21.12.(4)

假设该式不是重言式，则存在u 和 φ,使

$V(\forall x(p \to q) \to (p \to (\forall xq)))=0$

即$\begin{cases} v(\forall x(p \to q)) = 1 \\ v(p \to \forall xq) = 0 \end{cases}$

即$\begin{cases} v(\forall x(p \to q)) = 1 \\ v(p) = 1 \\ v(\forall xq) = 0 \end{cases}$

则在u 和 φ 下，取$x' \notin X \cup C$

对x'的任一指派$\varphi_0{}'(x') = \alpha \in U$ 有

$V'(p(x')) \to q(x')) = 1$

而 $V(\forall xq)=0$ 表示存在对 x'的某一指派$\varphi_0{}^*$

使 $V'(q(x'))=0$,而对该指派，同样有

$V'(p(x')) \to q(x')) = 1$

即 $1+v'(p(x'))=1$

所以 $V'(p(x'))=0$

而 $V(p(x))=1$,且由 $x \notin var(p)$

有 $p(x)=p(x')=p, v'(p(x'))=v(p)$

所以 $v(p)=0$ 与 $V(p)=1$ 矛盾

所以假设不成立

所以$\vdash \forall x(p \to q) \to (p \to \forall xq)$, 这里 $x \notin var(p)$

21.12(5)

假设该式不是重言式，则存在u 和 φ

使$V\left(\forall x(p(x))\right) \to p(t)) = 0$

即$\begin{cases} V(\forall xp(x)) = 1 \\ V(p(t)) = 0 \end{cases}$

$V(\forall(xp(x))=1$,即在u 和 φ 下，取$x' \notin X \cup C$,

对 x'的任一指派$\varphi_0{}'(x') = \alpha \in U$ 有

$V'(p(x'))=1$

取$\varphi_0{}^*$使$\varphi_0{}^*(x') = \varphi(t)$

则对 p(x)中所有包含自由出现的 x 的原子公式$R_k{}^i(t_1, t_{2,\ldots\ldots}, t_{m,} t_{m+1,\ldots\ldots,} t_k)$

其中$t_m = f_{k_2}^j(\ldots, x, \ldots.)$

在 p(x')中被替换为$t'_m = f_{k_2}^j(\ldots, x, \ldots.)$

则$\varphi_0{}^*(t'_m) = \overline{f_{k_2}^j}(\ldots, \varphi_0{}^*(x'), \ldots.)$

$= \overline{f_{k_2}^j}(\ldots \varphi(t), \ldots.)$

又 t 对 x 自由，所以 t 的所有个体变元在 p(t)中的所有项$t''_m = f_{k_2}^j(\ldots, t, \ldots.)$内是自由出现

则$t''_m = \frac{}{f^j_{k_2}}(\ldots\varphi(t),\ldots.) = \varphi_0{}^*(t'_m)$

所以 v’($R^i_k(t'_1, t'_2 \ldots . t'_k)$)=v($R^i_k(t''_1, t''_2 \ldots . t'_k{}')$)

并且 P(t)中所有来自替换的 t 的个体变元在替换处均为自由出现

所以 V’(p(x’))=v(p(t))这与 v’(p(x’))=1,v(p(t))=0 矛盾

所以$\vdash \forall x\big(p(x)\big) \rightarrow p(t)$,对 p 中的 x 是自由的

21.13(1)错误

取 P(x)=$R'_2(x, c)$ 取u 和 φ 令 $U = \{1,2,3\}, \varphi_3(R'_2)$为等于，$\varphi_1(c) = 2, \varphi_0(x) = 2$

则 v(p(x))=1

而取 x’ $\notin X \cup C$，令$\varphi'_0(x') = 3$

则 v’(p(x’))=0,则 v($\forall x\big(p(x)\big)$)=0

所以 p(x)$\Vdash \forall x\big(p(x)\big)$并不正确

(2)错误

取与(1)相同的 p(x)及u，φ则 v(p(x))=1，v($\forall x\big(p(x)\big)$)=0

所以 v(p(x)) $\rightarrow \forall x p(x)) = 0$

得证

(3)错误

取 p(x)=$R'_2(x, c)$

取 u 及φ 使 $U = \{1,2,3\}, \varphi_3(R'_2)$为等于，$\varphi_1(c) = 2, \varphi_0(x) = 3$

则 p(x)=0

而$\exists x p(x) = \urcorner\forall x\urcorner p(x)$

$=\forall x\urcorner p(x) \rightarrow F$

取 x’ $\notin X \cup C$，令$\varphi'_0(x') = 2$

则 v’(p(x’))=1，v’($\urcorner(p(x'))$) =0

所以 v($\forall x\urcorner p(x)$)=0，则 V($\urcorner\forall x\urcorner p(x)$)=1

则 v($\exists x p(x)$)=1

得证

(4)正确

假设存在u 及 φ 使

V((p$\rightarrow \forall x q(x)) \rightarrow \forall x(p \rightarrow q)$)=0

即$\begin{cases} v\big(p \rightarrow \forall x q(x)\big) = 1 \\ v\big(\forall x(p \rightarrow q)\big) = 0 \end{cases}$

其中$v\big(\forall x(p \rightarrow q)\big) = 0$ 即取$x' \notin X \cup C$,存在x'的某一指派$\varphi_0{}^*$

使 v’(p(x’) $\rightarrow q(x')$)=0,又由于 x $\notin var(p)$

所以此即 v’(p$\rightarrow q(x')$)=0

则 v(p)=1,v’(q(x’))=0

即在该指派下，v’(q(x’))=0, 则 v($\forall x q(x)$)=0

所以$v\big(p \rightarrow \forall x q(x)\big) = 0$

这与$v\big(p \rightarrow \forall x q(x)\big) = 1$ 矛盾

所以$\vdash \left(p \to \forall xq(x)\right) \to \forall x(p \to q)$

(5)正确

假设存在u 和 φ 使 $v\left(\left(p \to \exists xq(x)\right) \to \exists x(p \to q)\right) = 0$

即$\begin{cases} v\left(p \to \exists x\left(q(x)\right)\right) = 1 \\ v\left(\exists x(p \to q)\right) = 0 \end{cases}$

其中$\exists x(p \to q) = \neg\forall x\neg(p \to q)$

则 $v(\forall x\neg(p \to q))=1$

即取 $x' \notin X \cup C$,对 x'的任意指派φ_0'有

$V'(\neg(p(x') \to q(x')))=1$

即 $v'(p(x') \to q(x'))=0$

即$\begin{cases} v'\left(p(x')\right) = 1 \\ v'\left(q(x')\right) = 0 \end{cases}$

又 $x' \notin var(p)$, 则 $p(x') = p$ 所以 $v(p) = 1$

则由 $v(p\ p \to \exists x\left(q(x)\right))=1$

有 $V(\exists xq(x))=1$

即 $v(\forall x)\neg q(x) = 0$

即存在对 x'的指派$\varphi_0{}^*$使$v'\left(\neg q(x')\right) = 0$

而该指派下仍有 $v'(q(x'))=0$

但 $v'((\neg q(x'))=0$ 即 $v'(q(x'))=1$ 矛盾

得证

(6)正确。假设存在解释域 u 和项解释φ

使$v\left(\forall xp(x) \to p(t)\right) = 0$

$\therefore v\left(\forall xp(x)\right) = 1\ \ v\left(p(t)\right) = 0$

∴ 在 u 和 φ 下，取 $x' \notin X \cup C$，存在x'的任一指派$\varphi_0(x')$

有 $v'(p(x'))=1$

$\because \exists$指派$\varphi_0(x') = \varphi(t)$

$\therefore v\left(p(t)\right) = v'\left(p(x')\right) = 1$

∴ 矛盾 ∴ 命题得证

21.14

$\because \vDash p(x)$

∴ 对于任意项解释 φ 的解释域 V，$V\left(p(x)\right) = 1$

∴ 在任一 φ 和 U 下，取$x' \notin x \cup c$, 则对x'的任意指派$\varphi_0(x')$有$v'\left(p(x')\right) = 1$

$\therefore v\left(\forall xp(x)\right) = 1$

又由 21.13(1)得 $v\left(\forall xp(x) \to p(t)\right) = 1$

$\therefore v\left(p(t)\right) = 1$

$\therefore \vDash p(t)$

21.18

(1) $p_1 = \forall y\forall xp(x,y)$ (A)

$p_2 = \forall y\forall xp(x,y) \to \forall xp(x,y)$ (A_5)

$p_3 = \forall xp(x,y)$ $(p_2 = p_1 \to p_3)$

$p_4 = \forall x p(x,y) \to p(x,y)$ (A_5)

$p_5 = p(x,y)$ $(p_4 = p_3 \to p_5)$

$p_6 = \forall y p(x,y)$ $(p_5, y \notin var(A), G)$

$p_7 = \forall x \forall y p(x,y)$ $(p_6, x \notin var(A), G)$

(2) $p_1 = \forall x\big(p(x) \to q(x)\big)$

$p_2 = \forall x\big(p(x) \to q(x)\big) \to \big(p(x) \to q(x)\big)$ (A_5)

$p_3 = p(x) \to q(x)$ $(\ Mp)$

$p_4 = \big(\forall x \neg q(x) \to (\forall x \neg p(x) \to F)\big)$

$\to \Big(\big(\forall x \neg q(x) \to \forall x \neg p(x)\big) \to (\forall x \neg q(x) \to F)\Big)$ (A)

$p_5 = \big(\forall x \neg q(x) \to \forall x \neg p(x)\big) \to (\forall x \neg q(x) \to F)$ $(p_3 p_4 Mp)$

$p_6 = \forall x\big(p(x) \to q(x)\big)$ (A)

$p_7 = \forall x\big(p(x) \to q(x)\big) \to \big(p(x) \to q(x)\big)$ (A_5)

$p_8 = p(x) \to q(x)$ $(p_6 p_7 Mp)$

$p_9 = \big(p(x) \to q(x)\big) \to \big(q(x) \to \neg p(x)\big)$

$p_{10} = \neg q(x) \to \neg p(x)$ $(p_8 p_9 Mp)$

$p_{11} = \forall x(\neg q(x) \to \neg p(x))$ (G)

$p_{12} = \forall x\big(\neg q(x) \to \neg p(x)\big) \to (\forall x \neg q(x) \to \forall x \neg p(x))$

$p_{13} = \forall x \neg q(x) \to \forall x \neg p(x)$

$p_{14} = \forall x \neg q(x) \to F = \exists x q(x)$

(3) $A = \{\forall x\big(p(x) \to q(x)\big), \forall x p(x)\}$

$p_1 = \forall x\big(p(x) \to q(x)\big)$

$p_2 = \forall x\big(p(x) \to q(x)\big) \to \big(p(x) \to q(x)\big)$ (A_5)

$p_3 = p(x) \to q(x)$ $(\ Mp)$

$p_4 = \forall x p(x)$ (A_5)

$p_5 = \forall x p(x) \to p(x)$ (A_5)

$p_6 = p(x)$ $(p_4 p_5 Mp)$

$p_7 = q(x)$ $(p_6 p_3 Mp)$

$p_8 = \forall x q(x)$ $(G, x \notin var(\forall x\big(p(x) \to q(x)\big))) \cup var(\forall x p(x))$

$\therefore$ 由演绎定理 $\vdash \forall x\big(p(x) \to q(x)\big) \to (\forall x p(x) \to \forall x q(x))$

(4) $p_1 = \forall x q \to q$ (A_5)

$p_2 = (\forall x q \to q) \to (p \to (\forall x q \to q))$ $(\ A_1)$

$p_3 = p \to (\forall x q \to q)$ (p_1, p_2, Mp)

$P_4 = \big(p \to (\forall x q \to q)\big) \to \big((p \to \forall x q)\big) \to (p \to q\ ))$ $(\ A_2)$

$p_5 = (p \to \forall x q) \to (p \to q)$ (p_3, P_4, Mp)

$p_6 = p \to \forall x q$ A

$p_7 = p \to q$ $(\ \ p_6\ \ p_5 Mp)$

$p_8 = \forall x(p \to q)$

由演绎定理 $\vdash \big(p \to \forall x q(x)\big) \to \forall x(p \to q)$

(5) A={ $p \to \exists x q$}

$p_1 = \neg\neg(p \to \exists x q) \to \exists x(p \to q)$

$p_2 = p \to \exists x q$

$p_3 = (p \to \exists x q) \to \neg\neg(p \to \exists x q)$

$P_4 = \neg\neg(p \to \exists xq)$

$P_5 = \exists x(p \to q)$

$A \vdash \exists x(p \to q)$

所以$\vdash (p \to \exists xq) \to \exists x(p \to q)$

21.19 正确 $\because A \cup \{p(x)\} \vdash q$

$\therefore \vdash p(x) \to q$

$A' = \{ p \to q, \neg q \cup A$

$p_1 = \big(p \to (q \to F)\big) \to \big((p \to q) \to (p \to F)\big)$ A_2

$P_2 = p \to q$

$P_3 = (q \to F) \to \big(p \to (q \to F)\big)$

$P_4 = \neg q = q \to F$

$P_5 = p \to (q \to F)$

$P_6 = (p \to q) \to (p \to F)$

$P_7 = p \to F = \neg p$

$p_8 = \forall x \neg p$

$\therefore A \vdash (p \to q) \to (\neg q \to \forall x \neg p)$

$\therefore A \vdash (\neg q \to \forall x \neg p)$

$\therefore A \vdash (\neg\forall x \neg p \to \neg\neg q)$

$\therefore A \vdash \exists xq \to q$ $\therefore A \cup \{\exists xq\} \vdash q$