Deep Inference for Graphical Theorem Proving

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PhD defense, Palaiseau

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Introduction

Context

Logic

- Study of *sound* reasoning

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- Example of everyday life deduction:

$$\frac{\text{it rains 🌧️} \qquad \text{you don't have ☂️}}{\text{you are wet 💧}}$$

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premisses it rains 🌧️ you don't have ☂️

conclusion you are wet 💧

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you are not under a bus shelter ...

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- Hidden assumptions ⇒ lack of **certainty**

Formal logic

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$$\frac{x \text{ is } P \qquad \text{All } P \text{ are } Q}{x \text{ is } Q}$$

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↳ Formal essence of logical reasoning

Inference rules

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- Formalist school (Hilbert – 20th century):

Maths as a huge **game**

Goal: to prove theorems by following rules

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Goal: to prove theorems by following rules

- **Proof theory:** design & study of *rule systems* capturing maths

Proof assistants

- Inference rules represented with **symbols**

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↳ Teach computers how to do maths with proof theory!

- Problem: maths is *hard* ⇒ need for a **human** in the loop

↳ Interactive Theorem Provers (ITPs)

Contributions

Textual vs. Graphical

State-of-the art: build proofs by writing **textual** commands

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🙇 : “Please apply *this* rule”

💻 : “✅ OK here is the result!”

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🙇 : “Please apply *this* rule”

💻 : “❌ ERROR: dkfsljfjdklsfjdkfjsldjfkdlsfj”

Textual vs. Graphical

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🙇 : "Please apply *this* rule"

💻 : "❌ ERROR: dkfsljfjdklsfjdkfjsldjfkdlsfj"

1st contribution: build proofs by direct manipulation of *formulas*

↳ No need to *memorize* the rules

↳ No risks of *errors*

Symbolic vs. Iconic

- **Symbols** are hard to:
  - *learn* ⇒ purely conventional meaning
  - *manipulate* ⇒ need for very precise gestures

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  - *manipulate* $\Rightarrow$ need for very precise gestures
- Formulas can **interact** by being *moved* in the same space

2nd contribution: replace logical symbols by geometrical diagrams

Symbolic Manipulations

Proof-by-Action

*A **demo** is worth a thousand words!*

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- Fully graphical: no textual proof language

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proof = gesture sequence

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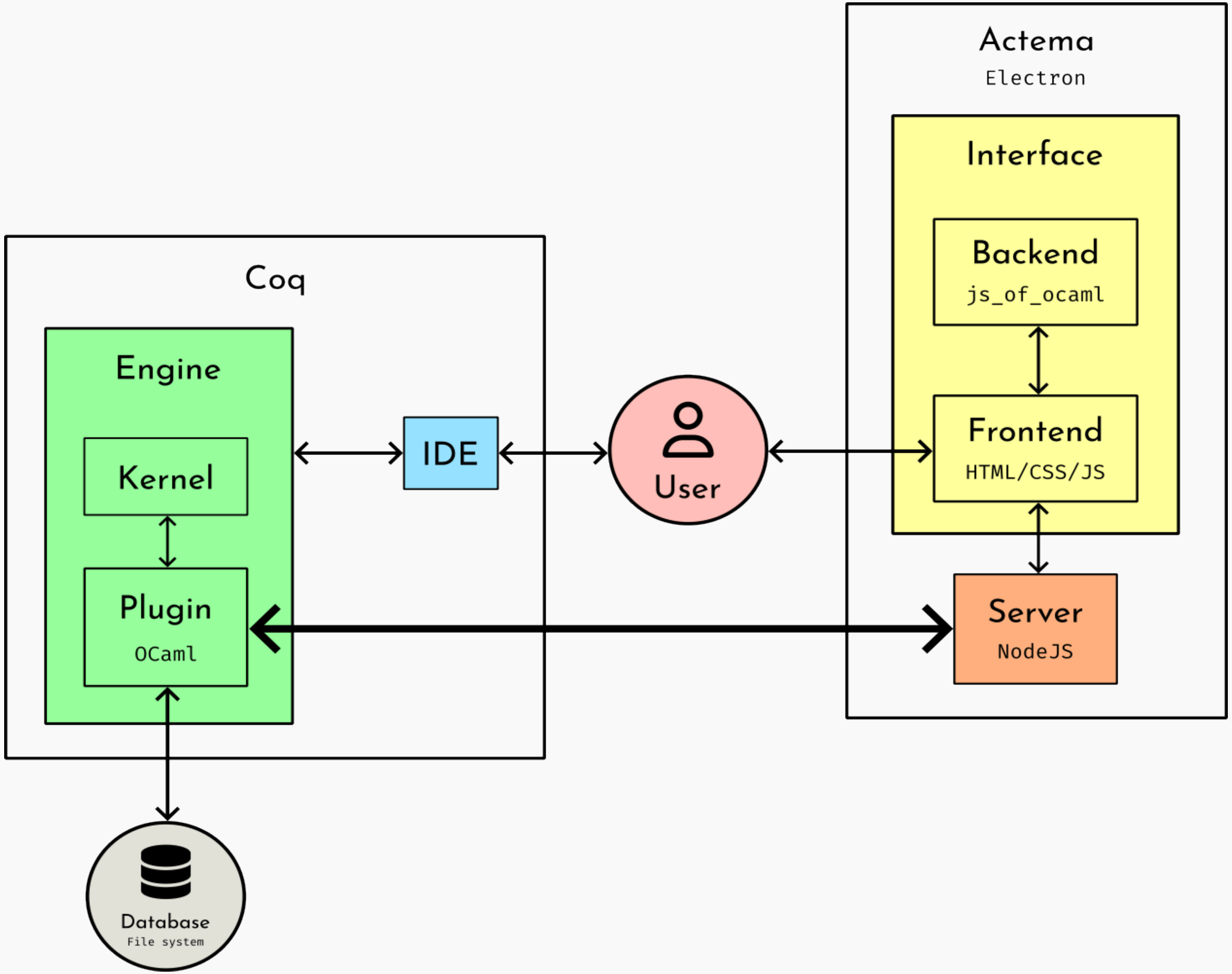
- Fully graphical: no textual proof language
- Both spatial and temporal:

$$\text{proof} \quad = \quad \text{gesture sequence}$$

- **Different modes** of reasoning with a **single "syntax"**:

$$\begin{array}{rcl} \text{Click} & \Longleftrightarrow & \text{introduction/elimination} \\ \text{Drag-and-Drop} & \Longleftrightarrow & \text{backward/forward} \end{array}$$

coq-actema


Actema
Electron
Interface
Backend
js_of_ocaml
Frontend
HTML/CSS/JS
Server
NodeJS
Coq
Engine
Kernel
Plugin
OCaml
IDE
User
Database
File system

Semantics of Drag-and-Drop

Subformula Linking (Chaudhuri 2013)

Idea: bring matching subformulas through switch rules

$$
\text{switch}\left\{
\begin{aligned}
&\underline{A} \land B ⧁ B \land (\underline{A} \lor C) \land D \\
\to\ & B \land (\underline{A} \land B ⧁ (\underline{A} \lor C) \land D) \\
\to\ & B \land (\underline{A} \land B ⧁ \underline{A} \lor C) \land D \\
\to\ & B \land ((\underline{A} \land B ⧁ \underline{A}) \lor C) \land D \\
\to\ & B \land ((B \Rightarrow (\underline{A} ⧁ \underline{A})) \lor C) \land D
\end{aligned}
\right.
$$

$$
\text{identity}\left\{ \to B \land ((B \Rightarrow \top) \lor C) \land D \right.
$$

$$
\text{unit elimination}\left\{
\begin{aligned}
\to\ & B \land (\top \lor C) \land D \\
\to\ & B \land \top \land D \\
\to\ & B \land D
\end{aligned}
\right.
$$

Variant of the **Calculus of Structures** (Guglielmi 1999)

Linking under quantifiers (Donato, Strub, and Werner 2022)

$$\exists y.\ \forall x.\underline{R(x,y)} \otimes \forall a.\exists b.\underline{R(a,b)}$$

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- Unify linked subformulas

$$\exists y.\ \forall x.\underline{R(x,y)} \otimes \forall a.\exists b.\underline{R(a,b)}$$

$$x \longmapsto a$$

$$y \longleftarrow\!\dashv b$$

Linking under quantifiers (Donato, Strub, and Werner 2022)

- Unify linked subformulas
- **Check** for $\forall\exists$ dependency cycles

$$\exists y.\, \forall x. \underline{R(x, y)} \ogreaterthan \forall a. \exists b. \underline{R(a, b)}$$

$$x \longmapsto a$$

$$y \longleftarrow\!\dashv b$$

$$\checkmark$$

Linking under quantifiers (Donato, Strub, and Werner 2022)

- Unify linked subformulas
- **Check** for ∀∃ dependency cycles
- **Switch** uninstantiated quantifiers

$$\exists y.\,\forall x.\underline{R(x,y)} \otimes \forall a.\exists b.\underline{R(a,b)}$$

$$\to \forall y.\big(\forall x.\underline{R(x,y)} \otimes \forall a.\,\exists b.\underline{R(a,b)}\big)$$

$$\to \forall y.\forall a.\big(\forall x.\underline{R(x,y)} \otimes \exists b.\,\underline{R(a,b)}\big)$$

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$$y \longleftarrow\!\!\dashv b$$

✓

Linking under quantifiers

(Donato, Strub, and Werner 2022)

- Unify linked subformulas
- Check for $\forall\exists$ dependency cycles
- Switch uninstantiated quantifiers
- Instantiate unified variables

$$\exists y.\forall x.R(x,y) \otimes \forall a.\exists b.R(a,b)$$
$$\to \forall y.(\forall x.R(x,y) \otimes \forall a.\exists b.R(a,b))$$
$$\to \forall y.\forall a.(\forall x.R(x,y) \otimes \exists b.R(a,b))$$
$$\to \forall y.\forall a.(\forall x.R(x,y) \otimes R(a,y))$$
$$\to \forall y.\forall a.(R(a,y) \otimes R(a,y))$$
$$\to^* \top$$

$$x \longmapsto a$$

$$y \longleftarrow\!\!\dashv b$$

✓

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- Unify linked subformulas

$$\forall a.\exists b.\underline{R(a,b)} \oslash \exists y.\forall x.\underline{R(x,y)}$$

$$x \longleftarrow\!\dashv a$$

$$y \longmapsto b$$

Linking under quantifiers (Donato, Strub, and Werner 2022)

- Unify linked subformulas
- **Check** for ∀∃ dependency cycles

$$\forall a.\exists b.\underline{R(a,b)} \ogreaterthan \exists y.\forall x.\underline{R(x,y)}$$

$$\begin{tikzcd} x \arrow[d, bend right] & a \arrow[l, mapsto] \\ y \arrow[r, mapsto] & b \arrow[u, bend right] \end{tikzcd}$$

$\times$

Rewriting equalities (Donato, Strub, and Werner 2022)

Add 4 rules $\Longrightarrow$ `rewrite` tactic for free!

$$\underline{t} = u \circledgt A \quad \to \quad A[u/t] \qquad\qquad t = \underline{u} \circledgt A \quad \to \quad A[t/u]$$

$$\underline{t} = u \circledast A \quad \to \quad A[u/t] \qquad\qquad t = \underline{u} \circledast A \quad \to \quad A[t/u]$$

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Compositional with semantics of **connectives**:

- **Quantifiers:** rewrite modulo *unification*
- **Implication:** *conditional* rewrite
- **Arbitrary** combinations are possible:

$$\forall x. x \neq 0 \Rightarrow \underline{f(x)} = g(x) \circledcirc \exists y. A\left(\underline{f(y)}\right) \vee B(y)$$

$$\rightarrow^{*} \ \exists y.(y \neq 0 \wedge A(g(y))) \vee B(y)$$

Completeness

Add the following rules:

- **Init** $C^{+}\boxed{A \Rightarrow B} \to C^{+}\boxed{A \otimes B}$ $C^{-}\boxed{A \wedge B} \to C^{-}\boxed{A \circledast B}$
- **Release** $C^{+}\boxed{A \otimes B} \to C^{+}\boxed{A \Rightarrow B}$ $C^{-}\boxed{A \circledast B} \to C^{-}\boxed{A \wedge B}$
- **Contraction** $C^{-}\boxed{A} \to C^{-}\boxed{A \wedge A}$

Theorem (Completeness): If $\Gamma \vdash A$ is provable in sequent calculus, then

$$\bigwedge \Gamma \Rightarrow A \to^{*} \top$$

Conclusion

- Based on the solid proof theory of SFL
- `coq-actema` still in development, but *already usable*
  ↳ follow install instructions on GitHub!
- Next step: exposure to *real users*
  - **Beginners/students:** introductory logic/proof assistants course
  - **Experts:** real maths codebases

Iconic Manipulations

Bubble Calculi

The chemical metaphor

Item	$\Longleftrightarrow$	Ion
Color	$\Longleftrightarrow$	Polarity
Logical connective	$\Longleftrightarrow$	Chemical bond
Click	$\Longleftrightarrow$	Heating
Drag-and-Drop	$\Longleftrightarrow$	Bimolecular reaction

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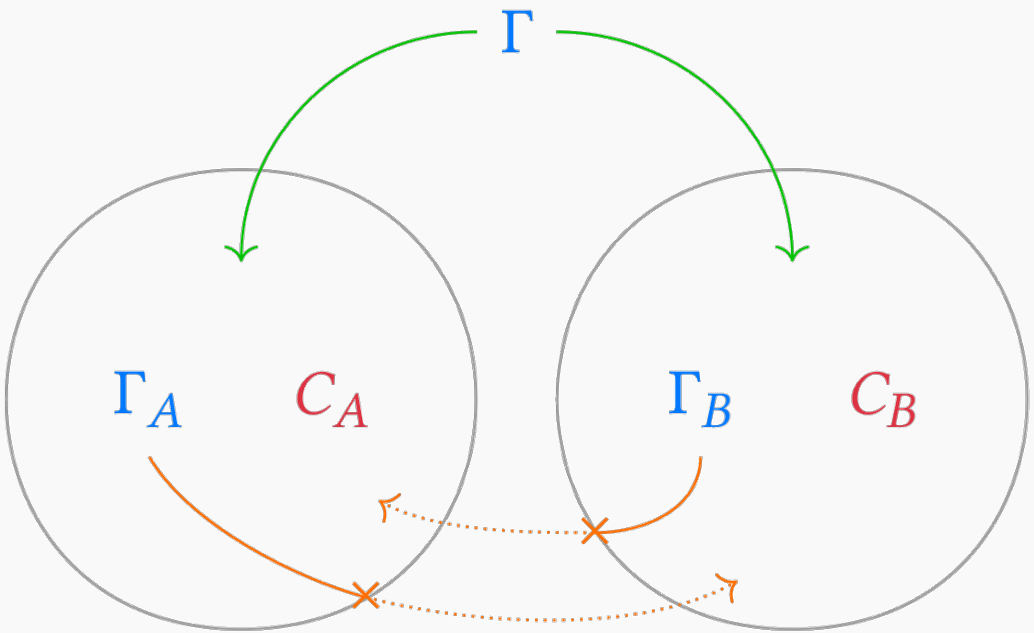
Breaks on rules that create subgoals (e.g. click on $\wedge$)

Context scoping



Two main inspirations:

- The **chemical abstract machine** (Berry and Boudol 1989)
- **Nested sequents** (Brünnler 2009)

Example proof

$$(A \vee B \Rightarrow C) \Rightarrow (A \Rightarrow C) \wedge (B \Rightarrow C)$$

Example proof

$A \vee B \Rightarrow C$ $\quad\quad$ $(A \Rightarrow C) \wedge (B \Rightarrow C)$

Example proof

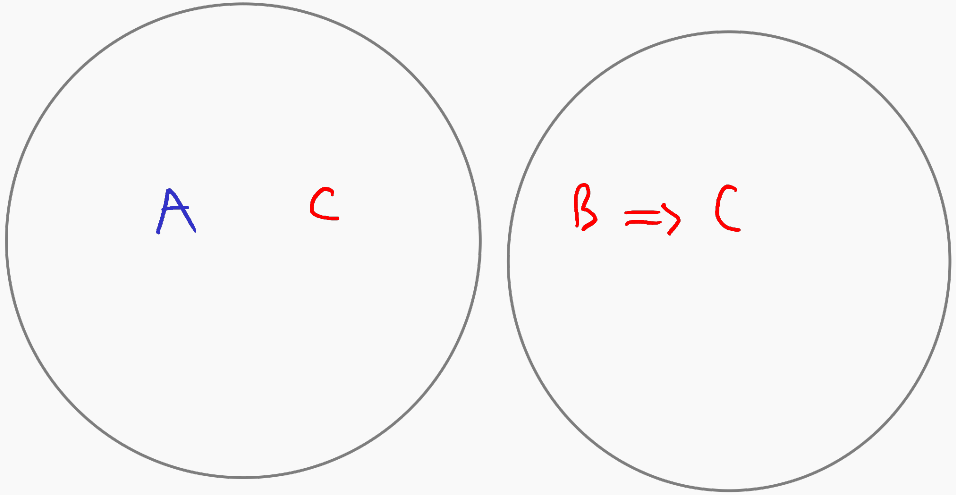
$A \vee B \Rightarrow C$

$A \Rightarrow C$

$B \Rightarrow C$

Example proof

$A \lor B \Rightarrow C$



Example proof

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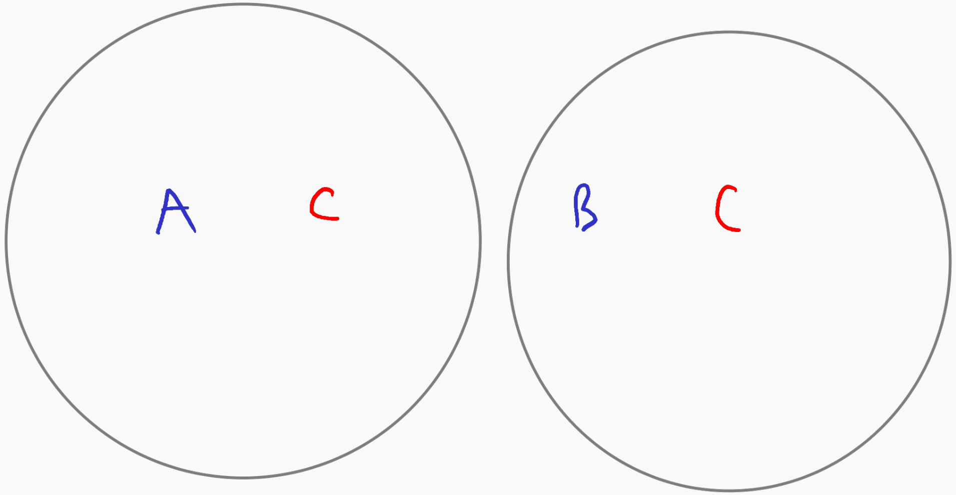

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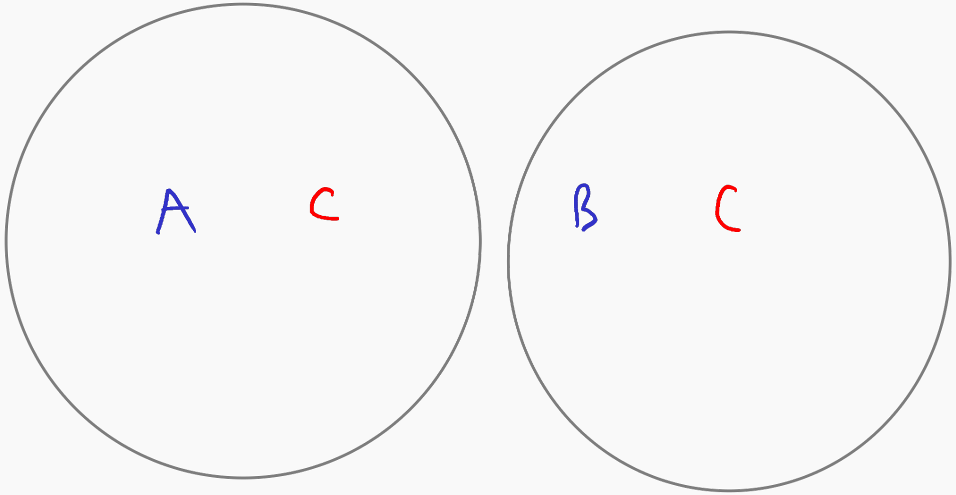

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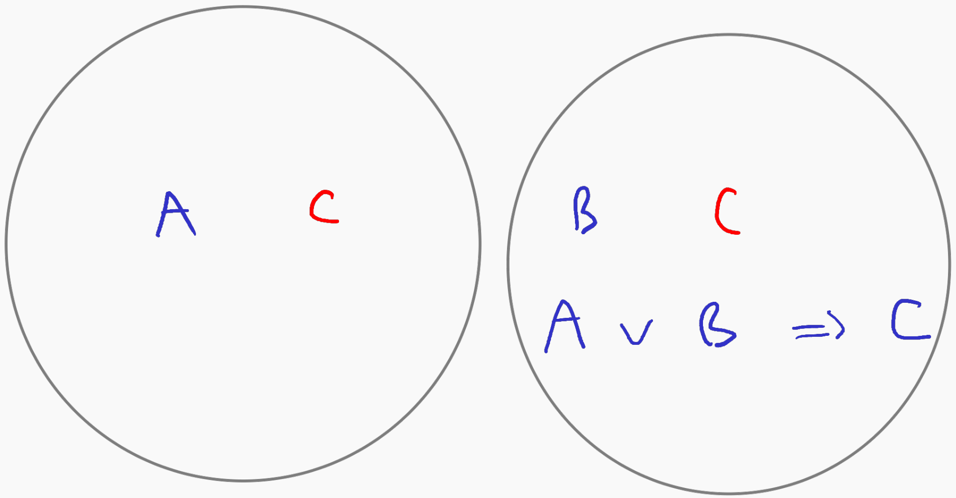

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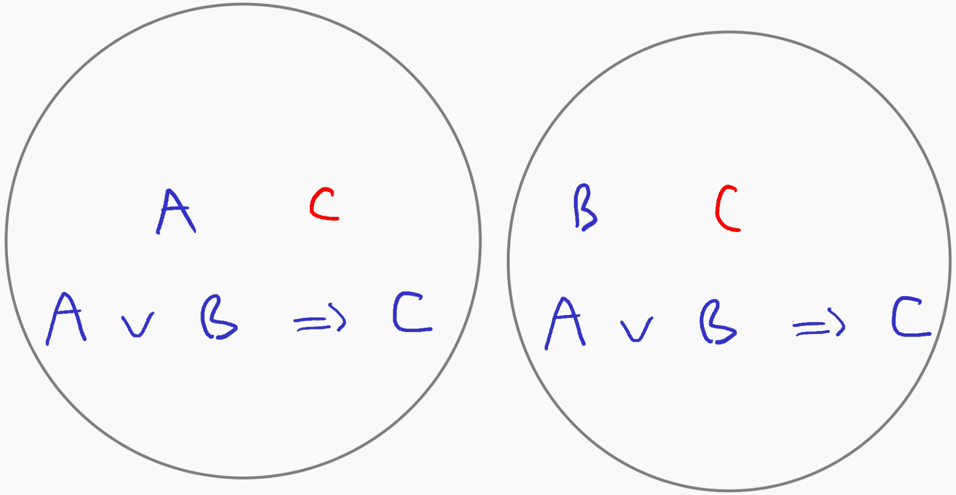

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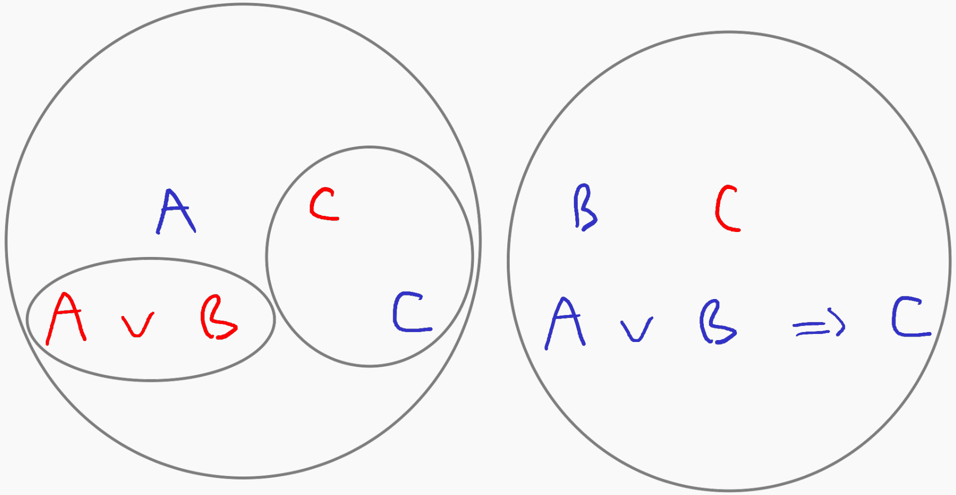

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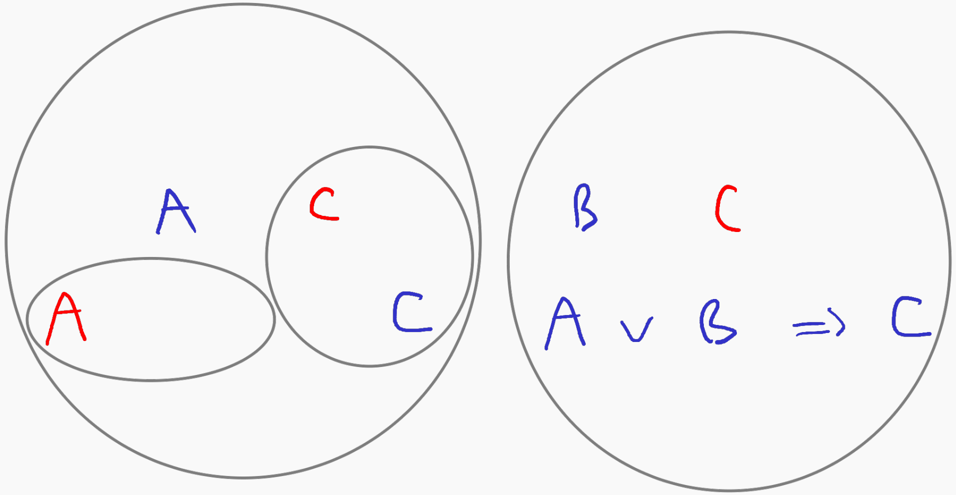

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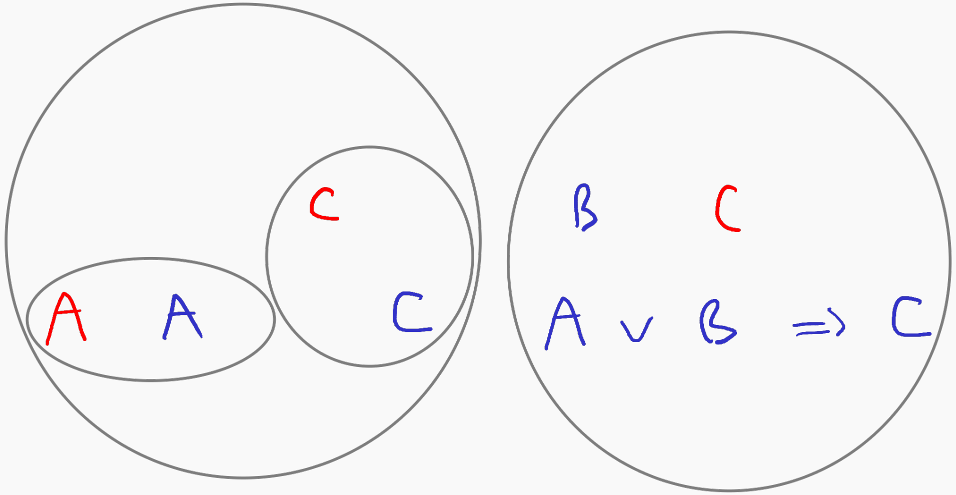

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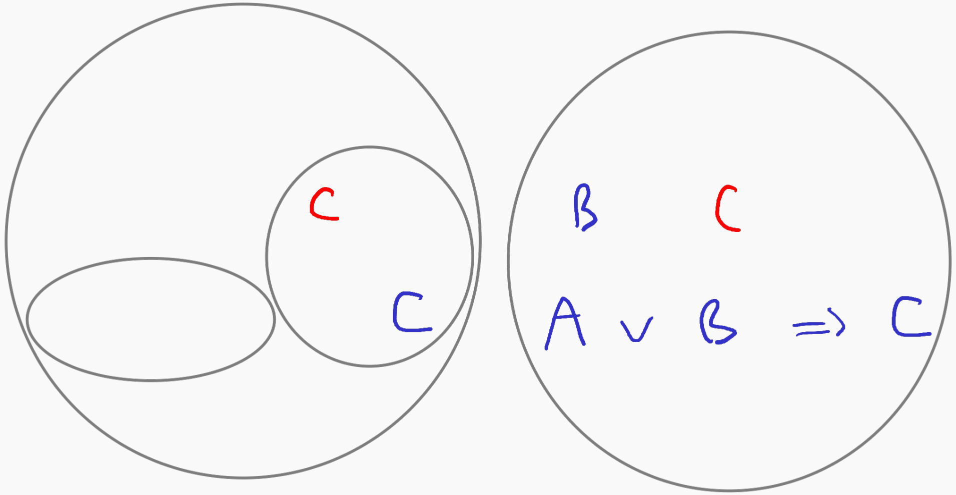

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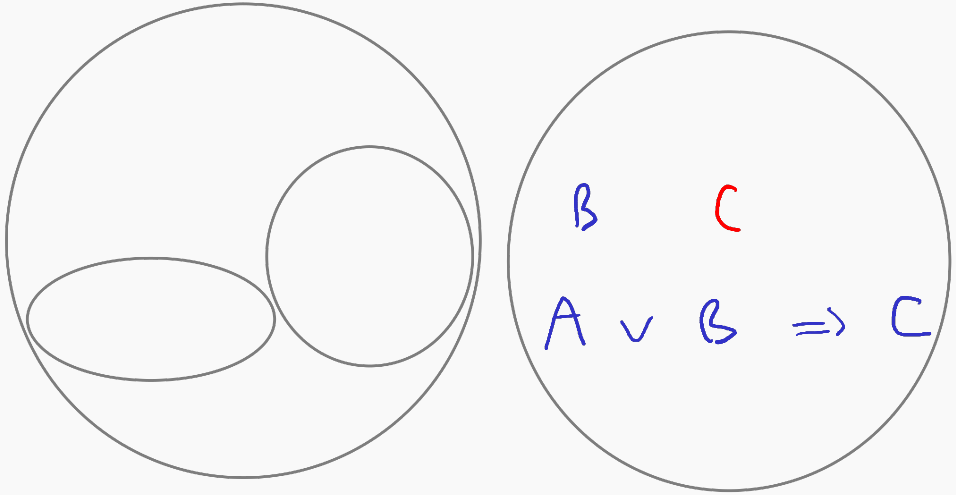

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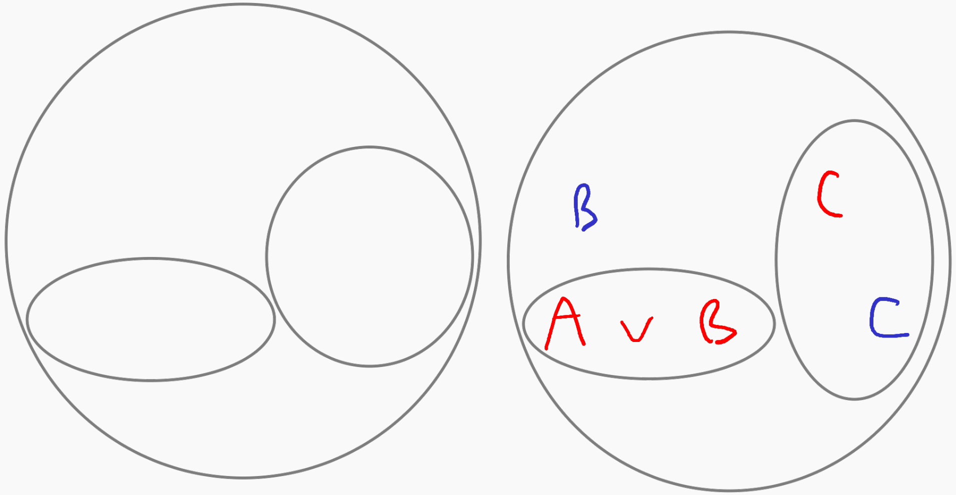

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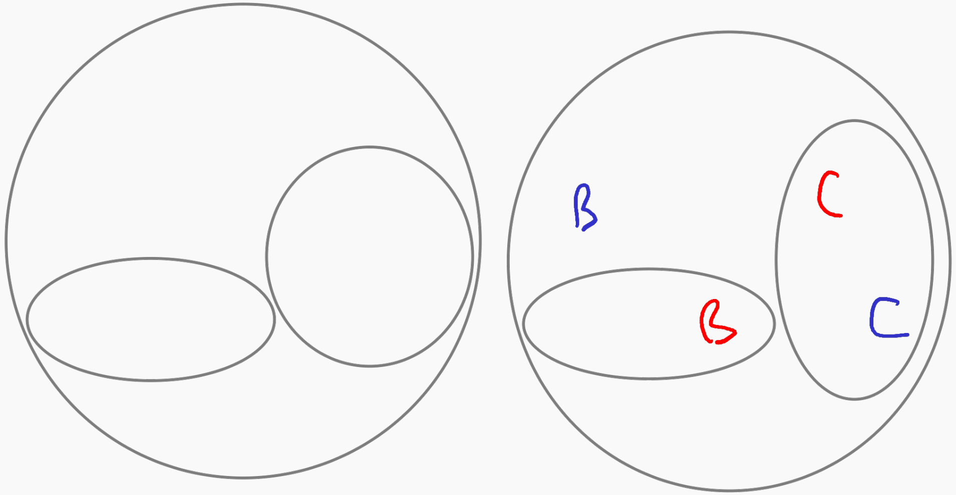

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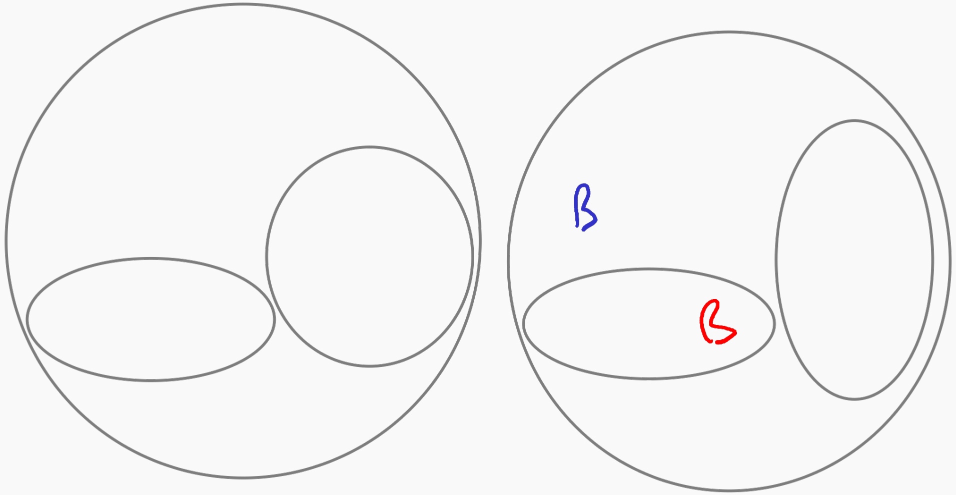

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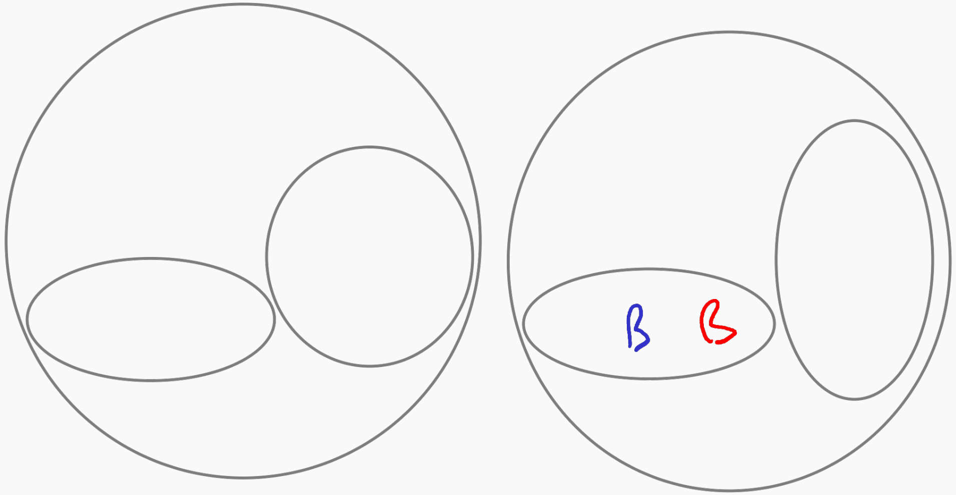

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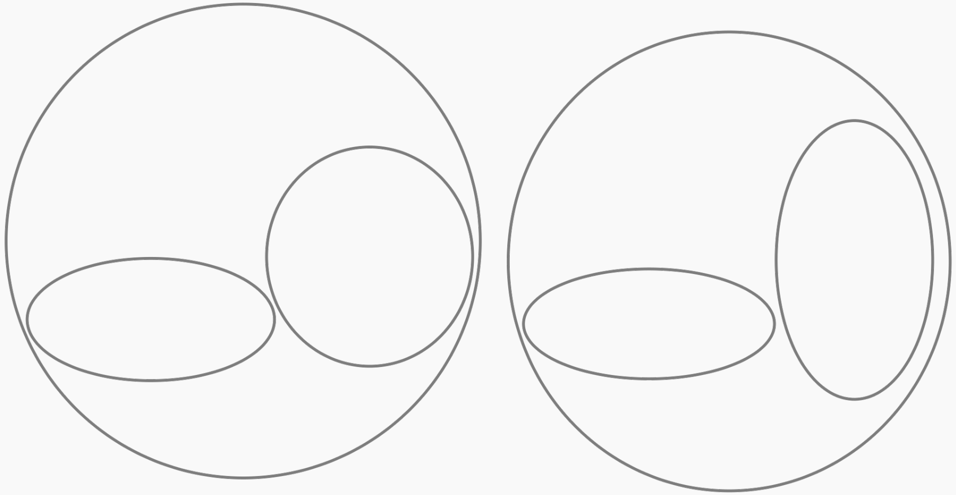
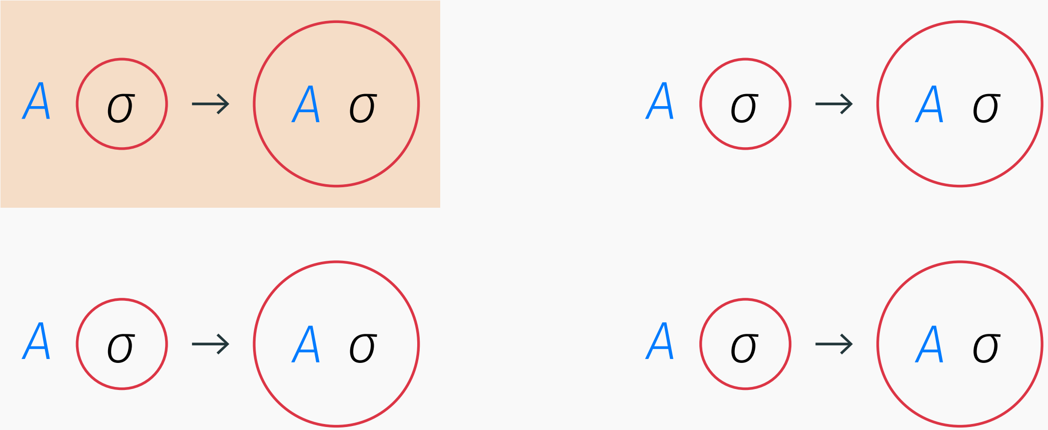


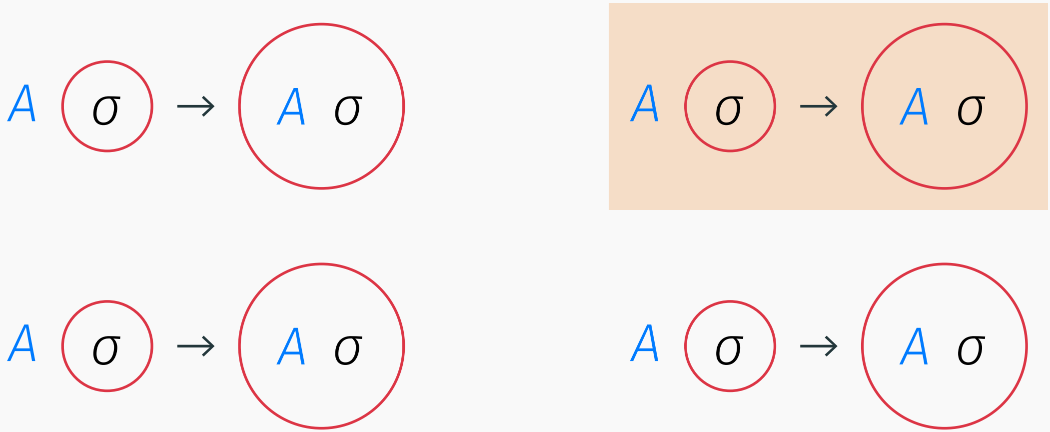
Example proof

Polarized bubbles



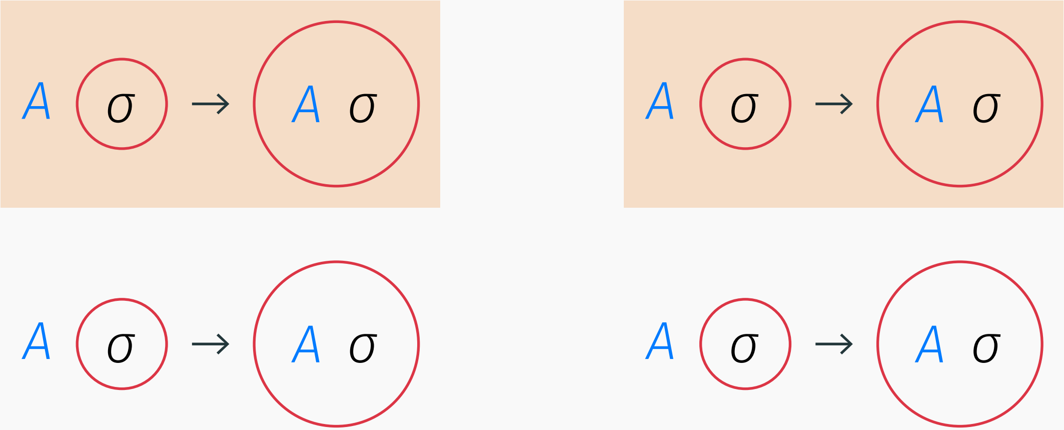
Intuitionistic logic

Polarized bubbles



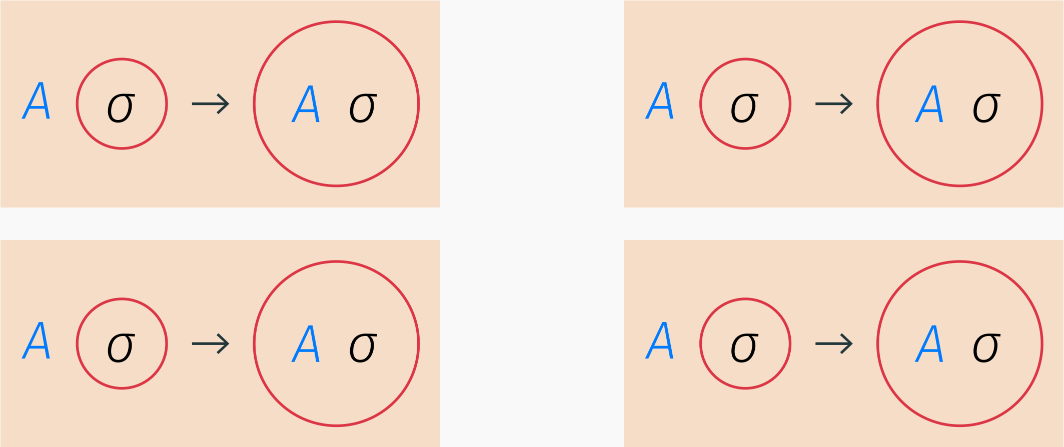
Dual-intuitionistic logic

Polarized bubbles



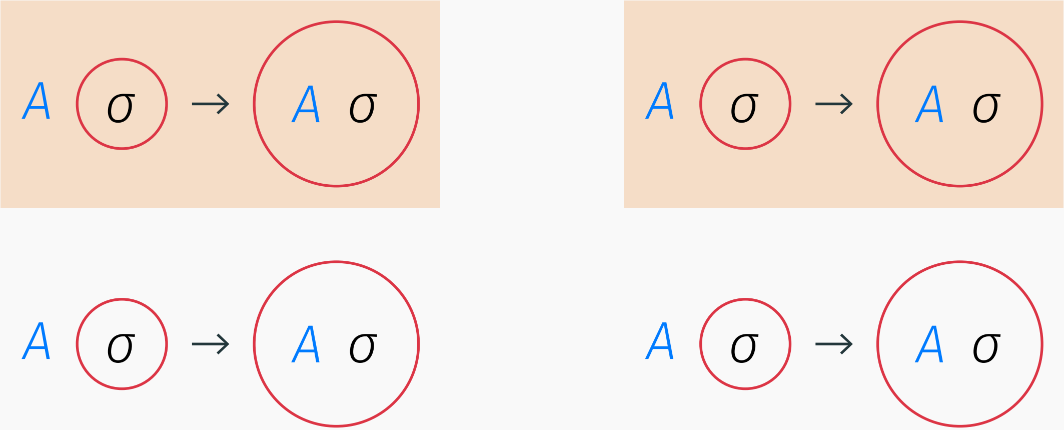
Bi-intuitionistic logic

Polarized bubbles



Classical logic

Polarized bubbles



Intuitionism = same polarities repel eachother

Flower Calculus

Polarity meets Space

- Bubble calculi are not **fully iconic** (need for *symbolic* connectives)

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Key idea: **space** is *polarized*, not **objects**

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- (Oostra 2010; Ma and Pietarinen 2019): EGs for *intuitionistic* logic

Polarity meets Space

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Key idea: **space** is *polarized*, not **objects**

- (Peirce, 1896): **existential graphs (EGs)** for *classical* logic
- (Oostra 2010; Ma and Pietarinen 2019): EGs for *intuitionistic* logic

↳ **Flower calculus**: intuitionistic variant that is analytic

Existential Graphs

- **Diagrammatic** proof system invented by C. S. Peirce around 1890
- Topological representation of **negation** as nested "*cuts*" (Jordan curves):

$A \quad B$	A
$A \wedge B$	$\neg A$

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(Peirce 1906)

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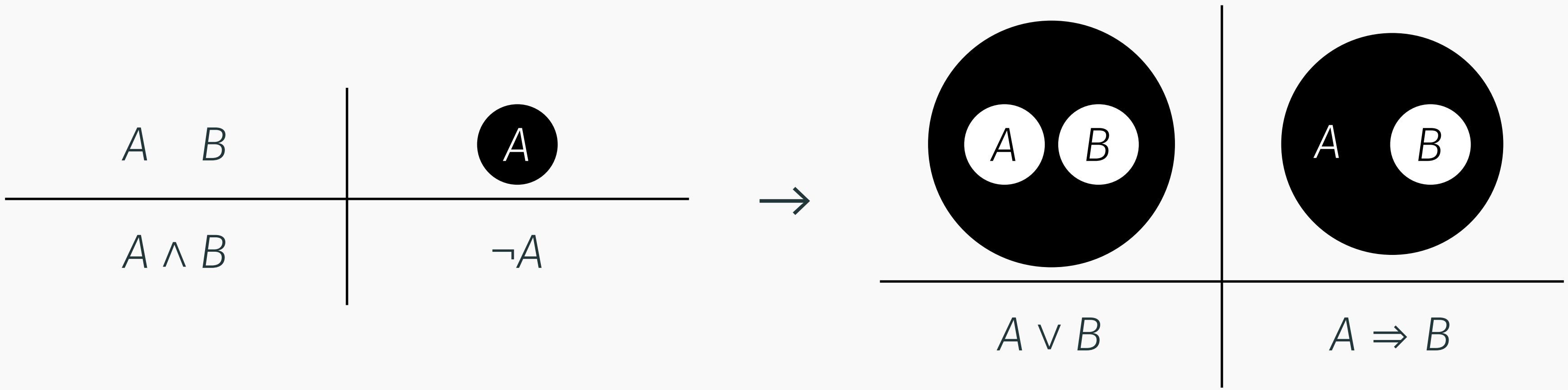
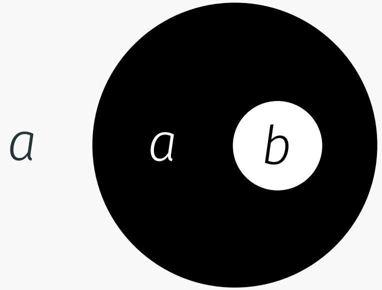


Illative transformations

Inference rules on locations



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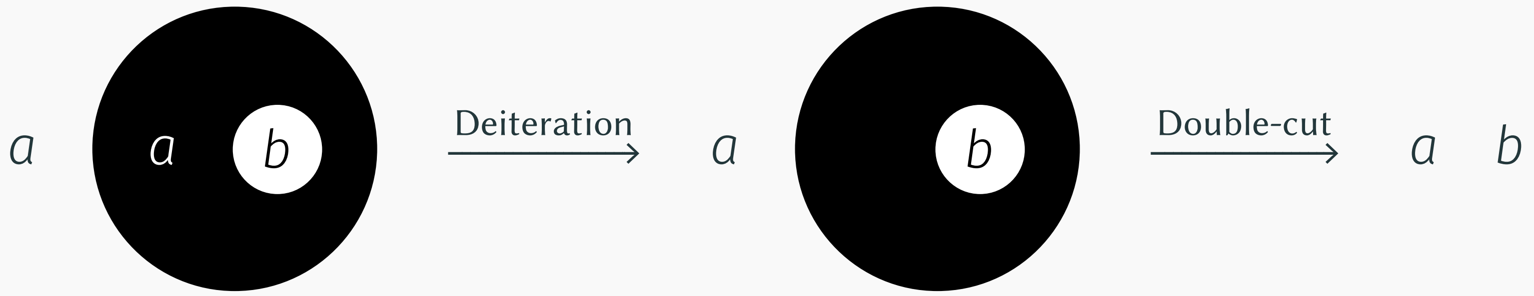

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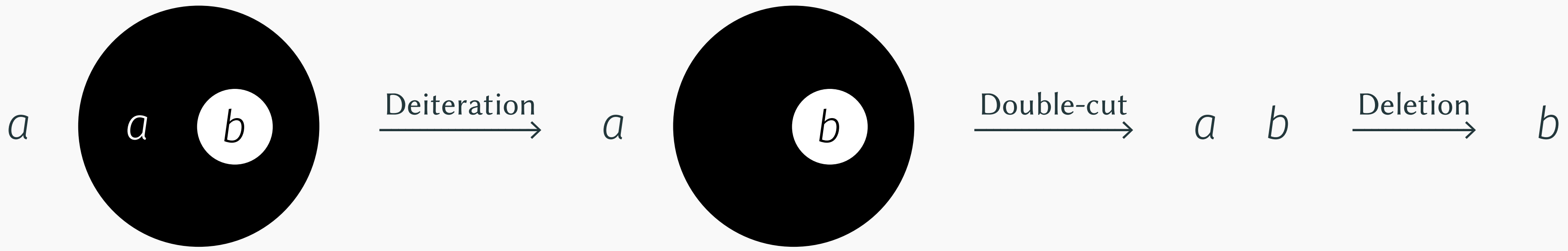
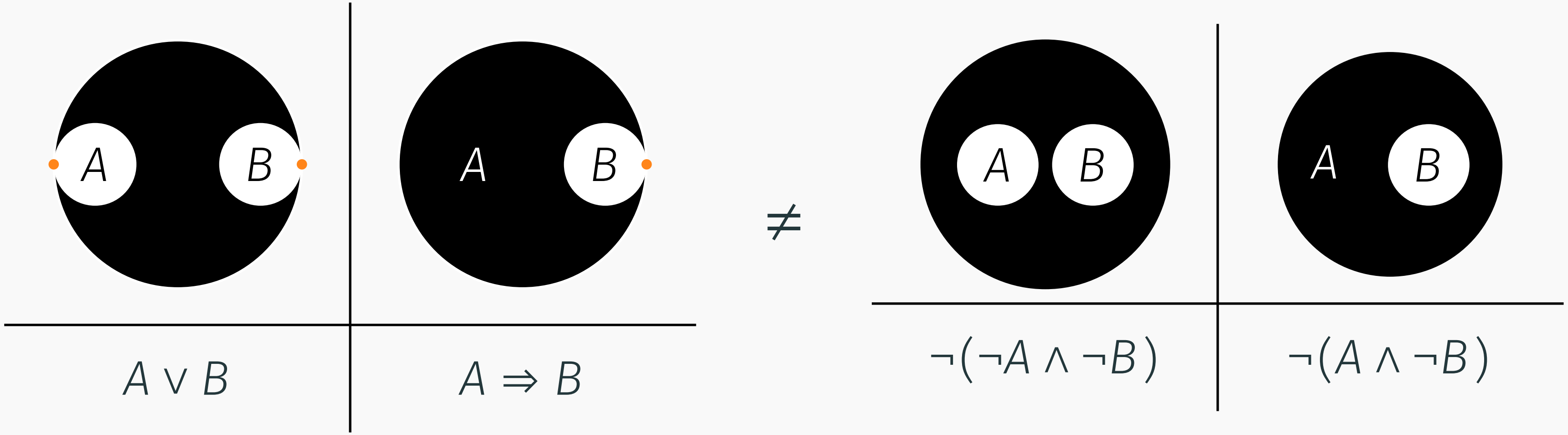
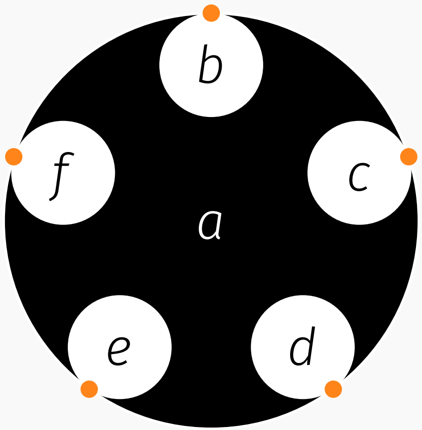


Intuitionistic Existential Graphs (Oostra 2011)

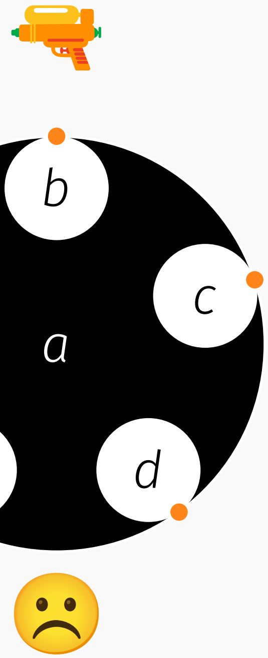
- Topological representation of **implication** with Peirce's "*scroll*"
- Scroll = continuously joined nested cuts:



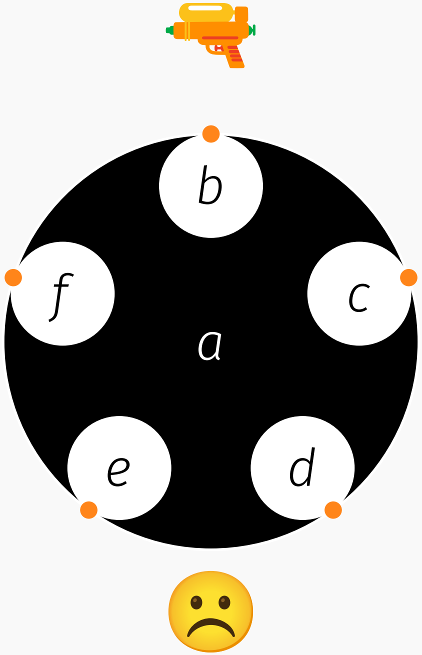
Blooming

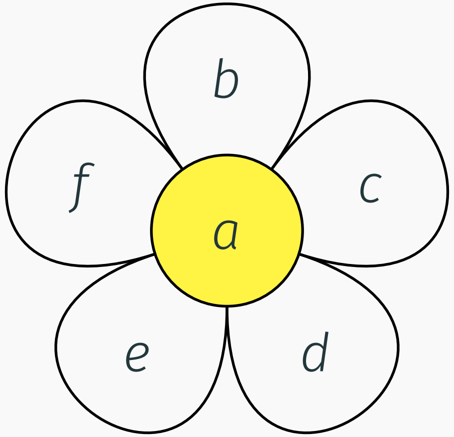


Blooming



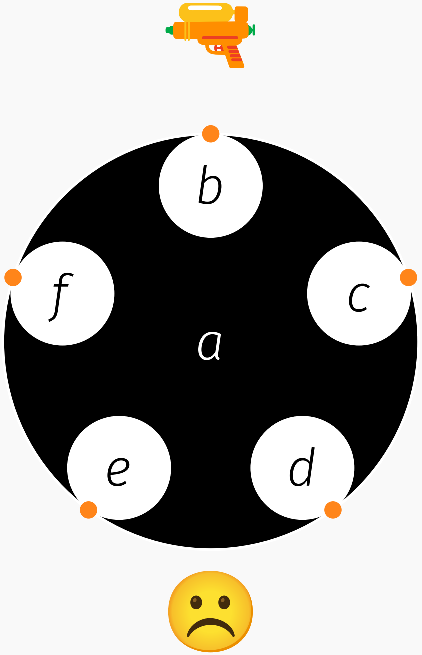
Blooming

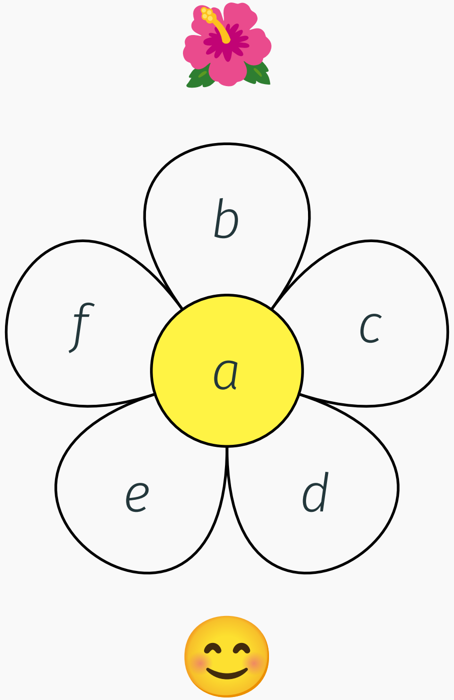




Turn **inloops** into petals.

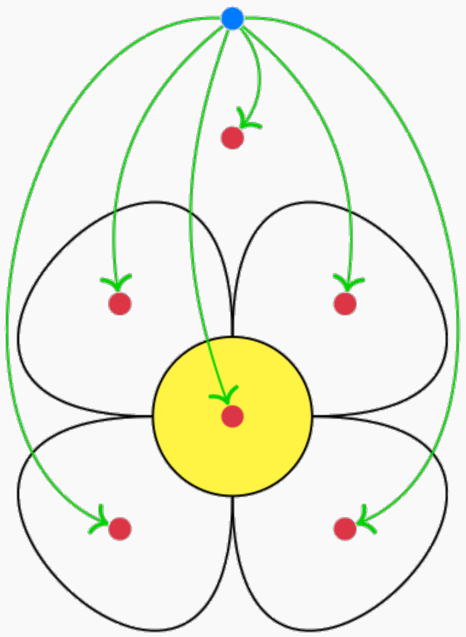
Blooming



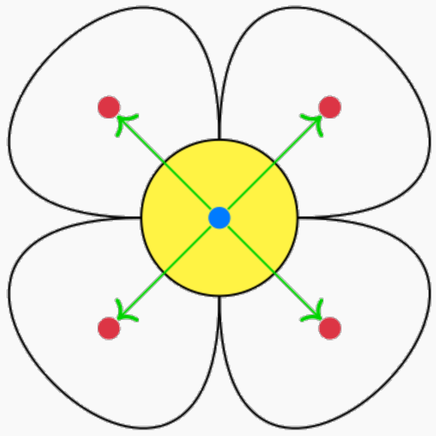


“Make love, not war"

Pollination



Cross-pollination



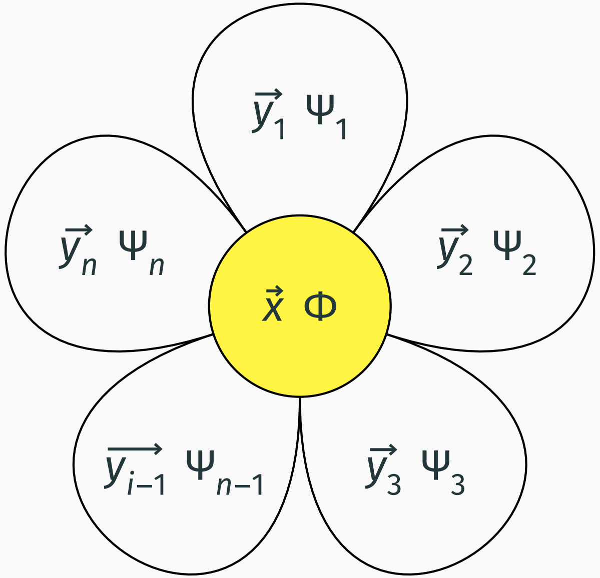
Self-pollination

Flower Calculus

(Donato 2024)

- Support for **quantification** with *binders* $\vec{x}$
- Interpretation as geometric formulas from *topos theory*
- Inference rules divided in two fragments:
  - **Nature** ❀ = analytic and invertible
  - **Culture** ✂ = non-invertible

Theorem (Analytic completeness): If a flower is *valid* (i.e. true in every Kripke model), then it is ❀-provable.



$$\forall \vec{x}. \left(\bigwedge \Phi \Rightarrow \bigvee_i \exists \vec{y}_i . \Psi_i \right)$$

Flower Prover (mascarpone cream)

GUI in the Proof-by-Action paradigm based on the flower calculus

- Represent flowers as nested **boxes**
- **Modal interface** to interpret gestural actions:

Proof mode $\Longleftrightarrow$ Natural (invertible and analytic) rules

Edit mode $\Longleftrightarrow$ Cultural (non-invertible) rules

Navigation mode $\Longleftrightarrow$ Contextual closure (functoriality)

Thank you!

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