ME 5224: Signals and Signals
Spring: 2023
Instructor: Dr. I-Fan Lin

Homework 6 - Fourier Series Expansion

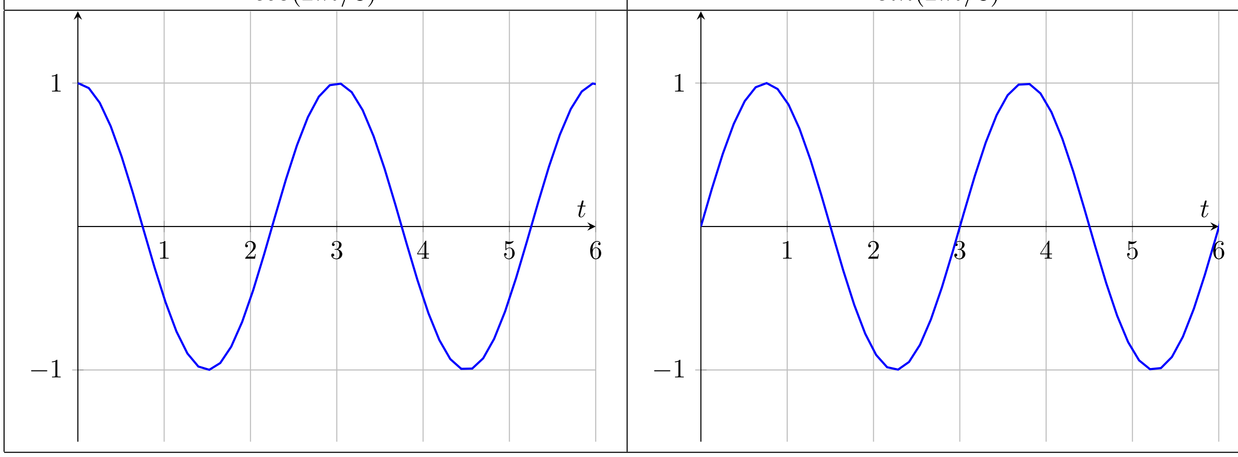
Spring 2023

Exercise 1. Determining Fourier Series Coefficients

Each of the following functions is periodic with period T. For each function sketch the real and imaginary parts of the function on the interval $[0, 2T]$ and calculate the Fourier series coefficients.

(a) $x(t) = e^{j2\pi t/3}$ with period $T = 3$.
$x(t) = e^{j2\pi t/3} = cos(2\pi t/3) + jsin(2\pi t/3)$, $\omega_0 = 2\pi/T = 2\pi/3$

REAL	**IMAGINARY**
$cos(2\pi t/3)$	$sin(2\pi t/3)$

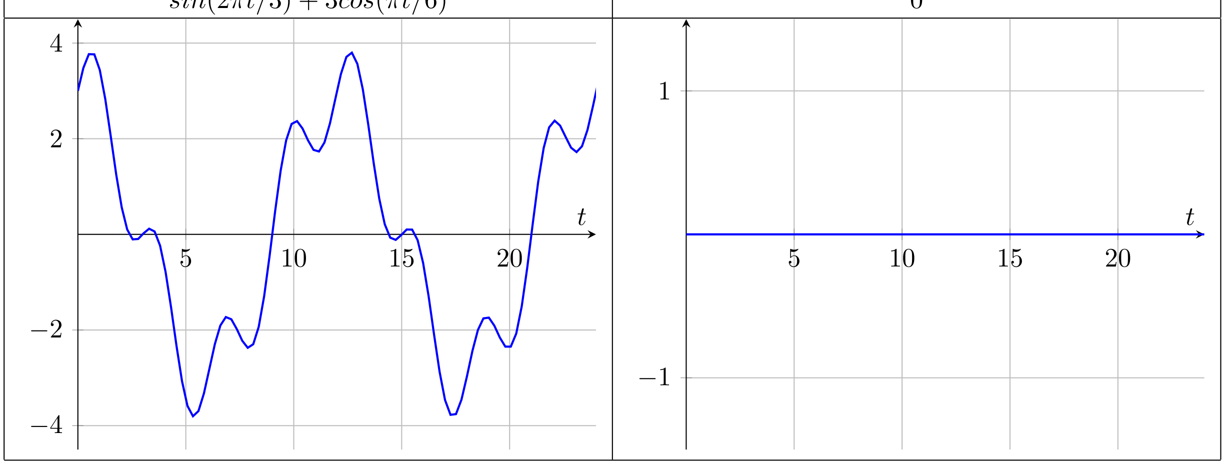


$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt = \frac{1}{3}\int_0^3 e^{jt\frac{2}{3}\pi(1-k)}dt = \frac{1}{3}\left.\frac{e^{jt\frac{2}{3}\pi(1-k)}}{j\frac{2}{3}\pi(1-k)}\right|_0^3 = \frac{e^{j2\pi(1-k)}-1}{j2\pi(1-k)}$$

$$\Longrightarrow a_k = \begin{cases} 1 & , k = 1 \\ 0 & , \text{else} \end{cases} = \delta[k-1]$$

(b) $x(t) = sin(2\pi t/3) + 3cos(\pi t/6)$ with period $T = 12$. $\Longrightarrow \omega_0 = 2\pi/T = \pi/6$

REAL	**IMAGINARY**
$sin(2\pi t/3) + 3cos(\pi t/6)$	0



$$a_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_0 t}dt = \frac{1}{12}\int_0^{12}\left(sin(\frac{2}{3}\pi t) + 3cos(\frac{\pi t}{6})\right)e^{-jk\frac{\pi}{6}t}dt$$

$$= \frac{1}{12}\int_0^{12}\left(\frac{1}{2j}\left(e^{j\frac{2}{3}\pi t} - e^{-j\frac{2}{3}\pi t}\right) + \frac{3}{2}\left(e^{j\frac{1}{6}\pi t} + e^{-j\frac{1}{6}\pi t}\right)\right)e^{-j\frac{1}{6}\pi t}dt$$

$$= \frac{1}{12}\int_0^{12}\left(\frac{1}{2j}\left(e^{j\pi t(\frac{2}{3}-\frac{k}{6})} - e^{-j\pi t(\frac{2}{3}+\frac{k}{6})}\right) + \frac{3}{2}\left(e^{j\pi t(\frac{1}{6}-\frac{k}{6})} + e^{-j\pi t(\frac{1}{6}+\frac{k}{6})}\right)\right)dt$$

$$a_k = \begin{cases} \frac{1}{2j} & ,k=4 \\ -\frac{1}{2j} & ,k=-4 \\ \frac{3}{2} & ,k=\pm 1 \\ 0 & ,\text{else} \end{cases}$$

(c) $x(t) = rect(t)$ for $|t| < T/2$ with period $T = 2$. (put in the simplest form)

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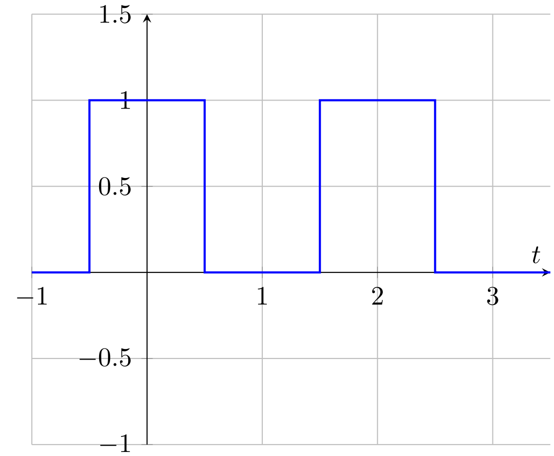
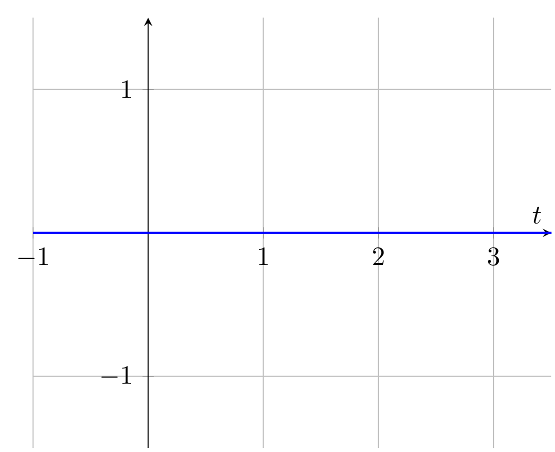
<tr><th>REAL</th><th>IMAGINARY</th></tr>
<tr><td>$rect(t)$</td><td>0</td></tr>
<tr><td>
</td><td>
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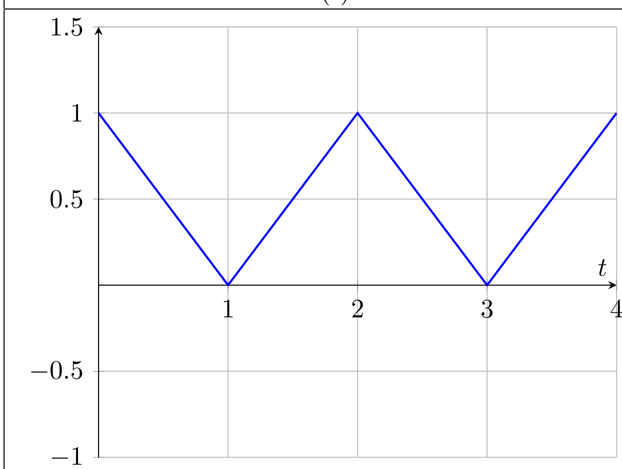
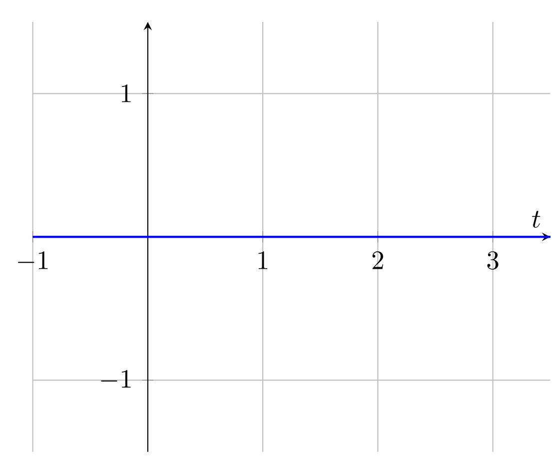
$$a_k = \frac{1}{2}\int_{-1}^{1} rect(t)e^{-jk\pi t}dt = \frac{1}{2}\int_{-1/2}^{1/2} e^{-jk\pi t}dt = \frac{1}{-2jk\pi}e^{-jk\pi t}\Big|_{-1/2}^{1/2} = \frac{1}{-2jk\pi}\left(e^{-j\frac{\pi}{2}k} - e^{j\frac{\pi}{2}k}\right)$$

$$= \frac{1}{k\pi}\frac{e^{-j\frac{\pi}{2}k} - e^{j\frac{\pi}{2}k}}{2j} = \frac{sin(\frac{\pi}{2}k)}{k\pi}$$

(d) $x(t) = \Lambda(t)$ for $|t| < T/2$ with period $T = 2$. (put in the simplest form)

<table>
<tr><th>REAL</th><th>IMAGINARY</th></tr>
<tr><td>$\Lambda(t)$</td><td>0</td></tr>
<tr><td>
</td><td>
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$$
\begin{aligned}
a_k &= \frac{1}{2}\int_0^2 \Lambda(t)e^{-jk\pi t}dt = \frac{1}{2}\Big(\int_0^1 (1-t)e^{-jk\pi t}dt + \int_1^2 (t-1)e^{-jk\pi t}dt\Big) \\
&= \frac{1}{2}\Big(\frac{e^{-jk\pi t}}{-jk\pi}\Big|_0^1 - \Big(\frac{te^{-jk\pi t}}{-jk\pi}\Big|_0^1 - \int_0^1 \frac{e^{-jk\pi t}}{-jk\pi}dt\Big) + \Big(\frac{te^{-jk\pi t}}{-jk\pi}\Big|_1^2 - \int_1^2 \frac{e^{-jk\pi t}}{-jk\pi}dt\Big) - \frac{e^{-jk\pi t}}{-jk\pi}\Big|_1^2\Big) \\
&= \frac{1}{-j2k\pi}\Big(e^{-jk\pi} - 1 - e^{-jk\pi} - \frac{e^{-jk\pi}-1}{jk\pi} + 2e^{-j2k\pi} - e^{-jk\pi} + \frac{e^{-j2k\pi}-e^{-jk\pi}}{jk\pi} - e^{-j2k\pi} + e^{-jk\pi}\Big) \\
&= \frac{1}{-j2k\pi}\Big(\frac{2(1-e^{-jk\pi})}{jk\pi}\Big) = \frac{(1-(-1)^k)}{k^2\pi^2} \quad \text{(derived from } e^{j\theta} = cos(\theta) + jsin(\theta)\text{)}
\end{aligned}
$$

Exercise 2. Properties of Fourier Series

Suppose that the Fourier series coefficients for the function $x(t)$ with period T are given as a_k, and the Fourier series coefficients for the function $y(t)$ with period T are given as b_k. Prove the following relationships.

(a) If $y(t) = \dfrac{dx(t)}{dt}$, then $b_k = jk\dfrac{2\pi}{T}a_k$

$$
\begin{cases} x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \\ y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\frac{2\pi}{T}t} \end{cases} \Longrightarrow y(t) = \frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} a_k \times jk\frac{2\pi}{T}e^{jk\frac{2\pi}{T}t} = \sum_{k=-\infty}^{\infty} b_k e^{jk\frac{2\pi}{T}t}
$$

$\Longrightarrow b_k = jk\dfrac{2\pi}{T}a_k$

(b) If $y(t) = x(-t)$, then $b_k = a_{-k}$

$$
\begin{cases} x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} \\ y(t) = \sum_{k=-\infty}^{\infty} b_k e^{jk\frac{2\pi}{T}t} \end{cases} \Longrightarrow x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}(-t)}, \text{ change } k \text{ to } -k
$$

$$
\Longrightarrow x(-t) = \sum_{k=\infty}^{-\infty} a_{-k}e^{jk\frac{2\pi}{T}t} = \sum_{k=-\infty}^{\infty} a_{-k}e^{jk\frac{2\pi}{T}t} = y(t) \Longrightarrow b_k = a_{-k}
$$

(c) If $x(t)$ is real, then $a_k = a^*_{-k}$

$\because x(t)$ is real $\Longrightarrow x(t) = x^*(t)$

$\therefore x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} = x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\frac{2\pi}{T}t}$ and use the method used in (b) $\Longrightarrow a^*_{-k} = a_k$

(d) If $x(t)$ is real and $x(t) = x(-t)$, then a_k are real and $a_k = a_{-k}$

First, we modify (b), change $y(t)$ to $x(t)$, then we'll get the result that $a_k = a_{-k}$

Then, we use the result from (c) and the result above $\Longrightarrow a_k = a^*_{-k} = (a_{-k})^* = (a_k)^*$

Since $a_k = (a_k)^* \Longrightarrow a_k$ are real.

Exercise 3. Reconstructing Signals from Fourier Series Coefficients

In each of the following, the Fourier series coefficients and the period of a signal are specified. Determine the signal $x(t)$ in each case.

(a) $a_k = (\dfrac{1}{2})^{|k|}$ and $T = 2$

$$
\begin{aligned}
x(t) &= \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} = \sum_{k=-\infty}^{\infty} (\frac{1}{2})^{|k|}e^{jk\pi t} = \sum_{k=0}^{\infty}\Big(\frac{e^{j\pi t}}{2}\Big)^k + \sum_{k=-\infty}^{0}\Big(2e^{j\pi t}\Big)^k - 1 \\
&= \sum_{k=0}^{\infty}\Big(\frac{e^{j\pi t}}{2}\Big)^k + \sum_{k=0}^{\infty}\Big(\frac{e^{-j\pi t}}{2}\Big)^k - 1 = \frac{1}{1-\frac{1}{2}e^{j\pi t}} + \frac{1}{1-\frac{1}{2}e^{-j\pi t}} - 1 \\
&= \frac{2-\frac{1}{2}e^{j\pi t}-\frac{1}{2}e^{-j\pi t}}{1-\frac{1}{2}e^{j\pi t}-\frac{1}{2}e^{-j\pi t}+\frac{1}{4}} - 1 = \frac{2-\frac{1}{2}(e^{j\pi t}+e^{-j\pi t})}{\frac{5}{4}-\frac{1}{2}(e^{j\pi t}+e^{-j\pi t})} - 1 = \frac{2-cos(\pi t)}{\frac{5}{4}-cos(\pi t)} - 1 = \frac{3}{5-4cos(\pi t)}
\end{aligned}
$$

(b) $a_k = \begin{cases} jk, & |k| < 3 \\ 0, & \text{otherwise} \end{cases}$ and $T = 4$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} = \sum_{k=-2}^{2} jke^{jk\frac{2\pi}{4}t} = -2je^{-j\frac{4\pi}{4}t} - je^{-j\frac{2\pi}{4}t} + je^{j\frac{2\pi}{4}t} + 2je^{j\frac{4\pi}{4}t}$$

$$= (2j)^2\Big(\frac{e^{j\pi t} - e^{-j\pi t}}{2j}\Big) + j \times 2j\Big(\frac{e^{j\frac{\pi}{2}t} - e^{-j\frac{\pi}{2}t}}{2j}\Big) = -4sin(\pi t) - 2sin\Big(\frac{\pi t}{2}\Big)$$

(c) $a_k = cos(\pi k/4)$ and $T = 4$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\frac{2\pi}{T}t} = \sum_{k=-\infty}^{\infty} cos(\frac{\pi k}{4})e^{jk\frac{2\pi}{T}t} = \sum_{k=-\infty}^{\infty} \frac{1}{2}(e^{j\frac{\pi k}{4}t} + e^{-j\frac{\pi k}{4}t})e^{jk\frac{2\pi}{T}t}$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{2}e^{jk\frac{3\pi}{4}t} + \frac{1}{2}e^{jk\frac{\pi}{4}t} =$$

Exercise 4. Fourier Series and LTI Systems

Suppose that the signal $x(t)$ is periodic with period T and Fourier coefficients a_k. Let $y(t) = h(t) * x(t)$ where $h(t)$ is the impulse response of an LTI system.

(a) Show that $y(t)$ is also periodic with period T

$x(t)$ is periodic with $T \Longrightarrow x(t) = x(t+T)$, and $h(t)$ is LTI $\Longrightarrow y(t+T) = h(t) * x(t+T)$
$= h(t) * x(t) = y(t) \Longrightarrow y(t)$ is periodic.

(b) Show that the Fourier series coefficients of $y(t)$ have the form $b_k = c_k a_k$ where c_k are multiplicative constants.

$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(t-\tau)x(\tau)d\tau = \int_{-\infty}^{\infty} h(t-\tau)\sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 k\tau}d\tau$$

$$= \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} h(t-\tau)e^{j\omega_0 k\tau}d\tau = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{\infty} h(t)e^{j\omega_0 k(t-\tau)}d\tau$$

$$= \sum_{k=-\infty}^{\infty} a_k e^{j\omega_0 kt} \underbrace{\int_{-\infty}^{\infty} h(\tau)e^{-j\omega_0 k\tau}d\tau}_{c_k} = \sum_{k=-\infty}^{\infty} a_k c_k e^{j\omega_0 kt} = \sum_{k=-\infty}^{\infty} b_k e^{j\omega_0 kt} \Longrightarrow b_k = a_k c_k$$

(c) Derive an expression for the multiplicative constants c_k

From the result above $\Longrightarrow c_k = \displaystyle\int_{-\infty}^{\infty} h(\tau)e^{-j\omega_0 k\tau}d\tau$