One dimensional heat equation

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1 The model

The shifted 1D heat equation is given by

$$z_t = \mu z_{xx} + \alpha z, \quad x \in (0,1) \times (0,T)$$

with boundary conditions

$$z(0,t) = 0, \quad z(1,t) = u(t)$$

and initial condition

$$z(x,0) = z_0(x), \quad x \in (0,1)$$

Here $\alpha \geq 0$ and $\mu > 0$.

1.1 Weak formulation

We assume $z_0 \in L^2(0,1)$. We wish to find $z \in L^2(0,T; H^1_{\{0\}}(0,1))$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(z(t),\phi)_{L^2} = -\mu \int_0^1 \frac{\partial z}{\partial x}\frac{\mathrm{d}\phi}{\mathrm{d}x}\, dx + \alpha \int_0^1 z(x,t)\phi\, dx, \quad \forall \phi \in H^1_0(0,1)$$

$$z(1,t) = u(t)$$

$$(z(0),\phi)_{L^2} = (z_0,\phi)_{L^2}$$

1.2 FEM approximation

Consider a regular subdivision of $[0,1]$

$$0 = x_0 < x_1 < ... < x_N = 1, \quad h = \frac{1}{N}, \quad x_k = kh \quad \forall k \in \{0, ..., N\}$$

We work with the finite dimensional subspaces of $H_0^1(0,1)$

$$Z_h = \{\phi \in C([0,1]) : \phi|_{[x_{i-1},x_i]} \in P_1, \quad \phi(0) = 0\}$$
$$Z_{h,0} = \{\phi \in C([0,1]) : \phi|_{[x_{i-1},x_i]} \in P_1, \quad \phi(0) = \phi(1) = 0\}$$

We denote the standard P_1 bases for $Z_{h,0}$ and Z_h by $\{\phi_1, \phi_2, ..., \phi_{N-1}\}$ and $\{\phi_1, \phi_2, ..., \phi_N\}$ respectively, with $\phi_j(x_i) = \delta_{ij}$. The test function are chosen from $Z_{h,0}$. The approximate weak formulation is given as

$$\text{Find } z = \sum_{i=1}^{N} z_i \phi_i = \sum_{i=1}^{N-1} z_i \phi_i + u(t)\phi_N \in Z_h, \quad z_i \in L^2(0,T) \text{ such that}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(z(t), \phi_k)_{L^2} = -\mu \int_0^1 \frac{\partial z}{\partial x} \frac{\mathrm{d}\phi_k}{\mathrm{d}x}\, dx + \alpha \int_0^1 z(x,t)\phi_k\, dx, \quad \forall k \in \{1, ..., N-1\}$$
$$z_N(t) = u(t)$$
$$(z(0), \phi_k)_{L^2} = (z_0, \phi_k)_{L^2}, \quad \forall k \in \{1, ..., N-1\}$$

Since $z_N(t)$ is known, we obtain a a differential system

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}_\alpha \mathbf{z} + \mathbf{B}u, \quad \mathbf{M}\mathbf{z}(0) = ((z_0, \phi_i)_{L^2})_{1 \le i \le N-1}$$

where

- $\mathbf{z} = (z_1, z_2, ..., z_{N-1})^\top$
- $\mathbf{M} \in \mathbb{R}^{(N-1)\times(N-1)}, \quad M_{i,j} = \int_0^1 \phi_i \phi_j\, dx$
- $\mathbf{A} \in \mathbb{R}^{(N-1)\times(N-1)}, \quad A_{i,j} = -\mu \int_0^1 \frac{\mathrm{d}\phi_i}{\mathrm{d}x} \frac{\mathrm{d}\phi_j}{\mathrm{d}x}\, dx$
- $\mathbf{A}_\alpha = \mathbf{A} + \alpha \mathbf{M}$

- $\mathbf{B} = \frac{\mu}{h}(0, 0, ..., 0, 1)^\top \in \mathbb{R}^{N-1}$
- $u \in \mathbb{R}$

Using exact integration for $\mathbf{M}$ and $\mathbf{A}$ gives

$$\mathbf{M} = h \begin{bmatrix} \frac{2}{3} & \frac{1}{6} & 0 & \cdots & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \vdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & \frac{1}{6} & \frac{2}{3} & \frac{1}{6} \\ 0 & \cdots & 0 & \frac{1}{6} & \frac{2}{3} \end{bmatrix}, \quad \mathbf{A} = \frac{\mu}{h} \begin{bmatrix} 2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \vdots & 0 \\ 0 & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -2 \end{bmatrix}$$

We use the trapezoidal rule for $u'(t) \int_0^1 \phi_N \phi_{N-1}$ which will evaluate out as zero.

2 Time integration using BDF

Assume for now that the feedback control is of the form

$$u(t) = -\mathbf{K}\mathbf{z}(t), \quad \mathbf{K} \in \mathbb{R}^{1\times(N-1)}$$

We shall use a Backward Differentiation Formula (BDF) for time integration. For the first time step we use BDF1 (backward Euler scheme)

$$\mathbf{M}\frac{\mathbf{z}^1 - \mathbf{z}^0}{\Delta t} = (\mathbf{A}_\alpha - \mathbf{B}\mathbf{K})\mathbf{z}^1$$

or

$$\left[\frac{\mathbf{M}}{\Delta t} - (\mathbf{A}_\alpha - \mathbf{B}\mathbf{K})\right]\mathbf{z}^1 = \frac{\mathbf{M}}{\Delta t}\mathbf{z}^0$$

For the remaining time steps we use BDF2

$$\mathbf{M}\frac{\mathbf{z}^{n+1} - \frac{4}{3}\mathbf{z}^n + \frac{1}{3}\mathbf{z}^{n-1}}{\frac{2}{3}\Delta t} = (\mathbf{A}_\alpha - \mathbf{B}\mathbf{K})\mathbf{z}^1$$

or

$$\left[\frac{\mathbf{M}}{\frac{2}{3}\Delta t} - (\mathbf{A}_\alpha - \mathbf{B}\mathbf{K})\right]\mathbf{z}^{n+1} = \frac{\mathbf{M}}{\frac{2}{3}\Delta t}\left(\frac{4}{3}\mathbf{z}^n - \frac{1}{3}\mathbf{z}^{n-1}\right)$$

Excercises

1. Run the code `heat.m` with $\alpha = 0$

```
alpha = 0;
mu = 1;
```

   Are all the eigenvalues stable? How do the solution and energy behave?

2. Evaluate the decay rate of the solution via the following steps

   - Declare an array

     ```
     diff = zeros(nT+1,1)
     ```

   - Store the norm of the solution at each time step

     ```
     diff(i) = norm([0,z(:,i)',0])
     ```

     Find the coefficients of the linear polynomial fit of the data set `(t,log(diff))`

     ```
     polyfit(t,log(diff)',1)
     ```

     This will give you the decay rate.

3. Run the code with

```
alpha = 0.4 + pi^2 * mu
```

   Are all the eigenvalues stable? How do the solution and energy behave this time?

4. Change the initial condition to

```
z0 = sqrt(2)*sin(pi*x)
```

   which is the eigenfunction corresponding to the unstable eigenvalue. What happens to the solution?

5. For each eigenvalue λ, compute eigenvector $\mathbf{V}$ of $(\mathbf{A}_\alpha^\top, \mathbf{M}^\top)$

$$\mathbf{A}_\alpha^\top \mathbf{V} = \lambda \mathbf{M}^\top \mathbf{V}$$

   and check if

$$\mathbf{B}^\top \mathbf{V} \neq 0$$

   for each unstable eigenvalue (Hautus criterion).

3 Minimal norm feedback control

The minimal norm control is given by

$$u(t) = -\mathbf{K}z(t)$$

The feedback matrix $\mathbf{K}$ is given by

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^\top\mathbf{P}\mathbf{M}$$

where $\mathbf{P}$ is solution of algebraic Riccati equation (ARE)

$$\mathbf{A}_\alpha^\top\mathbf{P}\mathbf{M} + \mathbf{M}^\top\mathbf{P}\mathbf{A}_\alpha - \mathbf{M}^\top\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{P}\mathbf{M} = 0$$

This can be done using the `care` function

```
[P,L,K] = care(A,B,R,Q,S,M);
```

which solves the more general ARE

$$\mathbf{A}^\top\mathbf{P}\mathbf{M} + \mathbf{M}^\top\mathbf{P}\mathbf{A} - (\mathbf{M}^\top\mathbf{P}\mathbf{B} \; + \mathbf{S}^\top)\mathbf{R}^{-1}(\mathbf{B}^\top\mathbf{P}\mathbf{M} + \mathbf{S}) + \mathbf{Q} = 0$$

where $\mathbf{L}$ gives the eigenvalues of $(\mathbf{A} - \mathbf{B}\mathbf{K}, \mathbf{M})$. Since $\mathbf{S} = 0$ for our problem, we can simply replace it with `[]` while calling the `care` function. Furthermore, for the minimal norm feedback, we take $\mathbf{Q} = 0$, which can be set in `matrix_fem.m`

Excercises

1. Run the code `heat_lqr.m` and observe the modification in the eigenvalues. Has the unstable eigenvalue observed earlier changed? What about the remaining stable eigenvalues?

2. Change the initial condition to

   ```
   z0 = sqrt(2)*sin(pi*x)
   ```

   What happens to the solution with the above control?

3. Plot the intial condition and the numerical solution after one time step.

4. Save the feedback matrix evaluated

   ```
   save('feedback.mat','K')
   ```

   Load $\mathbf{K}$ into `heat.m`, and for `alpha = 0` (no shift) introduce control using the feedback matrix

   ```
   load('feedback');
   A = A-B*sparse(K);
   ```

5. Find the decay rate in this case. Has it improved? Is it in the expected range?

4 Feedback control using LQR approach

Measurement

$$y_m = Cz, \qquad \text{e.g.} \qquad C = I$$

Performance measure

$$\begin{aligned} J &= \frac{1}{2}\int_0^\infty y_m^\top Q_m y_m \mathrm{d}t + \frac{1}{2}\int_0^\infty u^\top R u \mathrm{d}t \\ &= \frac{1}{2}\int_0^\infty z^\top Q z \mathrm{d}t + \frac{1}{2}\int_0^\infty u^\top R u \mathrm{d}t, \qquad Q = C^\top Q_m C \end{aligned}$$

Find feedback law

$$u = -\mathbf{K}\mathbf{z}$$

which minimizes J. The feedback matrix $\mathbf{K}$ is given by

$$\mathbf{K} = \mathbf{R}^{-1}\mathbf{B}^\top\mathbf{P}\mathbf{M}$$

where $\mathbf{P}$ is solution of algebraic Riccati equation (ARE)

$$\mathbf{A}_\alpha^\top\mathbf{P}\mathbf{M} + \mathbf{M}^\top\mathbf{P}\mathbf{A}_\alpha - \mathbf{M}^\top\mathbf{P}\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^\top\mathbf{P}\mathbf{M} + \mathbf{Q} = 0$$

Excercises

1. In `matrix_fem.m`, put $\mathbf{Q} = \mathbf{M}$ (which corresponds to $Q = I$). Run `heat_lqr.m`. How do the eigenvalues change this time? What is the behaviour of the solution?

2. Plot the intial condition and the numerical solution after one time step. How does it differ from the previous Bernoulli control case?

3. How does the solution and control vary as `mu` is decreased?

4. In `matrix_fem.m`, vary the value of $\mathbf{R}$ in (0.01, 50) (choose 6-7 values in the range). How does $\mathbf{K}$ vary with $\mathbf{R}$?

5. As before, save the feedback matrix and load it into `heat.m`. Is the decay rate as expected?

5 Control based on stabilizing only the unstable components

Let the us denote by λ_i and $\mathbf{V}_i$ the eigenvalues and normalized eigenvectors such that

$$\mathbf{A}\mathbf{V}_i = \lambda_i \mathbf{M}\mathbf{V}_i$$

We assume that α is chosen such that

$$\lambda_{N-1} < .... < \lambda_{N_\alpha+1} < -\alpha < \lambda_{N_\alpha} < ... < \lambda_1$$

Thus in the shifted system for the heat equation, there will be N_α unstable eigenvalues. The eigenvectors $\mathbf{V}_i$ form a basis for $\mathbb{R}^{N-1}$ and have the property

$$\mathbf{V}_i^\top \mathbf{M}\mathbf{V}_j = \delta_{i,j}, \quad \mathbf{V}_i^\top \mathbf{A}\mathbf{V}_j = \delta_{i,j}\lambda_i$$

Let $\Sigma \in \mathbb{R}^{(N-1)\times(N-1)}$ with $\mathbf{V}_i$ forming the columns, and consider the variable change

$$\mathbf{z} = \Sigma \boldsymbol{\zeta}$$

Thus the shifted system can be written as

$$\frac{\mathrm{d}\boldsymbol{\zeta}}{\mathrm{d}t} = \Lambda_\alpha \boldsymbol{\zeta} + \mathbb{B}u$$

where

$$\Lambda_\alpha = \begin{bmatrix} \alpha+\lambda_1 & 0 & \cdots & 0 \\ 0 & \alpha+\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \alpha+\lambda_{N-1} \end{bmatrix}, \qquad [\mathbb{B}]_i = V_i^\top \mathbf{B}, \quad \forall 1 \le i \le N-1$$

Next we project the system onto the unstable subspace using the projection operator $\mathbf{\Pi}_{\alpha,\mathbf{u}} \in \mathcal{L}(\mathbb{R}^{N-1}, \mathbb{R}^{N_\alpha})$

$$\frac{\mathrm{d}\boldsymbol{\zeta}_u}{\mathrm{d}t} = \Lambda_{\alpha,u} \boldsymbol{\zeta}_u + \mathbb{B}_u u$$

where

$$\Lambda_{\alpha,u} = \begin{bmatrix} \alpha+\lambda_1 & 0 & \cdots & 0 \\ 0 & \alpha+\lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \\ 0 & 0 & 0 & \alpha+\lambda_{N_\alpha} \end{bmatrix}, \qquad [\mathbb{B}_u]_i = V_i^\top \mathbf{B}, \quad \forall 1 \le i \le N_\alpha$$

and $\boldsymbol{\zeta}_u = \mathbf{\Pi}_{\alpha,\mathbf{u}}\boldsymbol{\zeta}$ are the first N_α components. We find the feedback matrix for this reduced system, by solving the Bernoulli equation

$$\mathbb{P}_u \Lambda_{\alpha,u} + \Lambda_{\alpha,u}^\top \mathbb{P}_u - \mathbb{P}_u \mathbb{B}\mathbb{B}^\top \mathbb{P}_u = 0$$

$$\Lambda_{\alpha,u} - \mathbb{B}_u \mathbb{B}_u^\top \mathbb{P}_u \text{ is stable}$$

The corresponding matrix $\mathbf{P} \in \mathcal{L}(\mathbb{R}^{N-1})$ such that $(\mathbf{M}, \mathbf{A}_\alpha - \mathbf{B}\mathbf{B}^\top \mathbf{P}\mathbf{M})$ is stable is given by

$$\mathbf{P} = \Sigma_u \mathbb{P}_u \Sigma_u^\top$$

where Σ_u is the matrix of eigenvectors corresponding to the unstable eigenvalues. This is implemented in `heat_lqru.m`

Excercises

1. In `matrix_fem.m`, put $\mathbf{Q} = 0$. Run `heat_lqru.m`. How do the eigenvalues change this time? What is the behaviour of the solution?

2. Plot the intial condition and the numerical solution after one time step. How does it differ from the previous two kinds of control?

3. For the same value of `alpha`, which of the three types of controls discussed so far performs the best?

4. Lecture 5 has the exact solutions for this problem, with `alpha = 10`. Take
```
z0 = sqrt(2)*sin(pi* x)
```
   in the equation and find the expressions for exact control and solution. Plot and compare

   - evolution of the numerical control and the exact control
   - numerical and exact solutions at the final time

5. Save the feedback matrix and load it into `heat.m`. Is the decay rate as expected?

6 System with noise and partial information

Consider the system with noise in the model and initial condition

$$z_t = \mu z_{xx} + \alpha z + w, \qquad z(x,0) = z_0(x) + \eta$$

where w and η are error/noise terms. The boundary conditions are as before

$$z(0,t) = 0, \quad z(1,t) = u(t)$$

We may not have access to the full state but only some partial information which is also corrupted by noise.

$$y = Hz + v$$

where H is a suitable measure. We shall consider the case where H is given by

$$Hz(t) = z_x(0,t)$$

In the FEM setup, we get the system

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}_\alpha \mathbf{z} + \mathbf{B}u + \mathbf{w}, \quad \mathbf{M}\mathbf{z}(0) = ((z_0, \phi_i)_{L^2})_{1 \le i \le N-1}$$

$$\mathbf{y} = \mathbf{H}\mathbf{z} + \mathbf{v}$$

where

$$\mathbf{H} = \frac{\mu}{h}(2, 0.5, 0, ..., 0)^\top \in \mathbb{R}^{N-1}$$

6.1 Estimation problem

Linear estimator

$$\frac{\mathrm{d}z_e}{\mathrm{d}t} = A_\alpha z_e + Bu + L(y - Hz_e)$$

We determine the filtering gain L by minimizing

$$J = \frac{1}{2}\int_0^\infty (y - Hz_e)^\top R_v^{-1}(y - Hz_e)\mathrm{d}t + \frac{1}{2}\int_0^\infty w^\top R_w^{-1} w \mathrm{d}t$$

The solution is given by

$$L = -P_e H^\top R_v^{-1}$$

where P_e is solution of

$$A_\alpha P_e + P_e A_\alpha^\top - P_e H^\top R_v^{-1} H P_e + R_w = 0$$

The operators R_w and R_v are chosen according to the apriori knowledge we have on the model noise and the measurement noise. In the FEM setup, we have

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}_e}{\mathrm{d}t} = \mathbf{A}_\alpha \mathbf{z}_e + \mathbf{B}u + \mathbf{L}(\mathbf{y} - \mathbf{H}\mathbf{z}_e)$$

$$\mathbf{L} = -\mathbf{M}\mathbf{P}_e\mathbf{H}^\top\mathbf{R}_\mathbf{v}^{-1}$$

where $\mathbf{P}_e$ is solution of

$$\mathbf{A}_\alpha\mathbf{P}_e\mathbf{M} + \mathbf{M}\mathbf{P}_e\mathbf{A}_\alpha^\top - \mathbf{M}\mathbf{P}_e\mathbf{H}^\top\mathbf{R}_\mathbf{v}^{-1}\mathbf{H}\mathbf{P}_e\mathbf{M} + \mathbf{R}_\mathbf{w} = 0$$

$$(\mathbf{M}, \mathbf{A}_\alpha - \mathbf{L}\mathbf{H}) \text{ is stable}$$

6.2 Coupled linear system

The feedback is based on estimated solution $u = -\mathbf{K}\mathbf{z}_e$

$$\begin{aligned}
\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} &= \mathbf{A}_\alpha\mathbf{z} - \mathbf{B}\mathbf{K}\mathbf{z}_e + \mathbf{w} \\
\mathbf{M}\frac{\mathrm{d}\mathbf{z}_e}{\mathrm{d}t} &= \mathbf{L}\mathbf{H}\mathbf{z} + (\mathbf{A}_\alpha - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{H})\mathbf{z}_e + \mathbf{L}\mathbf{v}
\end{aligned}$$

or in matrix form

$$\begin{bmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}_e \end{bmatrix} = \begin{bmatrix} \mathbf{A}_\alpha & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{H} & \mathbf{A}_\alpha - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}_e \end{bmatrix} + \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{L} \end{bmatrix} \begin{bmatrix} \mathbf{w} \\ \mathbf{v} \end{bmatrix}$$

The initial condition is given by

$$\mathbf{z}(0) = \mathbf{z}_0 + \eta, \qquad \mathbf{z}_e(0) = \mathbf{z}_0$$

This is implemented in program `heat_est.m`

Excercises

1. Run program `heat_est.m`
2. Plot the eigenvalues of the coupled system. Are they stable?
3. The matrices $\mathbf{K}$ and $\mathbf{L}$ should improve estimation and decay rate of the unshifted problem. Check this via the following steps
   - Save the matrices $\mathbf{K}$ and $\mathbf{L}$
   - Set `alpha = 0` and evaluate the decay rate of $\mathbf{z}$ with $\mathbf{K} = 0$ and $\mathbf{L} = 0$.
   - Load the saved matrices and evaluate the decay rate. Has it improved?

6.3 Stabilizing the unstable component

As done before, the feedback gain matrix $\mathbf{K}$ can be evaluated by considering the projected system

$$\frac{\mathrm{d}\boldsymbol{\zeta}_u}{\mathrm{d}t} = \Lambda_{\alpha,u}\boldsymbol{\zeta}_u + \mathbb{B}_u u$$

We use the variable change

$$\mathbf{z} = \Sigma\boldsymbol{\zeta}$$

and define the operator $\mathbb{H}$ by

$$\mathbb{H}\boldsymbol{\zeta} = \mathbf{H}\mathbf{z}$$

We shall work with the projected variable $\boldsymbol{\zeta}_u = \boldsymbol{\Pi}_{\alpha,\mathbf{u}}\boldsymbol{\zeta}$, and thus define the corresponding measure operator

$$\mathbb{H}_u = \mathbb{H}\boldsymbol{\Pi}_{\alpha,\mathbf{u}}$$

The filtering gain $\mathbb{P}_e$ for the reduced projected system is obtained as the solution of the ARE

$$\mathbb{P}_e\Lambda_{\alpha,u}^\top + \Lambda_{\alpha,u}\mathbb{P}_e - \mathbb{P}_e\mathbb{H}_u^\top\mathbf{R}_\mathbf{v}^{-1}\mathbb{H}_u\mathbb{P}_e + \mathbb{Q}_\mathbf{w}$$

$$\Lambda_{\alpha,u} - \mathbb{P}_e\mathbb{H}_u^\top\mathbf{R}_\mathbf{v}^{-1}\mathbb{H}_u \text{ is stable}$$

where $\mathbb{Q}_\mathbf{w} = \Sigma_u^\top\mathbf{R}_\mathbf{w}\Sigma_u$. The corresponding $\mathbf{P}_e$ such that $(\mathbf{M}, \mathbf{A}_\alpha - \mathbf{M}\mathbf{P}_e\mathbf{H}^\top\mathbf{R}_\mathbf{v}^{-1}\mathbf{H})$ is stable is given by

$$\mathbf{P}_e = \Sigma_u\mathbb{P}_e\Sigma_u^\top$$

Excercises

1. Write a code `heat_estu.m` for the above via the following steps

   - Copy the contents of `heat_est.m`
   - Evaluate the feedback gain $\mathbf{K}$ as done in `heat_lqru.m`
   - Evaluate the projected matrix $\mathbb{H}_u$

     ```
     Hu = full(H)*V;
     ```

     where `V` is the matrix of unstable eigenvectors (you would have already evaluated this while obtaining $\mathbf{K}$)
   - Evaluate $\mathbb{Q}_\mathbf{w}$

     ```
     Rwu = V'*full(Rw)*V;
     ```

   - Solve the ARE for the reduced system to obtain $\mathbb{P}_e$

     ```
     Peu = care(D',full(Hu)',full(Rwu),full(Rv));
     ```

where D is the diagonalized matrix of the reduced system (also evaluated while finding $\mathbf{K}$)

- Find $\mathbf{P}_e$ for the original system

```
Pe = V*Peu*V';
```

- Finally evaluate $\mathbf{L}$ by

```
L = M*Pe*H'/Rv;
```

- Run `heat_estu.m` with `alpha = 10`. How do the solution and energy behave?

2. Check whether the $\mathbf{K}$ and $\mathbf{L}$ improve the estimation and decay rate of the unshifted problem

   - Save the matrices $\mathbf{K}$ and $\mathbf{L}$
   - Set `alpha = 0` and evaluate the decay rate of `z` with $\mathbf{K} = 0$ and $\mathbf{L} = 0$. It would be easier to copy the code of `heat_estu.m` into a new file, say `heat_estu0.m` and remove all evaluations of $\mathbf{P}$ and $\mathbf{P}_e$.
   - Load the saved matrices and evaluate the decay rate. Has it improved?

7 List of Programs

1. `matrix_fem.m`: Computes FEM matrices
2. `hautus.m`: Checks the stabilizability of the system using Hautus criterion
3. `feedback_matrix.m`: Computes feedback matrix
4. `heat.m`: Solves the heat equation for a given α, without feedback
5. `heat_lqr.m`: Solves the heat equation for a given α, with feedback
6. `heat_lqr.m`: Solves the heat equation for a given α, stabilizing only the unstable components
7. `heat_est.m`: Solves the coupled estimation and control problem for the heat equation for a given α
8. `heat_estu.m`: Solves the coupled estimation and control problem for the heat equation for a given α, stablilizing only the unstable components
9. `heat_estu0.m`: Solves the coupled estimation and control problem for the heat equation for $\alpha = 0$, using matrice from `heat_estu.m`