Week 15 Limits and differentiation Reading note 2

Notebook:	Computational Mathematics		
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Cornell Notes	**Topic:** Limits and differentiation	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: July 22, 2020

Essential Question:

What are limits and derivatives and how do they relate to the notion of continuity of a function?

Questions/Cues:

- What is the gradient of a curve at a point?
- What is the formula for the gradient function?
- What is the formula for the gradient function of $y = x^n$?
- What are the gradient functions of some of the most common functions?
- What are some rules for finding gradient functions?
- What is the definition of the second derivative?
- What are Stationary points?
- What is the second derivative test for maximum and minimum points?
- What is the product rule?
- What is the quotient rule?
- What is the chain rule?

Notes

- Gradient of a curve at a point = is the gradient of the tangent at that point

Given a function $y = f(x)$ we denote its gradient function by $\frac{dy}{dx}$ or simply by y'.

  - Also called the first derivative or simply the derivative. Process of finding the gradient function is known as differentiation.
    - Because the gradient function measures how rapidly a graph is changing, it's also referred to as the rate of change of y.

For any function of the form $y = x^n$ the gradient function is found from the following formula:

Key point

If $y = x^n$ then $y' = nx^{n-1}$.

WORKED EXAMPLES

34.1 Find the gradient function of (a) $y = x^3$, (b) $y = x^4$.

Solution

(a) Comparing $y = x^3$ with $y = x^n$ we see that $n = 3$. Then $y' = 3x^{3-1} = 3x^2$.

(b) Applying the formula with $n = 4$ we find that if $y = x^4$ then $y' = 4x^{4-1} = 4x^3$.

34.2 Find the gradient function of (a) $y = x^2$, (b) $y = x$.

Solution

(a) Applying the formula with $n = 2$ we find that if $y = x^2$ then $y' = 2x^{2-1} = 2x^1$. Because x^1 is simply x we find that the gradient function is $y' = 2x$.

(b) Applying the formula with $n = 1$ we find that if $y = x^1$ then $y' = 1x^{1-1} = 1x^0$. Because x^0 is simply 1 we find that the gradient function is $y' = 1$.

WORKED EXAMPLES

34.3 Find the gradient of $y = x^2$ at the points where

(a) $x = -1$ (b) $x = 0$ (c) $x = 2$ (d) $x = 3$

Solution

From Worked Example 34.2, or from the formula, we know that the gradient function of $y = x^2$ is given by $y' = 2x$.

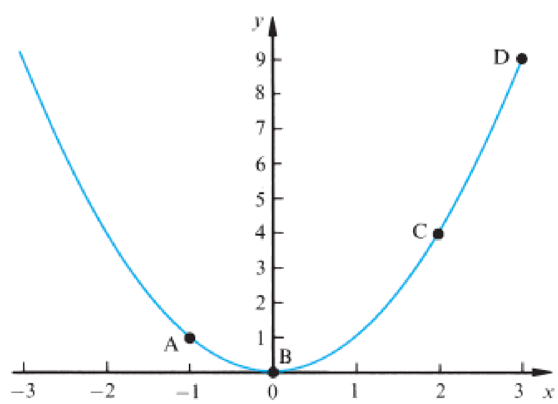
(a) When $x = -1$ the gradient of the graph is then $y'(-1) = 2(-1) = -2$. The fact that the gradient is negative means that the curve is falling at the point.

(b) When $x = 0$ the gradient is $y'(0) = 2(0) = 0$. The gradient of the curve is zero at this point. This means that the curve is neither falling nor rising.

(c) When $x = 2$ the gradient is $y'(2) = 2(2) = 4$. The fact that the gradient is positive means that the curve is rising.

(d) When $x = 3$ the gradient is $y'(3) = 2(3) = 6$ and so the curve is rising here. Comparing this answer with that of part (c) we conclude that the curve is rising more rapidly at $x = 3$ than at $x = 2$, where the gradient was found to be 4.

Figure 34.2
Graph of $y = x^2$



The graph of $y = x^2$ is shown in Figure 34.2. If we compare our results with the graph we see that the curve is indeed falling when $x = -1$ (point A) and is rising when $x = 2$ (point C) and $x = 3$ (point D). At the point where $x = 0$ (point B) the curve is neither rising nor falling.

34.4 Find the gradient function of $y = x^{-3}$. Hence find the gradient of $y = x^{-3}$ when $x = 4$.

Solution Using the formula with $n = -3$ we find that if $y = x^{-3}$ then $y' = -3x^{-3-1} = -3x^{-4}$. Because x^{-4} can also be written as $1/x^4$ we could write $y' = -3(1/x^4)$ or $-3/x^4$. When $x = 4$ we find $y'(4) = -3/4^4 = -\frac{3}{256}$. This number is very small and negative, which means that when $x = 4$ the curve is falling, but only slowly.

34.5 Find the gradient function of $y = 1$.

Solution Before we use the formula to calculate the gradient function let us think about the graph of $y = 1$. Whatever the value of x, this function takes the value 1. Its graph must then be a horizontal line – it neither rises nor falls. We conclude that the gradient function must be zero, that is $y' = 0$. To obtain the same result using the formula we must rewrite 1 as x^0. Then, using the formula with $n = 0$, we find that if $y = x^0$ then $y' = 0x^{0-1} = 0$.

$y = f(x)$	$y' = f'(x)$	Notes
constant	0	
x	1	
x^2	$2x$	
x^n	nx^{n-1}	
e^x	e^x	
e^{kx}	ke^{kx}	k is a constant
$\sin x$	$\cos x$	
$\cos x$	$-\sin x$	
$\sin kx$	$k \cos kx$	k is a constant
$\cos kx$	$-k \sin kx$	k is a constant
$\ln kx$	$1/x$	k is a constant

- When finding the gradient functions of trigonometric functions, the angle x must always be measured in radians

WORKED EXAMPLES

34.6 Use Table 34.1 to find the gradient function y' when y is

(a) $\sin x$ (b) $\sin 2x$ (c) $\cos 3x$ (d) e^x

Solution

(a) Directly from the table we see that if $y = \sin x$ then its gradient function is given by $y' = \cos x$. This result occurs frequently and is worth remembering.

(b) Using the table and taking $k = 2$ we find that if $y = \sin 2x$ then $y' = 2\cos 2x$.

(c) Using the table and taking $k = 3$ we find that if $y = \cos 3x$ then $y' = -3 \sin 3x$.

(d) Using the table we see directly that if $y = e^x$ then $y' = e^x$. Note that the exponential function e^x is very special because its gradient function y' is the same as y.

34.7 Find the gradient function of $y = e^{-x}$. Hence find the gradient of the graph of y at the point where $x = 1$.

Solution

Noting that $e^{-x} = e^{-1x}$ and using Table 34.1 with $k = -1$ we find that if $y = e^{-x}$ then $y' = -1e^{-x} = -e^{-x}$. Using a calculator to evaluate this when $x = 1$ we find $y'(1) = -e^{-1} = -0.368$.

34.8 Find the gradient function of $y = \sin 4x$ where $x = 0.3$.

Solution

Using Table 34.1 with $k = 4$ gives $y' = 4\cos 4x$. Remembering to measure x in radians we evaluate this when $x = 0.3$:

$$y'(0.3) = 4\cos 4(0.3) = 4\cos 1.2 = 1.4494$$

Key point

Rule 1: If $y = f(x) + g(x)$ then $y' = f'(x) + g'(x)$.

In words this says that to find the gradient function of a sum of two functions we simply find the two gradient functions separately and add these together.

WORKED EXAMPLE

34.9 Find the gradient function of $y = x^2 + x^4$.

Solution

The gradient function of x^2 is $2x$. The gradient function of x^4 is $4x^3$. Therefore

$$\text{if } y = x^2 + x^4 \quad \text{then} \quad y' = 2x + 4x^3$$

Key point

Rule 2:

If $y = f(x) - g(x)$ then $y' = f'(x) - g'(x)$.

WORKED EXAMPLE

34.10 Find the gradient function of $y = x^5 - x^7$.

Solution

We find the gradient function of each term separately and subtract them. That is, $y' = 5x^4 - 7x^6$.

Key point

Rule 3: If $y = kf(x)$, where k is a number, then $y' = kf'(x)$.

WORKED EXAMPLE

34.11 Find the gradient function of $y = 3x^2$.

Solution

This function is 3 times x^2. The gradient function of x^2 is $2x$. Therefore, using Rule 3, we find that

$$\text{if } y = 3x^2 \qquad \text{then } y' = 3(2x) = 6x$$

WORKED EXAMPLES

34.12 Find the derivative of $y = 4x^2 + 3x^{-3}$.

Solution The derivative of $4x^2$ is $4(2x) = 8x$. The derivative of $3x^{-3}$ is $3(-3x^{-4}) = -9x^{-4}$. Therefore, if $y = 4x^2 + 3x^{-3}$ then $y' = 8x - 9x^{-4}$.

34.13 Find the derivative of

(a) $y = 4 \sin t - 3 \cos 2t$ (b) $y = \dfrac{e^{2t}}{3} + 6 + \dfrac{\ln(2t)}{5}$

Solution (a) We differentiate each quantity in turn using Table 34.1:

$$y' = 4 \cos t - 3(-2 \sin 2t) = 4 \cos t + 6 \sin 2t$$

(b) Writing y as $\frac{1}{3}e^{2t} + 6 + \frac{1}{5} \ln(2t)$ we find

$$y' = \frac{2}{3} e^{2t} + 0 + \frac{1}{5}\left(\frac{1}{t}\right) = \frac{2e^{2t}}{3} + \frac{1}{5t}$$

Key point

y'' or $\dfrac{d^2y}{dx^2}$ is found by differentiating y'.

WORKED EXAMPLES

34.14 Find the first and second derivatives of $y = x^4$.

Solution From Table 34.1, if $y = x^4$ then $y' = 4x^3$. The second derivative is found by differentiating the first derivative. Therefore

$$\text{if } y' = 4x^3 \qquad \text{then } y'' = 4(3x^2) = 12x^2$$

34.15 Find the first and second derivatives of $y = 3x^2 - 7x + 2$.

Solution Using Table 34.1 we find $y' = 6x - 7$. The derivative of the constant 2 equals 0. Differentiating again we find $y'' = 6$, because the derivative of the constant -7 equals 0.

34.16 If $y = \sin x$, find

(a) $\dfrac{dy}{dx}$ (b) $\dfrac{d^2y}{dx^2}$

Solution (a) Recall that $\frac{dy}{dx}$ is the first derivative of y. From Table 34.1 this is given by

$$\frac{dy}{dx} = \cos x$$

(b) $\frac{d^2y}{dx^2}$ is the second derivative of y. This is found by differentiating the first derivative. From Table 34.1 we find that the derivative of $\cos x$ is $-\sin x$, and so

$$\frac{d^2y}{dx^2} = -\sin x$$

Key point

Stationary points are located by setting the gradient function equal to zero, that is $y' = 0$.

- If to the left of the point the curve is rising and to the right it's falling, then the point is a maximum turning point or simply a maximum
- If to the left of the point the curve is falling and to the right it's rising, then the point is a minimum turning point or simply a minimum
- If at the point, the slope of the curve is momentarily zero and then it continues rising or falling, then the point is a point of inflexion or inflection

WORKED EXAMPLE

34.17 Find the stationary points of $y = 3x^2 - 6x + 8$.

Solution We first determine the gradient function y' by differentiating y. This is found to be $y' = 6x - 6$. Stationary points occur when the gradient is zero, that is when $y' = 6x - 6 = 0$. From this

$$6x - 6 = 0$$
$$6x = 6$$
$$x = 1$$

When $x = 1$ the gradient is zero. When $x = 1$, $y = 5$. Therefore $(1, 5)$ is a stationary point. At this stage we cannot tell whether to expect a maximum, minimum or point of inflexion; all we know is that one of these occurs when $x = 1$. However, a sketch of the graph of $y = 3x^2 - 6x + 8$ is shown in Figure 34.4, which reveals that the point is a minimum.

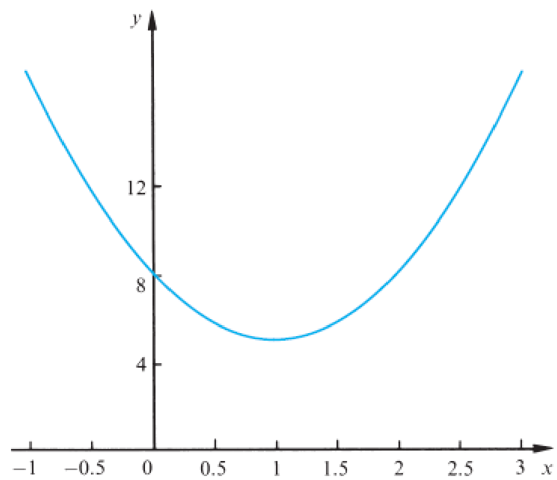


Figure 34.4 There is a minimum turning point at $x = 1$

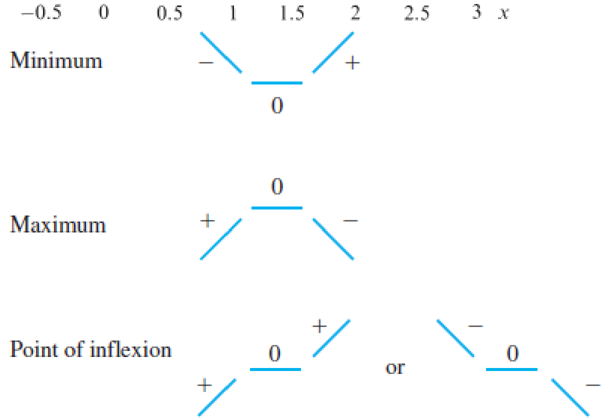


Figure 34.5 The sign of the gradient function close to a stationary point

34.18 Find the location and nature of the stationary points of $y = 2x^3 - 6x^2 - 18x$.

Solution In order to find the location of the stationary points we must calculate the gradient function. This is $y' = 6x^2 - 12x - 18$. At a stationary point $y' = 0$ and so the equation that must be solved is $6x^2 - 12x - 18 = 0$. This quadratic equation is solved as follows:

$$6x^2 - 12x - 18 = 0$$

Factorising:

$$6(x^2 - 2x - 3) = 0$$

$$6(x + 1)(x - 3) = 0$$

$$\text{so that } x = -1 \text{ and } 3$$

The stationary points are located at $x = -1$ and $x = 3$. At these values, $y = 10$ and $y = -54$ respectively. We now find their nature:

When $x = -1$ A little to the left of $x = -1$, say at $x = -2$, we calculate the gradient of the graph using the gradient function. That is, $y'(-2) = 6(-2)^2 - 12(-2) - 18 = 24 + 24 - 18 = 30$. Therefore the graph is rising when $x = -2$. A little to the right of $x = -1$, say at $x = 0$, we calculate the gradient of the graph using the gradient function. That is, $y'(0) = 6(0)^2 - 12(0) - 18 = -18$. Therefore the graph is falling when $x = 0$. From this information we conclude that the turning point at $x = -1$ must be a maximum.

When $x = 3$ A little to the left of $x = 3$, say at $x = 2$, we calculate the gradient of the graph using the gradient function. That is, $y'(2) = 6(2)^2 - 12(2) - 18 = 24 - 24 - 18 = -18$. Therefore the graph is falling when $x = 2$. A little to the right of $x = 3$, say at $x = 4$, we calculate the gradient of the graph using the gradient function. That is, $y'(4) = 6(4)^2 - 12(4) - 18 = 96 - 48 - 18 = 30$. Therefore the graph is rising when $x = 4$. From this information we conclude that the turning point at $x = 3$ must be a minimum.

To show this behaviour a graph of $y = 2x^3 - 6x^2 - 18x$ is shown in Figure 34.6 where these turning points can be clearly seen.

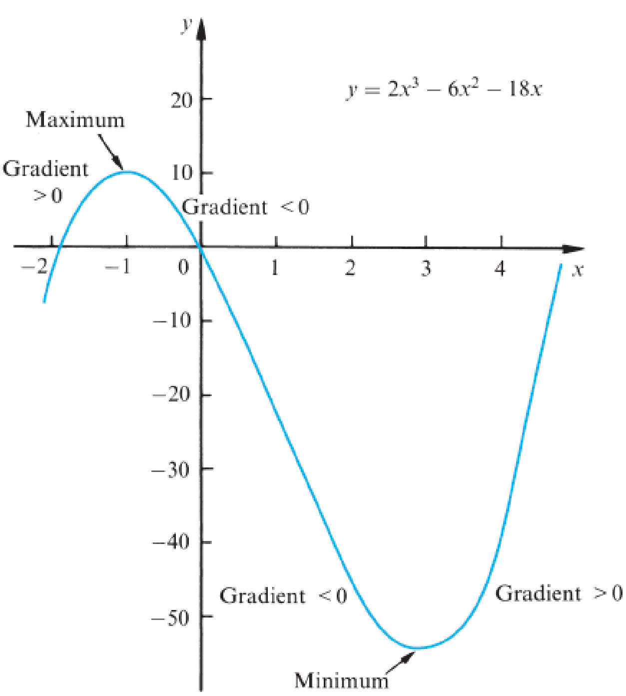


Figure 34.6 Graph of $y = 2x^3 - 6x^2 - 18x$

34.19 Find the location and nature of the stationary points of $y = x^3 - 3x^2 + 3x - 1$.

Solution First we find the gradient function: $y' = 3x^2 - 6x + 3$. At a stationary point $y' = 0$ and so

$$\begin{aligned} 3x^2 - 6x + 3 &= 0 \\ 3(x^2 - 2x + 1) &= 0 \\ 3(x-1)(x-1) &= 0 \\ \text{and so } x &= 1 \end{aligned}$$

When $x = 1$, $y = 0$.

We conclude that there is only one stationary point, and this is at $(1, 0)$. To determine its nature we look at the gradient function on either side of this point. At $x = 0$, say, $y'(0) = 3$. At $x = 2$, say, $y'(2) = 3(2^2) - 6(2) + 3 = 3$. The gradient function changes from positive to zero to positive as we move through the stationary point. We conclude that the stationary point is a point of inflexion. For completeness a graph is sketched in Figure 34.7.

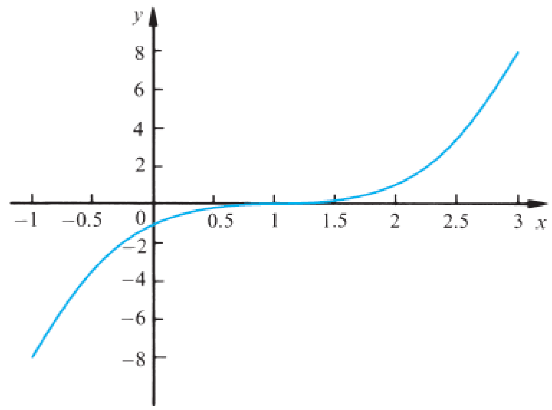


Figure 34.7 Graph of $y = x^3 - 3x^2 + 3x - 1$ showing the point of inflexion

Key point

If y'' is positive at the stationary point, the point is a minimum.
If y'' is negative at the stationary point, the point is a maximum.
If y'' is equal to zero, this test does not tell us anything and the previous method should be used.

WORKED EXAMPLE

34.20 Locate the stationary points of

$$y = \frac{x^3}{3} + \frac{x^2}{2} - 12x + 5$$

and determine their nature using the second-derivative test.

Solution We first find the gradient function: $y' = x^2 + x - 12$. Setting this equal to zero to locate the stationary points we find

$$x^2 + x - 12 = 0$$

$$(x+4)(x-3) = 0$$

$$x = -4 \text{ and } 3$$

When $x = -4$, $y = \frac{119}{3}$. When $x = 3$, $y = -\frac{35}{2}$.

There are two stationary points: at $(-4, \frac{119}{3})$ and $(3, -\frac{35}{2})$. To apply the second-derivative test we need to find y''. This is found by differentiating y' to give $y'' = 2x + 1$. We now evaluate this at each stationary point in turn:

$x = -4$ When $x = -4$, $y''(-4) = 2(-4) + 1 = -7$. This is negative and we conclude from the second-derivative test that the point is a maximum point.

$x = 3$ When $x = 3$, $y''(3) = 2(3) + 1 = 7$. This is positive and we conclude from the second-derivative test that the point is a minimum point. Remember, if in doubt, that these results could be confirmed by sketching a graph of the function.

Key point

The product rule: if $y = uv$ then

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}$$

35.1 Use the product rule to differentiate $y = x^3 \sin 2x$.

Solution We let $u(x) = x^3$ and $v(x) = \sin 2x$.

Then clearly

$$y = uv$$

and using Table 34.1 we have

$$\frac{du}{dx} = 3x^2 \qquad \frac{dv}{dx} = 2\cos 2x$$

Using the product rule we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx}\\ &= 3x^2 \sin 2x + x^3(2\cos 2x)\\ &= x^2(3\sin 2x + 2x\cos 2x)\end{aligned}$$

35.2 Use the product rule to differentiate $y = e^{2x} \cos x$.

Solution We let

$$u(x) = e^{2x} \qquad v(x) = \cos x$$

so that

$$y = e^{2x} \cos x = uv$$

Using Table 34.1 we have

$$\frac{du}{dx} = 2e^{2x} \qquad \frac{dv}{dx} = -\sin x$$

Using the product rule we have

$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{dx} v + u \frac{dv}{dx} \\ &= 2e^{2x} \cos x + e^{2x}(-\sin x) \\ &= e^{2x}(2\cos x - \sin x) \end{aligned}$$

35.3 Use the product rule to differentiate $y = x \ln x$.

Solution Let $u = x$, $v = \ln x$. Then

$$y = x \ln x = uv$$

and

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = \frac{1}{x}$$

So using the product rule

$$\begin{aligned} \frac{dy}{dx} &= \frac{du}{dx} v + u \frac{dv}{dx} \\ &= 1 \ln x + x \left(\frac{1}{x} \right) \\ &= \ln x + 1 \end{aligned}$$

35.4 Find the gradient of $y = 2x^2 e^{-x}$ when $x = 1$.

Solution Let $u = 2x^2$, $v = e^{-x}$ so that

$$y = 2x^2 e^{-x} = uv$$

and

$$\frac{du}{dx} = 4x \qquad \frac{dv}{dx} = -e^{-x}$$

Using the product rule we see

$$\begin{aligned}\frac{dy}{dx} &= \frac{du}{dx}v + u\frac{dv}{dx} \\ &= 4xe^{-x} + 2x^2(-e^{-x}) \\ &= 2xe^{-x}(2 - x)\end{aligned}$$

Recall that the derivative, $\frac{dy}{dx}$, is the gradient of y. When $x = 1$

$$\frac{dy}{dx} = 2e^{-1}(2 - 1) = 0.7358$$

so that the gradient of $y = 2x^2 e^{-x}$ when $x = 1$ is 0.7358.

Key point

The quotient rule: if

$$y = \frac{u}{v}$$

then

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

WORKED EXAMPLES

35.5 Use the quotient rule to differentiate

$$y = \frac{\sin x}{x^2}$$

Solution We let $u(x) = \sin x$ and $v(x) = x^2$. Then

$$y = \frac{u}{v} \qquad \text{and} \qquad \frac{du}{dx} = \cos x \qquad \frac{dv}{dx} = 2x$$

Using the quotient rule we have

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{x^2 \cos x - \sin x(2x)}{(x^2)^2}$$

$$= \frac{x(x \cos x - 2 \sin x)}{x^4}$$

$$= \frac{x \cos x - 2 \sin x}{x^3}$$

35.6 Use the quotient rule to differentiate $y = \dfrac{x+1}{x^2+1}$.

Solution We let

$$u(x) = x + 1 \qquad v(x) = x^2 + 1$$

and so clearly

$$y = \frac{u}{v}$$

Now

$$\frac{du}{dx} = 1 \qquad \frac{dv}{dx} = 2x$$

Applying the quotient rule we have

$$\begin{aligned}\frac{dy}{dx} &= \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \\ &= \frac{(x^2+1)1 - (x+1)2x}{(x^2+1)^2} \\ &= \frac{x^2+1-2x^2-2x}{(x^2+1)^2} \\ &= \frac{1-2x-x^2}{(x^2+1)^2}\end{aligned}$$

35.7 Use the quotient rule to differentiate $y = \dfrac{e^x + x}{\sin x}$.

Solution We let

$$u(x) = e^x + x \qquad v(x) = \sin x$$

and so

$$\frac{du}{dx} = e^x + 1 \qquad \frac{dv}{dx} = \cos x$$

Clearly

$$y = \frac{u}{v}$$

and so

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2}$$

$$= \frac{\sin x(e^x + 1) - (e^x + x)\cos x}{(\sin x)^2}$$

The chain rule

Suppose we are given a function $y(x)$ where the variable x is itself a function of another variable, t, say. We say that y is a **function of a function**. For example, suppose

$$y(x) = \cos x \qquad \text{and} \qquad x(t) = t^2$$

Then we can write

$$y = \cos(t^2)$$

There will be occasions when it is necessary to calculate $\frac{dy}{dt}$. This can be done by first finding $\frac{dy}{dx}$ and $\frac{dx}{dt}$ and then using the chain rule.

Key point

The **chain rule** states that if $y = y(x)$ and $x = x(t)$, then

$$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$$

WORKED EXAMPLE

35.8 Use the chain rule to find $\frac{dy}{dt}$ when $y = \cos x$ and $x = t^2$.

Solution When $y = \cos x$ our previous knowledge of differentiation (or use of Table 34.1) tells us that $\frac{dy}{dx} = -\sin x$. Also, when $x = t^2$, $\frac{dx}{dt} = 2t$. Using the chain rule,

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx} \times \frac{dx}{dt} \\ &= (-\sin x) \times 2t\end{aligned}$$

Using the fact that $x = t^2$ enables us to write this as

$$\frac{dy}{dt} = -2t \sin t^2$$

WORKED EXAMPLES

35.9 If $y = (7t + 3)^4$ find $\dfrac{dy}{dt}$.

Solution Note that if we introduce a new variable x and make the substitution $7t + 3 = x$ then y takes the much simpler form $y = x^4$. Then, from $y = x^4$,

$$\frac{dy}{dx} = 4x^3$$

and from $x = 7t + 3$,

$$\frac{dx}{dt} = 7$$

Using the chain rule

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx} \times \frac{dx}{dt} \\ &= 4x^3 \times 7 \\ &= 28x^3 \\ &= 28(7t + 3)^3\end{aligned}$$

35.10 If $y = (3t^2 + 4)^{1/2}$, find the derivative $\dfrac{dy}{dt}$.

Solution If we let $3t^2 + 4 = x$, then y looks much simpler: $y = x^{1/2}$. Then

$$\frac{dy}{dx} = \frac{1}{2}x^{-1/2} \qquad \text{and} \qquad \frac{dx}{dt} = 6t$$

Using the chain rule,

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx} \times \frac{dx}{dt} \\ &= \frac{1}{2}x^{-1/2} \times 6t \\ &= 3t(3t^2 + 4)^{-1/2}\end{aligned}$$

35.11 Find the gradient of the function $y = e^{(t^2)}$ when $t = 0.5$.

Solution Note that by writing $t^2 = x$ then $y = e^x$. Using the chain rule,

$$\begin{aligned}\frac{dy}{dt} &= \frac{dy}{dx} \times \frac{dx}{dt}\\ &= e^x \times 2t\\ &= 2te^{(t^2)}\end{aligned}$$

This is the gradient function of $y = e^{(t^2)}$. So, when $t = 0.5$ the value of the gradient is $(2)(0.5)e^{(0.5^2)} = e^{0.25} = 1.284$.

Summary

In this week, we learned about gradient of a curve at a point, the formula for the gradient function, the gradient functions of some of the most common functions, the second derivative, stationary points, the second derivative test, product rule, quotient rule and finally the chain rule.