Two dimensional heat equation

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1 The model

Let $z = z(x, y, t)$ denote the temperature. The shifted 2-D heat equation is given by

$$z_t = \mu \Delta z + \alpha z, \quad (x, y) \in \Omega = (0, 1) \times (0, 1), \qquad t \in [0, T]$$

with boundary conditions

$$z(x, 0, t) = z(x, 1, t) = 0, \quad z(1, y, t) = u(y, t), \qquad \frac{\partial z}{\partial x}(0, y, t) = 0$$

and initial condition

$$z(x, y, 0) = z_0(x, y)$$

Here $\alpha \geq 0$ and $\mu > 0$. Let us denote the Dirichlet part of the boundary by Γ_D

$$\Gamma_D = \{y = 0\} \cup \{y = 1\} \cup \{x = 1\}$$

the Neumann part as

$$\Gamma_N = \{x = 0\}$$

and the part on which the control is applied as

$$\Gamma_c = \{x = 1\}$$

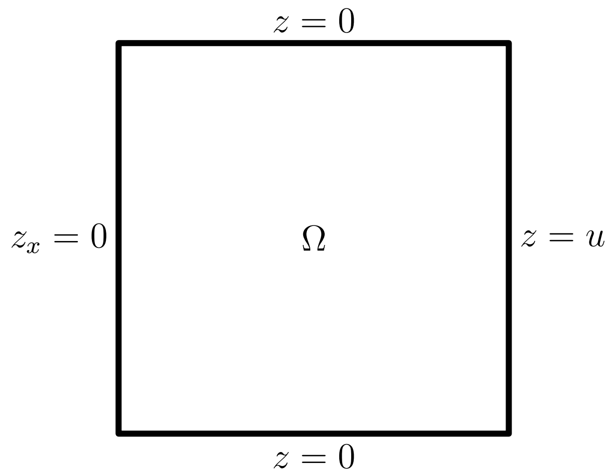

Figure 1: Problem definition

1.1 Observations

We will measure an average value of the temperature on strips along the left vertical boundary

$$I_i = [a_i, b_i]$$

as shown in figure. Thus the observations are

$$y_i(t) = \frac{1}{b_i - a_i} \int_{a_i}^{b_i} z(0, y, t) \mathrm{d}y \tag{1}$$

1.2 Weak formulation

We assume $z_0 \in L^2(\Omega)$. We wish to find $z \in L^2(0, T; H^1(\Omega))$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}(z(t), \phi)_{L^2} = -\mu \int_\Omega \nabla z \cdot \nabla \phi \mathrm{d}x + \alpha \int_\Omega z \phi \mathrm{d}x, \quad \forall \phi \in H^1_{\Gamma_D}(\Omega)$$

$$z(x, 0, t) = z(x, 1, t) = 0, \quad z(1, y, t) = u(y, t)$$

$$(z(0), \phi)_{L^2} = (z_0, \phi)_{L^2}$$

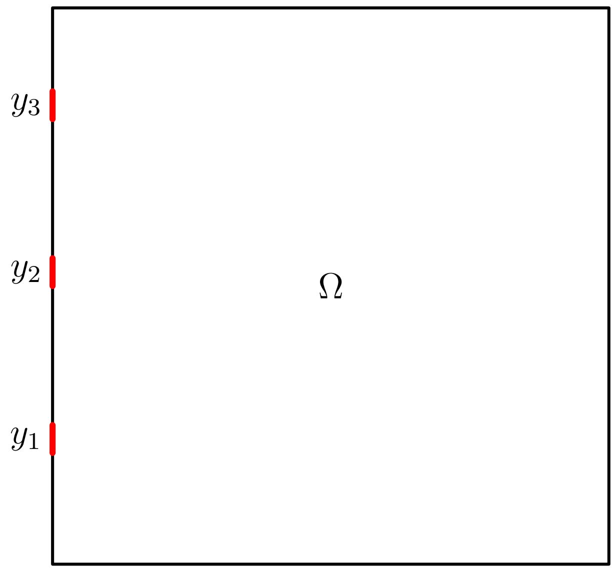

Figure 2: Observations

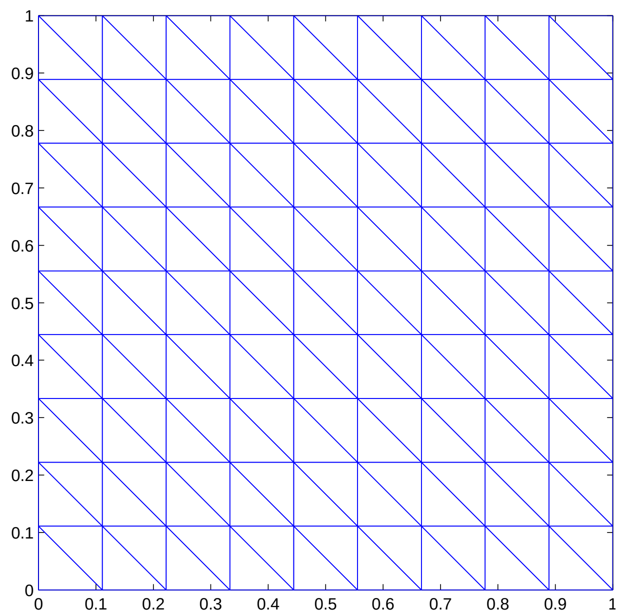

Figure 3: Example of a finite element mesh

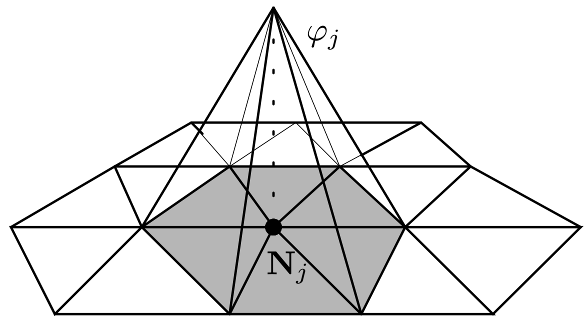

Figure 4: Piecewise affine basis functions

2 FEM approximation

Consider a division of Ω into disjoint triangles as shown in figure (xxx). We will assume that the vertices of the mesh are numbered in some manner. Let us define the following sets of vertices

$$\begin{aligned} N_c &= \text{vertices on } \Gamma_c \\ N_d &= \text{vertices on } \{y=0\} \cup \{y=1\} \\ N_f &= \text{remaining vertices} \end{aligned}$$

For each vertex i, define the piecewise affine functions $\phi_i(x,y)$ with the property that

$$\phi_i(x_j, y_j) = \delta_{ij}$$

We will take the control to be of the form

$$u(y,t) = v(t)\sin(\pi y)$$

Then the finite element solution is of the form

$$z(x,y,t) = \sum_{j \in N_f} z_j(t)\phi_j(x,y) + v(t) \sum_{j \in N_c} \sin(\pi y_j)\phi_j(x,y)$$

The approximate weak formulation is given as

$$\frac{\mathrm{d}}{\mathrm{d}t}(z(t), \phi_i)_{L^2} = -\mu \int_\Omega \nabla z \cdot \nabla \phi_i \mathrm{d}x + \alpha \int_\Omega z \phi_i \mathrm{d}x, \quad \forall i \in N_f$$

i.e.,

$$\begin{aligned}
& \sum_{j\in N_f} \frac{\mathrm{d}z_j}{\mathrm{d}t} \int_\Omega \phi_j\phi_i + \frac{\mathrm{d}v}{\mathrm{d}t} \sum_{j\in N_c} \sin(\pi y_j) \int_\Omega \phi_j\phi_i \\
= \ & -\mu \sum_{j\in N_f} z_j \int_\Omega \nabla\phi_j\cdot\nabla\phi_i - \mu v \sum_{j\in N_c} \sin(\pi y_j) \int_\Omega \nabla\phi_j\cdot\nabla\phi_i \\
& +\alpha \sum_{j\in N_f} z_j \int_\Omega \phi_j\phi_i + \alpha v \sum_{j\in N_c} \sin(\pi y_j) \int_\Omega \phi_j\phi_i, \qquad \forall i\in N_f
\end{aligned}$$

In order to simplify the presentation we will ignore the term containing $\frac{\mathrm{d}v}{\mathrm{d}t}$[1] and then we can write the FEM formulation as

$$\begin{aligned}
& \sum_{j\in N_f} \frac{\mathrm{d}z_j}{\mathrm{d}t} \int_\Omega \phi_j\phi_i \\
= \ & \sum_{j\in N_f} z_j \left[-\mu \int_\Omega \nabla\phi_j\cdot\nabla\phi_i + \alpha \int_\Omega \phi_j\phi_i\right] \\
& v \sum_{j\in N_c} \left[-\mu \sin(\pi y_j) \int_\Omega \nabla\phi_j\cdot\nabla\phi_i + \alpha \sin(\pi y_j) \int_\Omega \phi_j\phi_i\right] \qquad \forall i\in N_f
\end{aligned}$$

This can be written as a system of differential equations

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}\mathbf{z} + \mathbf{B}v$$

2.1 Finite element assembly

The finite element basis functions have compact support. Hence we can compute the integrals by adding the contributions from a small number of triangles. For example, the elements of the mass matrix can be computed as

$$\int_\Omega \phi_i\phi_j = \sum_{K\ :\ i,j\in K} \int_K \phi_i\phi_j$$

and similarly the stiffness matrix is computed as

$$\int_\Omega \nabla\phi_i\cdot\nabla\phi_j = \sum_{K\ :\ i,j\in K} \int_K \nabla\phi_i\cdot\nabla\phi_j$$

[1] This term vanishes if we use the trapezoidal rule for integration.

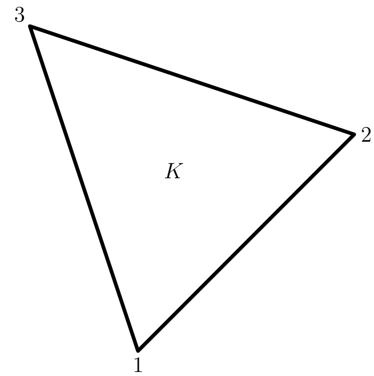

Figure 5: Triangle

The integrals on each triangle K will be evaluated exactly. For a triangle K with vertices labelled 1, 2, 3 as in figure (5), the local mass and stiffness matrices are given by

$$M^K = \frac{1}{24}\det\begin{bmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

$$A^K = \frac{1}{2}\det\begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} GG^\top \qquad \text{where} \qquad G = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix}^{-1}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

2.2 Computing the observation

We will assume that the intervals $[a_i, b_i]$ on which the observation is computed is exactly covered by the edges of the finite element mesh. Let E_i denote the edges on $[a_i, b_i]$

$$\begin{aligned} y_i &= \frac{1}{b_i - a_i}\int_{a_i}^{b_i} z(0, y, t)\mathrm{d}y \\ &= \frac{1}{b_i - a_i}\sum_{e \in E_i}\int_e z(0, y, t)\mathrm{d}y \\ &= \frac{1}{b_i - a_i}\sum_{e \in E_i}\frac{1}{2}(z_{e_1} + z_{e_2})|e| \end{aligned}$$

The set of observations can be written as

$$\mathbf{y} = \mathbf{H}\mathbf{z}$$

2.3 Grid information