Week 7

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CS21BTECH11012

1) From wikipedia,

A pseudo inverse satisfies the following conditions:

$$AA^+A = A$$
$$A^+AA^+ = A^+$$
$$(AA^+)^T = AA^+$$
$$(A^+A)^T = A^+A$$

We can check that, these infact hold true for the A^+ defined in class as.

$$A^+ = \sum_{i=1}^{n} \frac{1}{\lambda_i} v_i v_i^T \quad (\text{Assuming } \lambda_i \neq 0)$$

$\lambda_i \downarrow$ eigenvalue of A.

$$AA^+A = \left(\sum_{i=1}^{n} \lambda_i v_i v_i^T\right)\underbrace{\left(\sum_{j=1}^{n} \frac{1}{\lambda_j} v_j v_j^T\right)\left(\sum_{k=1}^{n} \lambda_k v_k v_k^T\right)}$$

$$= \left(\sum_{i:\{\lambda_i \neq 0\}} v_i v_i^T\right)\left(\sum_{k=1}^{n} \lambda_k v_k v_k^T\right)$$

$$= \sum_{i:\lambda_i=0} \lambda_i v_i v_i^T + \sum_{i:\{\lambda_i \neq 0\}} \lambda_i v_i v_i^T$$

$$= \sum_{i=1}^{n} \lambda_i v_i v_i^T$$

$$= A.$$

(a) $Ax = b$

In general, if y belongs to column space of A, then $\exists$ 'z' such that $Az = y$ (dimensions accordingly)

$$Az = y$$
$$AA^{+}\underbrace{Az} = y. \quad (\text{From } A = AA^{+}A)$$
$$AA^{+}y = y.$$

So, whenever y belongs to columnspace of A, we have $A(A^{+}y) = y$.

Now, since $Ax = b$ has a solution, 'b' belongs to column space of A.

$$\therefore \cancel{A^{+}b} \quad A(A^{+}b) = b.$$
$$x = A^{+}b \text{ is a solution.}$$

(e.) $L = I - W$.

~~Taking~~ $LL^{+}L = L$

Taking $L^{+} = (I-W)^{+} = \frac{1}{2}\left(I - J + (I+W)(I-W^2)^{+}(I+W)\right)$

$$LL^{+}L = (I-W)\times\frac{1}{2}\left(I - J + (I+W)(I-W^2)^{+}(I+W)\right)(I-W)$$
$$= \frac{1}{2}\left(I - W - J + JW + (I-W^2)(I-W^2)^{+}(I+W)\right)(I-W)$$
$$\cancel{= \frac{1}{2}(I - W - J + -}$$
$$= \frac{1}{2}\left(I - W - W + W^2 - J + JW + JW - JW^2 + \underbrace{(I-W^2)(I-W^2)^{+}(I-W^2)}_{I-W^2}\right)$$
$$= \frac{1}{2}\left(I - 2W + \cancel{W^2} - J + 2JW - JW^2 + I - \cancel{W^2}\right)$$
$$= \frac{1}{2}\left(2I - 2W + \left(2J\left(\frac{A}{d}\right) - J\left(\frac{A}{d}\right)^2 - J\right)\right) \qquad \boxed{JA = dJ}$$
$$= \frac{1}{2}\left(2I - 2W + \frac{(2dJA - JA^2 - d^2J)}{d^2}\right)$$
$$= \cancel{\frac{1}{2}(2I - 2W +}$$

$$JA = dJ$$
$$JA^2 = (JA)(A) = dJA = d^2J$$

$$= \frac{1}{2}\left(2I - 2W + \frac{2d^2J - d^2J - d^2J}{d^2}\right)$$

$$= \frac{1}{2}(2I - 2W)$$

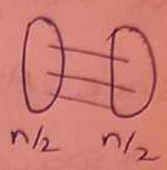
$$= I - W$$

$$= \mathcal{L}$$

Since $\mathcal{L}^+$ is unique,

$\mathcal{L}^+$ has to be $\frac{1}{2}\left(I - J + (I+W)(I-W^2)^+(I+W)\right)$.

2) ~~Taking~~ Let's consider 2 sets ~~M, N~~ P, Q both consisting of $n/2$ vertices



$$\sum_{i,j} A_{ij}(x_i - x_j)^2 = \sum_{i,j} A_{ij} x_i^2 + \sum_{i,j} A_{ij} x_j^2 - 2\sum_{i,j} A_{ij} x_i x_j$$

We take the vector x as

$$x_i = \begin{cases} 1 & \text{if } i \in P \\ -1 & \text{if } i \in Q \end{cases}$$

$$\sum_{i,j} A_{ij}(x_i - x_j)^2 = 2 \times (|\partial S| \times 4) = 8|\partial S|$$

$$\sum_{i,j} A_{ij} x_i^2 = \sum_{i,j} A_{ij} x_j^2 = nd$$

$$8|\partial S| = 2nd - 2\sum_{i,j} A_{ij} x_i x_j$$

We can take the subsets such that $\sum_{i,j} A_{ij} x_i x_j \geq 0$

This is possible ~~when~~, as follows:

$$\sum_{i,j} A_{ij} x_i x_j = \underbrace{\sum_{\substack{(i,j)\in E \\ i,j \text{ belong} \\ \text{to same} \\ \text{subset}}} x_i x_j}_{\substack{\text{let magnitude} \\ \text{of this quantity} \\ \text{be } a}} + \underbrace{\sum_{\substack{(i,j)\in E \\ i,j \text{ belong} \\ \text{to different} \\ \text{subsets}}} x_i x_j}_{\substack{\text{let magnitude} \\ \text{of this quantity} \\ \text{be } b.}}$$

We want $a-b \geq 0$.

$a+b = \frac{nd}{2}$.

Total No. of edges within the same subset have to be greater than those b/w the subsets.

If we take $a \geq \lceil \frac{nd}{4} \rceil$ and $b \leq \lfloor \frac{nd}{4} \rfloor$, then it is possible that $a-b \geq 0$.

$\sum_{i,j} A_{ij} x_i x_j \geq 0$ for these kind of subsets.

$$8|\partial S| = 2nd - 2\sum_{i,j} A_{ij} x_i x_j$$

$$\Rightarrow 8|\partial S| \leq 2nd.$$

$$|\partial S| \leq \frac{dn}{4}$$

$(\frac{n}{2}, \rho)$ edge expander means for every subset S of vertices s.t $|S| \leq n/2$,

$$|\partial S| \geq \rho d |S|$$

$$\frac{|\partial S|}{d|S|} \geq \rho.$$

From above we got a subset of $n/2$ vertices such that $\frac{|\partial S|}{d(\frac{n}{2})} \leq \frac{1}{2}$

If $\rho > \frac{1}{2}$, then $\frac{|\partial S|}{d|S|} > \frac{1}{2}$ for every subset $|S| \leq \frac{n}{2}$.

This is clearly a contradiction ~~for the~~ since we have a subset ~~for which~~ of $n/2$ vertices for which this is not holding true.

$\therefore$ There does not exist a $(n/2, \rho)$ edge expander for $\rho > 1/2$