APPM 2360

Lab #1: Credit Model

1 Instructions

Labs may be done in groups of **3** or less. You may use any program, however, the TA's will only answer coding questions in MATLAB or MVT. One report must be turned in for each group and must be in **PDF** format. Labs must include each student's:

- Name
- Student number
- Section number
- Recitation number

This lab is due on **Friday, October 3, 2014** at **5pm**. Each lab must be turned in through D2L on the course page. When you submit the lab please include each group members information (name, student number, section number and recitation number) in the comments. This allows us to search for a students report. **Late labs will not be accepted**. Once the labs are graded you need to specifically check that your grade was entered. If you are missing a grade please bring the issue up with your TA within a week of grading.

The report must be typed (including equations). Be sure to label all graph axes and curves so that, independent of the text, it is clear what is in the graph. Simply answering the lab questions will not earn you a good grade. Take time to write your report as up to 20% of your grade may be based on organization, structure, style, grammar, and spelling. Project office hours will be held in ECCR 143 from Monday, September 29, 2014 to Friday, October 3, 2014.

2 Introduction

The goals of this lab are:

- Interpret a model stated in terms of a differential equation by
  - Solving an initial value problem
  - Plotting solutions and slope fields
  - Comparing discrete and continuous models

3 Model

If we deposit money in a savings account, take out a car loan or maintain a balance on a credit card, then we face the implications of compound interest. Consider using a credit card. A person obtains the card, usually through a bank or credit union, and agrees to reimburse the credit company each time the card is used. As the person uses the card, a balance accumulates in his or her account.

At the end of the month, if the balance is not completely paid, then the credit company applies interest to the account. At the end of the following month interest is applied again to any remaining balance and so on.

Usually, the credit company expects a monthly payment, however, in order to derive a model, let us suppose that no payment is expected and interest rates never change. Let

- $A(t)$ = account balance (in dollars)
- r = annual interest rate, note $0 < r < 1$.
- n = number of times per year that interest is applied to the account
- t = time (in years)
- $A(0) = A_0$ = initial balance

First, suppose that interest is applied annually. Then the balance at the end of the year is

$$A(1) = A_0 + A_0 r = A_0(1 + r).$$

Now suppose that interest is applied biannually. Then the balance at the end of the year is

$$A(1) = A_0(1 + r/2) + A_0(1 + r/2)(r/2) = A_0(1 + r/2)^2.$$

Similarly, if interest is applied quarterly, then the balance at the end of the year is

$$A(1) = A_0(1 + r/4)^4.$$

Continuing, we see that if the interest is applied n times per year, then the balance at the end of the year is

$$A(1) = A_0(1 + r/n)^n.$$

The integer n is termed the compounding rate.

So far it seems that time has not been considered in the compound interest model. However, this notion is misleading since, up to this point in the derivation, $t = 1$ year.

Consider the following argument. The balance at the end of the second year is

$$A(2) = [A_0(1 + r/n)^n](1 + r/n)^n = A_0(1 + r/n)^{2n}.$$

Similarly, the balance at the end of three years is

$$A(3) = A_0(1 + r/n)^{3n}.$$

Continuing, we see that the discrete model for the balance at time t (where t may not be an integer) is given by

$$A_d(t) = A_0(1 + r/n)^{nt}. \tag{1}$$

It can be shown that taking the compounding rate n to infinity, that is, interest is compounded continuously, we obtain a new model

$$A_c(t) = A_0 e^{rt}, \tag{2}$$

where e is Euler's number. We will refer to (1) as the discrete model and to (2) as the continuous model. By differentiating (2) with respect to t, we notice the following relationship

$$\frac{dA_c}{dt} = rA_c. \tag{3}$$

So the rate of change of the balance is proportional to the balance itself. This notion provides us with a model for **installment debt**, debt that is reduced by making installment payments. Suppose that at time t we pay off the debt at the constant rate P in dollars per year. Then the change in the account balance is proportional to itself minus the payment rate, in other words

$$\frac{dA_c}{dt} = rA_c - P. \tag{4}$$

Notice that we are assuming that no additional charges are brought against the account and that interest rates never change. Also, recall that the initial balance is $A_c(0) = A_0$. Hence, (4), along with the initial balance, give us an initial value problem (IVP).

4 Questions

Interpret the installment debt model (4) by analyzing its behavior (analytically and graphically), comparing it to other models, experimenting with some specific scenarios, etc. Below are questions to consider in helping you to interpret (4).

1. Classify model (4). Also, describe each term in the model and discuss the units involved.
2. Show that taking the compounding rate to infinity, model (1) converges to model (2).
3. Compare (1) to (2).
   - In (1) and (2), solve for time, t.
   - Create four graphs, one for each $n = 1, 2, 4, 12$, each graph containing both the continuous and discrete model for $t \in [0, 100]$. Set $A_0 = 1$ and $r = 10\%$; label each curve and the plot axes.
   - Repeat the above plot, but this time set the interest rate to $r = 20\%$. How did this change affect the comparison? Try different interest rates in order to determine the impact of this parameter on the model behavior.
   - Is (2) an accurate model for (1)? How accurate or inaccurate is it for the varying values of n and r?
4. Determine if (2) is a reasonable model for (1) when interest is compounded daily, $n = 365$. Recall problem 2.
   - In what year will the difference in models (1) and (2) be greater than a dollar? Assume standard rounding and you may set $A_0 = 1$. Try rates of 6.9%, 12.99%, 19.99%.
5. Find the equilibrium solution for (4). What is the significance of this solution? What does it represent?

6. Let $r = 10\%$ and $P = 20$ in model (4). Plot dA_c/dt versus A_c for $0 < A_c < 300$. Interpret this graph by answering:
   - What does $dA_c/dt < 0$ mean in terms of debt?
   - What does $dA_c/dt > 0$ mean in terms of debt?
7. Let $r = 10\%$ and $P = 20$. Plot the slope field for (4) along with some solution curves. Describe this plot in detail.
   - What do the slopes or "arrows" represent?
   - What is the behavior of the solution curves with respect to the direction field?
   - Does the plot exhibit any asymptotic behavior? If so, describe it.
8. Solve (4) as an IVP with initial balance $A_c(0) = A_0$.
9. Given the solution that you obtained in 8, solve for the time, t. Now plot time as a function of P. Use $A_0 = 1$, $r = 10\%$ and assume that the balance is held fixed at $A_c(t) = 2$; if your graph seems strange (i.e. time is negative), recall problem 6. Interpret this plot.
10. In (4) find a value for P so that the debt is paid within T years. Explain the usefulness of this formula.
11. Plot P as a function of t; let $r = 1$ and $A_0 = 1$. Interpret this plot.
12. Is (4) a reasonable model? Discuss strengths and shortcomings of the model. Offer suggestions for improvement.
13. Write down a model for installment savings, savings that is increased by making installment deposits. Call this model (5). Solve (5) as an IVP with initial balance A_0.