CS5100: Quantum Computing

C. Akshay Santoshi
CS21BTECH11012

Problem Set 1

1. (a) $|\phi\rangle|0\rangle \rightarrow |\phi\rangle|\phi\rangle$ for every qubit $|\phi\rangle$?

$$U(|\phi\rangle|0\rangle) = |\phi\rangle|\phi\rangle$$

Taking $|\phi\rangle = |0\rangle$

$$U(|0\rangle|0\rangle) = |0\rangle|0\rangle \quad \text{——— (1)}$$

Taking $|\phi\rangle = |1\rangle$

$$U(|1\rangle|0\rangle) = |1\rangle|1\rangle \quad \text{——— (2)}$$

Taking $|\phi\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}$

$$\underbrace{U(|+\rangle|0\rangle)}_{\text{LHS}} = \underbrace{|+\rangle|+\rangle}_{\text{RHS.}}$$

Simplifying LHS:

$$U\left(\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) \otimes |0\rangle\right) = U\left(\frac{|00\rangle + |1\rangle|0\rangle}{\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}} U(|0\rangle|0\rangle) + \frac{1}{\sqrt{2}} U(|1\rangle|0\rangle)$$

Using (1) and (2),

$$= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle) + \frac{1}{\sqrt{2}}(|1\rangle|1\rangle) \quad \text{——— (3)}$$

Simplifying RHS:

$$|+\rangle|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$= \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) \quad \text{——— (4)}$$

LHS ≠ RHS

Contradiction.

$\therefore$ There doesn't exist a 2-qubit unitary U that maps $|\phi\rangle|0\rangle \rightarrow |\phi\rangle|\phi\rangle$ for every qubit $|\phi\rangle$.

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1.(b). $|0\rangle|0\rangle \longrightarrow |0\rangle|0\rangle$ and $|+\rangle|0\rangle \rightarrow |+\rangle|+\rangle$?

$U(|0\rangle|0\rangle) = |0\rangle|0\rangle$

$U(|+\rangle|0\rangle) = |+\rangle|+\rangle$.

Check whether U preserves the inner product b/w two states:

states:- ϕ_1, ϕ_2

$$\langle\phi_1|\phi_2\rangle = \langle U\phi_1|U\phi_2\rangle$$

LHS:

$\phi_1 = |0\rangle \otimes |0\rangle$

$\phi_2 = |+\rangle \otimes |0\rangle$

$$\phi_1 = \begin{pmatrix}1\\0\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\phi_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix}$$

$$\langle\phi_1|\phi_2\rangle = (1\,0\,0\,0) \times \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\1\\0\end{pmatrix} = \frac{1}{\sqrt{2}} \quad \text{---} \ (1).$$

RHS:

$$U\phi_1 = U(|0\rangle|0\rangle) = |0\rangle|0\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}.$$

$$U\phi_2 = U(|+\rangle|0\rangle) = |+\rangle|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle+|1\rangle)$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$$

$$\langle U\phi_1|U\phi_2\rangle = (1\,0\,0\,0) \times \frac{1}{2}\begin{pmatrix}1\\1\\1\\1\end{pmatrix} = \frac{1}{2} \quad \text{---} \ (2)$$

$(1) \neq (2)$

Here U is not preserving the inner product b/w the two states

$\therefore$ Such type of U doesn't exist. 15

2.(a) $\underline{ab = 00}$

Does nothing here. So ~~EPR pair~~ it remains unchanged

Since both are zero.

$$\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

(b) $\underline{ab = 10}$

$a=1$, so applies NOT gate to her qubit

" $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$$= \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$$

(c) $\underline{ab = 01}$

$b=1$, so applies Z(phaseflip) gate to her qubit

$|0\rangle \to |0\rangle$

$|1\rangle \to -|1\rangle$ $\leftarrow$ " $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$= \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

(d) $\underline{ab = 11}$

First, $a=1$, so applies NOT gate, then 2-qubit state becomes $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$.

~~The~~ Next, $b=1$, so applies Z (phase flip) gate

$|0\rangle \to |0\rangle$, $|1\rangle \to -|1\rangle$

$$\frac{1}{\sqrt{2}}(-|10\rangle + |01\rangle)$$

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$\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$ and $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$ are not different states. Mathematically they are equivalent.

Therefore,

$$ab = 00 \Rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

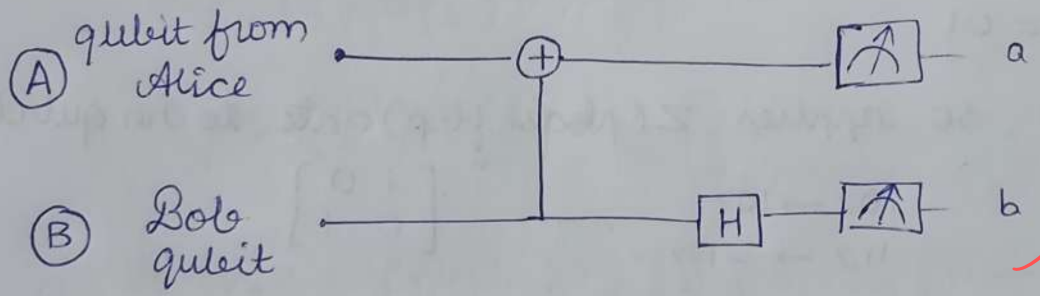
$$ab = 10 \Rightarrow \frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$$

$$ab = 01 \Rightarrow \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$ab = 11 \Rightarrow \frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$$

2.(b). Bob now has two qubits.

To recover Alice's message, Bob performs the following operations ~~on~~ on the qubits



- CNOT operation with B as control ~~bit~~ qubit and A as target ~~bit~~ qubit.

- H gate applied to B qubit.

If ~~A~~ 2-qubit state is $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$.

CNOT gate: $\frac{1}{\sqrt{2}}(|00\rangle + |01\rangle)$

H gate :- $\frac{1}{\sqrt{2}}\left(\frac{|00\rangle+|\cancel{0}1\rangle}{\sqrt{2}} + \frac{|00\rangle-|\cancel{0}1\rangle}{\sqrt{2}}\right)$

$= |00\rangle$

If 2-qubit state is $\frac{1}{\sqrt{2}}(|10\rangle + |01\rangle)$

CNOT gate : $\frac{1}{\sqrt{2}}(|10\rangle + |11\rangle)$

H gate : $\frac{1}{\sqrt{2}}\left(\frac{|10\rangle+|1\cancel{1}\rangle}{\sqrt{2}} + \frac{|10\rangle-|1\cancel{1}\rangle}{\sqrt{2}}\right)$

$= |10\rangle$

If 2-qubit state is $\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

CNOT gate : $\frac{1}{\sqrt{2}}(|00\rangle - |01\rangle)$

H gate : $\frac{1}{\sqrt{2}}\left(\frac{|0\cancel{0}\rangle+|01\rangle}{\sqrt{2}} - \left(\frac{|0\cancel{0}\rangle-|01\rangle}{\sqrt{2}}\right)\right)$

$= |01\rangle$

If 2-qubit state is $\frac{1}{\sqrt{2}}(|10\rangle - |01\rangle)$

CNOT gate : $\frac{1}{\sqrt{2}}(|10\rangle - |11\rangle)$

H gate : $\frac{1}{\sqrt{2}}\left(\frac{|1\cancel{0}\rangle+|11\rangle}{\sqrt{2}} - \left(\frac{|1\cancel{0}\rangle-|11\rangle}{\sqrt{2}}\right)\right)$

$= |11\rangle$

$\therefore$ Classical bits 'a' and 'b' can be recovered using the CNOT and H gates in this manner.

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3.(a) $U_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ $\quad |\phi\rangle = U_\theta |0\rangle$

$|\phi^\perp\rangle = U_\theta |1\rangle$.

$ZX|\phi^\perp\rangle = |\phi\rangle$

$$|\phi\rangle = U_\theta|0\rangle = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

$$= \cos\theta|0\rangle + \sin\theta|1\rangle.$$

$$|\phi^\perp\rangle = U_\theta|1\rangle = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$$

$$= -\sin\theta|0\rangle + \cos\theta|1\rangle$$

$$X|\phi^\perp\rangle = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix}$$

$$= \cos\theta|0\rangle - \sin\theta|1\rangle.$$

$$ZX|\phi^\perp\rangle = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} \cos\theta \\ -\sin\theta \end{bmatrix} = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$= \cos\theta|0\rangle + \sin\theta|1\rangle.$$

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$\therefore \; ZX|\phi^\perp\rangle = |\phi\rangle.$

3.(b) $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ as $\frac{1}{\sqrt{2}}(|\phi\rangle|\phi\rangle + |\phi^\perp\rangle|\phi^\perp\rangle)$

$$|\phi\rangle|\phi\rangle = \cancel{\begin{pmatrix} \cos\theta \\ \cos\theta \end{pmatrix}} \; (\cos\theta|0\rangle + \sin\theta|1\rangle) \otimes (\cos\theta|0\rangle + \sin\theta|1\rangle)$$

$$= \cancel{(\cos^2\theta|00\rangle + \cos\theta\sin\theta \overset{|01\rangle}{|10\rangle} + \sin\theta\cos\theta}$$

$$= (\cos^2\theta|00\rangle + \cos\theta\sin\theta|01\rangle + \sin\theta\cos\theta|10\rangle + \sin^2\theta|11\rangle)$$

$$|\phi^\perp\rangle|\phi^\perp\rangle = (-\sin\theta|0\rangle + \cos\theta|1\rangle) \otimes (-\sin\theta|0\rangle + \cos\theta|1\rangle)$$

$= (\sin^2\theta|00\rangle - \sin\theta\cos\theta|01\rangle - \cos\theta\sin\theta|10\rangle +$
$\cos^2\theta|11\rangle)$

$|\phi\rangle|\phi\rangle + |\phi^\perp\rangle|\phi^\perp\rangle = (\cos^2\theta + \sin^2\theta)|00\rangle + (\sin^2\theta + \cos^2\theta)|11\rangle$

$\Rightarrow \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|\phi\rangle|\phi\rangle + |\phi^\perp\rangle|\phi^\perp\rangle)$ 10

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3. (c). $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$U_\theta^{-1} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

Alice applies U_θ^{-1} to her qubit

$= \frac{1}{\sqrt{2}}(U_\theta^{-1}|0\rangle \otimes |0\rangle + U_\theta^{-1}|1\rangle \otimes |1\rangle)$ —— (1)

$U_\theta^{-1}|0\rangle = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}$

$= \cos\theta|0\rangle - \sin\theta|1\rangle.$

$U_\theta^{-1}|1\rangle = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$

$= \sin\theta|0\rangle + \cos\theta|1\rangle.$

(1) becomes.

$\frac{1}{\sqrt{2}}\left((\cos\theta|0\rangle - \sin\theta|1\rangle) \otimes |0\rangle + (\sin\theta|0\rangle + \cos\theta|1\rangle) \otimes |1\rangle\right)$

$= \frac{1}{\sqrt{2}}(\cos\theta|00\rangle - \sin\theta|10\rangle + \sin\theta|01\rangle + \cos\theta|11\rangle)$

If Alice measures 0:

State collapses to $\cos\theta|0\rangle+\sin\theta|1\rangle = |\phi\rangle$

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If Bob measures 1:

State collapses to $-\sin\theta|0\rangle+\cos\theta|1\rangle = |\phi^{\perp}\rangle$

3.(d) Alice and Bob share the EPR pair

$$\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)$$

On her qubit, Alice applies U_θ^{-1}

$$\frac{1}{\sqrt{2}}\left((U_\theta^{-1}|0\rangle)\otimes|0\rangle + (U_\theta^{-1}|1\rangle)\otimes|1\rangle\right)$$

From 3(c), we know that it becomes

$$\frac{1}{\sqrt{2}}(\cos\theta|00\rangle - \sin\theta|10\rangle + \sin\theta|01\rangle + \cos\theta|11\rangle)$$

We also got that,

If Alice measures "0" and sends it to Bob, he knows that the state collapses to

$\cos\theta|0\rangle+\sin\theta|1\rangle$ which is nothing but $|\phi\rangle$.

If Alice measures '1' and sends it to Bob, he knows that the state collapses to

$-\sin\theta|0\rangle+\cos\theta|1\rangle = |\phi^{\perp}\rangle$.

From 3(a), we know that $ZX|\phi^{\perp}\rangle = |\phi\rangle$, so he can apply the Z and X on the state to end up with $|\phi\rangle$.

So, here we can see that the protocol uses one EPR-pair and 1 classical bit.

If classical bit is 0, the qubit state itself is $|\phi\rangle$.

If classical bit is 1, apply unitary transformation of Z and X on the state to end up with $|\phi\rangle$.

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4. (a). $IP_2(x_1, \ldots, x_n, y_1, y_2, \ldots, y_n) = (x_1y_1 + x_2y_2 + \ldots + x_ny_n) \pmod 2$

$x_i \wedge y_i = x_i \cdot y_i \pmod 2$

Alice gives n bits $x_1, \ldots, x_n \in \{0,1\}$ to her n magical non-local boxes such that each box gets one x_i

Bob gives n bits $y_1, \ldots, y_n \in \{0,1\}$ to ~~her~~ his n magical non-local boxes such that each box gets one y_i in the following way:

Alice gives x_i to ~~the~~ a box and Bob gives y_i to the box. The box outputs a_i to Alice and b_i to Bob so that

$$a_i \oplus b_i = x_i \wedge y_i$$

So $IP_2(x_1, \ldots, x_n, y_1, \ldots, y_n) = ((x_1 \wedge y_1) + (x_2 \wedge y_2) + \ldots + (x_n \wedge y_n)) \pmod 2$

$$= ((a_1 \oplus b_1) + (a_2 \oplus b_2) + \ldots + (a_n \oplus b_n)) \pmod 2$$

$$IP(x_1,\ldots,x_n) = ((a_1 \oplus b_1) \oplus (a_2 \oplus b_2) \oplus \cdots \oplus (a_n \oplus b_n))$$

$$= ((a_1 \oplus a_2 \oplus \cdots \oplus a_n) \oplus (b_1 \oplus b_2 \oplus \cdots \oplus b_n)) \quad \text{——— (1)}$$

Alice will XOR all the bits she got from the magic box, i.e., $(a_1 \oplus a_2 \oplus \ldots \oplus a_n)$ and she will send this one classical bit to Bob.

Bob will XOR all the bits he got from the magic box, i.e, $(b_1 \oplus b_2 \oplus \ldots + b_n)$ and ~~after~~ then he XORs this value with the classical bit he received from Alice.

The final value, he ends up with is the value of $IP_2(x_1,\ldots,x_n, y_1,\ldots,y_n)$. This follows from (1).

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4.(b). $(x \wedge y) = (x \cdot y) (\bmod 2)$

$(\neg x) = (1+x) (\bmod 2)$

$(x \vee y) = \neg(\neg x \wedge \neg y)$

$= (1 + ((1+x)(1+y))) \bmod 2$

$= (2 + x + y + xy) (\bmod 2)$

$= (x + y + xy) (\bmod 2)$

~~Ext~~ Any Boolean function $f: \{0,1\}^n \to \{0,1\}$ can be written in as boolean circuit using AND and NOT gates and hence these can be converted to a polynomial over $\mathbb{F}_2$ ~~with~~ using previous equations.

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Also, as seen in class, every $f: \{0,1\}^n \to \{0,1\}$ can be uniquely represented by a multilinear polynomial $p(x_1, x_2, \ldots, x_n)$ s.t $p(x) = f(x)$.

$$f(x_1, \ldots, x_n) = \sum_{S \subseteq [n]} c_S \prod_{i \in S} x_i \qquad c_S \in \{0,1\}$$

why?

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4. (c). $f(x_1, \ldots, x_n, y_1, \ldots, y_n) : \{0,1\}^{2n} \to \{0,1\}$.

$$\cancel{f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{S \subseteq \{x_1 \ldots x_n, y_1, \ldots y_n\}} c_S \prod}$$

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{\substack{\text{subset} \leftarrow I \in [n] \\ \text{subset} \leftarrow J \in [n]}} c_S \prod_{i \in I} x_i \prod_{j \in J} y_j$$

$\subseteq$ $\subseteq$

Consider a subset S. The term in RHS corresponding to this subset is $\boxed{c_S G_S(x) H_S(y)}$ where $c_S \in \{0,1\}$, $G_S(x) = \prod_{\substack{i \in [n] \\ \text{and } x_i \in S}} x_i$ (1, if S does not contain atleast one x_i) and

$H_S(y) = \prod_{\substack{i \in [n] \\ \text{and } y_i \in S}} y_i$ (1, if S does not contain atleast one y_i)

There are also terms $G_S(x) H_{S'}(y)$

$\prod_{i \in [n]} x_i$ is a monomial.

So, there can be 2^n such monomials in x since there are n variables in x. $\{x_1, \ldots, x_n\}$.

Suppose there are two (or more) subsets S and S' such that they have the same monomial in x, i.e., $G_S(x) = G_{S'}(x)$, we can combine those two terms as

This implies $S = S'$ which would then imple $H_S(y) = H_{S'}(y)$!!

$$\underbrace{G_S(x) \left[H_S(y) + H_{S'}(y)\right]}$$

This term is a polynomial in y.

Let us denote $H'(y)$ as the polynomial in y.

Now RHS becomes $\sum_{i=1}^{2^n} G_{S_i}(x) H'_i(y)$

where $G_{S_i}(x)$ is a monomial in $x \in \{0,1\}^n$ and $H'(y)$ is a polynomial in $y \in \{0,1\}^n$.

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^{2^n} G_{S_i}(x) H'_i(y)$$

Both $G_{S_i}(x)$ and $H'_i(y)$ can be evaluated in (mod 2).

Now Alice takes each $G_{S_i}(x)$ term, evaluates the (mod 2) of it and gives it as input to a ~~~~ magical non-local box.

Bob evaluates the value of $H'_i(y)$ in (mod 2) and gives it as input to this magical non-local box.

The non-local box outputs a_i to Alice and b_i to Bob such that

$$a_i \oplus b_i = (G_{s_i}(x) \cdot H_i'(y)) \bmod 2.$$

So now.

$$f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \left(\sum_{i=1}^{2^n} G_{s_i}(x)\, H_i'(y)\right) \pmod 2$$

$$= \sum_{i=1}^{2^n} (a_i \oplus b_i) \pmod 2$$

Alice computes the value of $\sum_{i=1}^{2^n} a_i \pmod 2$ by XORing all the a_i's and sends this classical bit to Bob.

Now, Bob also computes the value of $\left(\sum_{i=1}^{2^n} b_i\right) \pmod 2$ by XORing all the b_i's and XORs this value with the classical bit sent by Alice to evaluate the value of $f(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Similarly, Alice can know the value of $f(x_1, \ldots, x_n, y_1, \ldots, y_n)$ once she receives a classical bit from Bob which is the XOR of all b_i's. She can use this bit to XOR with the already computed value of XOR of all a_i's to learn the value of $f(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

Therefore using two classical bits of communication both of them can learn the value of $f(x_1, \ldots, x_n, y_1, \ldots, y_n)$.

The number of non-local boxes used in this protocol is 2^n.

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5). Alice has prepared a two-qubit entangled state $|\phi\rangle = \alpha|00\rangle + \beta|11\rangle$.

Let Alice and Bob share an entangled Bell state $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

Alice wants to ~~send~~ teleport the second qubit of $|\phi\rangle$ to Bob.

Let the first qubit of the entangled pair shared between Alice and Bob correspond to that of Alice i.e., $\frac{1}{\sqrt{2}}|0_A 0_B\rangle + \frac{1}{\sqrt{2}}|1_A 1_B\rangle$

Initial state: $\left(\alpha|00\rangle + \beta|11\rangle\right) \otimes \frac{1}{\sqrt{2}}\left(|0_A 0_B\rangle + |1_A 1_B\rangle\right)$

$$= \frac{\alpha}{\sqrt{2}}|0000\rangle + \frac{\alpha}{\sqrt{2}}|0011\rangle + \frac{\beta}{\sqrt{2}}|1100\rangle + \frac{\beta}{\sqrt{2}}|1111\rangle.$$

~~Here, 1st qubit belongs to Alice (the entangled sto~~

Here, the first two qubits belong to Alice of the entangled state she created.

Third qubit is the Alice's qubit of the EPR pair shared between Alice and Bob.

Fourth qubit is Bob's qubit of the EPR pair shared between Alice and Bob.

Suppose that Alice wants to measure second and third qubit in the Bell basis.

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \qquad |\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \qquad |\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

$|00\rangle = \frac{|\phi^+\rangle + |\phi^-\rangle}{\sqrt{2}}$ $\quad |11\rangle = \frac{|\phi^+\rangle - |\phi^-\rangle}{\sqrt{2}}$

$|01\rangle = \frac{|\psi^+\rangle + |\psi^-\rangle}{\sqrt{2}}$ $\quad |10\rangle = \frac{|\psi^+\rangle - |\psi^-\rangle}{\sqrt{2}}$.

State: $\frac{\alpha}{\sqrt{2}}\left(|0\rangle \otimes \left(\frac{|\phi^+\rangle + |\phi^-\rangle}{\sqrt{2}}\right) \otimes |0\rangle\right) + \frac{\alpha}{\sqrt{2}}\left(|0\rangle \otimes \left(\frac{|\psi^+\rangle + |\psi^-\rangle}{\sqrt{2}}\right) \otimes |1\rangle\right)$

$+ \frac{\beta}{\sqrt{2}}\left(|1\rangle \otimes \left(\frac{|\psi^+\rangle - |\psi^-\rangle}{\sqrt{2}}\right) \otimes |0\rangle\right) + \frac{\beta}{\sqrt{2}}\left(|1\rangle \otimes \left(\frac{|\phi^+\rangle - |\phi^-\rangle}{\sqrt{2}}\right) \otimes |1\rangle\right)$

$= \frac{\alpha}{2\cancel{\sqrt{2}}}(|0\phi^+0\rangle) + \frac{\alpha}{2\cancel{\sqrt{2}}}|0\phi^-0\rangle + \frac{\alpha}{2\cancel{\sqrt{2}}}|0\psi^+1\rangle + \frac{\alpha}{2\cancel{\sqrt{2}}}|0\psi^-1\rangle$

$+ \frac{\beta}{2\cancel{\sqrt{2}}}|1\psi^+0\rangle - \frac{\beta}{2\cancel{\sqrt{2}}}|1\psi^-0\rangle + \frac{\beta}{2\cancel{\sqrt{2}}}|1\phi^+1\rangle - \frac{\beta}{2\cancel{\sqrt{2}}}|1\phi^-1\rangle$.

Outcome	Probability
$\lvert\phi^+\rangle$	$\frac{\alpha^2}{4} + \frac{\beta^2}{4} = \frac{1}{4}$
$\lvert\phi^-\rangle$	$\frac{\alpha^2}{4} + \frac{(-\beta)^2}{4} = \frac{1}{4}$
$\lvert\psi^+\rangle$	$\frac{\alpha^2}{4} + \frac{\beta^2}{4} = \frac{1}{4}$
$\lvert\psi^-\rangle$	$\frac{\alpha^2}{4} + \frac{(-\beta)^2}{4} = \frac{1}{4}$

If Alice says $|\phi^+\rangle$, then Bob's qubit is in the

entangled $\alpha|00\rangle + \beta|11\rangle$

state $\alpha|0\rangle + \beta|1\rangle = \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$

first qubit (Alice) fourth qubit (Bob)

If Alice says $|\phi^-\rangle$, then Bob's qubit is in the

entangled $\alpha|00\rangle - \beta|11\rangle$

state $\cancel{\alpha|0\rangle - \beta|1\rangle} = \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$

Bob's qubit

Applying Z gate $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ to $\cancel{\begin{bmatrix} \alpha \\ -\beta \end{bmatrix}}$ gives $\cancel{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}$.

$\alpha|00\rangle + \beta|11\rangle$

If Alice says $|\psi^+\rangle$, the Bob's qubit is in the

$\beta|10\rangle + \alpha|01\rangle$

state $\beta|0\rangle + \alpha|1\rangle = \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$

Bob's qubit

Applying NOT gate (X) $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ to $\cancel{\begin{bmatrix} \beta \\ \alpha \end{bmatrix}}$ gives $\cancel{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}$.

$\alpha|00\rangle + \beta|11\rangle$

Similar error's here

If Alice says $|\psi^-\rangle$, Bob's qubit is in the state $-\beta|0\rangle + \alpha|1\rangle = \begin{bmatrix} -\beta \\ \alpha \end{bmatrix}$

Applying ZX gate $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ to $\begin{bmatrix} -\beta \\ \alpha \end{bmatrix}$ gives $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$.

After performing the suitable unitary operation depending on what outcome Alice has sent, we can see that the first qubit and Bob's qubit now form $|\phi\rangle$, which is the initial two-qubit entangled state prepared by Alice.

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Now, follow similar procedure with ~~Carol~~ Charlie

Let Alice and ~~Carol~~ Charlie share an entangled Bell state (EPR pair) $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$.

Here measure the first and third ~~bi~~ qubits in the Bell basis for the initial state

$$\frac{\alpha}{\sqrt{2}}|0000\rangle + \frac{\alpha}{\sqrt{2}}|0011\rangle + \frac{\beta}{\sqrt{2}}|1100\rangle + \frac{\beta}{\sqrt{2}}|1111\rangle.$$

Report the outcome to ~~Carol~~ Charlie.

If Alice ~~s~~ gives $|\phi^+\rangle$, ~~Carol's~~ Charlie's qubit remains same.

If Alice gives $|\phi^-\rangle$, ~~Carol~~ Charlie applies Z gate on his qubit

If Alice gives $|\psi^+\rangle$, ~~Carol~~ Charlie applies X gate on his qubit.

If Alice gives $|\psi^-\rangle$, ~~Carol~~ Charlie applies ZX gate on his qubit.

After performing suitable operation, we can see that ~~Carol's~~ Charlie's qubit and Bob's qubit now form $|\phi\rangle$ entangled state.

Despite never physically interacting, Bob and Charlie now hold halves of the entangled state $|\phi\rangle$.

6) $|\psi\rangle = H \otimes H \left(\frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle \right)$

ⓐ $H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$H^2 = I.$

<u>If Alice = T , Bob T</u>

$|\psi\rangle = H \otimes H \left(\frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle \right)$

$= \frac{1}{\sqrt{3}} \left(\left(\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \right) + \left(\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \right) + \left(\left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \right) \right)$

$= \frac{1}{2\sqrt{3}} \left(|00\rangle + \cancel{|01\rangle} + \cancel{|10\rangle} + \cancel{|11\rangle} + |00\rangle - \cancel{|01\rangle} + |10\rangle - \cancel{|11\rangle} + |00\rangle + |01\rangle - \cancel{|10\rangle} - |11\rangle \right)$

$= \frac{1}{2\sqrt{3}} \left(3|00\rangle + |10\rangle + |01\rangle - |11\rangle \right)$

It is possible for A & B to measure 1 and 1 respectively with probability $\frac{1}{12}$.

<u>If Alice = T, Bob = H</u>

$|\psi\rangle = H \otimes I \left(\frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle \right)$

$= \frac{1}{\sqrt{3}} \left(\left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \otimes |0\rangle + \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}} \right) \otimes |1\rangle + \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}} \right) \otimes |0\rangle \right)$

$= \frac{1}{\sqrt{6}} \left(|00\rangle + \cancel{|10\rangle} + |01\rangle + |11\rangle + |00\rangle - \cancel{|10\rangle} \right)$

$= \frac{1}{\sqrt{6}} \left(2|00\rangle + |01\rangle + |11\rangle \right)$

There is no state ~~in which~~ of $|10\rangle$ in this overlapping.

Probability = 0

Therefore it is impossible for A & B to measure 1, 0 respectively.

~~If A = H, B~~

If Alice = H, Bob = T

$$|\psi\rangle = (I \otimes H)\left(\frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle\right)$$

$$= \frac{1}{\sqrt{3}}\left(|0\rangle \otimes \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right) + |0\rangle \otimes \left(\frac{|0\rangle-|1\rangle}{\sqrt{2}}\right) + |1\rangle \otimes \left(\frac{|0\rangle+|1\rangle}{\sqrt{2}}\right)\right)$$

$$= \frac{1}{\sqrt{6}}\left(|00\rangle + \cancel{|01\rangle} + |00\rangle - \cancel{|01\rangle} + |10\rangle + |11\rangle\right)$$

$$= \frac{1}{\sqrt{6}}\left(2|00\rangle + |10\rangle + |11\rangle\right)$$

There is no state of $|01\rangle$ in this overlapping.

Probability = 0

Therefore it is impossible for A & B to measure 0 and 1 respectively.

If Alice = ~~T~~ H, Bob = H

$$|\psi\rangle = (I \otimes I)\left(\frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle\right)$$

$$= \frac{1}{\sqrt{3}}|00\rangle + \frac{1}{\sqrt{3}}|01\rangle + \frac{1}{\sqrt{3}}|10\rangle.$$

There is no state of $|11\rangle$ in this overlapping.

Probability = 0

Therefore it is impossible for A & B to measure 1 and 1 respectively.

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