Understanding Analysis Solutions

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Chapter 1

Exercise 1.2.1. (a) *Proof.* AFSOC $\sqrt{3}$ is rational, so $\exists m, n \in \mathbb{Z}$ such that

$$\sqrt{3} = \frac{m}{n}, \tag{1}$$

where $\frac{m}{n}$ is in lowest reduced terms. Then we can square both sides, yielding $3 = \left(\frac{m}{n}\right)^2 \implies 3n^2 = m^2$. Now, we know m^2 is a multiple of 3 and thus m must also. Then, we can write $m = 3k$, and derive

$$\begin{aligned}(\sqrt{3})^2 &= \left(\frac{3k}{n}\right)^2 \\ 3n^2 &= 9k^2 \\ n^2 &= 3k^2\end{aligned}$$

Similar to before, we come to the conclusion that n is a multiple of 3. However, this is a contradiction since m, n are both multiples of 3 and we assumed $\frac{m}{n}$ was in lowest terms. Thus, we conclude $\sqrt{3}$ is irrational. □

(b) We cannot conclude that $\sqrt{4} = \frac{m}{n}$ implies that m is a multiple of 4, so we cannot reach our contradiction.

Exercise 1.2.2. (a) False. Consider

$$A_n = [0, \frac{1}{n}). \tag{2}$$

Then

$$\bigcap_{n=1}^{\infty} A_n = \{0\}. \tag{3}$$

(b) True

(c) False. Consider $A = \{1, 2\}, B = \{1\}, C = \{2, 3\}$.

(d) True

(e) True

Exercise 1.3.1. (a) We compute the additive inverse for each element in $\mathbb{Z}_5$.

$$\begin{aligned}0 + 0 &\equiv 0 \\ 1 + 4 &\equiv 0 \\ 2 + 3 &\equiv 0 \\ 3 + 2 &\equiv 0 \\ 4 + 1 &\equiv 0\end{aligned}$$

(b) We compute the multiplicative inverse for each element in $\mathbb{Z}_5$.

$$\begin{aligned}1 \times 1 &\equiv 1 \\ 2 \times 3 &\equiv 1 \\ 3 \times 2 &\equiv 1 \\ 4 \times 4 &\equiv 1\end{aligned}$$

(c) $\mathbb{Z}_4$ is not a field because multiplicative inverses do not exist for every single element. We conjecture that $\mathbb{Z}_n$ always has additive inverses and only has multiplicative inverses if n is prime.

Exercise 1.3.2. (a) $s = \inf A$ means

i) s is a lower bound for A

ii) if b is any lower bound for A, then $b \leq s$

(b) If $s \in \mathbb{R}$ is a lower bound for $A \subseteq \mathbb{R}$, then $s = \inf A$ iff $\forall \epsilon > 0, \exists a \in A$ such that $s + \epsilon > a$.

Proof. ($\Rightarrow$)If $s = \inf A$, then s is the greatest lower bound for A, meaning any $s + \epsilon$ for $\epsilon > 0$ will be greater than some element of A, otherwise $s + \epsilon$ is a greater lower bound and leads to a contradiction that $s \neq \inf A$. ($\Leftarrow$) If $\forall \epsilon > 0, \exists a \in A$ such that $s + \epsilon > a$, then since s is a lower bound, $\forall b > s$, b will not be a lower bound for A since $b > s \implies \exists a \in A \mid b > a$. Thus, all lower bounds b must be such that $b \leq s$, and we conclude $s = \inf A$. $\square$

Exercise 1.3.3. (a) $***$

(b) There might be a typo in this question. I think the question was meant to read "explain why there is no need to assert that the greatest *lower bound* in the Axiom of Completeness." In this case, the answer would be that the Axiom of Completeness already implies the greatest lower bound property, so there is no need to explicitly state it.

(c) We can take the negative of all elements in A, find $\sup A$, and then negate again to get $\inf A$.

Exercise 1.3.4. If $B \subseteq A$, then

$$\begin{aligned} \sup A = s &\geq a \in A \\ s &\geq b \in B && \text{(since } B \subseteq A\text{)} \\ \implies s &\geq \sup B. && \text{(since } s \text{ is an upper bound for } B\text{)} \end{aligned}$$

Exercise 1.3.5. (a)

$$\begin{aligned} &s = \sup(c + A) \\ \implies &s \text{ is the least upper bound for } c + A \\ \implies &s - c \text{ is the least upper bound for } A \\ \implies &s - c = \sup A \\ &s = c + \sup A \end{aligned}$$

(b)

$$\begin{aligned} &s = \sup(cA) \\ &\implies s \text{ is the least upper bound for } cA \\ &\implies \frac{s}{c} \text{ is the least upper bound for } A \\ &\implies \frac{s}{c} = \sup A \\ &s = c \sup A \end{aligned}$$

(c) If $c < 0$, $\sup(cA) = -c \sup(A)$.

Exercise 1.3.6. (a) $\sup, \sqrt{10}; \inf, 1$

(b) $\sup, 1; \inf, 0$

(c) $\sup, \frac{1}{2}; \inf, \frac{1}{3}$

(d) $\sup, \infty; \inf, -\infty$

Exercise 1.3.7. If $a \geq a', \forall a' \in A$, and $a \in A$, then

$$\forall \epsilon > 0, a - \epsilon < a, \tag{4}$$

so a is the least upper bound for A, and $a = \sup A$.

Exercise 1.3.8. Let

$$\epsilon = \sup B - \sup A > 0. \tag{5}$$

since $s_b = \sup B$, $\exists b \in B \mid b > s_b - \epsilon/2$. Since $s_b - \frac{\epsilon}{2} > \sup A$, then $b \geq \sup A$, so this $b \in B$ is an upper bound for A.

Exercise 1.3.9. (a) True (take the largest element)

(b) False $\sup(0,2) = 2$, but $2 > a \in (0,2)$, but $\sup A = 2 \not< 2 = L$.

(c) False $A = (0,2), B = [2,3)$. We have that $\sup A = \inf B$

(d) True

(e) False (take $A = B = (0,2)$)

Exercise 1.4.1. If $a < 0$, then we have two cases,

1. If $b > 0$, then $a < 0 < b$.
2. If $b = 0$, then we can take $-b, -a$, which satisfies $0 \leq -b < -a$, and apply Theorem 1.4.3.

Exercise 1.4.2. (a) If $a, b \in \mathbb{Q}$, then

$$\begin{aligned} a &= \frac{a_1}{a_2} \\ b &= \frac{b_1}{b_2} \\ \implies a + b &= \frac{a_1 b_2 + a_2 b_1}{a_2 b_2} \in \mathbb{Q} \end{aligned}$$

(b) AFSOC $at \in \mathbb{Q}$. But if $a = \frac{a_1}{a_2}$, this implies

$$t = \frac{a_2}{a_1}\frac{m}{n} \in \mathbb{Q},$$

which is a contradiction, so we must have that $at \in \mathbb{I}$

(c) $\mathbb{I}$ is not closed under addition or multiplication.

Exercise 1.4.3. We can apply Theorem 1.4.3, to find $a < q < b, q \in \mathbb{Q}$, and then subtract an irrational number such as $\sqrt{2}$ to end up at

$$a - \sqrt{2} < q - \sqrt{2} < b - \sqrt{2}, \tag{6}$$

where $q - \sqrt{2} \in \mathbb{I}$.

Exercise 1.4.4. Suppose $\exists b$ lower bound such that $b > 0$. Then by Archimedean Property of $\mathbb{R}$, $\exists n \in \mathbb{N}$ such that $\frac{1}{n} < b$, which means b is not a valid lower bound. Thus $b \leq 0$, and 0 is a valid lower bound so the inf is 0.

Exercise 1.4.5. AFSOC $\exists \alpha \in \bigcap_{n=1}^{\infty}(0, \frac{1}{n})$. Then $\alpha > 0$, but by Archimedean property of reals, we have that $\exists n \in \mathbb{N} \mid \frac{1}{n} < \alpha$. Since $\alpha \notin (0, \frac{1}{n}$, $\alpha \notin \bigcap_{n=1}^{\infty}(0, \frac{1}{n})$, a contradiction. Thus we conclude the set is empty.

Exercise 1.4.6. (a) If $\alpha^2 > 2$, then

$$\begin{aligned} \left(a - \frac{1}{n}\right)^2 &= \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} \\ &> \alpha^2 - \frac{2\alpha}{n} \end{aligned}$$

choose $\frac{1}{n_0} < \frac{\alpha^2-2}{2\alpha}$. Then

$$\begin{aligned}\left(a - \frac{1}{n_0}\right)^2 &> \alpha^2 - \frac{2\alpha}{2\alpha}(\alpha^2 - 2)\\ &> 2\end{aligned}$$

but $\alpha - \frac{1}{n_0} < \alpha$, so α is not the least upper bound for the set.

(b) Just replace $\sqrt{2}$ with $\sqrt{b}$

Exercise 1.4.7. Take the minimum from the set. Assign as i, then remove the minimum from the set. Repeat for $i+1$ and so on.

Exercise 1.4.8. (a) If both are finite, then their union is finite and trivially countable. If one is finite, then first enumerate elements of the finite set. Then map the rest of $\mathbb{N}$ to the countably infinite set. If both are countably infinite, map one set to odds and the other to evens.

(b) Induction only holds for finite integers, not infinity.

(c) We can arrange each A_n into row n of a $\mathbb{N} \times \mathbb{N}$ matrix. Then, we enumerate by diagonalization.

Exercise 1.4.9. (a) If $A \sim B$, then there is a 1-to-1 mapping. We can just take the inverse of the mapping to derive $B \sim A$.

(b) If we have $f : A \to B$, $g : B \to C$, then we can compose the functions so $g(f(x)) : A \to C$.

Exercise 1.4.10. The set of all finite subsets of $\mathbb{N}$ can be ordered in increasing order by the sum of each subset.

Exercise 1.4.11. (a) $f(x) = x$

(b) Interweave the decimal expansion of x, y, e.g.

$$f(x, y) = 0.x_1y_1x_2y_2x_3y_3\dots \tag{7}$$

Exercise 1.4.12. (a)

$$\begin{aligned}\sqrt{2} &: x^2 - 22 = 0\\ \sqrt[3]{2} &: x^3 - 2 = 0\end{aligned}$$

$\sqrt{3} + \sqrt{2}$ is not as trivial, so we will do it out in more steps.

There are two approaches to finding the integer coefficient polynomial. One is to take advantage of symmetry, and derive that

$$\prod(x - (\pm\sqrt{3} \pm \sqrt{2}) \tag{8}$$

will work (using loose notation of course). A more general technique is to notice that

$$\begin{aligned}x &= \sqrt{3} + \sqrt{2}\\ x^2 &= 5 + 2\sqrt{6}\\ (x^2 - 5)^2 &= 24\\ x^4 - 10x^2 + 1 &= 0.\end{aligned}$$

Notice that this is actually the exact same answer we get in (8) if you work it out.

(b) Each $|A_n| = |\mathbb{N}^n|$, which is countable

(c) We proved earlier in Theorem 1.4.13 that a countably infinite union of countable sets is countable. Since there are a countable number of algebraic numbers, and reals are uncountable, we conclude that transcendentals are also uncountable.

Exercise 1.4.13. (a) INCOMPLETE

Exercise 1.5.2. (a) Because b_1 differs from $f(1)$ in position 1

(b) b_i differs from $f(i)$ in position i.

(c) We reach a contradiction that we can enumerate the elements of $(0,1)$, and thus $(0,1)$ is uncountable.

Exercise 1.5.3. (a) $\frac{\sqrt{2}}{2} \in (0,1)$ but is irrational

(b) We can just define our decimal representations to never have an infinite string of 9s

Exercise 1.5.4. Suppose S is countable. Then we can enumerate the elements of S using the natural numbers. Now, consider some $s = (s_1, s_2, \dots)$, where

$$s_i = \begin{cases} 0, \text{if } f(i), \text{position } i = 1 1, \text{otherwise} \end{cases} \tag{9}$$

Then since $s \neq f(i) \forall i$, $s \notin S$. But this is a contradiction since s only contains elements 0 or 1, and thus should be in S. Thus, we conclude that S is uncountable.

Exercise 1.5.5. (a)

$$\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\} \tag{10}$$

(b) Each element has two choices when constructing a subset of A. To be, or not to be[1], in the set.

Exercise 1.5.6. (a) Many different answers.

$$\{(a, \{a\}), (b, \{b\}), (c, \{c\})\}$$
$$\{(a, \emptyset), (b, \{b\}), (c, \{c\})\}$$

(b)

$$\{(1, \{1\}), (2, \{2\}), (3, \{3\}), (4, \{4\})\}.$$

(c) Because $\left|\mathcal{P}(A)\right| > |A|$ and same for B.

Exercise 1.5.7. 1. $B = \emptyset$

2. $B = \{a, d\}$

Exercise 1.5.8. (a) AFSOC $a' \in B$. Then that means $a \notin f(a')$. But this is a contradiction since $a' \in B = f(a')$.

(b) AFSOC $a' \notin B = f(a')$. Then since $a' \notin f(a')$, $a' \in B$, but that is a contradiction.

Exercise 1.5.9. (a) This is the same as $\mathbb{N} \times \mathbb{N}$, which is countable.

(b) Uncountable, since $\mathcal{P}(\mathbb{N})$ is uncountable.

(c) Is this question asking for the number of antichains or if there is an antichain with uncountable cardinality? The latter is obvious, and *no* is the answer since any subset of $\mathbb{N}$ is countable. The first case probably uncountable???

[1] sorry, had to do it

2 Chapter 2

Exercise 2.2.1. (a) Let $\epsilon > 0$ be arbitrary. Then choose $n \in \mathbb{N}$ such that $n > \frac{1}{\sqrt{6\epsilon}}$. Then

$$\begin{aligned}\left|\frac{1}{6n^2+1}\right| &< \left|\frac{1}{6\frac{1}{6\epsilon}+1}\right| \\ &< \left|\frac{1}{\frac{1}{\epsilon}+1}\right| \\ &< \frac{\epsilon}{\epsilon+1} \\ &< \epsilon\end{aligned}$$

as desired.

(b) Choose $n > \frac{13}{2\epsilon} - \frac{5}{2}$

(c) Choose $n > \frac{4}{\epsilon^2} - 3$

Exercise 2.2.2. Consider the sequence

$$x_n = (-1)^n, n \geq 1. \tag{11}$$

Then for $\epsilon > 2$, it is true that $|x_n - 0| < 2, \forall n \geq 1$.

The *vercongent* definition describes a sequence that can be finitely bounded past some n.

Exercise 2.2.3. (a) We have to find one school with a student shorter than 7 feet.

(b) We would have to find a college with a grade lower than B.

(c) We just have to check every college for a student who is shorter than 6 feet.

Exercise 2.2.4. For $\epsilon > \frac{1}{2}$, we can find a suitable N, since we can claim the sequence "converges" to $\frac{1}{2}$. For $\epsilon \leq \frac{1}{2}$, there is no suitable response.

Exercise 2.2.5. (a) $\lim a_n = 0$. Take $n > 1$. Then

$$\begin{aligned}\left|\left[\left[\frac{1}{n}\right]\right]\right| &\leq 0 \\ &< \epsilon.\end{aligned}$$

(b) $\lim a_n = 0$. Take $n > 10$. Then

$$\begin{aligned}\left|\left[\left[\frac{10+n}{2n}\right]\right]\right| &= \left|\left[\left[\frac{5}{n}+\frac{1}{2}\right]\right]\right| \\ &\leq 0 \\ &< \epsilon.\end{aligned}$$

Usually, the sequence converges to some value by getting closer and closer eventually. Sometimes, the sequence converges to the exact value very fast, which means for some n, we don't need to choose a larger n. E.g. if we had the sequence of all 0s, we can choose any n and claim the sequence converges to 0.

Exercise 2.2.6. (a) Larger

(b) Larger

Exercise 2.2.7. (a) We say a sequence x_n *converges* to ∞ if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$ we have that $|x_n| > \epsilon$

(b) With our definition, we say this sequence diverges, but does not converge to ∞.

Exercise 2.2.8. (a) Frequently.

(b) Eventually is stronger, and implies frequently.

(c) We say that a sequence x_n converges to x if it eventually is in a neighborhood of radius ϵ of x for all $\epsilon > 0$.

(d) x_n is only necessarily frequently in $(1.9, 2.1)$.

Exercise 2.3.1. Let $\epsilon > 0$. Consider $n \geq 1$, then

$$|a - a| = 0 < \epsilon.$$

Exercise 2.3.2. (a) We are given $(x_n) \to 0$, so we can make $|x_n - 0|$ as small as we want. In particular, we choose N such that $|x_n| < \epsilon|\sqrt{x_n}|$, whenever $n \geq N$. To see that this N indeed works, observe that for all $n \geq N$,

$$|\sqrt{x_n}| = \frac{|x_n|}{|\sqrt{x_n}|} < \frac{1}{|\sqrt{x_n}|}\epsilon|\sqrt{x_n}| = \epsilon$$

so $(\sqrt{x_n}) \to 0$.

(b) We are given $(x_n) \to x$, so we can make $|x_n - x|$ as small as we want. We choose N such that

$$|x_n - x| < \epsilon|\sqrt{x_n} + \sqrt{x}|$$

whenever $n \geq N$. To see that this N works, notice that for all $n \geq N$,

$$|\sqrt{x_n} - \sqrt{x}| = \frac{|x_n - x|}{|\sqrt{x_n} + \sqrt{x}|} < \frac{1}{|\sqrt{x_n} + \sqrt{x}|}\epsilon|\sqrt{x_n} + \sqrt{x}| = \epsilon$$

Therefore, $(\sqrt{x_n}) \to \sqrt{x}$.

Exercise 2.3.3. By the Order Limit Theorem, since

$$\forall n, x_n \leq y_n \Rightarrow \lim_{n\to\infty} y_n \geq \lim_{n\to\infty} x_n = l$$
$$\forall n, z_n \leq y_n \Rightarrow \lim_{n\to\infty} y_n \leq \lim_{n\to\infty} z_n = l$$

so $l \leq \lim_{n\to\infty} y_n \leq l \Rightarrow \lim_{n\to\infty} y_n = 1$.

Exercise 2.3.4. AFSOC $\lim a_n = l_1$ and l_2, for $l_1 \neq l_2$. Then we have that $\forall \epsilon > 0$, for sufficiently large n, that

$$|a_n - l_1| < \epsilon$$
$$|a_n - l_2| < \epsilon$$

But this is a contradiction, since if we let $d = |l_1 - l_2|$, and $\epsilon = \frac{d}{2}$, then

$$|l_2 - l_1| \leq |a_n - l_1| + \big|-(a_n - l_2)\big| < 2\epsilon$$
$$d \leq |a_n - l_1| + \big|-(a_n - l_2)\big| < d,$$

which leads to $d < d$. Thus, we must conclude that $l_1 = l_2$, and limits are unique.

Exercise 2.3.5. (*Rightarrow*) If (z_n) is convergent to some l, then $\forall \epsilon > 0$, we have that $\exists N \in \mathbb{N}$ such that for $n \geq N$, that

$$|z_n - l| < \epsilon \Longrightarrow |x_n - l| < \epsilon, |y_n - l| < \epsilon, \tag{12}$$

because z_n appears before or at the same time as x_n and y_n in the sequence.

($\Leftarrow$) If $(x_n), (y_n)$ are both convergent to some limit l, then we have for some $n \geq N \in \mathbb{N}$, that

$$|x_n - l| < \epsilon$$
$$|y_n - l| < \epsilon.$$

Then choose $n' \geq 2N$, then we have two cases,

- If n' odd, then $z_{n'} = x_{(n'+1)/2}$. Since $\frac{n'+1}{2} \geq N$, $\left|x_{(n'+1)/2} - l\right| < \epsilon$.
- If n' even, then $z_{n'} = y_{n'/2}$. Since $\frac{n'}{2} \geq N$, $\left|y_{n'/2} - l\right| < \epsilon$.

In both cases we have that for every $\epsilon > 0$, we can find $n' \geq N' = 2N \in \mathbb{N}$ such that

$$|z_{n'} - l| < \epsilon, \tag{13}$$

so (z_n) is also convergent to l.

Exercise 2.3.6. (a) By triangle inequality, we have $\left||b_n| - |b|\right| \leq |b_n - b| < \epsilon$

(b) The converse is not true. Consider the sequence $a_n = (-1)^n$.

Exercise 2.3.7. (a) Since (a_n) is bounded, call M the upper bound of (a_n). Then since $|b_n|$ can get arbitrarily small, we choose $n \geq N$ such that $|b_n| < \frac{\epsilon}{M}$. Then we have

$$\begin{aligned} |a_n b_n| &\leq |a_n||b_n| \\ &< M\frac{\epsilon}{M} \\ &< \epsilon. \end{aligned}$$

We cannot use the Algebraic Limit Theorem because we are not given that (a_n) necessarily converges.

(b) No. For example, take $a_n = (-1)^n$, $b_n = 3$.

(c) When $a = 0$, we have

$$|a_n b_n - ab| \leq |b_n||a_n - a|.$$

We can bound $|b_n| \leq M$, and then choose n such that $|a_n - a| < \frac{\epsilon}{M}$. Then,

$$\begin{aligned} |a_n b_n - ab| &< M\frac{\epsilon}{M} \\ &< \epsilon. \end{aligned}$$

Exercise 2.3.8. (a) $x_n = (-1)^n, y_n = (-1)^{n-1}$

(b) Impossible by theorem ???

(c) $b_n = \frac{1}{n}$

(d) Impossible by theorem ???

(e) $a_n = 0, b_n = n$

Exercise 2.3.9. No. Consider $a_n = \frac{1}{n}, a_n > 0$. $\lim a_n = 0$, but $0 \not> 0$.

Exercise 2.3.10. Since $|a_n|$ gets arbitrarily small, we know for $n \geq N$,

$$|b_n - b| \leq |a_n| < \epsilon. \tag{14}$$

Exercise 2.3.11. Let $\lim x_n = x$. Then, for some $n_\epsilon \geq N$, we have $|x_n - x| < \epsilon/2$. Now,

$$\begin{aligned}
|y_n - x| &= \frac{1}{n}\left[\left|\sum_{i=1}^{n_\epsilon}(x_i - x)\right| + \left|\sum_{i=n_\epsilon}^{n}(x_i - x)\right|\right] \\
&= \frac{n_\epsilon}{n}\max_{i\in[1,n_\epsilon]}(x_i - x) + \frac{n - n_\epsilon}{n}\max_{i\in[n_\epsilon,n]}(x_i - x) \\
&= \frac{n_\epsilon}{n}\max_{i\in[1,n_\epsilon]}(x_i - x) + \frac{\epsilon}{2}
\end{aligned}$$

now if we choose $n > \frac{n_\epsilon \max_{i\in[1,n_\epsilon]}(x_i - x)}{\epsilon/2}$, then we can bound the RHS by ϵ.

Consider when $x_n = (-1)^n$. (x_n) does not converge but (y_n) does.

Exercise 2.3.12. (a) Intuitively, the limit should go to 1, since we have $\frac{\infty}{\infty}$.

$$\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} = 1$$
$$\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} = 0$$

(b) A sequence $(a_{m,n})$ converges to l if for every $\epsilon > 0$, $\exists N \in \mathbb{N}$ such that whenever $n \geq N$, we have that

$$\left|\lim_{n\to\infty}\lim_{m\to\infty} a_{m,n} - l\right| < \epsilon$$
$$\left|\lim_{m\to\infty}\lim_{n\to\infty} a_{m,n} - l\right| < \epsilon.$$

i.e. we approach the same limit no matter what permutation of the index variables we iterate through.