HYBRID FINITE-VOLUME/MESHLESS METHOD FOR HYPERBOLIC CONSERVATION LAWS

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Consider the scalar conservation law

$$\frac{\partial v}{\partial t} + \frac{\partial f}{\partial x} = 0 \tag{1}$$

where v is a conserved quantity and $f = f(v)$ is the flux.

1. Finite volume method

Let u denote the numerical solution of the conservation law. The finite volume semi-discretization of the conservation law is

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} + \frac{F_{i+1/2} - F_{i-1/2}}{h_i} = 0 \tag{2}$$

where $F_{i+1/2} = F(u_i, u_{i+1})$ is a numerical flux function and the above equation is solved using a time stepping scheme like Runge-Kutta method. We assume that the numerical flux function is an E-flux, i.e.,

$$\operatorname{sign}(u_{j+1} - u_j)(F_{i+1/2} - f(u)) \le 0, \quad \text{for all } u \text{ between } u_j, u_{j+1} \tag{3}$$

For the linear advection equation $f = au$, the above scheme is stable under the CFL condition

$$\frac{|a|\Delta t}{h_i} \le 1 \tag{4}$$

The second order scheme is obtained using the numerical flux $F_{i+1/2} = F(u_{i+1/2}^-, u_{i+1/2}^+)$. The interface values $u_{i+1/2}^\pm$ are obtained by reconstruction

$$u_{i+1/2}^- = u_i + \frac{1}{2}\Delta(u_{i-1}, u_i, u_{i+1}), \quad u_{i+1/2}^+ = u_{i+1} - \frac{1}{2}\Delta(u_i, u_{i+1}, u_{i+2}), \tag{5}$$

2. Interpolation scheme

Let point p be a point of a grid which cannot be updated using the finite volume method. The solution at p can be obtained by interpolating the solution from a donar grid.

2.1. **Linear interpolation.** Locate two points j, $j+1$ in the donar grid such that $x_j \le x_p \le x_{j+1}$. Then the linear interpolation is given by

$$u_p = \alpha u_j + (1-\alpha)u_{j+1}, \quad \alpha = \frac{x_{j+1} - x_p}{x_{j+1} - x_j} \tag{6}$$

2.2. **Quadratic interpolation.** As in the case of linear interpolation, locate two points j, $j+1$ from the donar grid. Then choose a third point which could either be $j-1$ or $j+2$ depending on which one is close to p. The quadratic interpolant can be obtained by fitting a polynomial $P(x) = p_o + p_1(x-x_p) + p_2(x-x_p)^2$ or using Lagrange interpolation formula.

2.3. **Cubic interpolation.** As in the case of linear interpolation, locate two points j, $j+1$ from the donar grid. The stencil for interpolation consists of $(j-1, j, j+1, j+2)$. The cubic interpolant can be obtained by fitting a polynomial $P(x) = p_o + p_1(x-x_p) + p_2(x-x_p)^2 + p_3(x-x_p)^3$ or using Lagrange interpolation formula.

3. Meshless method

We use a meshless method based on least squares for estimating the partial derivatives arising in the conservation law. Let point p be a boundary point which must be updated using interpolation. Instead we apply meshless method to this point. We select a point to the left j and right k of node p from a donar grid. The solution at p will be updated using the stencil (j,p,k). The semi-discrete meshless method is

$$\frac{\mathrm{d}u_p}{\mathrm{d}t} + 2\frac{w_{jp}(F_{jp}-f_p)(x_j-x_p) + w_{pk}(F_{pk}-f_p)(x_k-x_p)}{[w_{jp}(x_j-x_p)^2 + w_{pk}(x_k-x_p)^2]} = 0 \tag{7}$$

where $f_p = f(u_p)$ and F_{jp} is a numerical flux function evaluated at the middle of the interval $[x_j, x_p]$ and w is a positive weight function which is usually of the form $w_{jp} = |x_p - x_j|^{-s}$, $s \ge 0$. The same numerical flux function can be used in both the finite volume method and the meshless method.

3.1. **Stability analysis.** Since the numerical flux is an E-flux, we have

$$a_{jp} = \frac{F_{jp}-f_p}{u_p-u_j} \le 0, \quad a_{pk} = \frac{F_{pk}-f_p}{u_k-u_p} \le 0 \tag{8}$$

Equation (7) can be re-written as

$$\frac{\mathrm{d}u_p}{\mathrm{d}t} = A_{jp}(u_j-u_p) + B_{pk}(u_k-u_p) \tag{9}$$

where

$$A_{jp} = -\frac{2w_{jp}a_{jp}(x_p-x_j)}{w_{jp}(x_j-x_p)^2 + w_{pk}(x_k-x_p)^2} \ge 0$$

$$B_{jp} = -\frac{2w_{pk}a_{pk}(x_k-x_p)}{w_{jp}(x_j-x_p)^2 + w_{pk}(x_k-x_p)^2} \ge 0$$

The semi-discrete scheme is local extremum diminishing in the sense of Jameson, i.e., local minima do not decrease and local maxima do not increase. Using explicit time integration scheme

$$u_p^{n+1} = [1 - \Delta(A_{jp}+B_{pk})]u_p^n + \Delta t A_{jp} u_j^n + \Delta t B_{pk} u_k^n$$

The above is stable in the maximum norm if the following time-step condition is satisfied

$$\Delta t \le \frac{1}{A_{jp} + B_{pk}} \tag{10}$$

In the case of linear advection equation $a_{jp} = -a^+ = -\max(0,a)$ and $a_{pk} = a^- = \min(0,a)$. Let h be the grid spacing of the donar grid.

3.1.1. *Stencil* $(i, p, i+1)$. Let the stencil be $(x_j, x_p, x_k) = (x_i, x_p, x_{i+1})$. Let $x_p - x_j = \alpha h$ and $x_k - x_p = (1-\alpha)h$ so that $\alpha \in [0,1]$. Specializing to the case of linear advection $f(u) = au$, we obtain the following restriction on the time-step

$$\Delta t \le \frac{[\alpha^{2-p} + (1-\alpha)^{2-p}]h}{2[\alpha^{1-p}a^+ - (1-\alpha)^{1-p}a^-]} \tag{11}$$

Taking the case $a > 0$, we obtain the CFL condition

$$\frac{a\Delta t}{h} \le \frac{[\alpha^{2-p} + (1-\alpha)^{2-p}]}{2\alpha^{1-p}} \tag{12}$$

Let us look at this condition for different values of the weight power s which is shown in table (1). The weights $s = 0$ and $s = 1$ leads to a non-zero CFL condition for all values of α, while for $s > 1$, the CFL number can become very small, leading to very small time-steps.

s	CFL condition	$\alpha \to 0$	$\alpha \to 1$	Min CFL
0	$\frac{a\Delta t}{h} \le \frac{[\alpha^2+(1-\alpha)^2]}{2\alpha}$	∞	0.5	0.4142
1	$\frac{a\Delta t}{h} \le 0.5$	0.5	0.5	0.5
2	$\frac{a\Delta t}{h} \le \alpha$	0	1	0
3	$\frac{a\Delta t}{h} \le \frac{\alpha}{2(1-\alpha)}$	0	∞	0

TABLE 1. CFL condition for linear advection equation for different weights

3.1.2. *Enlarged stencil.* Assume that $x_i \le x_p < x_{i+1}$ and $0 \le \alpha \le 1/2$. We choose the stencil $(i-1, p, i+1)$ which leads to the time-step condition,

$$\Delta t \le \frac{[(1+\alpha)^{2-p} + (1-\alpha)^{2-p}]h}{2[(1+\alpha)^{1-p}a^+ - (1-\alpha)^{1-p}a^-]} \tag{13}$$

The CFL condition and minimum CFL number are shown in table (2). We see that with $s = 0$ and $s = 1$, the minimum CFL number is close to unity; in fact with $s = 1$ the scheme is stable under the same condition as the finite volume method.

4. TEST CASES

4.1. **Linear advection equation.** In this case the flux is linear, $f(v) = av$, and we take $a = 1$. The exact solution is obtained by advecting the initial condition at speed a.

4.2. **Burgers equation.** In this case the flux is quadratic, $f(v) = v^2/2$.

s	CFL condition	Min CFL	CFL condition	Min CFL
0	$\frac{a\Delta t}{h} \le \frac{[(1+\alpha)^2+(1-\alpha)^2]}{2(1+\alpha)}$	0.8284	$\frac{a\Delta t}{h} \le \frac{[(1+\alpha)^2+(1-\alpha)^2]}{2(1-\alpha)}$	1
1	$\frac{a\Delta t}{h} \le 1$	1	$\frac{a\Delta t}{h} \le 1$	1
2	$\frac{a\Delta t}{h} \le 1+\alpha$	1	$\frac{a\Delta t}{h} \le 1-\alpha$	0.5

TABLE 2. CFL condition for linear advection equation for different weights using enlarged stencil