

One dimensional Burgers' equation

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1 Non-linear Burgers' equation

We consider the non-linear viscous Burgers' equation

$$w_t - \nu w_{xx} + ww_x = f_s \quad \text{in } (0, 1) \times (0, \infty) \quad (1)$$

with boundary conditions

$$w(0, t) = u_s, \quad w_x(1, t) = g_s \quad (2)$$

and initial condition

$$w(x, 0) = w_0(x) \quad (3)$$

1.1 Stationary Burgers' equation

The stationary Burgers' equation is given by

$$-\nu w_{xx} + ww_x = f_s \quad \text{in } (0, 1) \quad (4)$$

with boundary conditions

$$w(0) = u_s, \quad w_x(1) = g_s \quad (5)$$

An unstable stationary solution can be obtained as

$$w_s = -\frac{\nu\pi}{2}(1 + \epsilon) \tan\left(\frac{\pi}{4}(1 + \epsilon) + C_0\right) \quad (6)$$

with the conditions

$$0 < \epsilon < 1, \quad -\frac{\pi}{4} < C_0 < 0, \quad (1 + \epsilon) \tan\left(\frac{\pi}{4}(1 + \epsilon) + C_0\right) = 1$$

We choose

$$\epsilon = 0.6 \quad \implies \quad C_0 = \arctan(1/(1 + \epsilon)) - \frac{\pi}{4}(1 + \epsilon) = -0.69803774609235$$

2 Nonlinear system without control

We set $z = w - w_s$. The equation satisfied by z is

$$\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{dw_s}{dx} + z \frac{dz}{dx} = \nu \frac{\partial^2 z}{\partial x^2} \quad \text{in } (0, 1) \times (0, \infty) \quad (7)$$

with boundary conditions

$$z(0, t) = 0, \quad z_x(1, t) = 0 \quad (8)$$

and initial conditions

$$z(x, 0) = w_0(x) - w_s(x) \quad (9)$$

We multiply by a test function ϕ with $\phi(0) = 0$ to obtain the weak formulation

$$\int_0^1 z_t \phi dx + \nu \int_0^1 \frac{\partial z}{\partial x} \frac{d\phi}{dx} dx + \int_0^1 w_s \frac{\partial z}{\partial x} \phi dx + \int_0^1 z \frac{\partial w_s}{\partial x} \phi dx + \int_0^1 z \frac{dz}{dx} \phi dx = 0$$

Divide domain into N elements with vertices

$$0 = x_0 < x_1 < \dots < x_N = 1, \quad x_i = ih, \quad h = \frac{1}{N}$$

The finite element solution can be written as

$$z(x, t) = \sum_{j=1}^N z_j(t) \phi_j(x)$$

while the stationary solution expressed as

$$w_s(x) = \sum_{j=1}^N w_s^j \phi_j(x) + u_s \phi_0(x)$$

where $w_s^j = w_s(x_j)$. Here the $\{\phi_j\}$ are the usual P_1 basis functions. The approximate weak formulation is

$$\begin{aligned} \int_0^1 z_t \phi_i dx &= -\nu \int_0^1 \frac{\partial z}{\partial x} \frac{d\phi_i}{dx} dx - \int_0^1 w_s \frac{\partial z}{\partial x} \phi_i dx - \int_0^1 z \frac{\partial w_s}{\partial x} \phi_i dx \\ &\quad - \int_0^1 z \frac{dz}{dx} \phi_i dx, \quad \forall i = 1, 2, \dots, N \end{aligned}$$

or

$$\begin{aligned} \sum_{j=1}^N z_j'(t) \int_0^1 \phi_i \phi_j dx &= - \sum_{j=1}^N \sum_{k=1}^N z_j(t) z_k(t) \int_0^1 \phi_j \phi_k' \phi_i dx \\ &\quad - \sum_{j=1}^N z_j(t) \left[\nu \int_0^1 \phi_i' \phi_j' dx + u_s \int_0^1 \phi_i (\phi_0 \phi_j)' dx \right] \\ &\quad - \sum_{j=1}^N z_j(t) \left[\sum_{k=1}^N w_s^k(t) \left(\int_0^1 \phi_j \phi_k' \phi_i dx + \int_0^1 \phi_k \phi_j' \phi_i dx \right) \right] \end{aligned}$$

We use the following notations:

- $\mathbf{z} = [z_1, z_2, \dots, z_N]^\top$
- $\mathbf{w}_s = [w_s^1, w_s^2, \dots, w_s^N]^\top$
- $\mathbf{M} \in \mathbb{R}^{N \times N}$ with $\mathbf{M}(i, j) = \int_0^1 \phi_i \phi_j dx$
- $\mathbf{A1} \in \mathbb{R}^{N \times N}$ with $\mathbf{A1}(i, j) = \int_0^1 \phi'_i \phi'_j dx$
- $\mathbf{A2} \in \mathbb{R}^{N \times N}$ with $\mathbf{A2}(i, j) = \int_0^1 \phi_i (\phi_0 \phi_j)' dx$
- $\mathbf{D1} \in \mathbb{R}^{N \times N \times N}$ with $\mathbf{D1}(i, j, k) = \int_0^1 \phi_i \phi'_j \phi_k dx$

Thus we have a non-linear system of first order ODEs in \mathbf{z}

$$\mathbf{M} \frac{d\mathbf{z}}{dt} = \mathbf{RHS}$$

where

$$\mathbf{RHS}(i) = -\mathbf{z}^\top \mathbf{D1}(:, :, i) \mathbf{z} - [\nu \mathbf{A1}(i, :) + u_s \mathbf{A2}(i, :) + \mathbf{w}_s^\top (\mathbf{D1}(:, :, i) + \mathbf{D1}(:, :, i)^\top)] \mathbf{z}$$

This is implemented and solved in `burger.m`

Warning It is not good computational practice to compute the three dimensional matrices like $\mathbf{D1}$ since it requires large storage. But we will use this approach here since we are only in 1-D case.

2.1 Exercises

1. The initial condition is chosen as

$$z(x, 0) = \delta \sin(\pi x)$$

Run program `burger.m` with `delta = 0`. The zero solution does not change with time. This corresponds to the stationary solution for the original Burgers's equation.

2. Take different initial conditions by varying `delta`, and convince yourself that the zero solution is unstable. Try `delta = 0.1` and `delta = -0.1` and see how the solution evolves.

3 Linearized Burgers' equation

3.1 Feedback control

Consider the linearized equation

$$\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{dw_s}{dx} = \nu \frac{\partial^2 z}{\partial x^2} \quad (10)$$

$$z(0, t) = u(t), \quad \frac{\partial z}{\partial x}(1, t) = 0 \quad (11)$$

$$z(x, 0) = w_0(x) - w_s(x) \quad (12)$$

The weak formulation is given by

$$\int_0^1 \left(\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{dw_s}{dx} \right) \phi dx = -\nu \int_0^1 \frac{\partial z}{\partial x} \frac{\partial \phi}{\partial x} dx$$

We look for a finite element solution of the form

$$z(x, t) = \sum_{j=1}^N z_j(t) \phi_j(x) + u(t) \phi_0(x)$$

The approximate weak formulation is

$$\int_0^1 \left(\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{dw_s}{dx} \right) \phi_i dx = -\nu \int_0^1 \frac{\partial z}{\partial x} \frac{\partial \phi_i}{\partial x} dx, \quad i = 1, 2, \dots, N$$

Ignoring the $\frac{du}{dt}$ term gives us

$$\begin{aligned} \sum_{j=1}^N \frac{dz_j}{dt} \int_0^1 \phi_i \phi_j dx &= \sum_{j=1}^N z_j \left[-\nu \int_0^1 \phi_i' \phi_j' dx - \int_0^1 w_s \phi_i \phi_j' dx - \int_0^1 \phi_i \phi_j \frac{dw_s}{dx} dx \right] \\ &+ u \left[-\int_0^1 w_s \phi_i \phi_0' dx - \int_0^1 \phi_i \phi_0 \frac{dw_s}{dx} dx - \nu \int_0^1 \phi_i' \phi_0' dx \right] \end{aligned}$$

Using the expression for w_s mentioned above, we get

$$\begin{aligned} \sum_{j=1}^N \frac{dz_j}{dt} \int_0^1 \phi_i \phi_j dx &= \sum_{j=1}^N z_j \left[-\nu \int_0^1 \phi_i' \phi_j' dx - u_s \int_0^1 \phi_i (\phi_0 \phi_j)' dx \right] \\ &+ \sum_{j=1}^N z_j \left[-\sum_{k=1}^N w_s^k \int_0^1 \phi_j \phi_k' \phi_i dx - \sum_{k=1}^N w_s^k \int_0^1 \phi_k \phi_j' \phi_i dx \right] \\ &+ u \left[-2u_s \int_0^1 \phi_i \phi_0 \phi_0' dx - \sum_{k=1}^N w_s^k \int_0^1 \phi_i (\phi_0 \phi_k)' dx - \nu \int_0^1 \phi_i' \phi_0' dx \right] \end{aligned}$$

We use the following notations:

- $\mathbf{z} = [z_1, z_2, \dots, z_N]^\top$
- $\mathbf{w}_s = [w_s^1, w_s^2, \dots, w_s^N]^\top$
- $\mathbf{M} \in \mathbb{R}^{N \times N}$ with $\mathbf{M}(i, j) = \int_0^1 \phi_i \phi_j dx$
- $\mathbf{A1} \in \mathbb{R}^{N \times N}$ with $\mathbf{A1}(i, j) = \int_0^1 \phi'_i \phi'_j dx$
- $\mathbf{A2} \in \mathbb{R}^{N \times N}$ with $\mathbf{A2}(i, j) = \int_0^1 \phi_i (\phi_0 \phi_j)' dx$
- $\mathbf{D1} \in \mathbb{R}^{N \times N \times N}$ with $\mathbf{D1}(i, j, k) = \int_0^1 \phi_i \phi'_j \phi_k dx$
- $\mathbf{A}(i, :) = -\nu \mathbf{A1}(i, :) - u_s \mathbf{A2}(i, :) - \mathbf{w}_s^\top [\mathbf{D1}(:, :, i) + \mathbf{D1}(:, :, i)^\top], \quad \forall i = 1, 2, \dots, N$
- $\mathbf{d1} \in \mathbb{R}^N$ with $\mathbf{d1}(i) = \int_0^1 \phi_i \phi_0 \phi'_0 dx$
- $\mathbf{d2} \in \mathbb{R}^N$ with $\mathbf{d2}(i) = \int_0^1 \phi'_i \phi'_0 dx$
- $\mathbf{B} = -2u_s \mathbf{d1} - \mathbf{A2} \cdot \mathbf{w}_s - \nu \mathbf{d2}$

This can be written in matrix form as

$$\mathbf{M} \frac{d\mathbf{z}}{dt} = \mathbf{A}\mathbf{z} + \mathbf{B}u$$

The control $u(t)$ is obtained in terms of the feedback matrix $\mathbf{K} = [K_1, K_2, \dots, K_N]$

$$u(t) = - \sum_{j=1}^N K_j z_j(t)$$

The feedback gain \mathbf{K} is given by

$$\mathbf{K} = \mathbf{R}^{-1} \mathbf{B}^\top \mathbf{P} \mathbf{M}$$

where \mathbf{P} is solution of algebraic Riccati equation (ARE)

$$\mathbf{A}^\top \mathbf{P} \mathbf{M} + \mathbf{M}^\top \mathbf{P} \mathbf{A} - \mathbf{M}^\top \mathbf{P} \mathbf{B} \mathbf{R}^{-1} \mathbf{B}^\top \mathbf{P} \mathbf{M} + \mathbf{Q} = 0$$

$$(\mathbf{M}, \mathbf{A} - \mathbf{B}\mathbf{K}) \text{ is stable}$$

3.2 Partial information with noise

Consider the linearized model with noise in the model

$$\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{dw_s}{dx} = \nu \frac{\partial^2 z}{\partial x^2} + \eta \quad (13)$$

$$z(0, t) = u(t), \quad \frac{\partial z}{\partial x}(1, t) = 0 \quad (14)$$

$$z(x, 0) = w_0(x) - w_s(x) \quad (15)$$

Assume that we have access to partial information corrupted by noise

$$y = Hz + \mu$$

H being a suitable observation operator. We shall consider the case where H is given by

$$Hz(t) = z(1, t)$$

In the FEM setup, we get

$$\begin{aligned} \mathbf{M} \frac{d\mathbf{z}}{dt} &= \mathbf{A}\mathbf{z} + \mathbf{B}u + \boldsymbol{\eta} \\ \mathbf{y} &= \mathbf{H}\mathbf{z} + \boldsymbol{\mu} \end{aligned}$$

where the observation operator is approximated as

$$\mathbf{H} = (0, 0, \dots, 0, 1)^\top \in \mathbb{R}^N$$

3.3 Estimation problem

Consider the

$$\frac{\partial z_e}{\partial t} + w_s \frac{\partial z_e}{\partial x} + z_e \frac{dw_s}{dx} = \nu \frac{\partial^2 z_e}{\partial x^2} + L(Hz - Hz_e) \quad (16)$$

$$z_e(0, t) = u(t), \quad \frac{\partial z_e}{\partial x}(1, t) = 0 \quad (17)$$

$$z_e(x, 0) = 0 \quad (18)$$

In the FEM setup, we have

$$\mathbf{M} \frac{d\mathbf{z}_e}{dt} = \mathbf{A}\mathbf{z}_e + \mathbf{B}u + \mathbf{L}(\mathbf{y} - \mathbf{H}\mathbf{z}_e)$$

The filtering gain \mathbf{L} is given by

$$\mathbf{L} = \mathbf{M}\mathbf{P}_e\mathbf{H}^\top\mathbf{R}_\mu^{-1}$$

where \mathbf{P}_e is solution of

$$\mathbf{A}\mathbf{P}_e\mathbf{M} + \mathbf{M}\mathbf{P}_e\mathbf{A}^\top - \mathbf{M}\mathbf{P}_e\mathbf{H}^\top\mathbf{R}_\mu^{-1}\mathbf{H}\mathbf{P}_e\mathbf{M} + \mathbf{R}_\eta = 0$$

$$(\mathbf{M}, \mathbf{A} - \mathbf{L}\mathbf{H}) \text{ is stable}$$

3.4 Coupled linear system

The feedback is based on estimated solution $u = -\mathbf{K}\mathbf{z}_e$

$$\begin{aligned}\mathbf{M}\frac{d\mathbf{z}}{dt} &= \mathbf{A}\mathbf{z} - \mathbf{B}\mathbf{K}\mathbf{z}_e + \boldsymbol{\eta} \\ \mathbf{M}\frac{d\mathbf{z}_e}{dt} &= \mathbf{L}\mathbf{H}\mathbf{z} + (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{H})\mathbf{z}_e + \mathbf{L}\boldsymbol{\mu}\end{aligned}$$

or in matrix form

$$\begin{bmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M} \end{bmatrix} \frac{d}{dt} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}_e \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{H} & \mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}_e \end{bmatrix} + \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{L} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\mu} \end{bmatrix}$$

The initial condition is given by

$$\mathbf{z}(0) = \mathbf{z}_0, \quad \mathbf{z}_e(0) = 0$$

This is implemented in program `lin_lqg.m`

3.5 Exercises

1. Run program `lin_lqg.m`
2. Check stabilizability of (\mathbf{A}, \mathbf{B}) using the Hautus criterion.
3. Set $(\mathbf{Q} = 0)$ which corresponds to minimal norm control. Run the code and observe the solution and control.
4. Set $(\mathbf{Q} = \mathbf{M})$ and run the code. How has the solution and control behaviour changed ?
5. Vary \mathbf{R} in the range $(0.01, 10)$ and observe how the feedback gain \mathbf{K} varies.
6. For $\mathbf{Q} = 0$, modify the code to solve the control and estimation problem for only the unstable components.

4 Non-linear system with control and estimation

We will solve the non-linear equation for the perturbation $z = w - w_s$.

$$\begin{aligned}\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{dw_s}{dx} + z \frac{dz}{dx} &= \nu \frac{\partial^2 z}{\partial x^2} \\ z(0, t) &= u(t), \quad z_x(1, t) = 0, \quad z(x, 0) = w^\delta(x)\end{aligned}$$

We multiply by a test function ϕ with $\phi(0) = 0$ to obtain the weak formulation

$$\int_0^1 z_t \phi dx + \nu \int_0^1 \frac{\partial z}{\partial x} \frac{d\phi}{dx} dx + \int_0^1 w_s \frac{\partial z}{\partial x} \phi dx + \int_0^1 z \frac{\partial w_s}{\partial x} \phi dx + \int_0^1 w \frac{dw}{dx} \phi dx = 0$$

The finite element solution can be written as

$$z(x, t) = \sum_{j=1}^N z_j(t) \phi_j(x) + u(t) \phi_0(x)$$

while the stationary solution expressed as

$$w_s(x) = \sum_{j=1}^N w_s^j \phi_j(x) + u_s \phi_0(x)$$

The Galerkin approximation is

$$\begin{aligned} \int_0^1 z_t \phi_i dx &= -\nu \int_0^1 \frac{\partial z}{\partial x} \frac{d\phi_i}{dx} dx - \int_0^1 w_s \frac{\partial z}{\partial x} \phi_i dx - \int_0^1 z \frac{\partial w_s}{\partial x} \phi_i dx \\ &\quad - \int_0^1 z \frac{dz}{dx} \phi_i dx, \quad \forall i = 1, 2, \dots, N \end{aligned}$$

Ignoring the $\frac{du}{dt}$ term and using the expression for w_s , we get

$$\begin{aligned} \sum_{j=1}^N z_j'(t) \int_0^1 \phi_i \phi_j dx &= - \sum_{j=1}^N \sum_{k=1}^N z_j(t) z_k(t) \int_0^1 \phi_j \phi_k' \phi_i dx \\ &\quad - \sum_{j=1}^N z_j(t) \left[\nu \int_0^1 \phi_i' \phi_j' dx + u_s \int_0^1 \phi_i (\phi_0 \phi_j)' dx \right] \\ &\quad - \sum_{j=1}^N z_j(t) \left[\sum_{k=1}^N w_s^k(t) \left(\int_0^1 \phi_j \phi_k' \phi_i dx + \int_0^1 \phi_k \phi_j' \phi_i dx \right) \right] \\ &\quad - u(t) \left[\sum_{j=1}^N z_j \int_0^1 \phi_i (\phi_0 \phi_j)' dx + \sum_{k=1}^N w_s^k \int_0^1 \phi_i (\phi_0 \phi_k)' dx \right] \\ &\quad - u(t) \left[\nu \int_0^1 \phi_i' \phi_0' dx + 2u_s \int_0^1 \phi_i \phi_0 \phi_0' dx \right] - u(t)^2 \int_0^1 \phi_i \phi_0 \phi_0' dx \end{aligned}$$

Thus we have the system

$$\mathbf{M} \frac{d\mathbf{z}}{dt} = \mathbf{RHS}_1 + \mathbf{RHS}_2$$

where

$$\begin{aligned}\mathbf{RHS}_1(i) &= -\mathbf{z}^\top \mathbf{D1}(:, :, i) \mathbf{z} - [\nu \mathbf{A1}(i, :) + u_s \mathbf{A2}(i, :) + \mathbf{w}_s^\top (\mathbf{D1}(:, :, i) + \mathbf{D1}(:, :, i)^\top)] \mathbf{z} \\ \mathbf{RHS}_2(i) &= -u(t) [\mathbf{A2}(\mathbf{z} + \mathbf{w}_s) + 2u_s \mathbf{d1} + \nu \mathbf{d2}] - u(t)^2 \mathbf{d1}\end{aligned}$$

The control $u(t)$ is obtained from the linear estimation problem

$$u(t) = - \sum_{j=1}^N K_j z_e^j$$

We solve the non-linear system and the linear estimator in a coupled manner.

$$\begin{aligned}\mathbf{M} \frac{d\mathbf{z}}{dt} &= \mathbf{RHS}_1 + \mathbf{RHS}_2 \\ \mathbf{M} \frac{dz_e}{dt} &= \mathbf{LH}\mathbf{z} + (\mathbf{A} - \mathbf{BK} - \mathbf{LH})\mathbf{z}_e \\ \mathbf{z}(0) &= \mathbf{z}_0, \quad \mathbf{z}_e(0) = 0\end{aligned}$$

This is implemented in `burger_lqg.m`

4.1 Exercises

1. Run program `burger_lqg.m`
2. Vary \mathbf{R} in the range (0.01, 10) and observe how the feedback gain \mathbf{K} changes. How does the solution and control change with \mathbf{R} .

5 List of Programs

1. `get_system_mat.m`: Computes FEM matrices
2. `feedback_matrix.m`: Computes feedback matrix
3. `stationarysol.m`: Computes exact unstable stationary solution
4. `burger.m`: Solves the non-linear model, without feedback
5. `lin_lqg.m`: Solves the coupled estimation and control system for the linear problem
6. `burger_lqg.m`: Solves the coupled estimation and control system for the non-linear problem
7. `rhs_burger.m`: Computes the right hand side for the non-linear problem, without feedback

8. `rhs_burger_lqg.m`: Computes the right hand side for coupled estimation and control system with the non-linear problem