Eigenvalue Problem

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1. Why $\mathbf{Ax} = \lambda\mathbf{x}$

2. How to Solve $\mathbf{Ax} = \lambda\mathbf{x}$

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Eigenvalue

- Eigenvalues have their best importance in dynamic problems
- The solution is changing with time-growing or decaying or oscillating
- Suppose we need the hundredth power $\mathbf{A}^{100}$

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- $\mathbf{A}^{100}$ was found using the eigenvalues of $\mathbf{A}$, not by multiplying 100 matrices

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  where the number λ is the “eigenvalue"
- We can also say that **Ax** is parallel to **x**

Steady State

$$\mathbf{A} = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix} \quad \mathbf{x}_1 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} \mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

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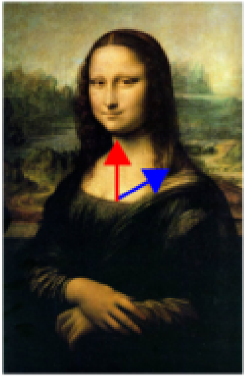
$$\text{Col}_1[\mathbf{A}^{99}] = \mathbf{x}_1 + .2(.5)^{99}\mathbf{x}_2 = \begin{bmatrix} .6 \\ .4 \end{bmatrix} + \begin{bmatrix} \text{very} \\ \text{small} \\ \text{vector} \end{bmatrix}$$

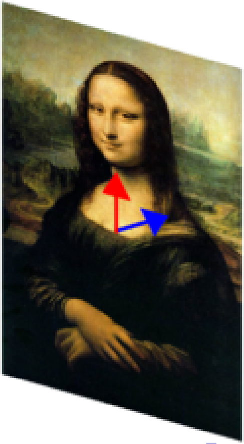
- the eigenvector $\mathbf{x}_1$ is a “steady state" that does not change
- the eigenvector $\mathbf{x}_2$ is a “decaying mode" that visually disappears
- the higher the power of $\mathbf{A}$, the closer its columns approach the steady state
- this particular $\mathbf{A}$ is a Markov matrix

Eigenvector - An Example

After a shear transformation

$$\mathbf{A} = \begin{bmatrix} 1 & 0 \\ -\frac{1}{2} & 1 \end{bmatrix}$$





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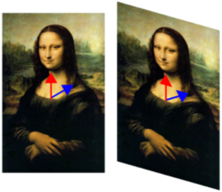

The shear transformation of the Mona Lisa

- the central vertical axis (red vector) was not modified, but the diagonal vector (blue) has changed direction

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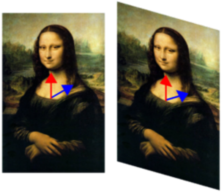

The shear transformation of the Mona Lisa

- the central vertical axis (red vector) was not modified, but the diagonal vector (blue) has changed direction
- the red vector is an eigenvector of the transformation with eigenvalue 1
- all vectors with the same vertical direction - i.e. parallel to this vector - are also eigenvectors, with the same eigenvalue

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Eigenvalue Equation

So far, we could try to solve $\mathbf{Ax} = \lambda\mathbf{x}$ by trial and error, but ... Note that

$$\mathbf{Ax} = \lambda\mathbf{x} \Rightarrow \ (\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = 0$$

- $\mathbf{A} - \lambda\mathbf{I}$ is not invertible
- the determinant of $\mathbf{A} - \lambda\mathbf{I}$ must be zero, i.e.,

  $$\det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

  involves only λ, not $\mathbf{x}$
- then, find the roots of this polynomial

Properties of Eigenvalues

- the product of the n eigenvalues equals the determinant **A**
- the sum of the n eigenvalues equals the sum of the n diagonal entries - the trace of **A**:

$$\lambda_1 + ... + \lambda_n = \text{trace} = a_{11} + ... + a_{nn}$$

Diagonalization

- suppose the n by n matrix **A** has n linearly independent eigenvectors and put them into the columns of an eigenvector matrix $\mathcal{S}$. We have

$$\mathcal{S}^{-1}\mathbf{A}\mathcal{S} = \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}$$

  the matrix **A** is “diagonalized"
- without n independent eigenvectors, we cannot diagonalize
- the matrices **A** and Λ have the same eigenvalues $\lambda_1, \ldots, \lambda_n$, but the eigenvectors are different

Markov Matrix: a Revisit

$$\mathbf{A} = \begin{bmatrix} .8 & .3 \\ .2 & .7 \end{bmatrix}$$

- its entries are positive and every column adds to 1
- the largest eigenvalue is $\lambda_1 = 1$ and the corresponding eigenvector is the steady state - which all columns of $\mathbf{A}^k$ will approach
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$$\mathbf{A}^2 = \mathcal{S}\Lambda\mathcal{S}^{-1}\mathcal{S}\Lambda\mathcal{S}^{-1} = \mathcal{S}\Lambda^2\mathcal{S}^{-1}$$
$$\mathbf{A}^k = \mathcal{S}\Lambda^k\mathcal{S}^{-1}$$