One dimensional Burgers' equation

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1 Non-linear Burgers' equation

We consider the non-linear viscous Burgers' equation

$$w_t - \nu w_{xx} + w w_x = f_s \qquad \text{in} \quad (0,1) \times (0,\infty) \tag{1}$$

with boundary conditions

$$w(0,t) = u_s, \qquad w_x(1,t) = g_s \tag{2}$$

and initial condition

$$w(x,0) = w_0(x) \tag{3}$$

1.1 Stationary Burgers' equation

The stationary Burgers' equation is given by

$$-\nu w_{xx} + w w_x = f_s \qquad \text{in} \quad (0,1) \tag{4}$$

with boundary conditions

$$w(0) = u_s, \qquad w_x(1) = g_s \tag{5}$$

An unstable stationary solution can be obtained as

$$w_s = -\frac{\nu\pi}{2}(1+\epsilon)\tan\left(\frac{\pi}{4}(1+\epsilon) + C_0\right) \tag{6}$$

with the conditions

$$0 < \epsilon < 1, \qquad -\frac{\pi}{4} < C_0 < 0, \qquad (1+\epsilon)\tan\left(\frac{\pi}{4}(1+\epsilon) + C_0\right) = 1$$

We choose $\epsilon = 0.6$ and evaluate the corresponding value for C_0, which turns out to be $C_0 \approx -0.698$. We shall use this in the following sections.

2 Nonlinear system without control

We set $z = w - w_s$. The equation satisfied by z is

$$\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{\mathrm{d}w_s}{\mathrm{d}x} + z \frac{\mathrm{d}z}{\mathrm{d}x} = \nu \frac{\partial^2 z}{\partial x^2} \qquad \text{in} \quad (0,1) \times (0,\infty) \tag{7}$$

with boundary conditions

$$z(0,t) = 0, \qquad z_x(1,t) = 0 \tag{8}$$

and initial conditions

$$z(x,0) = w_0(x) - w_s(x) \tag{9}$$

We multiply by a test function ϕ with $\phi(0) = 0$ to obtain the weak formulation

$$\int_0^1 z_t \phi \mathrm{d}x + \nu \int_0^1 \frac{\partial z}{\partial x} \frac{\mathrm{d}\phi}{\mathrm{d}x} \mathrm{d}x + \int_0^1 w_s \frac{\partial z}{\partial x} \phi \mathrm{d}x + \int_0^1 z \frac{\partial w_s}{\partial x} \phi \mathrm{d}x + \int_0^1 z \frac{\mathrm{d}z}{\mathrm{d}x} \phi \mathrm{d}x = 0$$

Divide domain into N elements with vertices

$$0 = x_0 < x_1 < \ldots < x_N = 1, \qquad x_i = ih, \qquad h = \frac{1}{N}$$

The finite element solution can be written as

$$z(x,t) = \sum_{j=1}^{N} z_j(t) \phi_j(x)$$

while the stationary solution expressed as

$$w_s(x) = \sum_{j=1}^{N} w_s^j \phi_j(x) + u_s \phi_0(x)$$

where $w_s^j = w_s(x_j)$. Here the $\{\phi_j\}$ are the usual P_1 basis functions. The approximate weak formulation is

$$\begin{aligned} \int_0^1 z_t \phi_i \mathrm{d}x &= -\nu \int_0^1 \frac{\partial z}{\partial x} \frac{\mathrm{d}\phi_i}{\mathrm{d}x} \mathrm{d}x - \int_0^1 w_s \frac{\partial z}{\partial x} \phi_i \mathrm{d}x - \int_0^1 z \frac{\partial w_s}{\partial x} \phi_i \mathrm{d}x \\ &\quad - \int_0^1 z \frac{\mathrm{d}z}{\mathrm{d}x} \phi_i \mathrm{d}x, \qquad \forall i = 1, 2, ..., N \end{aligned}$$

or

$$\begin{aligned}\sum_{j=1}^{N} z_j'(t)\int_0^1 \phi_i\phi_j \mathrm{d}x \;=\;& -\sum_{j=1}^{N}\sum_{k=1}^{N} z_j(t) z_k(t)\int_0^1 \phi_j\phi_k'\phi_i \mathrm{d}x\\ & -\sum_{j=1}^{N} z_j(t)\left[\nu\int_0^1 \phi_i'\phi_j' \mathrm{d}x + u_s\int_0^1 \phi_i(\phi_0\phi_j)'\mathrm{d}x\right]\\ & -\sum_{j=1}^{N} z_j(t)\left[\sum_{k=1}^{N} w_s^k(t)\left(\int_0^1 \phi_j\phi_k'\phi_i \mathrm{d}x + \int_0^1 \phi_k\phi_j'\phi_i \mathrm{d}x\right)\right]\end{aligned}$$

We use the following notations:

- $\mathbf{z} = [z_1, z_2, ..., z_N]^\top$
- $\mathbf{w}_s = [w_s^1, w_s^2, ..., w_s^N]^\top$
- $\mathbf{M} \in \mathbb{R}^{N\times N}$ with $\mathbf{M}(i,j) = \int_0^1 \phi_i\phi_j \mathrm{d}x$
- $\mathbf{A}1 \in \mathbb{R}^{N\times N}$ with $\mathbf{A}1(i,j) = \int_0^1 \phi_i'\phi_j' \mathrm{d}x$
- $\mathbf{A}2 \in \mathbb{R}^{N\times N}$ with $\mathbf{A}2(i,j) = \int_0^1 \phi_i(\phi_0\phi_j)' \mathrm{d}x$
- $\mathbf{D}1 \in \mathbb{R}^{N\times N\times N}$ with $\mathbf{D}1(i,j,k) = \int_0^1 \phi_i\phi_j'\phi_k \mathrm{d}x$

Thus we have a non-linear system of first order ODEs in $\mathbf{z}$

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{RHS}$$

where

$$\mathbf{RHS}(i) = -\mathbf{z}^\top\mathbf{D}1(:,:,i)\mathbf{z} - \left[\nu\mathbf{A}1(i,:) + u_s\mathbf{A}2(i,:) + \mathbf{w}_s^\top(\mathbf{D}1(:,:,i) + \mathbf{D}1(:,:,i)^\top)\right]\mathbf{z}$$

This is implemented and solved in `nlp.m`

Excercises

1. Run program `nlp.m`
2. Notice that for zero initial condition, i.e, `delta = 0`, the solution will be stationary. This corresponds to the stationary solution for the original Burgers's equation.

3. Take different initial conditions by varying `delta`, and convince yourself that the zero solution is unstable. Choose values for `delta` to observe the following behaviors
    - The solution blows up
    - The solution converges to another steady state

3 Linearized Burgers' equation

3.1 Feedback control

Consider the linearized equation

$$\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{\mathrm{d}w_s}{\mathrm{d}x} = \nu \frac{\partial^2 z}{\partial x^2} \tag{10}$$

$$z(0,t) = u(t), \qquad \frac{\partial z}{\partial x}(1,t) = 0 \tag{11}$$

$$z(x,0) = w_0(x) - w_s(x) \tag{12}$$

The weak formulation is given by

$$\int_0^1 \left(\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{\mathrm{d}w_s}{\mathrm{d}x} \right) \phi \mathrm{d}x = -\nu \int_0^1 \frac{\partial z}{\partial x} \frac{\partial \phi}{\partial x} \mathrm{d}x$$

We look for a finite element solution of the form

$$z(x,t) = \sum_{j=1}^{N} z_j(t)\phi_j(x) + u(t)\phi_0(x)$$

The approximate weak formulation is

$$\int_0^1 \left(\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{\mathrm{d}w_s}{\mathrm{d}x} \right) \phi_i \mathrm{d}x = -\nu \int_0^1 \frac{\partial z}{\partial x} \frac{\partial \phi_i}{\partial x} \mathrm{d}x, \qquad i = 1, 2, \ldots, N$$

Ignoring the $\frac{\mathrm{d}u}{\mathrm{d}t}$ term gives us

$$\begin{aligned} \sum_{j=1}^{N} \frac{\mathrm{d}z_j}{\mathrm{d}t} \int_0^1 \phi_i \phi_j \mathrm{d}x &= \sum_{j=1}^{N} z_j \left[-\nu \int_0^1 \phi_i' \phi_j' \mathrm{d}x - \int_0^1 w_s \phi_i \phi_j' \mathrm{d}x - \int_0^1 \phi_i \phi_j \frac{\mathrm{d}w_s}{\mathrm{d}x} \right] \\ &\quad + u \left[- \int_0^1 w_s \phi_i \phi_0' \mathrm{d}x - \int_0^1 \phi_i \phi_0 \frac{\mathrm{d}w_s}{\mathrm{d}x} \mathrm{d}x - \nu \int_0^1 \phi_i' \phi_0' \mathrm{d}x \right] \end{aligned}$$

Using the expression for w_s mentioned above, we get

$$\begin{aligned}\sum_{j=1}^{N}\frac{\mathrm{d}z_j}{\mathrm{d}t}\int_0^1\phi_i\phi_j\mathrm{d}x &= \sum_{j=1}^{N}z_j\left[-\nu\int_0^1\phi_i'\phi_j'\mathrm{d}x - u_s\int_0^1\phi_i(\phi_0\phi_j)'\mathrm{d}x\right]\\ &+\sum_{j=1}^{N}z_j\left[-\sum_{k=1}^{N}w_s^k\int_0^1\phi_j\phi_k'\phi_i\mathrm{d}x-\sum_{k=1}^{N}w_s^k\int_0^1\phi_k\phi_j'\phi_i\mathrm{d}x\right]\\ &+u\left[-2u_s\int_0^1\phi_i\phi_0\phi_0'\mathrm{d}x-\sum_{k=1}^{N}w_s^k\int_0^1\phi_i(\phi_0\phi_k)'\mathrm{d}x-\nu\int_0^1\phi_i'\phi_0'\mathrm{d}x\right]\end{aligned}$$

We use the following notations:

- $\mathbf{z} = [z_1, z_2, ..., z_N]^\top$
- $\mathbf{w}_s = [w_s^1, w_s^2, ..., w_s^N]^\top$
- $\mathbf{M} \in \mathbb{R}^{N\times N}$ with $\mathbf{M}(i,j) = \int_0^1 \phi_i\phi_j\mathrm{d}x$
- $\mathbf{A}1 \in \mathbb{R}^{N\times N}$ with $\mathbf{A}1(i,j) = \int_0^1 \phi_i'\phi_j'\mathrm{d}x$
- $\mathbf{A}2 \in \mathbb{R}^{N\times N}$ with $\mathbf{A}2(i,j) = \int_0^1 \phi_i(\phi_0\phi_j)'\mathrm{d}x$
- $\mathbf{D}1 \in \mathbb{R}^{N\times N\times N}$ with $\mathbf{D}1(i,j,k) = \int_0^1 \phi_i\phi_j'\phi_k\mathrm{d}x$
- $\mathbf{A}(i,:) = -\nu\mathbf{A}1(i,:) - u_s\mathbf{A}2(i,:) - \mathbf{w}_s^\top\left[\mathbf{D}1(:,:,i) + \mathbf{D}1(:,:,i)^\top\right], \quad \forall i = 1,2,...,N$
- $\mathbf{d}1 \in \mathbb{R}^N$ with $\mathbf{d}1(i) = \int_0^1 \phi_i\phi_0\phi_0'\mathrm{d}x$
- $\mathbf{d}2 \in \mathbb{R}^N$ with $\mathbf{d}2(i) = \int_0^1 \phi_i'\phi_0'\mathrm{d}x$
- $\mathbf{B} = -2u_s\mathbf{d}1 - \mathbf{A}2\cdot\mathbf{w}_s - \nu\mathbf{d}2$

This can be written in matrix form as

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

The control $u(t)$ is obtained in terms of the feedback matix $\mathbf{K} = [K_1, K_2, ..., K_N]$

$$u(t) = -\sum_{j=1}^{N} K_j z_j(t)$$

The feedback gain $\mathbf{K}$ is given by

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^\top\mathbf{P}\mathbf{M}$$

where $\mathbf{P}$ is solution of algebraic Riccati equation (ARE)

$$\mathbf{A}^\top\mathbf{P}\mathbf{M} + \mathbf{M}^\top\mathbf{P}\mathbf{A} - \mathbf{M}^\top\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{P}\mathbf{M} + \mathbf{Q} = 0$$

$$(\mathbf{M}, \mathbf{A} - \mathbf{B}\mathbf{K}) \text{ is stable}$$

3.2 Partial information with noise

Consider the linearized model with noise in the model

$$\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z\frac{\mathrm{d}w_s}{\mathrm{d}x} = \nu \frac{\partial^2 z}{\partial x^2} + \eta \tag{13}$$

$$z(0,t) = u(t), \qquad \frac{\partial z}{\partial x}(1,t) = 0 \tag{14}$$

$$z(x,0) = w_0(x) - w_s(x) \tag{15}$$

Assume that we have access to partial information corrupted by noise

$$y = Hz + \mu$$

H being a suitable observation operator. We shall consider the case where H is given by

$$Hz(t) = z(1,t)$$

In the FEM setup, we get

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}\mathbf{z} + \mathbf{B}u + \boldsymbol{\eta}$$
$$\mathbf{y} = \mathbf{H}\mathbf{z} + \boldsymbol{\mu}$$

where the observation operator is approximated as

$$\mathbf{H} = (0, 0, ..., 0, 1)^\top \in \mathbb{R}^{N-1}$$

3.3 Estimation problem

Consider the

$$\frac{\partial z_e}{\partial t} + w_s \frac{\partial z_e}{\partial x} + z_e \frac{\mathrm{d} w_s}{\mathrm{d} x} = \nu \frac{\partial^2 z_e}{\partial x^2} + L(Hz - Hz_e) \tag{16}$$

$$z_e(0,t) = u(t), \qquad \frac{\partial z_e}{\partial x}(1,t) = 0 \tag{17}$$

$$z_e(x,0) = 0 \tag{18}$$

In the FEM setup, we have

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}_e}{\mathrm{d}t} = \mathbf{A}\mathbf{z}_e + \mathbf{B}u + \mathbf{L}(\mathbf{y} - \mathbf{H}\mathbf{z}_e)$$

The filtering gain $\mathbf{L}$ is given by

$$\mathbf{L} = -\mathbf{M}\mathbf{P}_e\mathbf{H}^\top \mathbf{R}_{\boldsymbol{\mu}}^{-1}$$

where $\mathbf{P}_e$ is solution of

$$\mathbf{A}\mathbf{P}_e\mathbf{M} + \mathbf{M}\mathbf{P}_e\mathbf{A}^\top - \mathbf{M}\mathbf{P}_e\mathbf{H}^\top\mathbf{R}_{\boldsymbol{\mu}}^{-1}\mathbf{H}\mathbf{P}_e\mathbf{M} + \mathbf{R}_{\boldsymbol{\eta}} = 0$$

$$(\mathbf{M}, \mathbf{A} - \mathbf{L}\mathbf{H}) \text{ is stable}$$

3.4 Coupled linear system

The feedback is based on estimated solution $u = -\mathbf{K}\mathbf{z}_e$

$$\begin{aligned}
\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} &= \mathbf{A}\mathbf{z} - \mathbf{B}\mathbf{K}\mathbf{z}_e + \boldsymbol{\eta} \\
\mathbf{M}\frac{\mathrm{d}\mathbf{z}_e}{\mathrm{d}t} &= \mathbf{L}\mathbf{H}\mathbf{z} + (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{H})\mathbf{z}_e + \mathbf{L}\boldsymbol{\mu}
\end{aligned}$$

or in matrix form

$$\begin{bmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}_e \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{H} & \mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}_e \end{bmatrix} + \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{L} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\mu} \end{bmatrix}$$

The initial condition is given by

$$\mathbf{z}(0) = \mathbf{z}_0, \qquad \mathbf{z}_e(0) = 0$$

This is implemented in program `lp_est.m`

Excercises

1. Run program `lp_est.m`
2. Check stabilizabilty of $(\mathbf{A}, \mathbf{B})$ using the Hautus criterion.
3. Set $(\mathbf{Q} = 0)$ which corresponds to minimal norm control. Run the code and observe the solution and control.
4. Set $(\mathbf{Q} = I)$ and run the code. How has the solution and control beaviour changed?
5. Vary $\mathbf{R}$ in the range $(0.01, 10)$ and observe how the feedback gain $\mathbf{K}$ varies.
6. How does the solution and control change when ν is decreased?
7. For $\mathbf{Q} = 0$, modify the code to solve the control and estimation problem for only the unstable components. Refer to the codes used for the 1D heat problem.

4 Non-linear system with control

$$\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{\mathrm{d}w_s}{\mathrm{d}x} + z \frac{\mathrm{d}z}{\mathrm{d}x} = \nu \frac{\partial^2 z}{\partial x^2}$$

$$z(0,t) = u(t), \quad z_x(1,t) = 0, \quad z(x,0) = w^\delta(x)$$

We multiply by a test function ϕ with $\phi(0) = 0$ to obtain the weak formulation

$$\int_0^1 z_t \phi \mathrm{d}x + \nu \int_0^1 \frac{\partial z}{\partial x} \frac{\mathrm{d}\phi}{\mathrm{d}x} \mathrm{d}x + \int_0^1 w_s \frac{\partial z}{\partial x} \phi \mathrm{d}x + \int_0^1 z \frac{\partial w_s}{\partial x} \phi \mathrm{d}x + \int_0^1 w \frac{\mathrm{d}w}{\mathrm{d}x} \phi \mathrm{d}x = 0$$

The finite element solution can be written as

$$z(x,t) = \sum_{j=1}^{N} z_j(t) \phi_j(x) + u(t) \phi_0(x)$$

while the stationary solution expressed as

$$w_s(x) = \sum_{j=1}^{N} w_s^j \phi_j(x) + u_s \phi_0(x)$$

The Galerkin approximation is

$$\begin{aligned}\int_0^1 z_t\phi_i \mathrm{d}x &= -\nu\int_0^1 \frac{\partial z}{\partial x}\frac{\mathrm{d}\phi_i}{\mathrm{d}x}\mathrm{d}x - \int_0^1 w_s\frac{\partial z}{\partial x}\phi_i\mathrm{d}x - \int_0^1 z\frac{\partial w_s}{\partial x}\phi_i\mathrm{d}x \\ &\quad - \int_0^1 z\frac{\mathrm{d}z}{\mathrm{d}x}\phi_i\mathrm{d}x, \qquad \forall i = 1,2,...,N\end{aligned}$$

Ignoring the $\frac{\mathrm{d}u}{\mathrm{d}t}$ term and using the expression for w_s, we get

$$\begin{aligned}\sum_{j=1}^N z_j'(t)\int_0^1 \phi_i\phi_j\mathrm{d}x &= -\sum_{j=1}^N\sum_{k=1}^N z_j(t)z_k(t)\int_0^1 \phi_j\phi_k'\phi_i\mathrm{d}x \\ &\quad -\sum_{j=1}^N z_j(t)\left[\nu\int_0^1 \phi_i'\phi_j'\mathrm{d}x + u_s\int_0^1 \phi_i(\phi_0\phi_j)'\mathrm{d}x\right] \\ &\quad -\sum_{j=1}^N z_j(t)\left[\sum_{k=1}^N w_s^k(t)\left(\int_0^1 \phi_j\phi_k'\phi_i\mathrm{d}x + \int_0^1 \phi_k\phi_j'\phi_i\mathrm{d}x\right)\right] \\ &\quad -u(t)\left[\sum_{j=1}^N z_j\int_0^1 \phi_i(\phi_0\phi_j)'\mathrm{d}x + \sum_{k=1}^N w_s^k\int_0^1 \phi_i(\phi_0\phi_k)'\mathrm{d}x\right] \\ &\quad -u(t)\left[\nu\int_0^1 \phi_i'\phi_0'\mathrm{d}x + 2u_s\int_0^1 \phi_i\phi_0\phi_0'\mathrm{d}x\right] - u(t)^2\int_0^1 \phi_i\phi_0\phi_0'\mathrm{d}x\end{aligned}$$

Thus we have the system

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{RHS}_1 + \mathbf{RHS}_2$$

where

$$\begin{aligned}\mathbf{RHS}_1(i) &= -\mathbf{z}^\top\mathbf{D}1(:,:,i)\mathbf{z} - \left[\nu\mathbf{A}1(i,:) + u_s\mathbf{A}2(i,:) + \mathbf{w}_s^\top(\mathbf{D}1(:,:,i) + \mathbf{D}1(:,:,i)^\top)\right]\mathbf{z} \\ \mathbf{RHS}_2(i) &= -u(t)\left[\mathbf{A}2(\mathbf{z}+\mathbf{w}_s) + 2u_s\mathbf{d}1 + \nu\mathbf{d}2\right] - u(t)^2\mathbf{d}1\end{aligned}$$

The control $u(t)$ is obtained from the linear estimation problem

$$u(t) = -\sum_{j=1}^N K_j z_e^j$$

We solve the non-linear system and the linear estimator in a coupled manner.

$$\begin{aligned}\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} &= \mathbf{RHS}_1 + \mathbf{RHS}_2 \\ \mathbf{M}\frac{\mathrm{d}\mathbf{z}_e}{\mathrm{d}t} &= \mathbf{LHz} + (\mathbf{A} - \mathbf{BK} - \mathbf{LH})\mathbf{z}_e\end{aligned}$$

$$\mathbf{z}(0) = \mathbf{z}_0, \qquad \mathbf{z}_e(0) = 0$$

This is implemented in `nlp_est.m`

Excercises

1. Run program `nlp_est.m`
2. Vary $\mathbf{R}$ in the range $(0.01, 10)$ and observe how the feedback gain $\mathbf{K}$ varies. How does the solution and control vary with $\mathbf{R}$.
3. How does the solution and control change when ν is decreased?

5 List of Programs

1. `get_system_mat.m`: Computes FEM matrices
2. `feedback_matrix.m`: Computes feedback matrix
3. `stationarysol.m`: Computes exact unstable stationary solution
4. `nlp.m`: Solves the non-linear model, without feedback
5. `lp_est.m`: Solves the coupled estimation and control system for the linear problem
6. `nlp_est.m`: Solves the coupled estimation and control system for the non-linear problem
7. `rhs_nlp.m`: Computes the right hand side for the non-linear problem, without feedback
8. `rhs_nlpest.m`: Computes the right hand side for coupled estimation and control system with the non-linear problem