Week 18 Algebra, Vectors, and Matrices continued Lecture Note

Notebook:	Computational Mathematics		
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Cornell Notes	**Topic:** **Algebra, Vectors, and Matrices continued**	Course: BSc Computer Science Class: Computational Mathematics[Lecture] Date: July 29, 2020

Essential Question:

What are vectors and matrices?

Questions/Cues:

- What is the determinant of a 2 x 2 matrix?
- What is the determinant of a 3 x 3 matrix?
- What is the method to calculate the determinant of a matrix greater than 2 x 2 in general?
- What is the identity transformation and the identity matrix?
- What is the inverse transformation and the inverse matrix?
- How do we use matrices to solve a system of linear equations?
- What is Gauss-Jordan Elimination?

Notes

$$\begin{array}{cc} + & - \end{array}$$

$$M=\begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}$$

$$\det M= m_{11}m_{22}-m_{12}m_{21}$$

Example:

$$M=\begin{pmatrix} 1 & 4 \\ 8 & 0 \end{pmatrix}$$

$$\det M=0-32=-32$$

Three by three matrix

$$M=\begin{pmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{pmatrix}\begin{matrix} m_{11} & m_{12} \\ m_{21} & m_{22} \\ m_{31} & m_{32} \end{matrix}$$

$$\det M= m_{11}m_{22}m_{33}+m_{12}m_{23}m_{31}+m_{13}m_{21}m_{32}$$
$$-m_{13}m_{22}m_{31}-m_{11}m_{23}m_{32}-m_{12}m_{21}m_{33}$$

Example

$$M=\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 1 \end{pmatrix}\begin{matrix} 1 & 2 \\ 1 & 2 \\ 0 & 1 \end{matrix} \qquad \det M=2+0+0-0-3-2=-3$$

Three by three (and beyond) matrix, alternative method

$$M=\begin{pmatrix} 1 & 2 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 1^{+} & 2^{-} & 0^{+} \\ 1^{-} & 2^{+} & 3^{-} \\ 0^{+} & 1^{-} & 0^{+} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} 1^{+} & 2^{-} & 0^{+} \\ 1^{-} & 2^{+} & 3^{-} \\ 0^{+} & 1^{-} & 0^{+} \end{pmatrix} = +\,0 \det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}=0$$

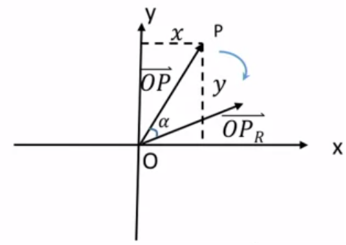
$$\Rightarrow \begin{pmatrix} 1^{+} & 2^{-} & 0^{+} \\ 1^{-} & 2^{+} & 3^{-} \\ 0^{+} & 1^{-} & 0^{+} \end{pmatrix} = -\,3 \det\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}=-3$$

$$\Rightarrow \begin{pmatrix} 1^{+} & 2^{-} & 0^{+} \\ 1^{-} & 2^{+} & 3^{-} \\ 0^{+} & 1^{-} & 0^{+} \end{pmatrix} = +\,0 \det\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}=0$$

$$\Rightarrow \quad \det M=0 -3 +0= -3$$

Identity Transformation and Identity Matrix

$\overrightarrow{OP}=\begin{pmatrix} x \\ y \end{pmatrix}$ $\boldsymbol{M}=\begin{pmatrix} \mathrm{Cos}(\alpha) & \mathrm{Sin}(\alpha) \\ -\mathrm{Sin}(\alpha) & \mathrm{Cos}(\alpha) \end{pmatrix} \Rightarrow \overrightarrow{OP_R}=\boldsymbol{M}\,\overrightarrow{OP}$



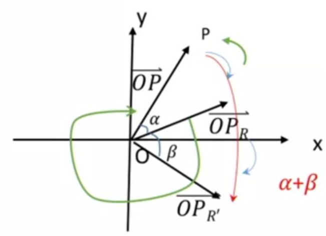
$\boldsymbol{M}\ (\alpha=0)=\begin{pmatrix} \mathrm{Cos}(0) & \mathrm{Sin}(0) \\ -\mathrm{Sin}(0) & \mathrm{Cos}(0) \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\boldsymbol{I}$

$$\overrightarrow{OP_R}=\boldsymbol{I}\,\overrightarrow{OP}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}=\overrightarrow{OP}$$

$$I=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \xrightarrow{\mathrm{n}\times\mathrm{n}} \begin{pmatrix} 1 & 0\cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & 0\cdots & 1 \end{pmatrix}$$

Inverse Transformation and Inverse Matrix

We saw that $\overrightarrow{OP_R}=\boldsymbol{M}\,\overrightarrow{OP}$ $\boldsymbol{M}=\begin{pmatrix} \mathrm{Cos}(\alpha) & \mathrm{Sin}(\alpha) \\ -\mathrm{Sin}(\alpha) & \mathrm{Cos}(\alpha) \end{pmatrix}$



$\overrightarrow{OP_{R'}}=\boldsymbol{M'}\,\overrightarrow{OP_R}$ with $\boldsymbol{M'}=\begin{pmatrix} \mathrm{Cos}(\beta) & \mathrm{Sin}(\beta) \\ -\mathrm{Sin}(\beta) & \mathrm{Cos}(\beta) \end{pmatrix}$

$\overrightarrow{OP_{R'}}=(\boldsymbol{M'}\cdot\boldsymbol{M})\,\overrightarrow{OP}=\boldsymbol{M_T}\overrightarrow{OP}$ $\boldsymbol{M_T}=(\boldsymbol{M'}\cdot\boldsymbol{M})=\begin{pmatrix} \mathrm{Cos}(\beta+\alpha) & \mathrm{Sin}(\beta+\alpha) \\ -\mathrm{Sin}(\beta+\alpha) & \mathrm{Cos}(\beta+\alpha) \end{pmatrix}$

Let $\beta=-\alpha$ or (equivalently $\beta=360^{o}-\alpha$) $\boldsymbol{M_T}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=\boldsymbol{I}$

$\Rightarrow$ $\overrightarrow{OP_{R'}}=\boldsymbol{M_T}\,\overrightarrow{OP}=(\boldsymbol{M'}\cdot\boldsymbol{M})\,\overrightarrow{OP}=\boldsymbol{I}\,\overrightarrow{OP}=\overrightarrow{OP}$

If M' exists such that M' ·M=M ·M'=I
M is invertible and M' is the <u>inverse</u> of M
(M' → M⁻¹)

$\boldsymbol{M}$ is in<u>vertible if and only if det$\boldsymbol{M}\neq0$</u> det $\boldsymbol{M}^{-1}=1/\det\boldsymbol{M}$

System of Linear Equations

x+2y=1
x-3y=-2 $\Longleftrightarrow \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x+2y \\ x-3y \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

$$\boldsymbol{M}=\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \Rightarrow \boldsymbol{M}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

- <u>***M*** is invertible if and only if det***M***≠0</u>
  if det ***M***=0 → ***M*** is not invertible
- <u>A linear system of equations has solutions if and only if det***M***≠0</u>

Let $\boldsymbol{M^{-1}}$ be the inverse of M

$$\boldsymbol{M}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \end{pmatrix} \quad \Rightarrow \quad \boldsymbol{M^{-1}M}\begin{pmatrix} x \\ y \end{pmatrix}= \boldsymbol{M^{-1}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

but $\boldsymbol{M^{-1}M=I} \Rightarrow \boldsymbol{I}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x \\ y \end{pmatrix}= \boldsymbol{M^{-1}}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Solving the system: Inverse Matrix

$$\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}=\begin{pmatrix} x+2y \\ x-3y \end{pmatrix}=\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

$$\boldsymbol{M}=\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix} \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix}= \boldsymbol{M}^{-1}\begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

How to find the inverse

If $\boldsymbol{M}= \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \Rightarrow \boldsymbol{M}^{-1}= \dfrac{1}{detM}\begin{pmatrix} m_{22} & -m_{12} \\ -m_{21} & m_{11} \end{pmatrix}$

if $\boldsymbol{M}=\begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}$ $\det\mathrm{M} = -5 \Rightarrow \boldsymbol{M}^{-1}= -\frac{1}{5}\begin{pmatrix} -3 & -2 \\ -1 & 1 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix}= \boldsymbol{M}^{-1}\begin{pmatrix} 1 \\ -2 \end{pmatrix}= \begin{pmatrix} 3/5 & 2/5 \\ 1/5 & -1/5 \end{pmatrix}\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \begin{pmatrix} \frac{3}{5}-\frac{4}{5} \\ \frac{1}{5}+\frac{2}{5} \end{pmatrix}= \begin{pmatrix} -1/5 \\ 3/5 \end{pmatrix}$$

x=-1/5 y=3/5

$$A\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} \qquad A=\begin{pmatrix} 1 & 1 & 0 \\ 3 & 1 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\det A = -1 \neq 0$$

$$\left(\begin{array}{ccc:ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right)$$

$$R_1 \to R_1 + \alpha R_2 + \beta R_3$$

$$R_1 \rightleftarrows R_2$$

$$R_1 \to \alpha R_1$$

- The aim of Gauss-Jordan Elimination is to obtain the identity matrix on the left hand side of the augmented matrix (of the set of colon). This can be achieved using the fact that rows can added/subtracted together and replaced, swapped and replaced or multiplied by a constant and replaced

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 3 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \quad R_2 - 3R_1 \to R_2$$

$$\begin{array}{cccccc} 3 & 1 & -1 & 0 & 1 & 0 \\ 3 & 3 & 0 & 3 & 0 & 0 \\ \hline 0 & -2 & -1 & -3 & 1 & 0 \end{array}$$

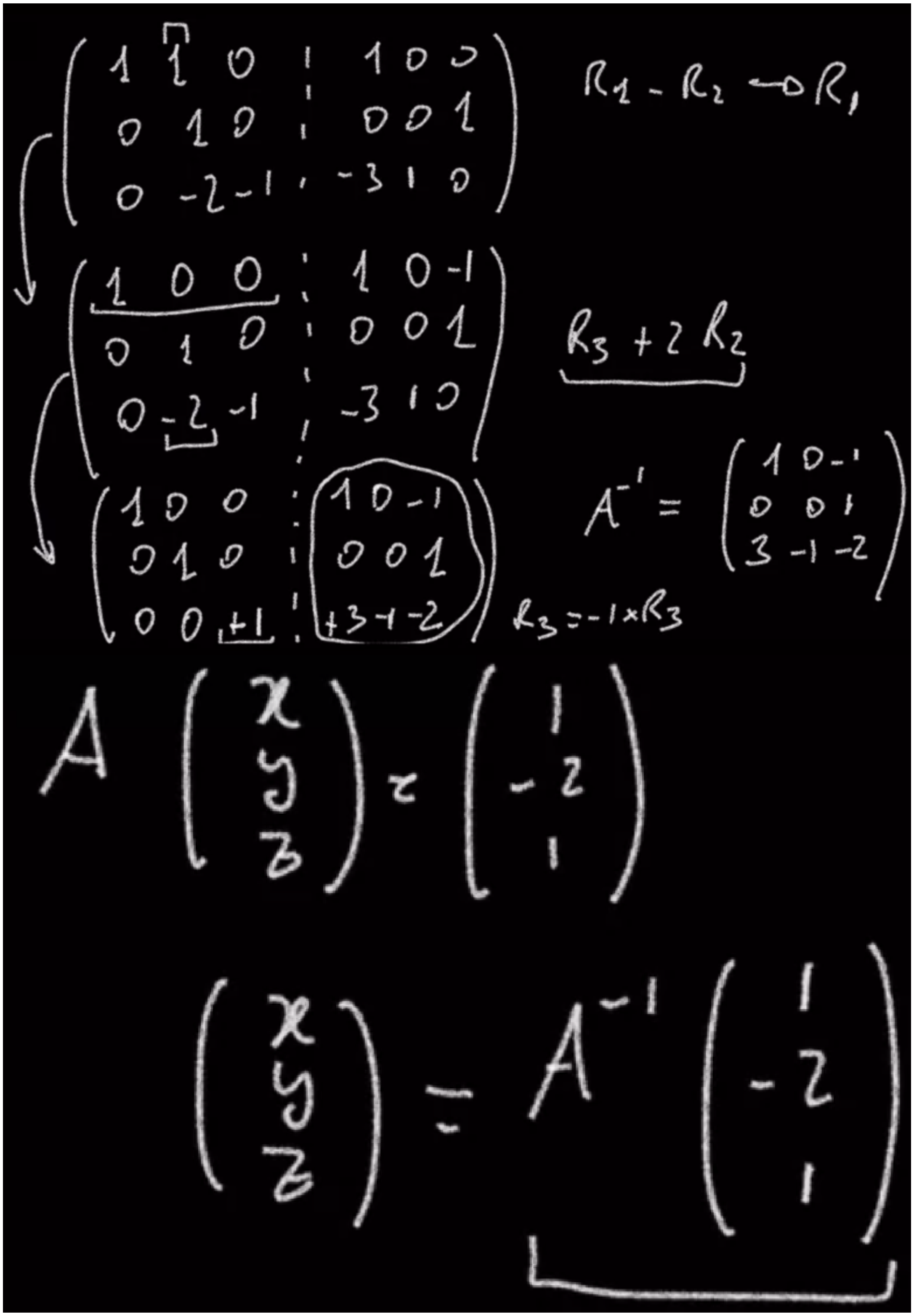
$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & -3 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right) \quad R_2 \rightleftarrows R_3$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & -1 & -3 & 1 & 0 \end{array}\right) \quad R_1 - R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & -1 & -3 & 1 & 0 \end{array}\right) \quad R_1 - R_2 \to R_1$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & -1 & -3 & 1 & 0 \end{array}\right) \quad R_3 + 2R_2$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & -3 & 1 & 2 \end{array}\right) \quad R_3 = -1 \times R_3$$



Summary

In this week, we learned about the determinant of a 2 x 2 and 3 x 3 matrix, the general method to calculate the determinant of a matrix greater than 2 x 2, the identity transformation and identity matrix, the inverse transformation and inverse matrix, and finally how to solve systems of linear equations by implementing Gauss-Jordan Elimination.