Report on Pablo Donato's PhD thesis: *Deep Inference for Graphical Theorem Proving*

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Context and significance

Proof theory is the study of mathematical proofs as formal objects in their own right. *Structural* proof theory is an umbrella term for proof theoretic activity where notions of inference step, normal forms, and equality of proofs have primary consideration. It is not a single area in its own right, but rather encompasses quite different technical developments seeking to understand and model proofs under the aforementioned considerations.

The foundational results of structural proof theory were Gentzen's famous normalisation theorems in the 1930s. His *natural deduction* systems underlie the famous *Curry-Howard* correspondence, identifying formal proofs with functional programs, itself the basis of modern proof assistants such as Coq.

Over the course of almost 100 years since Gentzen's work, countless alternative proof formalisms have been proposed, leading to now a rich landscape of systems within the discipline of structural proof theory. Two recent themes of notable interest are:

1. graphical formalisms (e.g. Pierce's existential graphs, Andrews' matings, Miller's expansion trees, Girard's proof nets, Guglielmi's atomic flows, Hughes' combinatorial proofs,...); and,

2. 'deep inference' formalisms (e.g. hyper/nested sequents, calculus of structures, open deduction,...).

Both of these themes have significantly advanced the capacity of structural proof theory, in particular providing alternative intuitions for formal proof building and, indeed, exhibiting *more* proofs than Gentzen's systems, in a formal sense. However, importing these ideas into the activity of *theorem proving* has remained underdeveloped, despite the promise they show to this end. Over the course of the last 20 years, only glimpses of such applications have appeared. This PhD thesis is a welcome shot in the arm with respect to these pursuits, amending an unfortunate oversight of the literature.

Summary of contributions

The guiding principle of this thesis is Proof-by-Action (PbA). This is inspired by the concept of *direct manipulation* in user interfaces dating back to the '80s, which has seen little development in the world of theorem provers. Nonetheless PbA seeks to bring together previous approaches such as Proof-by-Pointing and Proof-by-Linking. In this thesis there is particular focus on *click* actions and *drag-and-drop* actions. A prototype implementation *Actema* is given, a user interface incorporating click and DnD actions, and serves as a case study of recurring attention in the thesis.

Proof-by-Linking is based on the idea of subformula linking from *linear logic*. This thesis develops these ideas proof theoretically, in particular developing appropriate *deep inference* systems of intuitionistic and classical predicate logic. These systems appropriately model actions in Actema, thus establishing the soundness of that interface. The extension to classical logic incorporates modern ideas from the semantics of classical logic (as a compuational system), where disjunction is modelled as a sort of parallel interaction.

The thesis explores several use cases of Actema and techniques in the PbA paradigm. Notably it also develops the *Coq-Actema* system, recasting Actema as a front-end to the Coq proof assistant. This works by interpreting the actions of Actema as novel tactics in Coq by way of a plugin. While this system still has some way to go, its development is a welcome contribution and has the potential for significant future improvements.

In the second part, this thesis develops graphical calculi inspired by various forms of deep inference, notably nested sequents and Pierce's existential calculi. While motivated by the PbA paradigm, this part seems more focussed on theory building. It introduces two formalisms: the *bubble calculus* and the *flower calculus*. The development of both formalisms is governed by a discipline of 'iconic manipulation', favouring graphical and 'metaphorical' notations that are suggestive of the objects they represent.

The bubble calculus is inspired by Brünnler's nested sequents. A calculus BJ is given for intuitionistic logic, where the metaphor of bubbles is suggestive of context management. Notably, and reassuringly, the bubble calculus readily extends to desirable fragments and extensions, in particular classical logic and bi-intuitionistic logic. A general soundness result is given for these systems by interpretation within their algebraic formulations. Completeness for the bi-intuitionistic calculus is established by simulation of an existing system, whence completeness for the other calculi is recovered.

The flower calculus is an alternative calculus for intuitionistic logic, whose notations/concepts are given according to a horticultural taxonomy. It follows a rewriting style discipline inspired by Pierce's existential graphs, as well as deep inference in the main. This is preceded by a modern and comprehensive account of the theory of existential graphs themselves. Soundness and completeness of Kripke relational structures is established, the latter in particular employing semantic/model theoretic techniques. This actually simultaneously establishes *analytic* completeness, adapting an earlier argument of Hermant for cut-free sequent calculi. Some ideas for proof search algorithms are presented, and more notably an implementation of the flower calculus in the *Flower Prover* is given, exploiting the the metalogical adequacy of the flower calculus.

Evaluation

This PhD thesis has been a pleasure to read. Pablo's writing exhibits a level of maturity that even senior researchers rarely accomplish. In particular the careful structuring of the content, the deep literature survey, and generally helpful presentation (e.g. margin notes, use of colour/shading, internal hyperlinking), comprise academic writing of outstanding quality.

The merits of the thesis' presentation do not come at the cost of its content, with the manuscript standing at almost 300 pages! This is not simply a matter of quantity: Pablo exhibits deep dives into several quite different technical directions, and moreover develops new theory and tools at their interfaces. The breadth of material in this manuscript could easily cover two PhDs.

I would particularly highlight that Pablo's innovation in recasting ideas from graphical proof theory and deep inference as tactics and interfaces for proof assistants is very welcome. This is not a matter of simply plugging things together, rather requiring innovation all the way from foundations to implementation.

Further comments

Some of the more philosophical motivation underlying 'iconic manipulation' was somewhat lost on me. In particular, I am not wholly convinced that this discipline facilitates comprehension of proofs. For instance I could not easily appreciate the commitment to horticultural taxonomy for the flower calculus. Of course I look forward to debating this matter with Pablo soon.

On the proof theoretic side, it would be particularly interesting in future work to develop the Curry-Howard aspects of the calculi proposed, in particular reductive cut-elimination and term annotations. Pablo has already outlined some of these ideas for the flower calculus. Such considerations are generally underdeveloped for deep inference and graphical calculi as a whole.

Recommendation and conflict declaration

Pablo Donato should be awarded a PhD on the basis of the thesis submitted. I confirm that I, the undersigned, do not have any conflict of interest to declare in relation with this thesis.

Dr Anupam Das