ME 5224: Signals and Signals
Spring: 2023
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Homework 7 - The Continuous-Time Fourier Transform

Spring 2023

Exercise 1. Evaluating CTFTs

Calculate the continuous-time Fourier transform for the following signals:

(a) $x(t) = e^{-at}u(t)$ for $a > 0$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} e^{-(a+j\omega)t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty} = \frac{1}{a+j\omega}$$

(b) $x(t) = te^{-at}u(t)$ for $a > 0$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} te^{-at}u(t)e^{-j\omega t}dt = \int_{0}^{\infty} te^{-(a+j\omega)t}dt$$

$$\text{Let} \begin{cases} u = t & , du = dt \\ dv = e^{-(a+j\omega)t}dt & , v = -\dfrac{e^{-(a+j\omega)t}}{a+j\omega} \end{cases}$$

$$\Longrightarrow X(\omega) = --\frac{te^{-(a+j\omega)t}}{a+j\omega}\Big|_0^{\infty} - \int_0^{\infty} -\frac{e^{-(a+j\omega)t}}{a+j\omega}dt = 0 - 0 - \frac{e^{-(a+j\omega)t}}{(a+j\omega)^2}\Big|_0^{\infty} = \frac{1}{(a+j\omega)^2}$$

(c) $x(t) = rect(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} rect(t)e^{-j\omega t}dt = \int_{-1/2}^{1/2} e^{-j\omega t}dt = \frac{-e^{-j\omega t}}{j\omega}\Big|_{-1/2}^{1/2} = \frac{-1}{j\omega}(e^{-j\omega/2} - e^{j\omega/2})$$

$$= \frac{2}{\omega}\left(\frac{e^{j\frac{\omega}{2}} - e^{-j\frac{\omega}{2}}}{2j}\right) = \frac{2}{\omega}sin\left(\frac{\omega}{2}\right)$$

(d) $x(t) = rect\left(\frac{t-a}{b}\right)$ for any two real numbers a and b

Note: $rect\left(\frac{t}{\tau}\right) = u(t+\tau/2) - u(t-\tau/2)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} rect(\frac{t-a}{b})e^{-j\omega t}dt = \int_{-\infty}^{\infty}(u(t-(a-b/2)) - u(t-(a+b/2))e^{-j\omega t}dt$$

$$= \int_{a-b/2}^{a+b/2} e^{-j\omega t}dt = -\frac{1}{j\omega}e^{-j\omega t}\Big|_{a-b/2}^{a+b/2} = -\frac{1}{j\omega}(e^{-j\omega(a+b/2)} - e^{-j\omega(a-b/2)})$$

$$= \frac{2e^{-j\omega a}}{\omega}\left(\frac{e^{j\omega b/2} - e^{-j\omega b/2}}{2j}\right) = \frac{2e^{-j\omega a}}{\omega}sin\left(\frac{\omega b}{2}\right)$$

(e) $x(t) = \delta(t)$

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} \delta(t)e^{-j\omega t}dt = e^{-j\omega t}\Big|_{t=0} = 1$$

(f) $x(t) = a\delta(t-b)$ for any two real numbers a and b

$$X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} a\delta(t-b)e^{-j\omega t}dt = ae^{-j\omega t}\Big|_{t=b} = ae^{-j\omega b}$$

Exercise 2. Evaluating Inverse CTFTs

Calculate the **inverse CTFT** for the following signals.

(a) $X(\omega) = \delta(\omega)$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}e^{j\omega t}\Big|_{\omega=0} = \frac{1}{2\pi}$$

(b) $X(\omega) = \delta(\omega - \omega_0)$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \delta(\omega-\omega_0)e^{j\omega t}d\omega = \frac{1}{2\pi}e^{j\omega t}\Big|_{\omega=\omega_0} = \frac{e^{j\omega_0 t}}{2\pi}$$

(c) $X(\omega) = rect(\omega)$
$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} rect(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-1/2}^{1/2} e^{j\omega t}dt = \frac{1}{j2\pi t}e^{j\omega t}\Big|_{-1/2}^{1/2}$$
$$= \frac{1}{\pi t}\Big(\frac{e^{jt/2} - e^{-jt/2}}{2j}\Big) = \frac{sin(t/2)}{\pi t}$$

Exercise 3. Evaluating CTFTs

Use the CTFT properties to compute the CTFT's of the following signals.

(a) $x(t) = sinc(t)$
$$x(t) = sinc(t) = \frac{sin(\pi t)}{\pi t} \Longrightarrow X(\omega) = rect\Big(\frac{\omega}{2\pi}\Big) = \begin{cases} 1 & ,|\omega| < \pi \\ 0 & ,\text{otherwise} \end{cases}$$

(b) $x(t) = sinc\Big(\frac{t-a}{b}\Big)$ for any two real numbers a and b
Let $x_1(t) = sin\Big(\frac{t}{b}\Big) \Longrightarrow x(t) = x_1\Big(t - \frac{a}{b}\Big) \Longrightarrow X_1(\omega) = |b|rect\Big(\frac{b\omega}{2\pi}\Big) \Longrightarrow X(\omega) = e^{-j\omega a/b}|b|rect\Big(\frac{b\omega}{2\pi}\Big)$
Note: Here we used the scaling property first, then time shifting property.

(c) $x(t) = 1$
$x(t) = 1 \Longrightarrow X(\omega) = 2\pi\delta(\omega)$

(d) $x(t) = e^{j\omega_0 t}$
$x(t) = e^{j\omega_0 t} \Longrightarrow X(\omega) = 2\pi\delta(\omega - \omega_0)$

(e) $x(t) = cos(\omega_0 t)$
$$x(t) = cos(\omega_0 t) = \frac{1}{2}(e^{j\omega_0 t} + e^{-j\omega_0 t}) \Longrightarrow X(\omega) = \frac{2\pi}{2}(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)) = \pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0))$$

(f) $x(t) = sin(\omega_0 t)$
$$x(t) = sin(\omega_0 t) = \frac{1}{2}(e^{j\omega_0 t} - e^{-j\omega_0 t}) \Longrightarrow X(\omega) = \frac{2\pi}{2}(\delta(\omega-\omega_0)-\delta(\omega+\omega_0)) = \pi(\delta(\omega-\omega_0)-\delta(\omega+\omega_0))$$

Exercise 4. Properties of CTFTs

For the following problems, let $X(\omega)$ and $Y(\omega)$ be the CTFTs of $x(t)$ and $y(t)$, respectively. Calculate the CTFT of each function in terms of the function $x(t)$, $y(t)$, $X(\omega)$ and $Y(\omega)$.

(a) $5x(t-a)$
$\mathscr{F}\{5x(t-a)\} \Longrightarrow 5e^{-j\omega a}X(\omega)$

(b) $X(t)$
$\mathscr{F}\{X(t)\} \Longrightarrow 2\pi x(-\omega)$ (Duality property)

(c) $x(t) * y(t)$
$\mathscr{F}\{x(t) * y(t)\} \Longrightarrow X(\omega)Y(\omega)$

(d) $x(t)y(t)$
$\mathscr{F}\{x(t)y(t)\} \Longrightarrow \frac{1}{2\pi}X(\omega) * Y(\omega)$

(e) $x(-t)$
$\mathscr{F}\{x(-t)\} \Longrightarrow |\frac{1}{-1}|X(\frac{\omega}{-1}) = X(-\omega)$

(f) $x(t)e^{j\omega_o t}$
$\mathscr{F}\{x(t)e^{j\omega_0 t}\} \Longrightarrow \mathscr{F}\{x(t)\} * \mathscr{F}\{e^{j\omega_0 t}\} = \frac{1}{2\pi}X(\omega) * 2\pi\delta(\omega - \omega_0) = X(\omega - \omega_0)$

(g) $\frac{1}{|a|}X(\frac{\omega}{a})$
$\mathscr{F}\{\frac{1}{|a|}X(\frac{\omega}{a})\} \Longrightarrow x(at)$ (Time scaling property)

Exercise 5. Deriving CTFT Properties

Derive each of the following CTFT properties. Assume that in each case the CTFT of $x(t)$ and $y(t)$ are $X(\omega)$ and $Y(\omega)$, respectively.

(a) $x(-t) \Longleftrightarrow X(-\omega)$

$$\mathscr{F}\{x(-t)\} = \int_{-\infty}^{\infty} x(-t)e^{-j\omega t}dt, \quad \text{let } v = -t,\ dv = -dt$$
$$\Longrightarrow -\int_{\infty}^{-\infty} x(v)e^{j\omega v}dv = \int_{-\infty}^{\infty} x(v)e^{-j(-\omega)v}dv = X(-\omega)$$

(b) $x(t - t_0) \Longleftrightarrow X(\omega)e^{-j\omega t_0}$

$$\mathscr{F}\{x(t - t_0)\} = \int_{-\infty}^{\infty} x(t - t_0)e^{-j\omega t}dt, \quad \text{let } v = t - t_0,\ dv = dt$$
$$\Longrightarrow \int_{-\infty}^{\infty} x(v)e^{-j\omega(v+t_0)}dv = e^{-j\omega t_0}\int_{-\infty}^{\infty} x(v)e^{-j\omega v}dv = e^{-j\omega t_0}X(\omega)$$

(c) $x(at) \Longleftrightarrow \dfrac{1}{|a|}X\Big(\dfrac{\omega}{a}\Big)$

$$\mathscr{F}\{x(at)\} = \int_{-\infty}^{\infty} x(at)e^{-j\omega t}dt, \quad \text{let } v = at,\ dv = adt$$
$$\Longrightarrow \begin{cases} \displaystyle\int_{-\infty}^{\infty} x(v)e^{-j\frac{\omega}{a}v}\frac{1}{a}dv & , a > 0 \\ \displaystyle\int_{\infty}^{-\infty} x(v)e^{-j\frac{\omega}{a}v}\frac{1}{a}dv & , a < 0 \end{cases} = \frac{1}{|a|}X(\frac{\omega}{a})$$

(d) $X(\omega) = X^*(-\omega)$ if $x(t)$ is real

$$\mathscr{F}\{x(t)\} = X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \Longrightarrow X(-\omega) = \int_{-\infty}^{\infty} x(t)e^{j\omega t}dt \Longrightarrow X^*(-\omega) = \int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt$$

If $x(t)$ is real, $x^*(t) = x(t)$, then $X^*(-\omega) = \displaystyle\int_{-\infty}^{\infty} x^*(t)e^{-j\omega t}dt = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt = X(\omega)$

(e) $x(t)y(t) \Longleftrightarrow \dfrac{1}{2\pi}X(\omega) * Y(\omega)$

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2\pi}X(\omega) * Y(\omega)e^{j\omega t}d\omega = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(\omega - \tau)Y(\tau)d\tau e^{j\omega t}d\omega, \quad \text{let } u = \omega - \tau,\ du = d\omega$$
$$= \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(u)Y(\tau)e^{j(u+\tau)t}d\tau du = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(u)e^{jut}du \times \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\tau)e^{j\tau t}d\tau = x(t)y(t)$$

(f) $\dfrac{dx(t)}{dt} \Longleftrightarrow j\omega X(\omega)$

$$\mathscr{F}\{x(t)\} = \int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega \Longrightarrow \mathscr{F}\{\frac{dx(t)}{dt}\} = \frac{d}{dt}\int_{-\infty}^{\infty} X(\omega)e^{j\omega t}d\omega = \int_{-\infty}^{\infty} j\omega X(\omega)e^{j\omega t}d\omega = j\omega X(\omega)$$

Exercise 6. Symmetry Properties of the CTFT

For each of the following transforms, determine whether the corresponding time-domain signal is

(i) real, purely imaginary, or complex

(ii) even, odd. or neither even nor odd

Do this without evaluating the inverse **CTFT**.

$$\text{Note: } \begin{cases} X^*(-\omega) = X(\omega) \Longleftrightarrow \text{purely imaginary}, & X^*(-\omega) = -X(\omega) \Longleftrightarrow \text{purely real} \\ X(-\omega) = X(\omega) \Longleftrightarrow \text{even}, & X(-\omega) = -X(\omega) \Longleftrightarrow \text{odd} \end{cases}$$

(a) $X(\omega) = sin(2\omega)cos(3\omega)$
$X^*(-\omega) = sin(-2\omega)cos(-3\omega) = -sin(2\omega)cos(3\omega) = -X(\omega) \Longrightarrow$ **Purely Real**
$X(-\omega) = sin(-2\omega)cos(-3\omega) = -sin(2\omega)cos(3\omega) = -X(\omega) \Longrightarrow$ **Odd**

(b) $X(\omega) = sin(\omega)e^{j(2\omega+\pi/2)}$
$X^*(-\omega) = sin(-\omega)e^{-j(2(-\omega)+\pi/2)} = -sin(\omega)e^{j(2\omega-\pi/2)} \neq \pm X(\omega) \Longrightarrow$ **Complex**
$X(-\omega) = sin(-\omega)e^{j(2(-\omega)+\pi/2)} = -sin(\omega)e^{j(-2\omega+\pi/2)} \neq \pm X(\omega) \Longrightarrow$ **neither Even nor Odd**

(c) $X(\omega) = u(\omega) - u(\omega - 4\pi)$
$X^*(-\omega) = u(-\omega) - u(-\omega - 4\pi) = -(u(\omega) - u(\omega + 4\pi)) \neq \pm X(\omega) \Longrightarrow$ **Complex**
$X^(- \omega) = u(-\omega) - u(-\omega - 4\pi) = -(u(\omega) - u(\omega + 4\pi)) \neq \pm X(\omega) \Longrightarrow$ **neither Even nor Odd**