Two dimensional Burgers' equation

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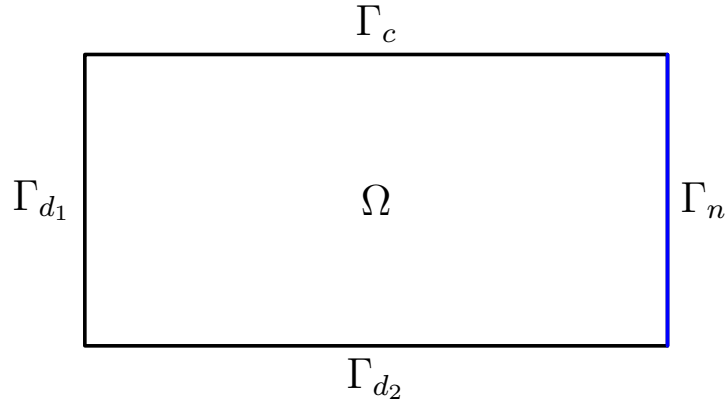
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1   Non-linear Burgers' equation

Figure 1: Problem domain

We consider the non-linear viscous Burgers' equation in two dimensions on the domain $\Omega = (0,1) \times (0,b)$

$$w_t - \nu \Delta w + w w_x = f_s \qquad \text{in} \quad \Omega \times (0, \infty)$$

with boundary conditions, as shown in figure (1)

$$
\begin{aligned}
w(0, y, t) &= u_s \sin(\pi y / b) && \text{on } \Gamma_{d_1} \\
w(x, 0, t) &= 0 && \text{on } \Gamma_{d_2} \\
w(x, b, t) &= u(x, t) && \text{on } \Gamma_c \\
\nu w_x(1, y, t) &= g_s \sin(\pi y / b) && \text{on } \Gamma_n
\end{aligned}
$$

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and initial condition

$$w(\mathbf{x}, 0) = w_0(\mathbf{x}) \quad \text{on } \Omega$$

Here we denote the Dirichlet boundary as

$$\Gamma_d = \Gamma_{d_1} \cup \Gamma_{d_2} \cup \Gamma_c$$

1.1 Observation

We will make a Dirichlet measurement on the strip $I = [b/6, b/2]$ along the right vertical boundary, as shown in figure (2). Thus the observation is given by
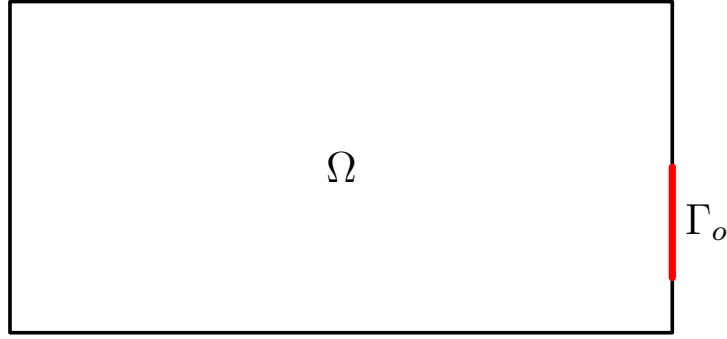
$$y_o(t) = \frac{1}{b/6} \int_{b/6}^{b/3} w(1, y, t) \mathrm{d}y$$

Figure 2: Observations

1.2 Stationary Burgers' equation

The stationary Burgers' equation is given by

$$-\nu \Delta w + w w_x = f_s \quad \text{in } \quad \Omega$$

with boundary conditions

$$
\begin{aligned}
w(0, y, t) &= u_s \sin(\pi y/b) && \text{on } \Gamma_{d_1} \\
w(x, 0, t) &= 0 && \text{on } \Gamma_{d_2} \\
w(x, b, t) &= 0 && \text{on } \Gamma_c \\
\nu w_x(1, y, t) &= g_s \sin(\pi y/b) && \text{on } \Gamma_n
\end{aligned}
$$

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An unstable stationary solution can be obtained as

$$w_s(x, y) = \tilde{w}_s(x) \sin\left(\frac{\pi y}{b}\right)$$

where $\tilde{w}_s(x)$ is the stationary solution evaluated for the one dimensional Burgers' equation

$$\tilde{w}_s(x) = -\frac{\nu\pi}{2}(1 + \epsilon) \tan\left[\frac{\pi}{4}(1 + \epsilon)x + C_0\right]$$

with

$$\epsilon = 0.6 \qquad \Longrightarrow \qquad C_0 = \arctan(1/(1 + \epsilon)) - \frac{\pi}{4}(1 + \epsilon) = -0.69803774609235$$

We then have

$$u_s = \tilde{w}_s(0), \qquad g_s = \nu\frac{\mathrm{d}\tilde{w}_s}{\mathrm{d}x}(1)$$

and f_s is given by the stationary Burgers' equation.

2 Nonlinear system without control

We set $z = w - w_s$. The equation satisfied by z is

$$\frac{\partial z}{\partial t} + w_s\frac{\partial z}{\partial x} + z\frac{\partial w_s}{\partial x} + z\frac{\partial z}{\partial x} = \nu\Delta z \qquad \text{in} \quad \Omega \times (0, \infty)$$

$$z = 0 \quad \text{on } \Gamma_d$$

$$\nu\frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma_n$$

$$z(\mathbf{x}, 0) = w_0(\mathbf{x}) - w_s(\mathbf{x}) \quad \text{on } \Omega$$

We wish to find $z \in L^2(0, T; H^1_{\Gamma_d}(\Omega))$ such that

$$\int_\Omega z_t\phi\mathrm{d}\mathbf{x} + \nu\int_\Omega \nabla z\cdot\nabla\phi\mathrm{d}\mathbf{x} + \int_\Omega w_s\frac{\partial z}{\partial x}\phi\mathrm{d}\mathbf{x} + \int_\Omega z\frac{\partial w_s}{\partial x}\phi\mathrm{d}\mathbf{x} + \int_\Omega z\frac{\partial z}{\partial x}\phi\mathrm{d}\mathbf{x} = 0$$

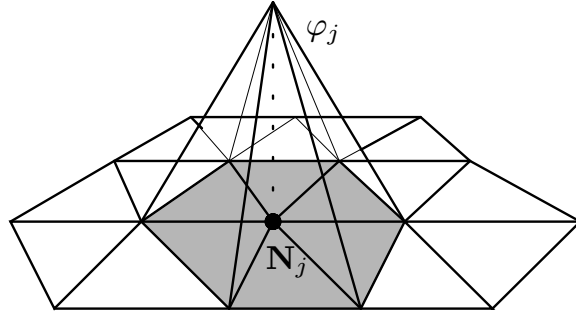for all $\phi \in H^1_{\Gamma_d}(\Omega)$

Figure 3: Piecewise affine basis functions

2.1 FEM approximation

Consider a division of Ω into disjoint triangles. We define the following sets of vertices

$$
\begin{aligned}
N_c &= \text{vertices on } \Gamma_c \\
N_f &= \text{interior vertices and those on } \Gamma_n \text{ (unknown degrees of freedom)}
\end{aligned}
$$

For each vertex i, define the piecewise affine functions $\phi_i(x, y)$, see figure (3), with the property that

$$\phi_i(x_j, y_j) = \delta_{ij}$$

Thus, the finite element solution is of the form

$$z(x, y, t) = \sum_{j \in N_f} z_j(t) \phi_j(x, y)$$

This function satisfies the Dirichlet boundary conditions

$$z(0, y, t) = z(x, 0, t) = z(x, b, t) = 0$$

The Galerkin method is given by

$$\int_\Omega z_t \phi_i \mathrm{d}\mathbf{x} + \nu \int_\Omega \nabla z \cdot \nabla \phi_i \mathrm{d}\mathbf{x} + \int_\Omega w_s \frac{\partial z}{\partial x} \phi_i \mathrm{d}\mathbf{x} + \int_\Omega z \frac{\partial w_s}{\partial x} \phi_i \mathrm{d}\mathbf{x} + \int_\Omega z \frac{\partial z}{\partial x} \phi_i \mathrm{d}\mathbf{x} = 0, \quad \forall i \in N_f$$

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i.e.,

$$\sum_{j \in N_f} z_j'(t) \int_\Omega \phi_i \phi_j = -\nu \sum_{j \in N_f} z_j(t) \int_\Omega \nabla \phi_i \cdot \nabla \phi_j - \sum_{j \in N_f} z_j \int_\Omega w_s \frac{\partial \phi_j}{\partial x} \phi_i$$

$$- \sum_{j \in N_f} z_j \int_\Omega \phi_j \frac{\partial w_s}{\partial x} \phi_i - \int_\Omega z \frac{\partial z}{\partial x} \phi_i$$

$$= \sum_{j \in N_f} A_{ij} z_j - \int_\Omega z \frac{\partial z}{\partial x} \phi_i$$

where

$$A_{ij} = -\nu \int_\Omega \nabla \phi_i \cdot \nabla \phi_j - \int_\Omega w_s \frac{\partial \phi_j}{\partial x} \phi_i - \int_\Omega \phi_j \frac{\partial w_s}{\partial x} \phi_i$$

We can evaluate the integrals as follows.

$$\int_\Omega w_s \frac{\partial \phi_j}{\partial x} \phi_i = \sum_{K:i,j \in K} \frac{\partial \phi_j}{\partial x}(K) \int_K w_s \phi_i$$

$$\approx \sum_{K:i,j \in K} \frac{\partial \phi_j}{\partial x}(K) w_s(K) \int_K \phi_i$$

$$= \sum_{K:i,j \in K} \frac{\partial \phi_j}{\partial x}(K) w_s(K) \frac{|K|}{3}$$

$$\int_\Omega \phi_j \frac{\partial w_s}{\partial x} \phi_i = \sum_{K:i,j \in K} \int_K \phi_j \frac{\partial w_s}{\partial x} \phi_i$$

$$\approx \sum_{K:i,j \in K} \frac{\partial w_s}{\partial x}(K) \int_K \phi_j \phi_i$$

$$\int_\Omega z \frac{\partial z}{\partial x} \phi_i = \sum_{K:i \in K} \int_K z \frac{\partial z}{\partial x} \phi_i$$

$$= \sum_{K:i \in K} \frac{\partial z}{\partial x}(K) \int_K z \phi_i$$

$$= \sum_{K:i \in K} \frac{\partial z}{\partial x}(K) \sum_{j \in K} z_j \int_K \phi_j \phi_i$$

Thus we have a non-linear system of first order ODEs in $\mathbf{z}$

$$\mathbf{M} \frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}\mathbf{z} + N(\mathbf{z})$$

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2.2 Excercises

3 Linearized Burgers' equation

3.1 Feedback control

$$\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{\partial w_s}{\partial x} = \nu \Delta z \qquad \text{in} \quad \Omega \times (0, \infty)$$

$$z = 0 \quad \text{on } \Gamma_d \backslash \Gamma_c, \qquad z = u \quad \text{on } \Gamma_c$$

$$\nu \frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma_n$$

$$z(\mathbf{x}, 0) = w_0(\mathbf{x}) - w_s(\mathbf{x}) \quad \text{on } \Omega$$

The finite element solution is of the form

$$z(x, y, t) = \sum_{j \in N_f} z_j(t) \phi_j(x, y) + \sum_{j \in N_c} u_j(t) \phi_j(x, y), \qquad z(x_i, y_i, t) = z_i(t), \quad \forall i \in N_f$$

This function already satisfies the Dirichlet boundary conditions

$$z(x, 0, t) = z(0, y, t) = 0, \qquad z(x, b, t) = \sum_{j \in N_c} u_j(t) \phi_j(x, b)$$

The Galerkin method is given by

$$\int_\Omega \left(\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{\partial w_s}{\partial x} \right) \phi_i = -\nu \int_\Omega \nabla z \cdot \nabla \phi_i, \qquad i \in N_f$$

Ignoring the $\frac{\mathrm{d}u_j}{\mathrm{d}t}$ term gives us

$$\sum_{j \in N_f} z_j'(t) \int_\Omega \phi_i \phi_j = \sum_{j \in N_f} A_{ij} z_j + \sum_{j \in N_c} A_{ij} u_j, \qquad i \in N_f$$

We will take the control to be of the form

$$u(x, t) = v(t) \sin(\pi x) \qquad \Longrightarrow \qquad u_j(t) = v(t) \sin(\pi x_j)$$

This is a one dimensional control which is compatible with the other boundary condition on Γ_{d_1}

$$u(0, t) = 0$$

Then we obtain

$$\sum_{j\in N_f} z'_j(t) \int_\Omega \phi_i\phi_j = \sum_{j\in N_f} A_{ij}z_j + \left(\sum_{j\in N_c} A_{ij}\sin(\pi x_j)\right) v, \qquad i \in N_f$$

This can be written in matrix form as

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}\mathbf{z} + \mathbf{B}v$$

$v(t)$ is obtained in terms of the feedback matix $\mathbf{K}$

$$v(t) = -\mathbf{K}\mathbf{z}$$

3.2 Partial information with noise

Consider the linearized model with noise in the model

$$\frac{\partial z}{\partial t} + w_s\frac{\partial z}{\partial x} + z\frac{\partial w_s}{\partial x} = \nu\Delta z + \eta \qquad \text{in} \quad \Omega \times (0,\infty)$$

$$z = 0 \quad \text{on } \Gamma_d\backslash\Gamma_c, \qquad z = u(t) \quad \text{on } \Gamma_c$$

$$\nu\frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma_n$$

$$z(\mathbf{x},0) = w_0(\mathbf{x}) - w_s(\mathbf{x}) \quad \text{on } \Omega$$

Assume that we have access to partial information corrupted by noise

$$y_o = Hz + \mu$$

Using the FEM, the observation can be written as

$$\mathbf{y} = \mathbf{H}\mathbf{z} + \boldsymbol{\mu}$$

3.3 Estimation problem

$$\frac{\partial z_e}{\partial t} + w_s\frac{\partial z_e}{\partial x} + z_e\frac{\partial w_s}{\partial x} = \nu\Delta z_e + L(y - Hz_e) \qquad \text{in} \quad \Omega \times (0,\infty)$$

$$z_e = 0 \quad \text{on } \Gamma_d\backslash\Gamma_c, \qquad z_e = u(t) \quad \text{on } \Gamma_c$$

$$\nu\frac{\partial z_e}{\partial n} = 0 \quad \text{on } \Gamma_n$$

$$z_e(\mathbf{x},0) = z_0(\mathbf{x}) \quad \text{on } \Omega$$

In the FEM setup, we have

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}_e}{\mathrm{d}t} = \mathbf{A}\mathbf{z}_e + \mathbf{B}v + \mathbf{L}(\mathbf{y} - \mathbf{H}\mathbf{z}_e)$$

3.4 Coupled linear system

The feedback is based on estimated solution $v = -\mathbf{K}\mathbf{z}_e$ which leads to the following coupled system

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}\mathbf{z} - \mathbf{B}\mathbf{K}\mathbf{z}_e + \boldsymbol{\eta}$$

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}_e}{\mathrm{d}t} = \mathbf{L}\mathbf{H}\mathbf{z} + (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{H})\mathbf{z}_e + \mathbf{L}\boldsymbol{\mu}$$

or in matrix form

$$\begin{bmatrix} \mathbf{M} & 0 \\ 0 & \mathbf{M} \end{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}_e \end{bmatrix} = \begin{bmatrix} \mathbf{A} & -\mathbf{B}\mathbf{K} \\ \mathbf{L}\mathbf{H} & \mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{H} \end{bmatrix} \begin{bmatrix} \mathbf{z} \\ \mathbf{z}_e \end{bmatrix} + \begin{bmatrix} \mathbf{I} & 0 \\ 0 & \mathbf{L} \end{bmatrix} \begin{bmatrix} \boldsymbol{\eta} \\ \boldsymbol{\mu} \end{bmatrix}$$

The initial condition is given by

$$\mathbf{z}(0) = \mathbf{z}_0, \qquad \mathbf{z}_e(0) = 0$$

3.5 Excercises

1. Compute feedback gain $\mathbf{K}$

2. Compute filtering gain $\mathbf{L}$

4 Non-linear system with control

$$\frac{\partial z}{\partial t} + w_s \frac{\partial z}{\partial x} + z \frac{\partial w_s}{\partial x} + z \frac{\partial z}{\partial x} = \nu \Delta z \quad \text{in} \quad \Omega \times (0, \infty)$$

$$z = 0 \quad \text{on } \Gamma_d \backslash \Gamma_c, \qquad z = u(t) \quad \text{on } \Gamma_c$$

$$\nu \frac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma_n$$

$$z(\mathbf{x}, 0) = w_0(\mathbf{x}) - w_s(\mathbf{x}) \quad \text{on } \Omega$$

The Galerkin method leads the following set of ODE

$$\mathbf{M}\frac{\mathrm{d}\mathbf{z}}{\mathrm{d}t} = \mathbf{A}\mathbf{z} + \mathbf{B}v + N(\mathbf{z})$$

Excercises

5 List of Programs