APPM 2360: Credit Model Project 1

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Abstract

This paper will analyze the behviour of different equations as possible models for credit card debt. Four different equations will be considered. The first two equations will be used to compare the accuracy of a discrete time model of debt where intrest is calculated a certain number of time each year against a continous model in which the debt is calculated an infinite number of times each year. These two models will be compared at various intrest rates. The final two models are differential equations, one of which defines the continous model and one which describes an "installment debt" model in which an account holder pays a set amount of debt off each year. These models will be considered for their properties and how long it will take to pay off a set debt. Finally an "installment savings" model will be created based on the reviewed models and then analyzed in the context of the observed properties of the previous models under consideration.

Contents

Part 1

Models under Consideration

1.1 Discrete Model for Account Balance

The first model being reviewed in this paper is a discrete model for a credit account balance assuming that the intrest is calculated "manually" a certain number of times in a time interval (one year in this case). The model is the below model 1.

$$A_d(t) = A_0(1 + \frac{r}{n})^{nt} \tag{1}$$

In the above model 1 $A_d(t)$ is the value of the debt accumulated on the account, r is the annual intrest rate (some value such that $0 \leq r \leq 1$), n is the number of times per year that the intrest is applied to the account, t is the time in years since the account was created, and A_o is the balance of the account at time 0 (initial balance).

1.2 Continous Model for Account Balance

The second model being reviewed in this paper is a continous model for a credit account balance assuming that intrest is calculated an infinite number of times in a time interval (one year in this case). The model is the below model 2.

$$A_c(t) = A_0 e^{rt} \tag{2}$$

In the above model 2 $A_c(t)$ is the value of the debt accumulated on the account, r is the annual intrest rate (some value such that $0 \leq r \leq 1$), t is the time in years since the account was created, and A_o is the balance of the account at time 0 (initial balance). The n term for the number of times per year that the intrest is applied to the account is missing as this model applies intrest an infinite number of times every year.

1.3 Rate of Change in Account Balance in Continous Model

The third model being reviewed in this paper is the derivative of model 2 which creates the below differential equation.

$$\frac{dA_c}{dt} = rA_c \tag{3}$$

In the above model 3 A_c is the value of the debt accumulated on the account, r is the annual intrest rate (some value such that $0 \leq r \leq 1$), t is the time in years since the account was created, and $\frac{dA_c}{dt}$ is rate of change of the balance of the account. This model will be used to compare model 4.

1.4 Installment Debt Rate Model

The fourth model being reviewed in this paper is an "installment debt" which calculates the debt accumulated or lost over time assuming that the account holder pays a set amount of debt off every year. This model is the below model 4.

$$\frac{dA_c}{dt} = rA_c - P \tag{4}$$

In the above model 3 A_c is the value of the debt accumulated on the account, r is the annual intrest rate (some value such that $0 \leq r \leq 1$), t is the time in years since the account was created, P is the amount of debt paid off each year, and $\frac{dA_c}{dt}$ is rate of change of the balance of the account. This model will be used to compare model 4.

1.4.1 Classification of model 4

Model 4 is a first order equation as the highest derivative is the second derivative, but it is not homogenous and there is a $-P$ term that comes from the amount being paid by the account holder each year. This model is autonomous as time does not affect the derivative, only the y-value has an effect. All of the terms in model 4 have constant coefficients as P is unchanging and r is unchanging. Nothing else can be said to classify model 4 as it is non-homogenous.

Part 2

Comparison of Discrete and Continous Account Balance Models

2.1 Comparison of Discrete model as compounding rate increases

In order to compare model 1 to model 2 it is useful to consider the behaviour of model 1 when n (the rate at which intrest is calculated) is increased to infinity such that model 1 should become model 2. The below calculations were performed:

$A_d(t) = A_o(1+\frac{r}{n})^{nt}$ ts taken to the limit $lim_{n\to\infty}A_o(1+\frac{r}{n})^{nt}$ it becomes $A_o lim_{n\to\infty}(1+\frac{r}{n})^{nt}$ so that we can then define $u = \frac{n}{r}$ where $u \to \infty$ so that we can then write $A_o lim_{u\to\infty}(1+\frac{1}{u})^{urt}$ and by definition $lim_{u\to\infty}(1+\frac{1}{u})^u = e$ so the overall equation becomes $A_o e^{rt}$ which is model 2 as originally expected, thus proving that when $n \to \infty$ model 1 becomes model 2.

2.2 Model 1 and 2 Solved for time

Using algebra with model 1 results in the below equation for t:

$$t = ln(\frac{\frac{A}{A_o}}{nln(\frac{r}{n}+1)})$$

Using algebra with mode 2 results in the below equation for t:

$$t = \frac{ln(\frac{A}{A_o})}{r}$$

These equations can be easily be reproduced by manipulating the model equations given in this paper. These solutions allow a debtor to calculate how long it will take for their debt to become a certain value based on the intrest rate, initial debt, and for model 1 the rate at which that debt is calculated.

2.3 Compounding rate calculated annually

In order to better understand the behaviors of model 1 and model 2 the difference between the two models was investigated using a variety of different conditions. In the below plots of the continous and discrete models' behavior over 100 years the account was assumed to have initial account balance of \$1, an intrest rate of 10% and 20%, with the assumption that the discrete model only had intrest applied each year.

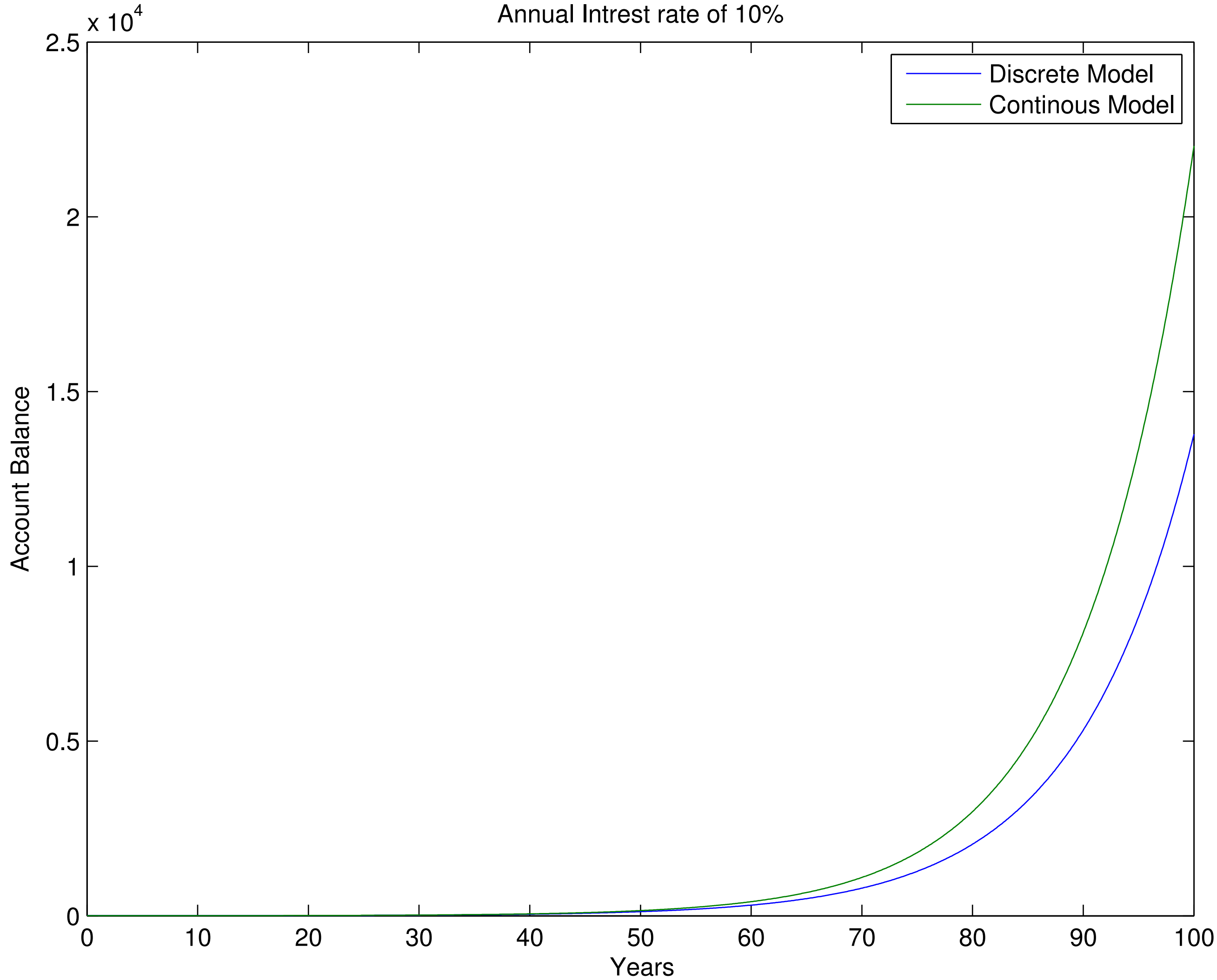

Figure 1: Annual Compouding Rate of 10%

As can be seen in the above figure 1 when the intrest was calculated yearly the discrete model differed from the continous model by nearly \$10,000. These models were then reviewed with an intrest rate of 20%.

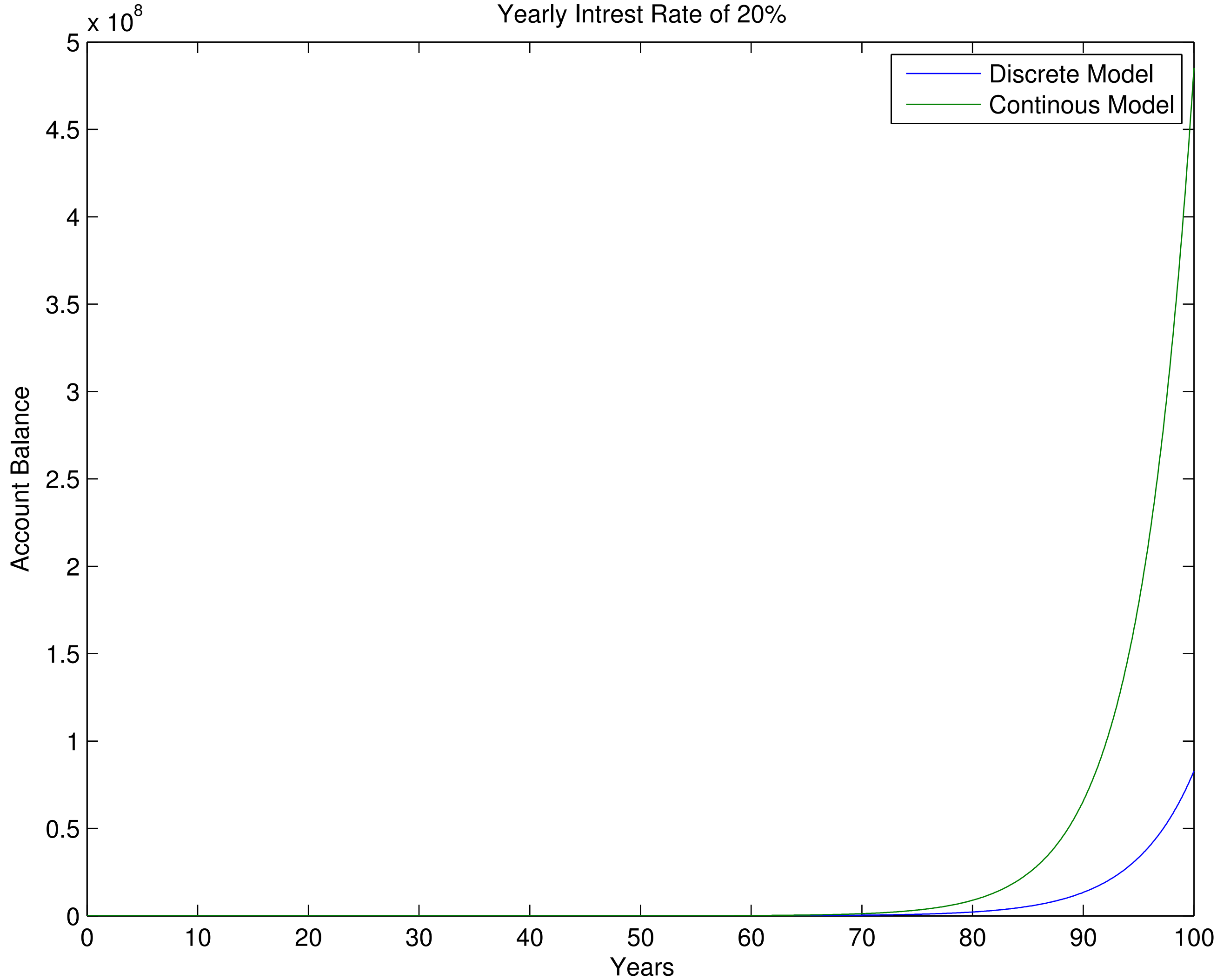

Figure 2: Annual Compounding Rate of 20%

In the above figure 2 we can see very clearly that the difference between the continous and discrete models is significantly greater when the intrest rate is increased to 20%. Based on the plot above it can be seen that when the discrete model is calculated each year it does not match the continous model, as should be expected as the continous model by definition will result in a greater account balance as it calculates intrest more frequently. This behavior matches the plots that have been created, thus validating the results in figures 1 and 2.

2.4 Compounding rate calculated biannually

In the below plots of the continous and discrete models' behavior over 100 years the account was assumed to have initial account balance of \$1, an intrest rate of 10% and 20%, with the assumption that the discrete model only had intrest applied twice each year.

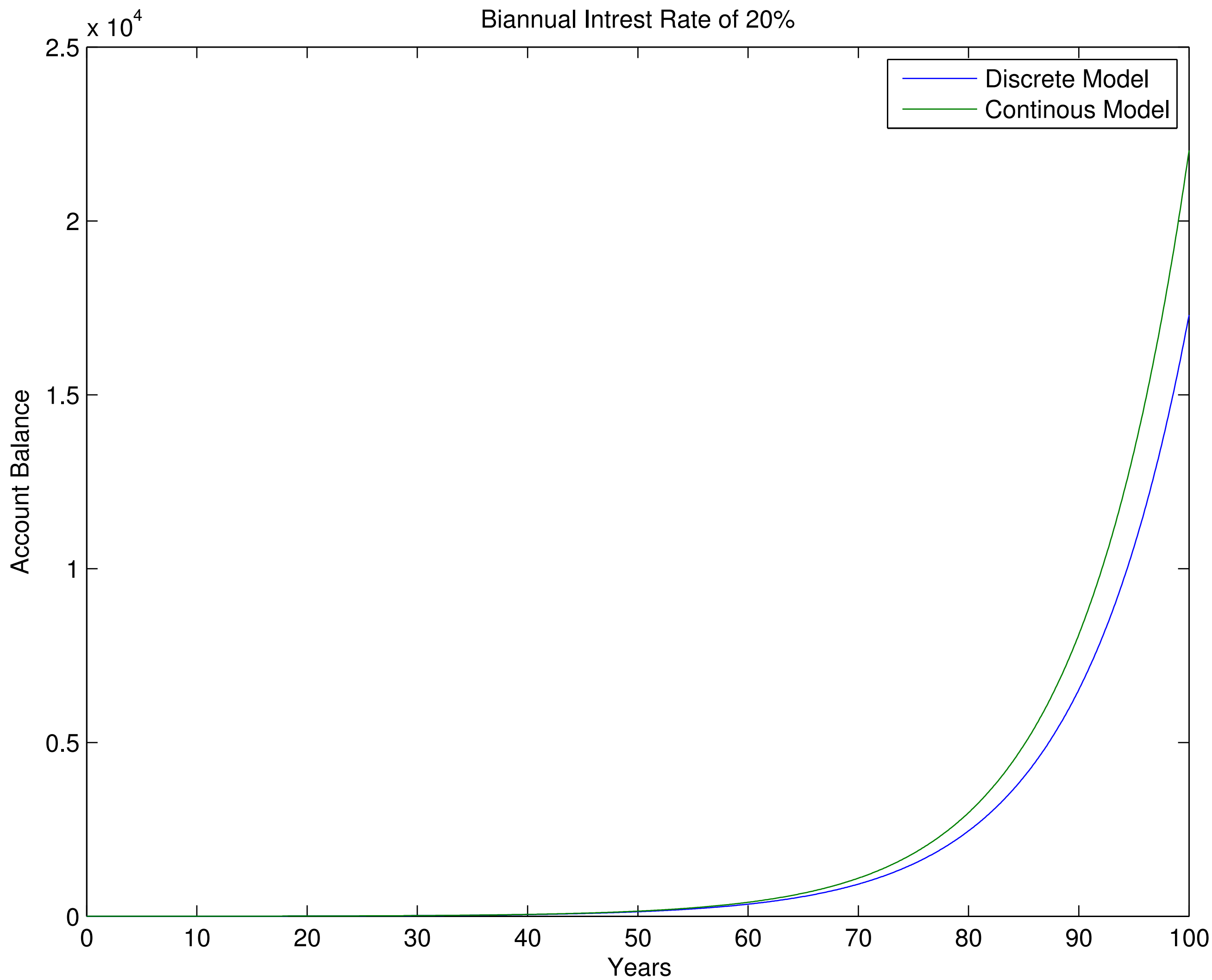

Figure 3: Biannual Compouding Rate of 10%

As can be seen in the above figure 3 when the intrest was calculated biannually the discrete model differed from the continous model by around \$5,000. These models were then reviewed with an intrest rate of 20%.

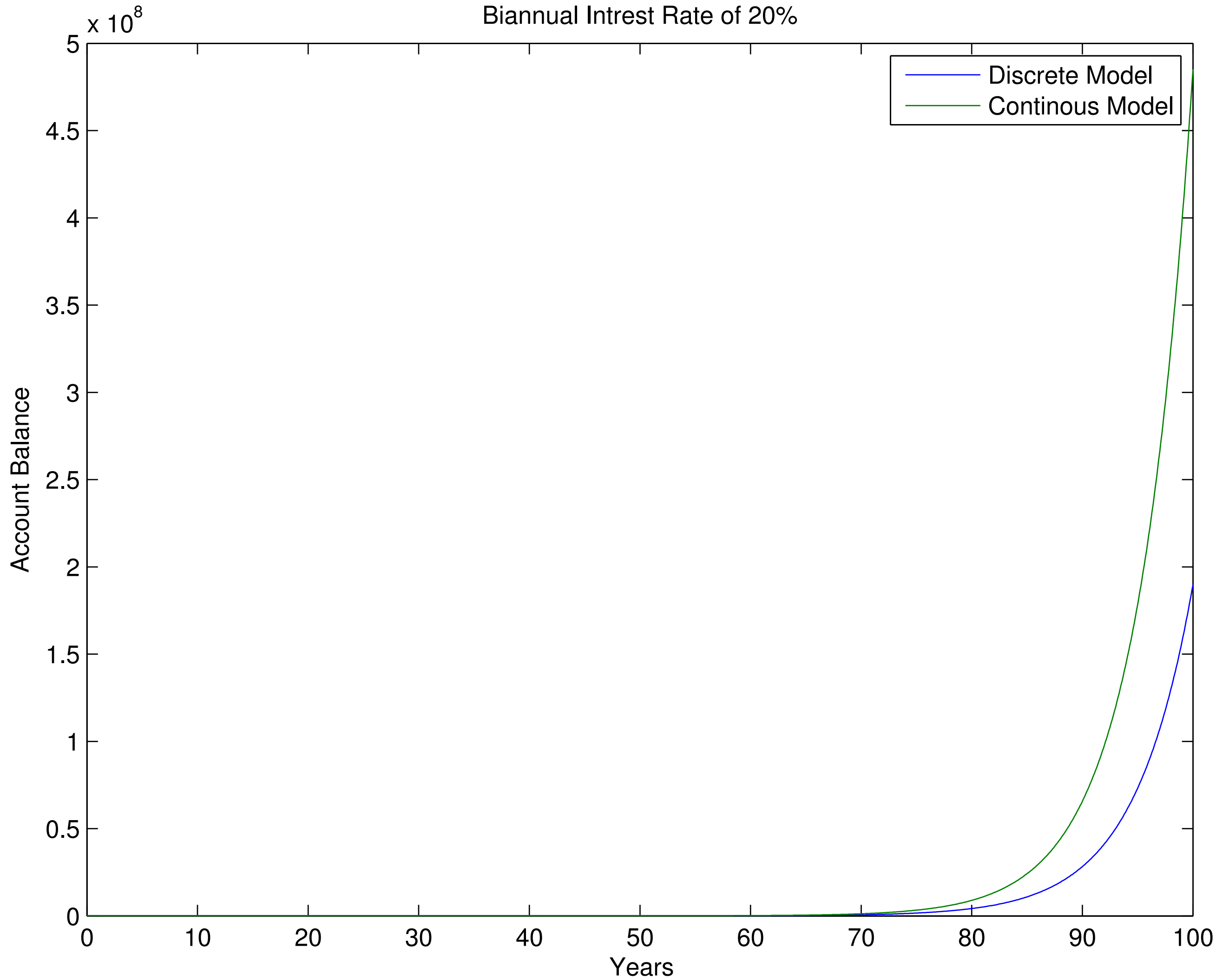

Figure 4: Biannual Compounding Rate of 20%

In the above figure 4 we can see very clearly that the difference between the continous and discrete models is significantly greater when the intrest rate is increased to 20%. The behavior of the plots in figures 3 and 4 matches the behavior in figures 1 and 2, which is to be expected as the model behavior should be similar. This behavior matches the plots that have been created, thus validating the results in figures 3 and 4.

2.5 Compounding rate calculated quarterly

In the below plots of the continous and discrete models' behavior over 100 years the account was assumed to have initial account balance of \$1, an intrest rate of 10% and 20%, with the assumption that the discrete model only had intrest applied quarterly.

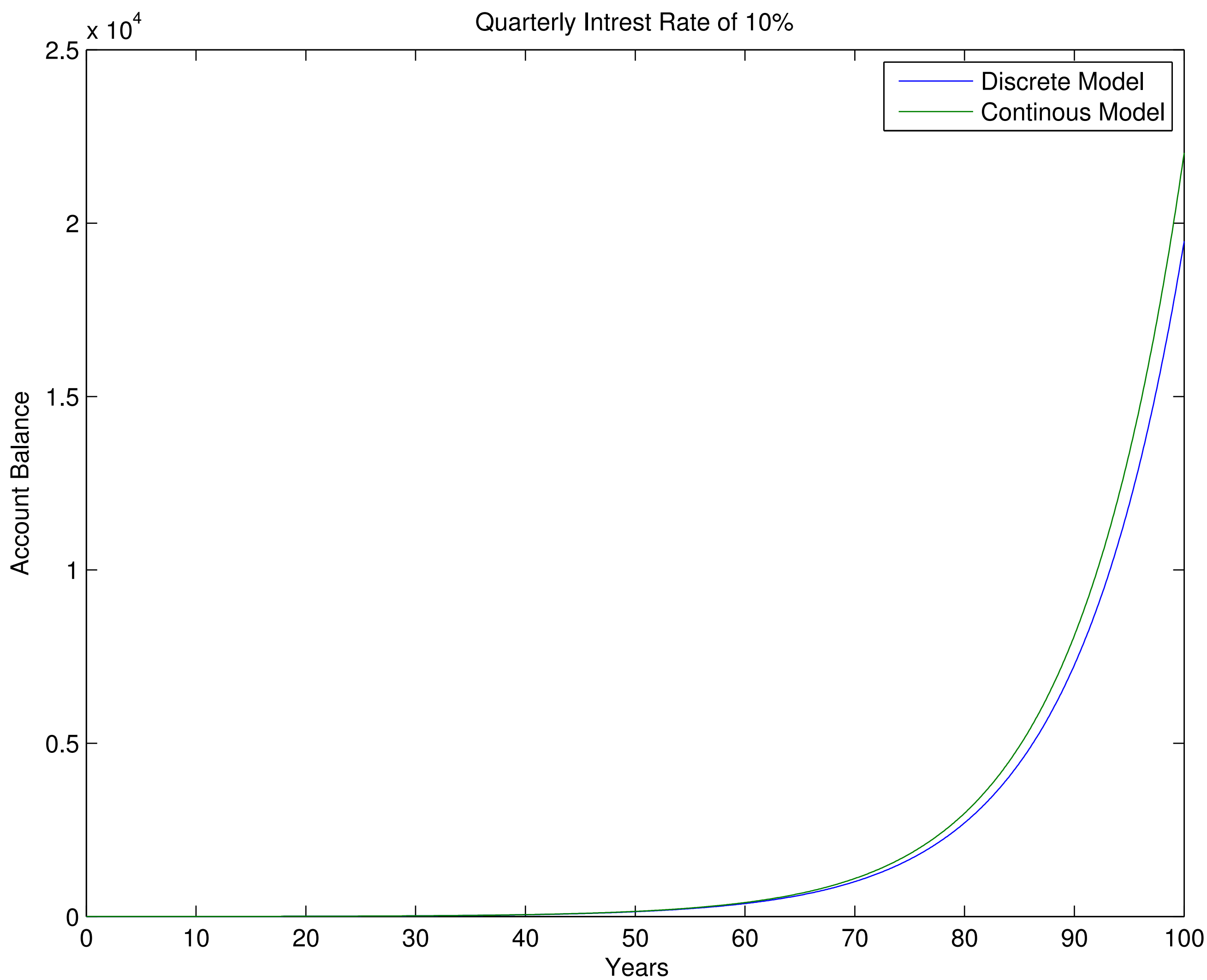

Figure 5: Quarterly Compouding Rate of 10%

As can be seen in the above figure 5 when the intrest was calculated biannually the discrete model differed from the continous model by around $1,000. These models were then reviewed with an intrest rate of 20%.

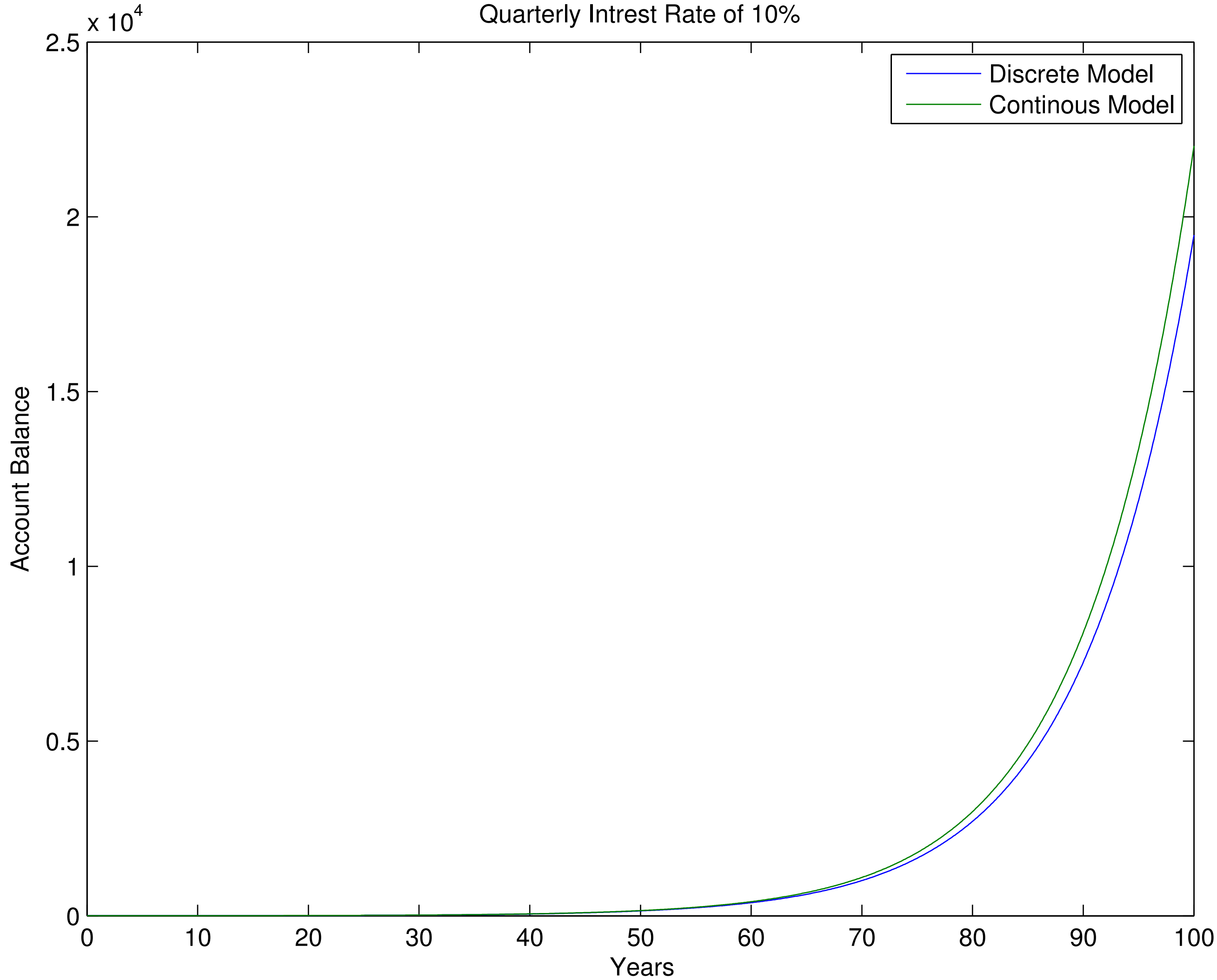

Figure 6: Quarterly Compounding Rate of 20%

In the above figure 6 we can see very clearly that the difference between the continous and discrete models is significantly greater when the intrest rate is increased to 20%. The behavior of the plots in figures 5 and 6 matches the behavior in figures 1 and 2, which is to be expected as the model behavior should be similar. This behavior matches the plots that have been created, thus validating the results in figures 5 and 6.

2.6 Compounding rate calculated monthy

In the below plots of the continous and discrete models' behavior over 100 years the account was assumed to have initial account balance of $1, an intrest rate of 10% and 20%, with the assumption that the discrete model only had intrest applied monthly.

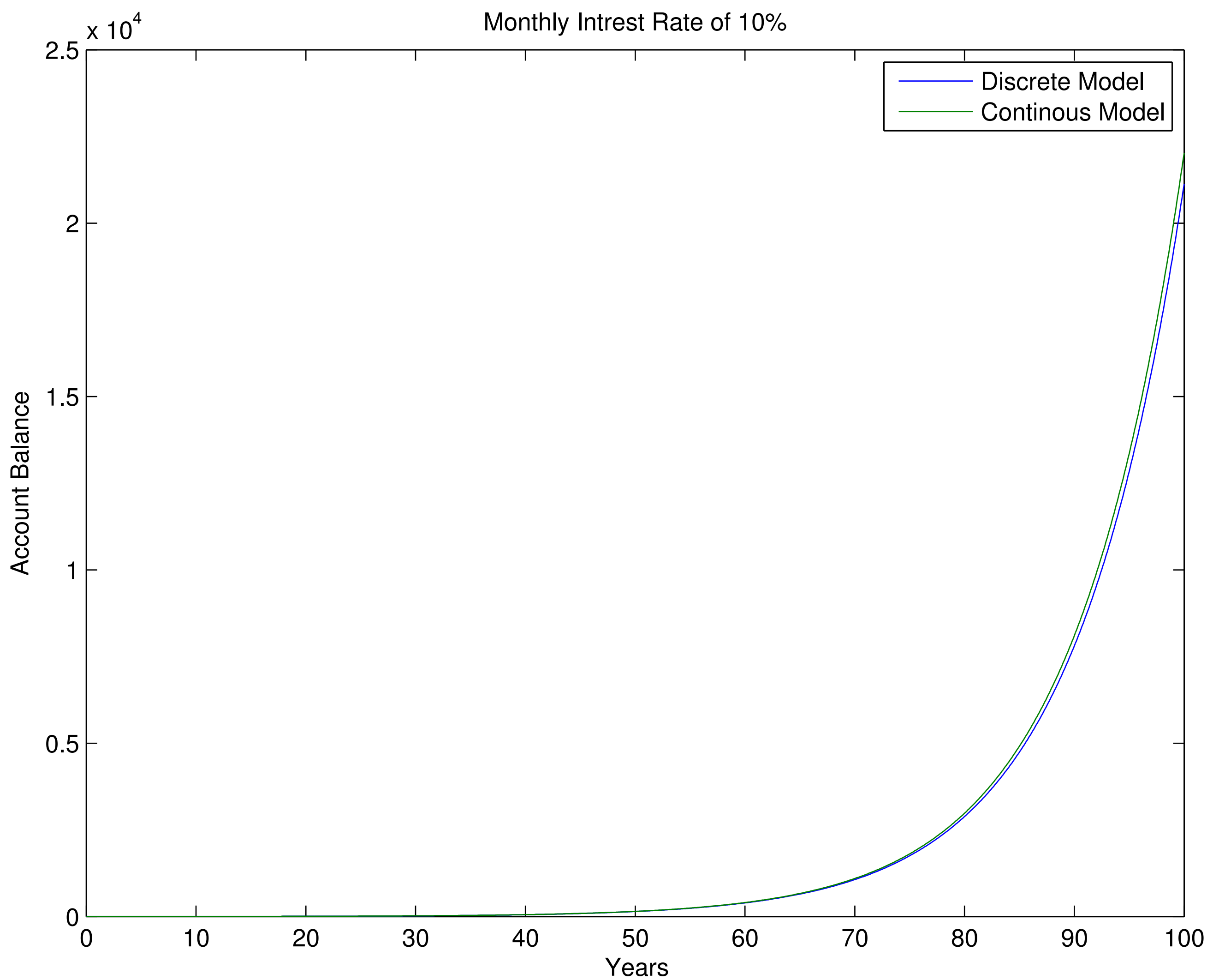

Figure 7: Monthly Compouding Rate of 10%

As can be seen in the above figure 7 when the intrest was calculated biannually the discrete model differed from the continous model by around \$500. These models were then reviewed with an intrest rate of 20%.

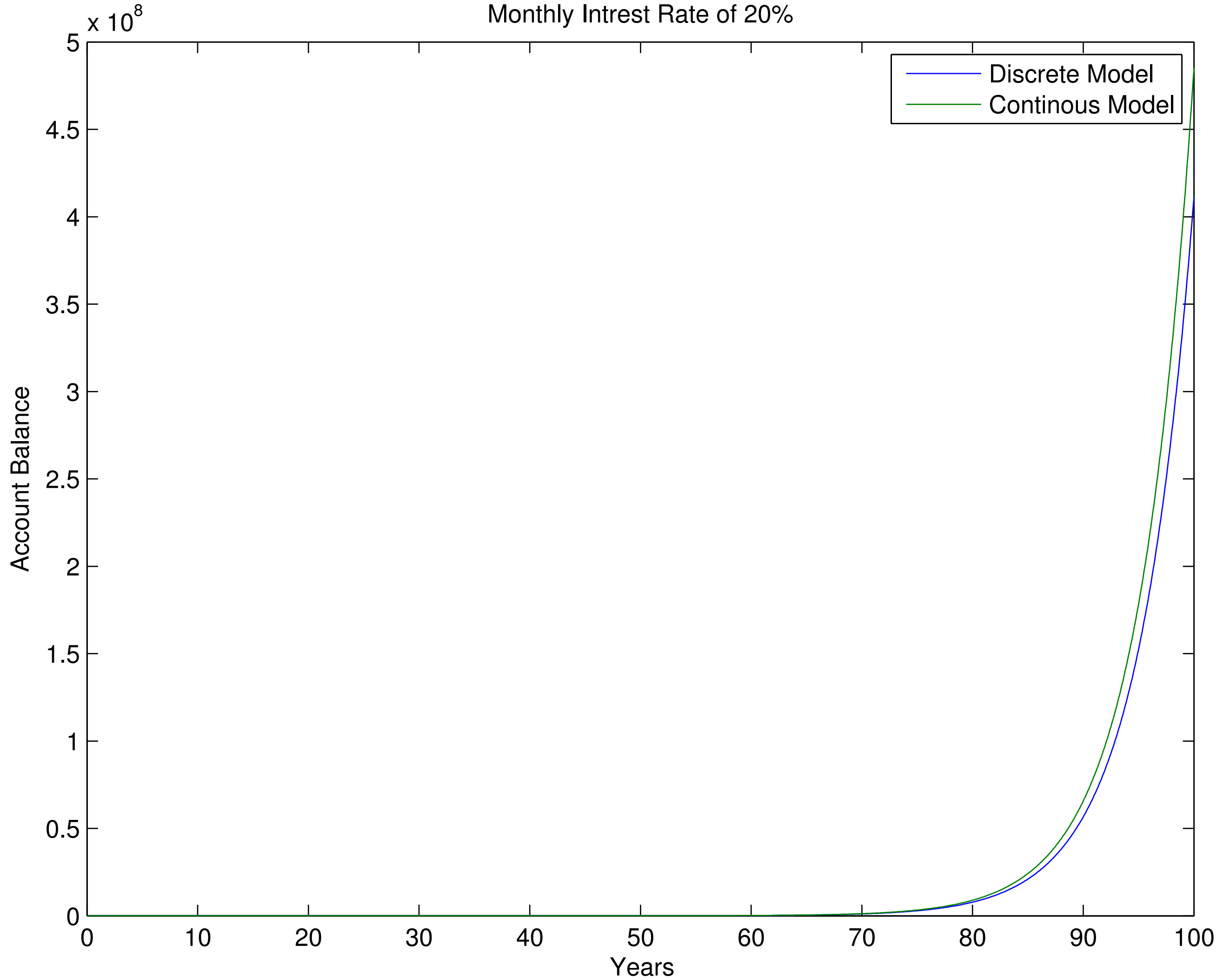

Figure 8: Monthly Compounding Rate of 20%

In the above figure 8 we can see very clearly that the difference between the continous and discrete models is significantly greater when the intrest rate is increased to 20%. The behavior of the plots in figures 7 and 8 matches the behavior in figures 1 and 2, which is to be expected as the model behavior should be similar. This behavior matches the plots that have been created, thus validating the results in figures 7 and 8.

2.7 Accuracy of Discrete model compared to Continous model

2.7.1 Qualitative Review

In the below plots of the continous and discrete models' behavior over 100 years the account was assumed to have initial account balance of \$1, an intrest rate of 10% and 20%, with the assumption that the discrete model only had intrest applied daily.

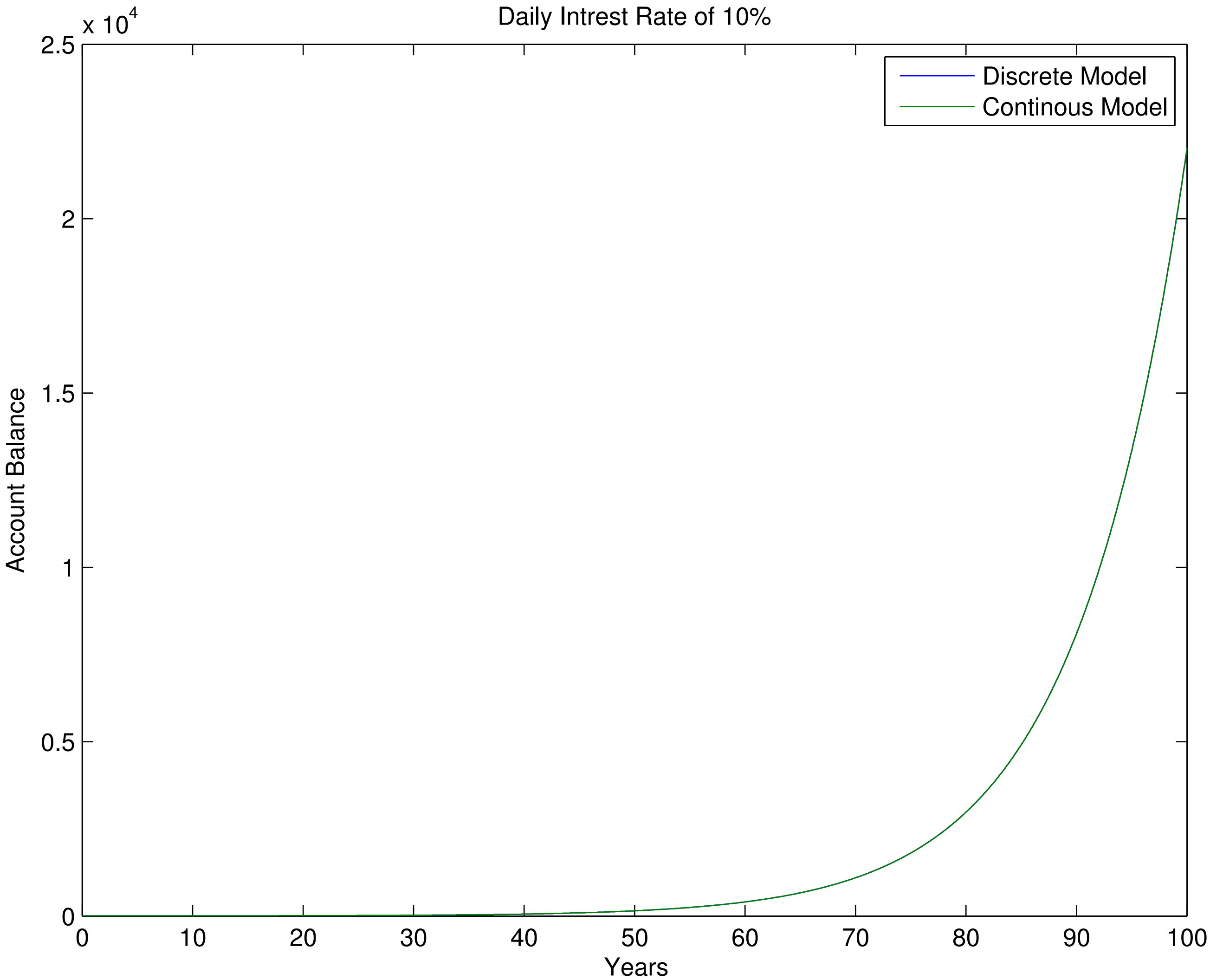

Figure 9: Behavior of Discrete Model when Daily 10% Intrest is calculated

As can be seen in the above figure 9 when the intrest was calculated biannually the discrete model differed from the continous model by around \$10. These models were then reviewed with an intrest rate of 20%.

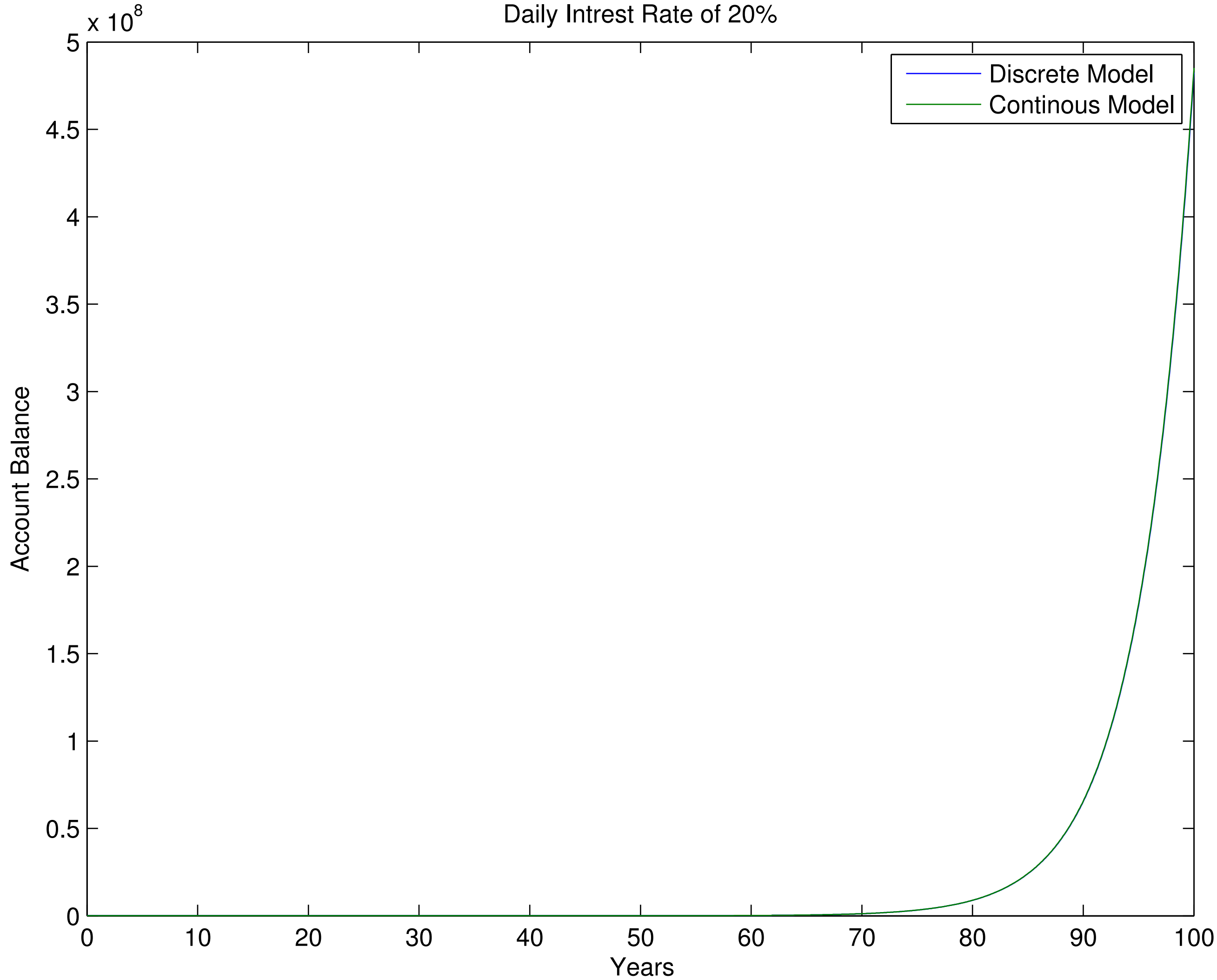

Figure 10: Behavior of Discrete Model when Daily 20% Intrest is calculated

The behavior of the plots in figures 9 and 10 matches the behavior in all previous plots, which is to be expected as the model behavior should be similar.

When reviewing the above plots we can see that the difference between the discrete and continous model increases when the intrest rate increases and when the discrete models compounding rate decreases. As such we can say that model 2 most accurately approximates model 1 when model 1 has a compounding rate that is large (in this case daily) and the intrest rate is relatively low. As such we can conclude that model 2 is an accurate approximation for the values showcased in this section.

Rate	Year
6.9%	107
12.99%	53
19.99%	33

Assumptions: $A_0 = 1$ and $n = 365$

Table 1: Year Discrete and Continous Models differ by one dollar

2.7.2 Quantitative Review

Models 1 and 2 were then reviewed to discovery in which year the difference between the two models became greater than a dollar assuming that the account started with \$1 of debt and that model 1 was calculated with a daily compounding rate. These models were calculated with intrest rates of 6.9%, 12.99%, and 19.99%. The results of these calculations are shown in the below table 1.

As can be clearly seen the pattern observed in section 2 still holds as the time it takes for the two models to differ y more than a dollar decreases as the intrest rate increases. These results support the findings showcased in the above figures and the conclusion discussed in section 2.

Part 3

Characterization of Installment Debt Model

3.1 Equilibrium Solution

This paper's examination of model 4 begins by considering its equilibrium solution. Based on the given model we can conclude that the equilibrium solution is $\frac{p}{r} = A_c$, as this is the only place for which $\frac{dA_c}{dt}$ is exactly zero for all values of t. This equilibrium solution is the point at which the account holder has enough debt that their rate of monthly payment exactly equals their intrest rate. This equilibrium solution is unstable as a value either slightly above or slightly below this value will result in exponentially increasing debt or exponentially decreasing debt respectively. As such they will be stuck with a static amount of debt as they will only every be paying off the intrest on the account.

3.2 Derivative Behavior

In order to better understand the behavior of model 4 the rate of change of the account balance was plotted against the account balance with the assumption that the account had an intrest rate of 10% and the account holder paid \$20 a year against the account. The resulting figure 11 is shown below.

As we can clearly see in the above figure 11 when the account has a balance of \$200 the account balance is unchanging in time as the rate of payment (\$20) is exactly equal to the intrest charged against the account. When the account balance is less than \$200 the account balance (debt) is decreasing ($\frac{dA_c}{dt} < 0$) as the rate of payment is greater than the intrest being charged to the account. When the account balance is greater than \$200 the account balance (debt) is increasing($\frac{dA_c}{dt} > 0$) as the rate of payment is less than the intrest being charged to the account.

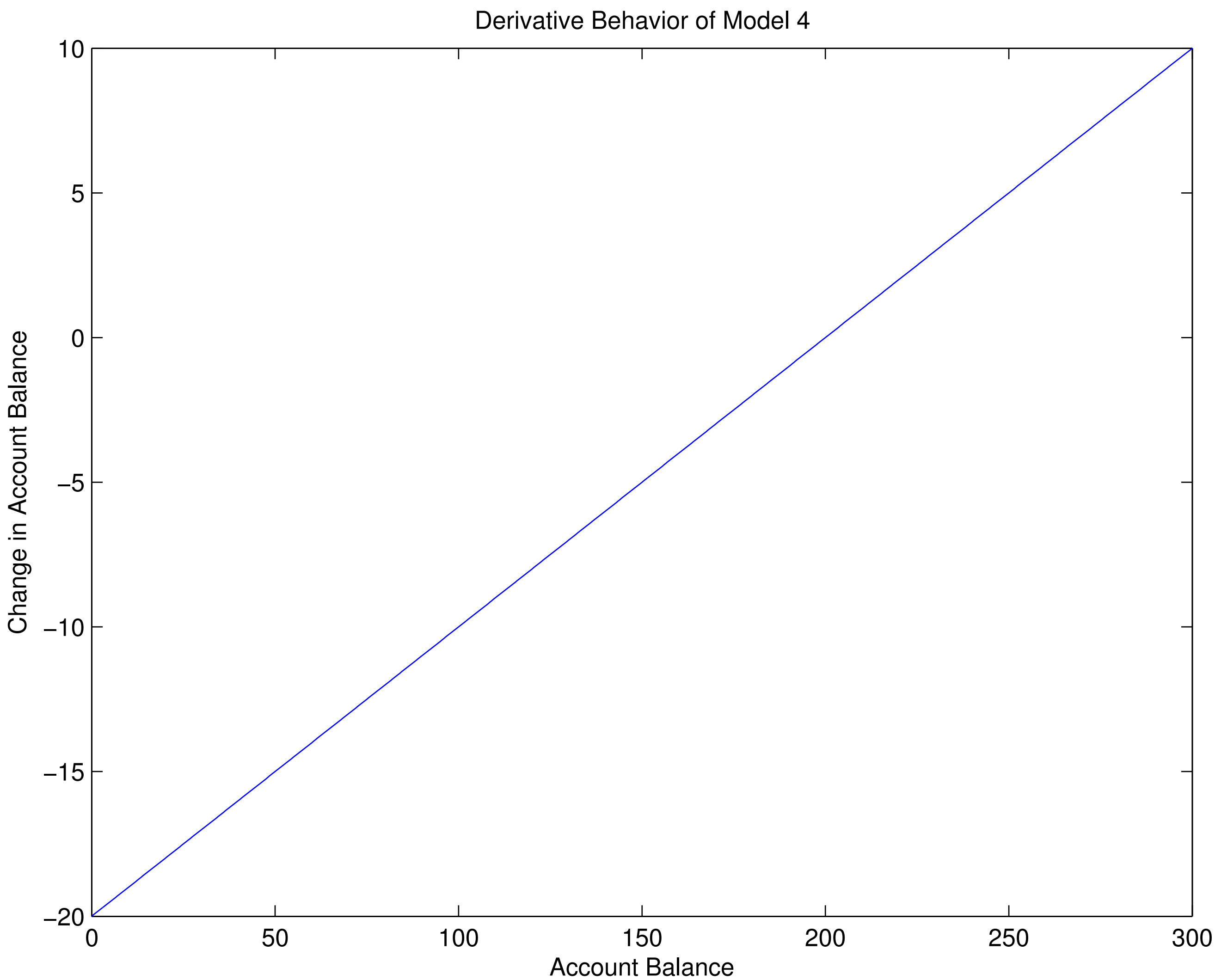

Figure 11: Derivative of Model 4 at account balance

3.3 Direction Field

In order to better understand the behavior of model 4 a direction field with a few solution curves was plotted under the same initial conditions as the plot of the slope in figure 11. The below figure 12 captures the resulting direction field.

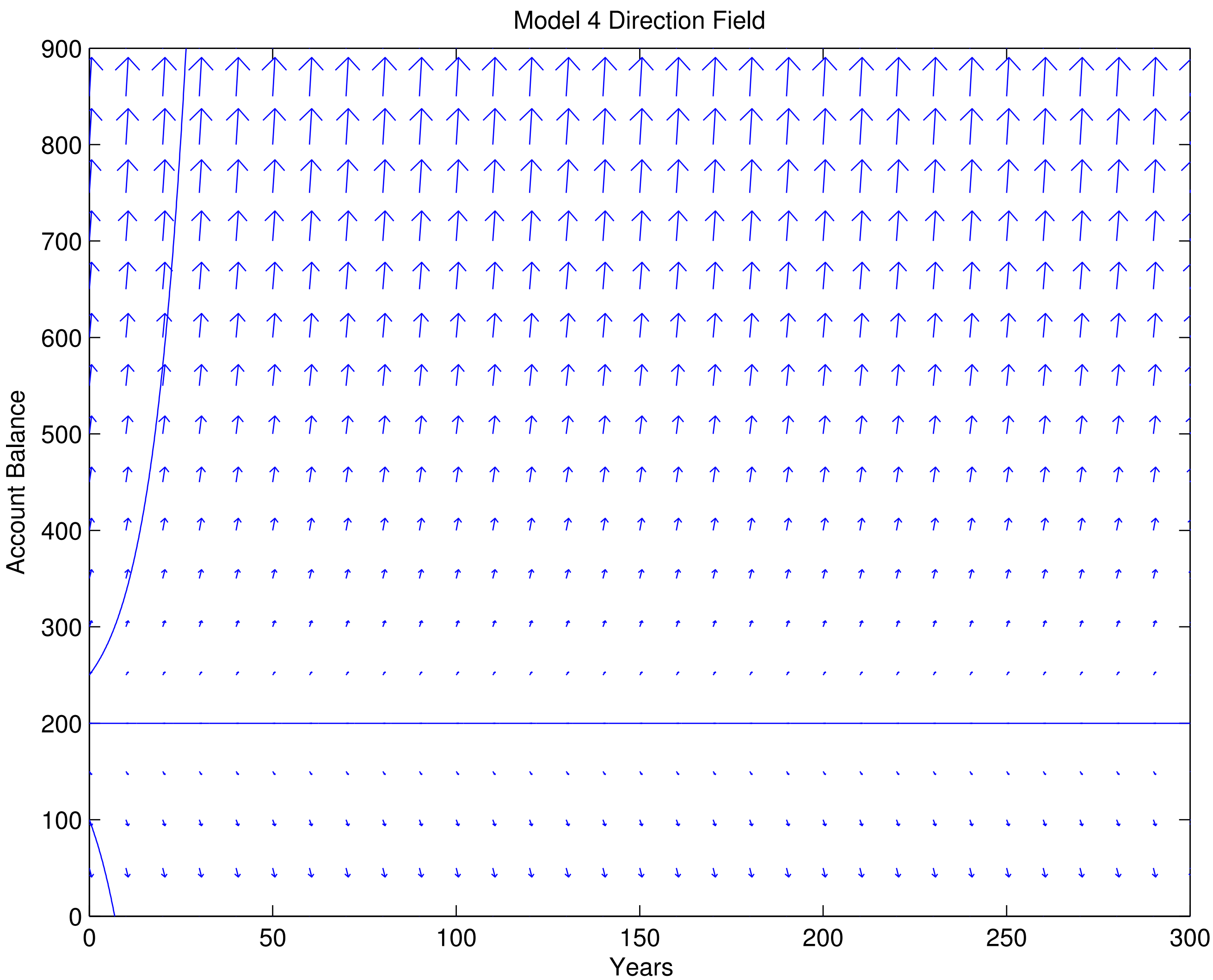

Figure 12: Direction Field of Model 4

In figure 12 above the arrows of the direction field represent the slope of the differential equation at that point. In this case it shows the rate at which the account holder's balance is increasing or decreasing as a result of the intrest charged and the payments made. When the arrows point downwards they indicate a negative slope, which means that solution cuves that move through fields with downwards arrows will move towards zero as the debt held by the account holder is decreasing until it vanishes. When the arrows point upwards they indicate a positive slope, which means that solution cuves that move through fields with upwards arrows will move towards infinity as the debt held by the account holder is constantly increasing. The plot exhibits asymptotic behavior when the slope is positive or negative as the exponential nature of the solution curves forces the account balance to near infinite values in a finite time when the slope is positive and forces the account balance to zero quickly when the slope is negative. When the account balance is exactly \$200 for the conditions described the equilibrium conditions are met and the slope is zero for all time.

3.4 Initial Value Problem

The next step in this analysis was to solve the differential equation in model 4. Using an integrating factor allowed the differential equation to be solved which resulted in the below equation where C is some constant:

$$A_c(t) = \frac{P - Ce^{rt}}{r}$$

The next step in analyzing this equation is to solve for C which was accomplished by assuming that $A_c(0) = A_o$ which meant that $C = P - A_o r$. When substituted back into the found equation the resulting solution to the differential equation in mode 4 is:

$$A_c(t) = \frac{P - (P - A_o r)e^{rt}}{r}$$

This solves model 4 for an initial value of A_o.

3.5 Behaviour of rate of payment over Time

3.5.1 Time to pay off debt at rate of payment

In order to better understand how model 4 be analyzed to best determine how to pay off debt in the best way. In order to best consider the behavior of the time it takes to pay off the debt as compared to the amount payed the equation that solves model 4 was solved for time. Simple algera resulted in the below equation:

$$t = \frac{ln(\frac{P - rA}{P - rA_o})}{r}$$

It was then assumed that the account holder wanted to maintain a debt of \$2 dollars on an account that originally had \$1 in it and their account has an intrest rate of 10%. Under these conditions the resulting time values were plotted for differing values of payment.

As we can see in the above figure 13 for payments between zero and \$1 it takes less and less time for the balance to reach the \$2 amount. Once the rate of payment reaches some value close to \$1 the time value becomes negative because at that point the rate of payment is larger than the rate at which intrest is being added, which means that the only time the account could have had a balance of \$2 was in the past as the debt is decreasing. As the payments increase however this low value of the time increases from approximatly 1 year in the past as the greater payments mean that the time when the account balance was \$2 for the

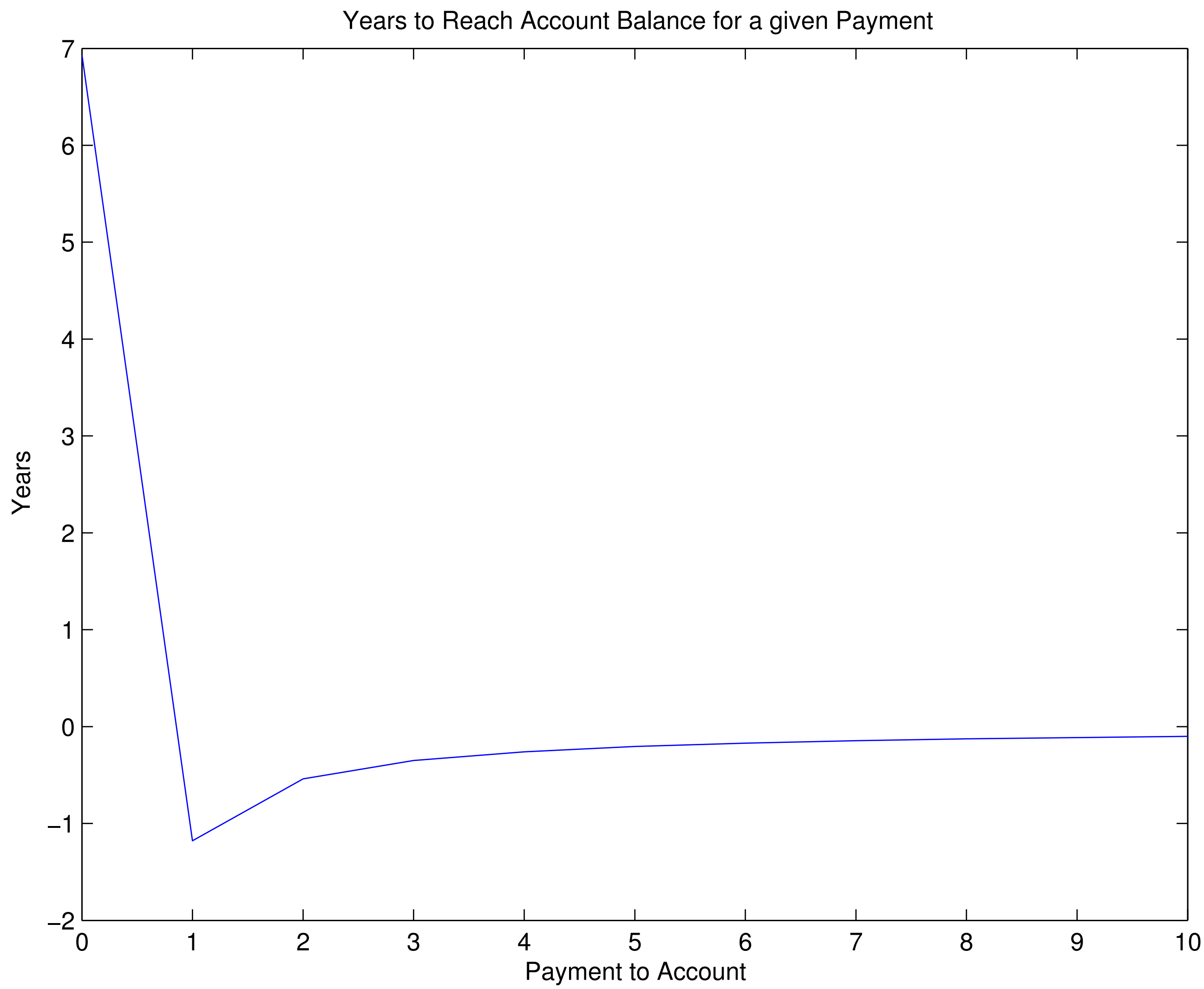

Figure 13: Time to pay off debt as a function of payment

initial condition of \$1 happened closer to the intial condition as the payment is greater and as such a smaller amount of time has passed since the point in the past the account balance was \$2 and the present when the account balance is \$1. As such we can see when the account balance will reach some point given and initial condition, the intrest rate, and then find a workable payment so settle the debt to that value in a set amount of time.

3.5.2 Payment needed to pay off debt in a given time

In order to better understand how model 4 be analyzed to best determine how to pay off debt in the best way. In order to best consider the behavior of how long it will take to pay off the time it takes to pay off the debt as compared to the amount payed the equation that solves model 4 was solved for time. Simple algera resulted in the below equation:

$$P = \frac{-r(A - A_o e^{rt})}{e^{rt}} - 1$$

This formula is useful because it allows the account holder to discover how much they will have to pay each period in order to pay off their debt in a set time.

It was then assumed that the account holder originally had \$1 in their account and that account has an intrest rate of 100%. Under these conditions the resulting payment values were plotted for differing values of time.

As can be seen in the above figure 14 when the goal is to reach the desired payment in a short amount of time (1-3 years) Then the payment slowly increases as you need a larger payment to counteract the intrest being generated to force the account balance to the desired value. However, as the time increases the value of the payment increases towards 1 because once the payment becomes 1 the intrest rate will exactly cancel the payments and the account balance will remain constant. As such this plot shows the trends in the payment needed to maintain an existing balance with an initial account condition of \$1.

3.6 Accuracy of Installment Debt Model

Based on the analysis conducted in the above sections it can be concluded that model 4 is a reasonable model for installment debt. This is true because model 4 exhibits the behavior we would expect of installment debt as it shows exponential debt increase when payments are less than the intrest on the account and exponential debt decrease when payments are more than the intrest on the account. All other patterns such as the behavior of the solution curve occurs as we would expect an installment debt model to work.

However, this model is severly flawed as it assumes that the account holder will not use the credit card. This is a severe limitation as most credit card holders will not incur an initial value of debt on the card and then not use that card while paying a set amount over a long period of time. As such this model does not help understand the behavior of the average credit card user. There is another large limitation of model 4 in that it assumes that the account holder will pay a set amount of the debt off every pay period. As such this model does not take into account a account holder that changes the amount they pay each month.

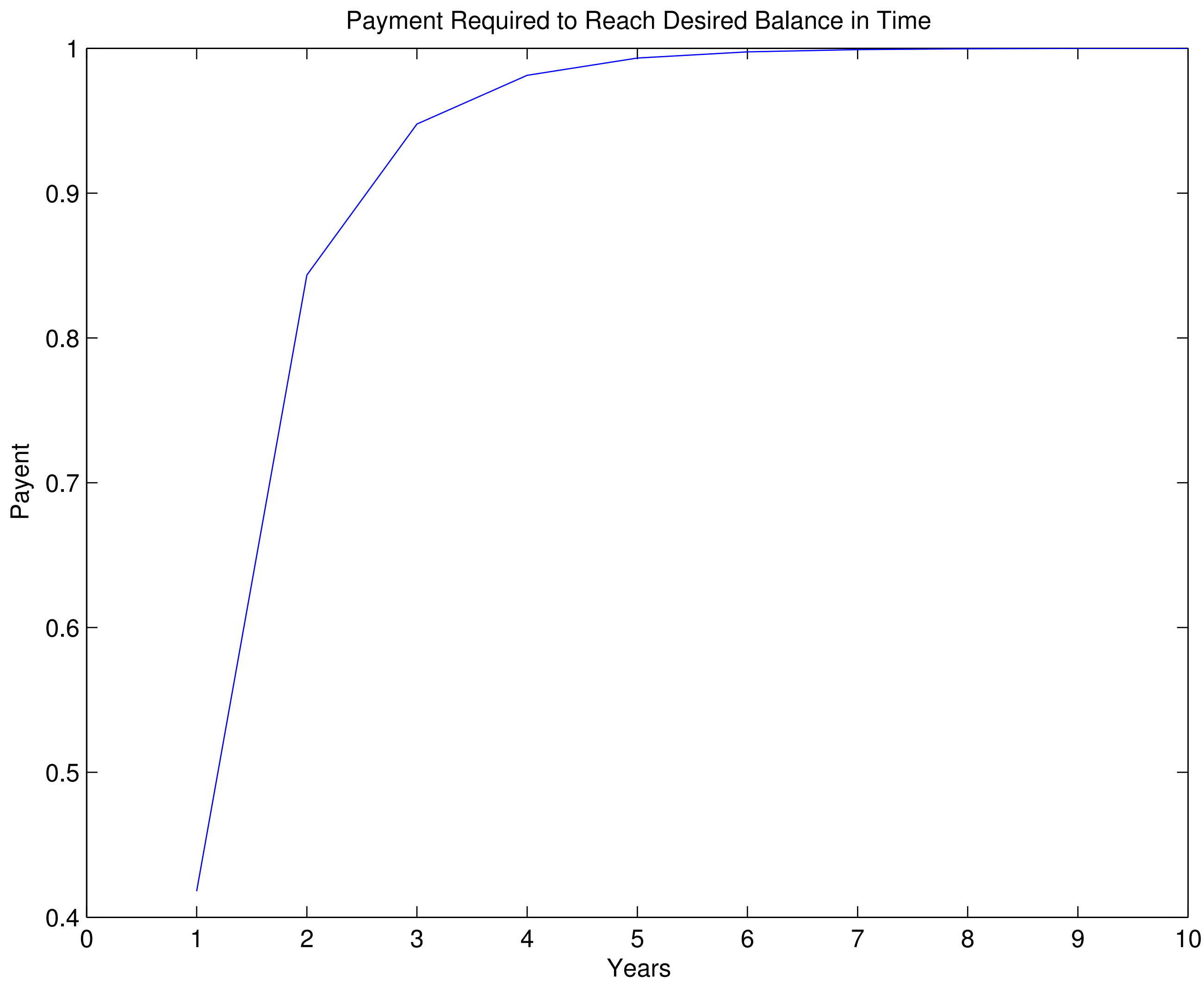

Figure 14: Payment to pay off debt as a function of years

Part 4
Installment Savings

In order to better understand the basic principles of models similar to the installment debt model this paper will now consider an installment savings model. Based on the installment debt model a model for the value of a savings account with constant payments being made to it is the below model 5.

$$\frac{dA_s}{dt} = rA_s + P \tag{5}$$

In the above model 5 A_s is the balance of the account, r is the annual intrest rate (some value such that $0 \leq r \leq 1$), t is the time in years since the account was created, P is the amount deposited in the account each year, and $\frac{dA_s}{dt}$ is rate of change of the balance of the account. This model exhibits all the behaviors we expect of the installment model as it adds the additional deposits to the account balance while also adding the intrest generated from the bank.

4.1 Initial Value Solution

The next step in this analysis was to solve the differential equation in model 4. Using an integrating factor allowed the differential equation to be solved which resulted in the below equation where C is some constant:

$$A_s(t) = \frac{-1(P - Ce^{rt})}{r}$$

The next step in analyzing this equation is to solve for C which was accomplished by assuming that $A_c(0) = A_o$ which meant that $C = P + A_o r$. When substituted back into the found equation the resulting solution to the differential equation in mode 5 is:

$$A_c(t) = \frac{-1(P - (P + A_o r)e^{rt})}{r}$$

Thus this paper can be concluded by presenting this model for savings which will exhbit similar behavior to the installment debt model due to their similar equations.