The Cauchy Distribution

1. The *Cauchy distribution* has PDF
$$f(x) = \frac{C}{1+x^2}$$
where C is an appropriate constant.

   (a) Find the value of C that makes f a valid PDF.

   (b) Find a formula for the CDF of a random variable with Cauchy distribution.

   (c) Find the *median* of the Cauchy distribution.

   (d) Nevertheless, show that the *mean* of a Cauchy distribution does not exist.

2. In R, the standard Cauchy distribution centered at 0 can be sampled n times using the function `rcauchy(n, 0, 1)`. Since the expected value for the Cauchy distribution is undefined (not 0, as symmetry might suggest), the arithmetic mean of a sample of a large number of independent Cauchy distributed random variables does not converge to a single number. Verify this by doing the following:

   (a) Generate a vector of 10,000 independent standard Cauchy random variables, and compute its arithmetic mean using the `mean(...)` function.

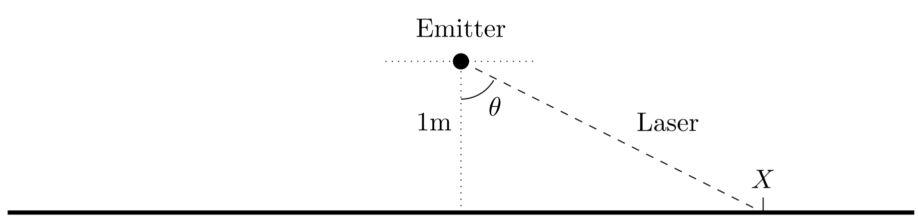
   (b) If the Cauchy distribution DID have an expected value, what what you anticipate would be the approximate arithmetic average of these 10,000 variates.

   (c) "One anecdote does not evidence make. Use the `replicate` function to repeat part (a) 10,000 times. Create a histogram of the results either using the `hist` function in base R, or a `geom_histogram` in `ggplot2`. Compute the standard deviation of your data set using the function `sd`.

   (d) If the Cauchy distribution DID have an expected value, what would we expect to be the approximate standard deviation of the set of 10,000 arithmetic averages of 10,000 variates?

3. (*) The Cauchy distribution isn't just a probabilistic boogeyman designed to scare you into carefully checking definitions, it actually shows up rather frequently in real life applications involving rotations (particularly in astrophysics, optics, and quantum mechanics).

   Consider a laser emitter which is positioned 1 meter away from an infinitely long wall (as shown below) and which rotates counterclockwise at a constant rate of 2π radians per second. Suppose you observe the position of the laser at a random moment in time. Conditioning on the event that the laser is actually pointing at the wall, then the angle θ made by the beam is uniformly distributed on the interval $(-\pi/2, \pi/2)$. Let X denote the location of the intersection of the laser beam with the wall (with $X = 0$ if the laser is perpendicular to the wall).



   (a) Use the change-of-variables formula to show that the PDF of X is that of the Cauchy distribution, $f(x) = \frac{1}{\pi(1+x^2)}$.

   (b) Use facts about the distribution of θ to argue that $P(-1 < X < 1) = 0.5$.

   (c) If you observed the position of the beam at many different (independently chosen) times, what would be the approximate arithemtic average of the *angle* of the beam? What would be the approximate arithmetic average of the *location* of the beam?