机器人学导论

Introduction to Robotics

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5 动力学分析

- 机器人动力学与加速度、负载、质量以及惯量有关；
- 运动方程 - 速度、加速度 - 力、力矩
- 根据力和力矩的大小来确定驱动器的出力和大小尺寸等
- 如果采取舵机驱动关节，虽然可以直接通过旋转角度来确定步态，但舵机的驱动力有限，也需要确定个关节的扭矩等动力学特性。
- **动力学正问题**：已知机械手各关节的作用力或力矩，求各关节的位移、速度、加速度、运动轨迹；
- **动力学逆问题**：已知机械手的运动轨迹，即各关节的位移、速度和加速度，求各关节的驱动力和力矩。

转动惯量与惯性张量

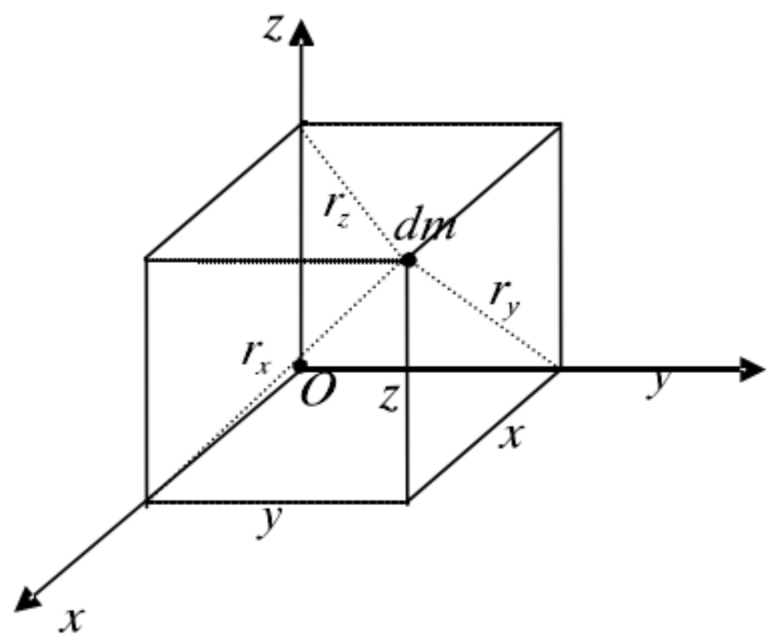
- 刚体的单位质量转动惯量与惯性张量

$$dI_{xx} = r_x^2 dm = (y^2 + z^2)dm$$

$$I_{xx} = \int_m r_x^2 dm = \int_m (y^2 + z^2)dm$$

$$I_{yy} = \int_m r_y^2 dm = \int_m (x^2 + z^2)dm$$

$$I_{zz} = \int_m r_z^2 dm = \int_m (x^2 + y^2)dm$$



- 单位质量的惯性积

质量微元的惯性积定义为质量微元与其到一系列两个正交平面的垂直距离的乘积

$$dI_{xy} = dI_{yx} = xydm$$

$$I_{xy} = I_{yx} = \int_m xydm$$

$$I_{yz} = I_{zy} = \int_m yzdm$$

$$I_{xz} = I_{zx} = \int_m xzdm$$

- 刚体的惯性属性-惯性张量

$$I=\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}$$

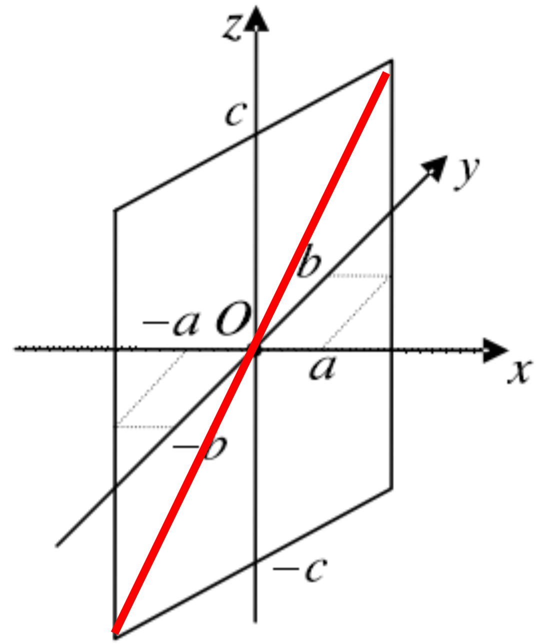
- 刚体的基准姿态-惯性张量“对角线化”

对于质量分布均匀的刚体，可以以其质心为原点O，确定坐标轴的方向时，可以使刚体对坐标轴的惯性积为零。

$$\bar{I}=\begin{bmatrix} I_{xx} & 0 & 0 \\ 0 & I_{yy} & 0 \\ 0 & 0 & I_{zz} \end{bmatrix}$$

机器人连杆

- 只有长度，且质量分布均匀的连杆，
  连杆的质量为m, 长度为l，
  以连杆质心为原点建立坐标系O系



$$I_{xx}=\int_m\left(y^2+z^2\right)dm=\int_{-\sqrt{b^2+c^2}}^{\sqrt{b^2+c^2}}\left(r^2\frac{m}{2\sqrt{b^2+c^2}}\right)dr=\frac{m}{3}\left(b^2+c^2\right)$$

$$I_{yy}=\int_m\left(x^2+z^2\right)dm=\frac{m}{3}\left(a^2+c^2\right)$$

$$I_{zz}=\int_m\left(x^2+y^2\right)dm=\frac{m}{3}\left(a^2+b^2\right)$$

$$I_{xy}=I_{yx}=\int_m xydm=\int_{-a}^{a}\left(x\cdot\frac{b}{a}\cdot x\cdot\frac{m}{2a}\right)dx=\frac{abm}{3}$$

$$I_{yz}=I_{zy}=\int_m yzdm=\frac{bcm}{3}$$

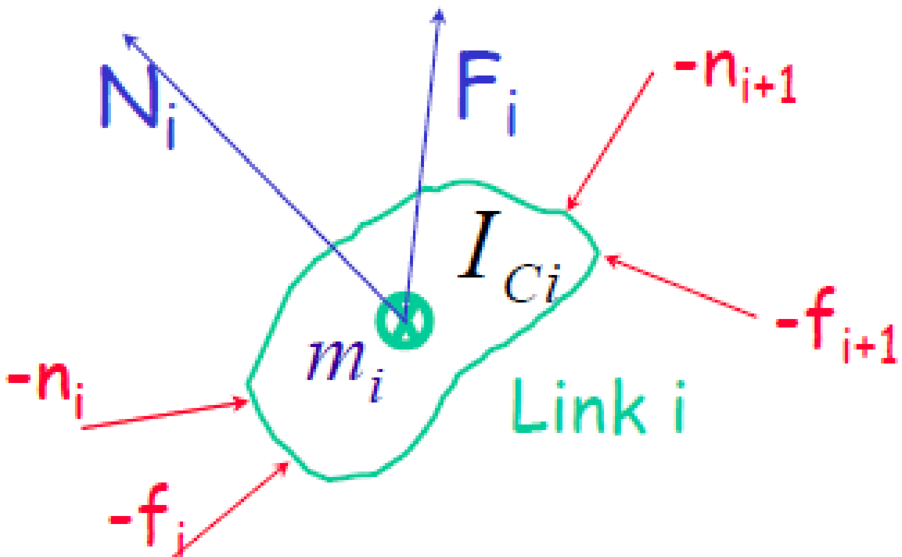
$$I_{xz}=I_{zx}=\int_m xzdm=\frac{acm}{3}$$

因此，连杆相对于O系的惯性张量为：

$$I=\begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix}=\begin{bmatrix} \frac{m}{3}\left(b^2+c^2\right) & -\frac{abm}{3} & -\frac{acm}{3} \\ -\frac{abm}{3} & \frac{m}{3}\left(a^2+c^2\right) & -\frac{bcm}{3} \\ -\frac{acm}{3} & -\frac{bcm}{3} & \frac{m}{3}\left(a^2+b^2\right) \end{bmatrix}$$

Formulations

Newton-Euler



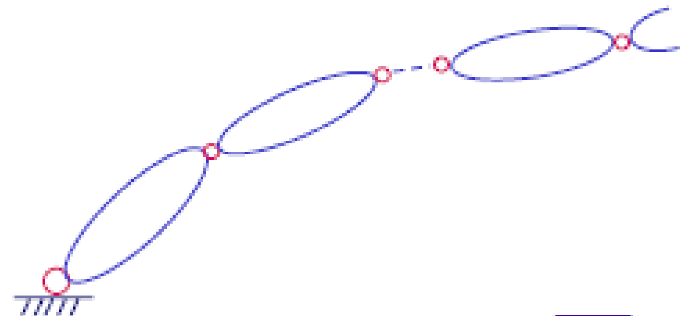
Newton: $m\dot{\mathbf{v}}_C = F$

Euler: $N_i = I_{C_i}\dot{\omega}_i + \omega_i \times I_{C_i}\omega_i$

Eliminate Internal Forces

$$\tau_i = \begin{cases} n_i^T . Z_i & \text{revolute} \\ f_i^T . Z_i & \text{prismatic} \end{cases}$$

Lagrange



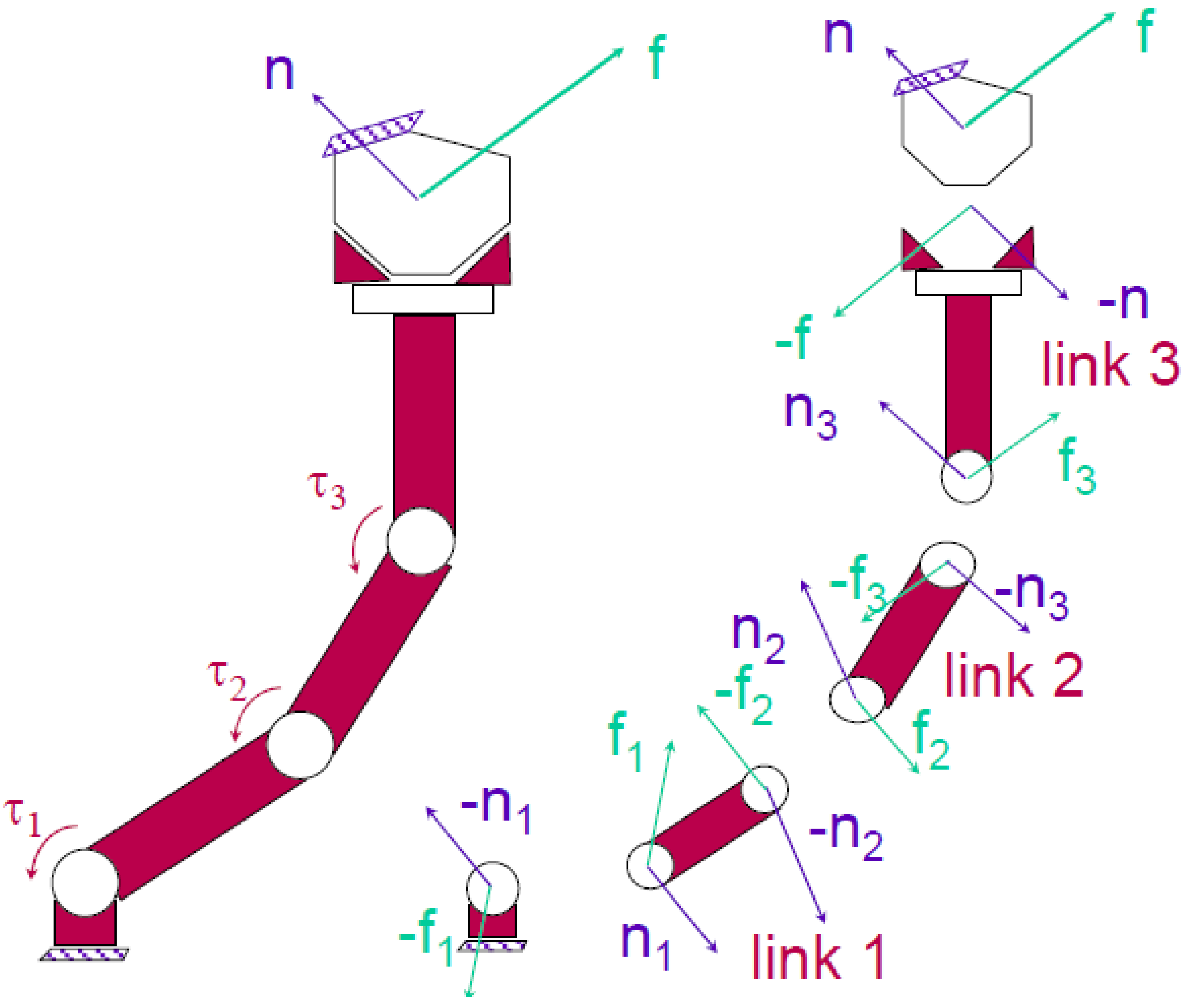
Kinetic Energy: $\sum_i K_i$

Potential Energy

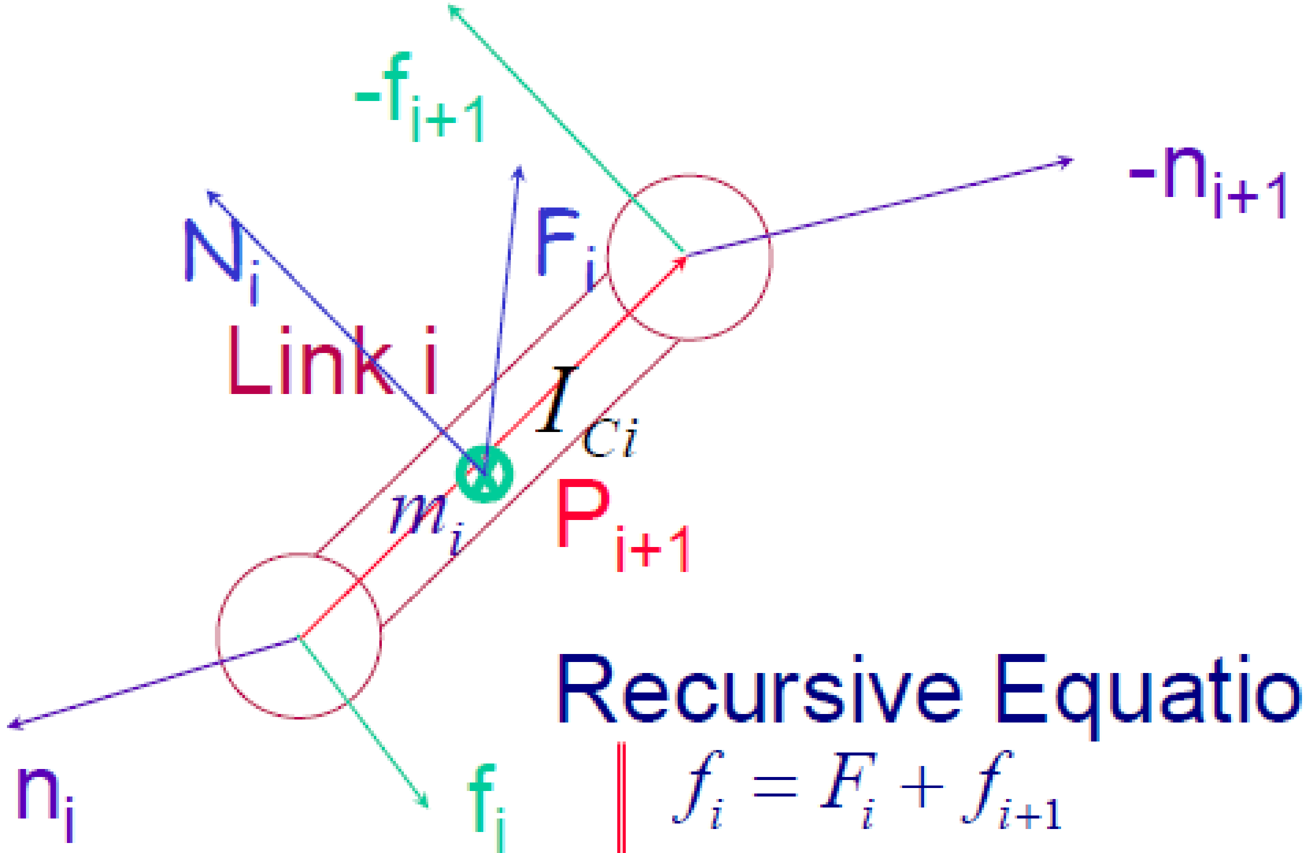
Generalized Coordinates

$$K = \frac{1}{2}\dot{q}^T M \dot{q}$$

$$M\ddot{q} + V + G = \tau$$



n
f
n
f
-f
-n
link 3
n_3
f_3
τ_3
$-f_3$
$-n_3$
n_2
link 2
τ_2
$-f_2$
f_2
f_1
τ_1
$-n_1$
$-n_2$
$-f_1$
n_1
link 1



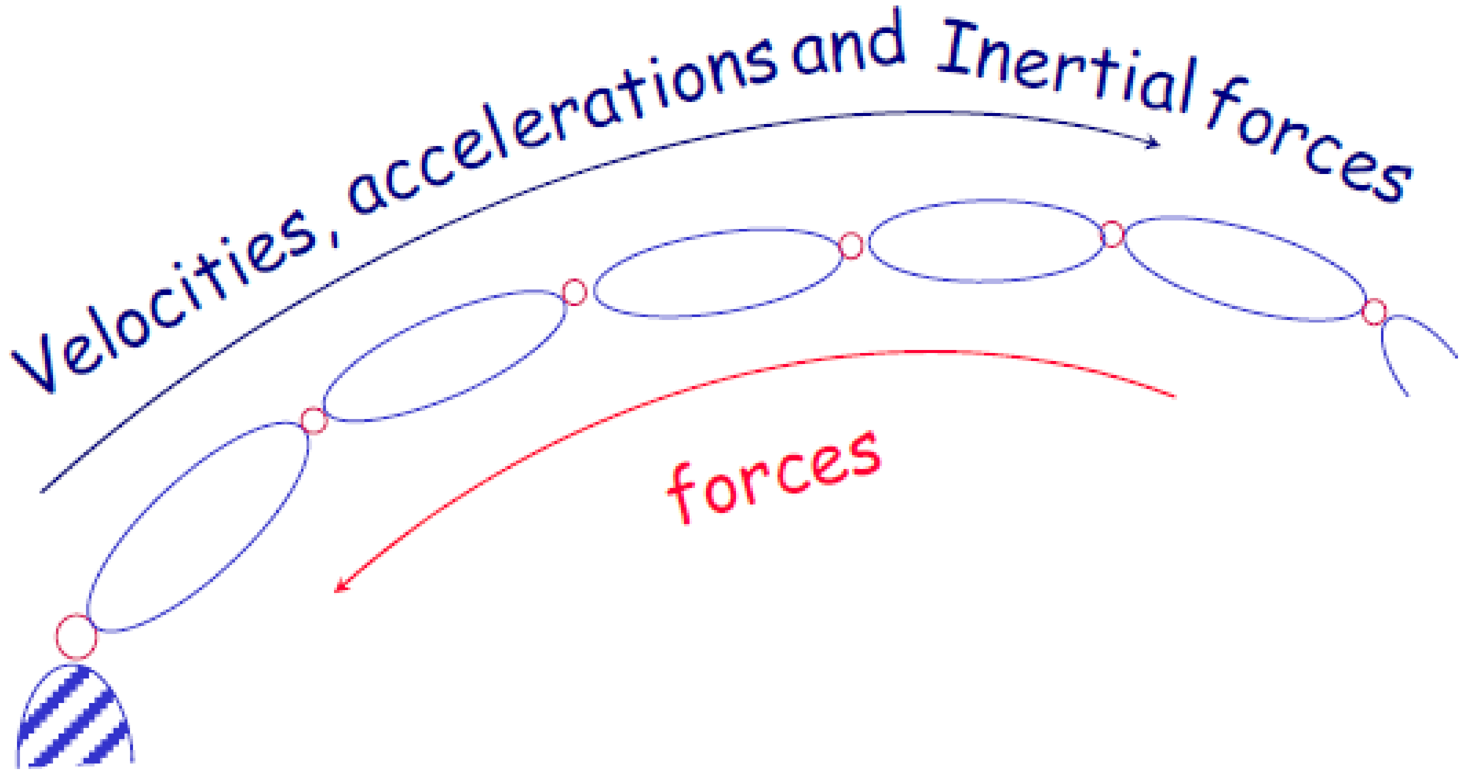
Recursive Equations

$$f_i = F_i + f_{i+1}$$

$$n_i = N_i + n_{i+1} + \mathbf{p}_{C_i} \times F_i + \mathbf{p}_{i+1} \times f_{i+1}$$

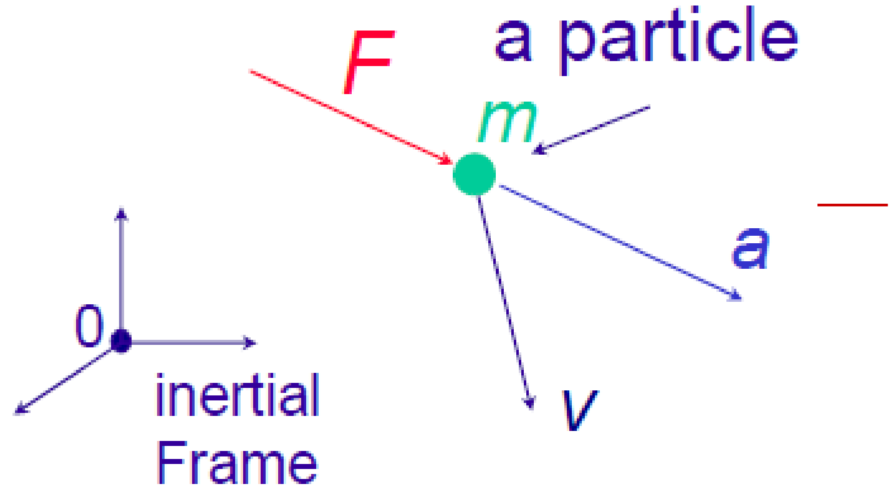
$$\tau_i = \begin{cases} n_i . Z_i & \text{revolute} \\ f_i . Z_i & \text{prismatic} \end{cases}$$

Newton-Euler Algorithm



Newton's Law

$$\underline{F} = m\underline{a}$$



$$\frac{d}{dt}(mv) = F$$

rate of change of the linear momentum is equal to the applied force

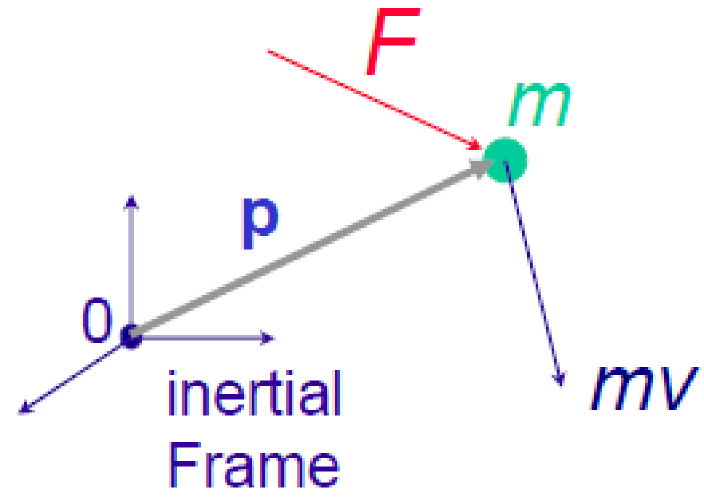
Linear Momentum

$$\underline{\varphi = mv}$$

Angular Momentum

$$m\dot{\mathbf{v}} = F$$

take the moment /0



$$\mathbf{p} \times m\dot{\mathbf{v}} = \mathbf{p} \times F$$

N

$$\frac{d}{dt}(\mathbf{p} \times m\mathbf{v}) = \mathbf{p} \times m\dot{\mathbf{v}} + \underbrace{\mathbf{v} \times m\mathbf{v}}_{=0} = \mathbf{p} \times m\dot{\mathbf{v}}$$

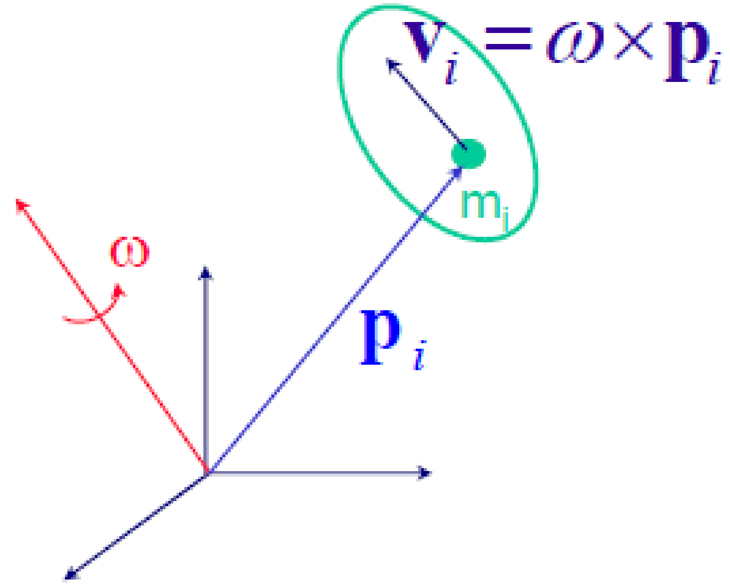
$$\boxed{\frac{d}{dt}(\mathbf{p} \times m\mathbf{v}) = N}$$

applied moment

angular momentum $\phi = p \times mv$

Rigid Body

Rotational Motion



$$\text{Angular Momentum} = \sum_i \mathbf{p}_i \times m_i \mathbf{v}_i$$

$$\phi = \sum_i m_i \mathbf{p}_i \times (\omega \times \mathbf{p}_i)$$

$$m_i \rightarrow \rho dv \qquad (\rho{:}density)$$

$$\phi = \int_V p \times (\omega \times p)\, \rho dv$$

$$\phi = \int_V p \times (\omega \times p)\rho dv$$

$$\mathbf{p} \times (\omega \times \mathbf{p}) = \hat{\mathbf{p}}(-\hat{\mathbf{p}})\omega$$

$$\phi = [\int_V -\hat{\mathbf{p}}\hat{\mathbf{p}}\rho dv]\omega$$

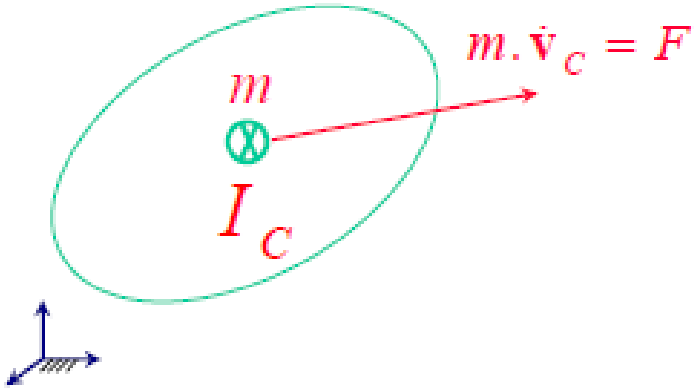
Inertia Tensor

$$\underline{\phi = I\omega}$$

Newton-Euler Equations

Translational Motion

$$m\dot{\mathbf{v}}_C = F$$

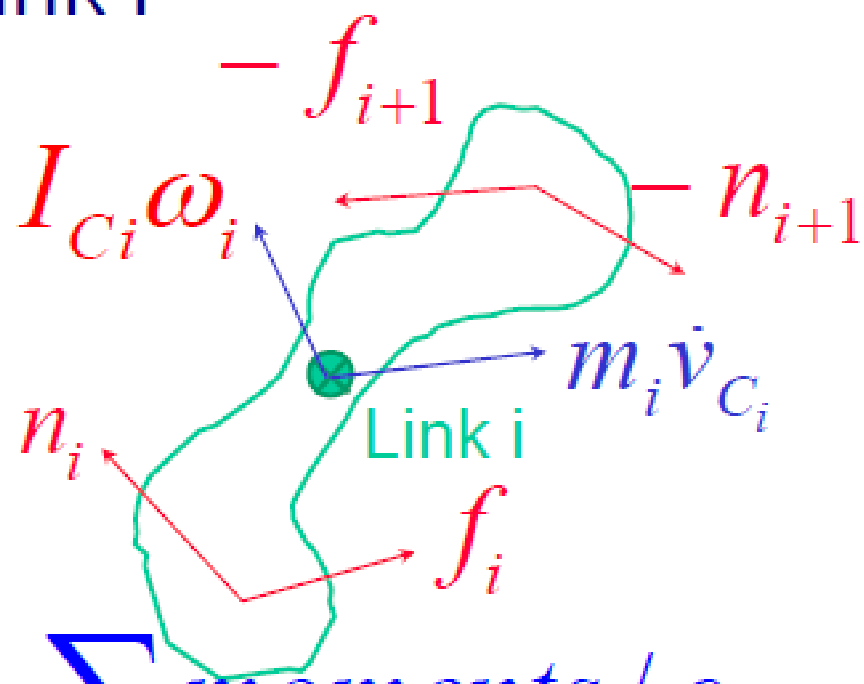


Rotational Motion

$$I_C\dot{\omega} + \omega \times I_C\omega = N$$

Dynamic forces on Link i

$$I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i$$



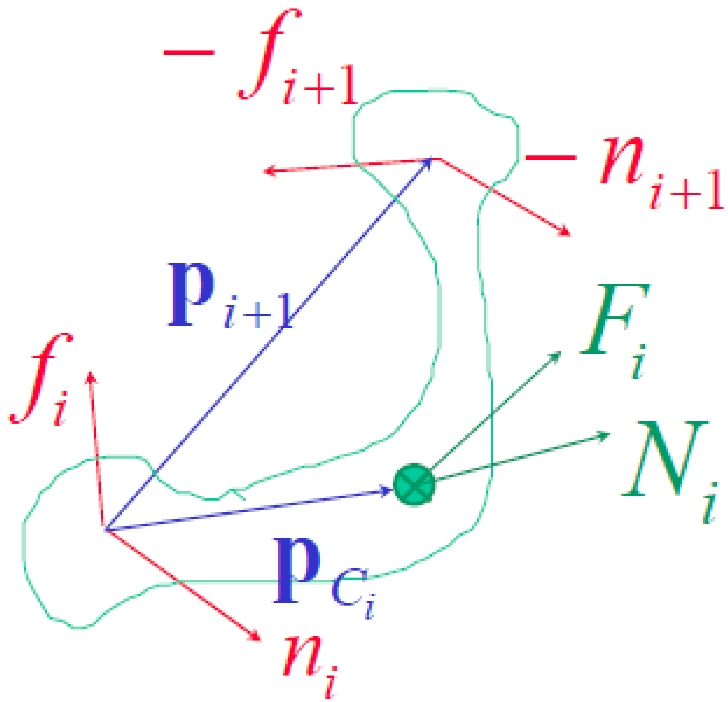
$$m_i \dot{\mathbf{v}}_{C_i} = \sum forces$$

$$I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i = \sum moments / c_i$$

Inertial forces/moments

$$F_i = m_i \dot{\mathbf{v}}_{C_i}$$

$$N_i = I_{Ci}\dot{\omega}_i + \omega_i \times I_{Ci}\omega_i$$



$$F_i = f_i - f_{i+1}$$

$$N_i = n_i - n_{i+1} + (-\mathbf{p}_{C_i}) \times f_i + (\mathbf{p}_{i+1} - \mathbf{p}_{C_i}) \times (-f_{i+1})$$

拉格朗日力学回顾

- 拉格朗日—分析力学的创立者
- 把力学体系的运动方程从以力为基本概念的牛顿形式，改变为以能量为基本概念的分析力学形式
- 方程中不包含约束力，便于分析和上机计算
- 机器人是具有分布质量的三维、多自由度机构，采用牛顿力学来确定动力学方程，非常困难。采用拉格朗日方程就非常容易。

□ 拉格朗日力学回顾(Cont.)

拉格朗日函数：

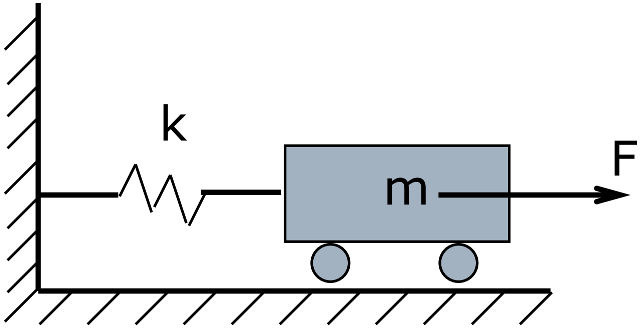
$$L = K - P$$

广义力：

$$F_i = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i}$$

$$T_i = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i}$$

- 拉格朗日方程的特点
  - 可按统一程序和步骤建立系统运动微分方程
  - 不需要考虑理想约束的约束反力
  - 只需分析速度，不需分析加速度
  - 标量方程



例4.1 小车－弹簧系统

$$F = m\ddot{x} + kx$$

$$F = ma + kx$$

- 推导图示2自由度系统的运动方程

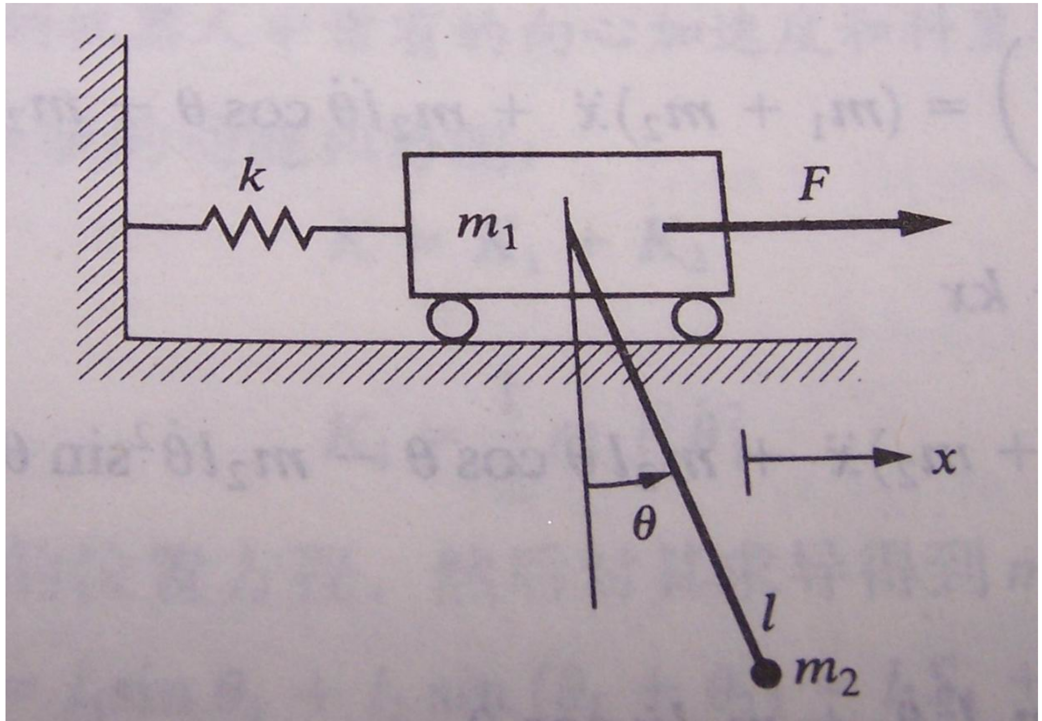
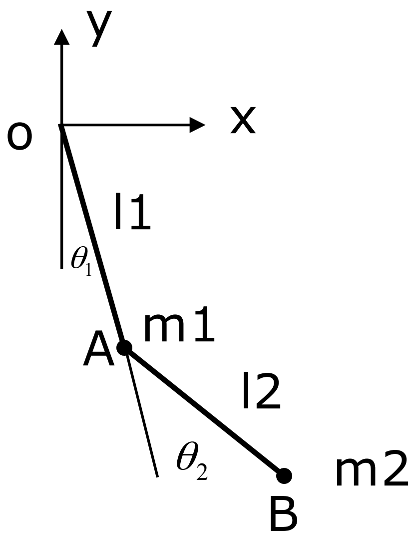


图 4.4　车－摆系统示意图



例4.3 集中质量的双连杆机构

$$K_1 = \frac{1}{2} m_1 v_1^2 = \frac{1}{2} m_1 l_1^2 \dot{\theta}_1^2$$

$$P_1 = -m_1 g y_1 = -m_1 g l_1 C_1$$

$$K_2 = \frac{1}{2} m_2 v_2^2 = \ldots$$

$$P_2 = -m g_2 y_2 = -m_2 g l_1 C_1 - m_2 g l_2 C_{12}$$

- 用拉格朗日法推导图示2自由度机器人手臂的运动方程。连杆的质心位于连杆的中心，其转动惯量分别是I1和I2。

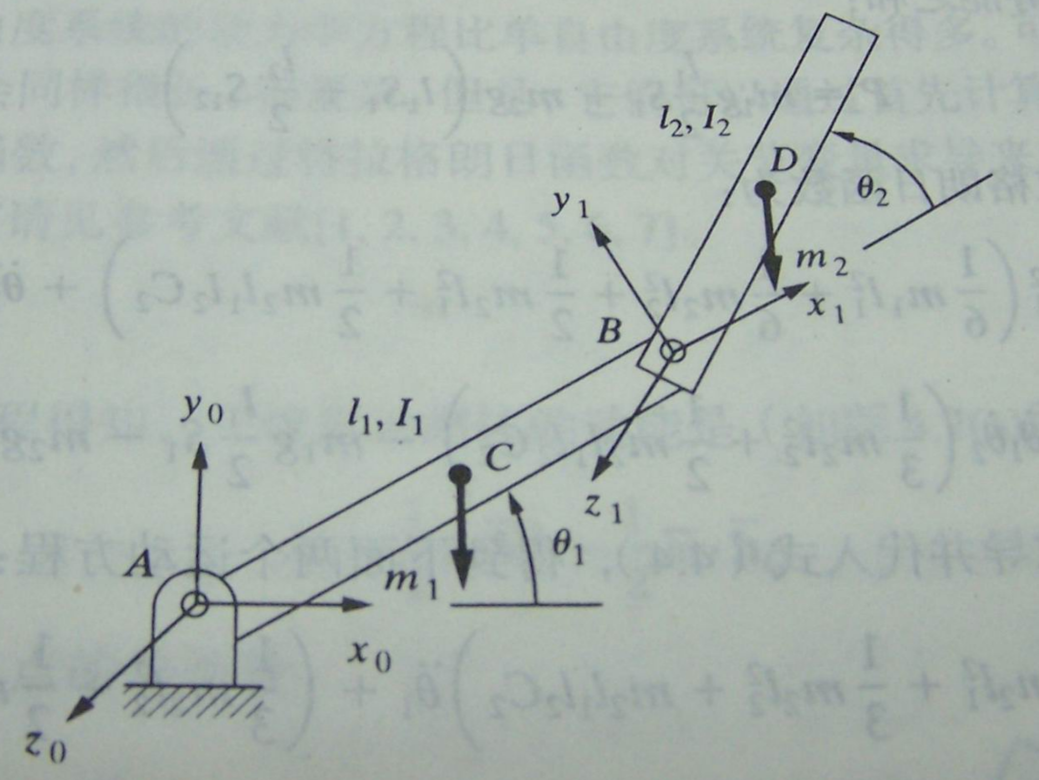


图 4.6　两自由度机器人手臂

多自由度机器人的动力学方程

- 计算连杆和关节的动能和势能来定义拉格朗日函数
- 对各关节变量进行求导

研究机械手的动力学方程建立的一般步骤：

- 计算任一连杆上任意一点的速度；
- 计算各连杆的动能和机械手的总动能；
- 计算各连杆的势能和机械手的总势能；
- 建立机械手系统的拉格朗日函数；
- 对拉格朗日函数求导，得到动力学方程。

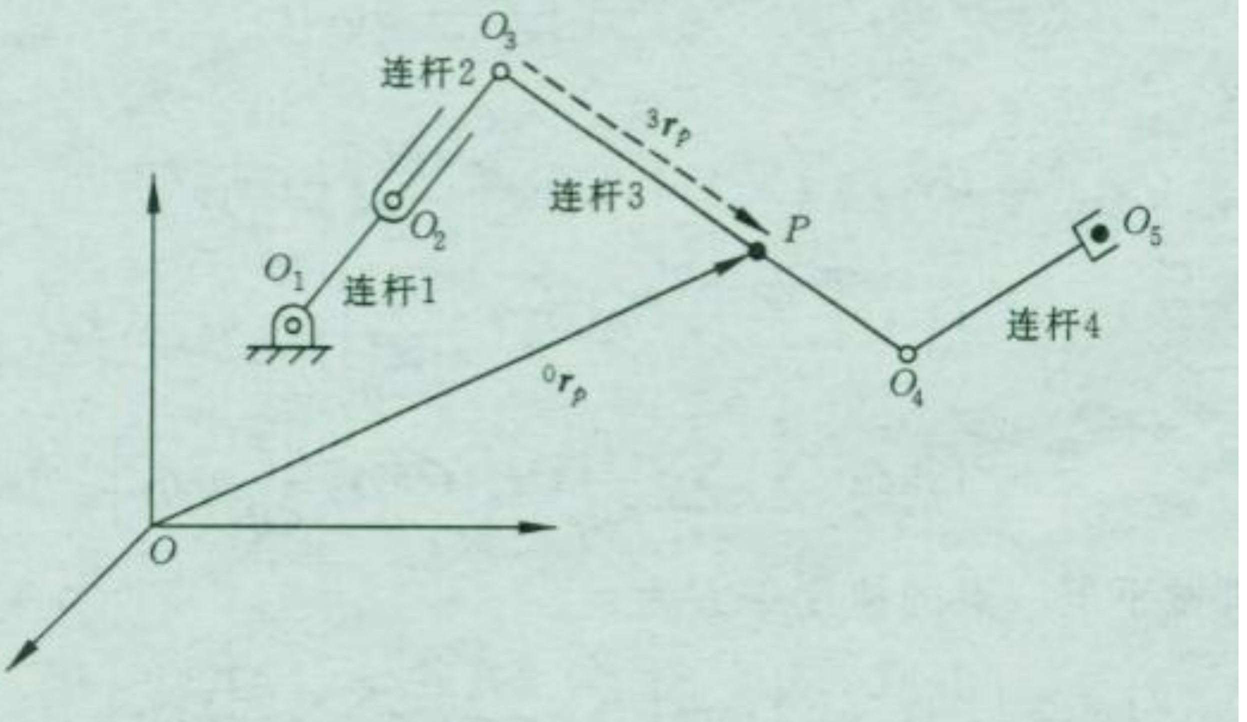


图 4.4　四连杆机械手

动能

三维运动刚体的动能：

$$K = \frac{1}{2} m \bar{V}^2 + \frac{1}{2} \bar{\omega} \bar{h}_G$$

$\bar{h}_G$:刚体关于G点的角动量

刚体作平面运动时：

$$K = \frac{1}{2} m \bar{V}^2 + \frac{1}{2} \bar{I} \omega^2$$

□ 位置方程求导来求连杆上某点的速度

机器人末端手坐标系和基座坐标系之间的变换 ${}^{o}T_i = A_1 A_2 A_3 \cdots A_n$

⇓

对每一矩阵进行关节变量的求导

⇓

某连杆上点的速度是所有关节关节变量求导的集

速度的计算（以四连杆机械手为例）

图中，连杆3上P点的位置为：

$$^{0}\mathbf{r}_{p}={}^{o}\mathbf{T}_{3}\,{}^{3}\mathbf{r}_{p}$$

$^{0}r_{p}$为基坐标系中P的位置矢量；

$^{3}r_{p}$为杆3坐标系中P的位置矢量（原点0_{3}）

T_{3}杆3的位姿矩阵；（即P点在上述两坐标系中坐标之间的变换矩阵。）

对任一连杆i上的一点，其位置为

$$^{0}\mathbf{r}={}^{o}\mathbf{T}_{i}\,{}^{i}\mathbf{r}$$

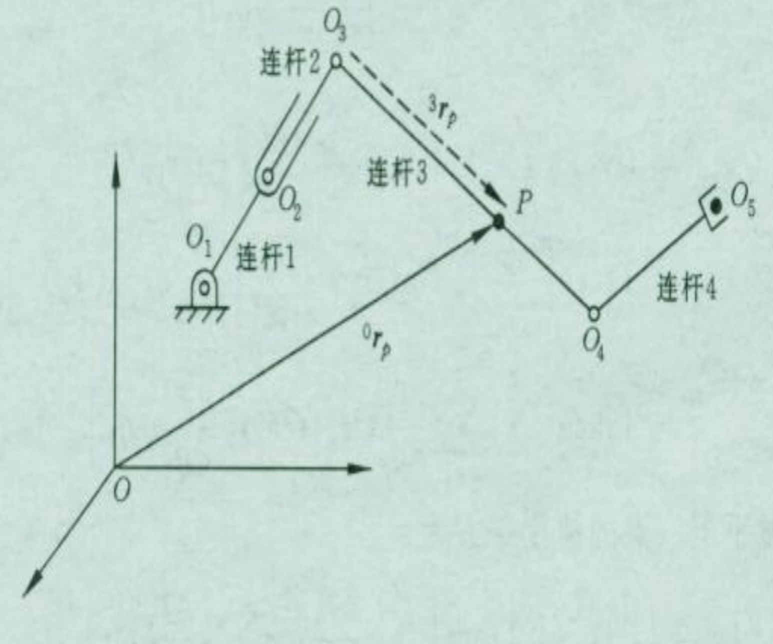


图 4.4 四连杆机械手

P点的速度为

$$^{0}\mathbf{v}_p = \frac{d}{dt}(^{0}\mathbf{r}_p) = \frac{d}{dt}(\mathbf{T}_3\,^{3}\mathbf{r}_p) = \dot{\mathbf{T}}_3\,^{3}\mathbf{r}_p$$

$$= \left(\sum_{j=1}^{3} \frac{\partial \mathbf{T}_3}{\partial q_j}\dot{q}_j\right)^{3}\mathbf{r}_p$$

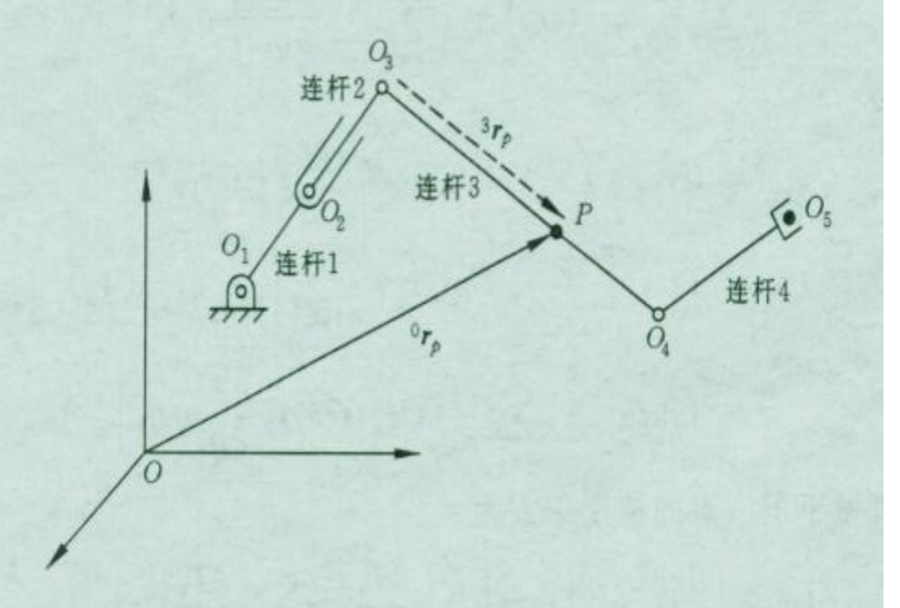


图 4.4 四连杆机械手

为何不对r求导？

对任一连杆i上的一点，其速度为

$$^{0}\mathbf{v} = \frac{d\mathbf{r}}{dt} = \left(\sum_{j=1}^{i} \frac{\partial \mathbf{T}_i}{\partial q_j}\dot{q}_j\right)^{i}\mathbf{r}$$

矩阵对关节变量的导数
（课上增加讲解）

P点的加速度为

$$ {}^0\mathbf{a}_p = \frac{d}{dt}({}^0\mathbf{v}_p) = \frac{d}{dt}(\dot{\mathrm{T}}_3\,{}^3\mathbf{r}_p) = \frac{d}{dt}\left(\left(\sum_{j=1}^{3} \frac{\partial \mathrm{T}_3}{\partial q_j}\dot{q}_j\right){}^3\mathbf{r}_p\right) $$

$$ = \left(\sum_{j=1}^{3} \frac{\partial \mathrm{T}_3}{\partial q_j}\frac{d}{dt}\dot{q}_j\right)({}^3\mathbf{r}_p) + \left(\frac{d}{dt}\left(\sum_{j=1}^{3} \frac{\partial \mathrm{T}_3}{\partial q_j}\right)\dot{q}_j\right)({}^3\mathbf{r}_p) $$

$$ = \left(\sum_{j=1}^{3} \frac{\partial \mathrm{T}_3}{\partial q_j}\ddot{q}_j\right)({}^3\mathbf{r}_p) + \left(\sum_{k=1}^{3}\sum_{j=1}^{3} \frac{\partial^2 \mathrm{T}_3}{\partial q_j \partial q_k}\dot{q}_k\dot{q}_j\right)({}^3\mathbf{r}_p) $$

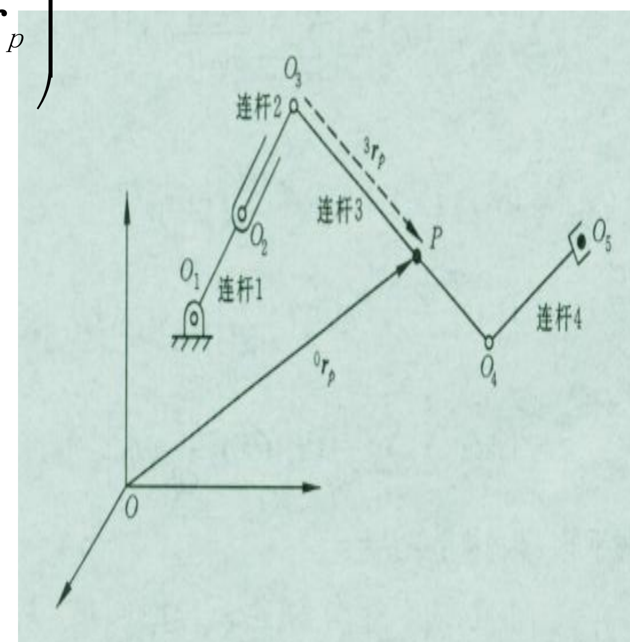


图 4.4 四连杆机械手

对任一连杆i上的一点，其加速度为

$$ {}^0\mathbf{a} = \left(\sum_{j=1}^{i} \frac{\partial \mathrm{T}_i}{\partial q_j}\ddot{q}_j\right){}^i\mathbf{r} + \left(\sum_{k=1}^{i}\sum_{j=1}^{i} \frac{\partial^2 \mathrm{T}_i}{\partial q_j \partial q_k}\dot{q}_k\dot{q}_j\right){}^i\mathbf{r} $$

P点速度的平方为

$$\left({}^0\mathbf{v}_p\right)^2 = Trace\left[\left({}^0\mathbf{v}_p\right)\cdot\left({}^0\mathbf{v}_p\right)^T\right]$$

$$= Trace\left[\left(\sum_{j=1}^{3}\frac{\partial \mathrm{T}_3}{\partial q_j}\dot{q}_j\right){}^3\mathbf{r}_p \bullet \left({}^3\mathbf{r}_p\right)^T\left(\sum_{k=1}^{3}\frac{\partial \mathrm{T}_3}{\partial q_k}\dot{q}_k\right)^T\right]$$

$$= Trace\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial \mathrm{T}_3}{\partial q_j}{}^3\mathbf{r}_p \bullet \left({}^3\mathbf{r}_p\right)^T\frac{\partial \mathrm{T}_3^T}{\partial q_k}\dot{q}_j\dot{q}_k\right]$$

对任一连杆i上的一点，其速度平方为

$$\left({}^0\mathbf{v}\right)^2 = Trace\left[\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial \mathbf{T}_i}{\partial q_j}{}^i\mathbf{r} \bullet \left({}^i\mathbf{r}\right)^T\frac{\partial \mathbf{T}_i^T}{\partial q_k}\dot{q}_j\dot{q}_k\right]$$

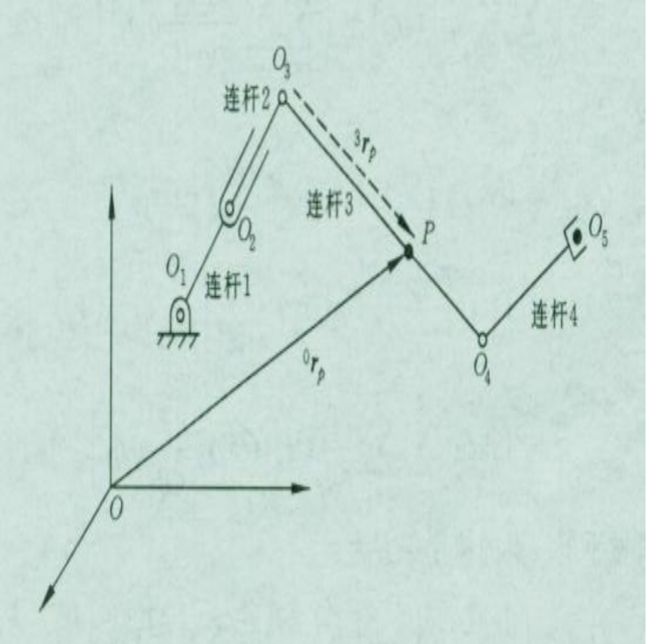


图 4.4 四连杆机械手

杆3上P点质量为dm的微元，其动能为：

$$dK_3 = \frac{1}{2}\left({}^0\mathbf{v}_p\right)^2 dm$$

$$= \frac{1}{2} Trace\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial \mathbf{T}_3}{\partial q_j}{}^3\mathbf{r}_p \bullet \left({}^3\mathbf{r}_p\right)^T \frac{\partial \mathbf{T}_3^T}{\partial q_k}\dot{q}_j\dot{q}_k\right]dm$$

$$= \frac{1}{2} Trace\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial \mathbf{T}_3}{\partial q_j}\left({}^3\mathbf{r}_p dm\left({}^3\mathbf{r}_p\right)^T\right)\frac{\partial \mathbf{T}_3^T}{\partial q_k}\dot{q}_j\dot{q}_k\right]$$

对任一连杆i上的质量为dm点，其动能为

$$dK_i = \frac{1}{2} Trace\left[\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial \mathbf{T}_i}{\partial q_j}\left({}^i\mathbf{r} dm\left({}^i\mathbf{r}\right)^T\right)\frac{\partial \mathbf{T}_i^T}{\partial q_k}\dot{q}_j\dot{q}_k\right]$$

杆3的动能为：

$$K_3 = \int_{\text{连杆3}} dK_3 = \frac{1}{2} Trace\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial \mathbf{T}_3}{\partial q_j}\left(\int_{\text{连杆3}} {}^3\mathbf{r}_p \left({}^3\mathbf{r}_p\right)^T dm\right)\frac{\partial \mathbf{T}_3^T}{\partial q_k}\dot{q}_j\dot{q}_k\right]$$

记 $I_3 = \int_{\text{连杆3}} {}^3\mathbf{r}_p\, {}^3\mathbf{r}_p^T dm$，并称之为连杆3的伪惯量矩阵，则

$$K_3 = \int_{\text{连杆3}} dK_3 = \frac{1}{2} Trace\left[\sum_{j=1}^{3}\sum_{k=1}^{3}\frac{\partial \mathbf{T}_3}{\partial q_j}\mathbf{I}_3\frac{\partial \mathbf{T}_3^T}{\partial q_k}\dot{q}_j\dot{q}_k\right]$$

对任连杆i，其动能为

$$K_i = \int_{\text{连杆}i} dK_i = \frac{1}{2} Trace\left[\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial \mathbf{T}_i}{\partial q_j}\mathbf{I}_i\frac{\partial \mathbf{T}_i^T}{\partial q_k}\dot{q}_j\dot{q}_k\right]$$

伪惯量矩阵I的一般形式为：

$$I_i = \int_{连杆i} {}^i\mathbf{r}\,{}^i\mathbf{r}^T dm = \begin{bmatrix} \int {}^ix^2 dm & \int {}^ix\,{}^iy dm & \int {}^ix\,{}^iz dm & \int {}^ix dm \\ \int {}^ix\,{}^iy dm & \int {}^iy^2 dm & \int {}^iy\,{}^iz dm & \int {}^iy dm \\ \int {}^ix\,{}^iz dm & \int {}^iy\,{}^iz dm & \int {}^iz^2 dm & \int {}^iz dm \\ \int {}^ix dm & \int {}^iy dm & \int {}^iz dm & \int dm \end{bmatrix}$$

$$I_{xx} = \int (y^2 + z^2) dm \quad I_{yy} = \int (x^2 + z^2) dm \quad I_{zz} = \int (x^2 + y^2) dm$$

$$I_{xy} = I_{yx} = \int xydm \quad I_{xz} = I_{zx} = \int xzdm \quad I_{yz} = I_{zy} = \int yzdm$$

$$mx = \int xdm \quad my = \int ydm \quad mz = \int zdm$$

则具有n个连杆的机械手的连杆总动能为：

$$K=\sum_{i=1}^{n}K_i=\frac{1}{2}\sum_{i=1}^{n}Trace\left[\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial \mathbf{T}_i}{\partial q_j}\mathbf{I}_i\frac{\partial \mathbf{T}_i^T}{\partial q_k}\dot{q}_j\dot{q}_k\right]$$

考虑传动装置的惯量，所有传动装置的总动能为：

$$K_a=\frac{1}{2}\sum_{i=1}^{n}I_{ai}\dot{q}_i^2$$

系统的总动能为

$$K_t=K+K_a=\frac{1}{2}\sum_{i=1}^{n}Trace\left[\sum_{j=1}^{i}\sum_{k=1}^{i}\frac{\partial \mathbf{T}_i}{\partial q_j}\mathbf{I}_i\frac{\partial \mathbf{T}_i^T}{\partial q_k}\dot{q}_j\dot{q}_k\right]+\frac{1}{2}\sum_{i=1}^{n}I_{ai}\dot{q}_i^2$$

势能： 质量m，高h的物体，其势能为

$$P = mgh$$

连杆i上位置 ${}^{i}\mathbf{r}$ 的质量dm的微元，势能为

$$dp_i = -dm g^{T}\,{}^{0}r = -g^{T} T_i\,{}^{i}r dm \qquad \mathbf{g}^{T} = \begin{bmatrix} g_x & g_y & g_z & 1 \end{bmatrix}$$

连杆i的总势能为

$$P_i = \int_{\text{连杆}i} dP_i = \int_{\text{连杆}i} -\mathbf{g}^{T} T_i\,{}^{i}\mathbf{r} dm = -\mathbf{g}^{T} T_i \int_{\text{连杆}i} {}^{i}\mathbf{r} dm$$

$$= -m_i \mathbf{g}^{T} T_i\,{}^{i}\mathbf{r}_i$$

系统的总位能为

$$P=\sum_{i=1}^{n}P_i=-\sum_{i=1}^{n}m_i\mathbf{g}^T T_i{}^i\mathbf{r}_i$$

式中，m_i 为连杆i的质量；

r_i 为连杆i对其前端关节坐标系的重心位置。

系统的拉格朗日函数为

$$L = K_t - P$$

$$= \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial \mathbf{T}_i}{\partial q_j}\mathbf{I}_i\frac{\partial \mathbf{T}_i^T}{\partial q_k}\right)\dot{q}_j\dot{q}_k + \frac{1}{2}\sum_{i=1}^{n} I_{ai}\dot{q}_i^2 + \sum_{i=1}^{n} m_i \mathbf{g}^T \mathbf{T}_i {}^i\mathbf{r}_i$$

$$n = 1,2,\cdots$$

$$\frac{\partial L}{\partial \dot{q}_p} = \frac{1}{2}\sum_{i=1}^{n}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_p}\mathbf{I}_i\frac{\partial T_i^T}{\partial q_k}\right)\dot{q}_k$$

$$+\frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_j}\mathbf{I}_i\frac{\partial T_i^T}{\partial q_p}\right)\dot{q}_j + I_{ap}\dot{q}_p$$

$$p = 1,2,\cdots,n$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_p} = \sum_{i=p}^{n}\sum_{k=1}^{i} Trace\left(\frac{\partial T_i}{\partial q_k}\mathbf{I}_i\frac{\partial T_i^T}{\partial q_p}\right)\ddot{q}_k + I_{ap}\ddot{q}_p$$

$$+2\sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_j \partial q_k}\mathbf{I}_i\frac{\partial T_i^T}{\partial q_p}\right)\dot{q}_j\dot{q}_k$$

$$p = 1,2,\cdots,n$$

$$\frac{\partial L}{\partial q_p} = \sum_{i=p}^{n}\sum_{j=1}^{i}\sum_{k=1}^{i} Trace\left(\frac{\partial^2 T_i}{\partial q_p \partial q_j}\mathbf{I}_i\frac{\partial T_i^T}{\partial q_k}\right)\dot{q}_j\dot{q}_k + \sum_{i=p}^{n} m_i \mathbf{g}^T \frac{\partial T_i}{\partial q_p}{}^i\mathbf{r}_i$$

系统的动力学方程为

$$T_i = \sum_{j=i}^{n}\sum_{k=1}^{j} Trace\left(\frac{\partial T_j}{\partial q_k}\mathbf{I}_j\frac{\partial T_j^T}{\partial q_i}\right)\ddot{q}_k + I_{ai}\ddot{q}_i$$

$$+\sum_{j=1}^{n}\sum_{k=1}^{j}\sum_{m=1}^{j} Trace\left(\frac{\partial^2 T_i}{\partial q_k \partial q_m}\mathbf{I}_j\frac{\partial T_j^T}{\partial q_i}\right)\dot{q}_k\dot{q}_m - \sum_{j=1}^{n} m_j \mathbf{g}^T\frac{\partial T_i}{\partial q_i}{}^i\mathbf{r}_i$$

$$=\sum_{j=1}^{n} D_{ij}\ddot{q}_j + \mathbf{I}_{ai}\ddot{q}_i + \sum_{j=1}^{n}\sum_{k=1}^{n} D_{ijk}\dot{q}_j\dot{q}_k + D_i$$

$$\begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix}\begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} D_{111} & D_{122} \\ D_{211} & D_{222} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} D_{112} & D_{121} \\ D_{212} & D_{221} \end{bmatrix}\begin{bmatrix} \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2\dot{\theta}_1 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix}$$

$$D_{ij} = \sum_{p=\max i,j}^{6} Trace\left(\frac{\partial T_p}{\partial q_j} \mathbf{I}_p \frac{\partial T_p^T}{\partial q_i} \right)$$

$$D_{ijk} = \sum_{p=\max i,j,k}^{6} Trace\left(\frac{\partial^2 T_p}{\partial q_j \partial q_k} \mathbf{I}_p \frac{\partial T_p^T}{\partial q_i} \right)$$

$$D_i = \sum_{p=i}^{6} \left(-m_p \mathbf{g}^T \frac{\partial T_p}{\partial q_i} {}^p\mathbf{r}_p \right)$$

动力学方程的简化

1、惯量项的简化

利用
$$\frac{\partial T_p}{\partial q_i} = T_p \, {}^{T_p}\Delta_i$$

$$D_{ij} = \sum_{p=\max i,j}^{6} Trace\left(\frac{\partial T_p}{\partial q_j} \mathbf{I}_p \frac{\partial T_p^T}{\partial q_i}\right)$$

$$D_{ijk} = \sum_{p=\max i,j,k}^{6} Trace\left(\frac{\partial^2 T_p}{\partial q_j \partial q_k} \mathbf{I}_p \frac{\partial T_p^T}{\partial q_i}\right)$$

$$D_{ij} = \sum_{p=i}^{6}\left(-m_p \mathbf{g}^T \frac{\partial T_p}{\partial q_i} {}^p\mathbf{r}_p\right)$$

记微分旋转和平移为： ${}^{T_p}\delta_i = {}^p\delta_i$ ${}^{T_p}d_i = {}^p d_i$

通过计算有：

$$D_{ij} = \sum_{p=\max i,j}^{6} m_p\left[{}^p\delta_i^{\,T}\,\mathbf{k}_p\,{}^p\delta_j + {}^p d_i \cdot {}^p d_j + {}^p\bar{\mathbf{r}}_p\left({}^p d_i \times {}^p\delta_j + {}^p d_j \times {}^p\delta_j\right)\right]$$

${}^p\bar{\mathbf{r}}_p$ 为质心矢量，$\mathbf{k}_p$ 为与惯量相关的矩阵，具有如下形式。

1、惯量项的简化

$$\mathbf{k}_p = \begin{bmatrix} k_{pxx}^2 & -k_{pxy}^2 & -k_{pxz}^2 \\ -k_{pxy}^2 & k_{pyy}^2 & -k_{pyz}^2 \\ -k_{pxz}^2 & -k_{pyz}^2 & k_{pzz}^2 \end{bmatrix}$$

当i=j时，有

$$D_{ii} = \sum_{p=i}^{6} m_p \left[{}^p\delta_{ix}^2 k_{pxx}^2 + {}^p\delta_{iy}^2 k_{pyy}^2 + {}^p\delta_{iz}^2 k_{pzz}^2 + {}^p d_i \cdot {}^p d_i + 2\, {}^p\bar{\mathbf{r}}_p \left({}^p d_i \times {}^p\delta_2 \right) \right]$$

动力学方程的简化

2、重力项的简化

$$D_{ij}=\sum_{p=\max i,j}^{6} Trace\left(\frac{\partial T_p}{\partial q_j}\mathbf{I}_p\frac{\partial T_p^T}{\partial q_i}\right)$$

$$D_{ijk}=\sum_{p=\max i,j,k}^{6} Trace\left(\frac{\partial^2 T_p}{\partial q_j \partial q_k}\mathbf{I}_p\frac{\partial T_p^T}{\partial q_i}\right)$$

$$D_{ij}=\sum_{p=i}^{6}\left(-m_p\mathbf{g}^T\frac{\partial T_p}{\partial q_i}{}^p\mathbf{r}_p\right)$$

将 $\frac{\partial T_p}{\partial q_i}=T_p{}^{T_p}\Delta_i$ 代入

$$D_i=\sum_{p=i}^{6}-m_p\mathbf{g}^T T_p{}^p\Delta_i{}^p\bar{\mathbf{r}}_p$$

机器人的静力分析

- 可以通过力控制来确定机器人的状态：如切割物体时，如果力过大或过少，意味着刀具切得过深或过浅；若让机器人在零件上攻螺纹，机器人不仅需要沿孔的方向施加一个轴向力，还要在丝锥上施加一定的力矩使其转动。

手坐标系中的力和力矩：$\left[{}^{H}F\right]=\left[f_x \quad f_y \quad f_z \quad m_x \quad m_y \quad m_z\right]^T$

手坐标系中的位移和转角：$\left[{}^{H}D\right]=\left[d_x \quad d_y \quad d_z \quad \delta_x \quad \delta_y \quad \delta_z\right]^T$

各关节处的力和力矩：$\left[T\right]=\left[T_1 \quad T_2 \quad T_3 \quad T_4 \quad T_5 \quad T_6\right]^T$

各关节处的微分运动：$\left[D_\theta\right]=\left[d_{\theta 1} \quad d_{\theta 2} \quad d_{\theta 3} \quad d_{\theta 4} \quad d_{\theta 5} \quad d_{\theta 6}\right]^T$

坐标系间力和力矩的变换

$$[F]^T = [f_x \quad f_y \quad f_z \quad m_x \quad m_y \quad m_z]$$

$$[D]^T = [d_x \quad d_y \quad d_z \quad \delta_x \quad \delta_y \quad \delta_z]$$

$$[{}^B F]^T = [{}^B f_x \quad {}^B f_y \quad {}^B f_z \quad {}^B m_x \quad {}^B m_y \quad {}^B m_z]$$

$$[{}^B F]^T = [{}^B f_x \quad {}^B f_y \quad {}^B f_z \quad {}^B m_x \quad {}^B m_y \quad {}^B m_z]$$

$$[{}^B D]^T = [{}^B d_x \quad {}^B d_y \quad {}^B d_z \quad {}^B \delta_x \quad {}^B \delta_y \quad {}^B \delta_z]$$