Synthetic Homotopy Theory Homework: The Hopf fibration

The goal of this homework is to construct the Hopf fibration. We use notations from the lecture notes freely. First we need an auxiliary definition.

Definition 1. *A span consists $A, B, C : \mathcal{U}$ with $f : B \to A$ and $g : B \to C$.*

Usually we will denote a span by a diagram:

$$A \xleftarrow{f} B \xrightarrow{g} C$$

We give an analysis of equality between spans.

Proposition 1. *Assume given two spans:*

$$A \xleftarrow{f} B \xrightarrow{g} C \qquad\qquad A' \xleftarrow{f'} B' \xrightarrow{g'} C'$$

then giving an equality between them is equivalent to giving three equivalences:

$$\epsilon_A : A \simeq A' \qquad \epsilon_B : B \simeq B' \qquad \epsilon_C : C \simeq C'$$

such that:

$$\epsilon_A \circ f \sim f' \circ \epsilon_B \qquad\qquad \epsilon_C \circ g \sim g' \circ \epsilon_B$$

Now we introduce a new higher inductive type.

Inductive Definition 1. *Assume given a span:*

$$A \xleftarrow{f} B \xrightarrow{g} C$$

- *There is a type $A \coprod_B C$ called the pushout of the span.*
- *For all $x : A$ we have:*

  $$\mathbf{left}(x) : A \coprod_B C$$

  For all $y : C$ we have:

  $$\mathbf{right}(y) : A \coprod_B C$$

  Moreover for all $z : B$ we have:

  $$\mathbf{quo}(z) : \mathbf{left}(f(z)) =_{A \coprod_B C} \mathbf{right}(g(z))$$

- *Assume given $P : A \coprod_B C \to \mathcal{U}$, in order to define: $h : (z : A \coprod_B C) \to P(z)$ it is enough to define:*

  $$h(\mathbf{left}(x)) :\equiv t_x$$

  with $t_x : P(\mathbf{left}(x))$ for $x : A$,

  $$h(\mathbf{right}(y)) :\equiv s_y$$

  with $s_y : P(\mathbf{right}(y))$ for $y : C$, and:

  $$\mathrm{apd}_h(\mathbf{quo}(z)) :\equiv r_z$$

  with $r_z : \mathbf{tr}^P_{\mathbf{quo}(z)}(t_{f(z)}) = s_{g(z)}$ for $z : B$.

Exercise 1 The join of two spaces

First we define the join of two spaces.

Definition 2. *Assume given two types A and B, then we define their join $A * B$ as the pushout of the span:*

$$A \xleftarrow{p_1} A \times B \xrightarrow{p_2} B$$

where p_1 and p_2 are projections.

Question 1 Let A be a type, show that:

$$\mathbf{2} * A \simeq \Sigma A$$

Question 2 (Optional) Show that the join operation is associative, meaning that for all $A, B, C : \mathcal{U}$ we have:

$$(A * B) * C \simeq A * (B * C)$$

Question 3 Using the two previous questions and the fact that $\Sigma S^n = S^{n+1}$, prove that:

$$S^1 * S^1 \simeq S^3$$

Exercise 2 Flattening Lemma for suspension

Assume given $A : \mathcal{U}$ and $P : \Sigma A \to \mathcal{U}$.

Question 1 (Optional) Prove that $(x : \Sigma A) \times P(x)$ is equivalent to the pushout of the span:

$$P(\mathbf{N}) \xleftarrow{p_1} P(\mathbf{N}) \times A \xrightarrow{\psi} P(\mathbf{S})$$

where ψ is defined for $q : P(\mathbf{N})$ and $a : A$ by:

$$\psi(q, a) :\equiv \mathbf{tr}^P_{\mathbf{merid}_a}(q)$$

Exercise 3 The Hopf construction

Assume given $A : \mathcal{U}$ with $\mu : A \times A \to A$ such that for all $a : A$ the maps

$$\mu(a, \_) \equiv \lambda x.\mu(a,x) : A \to A$$

$$\mu(\_, a) \equiv \lambda x.\mu(x,a) : A \to A$$

are equivalences.

We define $P : \Sigma A \to \mathcal{U}$ by:

$$P(\mathbf{N}) :\equiv A$$

$$P(\mathbf{S}) :\equiv A$$

and for $x : A$ we have:

$$\mathrm{ap}_P(\mathbf{merid}_x) :\equiv \mathbf{ua}^{-1}(\mu(\_, x))$$

where $\mathbf{ua}^{-1} : A \simeq B \to A =_{\mathcal{U}} B$ is the map assumed by univalence.

Question 1 Prove that for all $a, b : A$, we have $\mathbf{tr}^P_{\mathbf{merid}_b}(a) = \mu(a,b)$.

Question 2 Using the previous exercise, show that $(x : \Sigma) \times P(x)$ is equivalent to the pushout of the span:

$$A \xleftarrow{p_1} A \times A \xrightarrow{\mu} A$$

Question 3 Using the fact that $\mu(a, \_)$ is an equivalence, show that the span:

$$A \xleftarrow{p_1} A \times A \xrightarrow{\mu} A$$

is equal to the span:

$$A \xleftarrow{p_1} A \times A \xrightarrow{p_2} A$$

where $p_1(x, y) :\equiv x$ and $p_2(x, y) :\equiv y$.

Question 4 Conclude that we have:

$$(x : \Sigma A) \times P(x) \simeq A * A$$

Exercise 4 H-types

In this exercise we define H-types and show that connected H-types satisfy the hypothesis from the previous exercise, and then build a fiber sequence.

Definition 3. *An H-type consists of a type A with:*

- *An element $e : A$.*
- *A map:*

  $$\mu : A \times A \to A$$

  such that for all $x : A$ we have:

  $$\mu(x, e) = \mu(e, x) = x$$

As usual we identify an H-type and its underlying type. Let A be a connected H-type.

Question 1 Show that for all $x : A$, we have $|\mu(x, \_) = \mathbf{id}_A|$ and $|\mu(\_, x) = \mathbf{id}_A|$.

Question 2 From the previous question, prove that for all $x : A$, the maps $\mu(x, \_)$ and $\mu(\_, x)$ are equivalences.

Recall that given X a pointed type and $C : X \to \mathcal{U}$ with $*_C : C(*)$, we can build a fiber sequence:

$$C(*) \to_* (x : X) \times C(x) \to_* X$$

Question 3 Using the previous exercise, show that we have a fiber sequence:

$$A \to_* A * A \to_* \Sigma A$$

Exercise 5 S^1 is a H-type

In this exercise we build the Hopf fibration using the fact that S^1 is a H-type.

Question 1 (Optional) Define:

$$\psi' : (x : S^1) \to x =^1_S x$$

with $\psi'(\mathbf{base}) :\equiv \mathbf{loop}$.

We define $\mu : S^1 \to S^1 \to S^1$ by:

$$\mu(\mathbf{base}) :\equiv \mathbf{id}_{S^1} : S^1 \to S^1$$

$$\mathrm{ap}_\mu(\mathbf{loop}) :\equiv \psi$$

where $\psi : \mathbf{id}_{S^1} = \mathbf{id}_{S^1}$ is the image of ψ' by function extensionality.

Question 2 (Optional) Prove that for all $x : S^1$ we have:

$$\mu(x, \mathbf{base}) = x$$

$$\mu(\mathbf{base}, x) = x$$

Question 3 Using the previous exercises, conclude that we have a fiber sequence:

$$S^1 \to_* S^3 \to_* S^2$$

Question 4 Conclude that:

$$\pi_n(S^3) = \pi_n(S^2)$$

if $n > 2$.