Two dimensional Burgers' equation

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1 Non-linear Burgers' equation

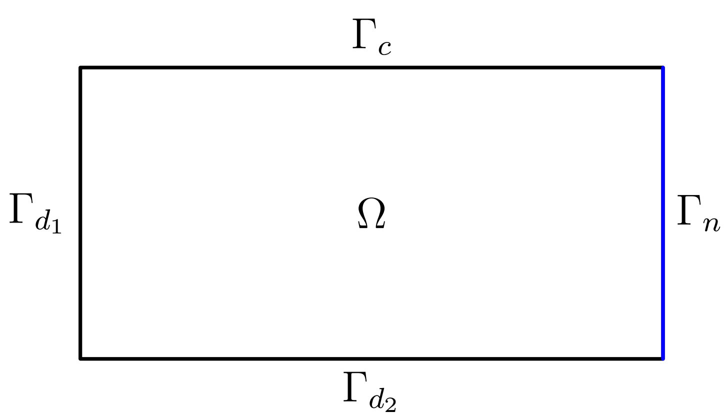

Figure 1: Problem domain

We consider the non-linear viscous Burgers' equation in two dimensions on the domain $\Omega = (0,1) \times (0,b)$

$$w_t - \nu \Delta w + w w_x = f_s \qquad \text{in} \quad \Omega \times (0,\infty)$$

with boundary conditions, as shown in figure (1)

$$\begin{aligned}
w(0,y,t) &= u_s \sin(\pi y/b) \qquad \text{on } \Gamma_{d_1} \\
w(x,0,t) &= 0 \qquad \text{on } \Gamma_{d_2} \\
w(x,b,t) &= u(x,t) \qquad \text{on } \Gamma_c \\
\nu w_x(1,y,t) &= g_s \sin(\pi y/b) \qquad \text{on } \Gamma_n
\end{aligned}$$

and initial condition

$$w(\mathbf{x},0) = w_0(\mathbf{x}) \quad \text{on } \Omega$$

Here we denote the Dirichlet boundary as

$$\Gamma_d = \Gamma_{d_1} \cup \Gamma_{d_2} \cup \Gamma_c$$

1.1 Observation

We will make a Dirichlet measurement on the strip $I = [b/6, b/2]$ along the right vertical boundary, as shown in figure (2). Thus the observation is given by

$$y_o(t) = \frac{1}{b/6} \int_{b/6}^{b/3} w(1,y,t) \mathrm{d}y$$

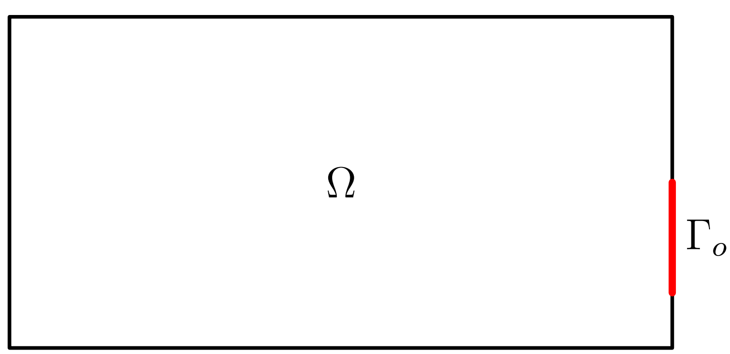

Figure 2: Observations

1.2 Stationary Burgers' equation

The stationary Burgers' equation is given by

$$-\nu\Delta w + w w_x = f_s \qquad \text{in} \quad \Omega$$

with boundary conditions

$$\begin{aligned} w(0,y,t) &= u_s \sin(\pi y/b) \qquad \text{on } \Gamma_{d_1} \\ w(x,0,t) &= 0 \qquad \text{on } \Gamma_{d_2} \\ w(x,b,t) &= 0 \qquad \text{on } \Gamma_c \\ \nu w_x(1,y,t) &= g_s \sin(\pi y/b) \qquad \text{on } \Gamma_n \end{aligned}$$

An unstable stationary solution can be obtained as

$$w_s(x,y) = \tilde{w}_s(x) \sin\left(\frac{\pi y}{b}\right)$$

where $\tilde{w}_s(x)$ is the stationary solution evaluated for the one dimensional Burgers' equation

$$\tilde{w}_s(x) = -\frac{\nu\pi}{2}(1+\epsilon)\tan\left[\frac{\pi}{4}(1+\epsilon)x + C_0\right]$$

with

$$\epsilon = 0.6 \qquad \Longrightarrow \qquad C_0 = \arctan(1/(1+\epsilon)) - \frac{\pi}{4}(1+\epsilon) = -0.69803774609235$$

We then have

$$u_s = \tilde{w}_s(0), \qquad g_s = \nu\frac{\mathrm{d}\tilde{w}_s}{\mathrm{d}x}(1)$$

and f_s is given by the stationary Burgers' equation.

2 Nonlinear system without control

We set $z = w - w_s$. The equation satisfied by z is

$$\frac{\partial z}{\partial t} + w_s\frac{\partial z}{\partial x} + z\frac{\partial w_s}{\partial x} + z\frac{\partial z}{\partial x} = \nu\Delta z \qquad \text{in} \quad \Omega\times(0,\infty)$$

$$z = 0 \quad \text{on } \Gamma_d$$

$$\nu\frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma_n$$

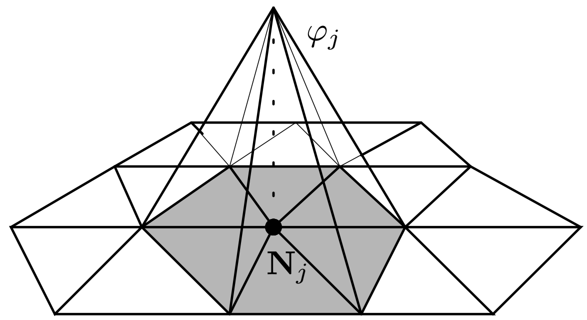

Figure 3: Piecewise affine basis functions

$$z(\mathbf{x},0) = w_0(\mathbf{x}) - w_s(\mathbf{x}) \quad \text{on } \Omega$$

We wish to find $z \in L^2(0,T; H^1_{\Gamma_d}(\Omega))$ such that

$$\int_\Omega z_t \phi \mathrm{d}\mathbf{x} + \nu \int_\Omega \nabla z \cdot \nabla \phi \mathrm{d}\mathbf{x} + \int_\Omega w_s \frac{\partial z}{\partial x} \phi \mathrm{d}\mathbf{x} + \int_\Omega z \frac{\partial w_s}{\partial x} \phi \mathrm{d}\mathbf{x} + \int_\Omega z \frac{\partial z}{\partial x} \phi \mathrm{d}\mathbf{x} = 0$$

for all $\phi \in H^1_{\Gamma_d}(\Omega)$

2.1 FEM approximation

Consider a division of Ω into disjoint triangles. We define the following sets of vertices

$$\begin{aligned} N_c &= \text{vertices on } \Gamma_c \\ N_f &= \text{interior vertices and those on } \Gamma_n \text{ (unknown degrees of freedom)} \end{aligned}$$

For each vertex i, define the piecewise affine functions $\phi_i(x,y)$, see figure (3), with the property that

$$\phi_i(x_j, y_j) = \delta_{ij}$$

Thus, the finite element solution is of the form

$$z(x,y,t) = \sum_{j \in N_f} z_j(t) \phi_j(x,y)$$

This function satisfies the Dirichlet boundary conditions

$$z(0,y,t) = z(x,0,t) = z(x,b,t) = 0$$

The Galerkin method is given by

$$\int_\Omega z_t \phi_i \mathrm{d}\mathbf{x} + \nu \int_\Omega \nabla z \cdot \nabla \phi_i \mathrm{d}\mathbf{x} + \int_\Omega w_s \frac{\partial z}{\partial x} \phi_i \mathrm{d}\mathbf{x} + \int_\Omega z \frac{\partial w_s}{\partial x} \phi_i \mathrm{d}\mathbf{x} + \int_\Omega z \frac{\partial z}{\partial x} \phi_i \mathrm{d}\mathbf{x} = 0, \quad \forall i \in N_f$$

i.e.,

$$\begin{aligned}
\sum_{j \in N_f} z_j'(t) \int_\Omega \phi_i \phi_j &= -\nu \sum_{j \in N_f} z_j(t) \int_\Omega \nabla \phi_i \cdot \nabla \phi_j - \sum_{j \in N_f} z_j \int_\Omega w_s \frac{\partial \phi_j}{\partial x} \phi_i \\
&\quad - \sum_{j \in N_f} z_j \int_\Omega \phi_j \frac{\partial w_s}{\partial x} \phi_i - \int_\Omega z \frac{\partial z}{\partial x} \phi_i \\
&= \sum_{j \in N_f} A_{ij} z_j - \int_\Omega z \frac{\partial z}{\partial x} \phi_i
\end{aligned}$$

where

$$A_{ij} = -\nu \int_\Omega \nabla \phi_i \cdot \nabla \phi_j - \int_\Omega w_s \frac{\partial \phi_j}{\partial x} \phi_i - \int_\Omega \phi_j \frac{\partial w_s}{\partial x} \phi_i$$

We can evaluate the integrals as follows.

$$\begin{aligned}
\int_\Omega w_s \frac{\partial \phi_j}{\partial x} \phi_i &= \sum_{K: i,j \in K} \frac{\partial \phi_j}{\partial x}(K) \int_K w_s \phi_i \\
&\approx \sum_{K: i,j \in K} \frac{\partial \phi_j}{\partial x}(K) w_s(K) \int_K \phi_i \\
&= \sum_{K: i,j \in K} \frac{\partial \phi_j}{\partial x}(K) w_s(K) \frac{|K|}{3}
\end{aligned}$$

$$\begin{aligned}
\int_\Omega \phi_j \frac{\partial w_s}{\partial x} \phi_i &= \sum_{K: i,j \in K} \int_K \phi_j \frac{\partial w_s}{\partial x} \phi_i \\
&\approx \sum_{K: i,j \in K} \frac{\partial w_s}{\partial x}(K) \int_K \phi_j \phi_i
\end{aligned}$$

$$\begin{aligned}
\int_\Omega z \frac{\partial z}{\partial x} \phi_i &= \sum_{K: i \in K} \int_K z \frac{\partial z}{\partial x} \phi_i \\
&= \sum_{K: i \in K} \frac{\partial z}{\partial x}(K) \int_K z \phi_i \\
&= \sum_{K: i \in K} \frac{\partial z}{\partial x}(K) \sum_{j \in K} z_j \int_K \phi_j \phi_i
\end{aligned}$$

Thus we have a non-linear system of first order ODEs in $\mathbf{z}$

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}\mathbf{z} + N(\mathbf{z})$$

3 Linearized Burgers' equation

3.1 Feedback control

$$\begin{gathered}
\frac{\partial z}{\partial t} + w_s\frac{\partial z}{\partial x} + z\frac{\partial w_s}{\partial x} = \nu\Delta z \qquad \text{in} \quad \Omega\times(0,\infty) \\
z = 0 \quad \text{ on } \Gamma_d\backslash\Gamma_c, \qquad z = u \quad \text{ on } \Gamma_c \\
\nu\frac{\partial z}{\partial n} = 0 \quad \text{ on } \Gamma_n \\
z(\mathbf{x},0) = w_0(\mathbf{x}) - w_s(\mathbf{x}) \quad \text{ on } \Omega
\end{gathered}$$

The finite element solution is of the form

$$z(x,y,t) = \sum_{j\in N_f} z_j(t)\phi_j(x,y) + \sum_{j\in N_c} u_j(t)\phi_j(x,y), \qquad z(x_i,y_i,t) = z_i(t), \quad \forall i\in N_f$$

This function already satisfies the Dirichlet boundary conditions

$$z(x,0,t) = z(0,y,t) = 0, \qquad z(x,b,t) = \sum_{j\in N_c} u_j(t)\phi_j(x,b)$$

The Galerkin method is given by

$$\int_\Omega \left(\frac{\partial z}{\partial t} + w_s\frac{\partial z}{\partial x} + z\frac{\partial w_s}{\partial x}\right)\phi_i = -\nu\int_\Omega \nabla z\cdot\nabla\phi_i, \qquad i\in N_f$$

Ignoring the $\frac{\mathrm{d}u_j}{\mathrm{d}t}$ term gives us

$$\sum_{j\in N_f} z_j'(t)\int_\Omega \phi_i\phi_j = \sum_{j\in N_f} A_{ij}z_j + \sum_{j\in N_c} A_{ij}u_j, \qquad i\in N_f$$

We will take the control to be of the form

$$u(x,t) = v(t)\sin(\pi x) \qquad \Longrightarrow \qquad u_j(t) = v(t)\sin(\pi x_j)$$

This is a one dimensional control which is compatible with the other boundary condition on Γ_{d_1}

$$u(0,t) = 0$$

Then we obtain

$$\sum_{j\in N_f} z_j'(t)\int_\Omega \phi_i\phi_j = \sum_{j\in N_f} A_{ij}z_j + \left(\sum_{j\in N_c} A_{ij}\sin(\pi x_j)\right)v, \qquad i\in N_f$$

This can be written in matrix form as

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}\mathbf{z} + \mathbf{B}v$$

$v(t)$ is obtained in terms of the feedback matix $\mathbf{K}$

$$v(t) = -\mathbf{K}\mathbf{z}$$

The computation of $\mathbf{K}$ is explained in the last section.

3.2 Partial information with noise

Consider the linearized model with noise in the model

$$\frac{\partial z}{\partial t} + w_s\frac{\partial z}{\partial x} + z\frac{\partial w_s}{\partial x} = \nu\Delta z + \eta \qquad \text{in} \quad \Omega\times(0,\infty)$$

$$z = 0 \quad \text{on } \Gamma_d\backslash\Gamma_c, \qquad z = u(t) \quad \text{on } \Gamma_c$$

$$\nu\frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma_n$$

$$z(\mathbf{x},0) = w_0(\mathbf{x}) - w_s(\mathbf{x}) \quad \text{on } \Omega$$

Assume that we have access to partial information corrupted by noise

$$y_o = Hz + \mu$$

Using the FEM, the observation can be written as

$$\mathbf{y} = \mathbf{H}\mathbf{z} + \boldsymbol{\mu}$$

3.3 Estimation problem

$$
\begin{gathered}
\frac{\partial z_e}{\partial t} + w_s \frac{\partial z_e}{\partial x} + z_e \frac{\partial w_s}{\partial x} = \nu \Delta z_e + L(y - H z_e) \qquad \text{in} \quad \Omega \times (0, \infty) \\
z_e = 0 \quad \text{on } \Gamma_d \backslash \Gamma_c, \qquad z_e = u(t) \quad \text{on } \Gamma_c \\
\nu \frac{\partial z_e}{\partial n} = 0 \quad \text{on } \Gamma_n \\
z_e(\mathbf{x}, 0) = z_0(\mathbf{x}) \quad \text{on } \Omega
\end{gathered}
$$

In the FEM setup, we have

$$
\mathbf{M} \frac{\mathrm{d}\mathbf{z}_e}{\mathrm{d}t} = \mathbf{A}\mathbf{z}_e + \mathbf{B}v + \mathbf{L}(\mathbf{y} - \mathbf{H}\mathbf{z}_e)
$$

The computation of $\mathbf{L}$ is explained in the last section.

3.4 Coupled linear system

The feedback is based on estimated solution $v = -\mathbf{K}\mathbf{z}_e$ which leads to the following coupled system

$$
\begin{aligned}
\mathbf{M} \frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} &= \mathbf{A}\mathbf{z} - \mathbf{B}\mathbf{K}\mathbf{z}_e + \boldsymbol{\eta} \\
\mathbf{M} \frac{\mathrm{d}\mathbf{z}_e}{\mathrm{d}t} &= \mathbf{L}\mathbf{H}\mathbf{z} + (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{H})\mathbf{z}_e + \mathbf{L}\boldsymbol{\mu}
\end{aligned}
$$

or in matrix form

$$
\begin{bmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}_e \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{H} & \mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}_e \end{bmatrix} + \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{L} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\mu} \end{bmatrix}
$$

The initial condition is given by

$$
\mathbf{z}(0) = \mathbf{z}_0, \qquad \mathbf{z}_e(0) = 0
$$

3.5 Excercises

1. Compute eigenvalues of $(\mathbf{A}, \mathbf{M})$ and check that the problem is unstable.
2. Check stabilizability and detectability using Hautus criterion.
3. Using the method given in the last section,
   (a) Compute feedback gain $\mathbf{K}$
   (b) Compute filtering gain $\mathbf{L}$
4. Solve the coupled linear system as in the 2-d heat equation case.

4 Non-linear system with control

$$\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{\partial w_s}{\partial x} + z \frac{\partial z}{\partial x} = \nu \Delta z \qquad \text{in} \quad \Omega \times (0, \infty)$$
$$z = 0 \quad \text{on } \Gamma_d \backslash \Gamma_c, \qquad z = u(t) \quad \text{on } \Gamma_c$$
$$\nu \frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma_n$$
$$z(\mathbf{x}, 0) = w_0(\mathbf{x}) - w_s(\mathbf{x}) \quad \text{on } \Omega$$

The Galerkin method leads the following set of ODE

$$\mathbf{M} \frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}\mathbf{z} + \mathbf{B}v + N(\mathbf{z}, v)$$

This model can be coupled with the linear estimator and the control is computed using the estimated solution, $v = -\mathbf{K}\mathbf{z}_e$.

5 Control and estimation based on the unstable components

Let λ_i be the eigenvalues and $\mathbf{V}_i, \mathbf{W}_i$ be the eigenvectors of the pair $(\mathbf{A}, \mathbf{M})$ and $(\mathbf{A}^\top, \mathbf{M}^\top)$ respectively.

$$\mathbf{A}\mathbf{V}_i = \lambda_i \mathbf{M}\mathbf{V}_i, \qquad \mathbf{A}^\top \mathbf{W}_i = \lambda_i \mathbf{M}^\top \mathbf{W}_i, \quad \forall i = 1, ..., N$$

Assume that the eigenvectors are normalized with respect to $\mathbf{M}$

$$\mathbf{W}_i^\top \mathbf{M}\mathbf{V}_j = \delta_{ij}, \qquad \mathbf{W}_i^\top \mathbf{A}\mathbf{V}_j = \lambda_i \delta_{ij} \tag{1}$$

Let there be N_u unstable eigenvalues

$$Real(\lambda_N) \le \ldots \le Real(\lambda_{N_u+1}) < 0 < Real(\lambda_{N_u}) \le \ldots \le Real(\lambda_1)$$

We will use the following notations

$$\begin{aligned}
\Lambda &= \text{Diagonal matrix of eigenvalues} \\
\Lambda_u &= \text{Diagonal matrix of unstable eigenvalues} \\
\mathbf{V} &= \text{Matrix with the right eigenvector as columns} \\
\mathbf{V}_u &= \text{Matrix with the unstable right eigenvector as columns} \\
\mathbf{W} &= \text{Matrix with the left eigenvector as columns} \\
\mathbf{W}_u &= \text{Matrix with the unstable left eigenvector as columns}
\end{aligned}$$

$$\mathbf{V} = [\mathbf{V}_1, \ldots, \mathbf{V}_N], \qquad \mathbf{V}_u = [\mathbf{V}_1, \ldots, \mathbf{V}_{N_u}], \qquad \mathbf{W} = [\mathbf{W}_1, \ldots, \mathbf{W}_N], \qquad \mathbf{W}_u = [\mathbf{W}_1, \ldots, \mathbf{W}_{N_u}]$$

Consider the variable change

$$\mathbf{z} = \mathbf{V}\boldsymbol{\zeta}$$

The system in terms of $\boldsymbol{\zeta}$ is written as

$$\mathbf{M}\mathbf{V}\frac{\mathrm{d}\boldsymbol{\zeta}}{\mathrm{d}t} = \mathbf{A}\mathbf{V}\boldsymbol{\zeta} + \mathbf{B}u$$

Premultiplying by $\mathbf{W}^\top$ gives us

$$\frac{\mathrm{d}\boldsymbol{\zeta}}{\mathrm{d}t} = \Lambda\boldsymbol{\zeta} + \mathbb{B}u$$

where

$$\mathbb{B} = \mathbf{W}^\top\mathbf{B}$$

Projecting the system onto the unstable subspace gives

$$\frac{\mathrm{d}\boldsymbol{\zeta}_u}{\mathrm{d}t} = \Lambda_u\boldsymbol{\zeta}_u + \mathbb{B}_u u$$

where

$$\mathbb{B}_u = \mathbf{W}_u^\top\mathbf{B}$$

5.1 Control

We find the feedback matrix for this reduced system, by solving the Bernoulli equation

$$\mathbb{P}_u\Lambda_u + \Lambda_u^\top\mathbb{P}_u - \mathbb{P}_u\mathbb{B}_u\mathbb{B}_u^\top\mathbb{P}_u = 0 \tag{2}$$

$$\Lambda_u - \mathbb{B}_u\mathbb{B}_u^\top\mathbb{P}_u \text{ is stable}$$

The corresponding matrix $\mathbf{P} \in \mathcal{L}(\mathbb{R}^N)$ such that $(\mathbf{M}, \mathbf{A} - \mathbf{B}\mathbf{B}^\top\mathbf{P}\mathbf{M})$ is stable is given by

$$\mathbf{P} = \mathbf{W}_u\mathbb{P}_u\mathbf{V}_u^\top$$

The feedback matrix $\mathbf{K}$ is given by

$$\mathbf{K} = \mathbf{B}^\top\mathbf{P}\mathbf{M}$$

5.2 Estimation

Define the operator $\mathbb{H}$ and $\mathbb{H}_u$ by

$$\mathbb{H}\boldsymbol{\zeta} = \mathbf{H}\mathbf{z}, \qquad \mathbb{H}_u = \mathbf{H}\mathbf{V}_u$$

The filtering gain $\mathbb{P}_e$ for the reduced projected system is obtained as the solution of the ARE

$$\mathbb{P}_e\Lambda_u^\top + \Lambda_u\mathbb{P}_e - \mathbb{P}_e\mathbb{H}_u^\top\mathbf{R}_{\boldsymbol{\mu}}^{-1}\mathbb{H}_u\mathbb{P}_e + \mathbb{Q}_\eta = 0$$
$$\Lambda_u - \mathbb{P}_e\mathbb{H}_u^\top\mathbf{R}_{\boldsymbol{\mu}}^{-1}\mathbb{H}_u \text{ is stable}$$

where $\mathbb{Q}_\eta = \mathbf{W}_u^\top\mathbf{R}_{\boldsymbol{\eta}}\mathbf{V}_u$. The corresponding $\mathbf{P}_e$ such that $(\mathbf{M}, \mathbf{A} - \mathbf{M}\mathbf{P}_e\mathbf{H}^\top\mathbf{R}_{\boldsymbol{\mu}}^{-1}\mathbf{H})$ is stable is given by

$$\mathbf{P}_e = \mathbf{W}_u\mathbb{P}_e\mathbf{V}_u^\top$$

Thus the filtering gain matrix $\mathbf{L}$ is

$$\mathbf{L} = \mathbf{M}\mathbf{P}_e\mathbf{H}^\top\mathbf{R}_{\boldsymbol{\mu}}^{-1}$$

Matlab tips To implement the above approach we need to compute only the unstable eigenvalues and eigenvectors of $(\mathbf{A}, \mathbf{M})$ and $(\mathbf{A}^\top, \mathbf{M}^\top)$. They must also satisfy the orthonormality condition as given in equation (1). In matlab, this condition may not be automatically satisfied and you need to scale the eigenvectors appropriately.