## 7.2 Canonical Form

The *canonical realization form*, otherwise known as *direct form II*, can be obtained from the direct form in the following way. Starting with the second-order filter of Eq. (7.1.1) we may group the five terms in the right-hand side of Eq. (7.1.2) into two subgroups: the recursive terms and the non-recursive ones, that is,

$$y_n = (b_0 x_n + b_1 x_{n-1} + b_2 x_{n-2}) + (-a_1 y_{n-1} - a_2 y_{n-2})$$

This regrouping corresponds to splitting the big adder of the direct form realization of Fig. 7.1.1 into two parts, as shown in Fig. 7.2.1.

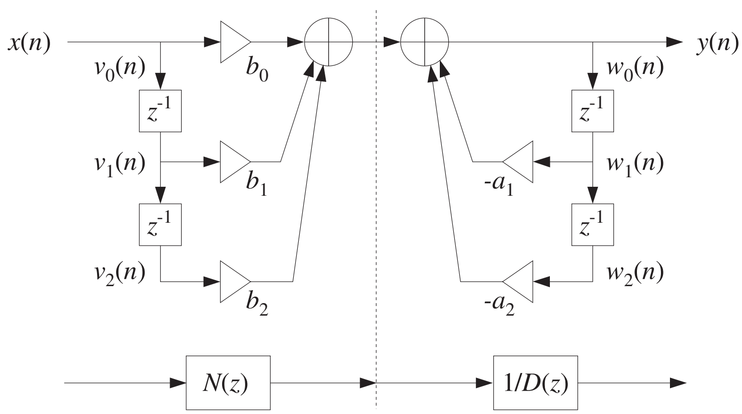


**Fig. 7.2.1** Regrouping of direct form terms.

We can think of the resulting realization as the cascade of two filters: one consisting only of the feed-forward terms and the other of the feedback terms. It is easy to verify that these two filters are the numerator $N(z)$ and the inverse of the denominator $1/D(z)$, so that their cascade will be

$$H(z) = N(z) \cdot \frac{1}{D(z)}$$

which is the original transfer function given in Eq. (7.1.1). Mathematically, the order of the cascade factors can be changed so that

$$H(z) = \frac{1}{D(z)} \cdot N(z)$$

which corresponds to changing the order of the block diagrams representing the factors $N(z)$ and $1/D(z)$, as shown in Fig. 7.2.2.

The output signal of the first filter $1/D(z)$ becomes the input to the second filter $N(z)$. If we denote that signal by $w(n)$, we observe that it gets delayed in the same way by the two sets of delays of the two filters, that is, the two sets of delays have the *same contents*, namely, the numbers $w(n-1)$, $w(n-2)$.

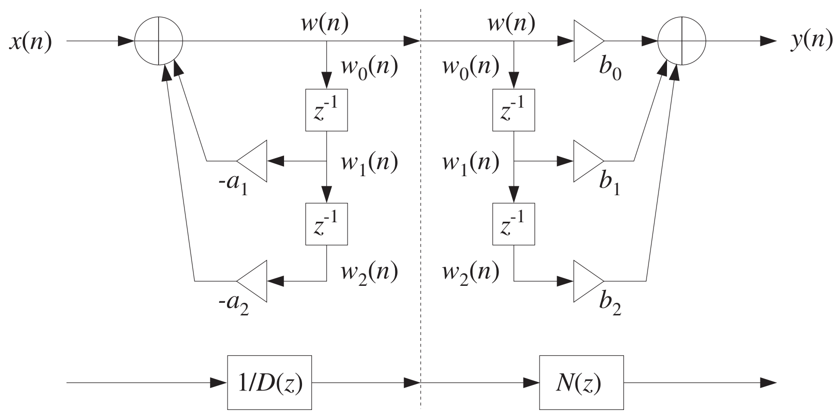


**Fig. 7.2.2** Interchanging $N(z)$ and $1/D(z)$.

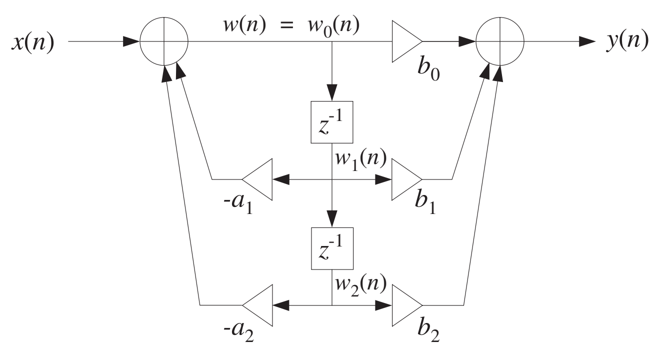


**Fig. 7.2.3** Canonical realization form of second-order IIR filter.

Therefore, there is no need to use two separate sets of delays. The two sets can be merged into one, shared by both the first and second filters $1/D(z)$ and $N(z)$. This leads to the *canonical realization form* depicted in Fig. 7.2.3.

The I/O difference equations describing the time-domain operation of this realization can be obtained by writing the conservation equations at each adder, with the input adder written first:

$$\boxed{\begin{aligned} w(n) &= x(n) - a_1 w(n-1) - a_2 w(n-2) \\ y(n) &= b_0 w(n) + b_1 w(n-1) + b_2 w(n-2) \end{aligned}} \tag{7.2.1}$$

The computed value of $w(n)$ from the first equation is passed into the second to compute the final output $y(n)$. It is instructive also to look at this system in the $z$-domain. Taking $z$-transforms of both sides, we find

$$\begin{aligned} W(z) &= X(z) - a_1 z^{-1} W(z) - a_2 z^{-2} W(z) \\ Y(z) &= b_0 W(z) + b_1 z^{-1} W(z) + b_2 z^{-2} W(z) \end{aligned}$$

which can be solved for $W(z)$ and $Y(z)$:

$$
\begin{aligned}
W(z) &= \frac{1}{1+a_1z^{-1}+a_2z^{-2}}X(z) = \frac{1}{D(z)}X(z) \\
Y(z) &= (b_0+b_1z^{-1}+b_2z^{-2})W(z) = N(z)W(z)
\end{aligned}
$$

Eliminating $W(z)$, we find that the transfer function from $X(z)$ to $Y(z)$ is the original one, namely, $N(z)/D(z)$:

$$Y(z) = N(z)W(z) = N(z)\frac{1}{D(z)}X(z) = \frac{N(z)}{D(z)}X(z)$$

At each time $n$, the quantities $w(n-1)$ and $w(n-2)$ in Eq. (7.2.1) are the contents of the two shared delay registers. Therefore, they are the internal states of the filter. To determine the corresponding sample processing algorithm, we redefine these internal states by:

$$
\begin{aligned}
w_0(n) &= w(n) \\
w_1(n) &= w(n-1) = w_0(n-1) \\
w_2(n) &= w(n-2) = w_1(n-1)
\end{aligned}
\quad\Rightarrow\quad
\begin{aligned}
w_1(n+1) &= w_0(n) \\
w_2(n+1) &= w_1(n)
\end{aligned}
$$

Therefore, the system (7.2.1) can be rewritten as:

$$
\boxed{
\begin{aligned}
&w_0(n) = x(n) - a_1w_1(n) - a_2w_2(n) \\
&y(n) = b_0w_0(n) + b_1w_1(n) + b_2w_2(n) \\
&w_2(n+1) = w_1(n) \\
&w_1(n+2) = w_0(n)
\end{aligned}
}
$$

which translates to the following sample processing algorithm:

$$
\boxed{
\begin{aligned}
&\textit{for each input sample } x \textit{ do:} \\
&\qquad w_0 = x - a_1w_1 - a_2w_2 \\
&\qquad y = b_0w_0 + b_1w_1 + b_2w_2 \\
&\qquad w_2 = w_1 \\
&\qquad w_1 = w_0
\end{aligned}
}
\tag{7.2.2}
$$

where, again, the states $w_2$ and $w_1$ must be updated in reverse order. The canonical form for the more general case of Eq. (7.1.4) is obtained following similar steps. That is, we define

$$Y(z) = N(z)W(z) \quad \text{and} \quad W(z) = \frac{1}{D(z)}X(z)$$

and rewrite them in the form:

$$
\begin{aligned}
&(1+a_1z^{-1}+a_2z^{-2}+\cdots+a_Mz^{-M})W(z) = X(z) \\
&Y(z) = (b_0+b_1z^{-1}+\cdots+b_Lz^{-L})W(z)
\end{aligned}
$$

or, equivalently

$$W(z)=X(z)-(a_1z^{-1}+a_2z^{-2}+\cdots+a_Mz^{-M})W(z)$$
$$Y(z)=(b_0+b_1z^{-1}+\cdots+b_Lz^{-L})W(z)$$

which become in the time domain:

$$\boxed{\begin{aligned} w(n) &= x(n)-a_1w(n-1)-\cdots-a_Mw(n-M) \\ y(n) &= b_0w(n)+b_1w(n-1)+\cdots+b_Lw(n-L) \end{aligned}} \tag{7.2.3}$$

The block diagram realization of this system is shown in Fig. 7.2.4 for the case $M = L$. If $M \neq L$ one must draw the *maximum* number of common delays, that is, $K = \max(M, L)$. Defining the internal states by

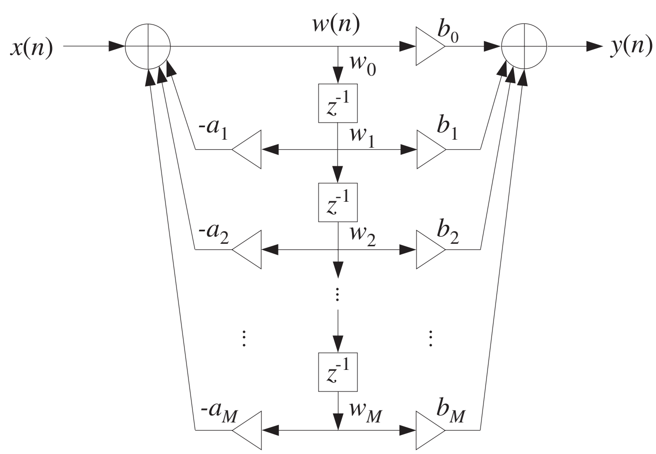


**Fig. 7.2.4** Canonical realization form of $M$th order IIR filter.

$$w_i(n)=w(n-i)=w_{i-1}(n-1),\quad i=0,1,\dots,K$$

we may rewrite the system (7.2.3) in the form:

$$\boxed{\begin{aligned} & w_0(n)=x(n)-a_1w_1(n)-\cdots-a_Mw_M(n) \\ & y(n)=b_0w_0(n)+b_1w_1(n)+\cdots+b_Lw_L(n) \\ & w_i(n+1)=w_{i-1}(n),\quad i=K,K-1,\dots,1 \end{aligned}} \tag{7.2.4}$$

This leads to the following sample-by-sample filtering algorithm:

$$\boxed{\begin{aligned} & \textit{for each input sample } x \textit{ do:} \\ & \qquad w_0=x-a_1w_1-a_2w_2-\cdots-a_Mw_M \\ & \qquad y=b_0w_0+b_1w_1+\cdots+b_Lw_L \\ & \qquad w_i=w_{i-1},\quad i=K,K-1,\dots,1 \end{aligned}} \tag{7.2.5}$$