Week 18 Algebra, Vectors, and Matrices continued Reading Note 3

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Cornell Notes	**Topic:** **Algebra, Vectors, and Matrices continued**	Course: BSc Computer Science Class: Computational Mathematics[Reading] Date: July 29, 2020

Essential Question:

What are vectors and matrices?

Questions/Cues:

- What is a matrix?
- What is a square matrix?
- What is a diagonal matrix?
- What is an identity matrix?
- How is the addition and subtraction of matrices performed?
- How do we multiply or divide a matrix by a number?
- How do we multiply two matrices together?
- What is the inverse of a 2 x 2 matrix?
- What is the determinant of a 2 x 2 matrix?
- How can matrices be used to solve simultaneous equations?

Notes

A **matrix** is a set of numbers arranged in the form of a rectangle and enclosed in curved brackets. The plural of matrix is **matrices**. For example,

$$\begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 6 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 2 \\ -3 & 4 & 5 \\ \frac{1}{2} & 2 & 1 \end{pmatrix}$$

are all matrices. Each number in a matrix is known as an **element**. To refer to a particular matrix we label it with a capital letter, so that we could write

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 4 & 6 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 2 \\ -4 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 1 & 2 \\ -3 & 4 & 5 \\ \frac{1}{2} & 2 & 1 \end{pmatrix}$$

We refer to the **size** of a matrix by giving its number of rows and number of columns, in that order. So matrix A above has two rows and three columns – we say it is a 'two by three' matrix, and write this as '2×3'. Similarly B has size 3×1 and C has size 3×3. Some particular types of matrix occur so frequently that they have been given special names.

A **square** matrix has the same number of rows as columns. So

$$\begin{pmatrix} 2 & -7 \\ -1 & 6 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 2 & -1 \\ 9 & 8 & 5 \\ 6 & -7 & 2 \end{pmatrix}$$

are both square matrices.

A **diagonal** matrix is a square matrix in which all the elements are 0 except those on the diagonal from the top left to the bottom right. This diagonal is called the **leading diagonal**. On the leading diagonal the elements can take any value including zero. So

$$\begin{pmatrix} 7 & 0 \\ 0 & 9 \end{pmatrix}, \quad \begin{pmatrix} 9 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -17 \end{pmatrix}$$

are all diagonal matrices.

An **identity** matrix is a diagonal matrix, all the diagonal entries of which are equal to 1. So

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

are all identity matrices.

Matrices of the same size can be added to, or subtracted from, one another. All matrices can be multiplied or divided by a scalar, that is a number. Under certain conditions, a matrix can be multiplied by another matrix. A matrix can never be divided by another matrix.

Addition and subtraction

Two matrices having the same size can be added or subtracted by simply adding or subtracting the corresponding elements. For example,

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1+5 & 2+2 \\ 3+1 & 4+0 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 4 & 4 \end{pmatrix}$$

Similarly,

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 1 & 7 \end{pmatrix} - \begin{pmatrix} 0 & 1 & 3 & 9 \\ 7 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1-0 & 2-1 & 3-3 & 4-9 \\ 2-7 & 1-0 & 1-0 & 7-1 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & 1 & 0 & -5 \\ -5 & 1 & 1 & 6 \end{pmatrix}$$

The matrices

$$\begin{pmatrix} 1 & 2 & 9 \\ -1 & \frac{1}{2} & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 0 \\ 7 \end{pmatrix}$$

can be neither added to nor subtracted from one another because they have different sizes.

Multiplying or dividing a matrix by a number

A matrix is multiplied (divided) by a number by multiplying (dividing) each element by that number. For example,

$$4\begin{pmatrix} 1 & 2 \\ 3 & -9 \end{pmatrix} = \begin{pmatrix} 4\times 1 & 4\times 2 \\ 4\times 3 & 4\times -9 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ 12 & -36 \end{pmatrix}$$

and

$$\frac{1}{4}\begin{pmatrix} 16 \\ 8 \end{pmatrix} = \begin{pmatrix} \frac{1}{4}\times 16 \\ \frac{1}{4}\times 8 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

WORKED EXAMPLE

27.1 If

$$A = \begin{pmatrix} 5 \\ 1 \\ 9 \end{pmatrix} \text{ and } B = \begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix}$$

find (a) $A + B$ (b) $A - B$ (c) $7A$ (d) $-\frac{1}{2}B$ (e) $3A + 2B$

Solution (a)

$$A + B = \begin{pmatrix} 5 \\ 1 \\ 9 \end{pmatrix} + \begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 13 \\ 3 \\ 3 \end{pmatrix}$$

(b)

$$A - B = \begin{pmatrix} 5 \\ 1 \\ 9 \end{pmatrix} - \begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} -3 \\ -1 \\ 15 \end{pmatrix}$$

(c)

$$7A = 7\begin{pmatrix} 5 \\ 1 \\ 9 \end{pmatrix} = \begin{pmatrix} 35 \\ 7 \\ 63 \end{pmatrix}$$

(d)

$$-\frac{1}{2}B = -\frac{1}{2}\begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} -4 \\ -1 \\ 3 \end{pmatrix}$$

(e)

$$3A + 2B = 3\begin{pmatrix} 5 \\ 1 \\ 9 \end{pmatrix} + 2\begin{pmatrix} 8 \\ 2 \\ -6 \end{pmatrix} = \begin{pmatrix} 15 \\ 3 \\ 27 \end{pmatrix} + \begin{pmatrix} 16 \\ 4 \\ -12 \end{pmatrix} = \begin{pmatrix} 31 \\ 7 \\ 15 \end{pmatrix}$$

Multiplying two matrices together

Two matrices can only be multiplied together if the number of columns in the first is the same as the number of rows in the second. The product of two such matrices is a matrix that has the same number of rows as the first matrix and the same number of columns as the second. In symbols, this states that if A has size $p \times q$ and B has size $q \times s$, then AB has size $p \times s$.

27.2 Find AB where

$$A = \begin{pmatrix} 1 & 4 \\ 6 & 3 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 2 \\ 5 \end{pmatrix}$$

Solution The size of A is 2×2. The size of B is 2×1. Therefore the number of columns in the first matrix, A, equals the number of rows in the second matrix B. We can therefore find the product AB. The result will be a 2×1 matrix. Let us call it C. The working is as follows:

$$C = \begin{pmatrix} 1 & 4 \\ 6 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \times 2 + 4 \times 5 \\ 6 \times 2 + 3 \times 5 \end{pmatrix} = \begin{pmatrix} 22 \\ 27 \end{pmatrix}$$

The first row of A multiplies the first column of B to give

$$(1 \times 2) + (4 \times 5) = 2 + 20 = 22$$

This is the element in row 1, column 1 of C. The second row of A then multiplies the first column of B to give

$$(6 \times 2) + (3 \times 5) = 12 + 15 = 27$$

This is the element in row 2, column 1 of C.

27.3 Using the same A and B as the previous example, is it possible to find BA?

Solution Recall that B has size 2×1 and A has size 2×2. When written in the order BA, the number of columns in the first matrix is 1 whereas the number of rows in the second is 2. It is not therefore possible to find BA.

27.4 Find, if possible,

$$\begin{pmatrix} 1 & 4 & 9 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 9 \\ 8 & 7 \\ -7 & 3 \end{pmatrix}$$

Solution The number of columns in the first matrix is 3 and this is the same as the number of rows in the second. We can therefore perform the multiplication and the answer will have size 2×2.

$$\begin{pmatrix} 1 & 4 & 9 \\ 2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 9 \\ 8 & 7 \\ -7 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \times 1 + 4 \times 8 + 9 \times -7 & 1 \times 9 + 4 \times 7 + 9 \times 3 \\ 2 \times 1 + 0 \times 8 + 1 \times -7 & 2 \times 9 + 0 \times 7 + 1 \times 3 \end{pmatrix}$$

$$= \begin{pmatrix} -30 & 64 \\ -5 & 21 \end{pmatrix}$$

27.5 Find $(1 \quad 5)\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

Solution

$$(1 \quad 5)\begin{pmatrix} 2 \\ 1 \end{pmatrix} = (1 \times 2 + 5 \times 1) = (7)$$

Note that in this example the result is a 1×1 matrix, that is a single number.

27.6 Find $\begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Solution

$$\begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3x + 2y \\ 5x + 3y \end{pmatrix}$$

27.7 Find IX where I is the 2×2 identity matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad X = \begin{pmatrix} a \\ b \end{pmatrix}$$

Solution

$$IX = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \times a + 0 \times b \\ 0 \times a + 1 \times b \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

Note that the effect of multiplying the matrix X by the identity matrix is to leave X unchanged. This is a very important and useful property of identity matrices, which we shall require later. It should be remembered. This property of identity matrices should remind you of the fact that multiplying a number by 1 leaves the number unchanged: for example, $7 \times 1 = 7$, $1 \times -9 = -9$. An identity matrix plays the same role for matrices as the number 1 does when dealing with ordinary arithmetic.

Consider the 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

An important matrix that is related to A is known as the **inverse** of A and is given the symbol A^{-1}. Here, the superscript '-1' should not be read as a power, but is meant purely as a notation for the inverse matrix. A^{-1} can be found from the following formula:

Key point

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A^{-1} = \dfrac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

This formula states that:

- the elements on the leading diagonal are interchanged
- the remaining elements change sign
- the resulting matrix is multiplied by $\dfrac{1}{ad - bc}$

The inverse matrix has the property that:

Key point

$A A^{-1} = A^{-1} A = I$

That is, when a 2×2 matrix and its inverse are multiplied together the result is the identity matrix.

WORKED EXAMPLE

27.8 Find the inverse of the matrix

$$A = \begin{pmatrix} 6 & 5 \\ 2 & 2 \end{pmatrix}$$

and verify that $A\,A^{-1} = A^{-1}\,A = I$.

Solution Using the formula for the inverse we find

$$A^{-1} = \frac{1}{(6)(2)-(5)(2)}\begin{pmatrix} 2 & -5 \\ -2 & 6 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & -5 \\ -2 & 6 \end{pmatrix} = \begin{pmatrix} 1 & -\frac{5}{2} \\ -1 & 3 \end{pmatrix}$$

The inverse of $\begin{pmatrix} 6 & 5 \\ 2 & 2 \end{pmatrix}$

is therefore $\begin{pmatrix} 1 & -\frac{5}{2} \\ -1 & 3 \end{pmatrix}$

Evaluating $A\,A^{-1}$ we find

$$\begin{pmatrix} 6 & 5 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 1 & -\frac{5}{2} \\ -1 & 3 \end{pmatrix} = \begin{pmatrix} 6\times 1 + 5\times -1 & 6\times(-\frac{5}{2}) + 5\times 3 \\ 2\times 1 + 2\times -1 & 2\times(-\frac{5}{2}) + 2\times 3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is the 2×2 identity matrix. Also evaluating $A^{-1}A$ we find

$$\begin{pmatrix} 1 & -\frac{5}{2} \\ -1 & 3 \end{pmatrix}\begin{pmatrix} 6 & 5 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 1\times 6 + (-\frac{5}{2})\times 2 & 1\times 5 + (-\frac{5}{2})\times 2 \\ -1\times 6 + 3\times 2 & -1\times 5 + 3\times 2 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

which is the 2×2 identity matrix. We have shown that $A\,A^{-1} = A^{-1}\,A = I$.

The quantity $ad - bc$ in the formula for the inverse is known as the **determinant** of the matrix A. We often write it as $|\,A\,|$, the vertical bars indicating that we mean the determinant of A and not the matrix itself.

Key point

If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then its determinant is

$$|\,A\,| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

WORKED EXAMPLES

27.9 Find the value of the determinant

$$\begin{vmatrix} 8 & 7 \\ 9 & 2 \end{vmatrix}$$

Solution The determinant is given by $(8)(2) - (7)(9) = 16 - 63 = -47$.

27.10 Find the determinant of each of the following matrices:

(a) $C = \begin{pmatrix} 7 & 2 \\ 4 & 9 \end{pmatrix}$ (b) $D = \begin{pmatrix} 4 & -2 \\ 9 & -1 \end{pmatrix}$ (c) $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

(d) $E = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix}$

Solution

(a) $|C| = (7)(9) - (2)(4) = 63 - 8 = 55$. Note that the determinant is always a single number.

(b) $|D| = (4)(-1) - (-2)(9) = -4 + 18 = 14$.

(c) $|I| = (1)(1) - (0)(0) = 1 - 0 = 1$. It is always true that the determinant of an identity matrix is 1.

(d) $|E| = (4)(1) - (2)(2) = 4 - 4 = 0$. In this example the determinant of the matrix E is zero.

As we have seen in the previous example, on some occasions a matrix may be such that $ad - bc = 0$, that is its determinant is zero. Such a matrix is called **singular**. When a matrix is singular it cannot have an inverse. This is because it is impossible to evaluate the quantity $1/(ad - bc)$, which appears in the formula for the inverse; the quantity $1/0$ has no meaning in mathematics.

WORKED EXAMPLE

27.11 Show that the matrix

$$P = \begin{pmatrix} 6 & -2 \\ -24 & 8 \end{pmatrix}$$

has no inverse.

Solution We first find the determinant of P:

$$|P| = (6)(8) - (-2)(-24) = 48 - 48 = 0$$

The determinant is zero and so the matrix P is singular. It does not have an inverse.

Matrices can be used to solve simultaneous equations. Suppose we wish to solve the simultaneous equations

$$3x + 2y = -3$$
$$5x + 3y = -4$$

First of all we rewrite them using matrices as

$$\begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

Writing

$$A = \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -3 \\ -4 \end{pmatrix}$$

we have $AX = B$. Note that we are trying to find x and y; in other words we must find X. Now, if

$$AX = B$$

then, provided A^{-1} exists, we multiply both sides by A^{-1} to obtain

$$A^{-1}AX = A^{-1}B$$

But $A^{-1}A = I$, so this becomes

$$IX = A^{-1}B$$

However, multiplying X by the identity matrix leaves X unaltered, so we can write

$$X = A^{-1}B$$

In other words, if we multiply B by the inverse of A we will have X as required. Now, given

$$A = \begin{pmatrix} 3 & 2 \\ 5 & 3 \end{pmatrix}$$

then

$$A^{-1} = \frac{1}{(3)(3) - (2)(5)}\begin{pmatrix} 3 & -2 \\ -5 & 3 \end{pmatrix} = \frac{1}{-1}\begin{pmatrix} 3 & -2 \\ -5 & 3 \end{pmatrix} = \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}$$

Finally,

$$X = A^{-1}B = \begin{pmatrix} -3 & 2 \\ 5 & -3 \end{pmatrix}\begin{pmatrix} -3 \\ -4 \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Therefore the solution of the simultaneous equations is $x = 1$ and $y = -3$.

We note that the crucial step is

Key point

$$AX = B$$
$$X = A^{-1}B \quad \text{provided } A^{-1} \text{ exists}$$

WORKED EXAMPLE

27.12 Solve the simultaneous equations

$$x + 2y = 13$$
$$2x - 5y = 8$$

Solution First of all we rewrite the equations using matrices as

$$\begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 13 \\ 8 \end{pmatrix}$$

Writing

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -5 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 13 \\ 8 \end{pmatrix}$$

we have $AX = B$. We must now find the matrix X. If $AX = B$ then multiplying both sides by A^{-1} gives $A^{-1}AX = A^{-1}B$. But $A^{-1}A = I$, so this becomes $IX = A^{-1}B$. It follows that

$$X = A^{-1}B$$

We must multiply B by the inverse of A. The inverse of A is

$$A^{-1} = \frac{1}{(1)(-5) - (2)(2)}\begin{pmatrix} -5 & -2 \\ -2 & 1 \end{pmatrix} = \frac{1}{-9}\begin{pmatrix} -5 & -2 \\ -2 & 1 \end{pmatrix}$$

Finally,

$$X = A^{-1}B = \frac{1}{-9}\begin{pmatrix} -5 & -2 \\ -2 & 1 \end{pmatrix}\begin{pmatrix} 13 \\ 8 \end{pmatrix} = -\frac{1}{9}\begin{pmatrix} -81 \\ -18 \end{pmatrix} = \begin{pmatrix} 9 \\ 2 \end{pmatrix}$$

Therefore the solution of the simultaneous equations is $x = 9$ and $y = 2$.

Summary

In this week, we learned about what a matrix is, how addition/subtraction on matrices is performed, how to multiply two matrices together, the inverse/determinant of a 2 x 2 matrix and finally how matrices can be used to solve simultaneous equations.