} \text{ and } \mathbf{C} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \boxed{1} & 5 & 8 & -1 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 9 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Why not?

In matrix $\mathbf{A}$ the third column contains a leading one but has a non-zero entry, 5.

In matrix $\mathbf{B}$ the leading ones do not go from top left to bottom right.

In matrix $\mathbf{C}$ the top row of zeros should be relegated to the bottom of the matrix as stated in condition (1) above.

However, matrix $\mathbf{A}$ is in row echelon form but not in reduced row echelon form. Matrices $\mathbf{B}$ and $\mathbf{C}$ are not in row echelon form.

The procedure which places an augmented matrix into row echelon form is called Gaussian elimination and the algorithm which places an augmented matrix into a reduced row echelon form is called **Gauss–Jordan** elimination.

- A matrix in reduced row-echelon form is unique. This means however we carry out our row operations we will always arrive at same matrix in reduced row-echelon form. However, a matrix in row-echelon form is not unique

Example 1.9

Place the augmented matrix $\begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left(\begin{array}{ccc|c} 1 & 5 & -3 & -9 \\ 0 & -13 & 5 & 37 \\ 0 & 0 & 5 & -15 \end{array}\right) \end{array}$ into reduced row echelon form.

Solution

Why should we want to place this matrix into reduced row echelon form?

In a nutshell, it's to avoid back substitution. If we look at the bottom row of the given augmented matrix we have $5z = -15$.

We need to divide by 5 in order to find the z value.

The reduced row echelon form, rref, gives us the values of the unknowns directly, and we do not need to carry out further manipulation or elimination.

What does reduced row echelon form mean in this case?

It means convert the given augmented matrix

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -9 \\ 0 & -13 & 5 & 37 \\ 0 & 0 & 5 & -15 \end{array}\right) \text{ into something like } \left(\begin{array}{ccc|c} 1 & 0 & 0 & * \\ 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \end{array}\right)$$

This means that we need to get 0 in place of the 5 in the second row, and 0's in place of the 5 and -3 in the first row. We also need a 1 in place of -13 in the middle row and 1 in place of the 5 in the bottom row.

How do we convert the 5 in the bottom row into 1?

Divide the last row by 5 (remember, this is the same as multiplying by $1/5$):

$$\begin{array}{r} R_1 \\ R_2 \\ R_3' = R_3/5 \end{array} \left(\begin{array}{ccc|c} 1 & 5 & -3 & -9 \\ 0 & -13 & 5 & 37 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

How do we get 0 in place of -3 in the first row and the 5 in the second row?

We execute the row operations $R_1 + 3R_3'$ and $R_2 - 5R_3'$:

$$\begin{array}{r} R_1^* = R_1 + 3R_3' \\ R_2^* = R_2 - 5R_3' \\ R_3' \end{array} \left(\begin{array}{ccc|c} 1+3(0) & 5+3(0) & -3+3(1) & -9+3(-3) \\ 0-5(0) & -13-5(0) & 5-5(1) & 37-5(-3) \\ 0 & 0 & 1 & -3 \end{array}\right)$$

Simplifying the entries gives

$$\begin{array}{c} R_1^* \\ R_2^* \\ R_3' \end{array} \left(\begin{array}{ccc|c} 1 & 5 & 0 & -18 \\ 0 & -13 & 0 & 52 \\ 0 & 0 & 1 & -3 \end{array}\right)$$

How do we get a 1 in place of -13?

Divide the middle row by -13 (remember, this is the same as multiplying by $-1/13$):

$$
\begin{array}{r}
R_1^* \\
R_2^{**} = R_2^*/(-13) \\
R_3'
\end{array}
\left(\begin{array}{ccc|l}
1 & 5 & 0 & -18 \\
0 & -13/(-13) & 0 & 52/(-13) \\
0 & 0 & 1 & -3
\end{array}\right)
$$

Simplifying the second row gives

$$
\begin{array}{r}
R_1^* \\
R_2^{**} \\
R_3'
\end{array}
\left(\begin{array}{ccc|r}
1 & 5 & 0 & -18 \\
0 & 1 & 0 & -4 \\
0 & 0 & 1 & -3
\end{array}\right)
$$

This matrix is now in row echelon form but not in reduced row echelon form. We need to convert the 5 in the top row into 0 to get it into reduced row echelon form.
How?
We carry out the row operation, $R_1^* - 5R_2^{**}$:

$$
\begin{array}{r}
R_1^{**} = R_1^* - 5R_2^{**} \\
R_2^{**} \\
R_3'
\end{array}
\left(\begin{array}{ccc|l}
1-5\,(0) & 5-5\,(1) & 0-5\,(0) & -18-5\,(-4) \\
0 & 1 & 0 & -4 \\
0 & 0 & 1 & -3
\end{array}\right)
$$

Simplifying the top row entries gives

$$
\begin{array}{r}
\\
R_1^{**} \\
R_2^{**} \\
R_3'
\end{array}
\begin{array}{c}
\begin{array}{ccc} x & y & z \end{array} \\
\left(\begin{array}{ccc|r}
1 & 0 & 0 & 2 \\
0 & 1 & 0 & -4 \\
0 & 0 & 1 & -3
\end{array}\right)
\end{array}
$$

Hence we have placed the given augmented matrix into reduced row echelon form.

3.1.1 Vector space

Let V be a non-empty set of elements called vectors. We define two operations on the set V – **vector addition** and **scalar multiplication**. Scalars are real numbers.

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in the set V. The set V is called a **vector space** if it satisfies the following 10 axioms.

1. The vector addition $\mathbf{u} + \mathbf{v}$ is also in the vector space V. Generally in mathematics we say that we have **closure** under vector addition if this property holds.
2. Commutative law: $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$.

3. Associative law: $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. Neutral element. There is a vector called the **zero** vector in V denoted by $\mathbf{O}$ which satisfies

$$\mathbf{u}+\mathbf{O}=\mathbf{u} \text{ for every vector } \mathbf{u} \text{ in } V$$

5. Additive inverse. For every vector $\mathbf{u}$ there is a vector $-\mathbf{u}$ (minus $\mathbf{u}$) which satisfies the following:

$$\mathbf{u}+(-\mathbf{u})=\mathbf{O}$$

6. Let k be a real scalar then $k\mathbf{u}$ is also in V. We say that we have **closure** under scalar multiplication if this axiom is satisfied.
7. Associative law for scalar multiplication. Let k and c be real scalars then

$$k(c\mathbf{u})=(kc)\mathbf{u}$$

8. Distributive law for vectors. Let k be a real scalar then

$$k(\mathbf{u}+\mathbf{v})=k\mathbf{u}+k\,\mathbf{v}$$

9. Distributive law for scalars. Let k and c be real scalars then

$$(k+c)\mathbf{u}=k\mathbf{u}+c\mathbf{u}$$

10. Identity element. For every vector $\mathbf{u}$ in V we have

$$1\mathbf{u}=\mathbf{u}$$

We say that if the elements of the set V satisfy the above 10 axioms then V is called a vector space and the elements are known as vectors.

3.2.2 Revision of linear combination

Linear combination combines the two fundamental operations of linear algebra – vector addition and scalar multiplication.

In the last chapter we introduced linear combination in $\mathbb{R}^n$. For example, we had

$$\mathbf{u} = k_1\mathbf{e}_1 + k_2\mathbf{e}_2 + \cdots + k_n\mathbf{e}_n \qquad (k\text{'s are scalars})$$

which is a **linear combination** of the standard unit vectors $\mathbf{e}_1,\ \mathbf{e}_2,\ \ldots$ and $\mathbf{e}_n$.

Similarly for general vector spaces we define linear combination as:

Definition (3.6). Let $\mathbf{v}_1,\ \mathbf{v}_2, \ldots$ and $\mathbf{v}_n$ be vectors in a vector space. If a vector $\mathbf{x}$ can be expressed as

$$\mathbf{x} = k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 + \cdots + k_n\mathbf{v}_n \text{ (where } k\text{'s are scalars)}$$

then we say $\mathbf{x}$ is a **linear combination** of the vectors $\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3,\ \ldots$ and $\mathbf{v}_n$.

Example 3.8

Let P_2 be the set of all polynomials of degree less than or equal to 2.

Let $\mathbf{v}_1 = t^2 - 1,\ \mathbf{v}_2 = t^2 + 3t - 5$ and $\mathbf{v}_3 = t$ be vectors in P_2.

Show that the quadratic polynomial

$$\mathbf{x} = 7t^2 - 15$$

is a linear combination of $\{\mathbf{v}_1,\ \mathbf{v}_2,\ \mathbf{v}_3\}$.

Solution

How do we show $\mathbf{x}$ is a linear combination of vectors $\mathbf{v}_1,\ \mathbf{v}_2$ and $\mathbf{v}_3$?

We need to find the values of the scalars $k_1,\ k_2$ and k_3 which satisfy

$$k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{x} \quad (*)$$

How can we determine these scalars?

By substituting $\mathbf{v}_1 = t^2 - 1,\ \mathbf{v}_2 = t^2 + 3t - 5,\ \mathbf{v}_3 = t$ and $\mathbf{x} = 7t^2 - 15$ into (*):

$$\begin{aligned} k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 &= k_1(t^2 - 1) + k_2(t^2 + 3t - 5) + k_3 t \\ &= k_1t^2 - k_1 + k_2t^2 + 3k_2t - 5k_2 + k_3t && \text{[expanding]} \\ &= (k_1 + k_2)t^2 + (3k_2 + k_3)t - (k_1 + 5k_2) && \text{[factorizing]} \\ &= 7t^2 - 15 && [\text{remember } \mathbf{x} = 7t^2 - 15] \end{aligned}$$

By equating coefficients of the last two lines

$$(k_1 + k_2)t^2 + (3k_2 + k_3)t - (k_1 + 5k_2) = 7t^2 - 15$$

gives

$$\begin{aligned} k_1 + k_2 &= 7 && [\text{equating } t^2] \\ 3k_2 + k_3 &= 0 && [\text{equating } t] \\ k_1 + 5k_2 &= 15 && \text{[equating constants]} \end{aligned}$$

Solving these equations gives the values of the scalars: $k_1 = 5$, $k_2 = 2$ and $k_3 = -6$. Substituting these into $k_1\mathbf{v}_1 + k_2\mathbf{v}_2 + k_3\mathbf{v}_3 = \mathbf{x}$:

$$5\mathbf{v}_1 + 2\mathbf{v}_2 - 6\mathbf{v}_3 = \mathbf{x}$$

This means that adding 5 lots of $\mathbf{v}_1$, 2 lots of $\mathbf{v}_2$ and -6 lots of $\mathbf{v}_3$ gives the vector $\mathbf{x}$:

$$5(t^2 - 1) + 2(t^2 + 3t - 5) - 6t = 7t^2 - 15$$

You may like to check the algebra.
We conclude that $\mathbf{x}$ is a linear combination of $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

The next proposition allows us to check that S is a subspace, by carrying out the test for scalar multiplication and vector addition in a single calculation.

Proposition (3.7). A non-empty subset S containing vectors $\mathbf{u}$ and $\mathbf{v}$ is a subspace of a vector space $V \Leftrightarrow$ any linear combination $k\mathbf{u} + c\,\mathbf{v}$ is also in S (k and c are scalars).

How do we prove this proposition?
Since it is a '$\Leftrightarrow$' statement we need to prove it both ways.

Proof.
($\Rightarrow$). Let S be a subspace of V, then S is a vector space. If $\mathbf{u}$ and $\mathbf{v}$ are in S then $k\mathbf{u} + c\mathbf{v}$ is also in S because the vector space S is closed under scalar multiplication and vector addition.
($\Leftarrow$). Assume $k\mathbf{u} + c\mathbf{v}$ is in S.
Substituting $k = c = 1$ into $k\mathbf{u} + c\mathbf{v}$ we have $\mathbf{u} + \mathbf{v}$ is also in S. Similarly for $c = 0$ we have $k\mathbf{u} + c\mathbf{v} = k\mathbf{u}$ is in S.
Hence we have closure under vector addition and scalar multiplication. By Proposition (3.5): S is subspace of $V \Leftrightarrow S$ is closed under vector addition and scalar multiplication.
We conclude that S is a subspace.

■

Proposition (3.7) is another test for a subspace. Hence a subspace of a vector space V is a non-empty subset S of V, such that for all vectors $\mathbf{u}$ and $\mathbf{v}$ in S and all scalars k and c we have $k\mathbf{u} + c\mathbf{v}$ is also in S.

Example 3.9

Let S be the subset of vectors of the form $(x \quad y \quad 0)^T$ in the vector space $\mathbb{R}^3$. Show that S is a subspace of $\mathbb{R}^3$.

(continued...)

Solution

How do we show that S is a subspace of $\mathbb{R}^3$?

We can use the above Proposition (3.7), which means we need to show that any linear combination $k\mathbf{u} + c\mathbf{v}$ is in S for any vectors $\mathbf{u}$ and $\mathbf{v}$ in S.

Let $\mathbf{u} = (a \quad b \quad 0)^T$ and $\mathbf{v} = (c \quad d \quad 0)^T$ be in S. Then for real scalars k_1 and k_2 we have

$$\begin{aligned} k_1\mathbf{u} + k_2\mathbf{v} &= k_1 \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} + k_2 \begin{pmatrix} c \\ d \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} k_1 a \\ k_1 b \\ 0 \end{pmatrix} + \begin{pmatrix} k_2 c \\ k_2 d \\ 0 \end{pmatrix} = \begin{pmatrix} k_1 a + k_2 c \\ k_1 b + k_2 d \\ 0 \end{pmatrix} \end{aligned}$$

Hence $k_1\mathbf{u} + k_2\mathbf{v}$ is also in S.

By the above Proposition (3.7):

S is subspace of $V \Leftrightarrow$ any linear combination $k\mathbf{u} + c\mathbf{v}$ is also in S.

We conclude that the given set S is a subspace of the vector space $\mathbb{R}^3$.

You might find it easier to use this test (3.7) rather than (3.5) because you only need to recall that the linear combination is closed in S.

Note that the given set S in the above example describes the xy plane in three-dimensional space $\mathbb{R}^3$ as shown in Fig. 3.10:

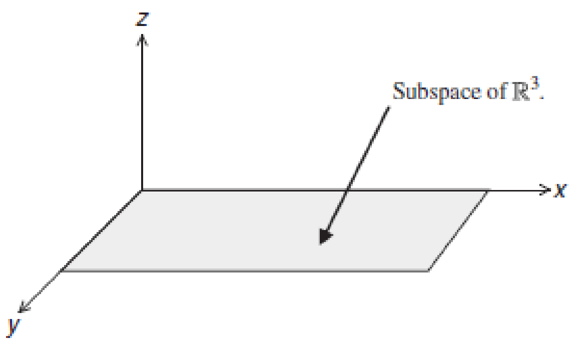


Figure 3.10

Sometimes we write $\begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$ as the set $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \; z = 0 \right\}$ where the vertical line in the set means 'such that'. That is, S contains the set of vectors in $\mathbb{R}^3$ such that the last entry is zero.

Summary

In this week, we learned about a linear equation means, what a system of linear equations is, possible solutions to a system of linear equations, what an augmented matrix is, what elementary row operations are, what Gaussian Elimination with Back Substitution is, what row-echelon and reduced row-echelon forms of a matrix are, the conditions for reduced row-echelon form, what a vector space is and finally what a linear combination is in terms of a vector space.