Week-9

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① For BPP, which has two-sided error algorithm, the algo has a probability of atmost $\left(\frac{1}{3}\right)$ of giving the wrong answer whether the answer is YES or NO.

Whereas in for one-sided error, like in RP, if correct answer is NO, it always returns NO. If the correct answer is YES, it returns YES with probability atleast $1/2$ (otherwise returns NO)

For BPP,
the KPS algo returns an error on YES instance, if vertex chosen at random v_0 has $\Gamma_c(v_0) \subseteq B_x$, where B_x is the bad input for the instance x.
The KPS algo returns an error on NO instance if vertex chosen at random v_0' has atleast one vertex v, which is reachable from v_0' in $\leq c$ steps that outputs YES.

$$\cancel{\text{RealBad}_x = \{ \underbrace{v \mid \Gamma_c(v) \subseteq B_x}_{\text{for YES instance}} \text{ or } \underbrace{v \mid \exists v' \in \Gamma_c(v) \text{ s.t } v' \notin B_x}_{\text{for NO instance}} \}}$$

~~Notice that both these quantities are complements of each other. So we get the whole vertex set of the graph.~~

~~Hence we cannot apply KPS technique to reduce the error probability for an algorithm with two-sided error.~~

$$\frac{|B_x|}{2^m} \leq \frac{1}{3}$$

2) (a)

$Pr[A_{KPS}$ makes an error on YES instance$] = \frac{|B|}{2^m}$

$B = \{v \mid \Gamma_c'(v) \subseteq Bad_x$ where $Bad_x = M(x,y) = \underset{NO}{0}\}$

$|Bad_x| \geq |\Gamma_c'(B)| \geq A^c |B|$

$\frac{|B|}{2^m} \leq \frac{|Bad_x|}{2^m \cdot A^c} \leq \frac{1}{3 \cdot A^c}$ (since $\frac{|Bad_x|}{2^m} \leq \frac{1}{3}$)

$Pr[A_{KPS}$ does not make an error on NO instance$] = \frac{|G|}{2^m}$

$G = \{v \mid \Gamma_c'(v) \subseteq Good_x$ where $Good_x = M(x,y) = \underset{NO}{0}\}$

$|Good_x| \geq \cancel{|\Gamma_c'(B)|}\ |\Gamma_c'(G)| \geq A^c |G|$

$\frac{|G|}{2^m} \leq \frac{|Good_x|}{2^m \cdot A^c}$ $\qquad \frac{|Good_x|}{2^m} \geq \frac{2}{3}$

$Pr[A_{KPS}$ makes an error on NO instance$] = 1 - \frac{|G|}{2^m}$

$\geq 1 - \frac{|Good_x|}{2^m \cdot A^c}$

Here, for the 'NO' instance, we cannot get any upperbound for the error, so that we could have taken (max(errorbound (upper) on 'YES', errorbound (upper) on 'NO')).

As such, we cannot ascertain anything here.

$\therefore$ KPS technique cannot be applied to reduce the error probability of two-sided error.

2) (a) Probability that Karger's algorithm returns a min-cut is $\geq \frac{1}{\binom{n}{2}}$

Let us assume that there are 'c' no. of cuts with minimum size and let the indices of their cuts be 1, 2, ..., c

Let p_i be the probability that Karger's Algo returns cut C_i.

Let all the cuts possible be $C_1, C_2, .. C_c, C_{c+1} ...$

$$\sum_i p_i = 1$$

$$\sum_{i=1}^{c} p_i = p_1 + p_2 + \dots + p_c$$

Karger's algo returns mincut with $p_i \geq \frac{1}{\binom{n}{2}}$ for $i \in [c]$.

$$\sum_{i=1}^{c} p_i \geq \left(\frac{1}{\binom{n}{2}}\right) c$$

$$1 \geq \sum_{i=1}^{c} p_i \geq \frac{c}{\binom{n}{2}} \Rightarrow c \leq \binom{n}{2}$$

$\therefore$ Atmost $\binom{n}{2}$ min cuts are present in a graph.

(b) Let T be the total no. of ~~iterations~~ times that the Karger's algorithm is ~~performed~~ run independently.

The probability that we ~~miss~~ fail to get a particular min-cut $\leq \left(1 - \frac{1}{\binom{n}{2}}\right)^T$

~~We can~~

As the Karger's contraction Algo, goes on, we can keep track of all the smallest cuts that we

See along.

Since there are at most $\binom{n}{2}$ min-cuts in a graph.

Probabitylity of missing any min-cut at all =

By union-bound,

$$\Pr[\text{miss any min-cut}] \le \binom{n}{2}\left(1 - \frac{1}{\binom{n}{2}}\right)^T$$

Take $T = \binom{n}{2} \ln \frac{\binom{n}{2}}{\delta}$ where $\delta \in (0,1)$

~~Pr[miss any min-cu~~

$$\left(1 - \frac{1}{\binom{n}{2}}\right)^T = \left(1 - \frac{1}{\binom{n}{2}}\right)^{\binom{n}{2}\ln\frac{\binom{n}{2}}{\delta}}$$

$$\le e^{-\frac{1}{\binom{n}{2}}\binom{n}{2}\ln\frac{\binom{n}{2}}{\delta}} \quad (\text{becoz } e^{-x} \ge 1-x \ \forall x \in \mathbb{R})$$

$$= e^{\ln\frac{\delta}{\binom{n}{2}}}$$

$$= \frac{\delta}{\binom{n}{2}}$$

$$\binom{n}{2}\left(1 - \frac{1}{\binom{n}{2}}\right)^T \le \binom{n}{2}\frac{\delta}{\binom{n}{2}} = \underline{\underline{\delta}}.$$

We have,

$\Pr[\text{miss any min-cut at all}] \le \delta$

$\therefore$ We can output all the minimum graphs with probability $\ge 1-\delta$ if we take $T = \binom{n}{2} \ln \frac{\binom{n}{2}}{\delta}$

If we want the answer to be correct with probability 0.999, $\delta = 0.001$

No. of iterations/repititions $= \binom{n}{2} \ln \frac{\binom{n}{2}}{\delta}$, $\delta = 0.001$.

Each run of the algo takes $O(n^2)$ time (edges $= O(n^2)$)

$\therefore$ Running time $= O\left(n^2 \times \binom{n}{2} \ln \frac{\binom{n}{2}}{\delta}\right)$ $\delta = 0.001$

We can write it as $O(n^4 \ln n)$, if we neglect the constants.