Week 5

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CS21BTECH11012

1. Weighted version of path inequality:

$$G_{1,n} \preceq \left(\sum_{a=1}^{n-1} \frac{1}{w_a}\right) \sum_{a=1}^{n-1} w_a G_{a,a+1}$$

$$G_{a,b} \preceq \left(\sum_{a=a_i}^{a_l} \frac{1}{w_a}\right) \sum_{a=a_i}^{a_l} w_a G_{a,a+1}$$

Here a_i is the starting vertex and a_l is the vertex just before b.

$\lambda_2(L_{T_n})$ for complete binary tree.

$$K_n = \sum_{a<b} G_{a,b}$$

$$\preceq \sum_{a,b} \left(\sum_{a=a_i}^{a_l} \frac{1}{w_a}\right) \sum_{a=a_i}^{a_l} w_a G_{a,a+1}$$

$T_d^{a,b}$ denotes the unique path in T from a to b.

We know, $G_{a,b} \preceq (b-a) P_{a,b}$

(d denotes depth) of tree

We can write $G_{a,a+1} \preceq T_d^{a,b}$.

$G_{a,b}$ is graph with only one edge $= (a,b)$

$$K_n \preceq \sum_{a,b} \left(\sum_{a=a_i}^{a_l} \frac{1}{w_a}\right) \sum_{a=a_i}^{a_l} w_a T_d^{a,b}$$

$$\preceq \sum_{a,b} \left(\sum_{a=a_i}^{a_l} \frac{1}{w_a}\right) \sum_{a=a_i}^{a_l} w_a T_d$$

wlog let's assume $b>a$, therefore there are ${}^nC_2$ ~~values of~~ choices of (a,b).

Let c' be the max of $\left(\left(\sum_{a=a_i}^{a_l} \frac{1}{w_a}\right) \sum_{a=a_i}^{a_l} w_a\right)$ over all those choices.

$$K_n \preceq \sum_{a,b} c' T_d$$

$$K_n \leq c' \frac{n(n-1)}{2} T_d$$

$$\lambda_2(L_{K_n}) \leq c' \frac{n(n-1)}{2} \lambda_2(L_{T_d})$$

$$\| $$

$$n \leq c' \frac{n(n-1)}{2} \lambda_2(L_{T_n})$$

$$\lambda_2(L_{T_n}) \geq \frac{2}{c'(n-1)} \geq \frac{2}{c'n}$$

Let $c = \frac{c'}{2}$

$\lambda_2(L_{T_n}) \geq \frac{1}{cn}$ for some absolute constant c

Hence proved.

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2. Prove $\frac{\lambda_2(\mathcal{L})}{2} \leq \Phi_G$

$$\mathcal{L} = D^{-1/2} L D^{-1/2} = I - D^{-1/2} A D^{-1/2}$$

$$\Phi_G = \min \frac{|\partial S|}{\mathrm{vol}(S)} \quad \mathrm{vol}(S) \leq |E|.$$

$$\mathcal{L} = D^{-1/2} L D^{-1/2}$$

If we take $v = D^{1/2}\mathbb{1} = \begin{pmatrix} \sqrt{d_1} \\ \vdots \\ \sqrt{d_n} \end{pmatrix}$

$$\mathcal{L}v = (D^{-1/2} L D^{-1/2})(D^{1/2}\mathbb{1})$$
$$= 0.$$

$\therefore$ v is the eigenvector corresponding to λ_1.

$$\lambda_2 = \min_{\substack{x \in \mathbb{R}^n \\ x \perp v}} \frac{x^T \mathcal{L} x}{x^T x}$$

$$x \perp v \Rightarrow \sum_{i=1}^{n} \sqrt{d_i}\, x(i) = 0$$

Let S be chosen set of vertices in V

$$vol(S) = \sum_{i \in S} deg(i)$$

Vertices belonging to S

Let us take $x =$ $\cancel{\begin{pmatrix} \frac{\sqrt{d_1}}{Vol(S)} \\ \frac{\sqrt{d_2}}{Vol(S)} \\ \vdots \end{pmatrix}}$

$$\begin{array}{cc} \begin{pmatrix} \frac{\sqrt{d_1}}{Vol(S)} \\ \frac{\sqrt{d_2}}{Vol(S)} \\ \vdots \\ -\frac{\sqrt{d_{n-1}}}{Vol(\bar{S})} \\ -\frac{\sqrt{d_n}}{Vol(\bar{S})} \end{pmatrix} & \begin{array}{c} \uparrow \\ S \\ \downarrow \\ \uparrow \\ \bar{S} \\ \downarrow \end{array} \end{array}$$

$$Vol(\bar{S}) = \sum_{i \notin S} deg(i)$$

Vertices belonging to $\bar{S}$.

$v^T x$ value

$$= (\sqrt{d_1}\ \sqrt{d_2} \cdots \sqrt{d_n})\, x.$$

$$= \left(\frac{d_1}{Vol(S)} + \frac{d_2}{Vol(S)} + \cdots\right) + \left(-\frac{d_{j+1}}{Vol(\bar{S})} - \frac{d_{j+2}}{Vol(\bar{S})} \cdots - \frac{d_n}{Vol(\bar{S})}\right)$$

$$= 1 + (-1)$$

$$= \underline{\underline{0}}$$

Valid eigen vector for λ_2.

$$\lambda_2 = \min_{\substack{x \in \mathbb{R}^n \\ x \perp v}} \frac{x^T L x}{x^T x} = \frac{x^T (I - D^{-1/2} A D^{-1/2}) x}{x^T x}$$

$$= \frac{x^T x - x^T D^{-1/2} A D^{-1/2} x}{x^T x}$$

$$x^T x = \left(\frac{\sqrt{d_1}}{Vol(S)} \cdots\cdots -\frac{\sqrt{d_n}}{Vol(\bar{S})}\right) \begin{pmatrix} \frac{\sqrt{d_1}}{Vol(S)} \\ \vdots \\ -\frac{\sqrt{d_n}}{Vol(\bar{S})} \end{pmatrix}$$

$$= \frac{d_1 + d_2 + \cdots + d_j}{(Vol(S))^2} + \left(\frac{d_{j+1} + \cdots d_n}{(Vol(\bar{S}))^2}\right)$$

$$= \frac{1}{Vol(S)} + \frac{1}{Vol(\bar{S})} = \frac{Vol(V)}{Vol(S)\, Vol(\bar{S})}$$

$$x^T x - x^T D^{-1/2} A D^{-1/2} x$$

$$= \sum_{i=1}^{n} x_i^2 - \sum_{i,j} \frac{x_i}{\sqrt{d_i}} a_{ij} \frac{x_j}{\sqrt{d_j}}$$

$$= \sum_{i=1}^{n} x_i^2 - \sum_{(i,j)\in E} \frac{x_i}{\sqrt{d_i}} \frac{x_j}{\sqrt{d_j}}$$

$$= \sum_{i=1}^{n} d_i \left(\frac{x_i}{\sqrt{d_i}}\right)^2 - \sum_{(i,j)\in E} \frac{x_i}{\sqrt{d_i}} \frac{x_j}{\sqrt{d_j}}$$

$$= \cancel{\sum_{i=1}} \sum_{(i,j)\in E} \left(\left(\frac{x_i}{\sqrt{d_i}}\right)^2 + \left(\frac{x_j}{\sqrt{d_j}}\right)^2 - 2\frac{x_i x_j}{\sqrt{d_i}\sqrt{d_j}} \right)$$

$$= \sum_{(i,j)\in E} \left(\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}}\right)^2$$

We can observe that edges which have vertices in S only or both vertices in $\bar{S}$ contribute zero since $\frac{x_i}{\sqrt{d_i}} - \frac{x_j}{\sqrt{d_j}}$ becomes zero for them for the taken vector x. $\left(\frac{\frac{\sqrt{d_i}}{vol(S)}}{\sqrt{d_i}} - \frac{\frac{\sqrt{d_j}}{vol(S)}}{\sqrt{d_j}} = 0 \right)$

Only edges which have one vertex in S and other in $\bar{S}$ remain.

$$x^T L x = \sum_{(i,j)\in \partial S} \left(\frac{\frac{\sqrt{d_i}}{vol(S)}}{\sqrt{d_i}} - \frac{\left(\frac{-\sqrt{d_j}}{vol(\bar{S})}\right)}{\sqrt{d_j}} \right)^2$$

$$= \sum_{(i,j)\in \partial S} \left(\frac{1}{vol(S)} + \frac{1}{vol(\bar{S})} \right)^2$$

$$= \sum_{(i,j)\in \partial S} \left(\frac{vol(V)}{vol(S)\, vol(\bar{S})} \right)^2$$

$$= |\partial S| \times \frac{(vol(V))^2}{(vol(S))^2 (vol(\bar{S}))^2}$$

$$\frac{x^T L x}{x^T x} = \frac{\dfrac{|\partial S| \times (Vol(V))^2}{(Vol(S))^2 (Vol(\bar{S}))^2}}{\dfrac{Vol(V)}{vol(S) vol(\bar{S})}} = \frac{|\partial S| \, Vol(V)}{Vol(S) \, Vol(\bar{S})}$$

We already have condition that

$$vol(S) \le |E|$$

$$= \frac{Vol(V)}{2}.$$

$$Vol(V) - Vol(\bar{S}) \le \frac{Vol(V)}{2}$$

$$\Rightarrow \frac{Vol(V)}{Vol(\bar{S})} \le 2.$$

$$\frac{x^T L x}{x^T x} \le \frac{2|\partial S|}{vol(S)} = 2\Phi(G).$$

$$vol(S) \le |E|$$

$$\Rightarrow \frac{\lambda_2(L)}{2} \le \Phi(G).$$

Hence proved.