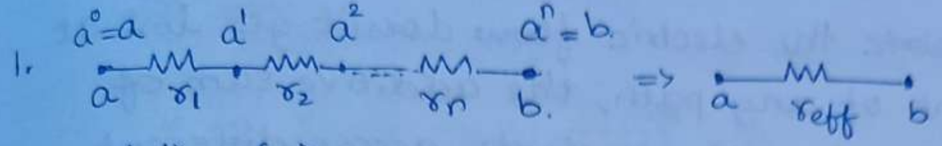
Week 13 Solutions

C. Akshay Santoshi
CS21BTECH11012

1.



$$\frac{p(a^1) - p(a)}{r_1} = f(a, a^1) = f(a,b)$$

$$\frac{p(a^2) - p(a^1)}{r_2} = f(a^1, a^2) = f(a,b)$$

$$\vdots$$

$$\frac{p(b) - p(a^{n-1})}{r_n} = f(a^{n-1}, b) = f(a,b)$$

$$\frac{p(b) - p(a)}{r_{eff}} = f(a,b) \quad \text{--- (1)}$$

For series resistances, the electric flow remains the same across the path.

Adding equations,

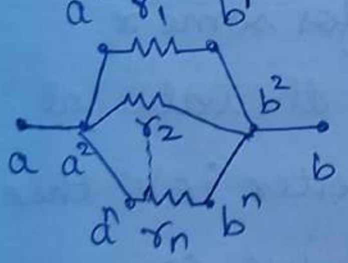
$$p(\cancel{a^1}) - p(a) + p(\cancel{a^2}) - p(\cancel{a^1}) + \dots + p(b) - p(\cancel{a^{n-1}}) = f(a,b)(r_1 + r_2 + \dots + r_n)$$

$$\Rightarrow p(b) - p(a) = f(a,b)(r_1 + r_2 + \dots + r_n) \quad \text{--- (2)}$$

From (1), $p(b) - p(a) = f(a,b)\, r_{eff}$.

From (1) and (2)

$$r_{eff} = r_1 + r_2 + \dots + r_n$$



For parallel

Note that $p(a^1) = p(a^2) = \dots = p(a^n) = \underline{\underline{p(a)}}$

$p(b^1) = p(b^2) = \dots = p(b^n) = p(b)$

For path ~~p(b)~~ $a^1 \to b^1$ $\quad \dfrac{p(b) - p(a)}{r_1} = f(a^1, b^1)$

For path $a^2 \to b^2$ $\quad \dfrac{p(b) - p(a)}{r_2} = f(a^2, b^2)$

$\vdots$

For path $a^n \to b^n$ $\quad \dfrac{p(b) - p(a)}{r_n} = f(a^n, b^n)$

~~If the whole system~~

Let us take $f(a,b)$ to be the electric flow which leaves point 'a' before splitting to the adjacent paths.

Since, ~~electric~~ the electric flow doesn't get lost at any point of any path, the additive sum of all the electric flows ~~of di~~ across different ~~resistors~~ resistances should equal $f(a,b)$.

$$\frac{p(b)-p(a)}{r_{eff}} = f(a,b)$$

$$f(a,b) = f(a^1,b^1) + f(a^2,b^2) + \dots + f(a^n,b^n)$$

$$\frac{p(b)-p(a)}{r_{eff}} = \frac{p(b)-p(a)}{r_1} + \frac{p(b)-p(a)}{r_2} + \dots + \frac{p(b)-p(a)}{r_n}$$

$$\Rightarrow r_{eff} = \frac{1}{\frac{1}{r_1}+\frac{1}{r_2}+\dots+\frac{1}{r_n}}$$

2) Let $X = \{x \in \text{Real Valued functions } V \to R \mid x(B) \text{ is fixed}\}$

$B \subset V$
$S = V - B$.

$$\mathcal{E}(x) = \sum_{\{i,j\} \in E} (x(i)-x(j))^2 \cdot w_{i,j}$$

$\mathcal{E}$ would have a global minimum for some x. because $\mathcal{E}(x) \geq 0$ and if we take the value at $|x(i)|$ to be very large for some vertex $i \in V$, then since we know that $x(B)$ is fixed, we can guarantee that there exist vertices j,k such that $(x(j)-x(k))^2$ is very large making $\mathcal{E}(x)$ very large.

To get the global minima, we can differentiate $\mathcal{E}$ wrt $x(a)$ for some $a \in S$.

$$\frac{\partial}{\partial x}\left(\sum_{\{i,j\} \in E} w_{i,j} (x(i) - x(j))^2 \right) = 0$$

$$\Rightarrow 2 \sum_{\{a,j\} \in E} w_{a,j} (x(a) - x(j)) = 0.$$

$$= x(a) \sum_{j:\{a,j\} \in E} w_{a,j} - \sum_{j:\{a,j\} \in E} w_{a,j} x(j) = 0$$

$$= x(a) d_a = \sum_{b:\{a,b\} \in E} w_{a,b} x(b).$$

$$\Rightarrow x(a) = \frac{1}{d_a} \sum_{b \sim a} w_{a,b} x(b).$$

This is true for all vertices present in S.

$\therefore$ Energy $\mathcal{E}(x)$ is minimized by setting $x(S)$ so that x is harmonic on S.

3) $\mathcal{E}(f) = p^T L_G p \quad p(s) - p(t) = r_{eff}(s,t)$

~~$x \in \mathbb{R}^n \quad x(s) - x(t) = r_{eff}(s,t)$~~

Consider all $X = \{x \in \mathbb{R}^n$ such that $x(s) - x(t) = r_{eff}(s,t)\}$

$$\mathcal{E}(v) = \sum_{\{i,j\} \in E} \frac{(v(i) - v(j))^2}{r_{ij}}$$

p is the potential that follows the KCL, KVL.

Consider $x \in X$ and y is a function defined as

$$y(i) = x(i) - p(i) \quad \forall\, i \in V. \longrightarrow x(i) = y(i) + p(i)$$

$$\mathcal{E}(x) = \sum_{\{i,j\} \in E} \frac{(x(i) - x(j))^2}{r_{ij}}$$

$$\mathcal{E}(x) = \sum_{\{i,j\}\in E} \frac{(x(i)-x(j))^2}{r_{ij}}$$

$$= \sum_{\{i,j\}\in E} \frac{((y(i)+p(i)) - (y(j)+p(j)))^2}{r_{ij}}$$

$$= \sum_{\{i,j\}\in E} \frac{(y(i)+p(i))^2 + (y(j)+p(j))^2 - 2(y(i)+p(i))(y(j)+p(j))}{r_{ij}}$$

$$= \sum_{\{i,j\}\in E} \left(\frac{(y(i))^2+(y(j))^2 - 2y(i)y(j)}{r_{ij}} + \frac{(p(i))^2+(p(j))^2 - 2p(i)p(j)}{r_{ij}} + \frac{(y(i)-y(j))(p(i)-p(j))}{r_{ij}} \right)$$

$$= \sum_{\{i,j\}\in E} \frac{(y(i)-y(j))^2}{r_{ij}} + \sum_{\{i,j\}\in E} \frac{(p(i)-p(j))^2}{r_{ij}} + \sum_{\{i,j\}\in E} \frac{(y(i)-y(j))(p(i)-p(j))}{r_{ij}}$$

$$= \mathcal{E}(y) + \mathcal{E}(p) + \sum_{\{i,j\}\in E} \frac{(y(i)-y(j))(p(i)-p(j))}{r_{ij}}$$

$$= \mathcal{E}(y) + \mathcal{E}(p) + \sum_{\{i,j\}\in E} f(i,j)(y(i)-y(j))$$

Note that ~~p(s)~~ $p(s)-p(t) = x(s)-x(t) = r_{eff}(s,t)$.

$$\therefore y(s)-y(t) = (x(s)-p(s)) - (x(t)-p(t))$$

$$= (x(s)-x(t)) - (p(s)-p(t))$$

$$= 0$$

$$\sum_{\{i,j\}\in E} f(i,j)(y(i)-y(j)) = (y(s)-y(t))f$$

$$= 0$$

$\therefore \mathcal{E}(x) = \mathcal{E}(y) + \mathcal{E}(p)$ (Terms are ≥ 0).

$\therefore \mathcal{E}(x) \geq \mathcal{E}_p$

$\mathcal{E}_p = p^T L_G p$.

$\mathcal{E}_x \geq p^T L_G p$.

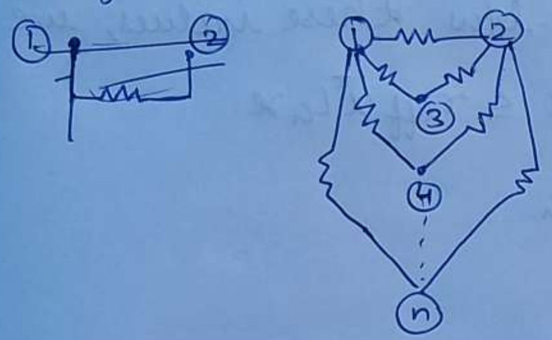
$\Rightarrow x^T L_G x \geq p^T L_G p$.

4) Let the vertices 1 and 2 b/w which resistance is measured. ~~be 1 and~~.

All the other vertices are symmetrices, in the sense that they have edges with all neighbours and ~~are~~ therefore their potential will be same.

Hence, while calculating effective resistance of the system, the resistances which are not ~~linked~~ ~~or~~ involved with either node 1 or node 2 can be ignored.

Diagram can be simplified to



~~Reff = 1~~

$$\frac{1}{R_{eff}} = \frac{1}{R} + \underbrace{\left(\frac{1}{2R} + \frac{1}{2R} + \cdots + \frac{1}{2R}\right)}_{n-2 \text{ times}}$$

$$= \frac{1}{R} + \frac{n-2}{2R} = \frac{n}{2R}$$

$$R_{eff} = \frac{2R}{n}$$

Unweighted, so $R_{eff} = \frac{2}{n}$.

5) $e = (i,j) \quad L_e = (e_i - e_j)(e_i - e_j)^T$

To prove $L_e \preceq r_{eff}(i,j) L_G$.

To show ~~$x^T L_G x \leq x^T r_{eff} x^T$~~

$$x^T L_e x \leq r_{eff}(i,j)\, x^T L_G x$$

$$(x_i - x_j)^2 \leq r_{eff}(i,j) \sum_{\{a,b\} \in E} (x_a - x_b)^2$$

~~If $x_i - x_j = 0$~~

If $x_i = x_j$, then we can see that it holds true trivially.

When ~~$x_i \neq x_j$~~ $x(i) - x(j) = r_{eff}(i,j)$

we get

~~$(r_{eff})^2 \le r_{eff}$~~

$$(r_{eff}(i,j))^2 \le r_{eff}(i,j) \sum_{\{a,b\}\in E} (x(a)-x(b))^2$$

$$\Rightarrow r_{eff}(i,j) \le x^T L_G x .$$ when $x(i)-x(j) = r_{eff}(i,j)$

This we have shown to be true in problem 3.

And since ~~tha~~ the difference b/w $x(i)$ and $x(j)$ can be scaled to be b/w these values, we can say that $x^T L_e x \le r_{eff}\, x^T L_G x$

$$\therefore L_e \preceq r_{eff}(i,j) L_G .$$

Method 2:-

If