Lab 2: Civil Engineering
APPM 2360

1 Instructions

Labs may be done in groups of **3** or less. You may use any programming language, but the TAs will only answer coding questions in MATLAB. One report must be turned in for each group and must be in **PDF** format. Labs must include each student's:

- Name
- Student number
- Section number
- Recitation number

This lab is due on **Friday, October 31, 2014** at **5pm**. Each lab must be turned in through D2L on the course page. When you submit the lab please include each group member's information (name, student number, section number, and recitation number) in the **comments**. This allows us to search for a studdent's report. **Late labs will not be accepted**. Once the labs are graded you need to specifically check that your grade was entered. If you are missing a grade please bring the issue up with your TA within a week of grading.

The report must be typed (including equations). Be sure to label all graph axes and curves so that, independent of the text, it is clear what is in the graph. Simply answering the lab questions will not earn you a good grade. Take time to write your report since up to 20% of your grade may be based on organization, structure, style, grammar, and spelling. Project office hours will be held in ECCR 143 from Monday, October 27 until Friday, October 31.

2 Introduction

In this lab we will consider two problems from civil engineering using techniques from linear algebra and differential equations. The first problem involves the distribution of forces in a simple bridge, and the second is concerned with the deflection of a beam under uniform loading.

3 Distribution of Forces in a Bridge

In bridge design, one is interested in connecting lightweight pieces together to make a truss which can support weight. The pieces of the truss (black lines in Figure 1) are connected together by pin joints (blue boxes) which are designed to rotate and transfer forces between the pieces.

In the diagram in Figure 1, each arrow indicates the direction of a force and is labeled according to its magnitude. Blue arrows indicate forces on the joints due to tension in the corresponding piece of the truss, and red arrows indicate external forces on the joints. In a given piece of the truss, a positive value for f_i indicates tension, while a negative value indicates compression. Pin joints 2 and 4 are held in place vertically by external forces F_1 and F_2, respectively, but are free to move horizontally. The remaining pin joints are free to move both horizontally and vertically. A load of $5,000$ newtons is placed at joint 3, and the problem is to determine the remaining unknown tensions and forces in the bridge.

In order for the bridge to be in equilibrium, both the horizontal and vertical components of force must sum to zero at each of the pin joints.

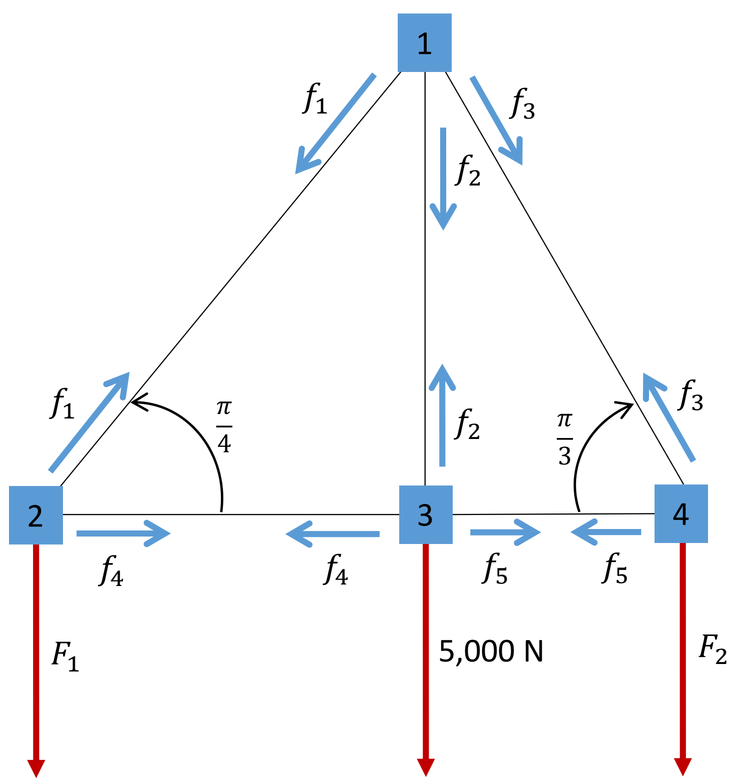

Figure 1: Distribution of the forces in a bridge.

For example, consider joint $\boxed{2}$. Balancing the forces in the horizontal and vertical directions gives two equations:

$$\text{Horizontal: } f_1 \cos\frac{\pi}{4} + f_4 = 0, \qquad \text{Vertical: } f_1 \sin\frac{\pi}{4} = F_1.$$

4 Deflection of a Uniform Beam

With the ends of a beam held in place, the second civil engineering problem is to estimate the deflection $y(x)$ of the beam under uniform loading (see Figure 2).

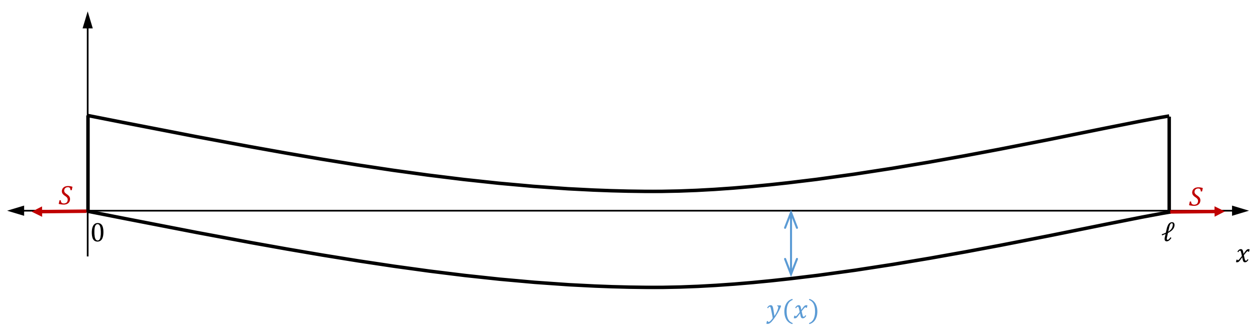

Figure 2: Deflection of a beam.

With some simplifying assumptions, the deflection of a beam supported at $x = 0$ and $x = \ell$ can be modeled by the boundary value problem (BVP)

$$y'' = \frac{S}{EI} y + \frac{Qx}{2EI}(x - \ell), \quad 0 \le x \le \ell, \quad y(0) = 0, \quad y(\ell) = 0, \tag{1}$$

where $y(x)$ is the deflection of the beam, ℓ is the length of the beam, Q is the intensity of the uniform load, E is the modulus of elasticity, S is the stress at the endpoints, and I is the central moment of inertia.

Our goal is to find an approximate solution for (1) using Matlab's `ode45`, but first we need to introduce second-order initial value problems (IVPs) and the method of linear shooting.

4.1 Second Order Initial Value Problems

We have seen plenty of first order IVPs, but you may not have seen many second order ones yet. In general, a second order IVP looks like

$$y'' = f(t, y, y'), \quad y(t_0) = y_0, \quad y'(t_0) = y_0'. \tag{2}$$

Notice that a second-order IVP has *two* initial conditions; one for y and one for y', and both given at the same initial time, t_0.

Written in this form, the solution to (2) *cannot* be directly approximated using one of Matlab's standard built-in ODE solvers, such as `ode45`. However, since Matlab's solvers are designed for systems of first-order IVPs, we would like to somehow write the IVP in (2) without using any second derivatives. To do this, define the new function $z(t) = y'(t)$, so that the original second order IVP can be written as a system of two first order IVPs:

$$\begin{bmatrix} y' \\ z' \end{bmatrix} = \begin{bmatrix} z \\ f(t, y, z) \end{bmatrix}, \qquad \begin{bmatrix} y(t_0) \\ z(t_0) \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}.$$

In this form, the solution *can* be approximated using one of Matlab's built-in ODE solvers.

4.2 Linear Shooting

The linear shooting method is used to find an approximate solution to a linear second-order boundary value problem (BVP), which looks like

$$y'' = p(x)y' + q(x)y + r(x), \quad a \le x \le b, \quad y(a) = \alpha, \quad y(b) = \beta. \tag{3}$$

Here, $p(x)$, $q(x)$, and $r(x)$ are continuous on $[a, b]$, and $q(x) > 0$ on $[a, b]$. The BVP is guaranteed to have a unique solution, but in its current form we don't have the tools to solve it, even approximately. We learned in Section 4.1 how to transform a second order IVP into a system of first order IVPs, but (3) is not an IVP at all.

To apply the linear shooting method, we temporarily set aside the BVP in (3), and instead consider two closely-related and cleverly-chosen IVPs:

$$y'' = p(x)y' + q(x)y + r(x), \quad a \le x \le b, \quad y(a) = \alpha, \quad y'(a) = 0, \tag{4}$$

$$y'' = p(x)y' + q(x)y, \quad a \le x \le b, \quad y(a) = 0, \quad y'(a) = 1. \tag{5}$$

Approximate solutions to (4) and (5) can be found using `ode45` because they are both second order IVPs (see Section 4.1). Now, let $y_1(x)$ denote the solution to (4) and let $y_2(x)$ denote the solution to (5). It turns out that y_1 and y_2 can be combined in a certain way to recover the solution to the original BVP given in (3). Define the new function $y(x)$ by

$$y(x) = y_1(x) + Cy_2(x). \tag{6}$$

It can be shown that, for the right choice of the constant C (See Problem 6b), $y(x)$ satisfies the ODE and boundary conditions given in (3). Once C is known, we can use ode45 to find approximate solutions to the IVPs in (4) and (5), combine them according to (6), and the result is an approximate solution to the original second-order BVP given in (3).

A Matlab function called `linearShooting.m` has been provided, and is currently configured to solve a specific BVP of the form given in (3) (you can run this code to see what it does). By modifying the "INPUTS" section of the code, the approximate solution for any BVP of the form given in (3) can be found and plotted.

5 Tasks

1. Using the diagram of the bridge in Figure 1, write down a set of linear equations for the unknown tensions ($f_1, f_2, f_3, f_4,$ and f_5) and external forces (F_1 and F_2). Display your equations by completing the following table:

Joint	Horizontal	Vertical
$\boxed{1}$		
$\boxed{2}$	$\frac{\sqrt{2}}{2}f_1 + f_4 = 0$	$\frac{\sqrt{2}}{2}f_1 = F_1$
$\boxed{3}$		
$\boxed{4}$		

2. (a) Write the linear system of equations from problem 1 in the form $\mathbf{A}\vec{\mathbf{f}} = \vec{\mathbf{b}}$, where

$$\vec{\mathbf{f}} = \begin{bmatrix} f_1 & f_2 & f_3 & f_4 & f_5 & F_1 & F_2 \end{bmatrix}^T.$$

   (b) What are the dimensions of $\mathbf{A}$ and $\vec{\mathbf{b}}$? Check your answer using the `size` function in Matlab.

   (c) Is it possible to calculate the determinant of $\mathbf{A}$? If so, find it. If not, explain why not.

3. (a) Form the augmented matrix for the linear system and put it into reduced-row-echelon form (you do not need to do this by hand).

   (b) What is the rank of $\mathbf{A}$?

   (c) Is the linear system $\mathbf{A}\vec{\mathbf{f}} = \vec{\mathbf{b}}$ consistent?

   (d) What are the tensions and external forces in the bridge?

   (e) Which piece of the truss is being stretched the most? Which piece is being compressed the most?

4. (a) Suppose the load of 5,000 newtons at joint 3 is replaced by the "free" force, F_3. Is the resulting linear system still solvable? If so, solve it. If not, explain why not.

   (b) If F_1 and F_2 turn out to be negative, what does this mean physically?

5. Classify the ODE for the deflection of a uniform beam, given in (1). What is the order of the ODE? Is it linear? If so, is it homogeneous or nonhomogeneous? Are the coefficients constant or variable?

6. (a) Explain how (6) is a manifestation of the nonhomogeneous principle.

   (b) Given that $y_1(x)$ and $y_2(x)$ are solutions to (4) and (5), respectively, find a value for the constant C so that (6) is the solution to the BVP given in (3).

7. (a) Set $z(x) = y'(x)$, and re-write (4) as a system of first-order ODEs, together with the appropriate initial conditions.

   (b) Similarly, re-write (5) as a system of first-order IVPs.

8. (a) Modify the "INPUTS" section of the given Matlab code, `linearShooting.m`, to approximate the deflection of a uniform beam every 12 inches with the following characteristics: length $\ell = 360$ in., intensity of uniform load $Q = 5$ lb/in., modulus of elasticity $E = 5 \times 10^7$ lb/in.^{2}, stress at ends $S = 900$ lb, and central moment of inertia $I = 600$ in.^{4}. Put the modified code in the appendix, and include a plot of the approximate solution in the body of the report.

   (b) What is the maximum deflection of the beam and where does it occur?

Written by Greg Barnett in the fall of 2014