Financial Frictions, Liquidity Constraint, and the Business Cycle

PS3: ...

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Overview

1. Fire sales and multiple equilibria
2. Impatient households and risky investments
3. Appendix
   - A note on the Ennis and Keister (2009) assumption

Setup

Model consists of:

- Two-period economy, $t \in \{0, 1\}$.
- Single consumption good (c),
- continuum of entrepreneurs (E) and financiers (E),
- fixed total supply of capital $\bar{k}$.
- F uses capital in concave production technology,
- E has constant return a.

Thus optimal allocation of k is such that

$$f'(k_E) = 1 - k_E = a = f'(k_F).$$

Thus optimally $k_E^* = 1 - a$ and $k_F^* = \bar{k} + a - 1$. Due to financial frictions this need not be the case!

Setup

The story goes:

- Temporary negative shock to cash flow for E $\Rightarrow$ no income from k_0.
- To repay initial debt b_0 E loads off capital k_0 at price q_0.
- To finance purchase of new capital E borrows b_1.

But!

- due to moral hazard, limited liability and so on, borrowing is constrained to what financiers know they can recoup at time 1. Thus $b_1 \leq q_1 k_1$.

New element today: Possible fire sales where a low q_0 can drive $k_1 = 0$!

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Entrepreneurial (E) problem

$$\max_{c_0,c_1,k_1,b_1} \quad c_0 + c_1 \tag{1}$$

$$\text{s.t.} \quad c_0 + q_0 k_1 \leq \max\{0,\ q_0 k_0 - b_0\} + b_1 \tag{2}$$

$$c_1 + b_1 \leq (\theta + a)k_1 \tag{3}$$

$$b_1 \leq \theta k_1 \tag{4}$$

- Maximizes consumption (risk-neutral),
- At time 0: Consumption and capital purchases corresponds to wealth endowment after repayment of debt and newly issued debt.

  Max statement: If E cannot repay initial debt, then contract is re-negotiated to 0.
- At time 1: Consumption and repayment of debt corresponds to return on capital (a) and resell value of capital (θ).
- Borrowing constraint by collateralized assets.

Q1

1. *State E's maximization problem. Characterize demand for capital.*

Entrepreneurial solution I: Capital demand

- If $q_0 < \theta$: E would accumulate infinite capital (arbitrage from investment),
- If $q_0 \geq \theta + \alpha$: E would never accumulate capital.
- If $q_0 < b_0/k_0$: E would default on initial debt and not be able to purchase k_1.
- Thus we will assume that

$$\max(\theta,\ b_0/k_0) < q_0 < \theta + \alpha. \tag{A1}$$

This ensures 1) No default on debt, 2) Finite capital demand due to liquidity constraint.

Q2

2. *Derive capital demand for entrepreneurs.*

Entrepreneurial solution II

Under (A1) all constraints bind and $c_0 = 0$ is optimal. Using this and borrowing constraint we have:

$$k_1 = \frac{q_0 k_0 - b_0 + \overbrace{b_1}^{=\theta k_1}}{q_0} \qquad \Rightarrow \qquad k_1 = \frac{q_0 k_0 - b_0}{q_0 - \theta}. \qquad (5)$$

Q3

3. *Discuss capital demand in different scenarios.*

Entrepreneurial solution III

- **Capital demand** rewritten slightly

$$k_1 = k_0 + \frac{k_0\theta - b_0}{q_0 - \theta}. \tag{6}$$

- **Effects from $\mathbf{q_0}$ on demand:**

  Price effect: $q_0 \uparrow$ price on $k_1 \uparrow$ and demand for $k_1 \downarrow$.

  Wealth effect: $q_0 \uparrow$ value of endowment $q_0 k_0 \uparrow$ and demand for $k_1 \uparrow$.

- Size of k_0 determines wealth effect, size of k_1 determines price effect.

Q4

4. *The problem of financiers and the market clearing condition for capital.*

Financiers (F) problem

$$\max_{\tilde{c}_0, \tilde{c}_1, \tilde{k}_1} \quad \tilde{c}_0 + \tilde{c}_1 \tag{7}$$

$$\text{s.t.} \quad \tilde{c}_0 + q_0 \tilde{k}_1 \leq \overbrace{\tilde{k}_0 - \frac{1}{2}\tilde{k}_0^2}^{\text{prod. tech}} + q_0 \tilde{k}_0 + e_0 \tag{8}$$

$$\tilde{c}_1 \leq \tilde{k}_1 - \frac{1}{2}\tilde{k}_1^2 + \theta \tilde{k}_1 + e_1. \tag{9}$$

Similar to E, except:

- Production technology from capital: Ensures finite capital demand.
- Note the language *having large endowments* (e_0, e_1):

  If e_0, e_1 were small, it still might be optimal to choose $\tilde{c}_0 < 0$ or $\tilde{c}_1 < 0$. We assume away this for simplicity.

Finaciers solution

Assuming interior consumption solution. Capital demand then follows from:

$$\max_{\tilde{k}_1 \geq 0} \; \tilde{k}_1 \left(1 + \theta - q_0\right) - \frac{1}{2}\tilde{k}_1^2,$$

yielding solution

$$\tilde{k}_1 = \max\left\{0,\; 1 + \theta - q_0\right\}. \tag{10}$$

With fixed supply of $\bar{k}$ this yields market clearing condition:

$$\bar{k} = k_1 + \tilde{k}_1. \tag{11}$$

For convenience we will rewrite this as an *residual supply* function for entrepreneurs:

$$k_1 = \min\left\{\bar{k},\; \bar{k} - \left(1 + \theta - q_0\right)\right\}. \tag{12}$$

Q5

5. *Show and depict graphically that when $b_0 < \theta k_0$ there is a unique equilibrium.*

Equilibrium on capital markets I

When $b_0 < \theta k_0$ note:

- **Supply curve:**

  Enters through $\min(\bar{k},\ \bar{k} - 1 - \theta)$ and increases linearly towards $\bar{k}$.

  Thus either constant function around $\bar{k}$ or a linearly increasing one in q_0.

- **Demand curve:**

  When $k_0\theta - b_0 > 0$ demand is decreasing towards the level $k_0 \leq \bar{k}$

  $\Rightarrow$ Always unique equilibrium (k_1, q_0).

Q6

6. *Show that when $b_0 > \theta k_0$ there can be multiple equilibria; a 'bad' and a 'good' one. Explain why.*

Equilibrium on capital markets II

1. Assumptions:

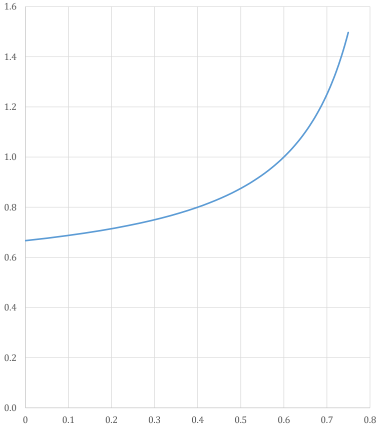
$$a + \theta > q_0 > \frac{b_0}{k_0} > \theta.$$

2. Demand bound by:

$$k^{max} = k_0 - \frac{b_0 - k_0\theta}{a},$$
$$k^{min} = 0.$$

3. Supply is linearly increasing.

Figure 1: Capital demand



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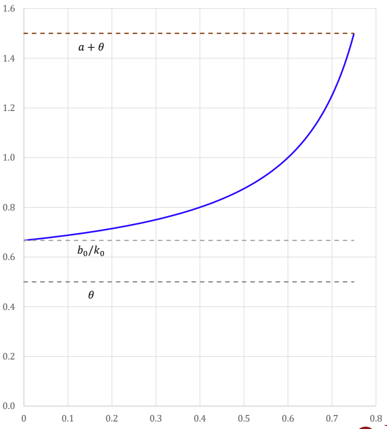

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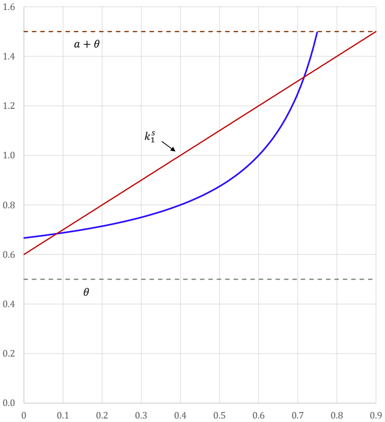
Figure 2: Equilibrium



Equilibrium on capital markets III

- **But**, if price q_0 drops below b_0/k_0 demand for k_1 drops to zero.

  This allows for two (stable) equilibria:

- The '**bad**' equilibrium with $k_1 = 0$ and $q_0 = 0.6$:

  E defaults on his debt $\Rightarrow$ zero net worth.

- The '**good**' equilibrium:

  A high q_0 implies wealthy E, which implies a large capital demand k_1, when $k_0 > k_1$.

Q7

7. *How can an appropriate reduction in $\mathfrak{b}_0$ eliminate the bad equilibrium?*

Reduction of debt to remove bad equilibrium

- Reduction in b_0 increases E demand for capital,
- as it increases wealth and thus relaxes borrowing constraint.
- This removes the bad (and the unstable intermediate equilibrium).

Q8

8. *Suppose there were multiple equilibria and the government stands ready to buy assets at the good equilibrium price. How many assets would it have to buy to implement this policy?*

Buy-out policy

Within our model: The government would not have to buy any assets.

- The 'good price equilibrium' is self-sustainable:

  If the announcement of the price is credible, then the market will clear itself.

- This is due to our simple model. Caveats?

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9. *Compare costs of Q7 and Q8 as policies working in the financial crisis: Q7 interpreted as injecting capital into the banking system and Q8 as a plan to buy toxic securities with government money. Discuss (briefly).*

Financial crisis policies

- **Within model evaluation:**

  Asset purchase (Q8) is free, capital injection (Q7) can be costly.

- **Why is it free in Q8**?

  In our model assets are not really toxic, it is only a matter of signaling the right price.

  Adjustment of to new market equilibrium is costless (no price or information frictions).

- **Adverse selection:**

  Government may end up buying the most toxic assets.

- **Moral Hazard:**

  Both strategies may induce unhealthy future risk-taking.

- More?

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The setup

DD-type of economy, now:

- Three periods $t = 0, 1, 2$.
- Two storage technologies:

  One-period investment with return 1,

  Two-period investment with return $R > 1$. Can be prematurely liquidated at return $L < 1$.
- Continuum of agents with unit endowment, where
- A fraction π will be impatient and a fraction $(1 - \pi)$ patient. Thus expected utility is presented by

$$u = \pi\sqrt{c_1} + (1 - \pi)\rho\sqrt{c_2}. \tag{13}$$

where $1, 2$ refers to time periods.

1. *Characterize optimal and market allocation.*

Market allocation

Without uncertainty the market allocation is straightforward:

$$c_1^M = 1, \qquad\qquad c_2^M = R. \tag{14}$$

This is obtained through issuing bonds at price p. In particular budgets become

$$c_1 = 1 - I + pRI$$
$$c_2 = \frac{1-I}{p} + RI,$$

where an equilibrium price of $p = 1/R$ yields (14).

The utility (from a utilitarian welfare function) is then

$$U^M = \pi + (1-\pi)\rho\sqrt{R}. \tag{15}$$

Social Planner allocation

A planner allocates I of the initial endowment in long-run investment and $1 - I$ in short run. This yields:

$$\pi c_1 = 1 - I \qquad\qquad (1-\pi)c_2 = RI,$$

or the combined budget

$$\pi c_1 + (1-\pi)\frac{c_2}{R} = 1. \tag{16}$$

In other words we solve

$$\max_{c_1, c_2} \; \pi\sqrt{c_1} + (1-\pi)\rho\sqrt{c_2} + \lambda\left[1 - \pi c_1 + (1-\pi)\frac{c_2}{R}\right]. \tag{17}$$

This yields optimality condition:

$$(\rho R)^2 c_1^* = c_2^*. \tag{18}$$

Intuitive result: If $(\rho R > 1)$ long run investing is worth it. Productivity more than offsets discounting ρ.

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Comparison of allocations

- **Same relative size:**

  Note $c_2 > c_1$ both in market and planner allocation.

- **Generally not the same:**

  Planner allocation can deviate in general. Note particularly that:

  $$\text{If } c_1^* = 1, \text{ then M allocation is optimal when } c_2^* = (\rho R)^2 = R,$$

  i.e. only optimal exactly when $R = 1/\rho^2$.

- If either R or ρ increases from this case, then $c_2^* > c_2^M$ and $c_1^* < c_1^M$:

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Q2

2. *Part 1: Is it possible to introduce financial intermediary (can he implement optimal allocation?). Part 2: Is it possible to have bank runs, and why are they inefficient?*

First best by financial intermediary

- **First best contract:**

  Offer c_1^* if withdraw and $t = 1$ and c_2^* if withdraw and $t = 2$.

- **Incentive compatible:**

  Two types have no incentive to lie and declare to be the other type.

  If patient household does so, then he can withdraw c_1^* early, use short-run technology and consume c_1^* instead of c_2^*.

  However, $c_2^* > c_1^*$ thus no incentive to lie.

Bank runs

- Assume a patient household expects all to withdraw early.

  If he benefits from withdrawing early (before other patient households even), we can have a bank 'run'.

- **In our model:**

  If I expect all to withdraw early, the bank can liquidate all assets and get return:

$$\pi c_1^* + (1 - \pi c_1^*)L. \tag{20}$$

  Recall: Bank invests πc_1^* in short technology and residual in 'long'.

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Bank runs

- Solving for c_1^* the bank run requirement becomes:

$$L\rho^2 R < 1. \tag{21}$$

- **Inefficient run** as liquidation of long-run investment earns less (L) than otherwise equivalent short-run technology (1).
- **Run equilibrium is pareto dominated**

  (more on this later)

Q3

3. *Assume from here on that* $\rho R < 1$. *Characterize optimal allocation and compare to market outcome.*

Optimal versus market allocation

The optimal allocation is unchanged:

$$c_1^* = \frac{1}{\pi + (1-\pi)\rho^2 R}, \qquad c_2^* = \frac{(\rho R)^2}{\pi + (1-\pi)\rho^2 R}.$$

- Now $c_1^* > c_1^M = 1$ and $c_2^* < c_2^m = R$.
- Long-run investment is not worth it: Discount-factor ρ more than offsets productivity R.

  $\Rightarrow$ Push consumption to $t = 1$.
- Why is cut-off not at $\rho R = 1$?
- Recall that utility is concave ($\sqrt{c}$) and $c_1^M = 1$ while $c_2^M = R$.

  I.e. optimal allocation redistributes to c_1 when $\rho R = 1$ to equalize marginal utility of patient and impatient households.

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4. *Part 1: Can intermediary implement optimal allocation? Part 2: Can we have bank runs now?*

Constrained optimal bank contract

- **Optimal contract problem:**

  Maximize utility subject to (1) budget and (2) IC constraint:

$$\max_{c_1,c_2} \ \pi\sqrt{c_1} + (1-\pi)\rho\sqrt{c_2} + \lambda\left[1 - \pi c_1 + (1-\pi)\frac{c_2}{R}\right] + \mu\left[c_2 - c_1\right], \tag{22}$$

  with the complementary slackness condition $\mu(c_2 - c_1) = 0$ and $\mu \geq 0$ measuring shadow-value of IC constraint.

- **Constrained solution:**

  When $\rho R < 1$ the unconstrained solution in (18) is not incentive compatible, $c_2^* < c_1^*$.

  $\Rightarrow$ constrained solution is given by imposing $c_1 = c_2 = c$ in budget:

$$\pi c_1 + (1-\pi)\frac{c}{R} = 1 \quad \Rightarrow \quad c = \frac{R}{R\pi + (1-\pi)} \geq 1. \tag{23}$$

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Bank runs?

Yes, it is always possible to have bank runs: Sufficient condition is still $L\rho^2 R < 1$.

Recall: Technical requirement is

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Q5

5. *Why can commitment to 'pre-announced suspension of convertibility' eliminate bank run equilibrium?*

Eliminating bank run equilibria

- **Strategic equilibrium:**

  Note that equilibria are *strategic*: Only optimal for patient household to withdraw early **if** all other patient households does so.

- **Pareto-dominated equilibrium:**

  All households are better off in non-withdraw equilibrium.

- Thus to avoid bank run equilibrium:

  Send credible signal that bank run cannot occur $\Rightarrow$ no patient household withdraws early.

Limiting paying total deposits to πc_1^* send such a signal: Each individual household can still withdraw early, but patient households will choose not to.

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Q6

6. *In absence of commitment, assume the CB can choose a threshold $\pi^B \geq \pi$ of deposits that get paid. What is optimal ex post policy (threshold)? Does it prevent bank runs?*

Optimal ex post policy

The setup of the policy is:

- Ex ante optimal:

  Implement Q5 policy and stop payments at πc in $t = 1$.

- Assume CB observes πc in withdrawals, but demand for early withdrawals is still present:

  $\Rightarrow$ (at least partial) bank run has occurred.

- If CB can choose a new $\pi^S \geq \pi$ what is then optimal?

The setup

- **Now 4 types of agents**:

  Impatient/patient and already withdrawn / not withdrawn.

- **Define** impatient share of the π already withdrawn households as $\phi_1(\pi^S)$.

- **Define** impatient share of the withdrawals when increasing π^s as $\phi_2(\pi^S)$.

- **Define** impatient share of households not withdrawing $\phi_3(\pi^S)$.

- **Implicit assumption:** If $\pi^S = 1$ all agents withdraw (everyone expect bank run to occur).

- **Consistency requirement:**

$$
\begin{aligned}
\pi\phi_1 + (\pi^S - \pi)\phi_2 + (1 - \pi^S)\phi_3 &= \pi \\
\pi(1 - \phi_1) + (\pi^S - \pi)(1 - \phi_2) + (1 - \pi^S)(1 - \phi_3) &= 1 - \pi.
\end{aligned}
$$

Note: The two consistency requirements only constitutes *one* identifying restriction (See appendix)

The objective function

Objective function:

$$\max \ \pi\Big[\phi_1\sqrt{c} + (1-\phi_1)\rho\sqrt{c}\Big] + (\pi^S - \pi)\Big[\phi_2\sqrt{c} + (1-\phi_2)\rho\sqrt{c}\Big]$$
$$+(1-\pi^S)\Big[\phi_3\sqrt{0} + (1-\phi_3)\rho\sqrt{c_2^S}\Big]. \tag{24}$$

This states that:

- The share π already withdrew c.

  Of these ϕ_1 were impatient consuming c and $1-\phi_1$ save using short run technology (w. return 1) and consume c when $t=2$.

- The share $\pi^S - \pi$ withdraws (where π^S is chosen).

  ϕ_2 are impatient households withdrawing, the rest patient households.

- The share $1-\pi^S$ cannot withdraw:

  ϕ_3 are impatient households, $1-\phi_3$ of patient households consume 'residual' (c_2^S) of bank's deposits.

The objective function

- To arrive at setup similar to Ennis and Keister (2009), assume for now that $\phi_1 = \phi_2 = \phi_3 = \pi$ and that $\rho = 1$.
- In this case maximizing (25) is equivalent to:

$$\max \ \pi^S \sqrt{c} + (1 - \pi^S)(1 - \pi)\sqrt{c_2^S}, \tag{25}$$

where c_2^S still needs to be specified.

The residual payout

So, what can the bank afford to pay out $(c_2^S(\pi^S))$?

- **Bank expenditures:**

$$\begin{aligned} \text{When } t = 1: &\quad I_1 = \pi c + (\pi^S - \pi)c \\ \text{When } t = 2: &\quad I_2 = (1-\pi^S)(1-\pi)c_2^S(\pi^S). \end{aligned}$$

  Implicitly assumed: If impatient households cannot withdraw when $t = 1$ then they are paid 0 when $t = 2$. (recall they get 0 utility from consumption when $t = 2$)

- **Bank income:**

$$\begin{aligned} \text{When } t = 1: &\quad E_1 = \pi c + \theta(1-\pi c)L \\ \text{When } t = 2: &\quad E_2 = (1-\theta)(1-\pi c)R. \end{aligned}$$

  To finance unexpected early withdrawals $(\pi^S - \pi)c$, the bank has to liquidate share θ of long run investments.

The residual payout

- From $I_1 = E_1$ we get

$$\theta = \frac{\pi^S - \pi}{1 - \pi c} \frac{c}{L}. \tag{26}$$

- Using this and $I_2 = E_2$ we have

$$c_2^S(\pi^S) = \frac{(1 - \pi c)R - (\pi^S - \pi)c\frac{R}{L}}{(1 - \pi^S)(1 - \pi)}. \tag{27}$$

- Thus optimal ex post policy is defined by

$$\max_{\pi^S} \ \pi^S \sqrt{c} + (1 - \pi^S)(1 - \pi)\rho\sqrt{\frac{(1 - \pi c)R - (\pi^S - \pi)c\frac{R}{L}}{(1 - \pi^S)(1 - \pi)}}. \tag{28}$$

The optimal policy I: Technical observations

- When $\pi^S = \pi$ (and using (23)):

$$c_2^S(\pi) = R\frac{1-\pi c}{(1-\pi)^2} = \frac{c}{1-\pi} > c. \tag{29}$$

- $c_2^S(\pi^S)$ is decreasing in π^S:

$$\frac{\partial c_2^S}{\partial \pi^S} = R\frac{1-\pi c-\frac{c}{L}(1-\pi)}{(1-\pi^s)^2(1-\pi)} < 0 \tag{30}$$

and clearly $\partial^2 c_2^S/\partial \pi^{S\,2} > 0$.

The optimal policy II

- **Optimal policy is time inconsistent:**
  1. When $\pi^S > \pi$ ex ante and ex post optimality does not coincide.
  2. $\Rightarrow$ can be rational not to 'trust' ex ante promised CB policy.
  3. If CB is credible always promise $\pi^S = \pi$ in which case bank run is never rational.
- **Optimal policy might still eliminate bank runs:**

  If optimal π^S still ensures $c_2^S(\pi^S) > c$, then bank run equilibrium is eliminated

  Start of $t = 1$ patient households recognize that CB even in case of a bank run, ensures a payoff $c_2^S > c$ by not withdrawing early.

  Thus they are protected from bank runs.

Overview

1. Fire sales and multiple equilibria
2. Impatient households and risky investments
3. Appendix
    - A note on the Ennis and Keister (2009) assumption

Identifying assumptions

Earlier we assumed $\phi_1 = 1 = \phi_2 = \phi_3$ and $= 1$ for simplicity and to arrive at a problem similar to that of Ennis and Keister (2009).

But what can we actually say about the ϕ_j parameters?

- π is a fundamental parameter and assumed constant.
- ϕ_1 is impatient share of already withdrawn households and must considered pre-determined, i.e. constant as well.
- Recognizing that consistency holds when changing π^S implicit differentiation yields

$$\phi_2 - \phi_3 + (\pi^S - \pi)\frac{\partial \phi_2}{\partial \pi^S} + (1 - \pi^S)\frac{\partial \phi_3}{\partial \pi^S} = 0. \tag{31}$$

- If we wish to consider ϕ_j constants this obviously entails $\phi_2 = \phi_3$. Thus generally we would have a problem of the type:

$$\pi^S \left[\phi\sqrt{c} + (1-\phi)\rho\sqrt{c}\right] + (1-\pi^S)(1-\phi)\rho\sqrt{c_2^S}, \quad \phi \equiv \frac{\pi(1-\phi_1)}{1-\pi}. \tag{32}$$

Identifying assumptions

Note that consistency requirement can be written in the form:

$$\begin{pmatrix} \pi & \pi^S - \pi & 1 - \pi^S \\ -\pi & -(\pi^S - \pi) & -(1 - \pi^S) \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} \pi \\ -\pi \end{pmatrix}.$$

Thus the two linear restrictions are linearly dependent and only constitutes one identifying restriction.