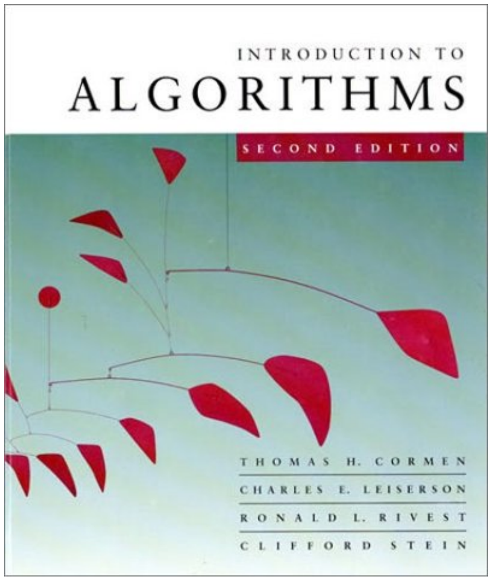
Design and Analysis of Algorithms

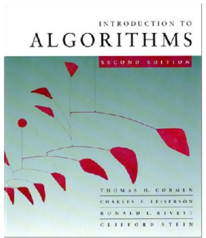
6.046J/18.401J



LECTURE 14

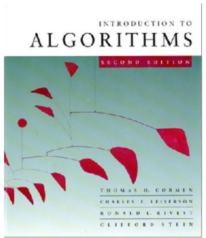
Network Flow I

- Flow networks
- Maximum-flow problem
- Flow notation
- Properties of flow
- Cuts
- Residual networks
- Augmenting paths



Flow networks

Definition. A ***flow network*** is a directed graph $G = (V, E)$ with two distinguished vertices: a ***source*** s and a ***sink*** t. Each edge $(u, v) \in E$ has a nonnegative ***capacity*** $c(u, v)$. If $(u, v) \notin E$, then $c(u, v) = 0$.



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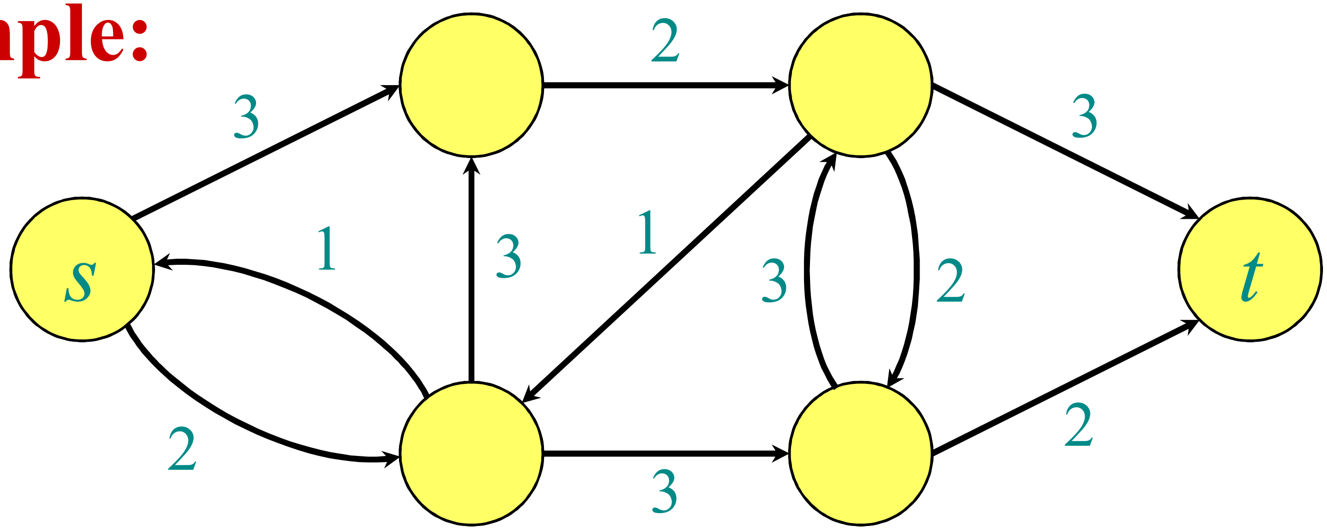
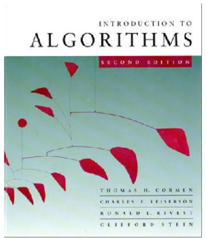
Example:





Flow networks

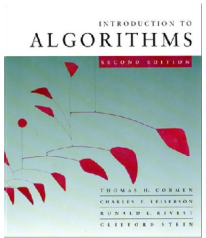
Definition. A ***positive flow*** on G is a function $p : V \times V \to \mathbb{R}$ satisfying the following:

- ***Capacity constraint:*** For all $u, v \in V$,

$$0 \le p(u, v) \le c(u, v).$$

- ***Flow conservation:*** For all $u \in V - \{s, t\}$,

$$\sum_{v \in V} p(u, v) - \sum_{v \in V} p(v, u) = 0 .$$



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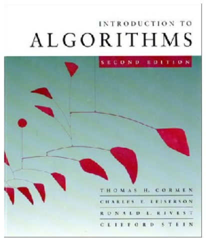
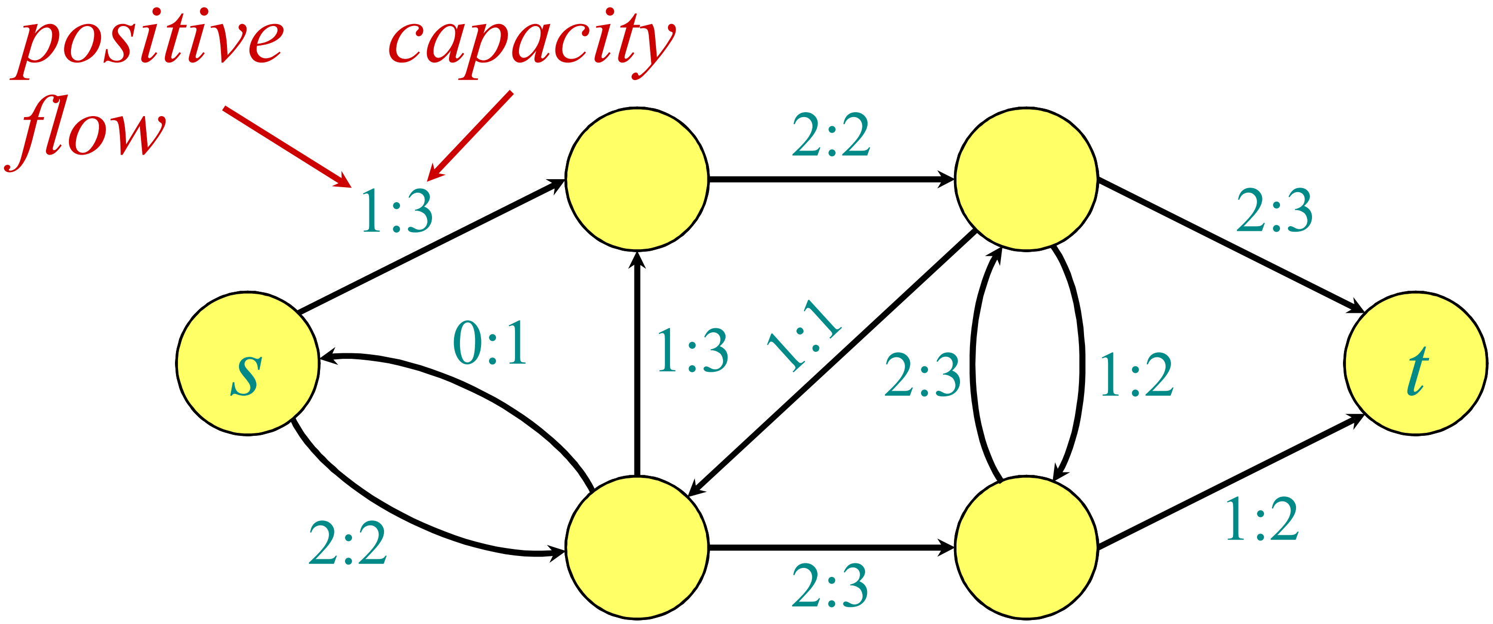
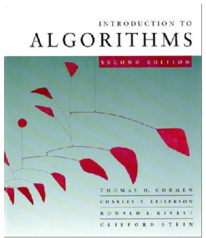
The ***value*** of a flow is the net flow out of the source:

$$\sum_{v \in V} p(s, v) - \sum_{v \in V} p(v, s).$$



A flow on a network





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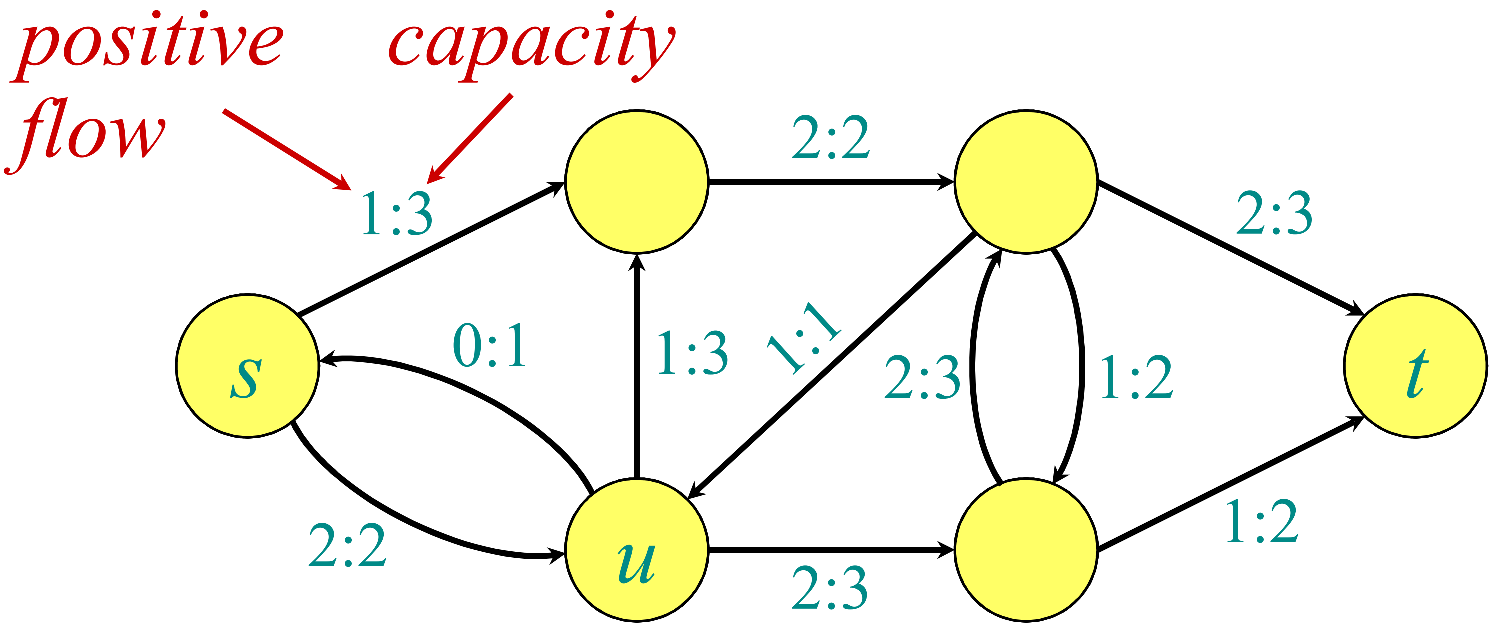


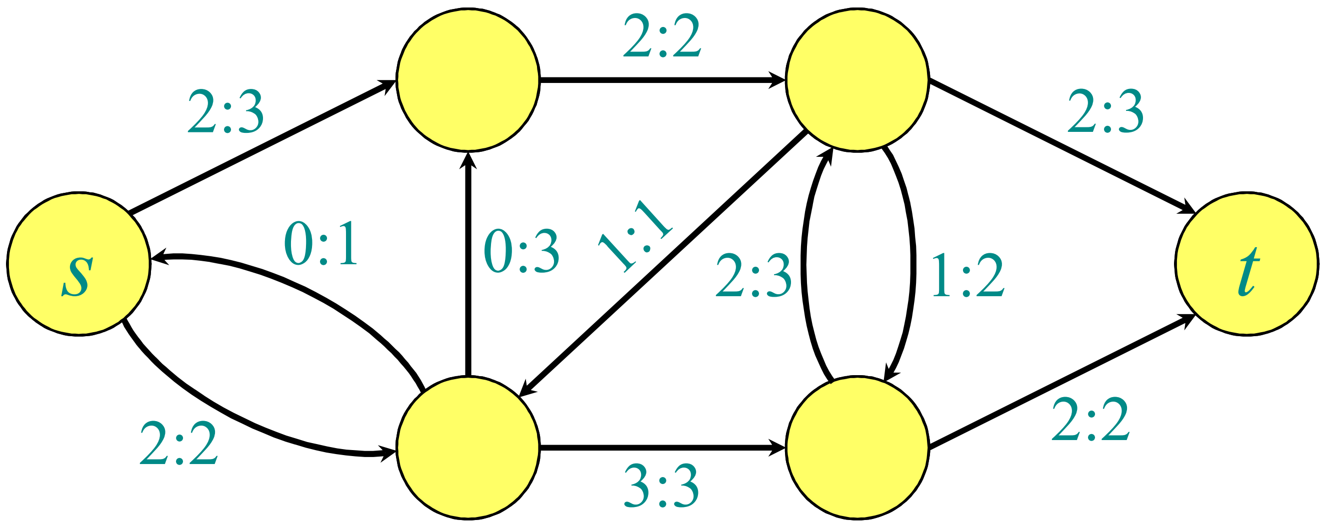
Flow conservation (like Kirchoff's current law):

- Flow into u is $2 + 1 = 3$.
- Flow out of u is $0 + 1 + 2 = 3$.

The value of this flow is $1 - 0 + 2 = 3$.

The maximum-flow problem

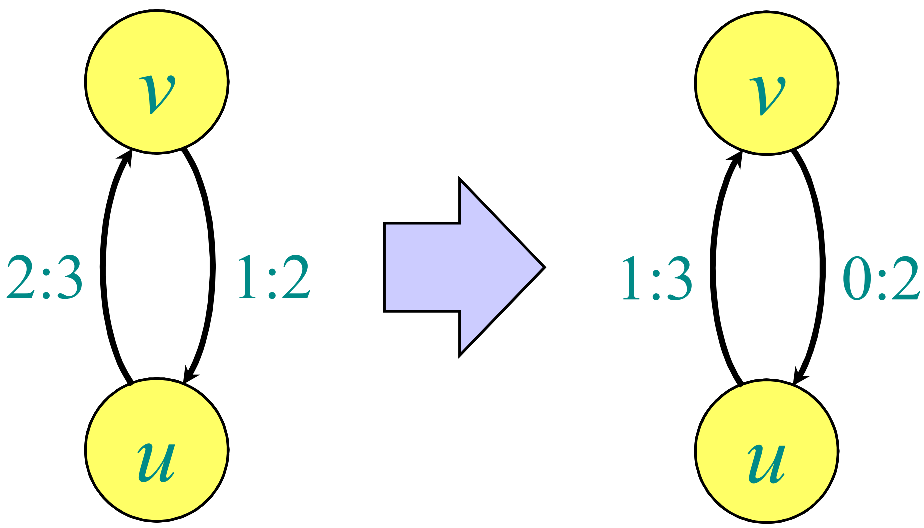
Maximum-flow problem: Given a flow network G, find a flow of maximum value on G.



The value of the maximum flow is 4.

Flow cancellation

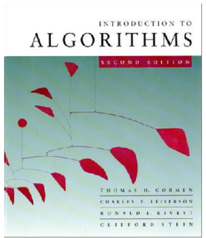
Without loss of generality, positive flow goes either from u to v, or from v to u, but not both.



Net flow from u to v in both cases is 1.

The capacity constraint and flow conservation are preserved by this transformation.

INTUITION: View flow as a *rate*, not a *quantity*.



A notational simplification

IDEA: Work with the net flow between two vertices, rather than with the positive flow.

Definition. A ***(net) flow*** on G is a function $f : V \times V \to \mathbb{R}$ satisfying the following:

- ***Capacity constraint:*** For all $u, v \in V$,

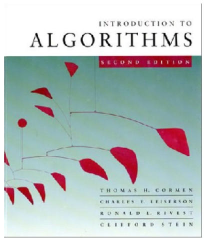
$$f(u, v) \le c(u, v).$$

- ***Flow conservation:*** For all $u \in V - \{s, t\}$,

$$\sum_{v \in V} f(u, v) = 0.$$

- ***Skew symmetry:*** For all $u, v \in V$,

$$f(u, v) = -f(v, u).$$



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- ***Flow conservation:*** For all $u \in V - \{s, t\}$,

$$\sum_{v \in V} f(u, v) = 0. \longleftarrow$$ *One summation instead of two.*

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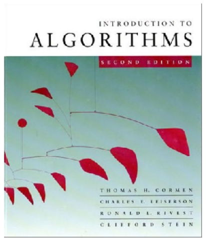
Equivalence of definitions

Theorem. The two definitions are equivalent.



Equivalence of definitions

Theorem. The two definitions are equivalent.

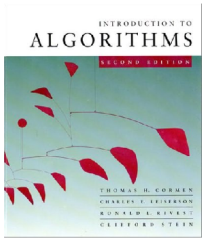
Proof. ($\Rightarrow$) Let $f(u, v) = p(u, v) - p(v, u)$.

- ***Capacity constraint:*** Since $p(u, v) \le c(u, v)$ and $p(v, u) \ge 0$, we have $f(u, v) \le c(u, v)$.
- ***Flow conservation:***

$$\begin{aligned}\sum_{v \in V} f(u,v) &= \sum_{v \in V} \big(p(u,v) - p(v,u)\big) \\ &= \sum_{v \in V} p(u,v) - \sum_{v \in V} p(v,u)\end{aligned}$$

- ***Skew symmetry:***

$$\begin{aligned}f(u, v) &= p(u, v) - p(v, u) \\ &= -\,(p(v, u) - p(u, v)) \\ &= -\,f(v, u).\end{aligned}$$



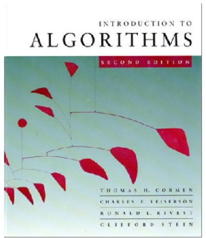
Proof (continued)

($\Leftarrow$) Let

$$p(u, v) = \begin{cases} f(u, v) & \text{if } f(u, v) > 0, \\ 0 & \text{if } f(u, v) \le 0. \end{cases}$$

- ***Capacity constraint:*** By definition, $p(u, v) \ge 0$. Since $f(u, v) \le c(u, v)$, it follows that $p(u, v) \le c(u, v)$.
- ***Flow conservation:*** If $f(u, v) > 0$, then $p(u, v) - p(v, u) = f(u, v)$. If $f(u, v) \le 0$, then $p(u, v) - p(v, u) = -f(v, u) = f(u, v)$ by skew symmetry. Therefore,

$$\sum_{v \in V} p(u,v) - \sum_{v \in V} p(v,u) = \sum_{v \in V} f(u,v). \quad \square$$



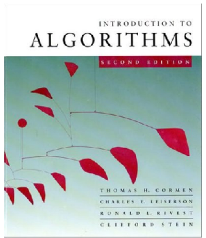
Notation

Definition. The ***value*** of a flow f, denoted by $|f|$, is given by

$$
\begin{aligned}
|f| &= \sum_{v \in V} f(s, v) \\
&= f(s, V).
\end{aligned}
$$

Implicit summation notation: A set used in an arithmetic formula represents a sum over the elements of the set.

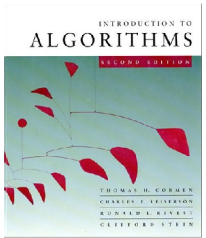
- **Example** — flow conservation: $f(u, V) = 0$ for all $u \in V - \{s, t\}$.



Simple properties of flow

Lemma.

- $f(X, X) = 0$,
- $f(X, Y) = -f(Y, X)$,
- $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ if $X \cap Y = \emptyset$. $\square$



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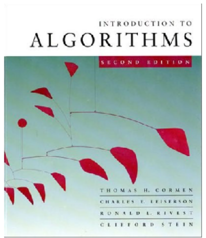
- $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$ if $X \cap Y = \varnothing$. $\square$

Theorem. $|f| = f(V, t)$.

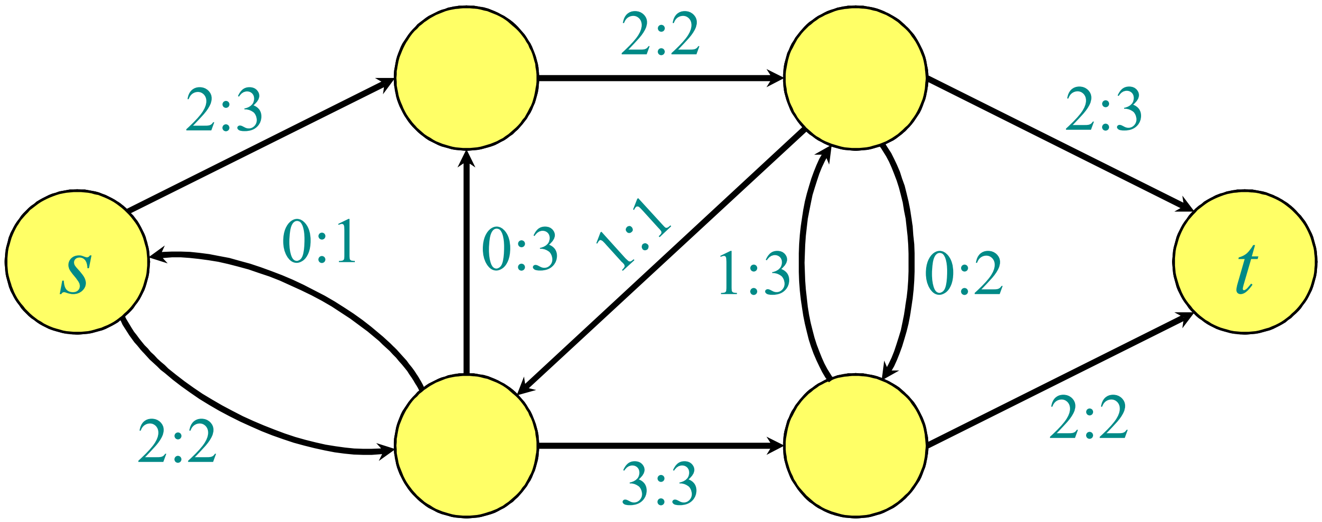
Proof.

$$
\begin{aligned}
|f| &= f(s, V) \\
&= f(V, V) - f(V-s, V) \\
&= f(V, V-s) \\
&= f(V, t) + f(V, V-s-t) \\
&= f(V, t). \quad \square
\end{aligned}
$$

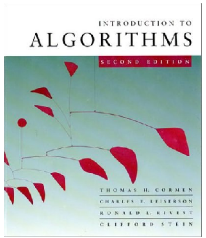
Omit braces.



Flow into the sink

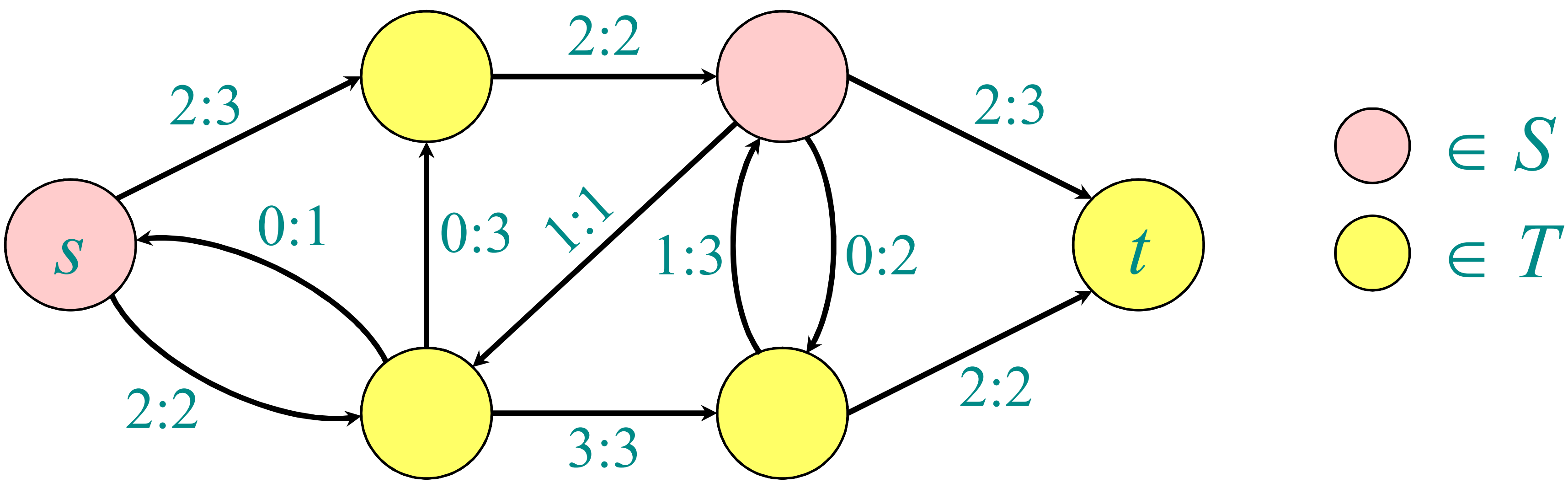


$|f| = f(s, V) = 4$ $\qquad\qquad$ $f(V, t) = 4$



Cuts

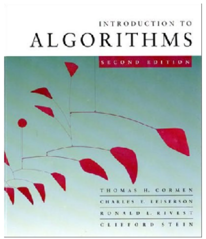
Definition. A ***cut*** (S, T) of a flow network $G = (V, E)$ is a partition of V such that $s \in S$ and $t \in T$. If f is a flow on G, then the ***flow across the cut*** is $f(S, T)$.



$$f(S, T) = (2 + 2) + (-2 + 1 - 1 + 2)$$
$$= 4$$

Another characterization of flow value

Lemma. For any flow f and any cut (S, T), we have $|f| = f(S, T)$.



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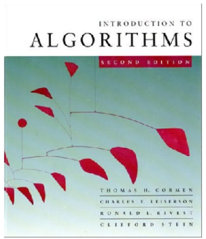
Lemma. For any flow f and any cut (S, T), we have $|f| = f(S, T)$.

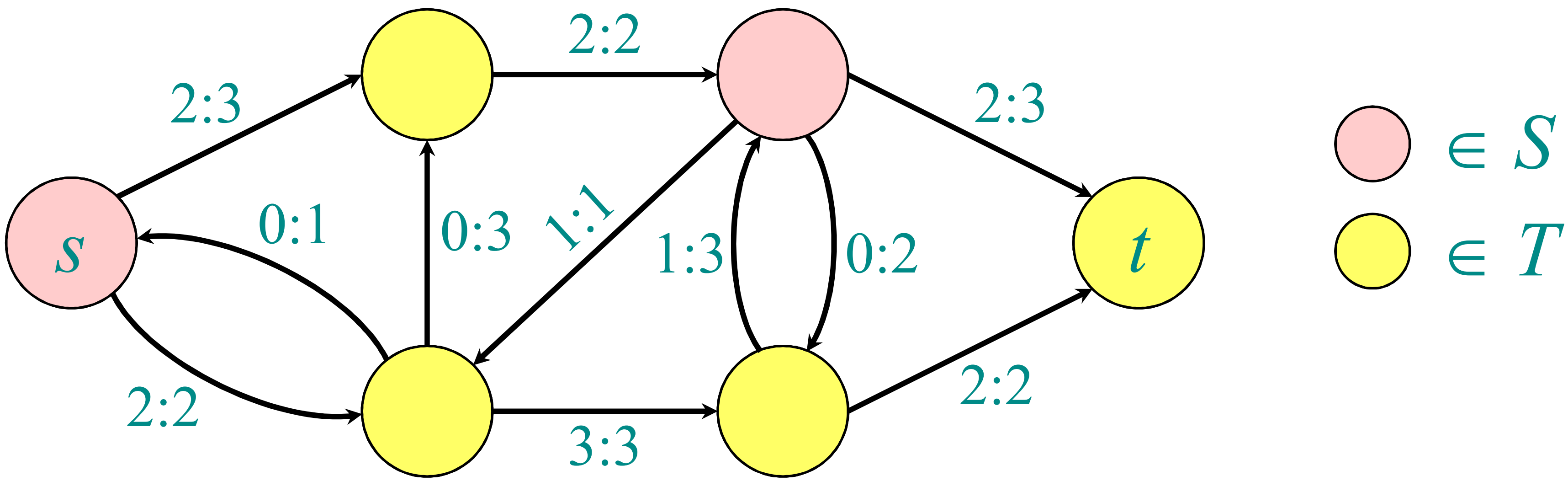
Proof.

$$
\begin{aligned}
f(S, T) &= f(S, V) - f(S, S) \\
&= f(S, V) \\
&= f(s, V) + f(S-s, V) \\
&= f(s, V) \\
&= |f|. \qquad \square
\end{aligned}
$$



Capacity of a cut

Definition. The ***capacity of a cut*** (S, T) is $c(S, T)$.



$$c(S, T) = (3 + 2) + (1 + 2 + 3)$$
$$= 11$$

Upper bound on the maximum flow value

Theorem. The value of any flow is bounded above by the capacity of any cut.

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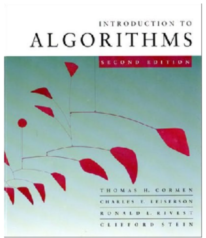

Upper bound on the maximum flow value

Theorem. The value of any flow is bounded above by the capacity of any cut.

Proof.

$$
\begin{aligned}
|f| &= f(S,T) \\
&= \sum_{u \in S} \sum_{v \in T} f(u,v) \\
&\le \sum_{u \in S} \sum_{v \in T} c(u,v) \\
&= c(S,T)\,. \quad \square
\end{aligned}
$$

Residual network

Definition. Let f be a flow on $G = (V, E)$. The ***residual network*** $G_f(V, E_f)$ is the graph with strictly positive ***residual capacities***

$$c_f(u, v) = c(u, v) - f(u, v) > 0.$$

Edges in E_f admit more flow.

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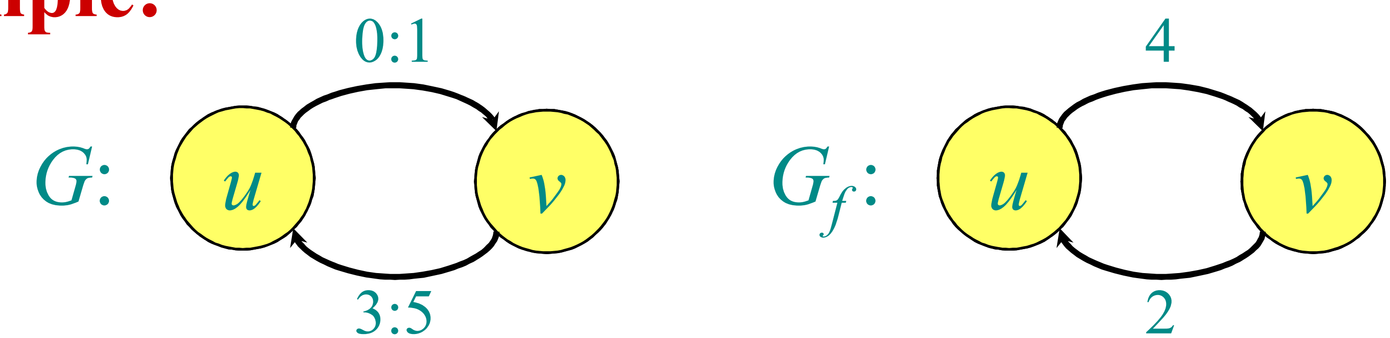
Example:



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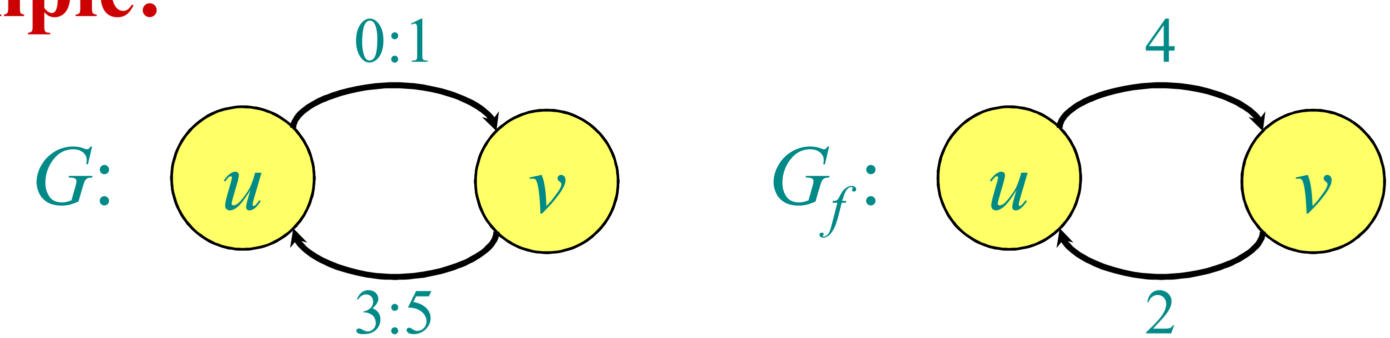
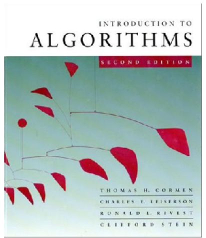


Lemma. $|E_f| \le 2|E|$. $\square$



Augmenting paths

Definition. Any path from s to t in G_f is an ***augmenting path*** in G with respect to f. The flow value can be increased along an augmenting path p by $c_f(p) = \min_{(u,v)\in p} \{c_f(u,v)\}$.

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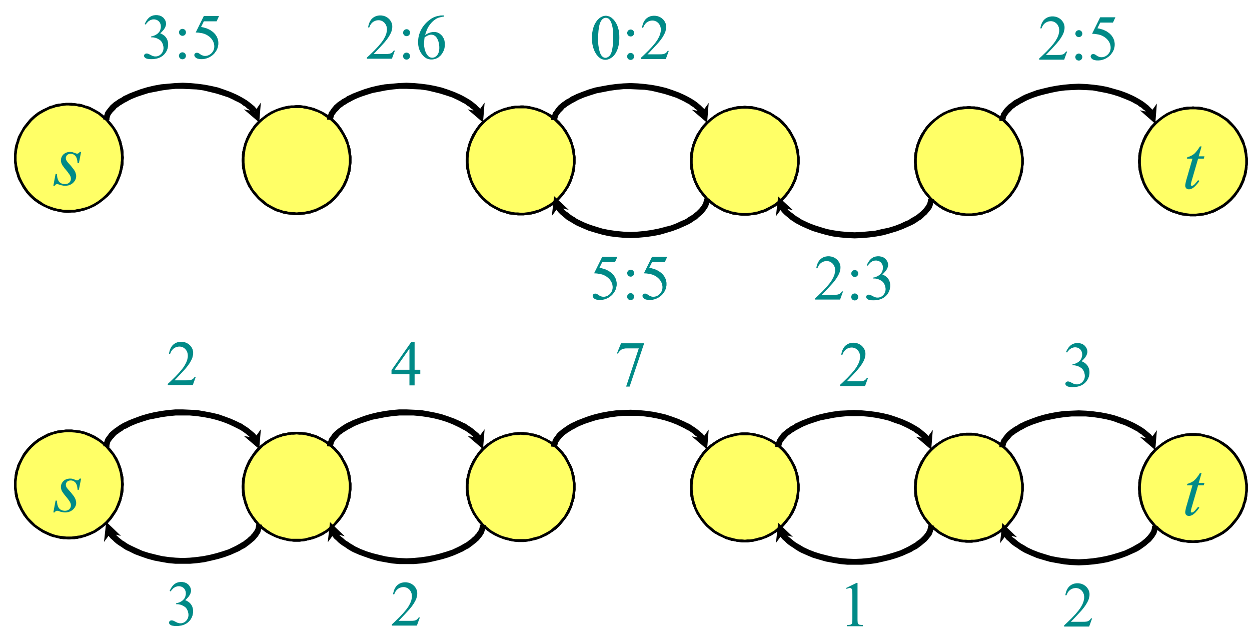
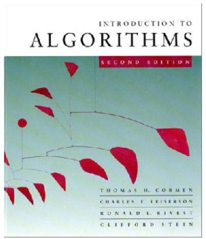
Ex.:

G:

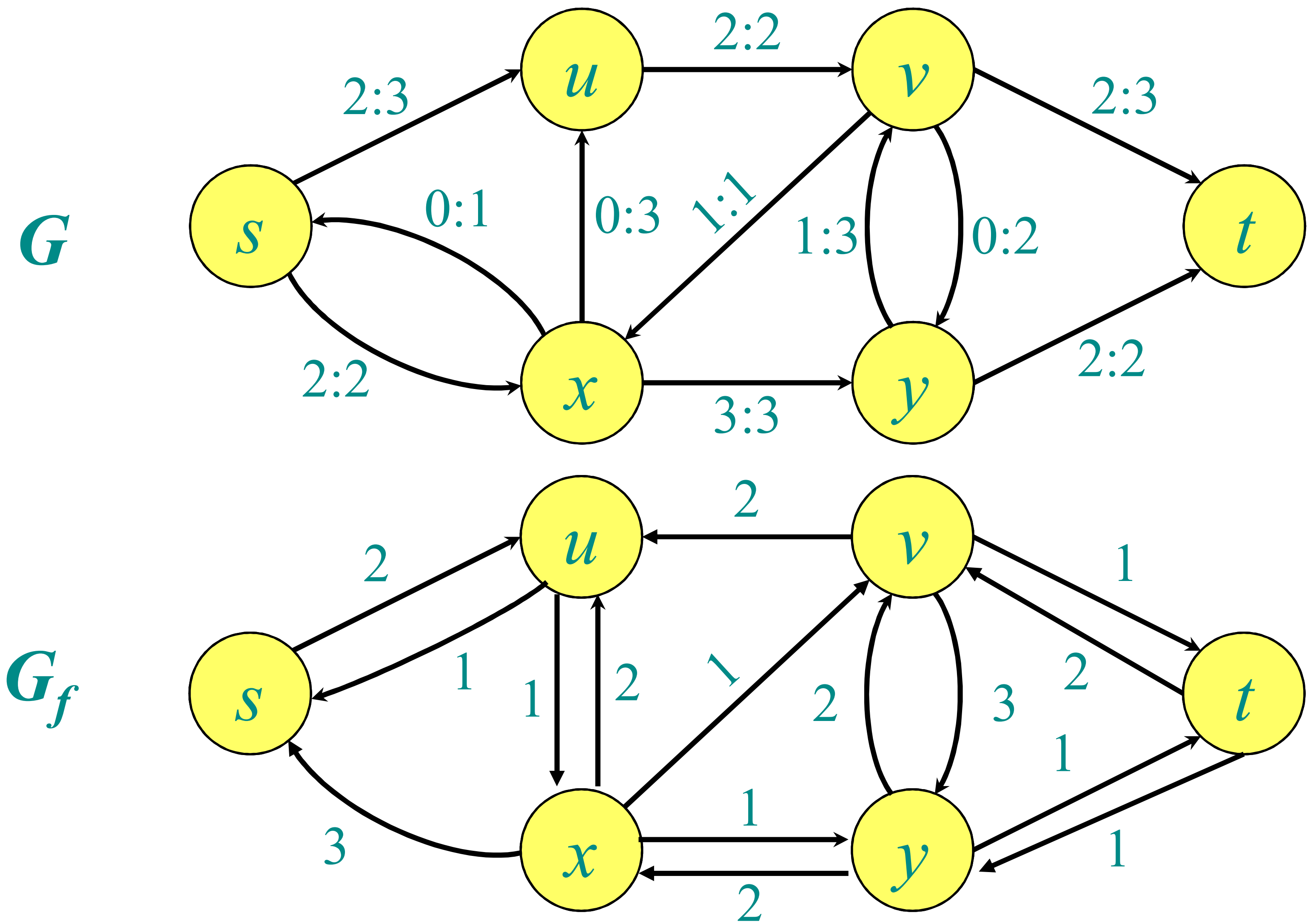


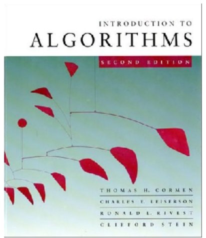
$c_f(p) = 2$

G_f:



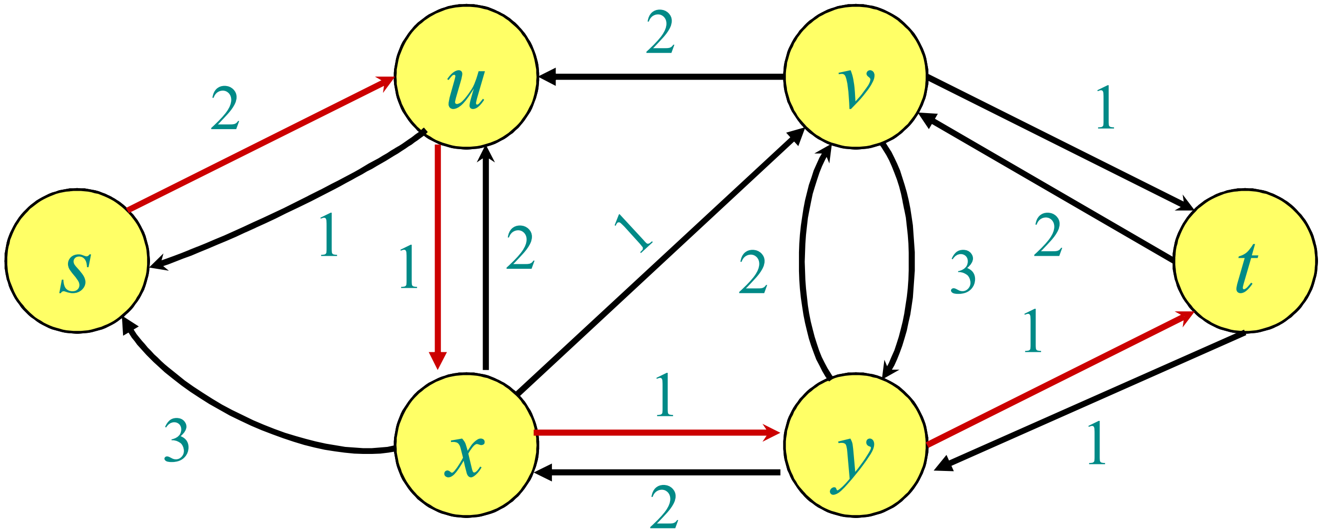
Flow and Residual Network





Residual Network and Augmenting Paths

G_f



$p = s, u, x, y, t \qquad c_f(p) = 1$

Also $q = s, u, x, v, t \qquad c_f(q) = 1$

Max-flow, min-cut theorem

Theorem. The following are equivalent:

1. f is a maximum flow.
2. G_f contains no augmenting paths.
3. $|f| = c(S, T)$ for some cut (S, T) of G.

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1. f is a maximum flow.
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Proof (and algorithms). Next time. □