ME 5224: Signals and Signals
Spring: 2023
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Homework 2 - Signal Properties

Spring 2023

Exercise 1. CT System Properties

Consider a CT system with input x(t) and output y(t). For each of the following systems, **i**) prove that it is linear or give a counter example, **ii**) prove that it is time-invariant or give a counter example, **iii**) determine whether it is causal or non-causal, and **iv**) determine if it is a memoryless or memory system.

(a) y(t) = u(t)x(t)

i) Is the system linear?
Let $y_1(t) = u(t)x_1(t), y_2(t) = u(t)x_2(t), x_3(t) = ax_1(t) + bx_2(t)$
$\Longrightarrow y_3(t) = u(t)x_3(t) = u(t)(ax_1(t) + bx_2(t)) = au(t)x_1(t) + bu(t)x_2(t) = ay_1(t) + by_2(t)$
$\Longrightarrow$ The system is **LINEAR**.

ii) Is the system time-invariant?
Let $y_1(t)$ be the output of the shifted input $x(t - t_0)$, and $y_2(t)$ be the shifted output.

$$\begin{cases} y_1(t) = u(t)x(t - t_0) \\ y_2(t) = y(t - t_0) = u(t - t_0)x(t - t_0) \end{cases}$$

$\because y_1(t) \neq y_2(t) \Longrightarrow$ The system is **TIME VARYING**.
Counter example:
Let $x(t) = u(t),\ t_0 = -1$

$$\Longrightarrow \begin{cases} y_1(t) = u(t)x(t + 1) = u(t)u(t + 1) = u(t) \\ y_2(t) = u(t + 1)x(t + 1) = u(t + 1)u(t + 1) = u(t + 1) \end{cases}$$

iii) Is the system causal or non-causal?
The system depends only on present time, so the system is **CAUSAL**.

iv) Does the system have memory or memoryless?
The system depends only on present time, so the system is **MEMORYLESS**.

(b) y(t) = x(sin(t))

i) Is the system linear?
Let $y_1(t) = x_1(sin(t))$, $y_2(t) = x_2(sin(t))$, $x_3(sin(t)) = ax_1(sin(t)) + bx_2(sin(t))$
$\Longrightarrow y_3(t) = x_3(sin(t)) = ax_1(sin(t)) + bx_2(sin(t)) = ay_1(t) + by_2(t)$
$\Longrightarrow$ The system is **LINEAR**.

ii) Is the system time-invariant?
Let $y_1(t)$ be the output of the shifted input $x(t - t_0)$, and $y_2(t)$ be the shifted output.

$$\begin{cases} y_1(t) = x(sin(t - t_0)) \\ y_2(t) = y(t - t_0) = x(sin(t) - t_0) \end{cases}$$

$\because y_1(t) \neq y_2(t) \Longrightarrow$ The system is **TIME VARYING**.
Counter example:
Let $x(t) = sin^{-1}(t),\ t_0 = 1$

$$\Longrightarrow \begin{cases} y_1(t) = x(sin(t - 1)) = sin^{-1}(sin(t - 1)) = t - 1 \\ y_2(t) = x(sin(t) - 1) = sin^{-1}(sin(t) - 1) \end{cases}$$

iii) Is the system causal or non-causal?
Since -1 $\leq sin(t) \leq$ 1, so for any t < -1, $sin(t)$ > t, therefore, the system is **NON-CAUSAL**.

iv) Does the system have memory or memoryless?
Since -1 $\leq sin(t) \leq$ 1, as t gets greater $x(t)$ will only depend on -1 $\leq t \leq$ 1, so the system has **MEMORY**.

(c) y(t) = sin(x(t))

i) Is the system linear?
Let $y_1(t) = sin(x_1(t))$, $y_2(t) = sin(x_2(t))$, $x_3(t) = ax_1(t) + bx_2(t)$
$\implies y_3(t) = sin(x_3(t)) = sin(ax_1(t) + bx_2(t)) \neq asin(x_1(t)) + bsin(x_2(t))$
$\implies$ The system is **NON-LINEAR**.
Counter example:
Let $x_1(t) = sin^{-1}(t)$, $x_2(t) = 0$, $x_3(t) = ax_1(t)$
$\implies y_3(t) = sin(x_3(t)) = sin(ax_1(t)) \neq ay_1(t) = asin(sin^{-1}(t)) = at$

ii) Is the system time-invariant?
Let $y_1(t)$ be the output of the shifted input $x(t-t_0)$, and $y_2(t)$ be the shifted output.

$$\begin{cases} y_1(t) = sin(x(t-t_0)) \\ y_2(t) = y(t-t_0) = sin(x(t-t_0)) \end{cases}$$

$\because y_1(t) = y_2(t) \implies$ The system is **TIME-INVARIANT**.

iii) Is the system causal or non-causal?
The system depends only on present time, so the system is **CAUSAL**.

iv) Does the system have memory or memoryless?
The system depends only on present time, so the system is **MEMORYLESS**.

(d) y(t) = $\frac{dx(t)}{dt}$

i) Is the system linear?
Let $y_1(t) = \frac{dx_1(t)}{dt}$, $y_2(t) = \frac{dx_2(t)}{dt}$, $x_3(t) = ax_1(t) + bx_2(t)$
$\implies y_3(t) = \frac{dx_3(t)}{dt} = \frac{d(ax_1(t)+bx_2(t))}{dt} = a\frac{dx_1(t)}{dt} + b\frac{dx_2(t)}{dt} = ay_1(t) + by_2(t)$
$\implies$ The system is **LINEAR**.

ii) Is the system time-invariant?
Let $y_1(t)$ be the output of the shifted input $x(t-t_0)$, and $y_2(t)$ be the shifted output.

$$\begin{cases} y_1(t) = \frac{dx(t-t_0)}{dt} \\ y_2(t) = y(t-t_0) = \frac{dx(t-t_0}{dt} \end{cases}$$

$\because y_1(t) = y_2(t) \implies$ The system is **TIME-INVARIANT**.

iii) Is the system causal or non-causal?
The system depends only on present time, so the system is **CAUSAL**.

iv) Does the system have memory or memoryless?
The system depends only on present time, so the system is **MEMORYLESS**.

(e) y(t) = x(2t)-x(t-1)

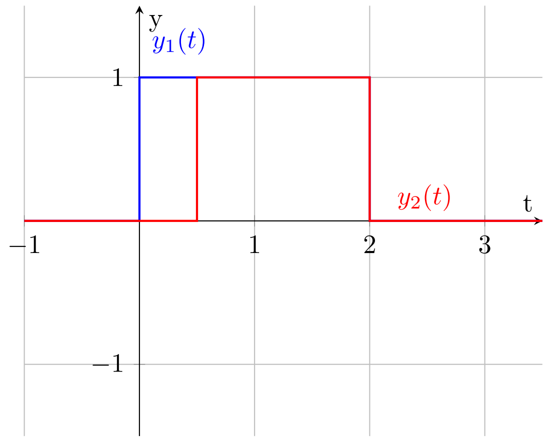
i) Is the system linear?
Let $y_1(t) = x_1(2t) - x_1(t-1)$, $y_2(t) = x_2(2t) - x_2(t-1)$, $x_3(t) = ax_1(t) + bx_2(t)$
$\implies y_3(t) = x_3(2t) - x_3(t-1) = (ax_1(2t) + bx_2(2t)) - (ax_1(t-1) + bx_2(t-1))$
$\implies y_3(t) = a(x_1(2t) - x_1(t-1)) + b(x_2(2t) - x_2(t-1)) = ay_1(t) + by_2(t)$
$\implies$ The system is **LINEAR**.

ii) Is the system time-invariant?
Let $y_1(t)$ be the output of the shifted input $x(t-t_0)$, and $y_2(t)$ be the shifted output.

$$\begin{cases} y_1(t) = x(2(t-t_0)) - x(t-t_0-1) \\ y_2(t) = x(2t-t_0) - x(t-t_0-1) \end{cases}$$

$\because y_1(t) \neq y_2(t) \implies$ The system is **TIME-VARYING**.
Counter example:
Let $x(t) = u(t)$, $t_0 = 1$

$$\implies \begin{cases} y_1(t) = x(2(t-1)) - x(t-2) = u(2(t-1)) - u(t-2) \\ y_2(t) = x(2t-1) - x(t-2) = u(2t-1) - u(t-2) \end{cases}$$



iii) Is the system causal or non-causal?
When t $>$ 0, the output depends on $2t$, so the system is **NON-CAUSAL**.

iv) Does the system have memory or memoryless?
The system depends on $t-1$, so the system has **MEMORY**.

(f) y(t) = x(0)

i) Is the system linear?
Let $y_1(t) = x_1(0)$, $y_2(t) = x_2(0)$, $x_3(0) = ax_1(0) + bx_2(0)$
$\implies y_3(t) = x_3(0) = ax_1(0) + bx_2(0) = ay_1(t) + by_2(t)$
$\implies$ The system is **LINEAR**.

ii) Is the system time-invariant?
Let $y_1(t)$ be the output of the shifted input $x(t-t_0)$, and $y_2(t)$ be the shifted output.

$$\begin{cases} y_1(t) = x(0 - t_0) = x(-t_0) \\ y_2(t) = y(t-t_0) = x(0) \end{cases}$$

$\because y_1(t) \neq y_2(t) \implies$ The system is **TIME-VARYING**.
Counter example:
Let $x(t) = u(t-1)$, $t_0 = -1$

$$\implies \begin{cases} y_1(t) = x(0-(-1)) = x(1) = u(0) = 1 \\ y_2(t) = y(t-(-1)) = x(0) = u(-1) = 0 \end{cases}$$

iii) Is the system causal or non-causal?
When t $<$ 0, the system depends on future value (t=0), so the system is **NON-CAUSAL**.

iv) Does the system have memory or memoryless?
When t $>$ 0, the system depends on previous value (t=0), so the system has **MEMORY**.

(g) y(t) = $\int_0^t$x(τ)dτ

i) Is the system linear?
Let $y_1(t) = \int_0^t x_1(\tau)d\tau$, $y_2(t) = \int_0^t x_2(\tau)d\tau$, $x_3(t) = ax_1(t) + bx_2(t)$
$\implies y_3(t) = \int_0^t x_3(\tau)d\tau = \int_0^t ax_1(\tau) + bx_2(\tau)d\tau = a\int_0^t x_1(\tau)d\tau + b\int_0^t x_2(\tau)d\tau = ay_1(t) + by_2(t)$
$\implies$ The system is **LINEAR**.

ii) Is the system time-invariant?
Let $y_1(t)$ be the output of the shifted input $x(t-t_0)$, and $y_2(t)$ be the shifted output.

$$\begin{cases} y_1(t) = \int_0^t x(\tau - t_0)d\tau \\ y_2(t) = y(t-t_0) = \int_0^{t-t_0} x(\tau)d\tau \end{cases}$$

$\because y_1(t) \neq y_2(t) \implies$ The system is **TIME-VARYING**.
Counter example:
Let $x(t) = u(t)$, $t_0 = -2$

$$\implies \begin{cases} y_1(t) = \int_0^t u(\tau - (-2))d\tau = t \\ y_2(t) = y(t-(-2)) = \int_0^{t-(-2)} u(\tau)d\tau = t+2 \end{cases}$$

iii) Is the system causal or non-causal?
The system depends only on present and past value, so the system is **CAUSAL**.

iv) Does the system have memory or memoryless?
When t $\neq$ 0, the system will depend on past values, so the system has **MEMORY**.

Exercise 2. Dt System Properties

Consider a DT system with input $x[n]$ and output $y[n]$. For each of the following system, **i**) prove that it is linear or give a counter example, **ii**) prove that it is time-invariant or give a counter example.

(a) y[n]=x[n]+1

i) Is the system linear?
Let $y_1[n] = x_1[n] + 1$, $y_2[n] = x_2[n] + 1$, $x_3[n] = ax_1[n] + bx_2[n]$
$\implies y_3[n] = x_3[n] + 1 = ax_1[n] + bx_2[n] + 1 \neq ay_1[n] + by_2[n]$
$\implies$ The system is **NON-LINEAR**.
Counter example:
Let $x_1[n] = u[n]$, $x_2[n] = u[n]$, $x_3[n] = x_1[n] + x_2[n] \implies x_3[n] = 2u[n]$
$\implies y_3[n] = x_3[n] + 1 = 2u[n] + 1 \neq 2u[n] + 2$

ii) Is the system time-invariant?
Let $y_1[n]$ be the output of the shifted input $x[n-n_0]$, and $y_2[n]$ be the shifted output.

$$\begin{cases} y_1[n] = x[n-n_0] + 1 \\ y_2[n] = y[n-n_0] = x[n-n_0] + 1 \end{cases}$$

$\because y_1[n] = y_2[n] \implies$ The system is **TIME-INVARIANT**.

(b) y[n]=x[2n] (This operation is known as *decimation*.)

i) Is the system linear?
Let $y_1[n] = x_1[2n]$, $y_2[n] = x_2[2n]$, $x_3[n] = ax_1[n] + bx_2[n]$
$\implies y_3[n] = x_3[2n] = ax_1[2n] + bx_2[2n] = ay_1[n] + by_2[n]$
$\implies$ The system is **LINEAR**.

ii) Is the system time-invariant?
Let $y_1[n]$ be the output of the shifted input $x[n-n_0]$, and $y_2[n]$ be the shifted output.

$$\begin{cases} y_1[n] = x[2(n-n_0)] \\ y_2[n] = y[n-n_0] = x[2n-n_0] \end{cases}$$

$\because y_1[n] \neq y_2[n] \implies$ The system is **TIME-VARYING**.

Counter example:

Let $x[n] = \begin{cases} 1, & \text{n=even} \\ 0, & \text{n=odd} \end{cases} \qquad n_0 = 1$

$$\implies \begin{cases} y_1[n] = x[2(n-1)] = 1 & \text{(since 2(n-1) is always even)} \\ y_2[n] = y[n-1] = x[2n-1] = 0 & \text{(since 2n-1 is always odd)} \end{cases}$$

(c) y[n] = $\begin{cases} x[n/2] & , n = \text{even} \\ 0 & , n = \text{odd} \end{cases}$

i) Is the system linear?

Let $y_1[n] = \begin{cases} x_1[n/2] & , n = \text{even} \\ 0 & , n = \text{odd} \end{cases}$, $y_2[n] = \begin{cases} x_2[n/2] & , n = \text{even} \\ 0 & , n = \text{odd} \end{cases}$

$\implies y_3[n] = \begin{cases} x_3[n/2] & , n = \text{even} \\ 0 & , n = \text{odd} \end{cases} = \begin{cases} ax_1[n/2] + bx_2[n/2] & , n = \text{even} \\ 0 & , n = \text{odd} \end{cases} = \begin{cases} ay_1[n] + by_2[n] & , n = \text{even} \\ 0 & , n = \text{odd} \end{cases}$

$\implies$ The system is **LINEAR**.

ii) Is the system time-invariant?

Let $y_1[n]$ be the output of the shifted input $x[n-n_0]$, and $y_2[n]$ be the shifted output.

$$\begin{cases} y_1[n] = \begin{cases} x[(n-n_0)/2] & , n = \text{even} \\ 0 & , n = \text{odd} \end{cases} \\ y_2[n] = y[n-n_0] = \begin{cases} x[n/2 - n_0] & , n = \text{even} \\ 0 & , n = \text{odd} \end{cases} \end{cases}$$

$\because y_1[n] \neq y_2[n] \implies$ The system is **TIME-VARYING**.

Counter example:

Let $x[n] = n$, $n_0 = 1$

$y_1[n] = \begin{cases} (n-1)/2 & , n = \text{even} \\ 0 & , n = \text{odd} \end{cases} , \qquad y_2[n] = y[n-1] = \begin{cases} n/2 - 1 & , n = \text{even} \\ 0 & , n = \text{odd} \end{cases}$

(d) y[n] = $\begin{cases} x[n] & , x[n] < 4 \\ 4 & , \text{else} \end{cases}$

i) Is the system linear?

Let $y_1[n] = \begin{cases} x_1[n] & , x_1[n] < 4 \\ 4 & , \text{else} \end{cases}$, $y_2[n] = \begin{cases} x_2[n] & , x_2[n] < 4 \\ 4 & , \text{else} \end{cases}$, $\quad x_3[n] = ax_1[n] + bx_2[n]$

$\implies y_3[n] = \begin{cases} x_3[n] & , x_3[n] < 4 \\ 4 & , \text{else} \end{cases} \implies \begin{cases} ax_1[n] + bx_2[n] & , ax_1[n] + bx_2[n] < 4 \\ 4 & , \text{else} \end{cases}$

$\implies y_3[n] = \begin{cases} ax_1[n] + bx_2[n] & , ax_1[n] + bx_2[n] < 4 \\ 4 & , \text{else} \end{cases} \neq a \begin{cases} x_1[n] & , x_1[n] < 4 \\ 4 & , \text{else} \end{cases} + b \begin{cases} x_2[n] & , x_2[n] < 4 \\ 4 & , \text{else} \end{cases}$

$\implies$ The system is **NON-LINEAR**.

Counter example:

Let $x_1[n] = u[n], \quad x_2[n] = 4u[n], \quad x_3[n] = x_1[n] + x_2[n] = 5u[n]$

$y_3[n] = \begin{cases} 0 & , n < 0 \\ 4 & , n \geq 0 \end{cases} \neq \begin{cases} 0 & , n < 0 \\ 1 & , n \geq 0 \end{cases} + \begin{cases} 0 & , n < 0 \\ 4 & , n \geq 0 \end{cases} = \begin{cases} 0 & , n < 0 \\ 5 & , n \geq 0 \end{cases}$

ii) Is the system time-invariant?

Let $y_1[n]$ be the output of the shifted input $x[n-n_0]$, and $y_2[n]$ be the shifted output.

$$\begin{cases} y_1[n] = \begin{cases} x[n-n_0] & , x[n-n_0] < 4 \\ 4 & , else \end{cases} \\ y_2[n] = y[n-n_0] = \begin{cases} x[n-n_0] & , x[n-n_0] < 4 \\ 4 & , else \end{cases} \end{cases}$$

$\because y_1[n] = y_2[n] \implies$ The system is **TIME-INVARIANT**.