Theoretical Computer Science

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1) Given $U \in \mathbb{R}^{n \times n}$ If $U^TU = I$, then $UU^T = I$

Let columns of U be $u_1, u_2, \cdots, u_n$.

These are linearly independent.

~~Suppos~~ To prove this, let us see the contradiction.

Suppose they aren't linearly independent,

$c_1u_1 + c_2u_2 + \cdots + c_nu_n = 0$ does not have every $c_i = 0$.

$c_1U^Tu_1 + c_2U^Tu_2 + \cdots + c_nU^Tu_n = U^T0 = 0$

U^Tu_i is ~~firs~~ i^{th} column of $I_{n \times n}$ because $U^TU = I$ (given)

Hence

$c_1e_1 + c_2e_2 + \cdots + c_ne_n = 0$ where e_i is i^{th} column of $I_{n\times n}$.

It is true only when all c_i's are zero.

Hence, columns of U are linearly independent.

Any $x \in$ ~~$\mathbb{R}^{n\times n}$~~ $\mathbb{R}^{n \times 1}$ can be written as

$x = y_1u_1 + y_2u_2 + \cdots + y_nu_n$ where y_i's are constants

$x = Uy$ where $y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$

$U^Tx = U^TUy = Iy = y$

$UU^Tx = Uy = x$

We got $(UU^T)x = x$ for any vector $x \in \mathbb{R}^{n\times 1}$.

$\therefore UU^T = I$

2) $L_G \succeq c L_H$

$L_G - c L_H \succeq 0$ ——— (1)

L_G and L_H are symmetric matrices.

Method 1:-

We use Rayleigh coefficient wrt symm matrix

$\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$ for both matrices. (order)

Let x_1 be the eigen vector corresponding to the eigenvalue $\lambda_1(G)$ for L_G.

Let x_1' be the eigen vector corresponding to the eigenvalue $\lambda_1(H)$ for L_H.

$$\lambda_1(H) = \max_{\substack{x \in \mathbb{R}^n \\ x \neq 0}} \frac{x^T L(H) x}{x^T x} = \frac{x_1'^T L(H) x_1'}{x_1'^T x_1'}$$

From eq(1) we get

$$\cancel{\frac{x_1'^T L(H) x_1'}{x_1'^T x_1'}} \quad \underline{\underline{\frac{x_1'^T L(G) x_1'}{x_1'^T x_1}}} \geq c \frac{x_1'^T L(H) x_1'}{x_1'^T x_1'}$$

This term $\leq \dfrac{x_1^T L(G) x_1^T}{x_1^T x_1}$ since x_1 is eigen vector for $L(G)$.

$\therefore$ We get $\cancel{\lambda_1 \geq c}$ $\lambda_1 L(G) \geq \lambda_1 L(H)$.

Now, let x_2 be the eigenvector corresponding to $\lambda_2(G)$ for L_G

and x_2' be the eigenvector corresponding to $\lambda_2(H)$ for L_H.

We generalise and show it to be true for any $k \geq 2$ now. (Like induction) on λk)

Now, there are two cases.

① $\mathrm{span}(x_1, x_2, \ldots x_{k-1}) = \mathrm{span}(x_1', x_2', \ldots x_{k-1}')$

② $\exists y$ which belongs to $\mathrm{span}(x_1', x_2', x_3', \ldots x_{k-1}')$ but not belongs to $\mathrm{span}(x_1, x_2, x_3, \ldots x_{k-1})$

Note that ~~one sub~~ vector ~~sub~~space span $(x_1', x_2', x_3' \ldots x_{k-1}')$ cannot be contained inside span $(x_1, x_2, \ldots x_{k-1})$ and vice versa without being equal because both the dimensions are same.

For case ①,

since both spans are same vector space, when we look for eigenvectors x_k (for $\lambda_k L(G)$) and x_k' (for $\lambda_k L(H)$), we are looking for "maximum vectors" perpendicular to the span.

Using the same tecnique which we used before we get

$$\frac{c\, x_k'^T L(H) x_k'}{x_k'^T x_k'} \le \frac{x_k'^T L(G) x_k'}{x_k'^T x_k'} \le \frac{x_k^T L(G) x_k}{x_k^T x_k}$$

$$\therefore\ c\lambda_k L(H) \le \lambda_k L(G)$$

For case ②

Since $\mathrm{span}(x_1, x_2, \ldots x_{k-1}) \neq \mathrm{span}(x_1', x_2', \ldots, x_{k-1}')$ there exists an eigen vector x_i' present in $\{x_1', x_2', \ldots, x_{k-1}'\}$ which is perpendicular to $\mathrm{Span}(x_1, x_2, \ldots, x_{k-1})$.

x_k — eigen vector for $\lambda_k L(G)$

x_k' — eigen vector for $\lambda_k L(H)$.

~~$c\lambda_k = c x_k^T L$~~

$$c\lambda_k L(H) = \frac{c\,x_k'^T L(H) x_k'}{x_k'^T x_k} \leq \frac{c\,x_i'^T L(H) x_i'}{x_i'^T x_i} = c\lambda_i$$

since $\lambda_k L(H) \leq \lambda_i L(H)$ by notation

$$\frac{c\,x_i'^T L(H) x_i'}{x_i'^T x_i'} \leq \frac{x_i'^T L(G) x_i'}{x_i'^T x_i'} \leq \max_{x \perp S} \frac{x^T L(G) x}{x^T x}$$

$S = \text{span}(x_1, x_2, \ldots, x_{k-1})$

Since $x_i' \perp \text{span}(x_1, x_2, \ldots, x_{k-1})$ as explained before

$$= \frac{x_k^T L(G) x_k}{x_k^T x_k} = \lambda_k L(G)$$

$$\therefore \; c\lambda_k L(H) \leq \lambda_k L(G) \;\; \forall\, k.$$

Hence proved.

<u>Note</u>:- c is considered positive here, so we are neglecting any reversal of inequalities. Even if $c \leq 0$, $\lambda_i(G) \geq c\lambda_i(H)$, holds true because eigen values of any laplacian matrix are non negative.

<u>Method 2</u> :-

Using Courant - Fischer Theorem.

$$\lambda_k(G) = \min_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S) = n-k+1}} \max_{x \in S} \frac{x^T L(G) x}{x^T x}$$

$$\geq \min_{\substack{S \subseteq \mathbb{R}^n \\ \dim(S) = n-k+1}} \max_{x \in S} c\,\frac{x^T L(H) x}{x^T x} = c\lambda_k L(H).$$

$$\therefore \; \lambda_k L(G) \geq c \cdot \lambda_k L(H).$$

~~Eigen~~ Interlacing theorem states that

3. Thm: Let A be a real symmetric $n\times n$ matrix and let B be a principal submatrix of A with order $m\times m$. Then, for $i=1,\dots,m$,

$$\lambda_{n-m+i}(A) \le \lambda_i(B) \le \lambda_i(A).$$

If $m=n-1$, we get

$$\lambda_1(A) \ge \lambda_1(B) \ge \lambda_2(A) \ge \lambda_2(B) \dots \ge \lambda_{n-1}(B) \ge \lambda_n(A)$$

Statement:- $(k-1)\lambda_{min}(A)+\lambda_{max}(A) \le \sum_i \lambda_{max}(A_{ii})$

We ~~using~~ use induction on dimension k.

Let $A = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$ $\quad M_{12}^T = M_{21}$ $(k=2)$

Base case:-

Let $X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$ be eigen vector of A corresponding to $\lambda_{max}(A)$. Let $\|X\|^2=1 \Rightarrow \|X_1\|^2+\|X_2\|^2=1$

Let $Y = \begin{pmatrix} aX_1 \\ bX_2 \end{pmatrix}$ be another vector (any)

$$\lambda_{min}(A) \le \frac{Y^TAY}{Y^TY} \quad \text{Let } \|Y\|^2=1 \Rightarrow a^2+b^2=1$$

$$\begin{aligned}\lambda_{max}(A)+\lambda_{min}(A) &\le X_1^TM_{11}X_1 + X_1^TM_{12}X_2 + X_2^TM_{21}X_1 \\ &\quad + X_2^TM_{22}X_2 \\ &\quad + a^2X_1^TM_{11}X_1 + abX_1^TM_{12}X_2 + \\ &\quad ab\,X_2^TM_{21}X_1 + b^2X_2^TM_{22}X_2 \\ &= (1+a^2)X_1^TM_{11}X_1 + (1+ab)X_1^TM_{12}X_2 \\ &\quad + (1+ab)X_2^TM_{21}X_1 + (1+b^2)X_2^TM_{22}X_2\end{aligned}$$

Let us take values of ~~ab~~ a, b such that $1+ab=0$ and $1+a^2 = \frac{1}{\|X_1\|^2}$, $1+b^2 = \frac{1}{\|X_2\|^2}$.

We get $a = \frac{\|X_2\|}{\|X_1\|}$, $b = \frac{-\|X_1\|}{\|X_2\|}$ as one possible pair, so such a vector exists and we take those values.

$$\therefore \lambda_{max}(A) + \lambda_{min}(A) \le \frac{x_1^T M_{11} x_1}{\|x_1\|^2} + \frac{x_2^T M_{22} x_2}{\|x_2\|^2}$$

$$\le \lambda_{max}(M_{11}) + \lambda_{max}(M_{22})$$

True for base case $k=2$.

Hypothesis:- Let us assume by inductive hypothesis that statement is true for $\{3,4,\dots,k-1\}$

To show for k:

$$A = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1k} \\ M_{21} & M_{22} & \cdots & M_{2k} \\ \vdots & & & \\ M_{k1} & M_{k2} & \cdots & M_{kk} \end{pmatrix}$$

Let $M'_{11} = \begin{pmatrix} M_{11} & M_{12} & \cdots & M_{1k-1} \\ M_{21} & M_{22} & \cdots & M_{2k-1} \\ \vdots & & & \\ M_{k-11} & M_{k-12} & \cdots & M_{k-1k-1} \end{pmatrix}$

Let $M'_{12} = \begin{pmatrix} M_{1k} \\ M_{2k} \\ \vdots \\ M_{k-1k} \end{pmatrix}$, $M'_{21} = \begin{pmatrix} M_{k1} & M_{k2} & \cdots & M_{kk-1} \end{pmatrix}$

Observe that $(M'_{12})^T = M'_{21}$

$M'_{22} = M_{kk}$.

$$A = \left(\begin{array}{c|c} M'_{11} & M'_{12} \\ \hline M'_{21} & M'_{22} \end{array}\right).$$

Same like $k=2$, we get

$$\lambda_{max}(A) + \lambda_{min}(A) \le \lambda_{max}(M'_{11}) + \lambda_{max}(M'_{22})$$

Observe M'_{11},

from induction hypothesis, we know that ~~it is true for~~ statement is true for $k-1$

So,

$$\cancel{(k-2)\lambda_{min}(M'_{11}) + \lambda_{max}(M'_{11}) \le \sum_i \lambda_{max}(M_{ii})}$$

So,

$$(k-2)\lambda_{min}(M_{11}') + \lambda_{max}(M_{11}') \le \lambda_{max}(M_{11}) + \lambda_{max}(M_{22}) + \dots + \lambda_{max}(M_{k-1k-1})$$
$$= \sum_{i=1}^{k-1} \lambda_{max}(M_{ii})$$

$$\Rightarrow \lambda_{max}(M_{11}') \le \sum_{i=1}^{k-1} \lambda_{max}(M_{ii}) - (k-2)\lambda_{min}(M_{11}')$$

Substituting back,

$$\lambda_{max}(A) + \lambda_{min}(A) \le \sum_{i=1}^{k-1} \lambda_{max}(M_{ii}) - (k-2)\lambda_{min}(M_{11}') + \lambda_{max}(M_{22}')$$
$$= \sum_{i=1}^{k} \lambda_{max}(M_{ii}) - (k-2)\lambda_{min}(M_{11}')$$

since $M_{22}' = A_{kk}$

M_{11}' is ~~obtained~~ a principal submatrix of A since it is obtained by removing the same indexed rows and columns in A.

From interlacing theorem, we know that

$$\lambda_{min}(A) \le \lambda_{min}(M_{11}').$$

So,

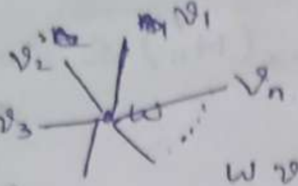
$$\lambda_{max}(A) + \lambda_{min}(A) \le \sum_{i=1}^{k} \lambda_{max}(M_{ii}) - (k-2)\lambda_{\cancel{max}\,min}(A).$$

$$\Rightarrow (k-1)\lambda_{min}(A) + \lambda_{max}(A) \le \sum_{i=1}^{k} \lambda_{max}(M_{ii}).$$

∴ Showed that true ~~by~~ for k.

∴ By induction, we proved that the statement is true.

4. Star graph



Adjacency matrix:

$$A = \begin{array}{c|c} & \begin{matrix} w & v_1 & v_2 & \cdots & v_n \end{matrix} \\ \begin{matrix} w \\ v_1 \\ v_2 \\ \vdots \\ v_n \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{array}$$

Let λ be an eigen value

$Ax = \lambda x$ where $x = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n+1} \end{pmatrix}$ is an eigenvector of λ.

~~Solving~~, we get equations

$$x_2 + x_3 + \dots + x_{n+1} = \lambda x_1$$
$$x_1 = \lambda x_2$$
$$x_1 = \lambda x_3$$
$$\vdots$$
$$x_1 = \lambda x_{n+1}.$$

Two cases:

① $\lambda = 0$.

$\Rightarrow x_1 = 0.$

$\Rightarrow \sum_{i=2}^{n+1} x_i = 0.$

② $x_2 = x_3 = \dots = x_{n+1}$

$\Rightarrow n x_2 = \lambda x_1$

$= \lambda^2 x_2$

$\lambda^2 = n$

$\lambda = \pm\sqrt{n}$

rank (A) + Nullity $(A) = \dim(A)$

$= n+1$

$2 + N(A) = n+1$

$N(A) = n-1.$

$\therefore$ Eigenspace of $\lambda = 0$ has dim $n-1$.

eigen Values $= \sqrt{n}, -\sqrt{n}, \underbrace{0, \dots, 0}_{n-1 \text{ times}}$

Eigen vector of $\sqrt{n}$ can be $k\begin{pmatrix}\sqrt{n}\\1\\\vdots\\1\end{pmatrix}$ k belongs to field

Eigen vector of $-\sqrt{n}$ can be $k\begin{pmatrix}-\sqrt{n}\\1\\\vdots\\1\end{pmatrix}$

Eigen vectors of 0 are of the form $\begin{pmatrix}0\\x_1\\\vdots\\x_{n-1}\\-\sum_{i=1}^{n-1}x_i\end{pmatrix}$

So, $n-1$ eigenvectors can be

$$\begin{pmatrix}0\\1\\0\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\\\vdots\\-1\end{pmatrix}, \cdots\cdots \begin{pmatrix}0\\0\\\vdots\\0\\1\\-1\end{pmatrix}$$

Laplacian matrix : $L(G) =$

$$\begin{array}{c|c} & w \quad v_1 \;\cdots\cdots\; v_n \\ \begin{matrix} w \\ v_1 \\ \vdots \\ v_n \end{matrix} & \begin{bmatrix} n & -1 & -1 & \cdots & -1 \\ -1 & 1 & & & \\ -1 & & \ddots & 0 & \\ \vdots & 0 & & \ddots & \\ -1 & & & & 1 \end{bmatrix} \end{array}$$

Let λ be an eigen value of $L(G)$

$Ax = \lambda x$ where $x = \begin{pmatrix}x_1\\x_2\\\vdots\\x_{n+1}\end{pmatrix}$ is an eigenvector of λ.

We get equations:

$$n x_1 - \sum_{i=2}^{n+1} x_i = \lambda x_1$$

$$\Rightarrow x_1 = x_2(1-\lambda)$$

$$x_1 = x_3(1-\lambda)$$

$$\vdots$$

$$x_1 = x_{n+1}(1-\lambda).$$

Solving, two cases.

① $\lambda = 1$

$\Rightarrow x_1 = 0$

$\sum_{i=2}^{n+1} x_i = 0$

② $x_2 = x_3 = \cdots = x_{n+1}$

$n x_1 - n x_2 = \lambda x_1$

$(n-\lambda) x_1 = n x_2$

$(n-\lambda)(1-\lambda) x_2 = n x_2$

$\lambda = 0$ or $\lambda = n+1$

For $\lambda=0$, eigen vector is of the form $k\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$

dim = 1

For $\lambda=(n+1)$, eigen vector is of the form $k\begin{pmatrix}-n\\1\\\vdots\\1\end{pmatrix}$

dim = 1

For $\lambda=1$, eigen vector is of the form

$$\begin{pmatrix}0\\x_1\\\vdots\\x_{n-1}\\-\sum_{i=1}^{n-1} x_i\end{pmatrix}$$ dim = $n-1$

We can take vectors like

$$\begin{pmatrix}0\\1\\\vdots\\0\\-1\end{pmatrix}, \begin{pmatrix}0\\0\\1\\\vdots\\0\\-1\end{pmatrix}, \ldots \begin{pmatrix}0\\0\\\vdots\\0\\1\\-1\end{pmatrix}.$$

eigen values are $0, n+1, \underbrace{1,1,1\ldots 1}_{n-1 \text{ times}}$.

5) We know that,

G is bipartite. $\Longleftrightarrow$ eigen values occur in $+\lambda, -\lambda$ pairs.

$(\Rightarrow)$

$\therefore$ If G is bipartite, then $\mu_1 = -\mu_n$.

~~Now, other side of eque~~

$(\Leftarrow)$.

If $\mu_1 = -\mu_n$, we have to show that G is bipartite.

Let x be the eigenvector corresponding to μ_n.

$\cancel{A\mu_n =}$ $Ax = \mu_n x$.

Let $\|x\|^2 = 1$

$$\mu_n = \frac{x^T A x}{\|x\|^2} = x^T A x$$

$$|\mu_n| = |x^T A x|$$

$$= \left|\sum_{i,j} A_{ij} x_j x_i\right|$$

$$\leq \sum_{i,j} A_{ij} |x_j||x_i|$$

let us take vector y to be $y = \begin{pmatrix} |x_1| \\ |x_2| \\ \vdots \\ |x_n| \end{pmatrix} \geq 0$

Since $x^T x = 1 \Rightarrow y^T y = 1$

$|\mu_n| = \mu_1$, since $\mu_1 = -\mu_n$ and $\mu_1 \geq \mu_n$.

$$\mu_1 \leq \sum_{i,j} A_{ij} y_j y_i$$

This should be an equality since

$$\mu_1 = \max_{x \in \mathbb{R}^n} \frac{x^T A x}{\|x\|^2} \quad (\|x\|^2 = 1)$$

$\therefore \mu_1 = \sum_{i,j} A_{ij} y_j y_i$ $\quad \therefore$ y is an eigenvector of μ_1

$$\Rightarrow \left|\sum_{i,j} A_{ij} x_j x_i\right| = \sum_{i,j} A_{ij} |x_j||x_i|$$

This holds true for two cases.

Case 1:- x_i, x_j have same sign whenever i and j have an edge.

Case 2:- x_i, x_j have opposite sign whenever i and j have an edge.

Case 1 is not possible because if it is true, since given that G is connected, all x_i's would have the same sign. This would give lead to vector y being a scalar multiple of vector x.

$$y = kx.$$

This is contradiction, since y is eigenvector of μ_1 and x is eigenvector of μ_n. This ~~would~~ could only hold true when $\mu_1 = \mu_n = 0$ but this leads to all eigenvalues being 0. (but G is connected).

Therefore Case 2 is the only correct case.

Therefore whenever there is an edge between i and j, x_i and x_j have opposite signs.

~~Now, we can take two partitions,~~
~~V_1 = Vertex set with index i such that $x_i > 0$.~~

So, initialise a ~~non~~-positive x_i with 1 And give all of its neighbours $(j \sim i)$ -1.
Now, repeat the process for all of the neighbours, giving their neighbours 1 if already not given.

This is possible, since equation holds true and case 2 is the only possible case.

Now, every index would have either 1 or -1 given to them since G is connected.

We take two vertex ~~sets~~ partitions.

$$V_1 = \{ i, \text{ such that } v_i \text{ is given } 1 \}$$

$$V_2 = \{ i, \text{ such that } v_i \text{ is given } -1 \}$$

In V_1, there are no edges in b/w them since that is how we assigned the signs.

Similarly for V_2.

~~But any~~ Any edge $\in G$ has one endpoint in V_1 and another end point in V_2.

$\therefore$ It is bipartite (G).

$\therefore$ G is connected.

G is bipartite iff $\mu_1 = -\mu_n$.

6. (a) ~~We prove~~

To show $\chi(G) \leq \lfloor \mu_1 \rfloor + 1$.

We use induction. on no. of vertices in graph.

Base case:- Graph with one vertex.

$\chi(G)$ would be 1 and $\mu_1 = 0$.

$1 \leq 0 + 1$.

Holds true.

~~Now let~~

For hypothesis, assume that the statement holds true for all graphs with $n-1$ vertices.

Now, take a graph with n vertices. (G)

We have seen in class that

$\mu_1 \geq d_{avg}$

avg degree of vertices.

Select a vertex v such that it has a degree atmost $\lfloor \mu_1 \rfloor$.

$G-\{v\}$ is the graph obtained by removing this vertex.

From our inductive hypothesis, we know that statement holds true on this graph ($G\setminus\{v\}$) since it has $n-1$ vertices.

$$\mu_{max}(A(G-\{v\})) \leq \mu_{max}(A(G))$$

This is from Cheeger's inequality.

So, $\chi(G\setminus\{v\}) \leq \lfloor \mu_1 \rfloor + 1$

$\therefore$ $G\setminus\{v\}$ has a colouring with atmost $\lfloor \mu_1 \rfloor + 1$ colours.

Now, when we add v into $G\setminus\{v\}$, we already know that v had at most $\lfloor \mu_1 \rfloor$ neighbours.

So, we can colour vertex v with some colour which belongs to $\{1, 2, \ldots \lfloor \mu_1 \rfloor + 1\}$ that is ~~neighbours are not coloured with~~ not assigned or coloured to any of the neighbours.

$\therefore$ G has a colouring with $\lfloor \mu_1 \rfloor + 1$ colours.

$\therefore$ By induction, we showed that

$$\chi(G) \leq \lfloor \mu_1 \rfloor + 1.$$

b) Show that for any graph G,

$$\chi(G) \geq 1 + \frac{\mu_1}{-\mu_n}$$

Let $\chi(G) = k$.

$V_1 =$ {set of vertices which ~~are co~~ have colour 1}

$V_2 =$ {set of vertices which have colour 2}

$\vdots$

$V_k =$ {set of vertices which have colour k}.

We re-label vertices in A(G) such that

$$A(G) = \begin{array}{cc} & \begin{array}{cccc} V_1 & V_2 & \cdots & V_k \end{array} \\ \begin{array}{c} V_1 \\ V_2 \\ \vdots \\ V_k \end{array} & \left[\begin{array}{cccc} 0 & M_{12} & \cdots & M_{1k} \\ M_{21} & 0 & \cdots & M_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ M_{k1} & M_{k2} & \cdots & 0 \end{array}\right] \end{array}$$

M_{ij} are all block matrices and

~~we can see~~ $\cancel{M_{12} =}$ $(M_{ij})^T = M_{ji}$

From Problem 3, we get

$$(k-1)\lambda_{min}(A) + \lambda_{max}(A) \leq \sum_i \lambda_{max}(M_{ii})$$
$$= 0.$$

$$(k-1)\mu_n + \mu_1 \leq 0.$$

$$k\mu_n \leq \mu_n - \mu_1$$

$$\chi(G) = k \geq 1 + \frac{\mu_1}{-\mu_n}$$

(μ_n is negative since trace is 0).

Hoffman's bound gives that if S is any independent set in a d-regular graph G,

then $$\frac{|S|}{n} \leq \frac{-\lambda_{min}(A(G))}{d - \lambda_{min}(A(G))}$$

Since $|S|$ is an independent set.

Let us take S such that it has maximum cardinality, then

$$\chi(G) \geq \frac{n}{|S|} \geq \frac{d - \lambda_{min}(A(G))}{-\lambda_{min}(A(G))}$$

$$\Rightarrow \chi(G) \geq 1 + \frac{d}{-\mu_n}$$

For a d-regular graph, $\lambda_1 = d$.

($\because \lambda_1 = d$. This can be seen by taking ~~A = d~~ $M = dI - A_G$ which is the laplacian matrix. It is positive semidefinite and has non-negative eigen values. So, eigen values of 'A' should be $\leq d$. We can easily see that d is an eigen value of $A(G)$. So $\mu_1 = d$.)

For complete graphs $\chi(G) = n$, $\mu_1 = n-1$, $\mu_n = -1$.

$$n = 1 + \frac{(n-1)}{-(-1)}$$

Equality holds true for complete graphs.