Deep-Learning Solution to Portfolio Selection with Serially-Dependent Returns

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Outline

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Introduction

Practical portfolio selection has to overcome the challenges arising from high-dimensionality, flexible modeling for risky asset returns and investment constraints. The literature has great success in developing theoretical results for models with serially-independent asset returns (Li and Ng, 2000).

Introduction

Tsang and Wong (2020) focuses on two classes of well-known time series models: autoregressive (AR) and the generalized autoregressive conditional heteroskedasticity (GARCH) models. The former describes serial-dependence through the drift terms of risky asset returns while the latter through the innovation terms.

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Problem Formulation

MDP

Tsang and Wong (2020) claimed that the classic Markov decision process (MDP) can be defined as follows in a filtered probability space $\left(\Omega, P, \{\mathcal{F}_k\}_{k\geq 0}\right)$, where k belongs to the integer set:

$$\begin{aligned} V_0(s_0) &= \min_{g_0,\ldots,g_{K-1}\in\mathcal{U}(\cdot)} \mathbb{E}\left[\Psi(s_K) \mid \mathcal{F}_0\right], \\ \text{s.t.} \quad s_{k+1} &= h\left(s_k, g_k(s_k), \eta_k\right), k = 0, 1, \ldots, K-1, \end{aligned} \tag{1}$$

where

$h(\cdot,\cdot,\cdot) : \mathcal{S} \times D \times \Omega \to \mathcal{S}$ is the transition function;
$\Psi(\cdot) : \mathcal{S} \to \mathbb{R}$ is the objective function;
The $\mathcal{F}_k$-measurable s_k is defined in the Markovian state space $\mathcal{S}$;
$\eta_k \in \Omega$ is a $\mathcal{F}_{k+1}$-measurable random variable;
$g_k : \mathcal{S} \to D$ is a feedback control function;
$\mathcal{U}(\cdot)$ is the set of admissible control functions satisfying some regularity conditions.

Portfolio Optimization

EUM

$$\begin{aligned} V_0(\cdot) = & \max_{\{g_k\}_{k=0}^{K-1}} E\left[U\left(W_K\right) \mid \mathcal{F}_0\right], \\ \text{s.t.} \quad & W_{k+1} = W_k\left(g_k^T R_k + R_f\right), k = 0, 1, \ldots, K-1, \end{aligned} \tag{2}$$

where

$U(\cdot) : \mathbb{R} \to \mathbb{R}$ is the utility function on the terminal wealth;
The $\mathcal{F}_k$-measurable $g_k \in D \subseteq \mathbb{R}^p$ is the investment policy on p risky assets at time k while D is the constrained investment opportunity set;
$R_k \in \mathbb{R}^p$ is the excess returns of risky assets between the time points k and $k+1$;
R_f is the constant risk-free return for each period.

When $\{R_k\}$ is serially-independent, W_k represents the state vector in (1).
When $\{R_k\}$ has serially-dependence, W_k is no longer Markovian in general and the state vector s_k in (1) contains elements beyond W_k.
In both cases, we still have $\Psi\left(s_K\right) = -U\left(W_K\right)$.

Portfolio Optimization

Tsang and Wong (2020) aims to solve the EUM problem (2) with bounding constraints and serially-dependent returns. For purposes of more clearly illustrating their results, Tsang and Wong (2020) focus on the quadratic utility function $U(W_K) = -\left(W_K - \frac{\gamma}{2}\right)^2$ with $\mathrm{AR}(1)$ and/or the $\mathrm{CCC\text{-}GARCH}(1,1)$ models for the asset returns.

Portfolio Optimization

AR(1)

$$R_k = \alpha + AR_{k-1} + \epsilon_k, k = 0, 1, \ldots, K-1 \tag{3}$$

where $\alpha \in \mathbb{R}^p, A \in \mathbb{R}^{p\times p}$ and ϵ_k is identically and independently distributed (i.i.d.) p-dimensional random variables.

CCC-GARCH(1, 1)

$$\begin{aligned} R_{k,i} &= \mu_i + z_{k,i}, \\ z_{k,i} &= \Sigma_{k,i}\epsilon_{k,i}, \\ \Sigma^2_{k,i} &= \omega_i + \alpha_i \Sigma^2_{k-1,i} + \beta_i z^2_{k-1,i}, i = 1, \ldots, p, k = 0, 1, \ldots, K-1, \end{aligned} \tag{4}$$

where

$\omega_i, \alpha_i, \beta_i \geq 0$ for $i = 1, \ldots, p$. $\mu = (\mu_1, \ldots, \mu_p) \in \mathbb{R}^p, z_k = (z_{k,1}, \ldots, z_{k,p}) \in \mathbb{R}^p$ and $\Sigma^2_k = \left(\Sigma^2_{k,1}, \ldots, \Sigma^2_{k,p}\right)$.$\epsilon_k = (\epsilon_{k,1}, \ldots, \epsilon_{k,p})$ are $i.i.d.$ random variables with constant correlation matrix Γ which is positive definite and satisfies $\Gamma_{ii} = 1, \Gamma_{ij} = \Gamma_{ji} = \rho_{ij}$ and $\rho_{ij} \in [-1, 1]$.

Portfolio Optimization

AR(1) MDP

$$\begin{aligned} V_0(s_0) &= \min_{g_0,\ldots,g_{K-1}\in\mathcal{U}(\cdot)} \mathbb{E}\left[-U(W_K) \mid \mathcal{F}_0\right] \\ \text{s.t.} \quad s_{k+1} &= \left(W_k\left(g_k(s_k)^T R_k + R_f\right), R_k\right), k = 0, 1, \ldots, K-1, \end{aligned} \tag{5}$$

where

$s_k \equiv (W_k, R_{k-1})$; $\eta_k \equiv \epsilon_k$; $R_k = \alpha + AR_{k-1} + \eta_k$.

CCC-GARCH(1,1) MDP

$$\begin{aligned} V_0(s_0) &= \min_{g_0,\ldots,g_{K-1}\in\mathcal{U}(\cdot)} \mathbb{E}\left[-U(W_K) \mid \mathcal{F}_0\right] \\ \text{s.t.} \quad s_{k+1} &= \left(W_k\left(g_k(s_k)^T R_k + R_f\right), \Sigma^2_{k+1}\right), k = 0, 1, \ldots, K-1, \end{aligned} \tag{6}$$

where

$s_k \equiv \left(W_k, \Sigma^2_k\right)$; $\eta_k \equiv z_k$; $R_k = \mu + \eta_k$; $\Sigma^2_{k+1,i} = \omega_i + \alpha_i \Sigma^2_{k,i} + \beta_i \eta^2_{k,i}$ for $i = 1, \ldots, p$.

Basics of Neural Network

Notation of NN

The mathematical expression for a multi-layer multi-output feedforward neural network (FNN) with L hidden layers is:

$$x \in \mathbb{R}^d \to f_{L+1}(x;\theta) \in \mathbb{R}^p,$$

where θ is the parameters of the network.

Basics of Neural Network

Let $\mathcal{A}^L = \left\{ f_{L+1,\theta} : \mathbb{R}^d \to \mathbb{R}^p : \theta \in \mathbb{R}^o \right\}$ be the class of functions computed by the standard multi-layer multi-output FNN with L hidden layers. $\mathcal{A}^L$ can be represented by successive composition of linear combination and activation function $\sigma_r(\cdot)$ (Tsang and Wong, 2020):

$$
\begin{aligned}
f_0 &= x \\
f_r &= \sigma_r \left(B_{r-1} f_{r-1} + C_{r-1} \right), r = 1, \ldots, L, \\
f_{L+1} &= B_L f_L + C_L
\end{aligned}
$$

where

B_r is the matrix weight and C_r is the vector bias.

$\theta = (B_r, C_r)_{r=0,\ldots,L}$.

$f_0 \in \mathbb{R}^d$ is called the input layer and $f_{L+1} \in \mathbb{R}^p$ is called the output layer.

$f_r \in \mathbb{R}^{\nu_r}$ is called the hidden layers,

where ν_r is the number of hidden units in layer$r, r = 1, \ldots, L$.

Basics of Neural Network

Tsang and Wong (2020) assume that the activation function $\sigma_r(\cdot)$ and the number of hidden units ν_r is the same for $r = 1, \ldots, L$, i.e. $\sigma_r(\cdot) \equiv \sigma(\cdot)$; $\nu_r \equiv \nu$, to simplify their notation. They rewrite $\mathcal{A}^L$ as $\mathcal{A}_v$ to emphasize its dependence on ν.

To handle the bounded control, Tsang and Wong (2020) apply the sigmoid function to the output of $\mathcal{A}_v$. Define as the following:

$$\mathcal{G}_\nu = \left\{ f : \mathbb{R}^d \to \mathbb{R}^p : f(x) = \sigma\left(f_{L+1}(x)\right), f_{L+1} \in \mathcal{A}_\nu \right\} \tag{7}$$

where

$\sigma(x) = \frac{1}{1+e^{-x}}$ is the sigmoid function with range of $[0, 1]$, and is applied to f_{L+1} elementwisely.

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Deep-Learning Solution

Tsang and Wong (2020) introduce their deep-learning solution for the MDP problem (1) in this section. Suppose $D = [0,1]^p$, denote $g_{(K-1)} = \{g_0, \ldots, g_{K-1}\}$ as a sequence of feedback control functions. The value function for the MDP problem (1) can be expressed as follows:

$$V_0(s_0) = \min_{\phi \in \mathcal{H}^{\mathcal{U}}} \mathbb{E}[\phi(\zeta)], \tag{8}$$

where

$\zeta = (\eta_0, \ldots, \eta_{K-1})$ contains all the random variables throughout the entire process,

$$\mathcal{H}^{\mathcal{U}} = \left\{ \zeta \to \Psi\left(s_K^{g_{(K-1)}}(\zeta)\right) : \{s_k\}_{k=0}^K \text{ subject to } (1), g_0, \ldots, g_{K-1} \in \mathcal{U}(\cdot) \right\} \tag{9}$$

denotes the class of controlled process with admissible control function in $\mathcal{U}(\cdot)$. The notation $s_K^{g_{(K-1)}}$ is used to emphasize that the state process adopts the control $g_{(K-1)}$.

Deep-Learning Solution

Assume that Tsang and Wong (2020) restrict their minimization over the feedback control policy for the functions $g_k(\cdot;\theta_k) \in \mathcal{G}_\nu$. They define the following relaxed problem:

$$\begin{aligned} V_0^*(s_0) &= \min_{g_0,\ldots,g_{K-1}\in\mathcal{G}_\nu} \mathbb{E}\left[\Psi(s_K) \mid \mathcal{F}_0\right] \\ &= \min_{\theta_0,\ldots,\theta_{K-1}} \mathbb{E}\left[\Psi(s_K) \mid \mathcal{F}_0\right] \\ &= \min_{\phi\in\mathcal{H}^{\mathcal{G}_\nu}} \mathbb{E}[\phi(\zeta)], \end{aligned} \tag{10}$$

where $\mathcal{G}_\nu$ is defined in (7). Based on the above relaxed problem, the class of controlled process $\mathcal{H}^{\mathcal{G}_\nu}$ considered in (10) can be described by a deep neural network architecture as follows (Tsang and Wong, 2020):

Deep-Learning Solution

1. Denote the computational unit $\{s_k\}_{k=0}^{K}, \{g_k\}_{k=0}^{K-1}, \{\eta_k\}_{k=0}^{K-1}$ and V_K.
   s_0 and $\{\eta_k\}_{k=0}^{K-1}$ are the input units (which means in-degree of node is 0).
   V_K is the only output unit of the network
   (which also means out-degree of this node is 0).

2. Define the following connections within the graph.
   (a) For $k = 0, 1, \ldots, K-1, (s_k, g_k, \eta_k) \to s_{k+1} : s_{k+1} = h(s_k, g_k, \eta_k)$ models the transition to the next timestep.
   (b) For $k = 0, 1, \ldots, K-1, s_k \to g_k : g_k = g_k(s_k; \theta_k)$ for some function $g_k(s_k; \theta_k) \in \mathcal{G}_\nu$.
   θ_k is the parameters of the subnetwork $s_k \to g_k$ at timestep k.
   $\mathcal{G}_\nu$ may have any number of hidden layers L ≥ 1.
   (c) $s_K \to V_K : V_K = \Psi(s_K)$ denotes the terminal network loss function.

Deep-Learning Solution

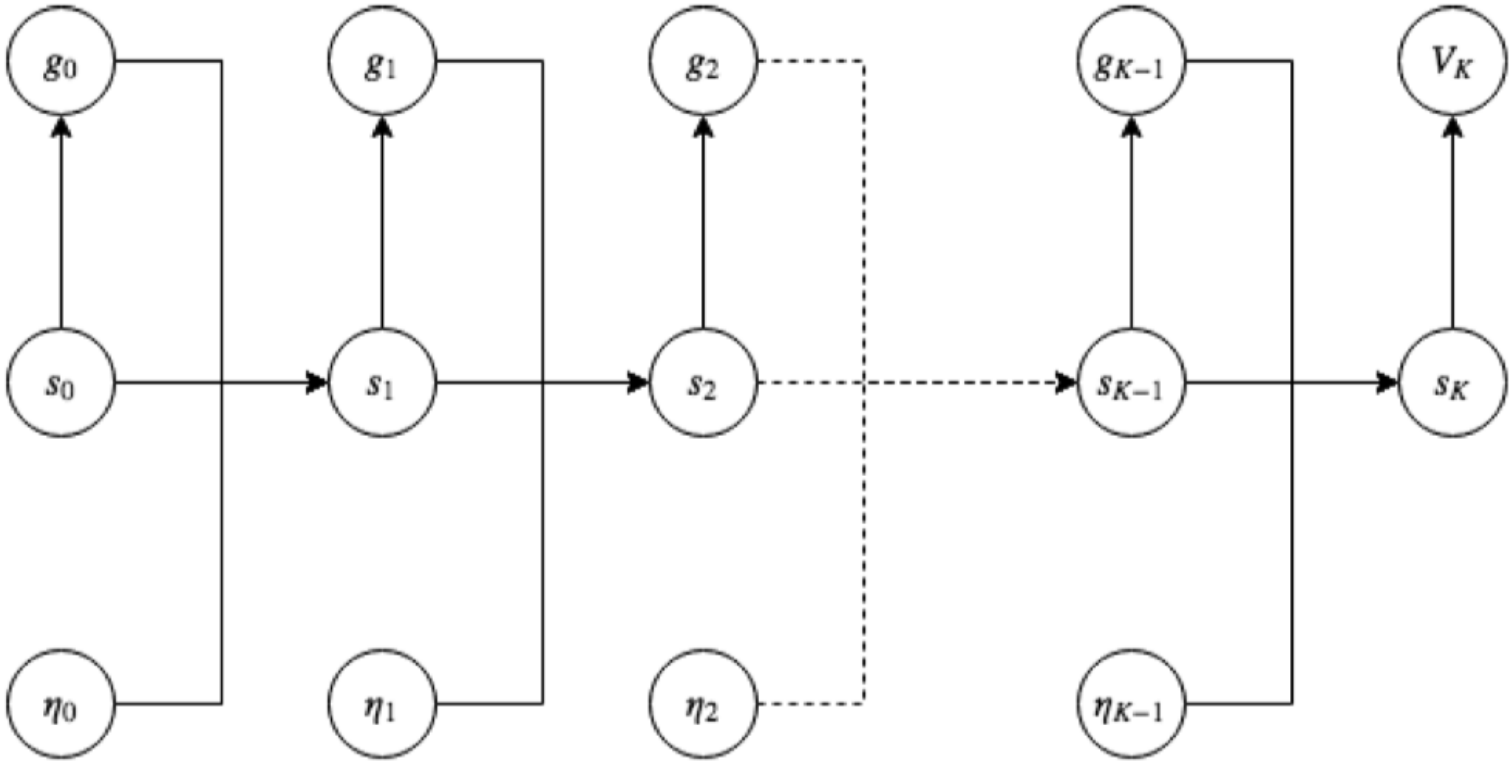

Figure 1: Graphical illustration of the proposed deep neural network (Tsang and Wong, 2020)

Deep-Learning Solution

Denote $\phi(\zeta;\Theta)$ as the output value V_K of the network with input s_0 and ζ for fixed s_0. The network training is described as, optimizing for $\hat{\Theta}$ which satisfies (Tsang and Wong, 2020),

$$n^{-1}\sum_{i=1}^{n}\phi\left(\zeta_i;\hat{\Theta}\right)=\min_{\Theta}n^{-1}\sum_{i=1}^{n}\phi\left(\zeta_i;\Theta\right) \tag{11}$$

Denote $\phi_\Theta(\zeta)=\phi(\zeta;\Theta)$. Observe that $\mathcal{H}^{\mathcal{G}_\nu}=\{\phi_\Theta(\zeta):\Theta\in\mathbb{R}^w\}$ where w is the total number of parameters. Tsang and Wong (2020) can then write $\mathcal{H}^{\mathcal{G}_\nu}$ as the class of functions computed by the DNN defined above. In addition, $\hat{\phi}\left(\zeta\right)=\phi(\zeta;\hat{\Theta})$ satisfies,

$$n^{-1}\sum_{i=1}^{n}\hat{\phi}\left(\zeta_i\right)=\min_{\phi\in\mathcal{H}^{\mathcal{G}_\nu}}n^{-1}\sum_{i=1}^{n}\phi\left(\zeta_i\right) \tag{12}$$

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Numerical Examples

In this section, the author mainly analyze and verify the convergence conclusion of AR(1) and CCC-GARCH(1,1) model where their utility function values will converge to a global minimum point.

The method to implement and get the deep-learning solution to this portfolio selection problem is Monte-Carlo simulation so that tons of data can be generated to feed into the deep-learning model, which can effectively handle high-dimensional problem according to the author. However, the procedure of generating random numbers, training models and finally plotting the capital allocation line is very time-consuming and computational expensive.

Numerical Examples - Preliminaries

I.I.D. Return Recurrence Relation

Under the i.i.d. return case, we assume that the asset prices follow the Black-Scholes model

$$\frac{dS_t}{S_t} = (r\mathbf{1} + \Sigma\lambda)\, dt + \Sigma dZ_t$$

where $r \in \mathbb{R}$, $\mathbf{1} = (1, \cdots, 1) \in \mathbb{R}^p$, $\lambda \in \mathbb{R}^p$, $\Sigma \in \mathbb{R}^{p\times p}$ and Z_t are p independent Wiener processes. With a constant rebalancing time period Δ, the risk free return $R_f = \exp(r\Delta) - 1$ and then we can derive the excess return vector R_k^e

$$S_k = \exp\left(\left(r\mathbf{1} + \Sigma\lambda - \frac{1}{2} diag\left(\Sigma\Sigma^T\right)\right)\Delta + \sqrt{\Delta}\Sigma\epsilon_k\right) S_{k-1}$$

$$\Rightarrow \quad R_k^e = \frac{S_k}{S_{k-1}} - \mathbf{1} - R_f\mathbf{1}$$

$$\Rightarrow \quad R_k^e = \exp\left(\left(r\mathbf{1} + \Sigma\lambda - \frac{1}{2} diag\left(\Sigma\Sigma^T\right)\right)\Delta + \sqrt{\Delta}\Sigma\epsilon_k\right) - \exp(r\Delta)\,\mathbf{1}$$

where $\epsilon_k \sim N_p(\mathbf{0}, I_p)$ are i.i.d. multivariate normal, I_p is the identity matrix. $diag(A) = (A_{11}, \cdots, A_{pp})$ is the diagonal elements vector of square matrix A.

Numerical Examples - Preliminaries

Single-period Models

In order to compare the performance of single-period model and multi-period model in the following sections, we have to figure out a way to derive the efficient frontier (capital allocation line) for the single-period case which is not clearly explained by the author.

We can use quadratic programming to solve the minimum-variance portfolio (MVP) problem with the box constraints of the portfolio weights given the expected return μ^*, and write it using the notation in this paper

$$\begin{aligned} &\min_{g_0} g_0^T \Sigma_{R_0^e} g_0 \\ \text{s.t.} \quad &g_0^T R_0^e = \mu^* - R_f \\ &g_0 \in [l, u]^p \end{aligned}$$

Numerical Examples - Preliminaries

Single-period Models

Alternatively, if there are no constraints on the weights of the portfolio, we can just use the conclusion in the textbook (Lai and Xing, 2008).

$$\min_{g_0} g_0^T \Sigma_{R_0^e} g_0 \quad \text{s.t.} \quad g_0^T \boldsymbol{\mu_0} + \left(1 - g_0^T \mathbf{1}\right) R_f = \mu^*$$

$$\Rightarrow \quad \min_{g_0} g_0^T \Sigma_{R_0^e} g_0 \quad \text{s.t.} \quad g_0^T R_0^e + R_f = \mu^*$$

$$g_0^* = \frac{(\mu^* - R_f)}{(\boldsymbol{\mu_0} - R_f \mathbf{1})^T \Sigma_{R_0^e}^{-1} (\boldsymbol{\mu_0} - R_f \mathbf{1})} \Sigma_{R_0^e}^{-1} (\boldsymbol{\mu_0} - R_f \mathbf{1})$$

$$\Rightarrow \quad g_0^* = \frac{(\mu^* - R_f) \Sigma_{R_0^e}^{-1} R_0^e}{R_0^{eT} \Sigma_{R_0^e}^{-1}}$$

Numerical Examples - Preliminaries

Single-period Models

Using these formulas, terminal wealth for the single-period case without constraints can be derived

$$W_{K+1} = W_0 \prod_{k=0}^{K} \left[(\mu^* - R_f) \frac{R_k^{eT} \Sigma_{R_0^e}^{-1} R_0^e}{R_0^{eT} \Sigma_{R_0^e}^{-1} R_0^e} + R_f + 1 \right]$$

and as well as the formula for single-period case with constraints

$$W_{K+1} = W_0 \prod_{k=0}^{K} \left(g_0^{*T} R_k^e + R_f + 1 \right)$$

In this way, mean and variance of the terminal wealth using simulated samples can be calculated.

Numerical Examples - Preliminaries

Multi-period Models

Multi-period portfolio optimization problems can be embedded into the MDP problem of expected utility maximization (EUM) in discrete-time framework.

Quadratic utility function will be used for analysis in this section like the author. And to simulate risky assets return paths for different multi-period models, which is one of the very important parts, the recurrence relation and dynamics should be paid attention to, while the objective for each problem is exactly the same

$$V_0(\cdot) = \min_{\{g_K\}_{K=0}^{K-1}} E\left[\left(W_K - \frac{\gamma}{2}\right)^2 | \mathcal{F}_0\right]$$
$$\text{s.t.} \quad W_{k+1} = W_k\left(g_k^T R_k^e + R_f + 1\right)$$

Numerical Examples - Preliminaries

Multi-period Models - I.I.D. Return Model

$$\text{s.t.} \quad R_k^e = \exp\left(\left(r\mathbf{1} + \Sigma\lambda - \frac{1}{2}diag\left(\Sigma\Sigma^T\right)\right)\Delta + \sqrt{\Delta}\Sigma\epsilon_k\right) - \exp\left(r\Delta\right)\mathbf{1},$$
$$k = 0, 1, \cdots, K-1$$

where the $\mathcal{F}_k$-measurable $g_k \in D \subseteq \mathbb{R}^p$ is the investment policy on p risky assets at time k while D is the constrained investment opportunity set. $R_k^e \in \mathbb{R}^p$ is the excess returns of risky assets between the time points k and $k+1$. R_f is the constant risk-free return for each period. In this case, $\{R_k\}$ is serially-independent, and the state vector only consists of W_k.

Multi-period Models - AR(1) Model

$$\begin{aligned} \text{s.t.} \quad & R_k = \alpha + AR_{k-1} + \epsilon_k \\ & R_k^e = R_k - R_f, \quad k = 0, 1, \cdots, K-1 \end{aligned}$$

where $\alpha \in \mathbb{R}^p$, $A \in \mathbb{R}^{p\times p}$ and ϵ_k is identically and independently distributed p-dimensional random variables.

Numerical Examples - Preliminaries

Multi-period Models - CCC-GARCH(1,1) Model

$$
\begin{aligned}
\text{s.t.} \quad & R_{k,i} = \mu_i + z_{k,i} \\
& z_{k,i} = \sigma_{k,i}\epsilon_{k,i} \\
& \sigma^2_{k,i} = \omega_i + \alpha_i \sigma^2_{k-1,i} + \beta_i z^2_{k-1,i} \\
& R^e_k = R_k - R_f, \quad i = 1, \cdots, p, \quad k = 0, 1, \cdots, K-1
\end{aligned}
$$

where ω_i, α_i, $\beta_i \geq 0$ for $i = 1, \cdots, p$. $\mu = (\mu_1, \cdots, \mu_p) \in \mathbb{R}^p$, $z_k = (z_{k,1}, \cdots, z_{k,p}) \in \mathbb{R}^p$ and $\sigma^2_k = \left(\sigma^2_{k,1}, \cdots, \sigma^2_{k,p}\right)$. $\epsilon_k = (\epsilon_{k,1}, \cdots, \epsilon_{k,p})$ are i.i.d. random variables with constant correlation matrix Γ which is positive definite and satisfies $\Gamma_{ii} = 1$, $\Gamma_{ij} = \Gamma_{ji} = \rho_{ij}$ and $\rho_{ij} \in [-1, 1]$.

Numerical Examples - Results

We will mainly analyze three aspects of the algorithm. The first one is its convergence in I.I.D case, the second one is the model dynamics by using CCC-GARCH(1,1) model, and the last one is the capital allocation line of the portfolio (efficient frontier in original paper) in both AR(1) and CCC-GARCH(1,1) case.

Numerical Examples - Results

Coding Environment

- Windows 10
- Python 3
- Google Colab
- Keras & Tensorflow (for constructing DNN model in multi-period model)
- cvxopt (for quadratic programming in single-period model with box constraints)

In terms of setting up a neuron network, following what the author did, we use tanh activation function and $L = 2$ for each subnetwork, and the neurons each hidden layer consists of varies in different examples. The model is trained without using GPU speed-up. The training is performed with mini-batch optimization with batch size 64 by ADAM optimizer (Kingma and Ba (2014)) with default setting. The parameters are initialized using Glorot uniform initializer (Glorot and Bengio (2010)) without pre-training.

Numerical Examples - Results

Data

The original data is from Yahoo Finance. They are from 2017-11-01 to 2022-11-01 (1259 observations) of 21 stocks in S&P 500 list, which will be used in analyzing CCC-GARCH(1,1) model. And data pre-processing procedure is done to convert the stock adjusted close prices to stock returns.

And all of the returns are fitted into univariate GARCH(1,1) models to get the parameters for CCC-GARCH(1,1) model, which will be used in generating random paths of stock returns. Table[1] shows the results.

Numerical Examples - Results

Stock - i	μ_i	α_i	β_i	ω_i	$\sigma_{0,i}$
AAL	-0.00004	0.00003	0.10000	0.88000	0.02530
AAPL	0.00220	0.00001	0.10000	0.88000	0.02885
ABBV	0.00069	0.00003	0.10000	0.80000	0.01823
ABC	0.00093	0.00004	0.10000	0.80000	0.01709
ABT	0.00077	0.00003	0.10000	0.80000	0.01795
ADBE	0.00162	0.00001	0.10027	0.87973	0.02042
ADI	0.00070	0.00001	0.09003	0.87894	0.02120
ADM	0.00140	0.00003	0.20000	0.70000	0.01532
AEE	0.00065	0.00001	0.10000	0.88000	0.01691
AEP	0.00068	0.00000	0.10000	0.88000	0.01747

Estimated parameters in CCC-GARCH(1,1) model.

Numerical Examples - Results

Stock - i	μ_i	α_i	β_i	ω_i	$\sigma_{0,i}$
AES	0.00141	0.00001	0.10000	0.88000	0.02149
AFL	0.00012	0.00001	0.20000	0.78000	0.01986
AIG	0.00037	0.00001	0.20000	0.78000	0.01851
AIV	0.00077	0.00001	0.09900	0.87120	0.02282
AJG	0.00108	0.00001	0.10000	0.88000	0.02286
AMD	0.00253	0.00013	0.10000	0.80000	0.03390
AME	0.00092	0.00001	0.10000	0.88000	0.01950
AMG	-0.00010	0.00001	0.05000	0.93000	0.02292
AMGN	0.00079	0.00003	0.10000	0.80000	0.01651
AMP	0.00119	0.00001	0.10000	0.88000	0.02808
AMT	0.00077	0.00001	0.10000	0.88000	0.02269

Table 1: Estimated parameters in CCC-GARCH(1,1) model.

Numerical Examples - Results (Convergence)

Firstly, the performance of the algorithm convergence will be tested when training sample size increases with different number of neurons v. For each p, we train 2 models with different values of v. Each model is trained by 100 steps described as follows. For each step, the model is trained with 10,000 simulated training samples (batch size $= 64$ and epochs $= 1$), and then the outsample score (loss) of the model is evaluated using a set of simulated testing samples. The same set of simulated testing samples is applied to the model at all steps. And Table[2] shows the other parameters.

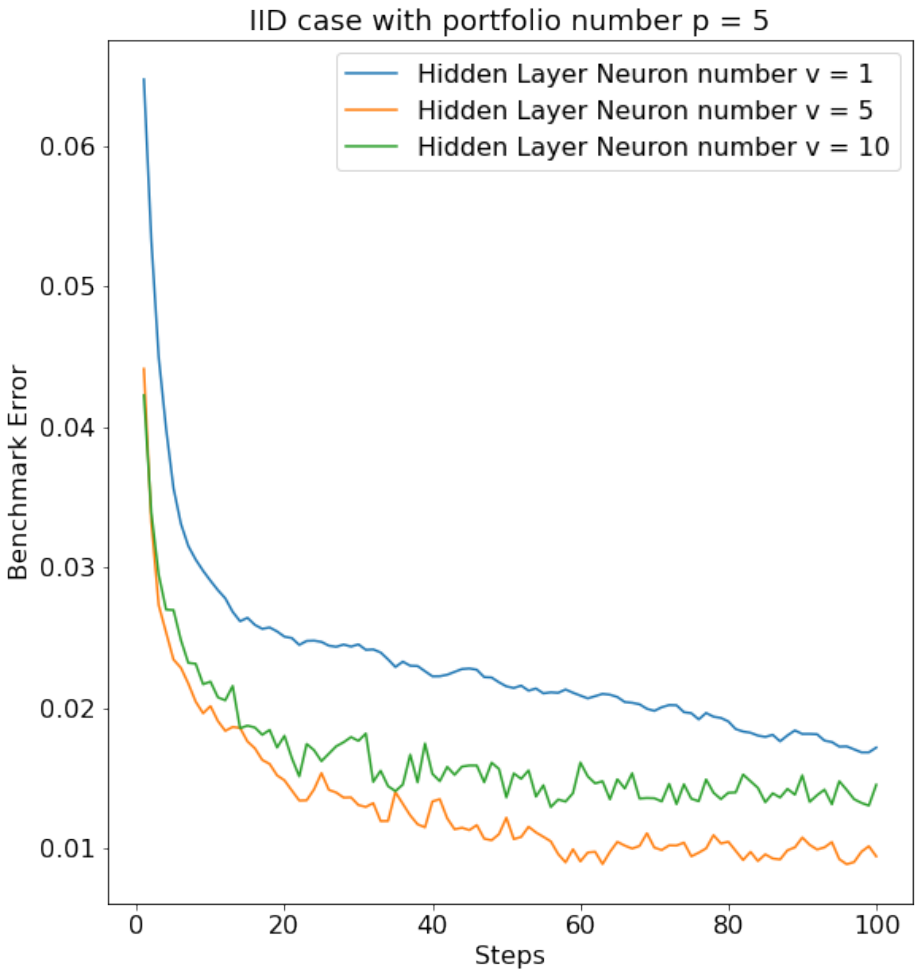
Numerical Examples - Results (Convergence)

Portfolio Number p	$p = 5$	$p = 100$
Continuous Risk-free Rate r	0.03	0.03
Rebalancing Time Δ	$1/40$	$1/30$
Multi-period Number K	40	30
Initial Wealth W_0	1	1
Control Level γ	4	6
Weights Domain D	$[0, 1.5]^p$	$[0, 0.5]^p$
Benchmark - HJB Solution V_0	0.821	3.460

Table 2: Parameters Setting for I.I.D. Return Case.

*Under $p = 5$, $\lambda_i = 0.1$ for $i = 1, 2$, $\lambda_i = 0.2$ for $i = 3, 4, 5$. $\Sigma_{ii} = 0.15$, $\Sigma_{ij} = 0.01$ for $i \neq j$. Under $p = 100$, $\lambda_i = 0.01$ for $i = 1, \cdots, 50$, $\lambda_i = 0.05$ for $i = 51, \cdots, 100$. $\Sigma_{ii} = 0.15$, $\Sigma_{ij} = 0.0025$ for $i \neq j$.

Numerical Examples - Results (Convergence)



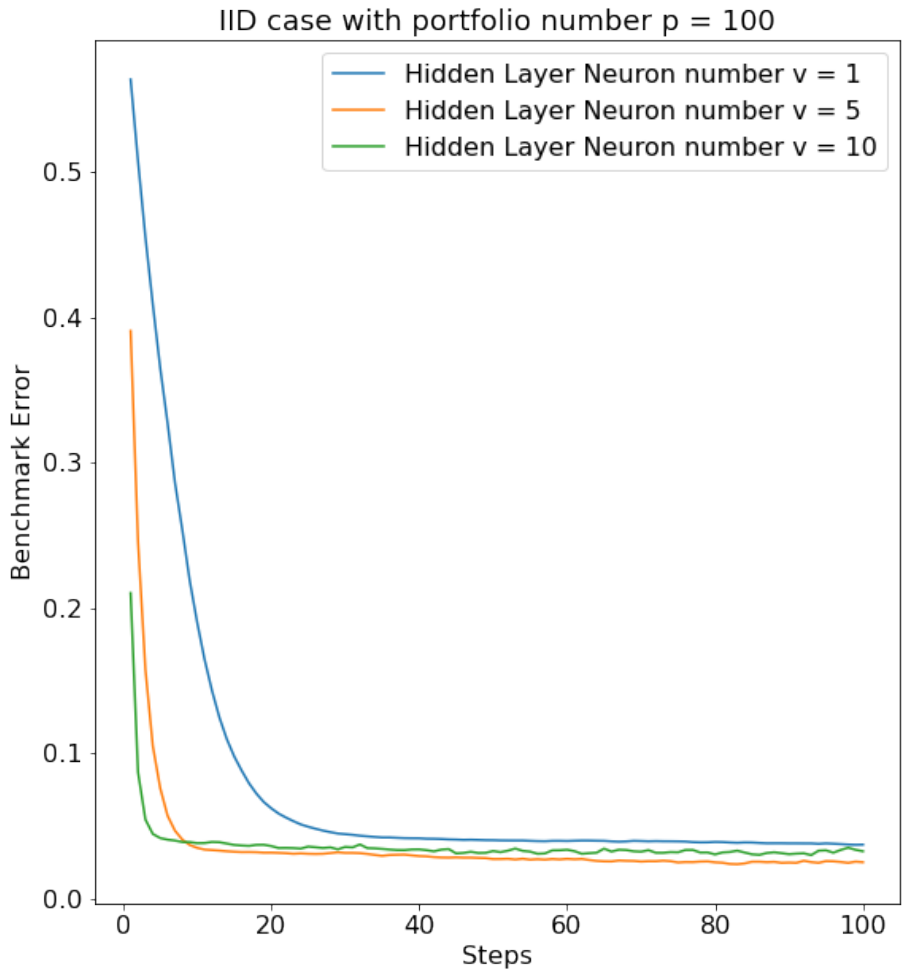


Figure 2: Convergence Analysis - IID case.

Numerical Examples - Results (Dynamics)

Model Index	1	2	3	4	5
Portfolio Number p	21				
Multi-period Number K	10				
Rebalancing Time Δ	$1/10$				
Initial Wealth W_0	1				
Continuous Risk-free Rate r	$\exp\left(\frac{0.03}{360}\right)$				
Model Type	Single-period		Multi-period		
Control Level	$\mu^* = 0.25$		$\gamma = 3$		
Weights Domain D	$[-\infty, \infty]^p$	$[0,1]^p$	$[0,1]^p$	$[-0.2,1]^p$	$[-1,1]$
Trained Samples	-		100,000		
Neuron Number v	-		50		

Table 3: Parameters Setting for Robustness checks of algorithm in CCC-GARCH(1,1) Case.

*Multi-period models may not give very accurate results since small numbers of trained samples are used.

Numerical Examples - Results (Dynamics)

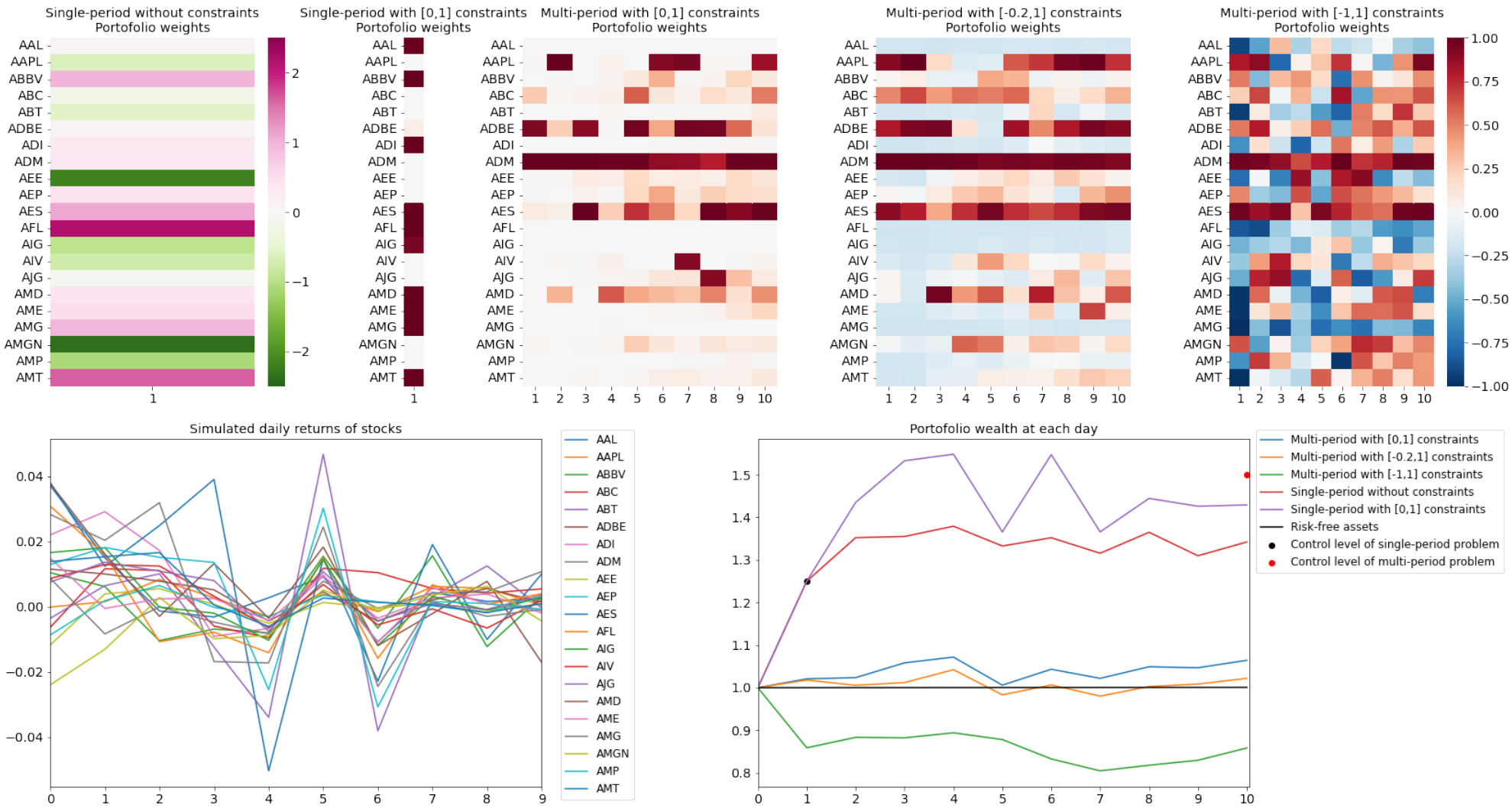


Figure 3: Robustness check of algorithm - CCC-GARCH(1,1) case.

Numerical Examples - Results (Dynamics)

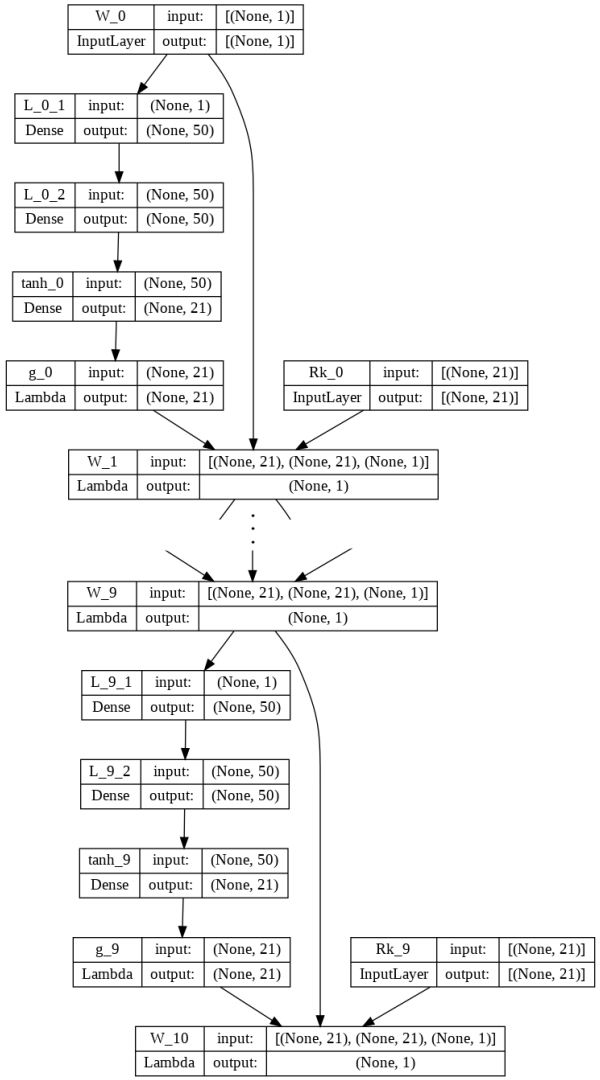


Figure 4: Graphical illustration of the deep neural network structure of CCC-GARCH(1,1) case.

Numerical Examples - Results (CAL)

Model	1	2	3
Portfolio Number p		30	
Multi-period Number K		10	
Rebalancing Time Δ		$1/10$	
Initial Wealth W_0		1	
Continuous Risk-free Rate r		0.03	
Model Type	Multi-period	Single-period	Single-period
Control Level	$\gamma \in [3, 15]$	$\mu^* \in [0.03, 0.5]$	$\mu^* \in [0.03, 1]$
Weights Domain D	$[0, 1]^p$	$[0, 1]^p$	$[-\infty, \infty]^p$
Trained Samples	1,000,000	100	100
Neuron Number v	50	-	-

Table 4: Parameters Setting for computing CAL in AR(1) Case.

*$\alpha = 0.015$ for $i = 1, \cdots, p$. $A_{ii} = -0.15$, $A_{ij} = 0$ for $i \neq j$. $\Sigma_{ii} = 0.0238$, $\Sigma_{ij} = 0.0027$ for $i \neq j$

**Both single-period and multi-period models may not give very accurate results since small numbers of trained samples are used.

Numerical Examples - Results (CAL)

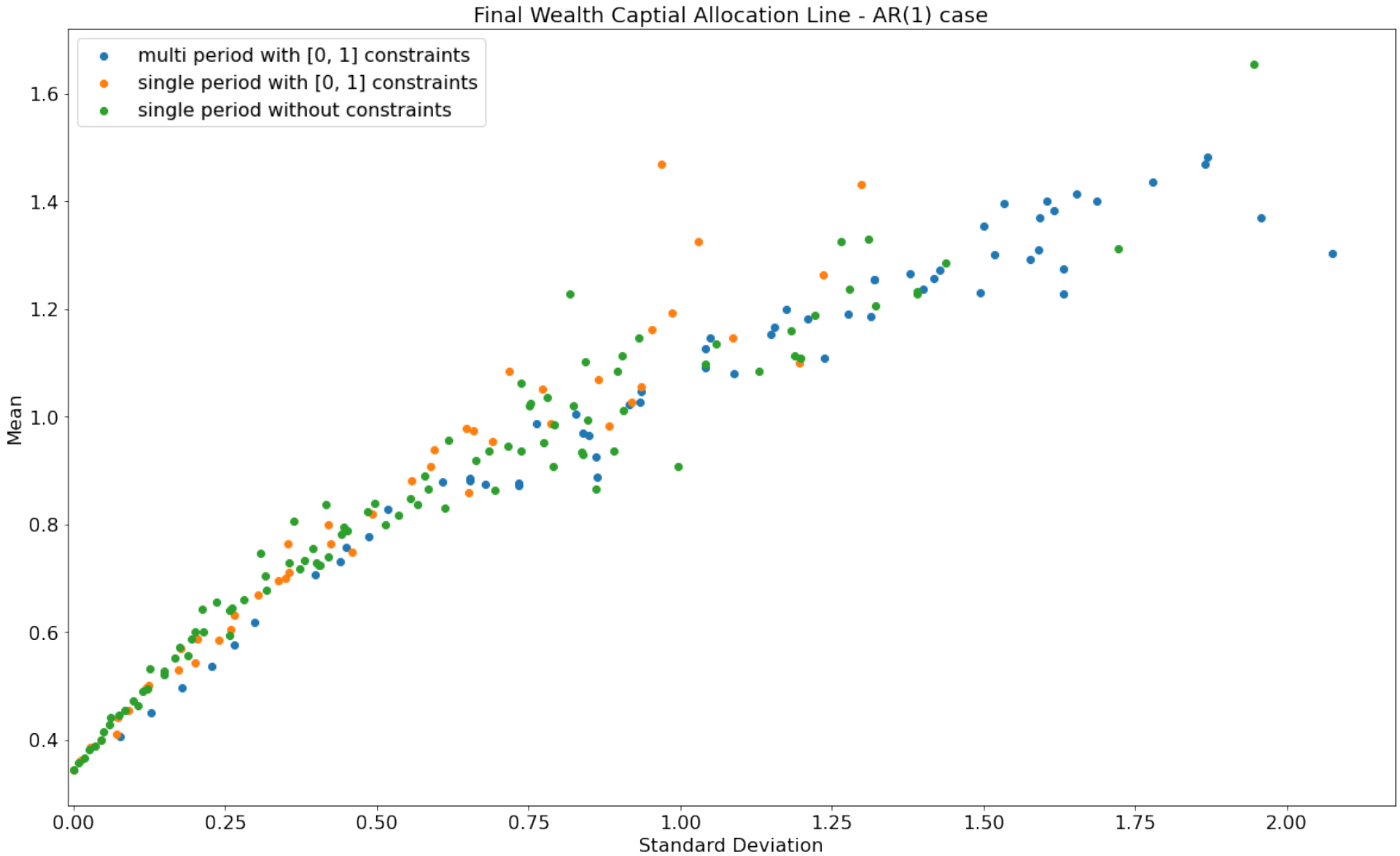


Figure 5: Final Wealth Capital Allocation Line - AR(1) case.

Numerical Examples - Results (CAL)

Weights Domain D	$[-\infty, \infty]^p$	$[0,1]^p$	$[-0.2,1]^p$	$[-1,1]^p$
Portfolio Number p		21		
Multi-period Number K		10		
Rebalancing Time Δ		$1/10$		
Initial Wealth W_0		1		
Continuous Risk-free Rate r		$\exp\left(\frac{0.03}{360}\right)$		
Model Type	Single-period	Multi-period		
Control Level	$\mu^* \in [0, 0.5]$	$\gamma \in [2.5, 8]$		
Trained Samples	1,000	100,000		
Neuron Number v	-	50		

Table 5: Parameters Setting for computing CAL in CCC-GARCH(1,1) Case.

*Both single-period and multi-period models may not give very accurate results since small numbers of trained samples are used.

Numerical Examples - Results (CAL)

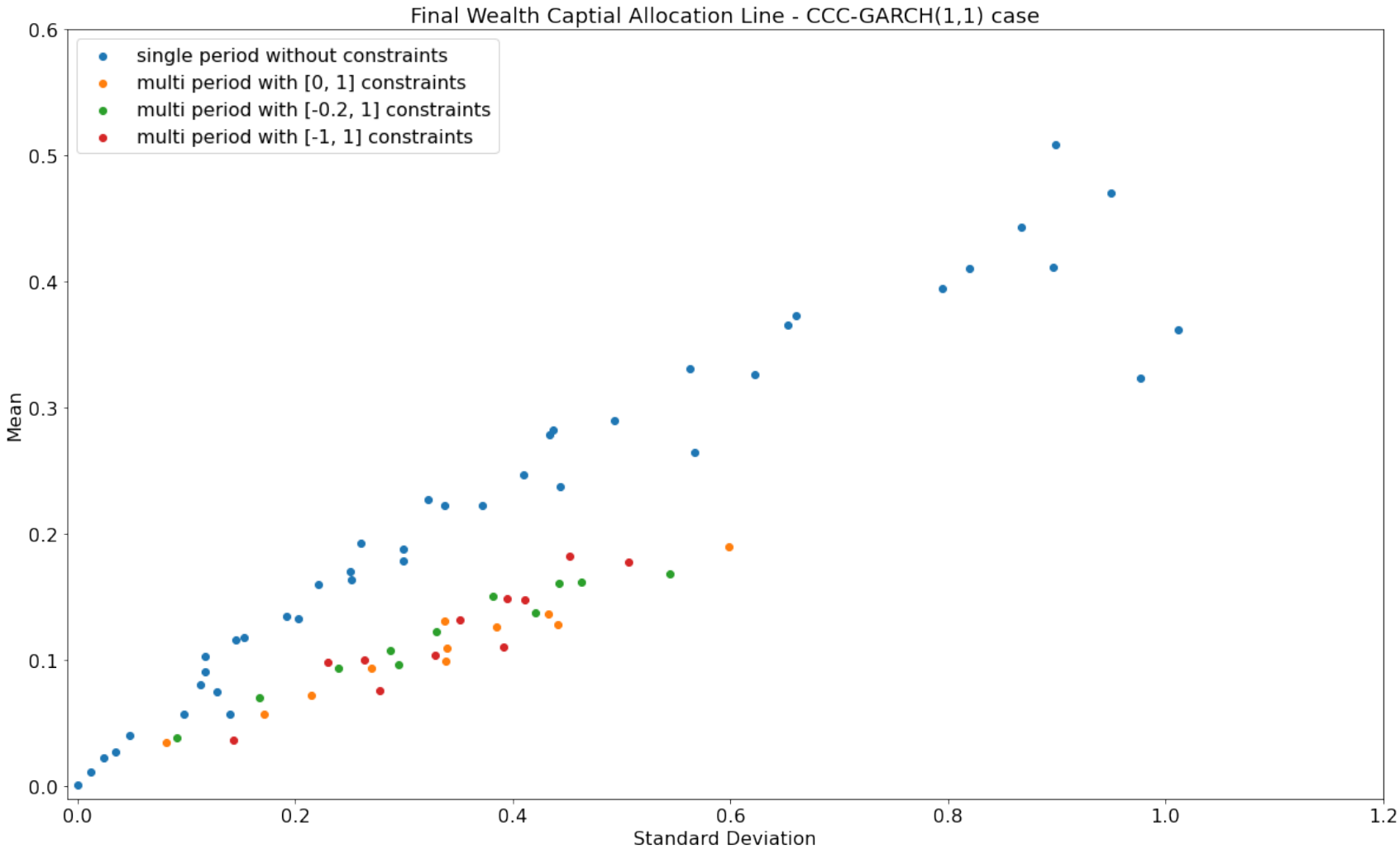


Figure 6: Final Wealth Capital Allocation Line - CCC-GARCH(1,1) case.

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Conclusions

Observations

From our results, the performance of multi-period models are not better than single-period ones, which is totally different from the author's results. On the one hand, there must be some logic problems in either the author's code or ours. On the other hand, if our code is right, we can conclude from our results that the multi-period models are likely to fit the noises in the excess return series and finally yields terrible results because of the problem of overfitting. There may be two things that can be done to improve the models. One is to reduce the number of parameter, i.e., we can reduce the number of hidden layer number or neuron number in each layer. The other one is that we can furtherly process the excess return to show their characteristics more clearly.

Conclusions

Observations

From last section, we can conclude that the convergence statements about the algorithms hold true in a general sense, which demonstrates that this algorithm is very efficient in handling high dimensionality case up to 100. However, as the number of dimension becomes large, the number of parameters will grow exponentially, which is likely to yield overfitting issue. Then it is not likely to get accurate utility values in very high dimensionality, which is an obvious drawback of this algorithm.

Conclusions

Criticism

Firstly, from our view, the aim to analyze the concept of efficient frontier in the author's paper is not appropriate. Since we only focus on the process of wealth accumulating, it does not have a lot of things to do with the current weights of the each assets, so that it does not make sense to analyze the efficient frontier in the last day. CAL(Capital Allocation Line) in the last day is more proper and can be compared in different setting of single- or multi-period. Therefore, we analyze the CAL of different models instead.

Conclusions

Criticism

Moreover, there might be something wrong in the implementation of single-period model. Since we can only control the portfolio wealth at the first day, and have no means to control it later on. And when [0,1] constraints are imposed, the maximum value of the return of the portfolio is to invest all of the stocks in weights of ones which have positive returns, which is actually bounded in terms of the maximum wealth at the first day. Thus, there must be some samples that can not satisfy the high expect return of the first day, which will yield a biased result if we simply discard these samples. So, we are highly suspicious for the accuracy of author's single-period results, where the author does not explain what he has done.

Finally, we can see that the curves of the author are extremely smooth, there must be some smoothing methods applied. However, the author does not say anything about them, which make us doubt the authenticity about how he get these curves.

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Thank you!