Week 6.

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CS21BTECH11012

① "Hard direction" of Cheeger's inequality.

λ_2 is the second ~~eige~~ smallest eigenvector of $\mathcal{L}$.

Let y be the eigenvector of $\mathcal{L}$ for λ_2

$$\lambda_2 = \frac{y^T \mathcal{L} y}{y^T y}$$

$$y \perp x = \begin{pmatrix} \sqrt{d_1} \\ \sqrt{d_2} \\ \vdots \\ \sqrt{d_n} \end{pmatrix} = D^{1/2} \mathbb{1}$$

Let $z = D^{-1/2} y - k\mathbb{1}$.

$$
\begin{aligned}
z^T L z &= (D^{-1/2} y - k\mathbb{1})^T L (D^{-1/2} y - k\mathbb{1}) \\
&= (y^T D^{-1/2} - k\mathbb{1}^T) L (D^{-1/2} y - k\mathbb{1}) \\
&= (y^T D^{-1/2} L - k\underbrace{\mathbb{1}^T L}_{=0})(D^{-1/2} y - k\mathbb{1}) \\
&= y^T D^{-1/2} L D^{-1/2} y - k\underbrace{y^T D^{-1/2} L \mathbb{1}}_{=0} \\
&= y^T D^{-1/2} L D^{-1/2} y \\
&= y^T \mathcal{L} y
\end{aligned}
$$

$$
\begin{aligned}
z^T D z &= (D^{-1/2} y - k\mathbb{1})^T D (D^{-1/2} y - k\mathbb{1}) \\
&= (y^T D^{-1/2} - k\mathbb{1}^T) D (D^{-1/2} y - k\mathbb{1}) \\
&= (y^T D^{1/2} - k\mathbb{1}^T D)(D^{-1/2} y - k\mathbb{1}) \\
&= y^T y - k\underbrace{y^T D^{1/2} \mathbb{1}}_{=0} - k\overbrace{\mathbb{1}^T D^{1/2} y}^{=0} + k^2 \mathbb{1}^T D \mathbb{1} \\
&= y^T y + k^2 \sum_{i=1}^{n} \deg(i) \\
&\geq y^T y.
\end{aligned}
$$

$$\frac{z^T L z}{z^T D z} \le \lambda_2$$

without loss of generality, we can order the coordinates of z so that $z_1 \le \cdots \le z_n$

we choose the value of 'k' so that $z_j = 0$ where j is the least no for which

$$\sum_{i=1}^{j} d(i) \ge d(V)/2 \qquad d(i) \rightarrow \text{degree of } i^{th} \text{ vertex}$$

$$d(V) \rightarrow vol(V)$$

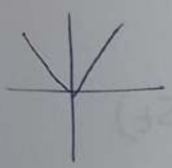
In other words

$$\sum_{i<j} d(i) < \frac{d(V)}{2} \qquad \sum_{i \le j} d(j) \ge \frac{d(V)}{2}$$

we even scale z so that

$$z_1^2 + z_n^2 = 1.$$

Let $t \in [z_1, z_n]$ be chosen with probability density $2|t|$. (distribution).



$$\int_{z_1}^{z_n} 2|t| = \int_{z_1}^{z_j=0} 2|t|\,dt + \int_{z_j=0}^{z_n} 2|t|\,dt$$

$$= \int_{z_1}^{0} -2t\,dt + \int_{0}^{z_n} 2t\,dt$$

$$= -(0^2 - z_1^2) + (z_n^2 - 0^2)$$

$$= z_1^2 + z_n^2$$

$$= \underline{\underline{1}} \longrightarrow \text{scaling of } z \text{ assumed before}$$

Now, for any $t \in [a, b]$
$a < b$.

$$Pr(t \in [a,b]) = \int_a^b 2|t|\,dt.$$

If $a<0<b$, $Pr(t\in[a,b]) = \int_a^0 -2t\,dt + \int_0^b 2t\,dt$

$= a^2+b^2$

If $0<a<b$ $Pr(t\in[a,b]) = \int_a^b 2t\,dt$

$= b^2-a^2$

If $a<b<0$ $Pr(t\in[a,b]) = \int_a^b -2t\,dt$

$= a^2-b^2$

We can see that $a^2+b^2 \le |b-a|(|a|+|b|)$

$b^2-a^2 \le |b-a|(|a|+|b|)$

$a^2-b^2 \le |b-a|(|a|+|b|)$

$\therefore Pr(t\in[a,b]) \le |b-a|(|a|+|b|)$

For any t, let $S_t = \{i : z_i \le t\}$

If $t<0$ $\min(d(S_t), d(V\setminus S_t)) = d(S_t)$

If $t\ge 0$ $\min(d(S_t), d(V\setminus S_t)) = d(V\setminus S_t)$

This is because we have chosen 'k' such that $z_j = 0$ for j s.t $\sum_{i<j} d(i) < \frac{d(V)}{2}$ and $\sum_{i\le j} d(i) \ge \frac{d(V)}{2}$.

~~$E[\min(d(S_t), d$~~

$E[\min(d(S_t), d(V\setminus S_t))]$

$= \sum_{i<j} Pr(i\in S_t \wedge t<0)\, d_i$

$+ \sum_{i\ge j} Pr(i\notin S_t \wedge t\ge 0)\, d_i$

$$E_t\left[\min(d(S_t), d(V\setminus S_t))\right] = \sum_{i<j} \Pr(z_i \le t \wedge t<0)\, d_i + \sum_{i\ge j} \Pr(z_i > t \wedge t \ge 0)\, d_i$$

$$\Pr(t\in[z_i,0)) = z_i^2$$
$$\Pr(t\in[0,z_i)) = z_i^2$$

$$E_t\left[\min(d(S_t), d(V\setminus S_t))\right] = \sum_{i<j} z_i^2 d_i + \sum_{i\ge j} z_i^2 d_i$$
$$= \sum_i z_i^2 d_i$$
$$= z^T D z$$

$$E_t\left[|\partial S_t|\right] = \sum_{(u,v)\in E} \Pr(u\in S_t \wedge v\notin S_t)$$
$$= \sum_{(u,v)\in E} \Pr(z_u \le t \wedge z_v > t)$$
$$= \sum_{(u,v)\in E} \Pr(z_u \le t < z_v)$$
$$\le \sum_{(u,v)\in E} |z_v - z_u|\,(|z_v|+|z_u|)$$
$$\le \sqrt{\sum_{(u,v)\in E} (z_v-z_u)^2}\ \sqrt{\sum_{(u,v)\in E} (|z_v|+|z_u|)^2}$$
(Cauchy-Schwartz)
$$\le \sqrt{z^T L z}\ \sqrt{2\sum_{(u,v)\in E} (|z_u|^2+|z_v|^2)}$$
$$= \sqrt{z^T L z}\ \sqrt{2 z^T D z}$$

We know that $\dfrac{z^T L z}{z^T D z} \le \lambda_2$.

$$E_t\left[|\partial S_t|\right] \le \sqrt{\lambda_2 (z^T D z)}\ \sqrt{2 z^T D z}$$
$$= \sqrt{2\lambda_2}\ z^T D z$$

$$E_t[|\partial S_t|] \le \sqrt{2\lambda_2}\, E[\min(d(S_t), d(V\setminus S_t))]$$

$$E_t[|\partial S_t| - \sqrt{2\lambda_2}\min(d(S_t), d(V\setminus S_t))] \le 0$$

If $E[X] \le 0$

then $\exists\, y \le 0$ s.t $Pr[X=y] > 0$.

$$\Rightarrow \frac{|\partial S_t|}{\min(d(S_t), d(V\setminus S_t))} \le \sqrt{2\lambda_2}$$

$$\Phi(S_t) \le \sqrt{2\lambda_2}.$$

$$\Rightarrow \Phi(G) \le \sqrt{2\lambda_2}$$

(References: Daniel Spielman Notes)

Hence proved.

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② For psd matrices all eigen values ≥ 0.

But for any matrix eigen values can be negative.

$$M = \sum_{i=1}^{n} \alpha_i v_i v_i^T$$

$$x = \sum_{i=1}^{n} a_i v_i$$

$a_i \rightarrow$ scalars.

$$Mx = \sum_{i=1}^{n} \alpha_i a_i v_i$$

$$M^k x = \sum_{i=1}^{n} \alpha_i^k a_i v_i$$

$$= \alpha_1^k a_1 v_1 + \alpha_2^k a_2 v_2 + \dots + \alpha_n^k a_n v_n$$

$$= \alpha_1^k a_1 \left(v_1 + \left(\frac{\alpha_2}{\alpha_1}\right)^k \left(\frac{a_2}{a_1}\right) v_2 + \dots + \left(\frac{\alpha_n}{\alpha_1}\right)^k \frac{a_n v_n}{a_1}\right)$$

This may not tend to '0' as $k \to \infty$.

$|\alpha_n|$ can be $\ge |\alpha_1|$

So, we cannot zero them.

In the power method analysis,

we let $l = \max\{i \mid \alpha_i \geq (1-\varepsilon)\alpha_1\}$

we wrote

$$
\begin{aligned}
y^T y &= \sum_{i=1}^{n} a_i^2 \alpha_i^{2k} \\
&= \sum_{i=1}^{l} a_i^2 \alpha_i^2 \alpha_i^{2k} + \sum_{i=l+1}^{n} a_i^2 \alpha_i^{2k} \\
&\leq \sum_{i=1}^{l} a_i^2 \alpha_i^{2k} + \underbrace{(1-\varepsilon)^{2k} \alpha_1^{2k} \sum_{i=l+1}^{n} a_i^2}_{\text{Not true for general matrices}}
\end{aligned}
$$

If matrix was not P.S.D.

$$|\alpha_n| \geq |\alpha_1(1-\varepsilon)|$$

There may exist eigenvalues which are negative whose magnitude might be greater than $\alpha_1(1-\varepsilon)$, which implies $\alpha_n^{2k} \geq \alpha_1^{2k}(1-\varepsilon)^{2k}$.

Hence we used the fact that M is PSD while writing that step.

For bipartite graphs:

$$\alpha_n = -\alpha_1 \quad \text{(Eigen values come in +ve, -ve pairs)}$$

$$\therefore M^k x = \alpha_1^k a_1 \left(v_1 + \left(\frac{\alpha_2}{\alpha_1}\right)^k \left(\frac{a_2}{a_1}\right) v_2 + \dots + \underbrace{\left(\frac{\alpha_n}{\alpha_1}\right)^k}_{-1} \left(\frac{a_n}{a_1}\right) v_n \right)$$

(Rest all terms would tend to 0 as $k \to \infty$)

So Algorithm would give for large values of k.

$$
\begin{aligned}
M^k x &\approx \alpha_1^k a_1 v_1 + \alpha_1^k (-1)^k a_n v_n \\
&\approx \alpha_1^k (a_1 v_1 + (-1)^k a_n v_n)
\end{aligned}
$$

If k is odd,

$$M^k x \approx \alpha_1^k (a_1 v_1 - a_n v_n)$$

If k is even

$$M^k x \approx \alpha_1^k (a_1 v_1 + a_n v_n)$$

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(3) (a) Let $\alpha_2^2 + \eta^2 \le d^2$ $\quad$ α_2 — A

$\alpha_1 \ge \alpha_2 \ge \dots \ge \alpha_n$

Let x_2 be eigen vector of A for α_2

$$A x_2 = \alpha x_2$$

$$\mathcal{L} x_2 = \left(I - \frac{A}{d}\right) x_2 = x_2 - \frac{\alpha_2 x_2}{d}$$

$$= \left(1 - \frac{\alpha_2}{d}\right) x_2$$

$\therefore \lambda_2 = 1 - \frac{\alpha_2}{d}$ $\quad$ $\underset{0}{\lambda_1} \le \lambda_2 \le \dots \le \lambda_n$ — $\mathcal{L}$

eigen value of $\mathcal{L}$

$$\frac{\alpha_2}{d} = 1 - \lambda_2.$$

$$\alpha_2^2 + \eta^2 \le d^2$$

$$\Rightarrow \cancel{d^2(1-\lambda_2)^2 + \eta^2 \le d^2} \quad \frac{\alpha^2}{d^2} + \frac{\eta^2}{d^2} \le 1$$

$$\Rightarrow \frac{\eta^2}{d^2} \le 1 - (1-\lambda_2)^2$$

$$= 2\lambda_2 - \lambda_2^2$$

$$\Rightarrow \frac{\eta^2}{d^2} \le 2\lambda_2$$

$$\frac{\eta}{d} \le \sqrt{2\lambda_2} \qquad \frac{\eta}{d} = \min_{\substack{S \subseteq V \\ |S| \le n/2}} \frac{|\partial S|}{d|S|} = \Phi(G)$$

$$\Phi(G) \le \sqrt{2\lambda_2}$$

$\mathcal{L} = \frac{L}{d}$ (for d-regular)

If λ_2' is eigen value of L', we get

$$\Phi(G) \le \sqrt{2\lambda_2'/d}$$

$\therefore$ Hence showed

(b) $B(y) \triangleq \sum_{\substack{i<j \\ (i,j)\in E}} (y_i^2 - y_j^2)$.

$$= \sum_{i<j\,(i,j)\in E} |y_i - y_j|\,|y_i + y_j|$$

$$\leq \sqrt{\sum_{\substack{i,j\in E \\ i<j}} (y_i - y_j)^2}\ \sqrt{\sum_{\substack{(i,j)\in E \\ i<j}} (y_i + y_j)^2} \quad \text{(Cauchy Schwartz)}$$

$$\sum_{\substack{(i,j)\in E \\ i<j}} (y_i + y_j)^2 = \sum_{(i,j)\in E} y_i^2 + y_j^2 + 2y_i y_j$$

$$= \sum_{(i,j)\in E} y_i^2 + y_j^2 + (y_i^2 + y_j^2 - (y_i - y_j)^2)$$

$$= 2\sum_{(i,j)\in E} y_i^2 + y_j^2 - \sum_{(i,j)\in E} (y_i - y_j)^2$$

$$= 2d \sum_i y_i^2 - y^T L y$$

$$B(y) \leq \sqrt{y^T L y}\ \sqrt{2d\|y\|_2^2 - y^T L y}$$

$$= \sqrt{2d\|y\|_2^2 (y^T L y) - (y^T L y)^2}$$

$$= \sqrt{2d\|y\|_2^2 (d\|y\|_2^2 - y^T A y) - (d^2\|y\|^4 + (y^T A y)^2 - 2d\|y\|^2 y^T A y)}$$

$$\cancel{= \sqrt{2d^2\|y\|_2^4 - 2d\|y\|_2^2 y^T A y - d^2\|y\|_2^4 - (y^T A y)^2 + 2d\|y\|^2 y^T A y}}$$

$$\cancel{= \sqrt{2d\|y\|_2^2 (d\|y\|_2^2 - y^T A y)} -}$$

$$= \sqrt{2d^2\|y\|_2^4 \cancel{- 2d\|y\|_2^2 y^T A y} - d^2\|y\|_2^4 - (y^T A y)^2 \cancel{+ 2d\|y\|_2^2 y^T A y}}$$

$$= \sqrt{d^2\|y\|_2^4 - (y^T A y)^2}$$

$$\Rightarrow B(y) \leq \sqrt{d^2\|y\|_2^4 - (y^T A y)^2}$$

(c) Show $B(y) \geq \eta \|y\|_2^2$

$$B(y) = \sum_{\substack{i<j \\ (i,j)\in E}} (y_i^2 - y_j^2)$$

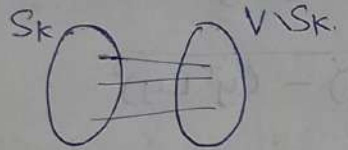
$$= \sum_{\substack{i<j \\ (i,j)\in E}} \sum_{k=i}^{j-1} y_k^2 - y_{k+1}^2$$

$$= \sum_{k=1}^{n-1} \sum_{\substack{i\leq k}} \sum_{\substack{j>k \\ (i,j)\in E}} (y_k^2 - y_{k+1}^2)$$

$$= \sum_{k=1}^{n-1} (y_k^2 - y_{k+1}^2) \cdot \sum_{i\leq k} \sum_{\substack{j>k \\ (i,j)\in E}} 1$$

For any $k \in \mathbb{N}$, we define $S_k = \{i : \underline{\underline{1 \leq i \leq k}}\}$

$= \{1, 2, \dots, k\}$.



$$B(y) = \sum_{k=1}^{n-1} (y_k^2 - y_{k+1}^2) \, |\underbrace{\partial(S_k)}_{\text{edges from } S_k \text{ to } V\setminus S_k}|$$

we have defined $y_i = x_i \cdot \mathbb{1}_{x_i > 0}$.

and $y_1 \geq y_2 \geq \dots \geq y_n$.

Let 'a' be the largest index such that $y_a > 0$

i.e $y_i = 0 \;\; \forall \; i \geq a+1 \qquad a \in \{1, \dots, n\}$

$$B(y) = \sum_{k=1}^{a} (y_k^2 - y_{k+1}^2) \, |\partial(S_k)|$$

Since ~~atmost~~ y has atmost half the entries as non-zero, $a \leq n/2$.

$$\cancel{B(y) \geq \sum_{k=1}^{a} (y_k^2 - y_{k+1}^2)}$$

$$B(y) = \sum_{k=1}^{a} (y_k^2 - y_{k+1}^2)\, k \frac{|\partial(S_k)|}{k}$$

$$\geq \sum_{k=1}^{a} (y_k^2 - y_{k+1}^2)(k)\,\eta \quad (\text{Since } a \leq n/2)$$

$$= \eta \sum_{k=1}^{a} (y_k^2 - y_{k+1}^2) . k$$

$$\cancel{= (y_1^2 - a y_a^2}$$

$$= (y_1^2 + y_2^2 + \dots + y_a^2 - a y_{a+1}^2)\eta$$

$$(y_{a+1} = 0)$$

$$= \left(\sum_{k=1}^{a} y_k^2\right)\eta$$

$$\cancel{= y^T y}$$

$$= \eta y^T y$$

$$\therefore\ B(y) \geq \eta \|y\|_2^2$$

(d) We have used the fact that y has atmost half the entries as non-zero when going from this step:

$$B(y) = \sum_{k=1}^{a} (y_k^2 - y_{k+1}^2)\, k \frac{|\partial(S_k)|}{k}$$

to

$$B(y) \geq \eta \sum_{k=1}^{a} k (y_k^2 - y_{k+1}^2).$$

This is because $a \leq n/2$.

'a' is the largest index such that $y_a > 0$.

$y_i = 0$ for $\forall\ i \geq a+1$.

(e)

To prove

$$\alpha_2^2 + \eta^2 \leq d^2.$$

We have,

$$\eta \|y\|_2^2 \le B(y) \le \sqrt{d^2\|y\|_2^4 - (y^TAy)^2}$$

$$\Rightarrow \eta^2 \|y\|_2^4 \le d^2\|y\|_2^4 - (y^TAy)^2$$

$$\Rightarrow \eta^2 \le d^2 - \left(\frac{y^TAy}{y^Ty}\right)^2$$

We already proved in class that

$$y^TLy \le \lambda_2 \|y\|^2$$

$$y^TLy = \sum_i y_i (Ly)_i$$

$$= \sum_i y_i (dy_i - \sum_{j\in N(i)} y_j)$$

$$= d\sum_i y_i^2 - \sum_{y_i>0} y_i \sum_{j\in N(i)} y_j$$

$\forall j$ ~~$y_j \ge x_j$~~

$$\cancel{\le d\sum_i x_i^2 - \sum_{i:x_i>0} x_i \left(\sum_{j\in N(i)} x_j\right)}$$

$\forall j \;\; y_j \ge x_j$

$$y^TLy \le d\sum_i y_i^2 - \sum_{y_i>0} y_i \left(\sum_{j\in N(i)} x_j\right)$$

$$y^TLy \le d\sum_{i:x_i\ge 0} x_i^2 - \sum_{i:x_i>0} x_i \sum_{j\sim N(i)} x_j$$

$$= \sum_{i:x_i>0} x_i \underbrace{\left(dx_i - \sum_{j\in N(i)} x_j\right)}_{(Lx)_i = \lambda_2 x_i}$$

$$= \sum_{i:x_i>0} x_i(\lambda_2 x_i) = \lambda_2 \|y\|_2^2$$

$$\frac{y^TLy}{y^Ty} \le \lambda_2$$

$\Rightarrow$ If x is eigenvector of L for λ_2

$$\frac{y^TLy}{y^Ty} \le \frac{x^TLx}{x^Tx}$$

$$\Rightarrow \frac{y^T(dI-A)y}{y^Ty} \le \frac{x^T(dI-A)x}{x^Tx}$$

$$\Rightarrow \cancel{d y^T} \; d - \frac{y^TAy}{y^Ty} \le d - \frac{x^TAx}{x^Tx}$$

(x is also eigenvector of A for α_2)

Since d-regular

$$\Rightarrow \frac{y^TAy}{y^Ty} \ge \alpha_2.$$

$$\eta^2 \le d^2 - \left(\frac{y^TAy}{y^Ty}\right)^2$$

$$\le d^2 - \alpha_2^2$$

$$\Rightarrow \eta^2 + \alpha_2^2 \le d^2$$

Hence proved.