System dynamics
-Final Project

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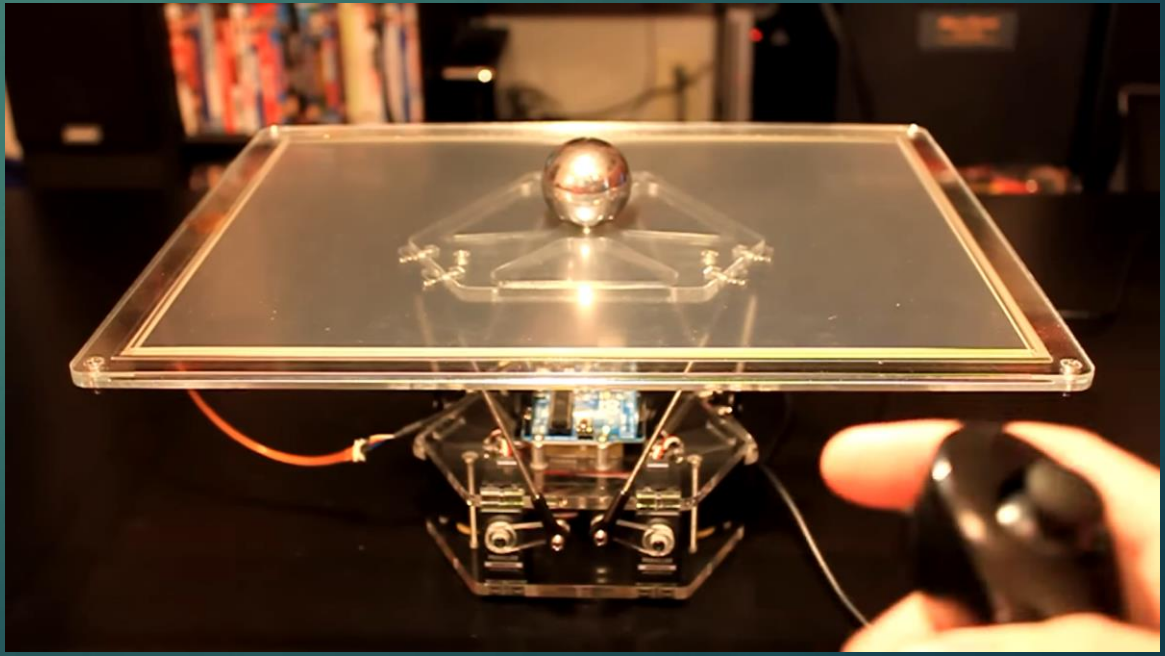
Outline

- Introduction
- Mathematical modelling
- State-space model
- Simulation

Introduction

- Topic: Modelling and Control of Ball-Plate System
- Goal: To control the ball movement by manipulating the inclination of the plate using system dynamics modelling skills.



Mathematical Modelling

Assumptions:

- There is no slipping for ball
- The ball is completely symmetric and homogeneous
- All frictions are neglected
- The ball and the plate contact all the time

Mathematical Modelling

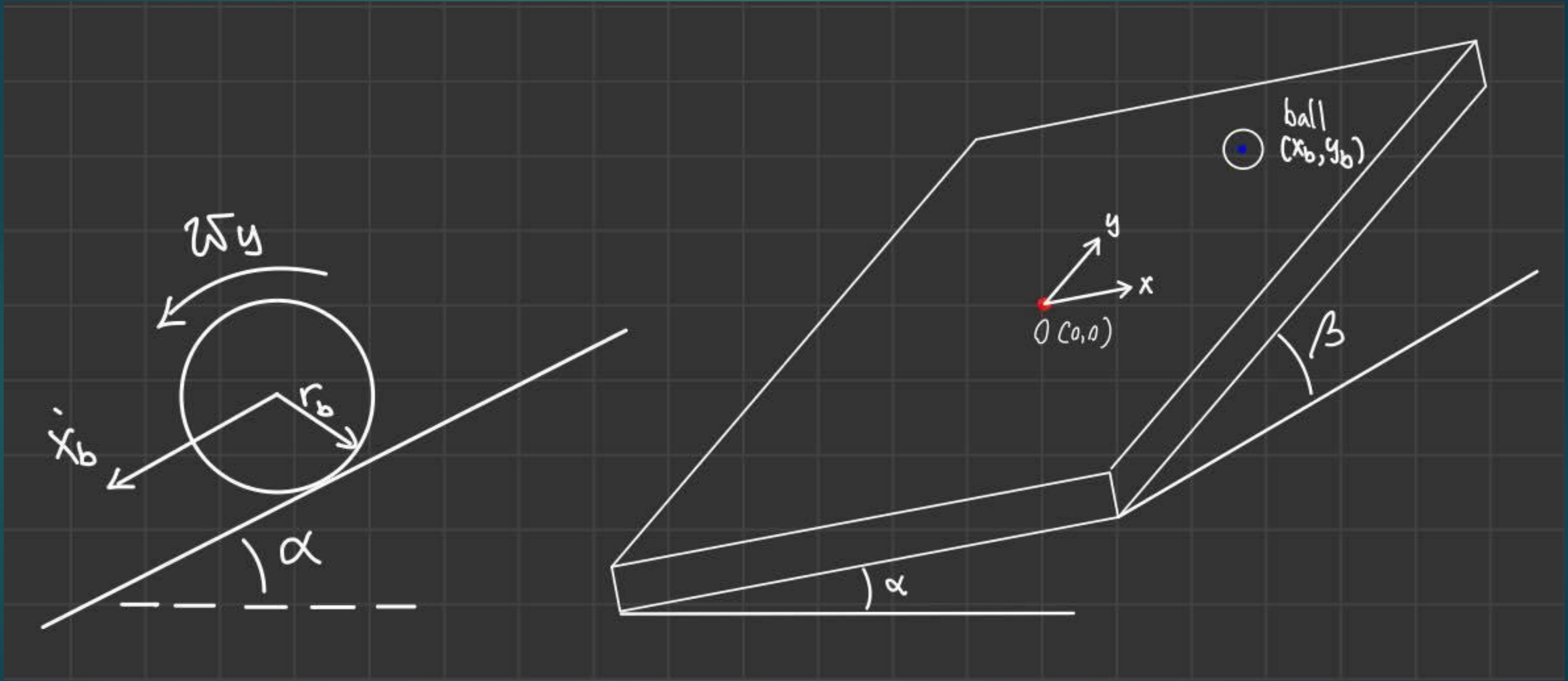
The system has 4 degree of freedom:

- x_b , y_b: The position of the ball on the plate

(The center of x-y coordinate is at the center of the plate)

- α , β: The inclination of the plate

Mathematical Modelling



Mathematical Modelling

- Kinetic energy of the ball:

$$T_b = \frac{1}{2} m_b\left(\dot{x}_b^2 + \dot{y}_b^2\right) + \frac{1}{2} I_b(\omega_x^2 + \omega_y^2)$$

- Relationship between translational and rotational velocity

$$\dot{x}_b = r_b\omega_y \quad , \dot{y}_b = r_b\omega_x$$

- Substitute the velocity relationship into kinetic energy

$$T_b = \frac{1}{2}(m_b + \frac{I_b}{{r_b}^2})\left(\dot{x}_b^2 + \dot{y}_b^2\right)$$

Mathematical Modeling

- Kinetic energy of the plate

$$T_p = \frac{1}{2}(I_p + I_b)(\dot{\alpha}^2 + \dot{\beta}^2) + \frac{1}{2}m_b(x_b\dot{\alpha} + y_b\dot{\beta})^2$$

- Kinetic energy of the system

$$T = T_b + T_p$$
$$= \frac{1}{2}\left(m_b + \frac{I_b}{{r_b}^2}\right)(\dot{x}_b^2 + \dot{y}_b^2) + \frac{1}{2}(I_p + I_b)(\dot{\alpha}^2 + \dot{\beta}^2) + \frac{1}{2}m_b(x_b\dot{\alpha} + y_b\dot{\beta})^2$$

Mathematical Modelling

- Potential energy of the ball (relative to the center of the inclined plate)

$$V_b = m_b g(x_b \sin\alpha + y_b \sin\beta)$$

- Derive the system's equation using Lagrange's Equation

$$L = T_b + T_p - V_b$$

$$\frac{\partial L}{\partial \dot{\alpha}} = (I_p + I_b)\dot{\alpha}_x + m_b x_b(x_b\dot{\alpha} + y_b\dot{\beta}) \quad , \quad \frac{\partial L}{\partial \alpha} = -m_b g x_b \cos\alpha$$

$$\frac{\partial L}{\partial \dot{\beta}} = (I_p + I_b)\dot{\beta}_x + m_b y_b(x_b\dot{\alpha} + y_b\dot{\beta}) \quad , \quad \frac{\partial L}{\partial \beta} = -m_b g y_b \cos\beta$$

Mathematical Modelling

- Conti.

$$\frac{\partial L}{\partial \dot{x_b}} = \left(m_b + \frac{I_b}{{r_b}^2}\right)\dot{x_b} \quad , \quad \frac{\partial L}{\partial x_b} = m_b\left(x_b\dot{\alpha} + y_b\dot{\beta}\right)\dot{\alpha} - m_b g\sin\alpha$$

$$\frac{\partial L}{\partial \dot{y_b}} = \left(m_b + \frac{I_b}{{r_b}^2}\right)\dot{y_b} \quad , \quad \frac{\partial L}{\partial y_b} = m_b\left(x_b\dot{\alpha} + y_b\dot{\beta}\right)\dot{\beta} - m_b g\sin\beta$$

Mathematical Modelling

- From Lagrange-Euler equation:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\alpha}}-\frac{\partial L}{\partial \alpha}$$
$$=\left(I_p+I_b\right)\ddot{\alpha}+2m_b x_b \dot{x}_b \dot{\alpha}+m_b x_b^2 \ddot{\alpha}+m_b x_b y_b \ddot{\beta}+m_b \dot{x}_b y_b \dot{\beta}$$
$$+m_b x_b \dot{y}_b \dot{\beta}+m_b g x_b \cos\alpha=\tau_x$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{\beta}}-\frac{\partial L}{\partial \beta}$$
$$=\left(I_p+I_b\right)\ddot{\beta}+2m_b y_b \dot{y}_b \dot{\beta}+m_b y_b^2 \ddot{\beta}+m_b x_b y_b \ddot{\alpha}+m_b \dot{x}_b y_b \dot{\alpha}$$
$$+m_b x_b \dot{y}_b \dot{\alpha}+m_b g y_b \cos\beta=\tau_y$$

Mathematical Modelling

- Conti.

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{x}_b} - \frac{\partial L}{\partial x_b} = \left(m_b + \frac{I_b}{r_b^2}\right)\ddot{x}_b - m_b x_b \dot{\alpha}^2 - m_b y_b \dot{\alpha}\dot{\beta} + m_b g \sin\alpha = 0$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{y}_b} - \frac{\partial L}{\partial y_b} = \left(m_b + \frac{I_b}{r_b^2}\right)\ddot{y}_b - m_b y_b \dot{\beta}^2 - m_b x_b \dot{\alpha}\dot{\beta} + m_b g \sin\beta = 0$$

Mathematical Modelling

- Non-linear differential equations:

$$\begin{cases} \left(m_b + \frac{I_b}{r_b^2}\right)\ddot{x_b} - m_b x_b \dot{\alpha}^2 - m_b y_b \dot{\alpha}\dot{\beta} + m_b g \sin\alpha = 0 \\ \left(m_b + \frac{I_b}{r_b^2}\right)\ddot{y_b} - m_b y_b \dot{\beta}^2 - m_b x_b \dot{\alpha}\dot{\beta} + m_b g \sin\beta = 0 \\ (I_p + I_b)\ddot{\alpha} + 2m_b x_b \dot{x}_b \dot{\alpha} + m_b x_b^2 \ddot{\alpha} + m_b x_b y_b \ddot{\beta} + m_b \dot{x}_b y_b \dot{\beta} + m_b x_b \dot{y}_b \dot{\beta} + m_b g x_b \cos\alpha = \tau_x \\ (I_p + I_b)\ddot{\beta} + 2m_b y_b \dot{y}_b \dot{\beta} + m_b y_b^2 \ddot{\beta} + m_b x_b y_b \ddot{\alpha} + m_b \dot{x}_b y_b \dot{\alpha} + m_b x_b \dot{y}_b \dot{\alpha} + m_b g y_b \cos\beta = \tau_y \end{cases}$$

State-space Model of System

$$B = m/(m + I_b/r_b^2)$$

- Define state variable

$$X = [x_1, x_2, x_{3,} x_4, x_5, x_6, x_7 x_8]^T = [x_b, \dot{x_b}, \alpha, \dot{\alpha}, y_b, \dot{y_b}, \beta, \dot{\beta}]^T$$

- Input

$$U = [u_x, u_y]^T = [\ddot{\alpha}, \ddot{\beta}]^T$$

- State-space equation: $\dot{x} = f(x, u)$

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \\ \dot{x_5} \\ \dot{x_6} \\ \dot{x_7} \\ \dot{x_8} \end{bmatrix} = \begin{bmatrix} x_2 \\ B(x_1 x_4^2 + x_4 x_5 x_8 - g \sin x_3) \\ x_4 \\ 0 \\ x_6 \\ B(x_5 x_8^2 + x_1 x_4 x_8 - g \sin x_7) \\ x_8 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_x \\ u_y \end{bmatrix}$$

Model Linearization

- The approximate value for a solid ball's momentum of inertia is
$$I_{ball} = \frac{2}{5} m_b r_b^2$$
- Small angle of inclination of the plate
max $\pm 5^{\circ}$
- Slow rate of change of the plate
$$\dot{\alpha} \cong 0, \dot{\beta} \cong 0$$

Model linearization

- Using the assumption:

$$\begin{cases} \frac{7}{5}\ddot{x_b} + g\alpha = 0 \\ \frac{7}{5}\ddot{y_b} + g\beta = 0 \end{cases}$$

- Transfer function:

$$P_x(s) = \frac{X_b(s)}{\alpha(s)} = \frac{g}{\frac{7}{5}s^2} \quad , P_y(s) = \frac{Y_b(s)}{\beta(s)} = \frac{g}{\frac{7}{5}s^2}$$

Model linearization

$$B = m/(m + I_b/r_b^2)$$

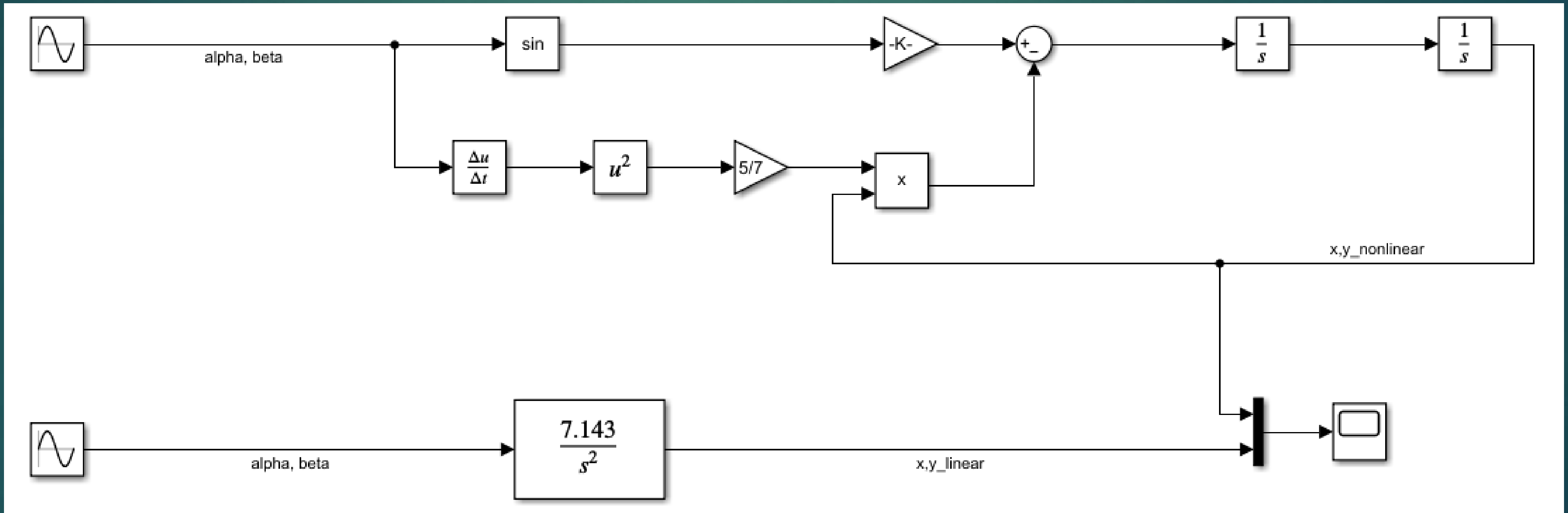
- Simplified State-space equations

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \\ \dot{x_3} \\ \dot{x_4} \end{bmatrix} = \begin{bmatrix} x_2 \\ B(x_1 x_4^2 - g \sin x_3) \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} [u_x]$$

$$\begin{bmatrix} \dot{x_5} \\ \dot{x_6} \\ \dot{x_7} \\ \dot{x_8} \end{bmatrix} = \begin{bmatrix} x_6 \\ B(x_5 x_8^2 - g \sin x_7) \\ x_8 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} [u_y]$$

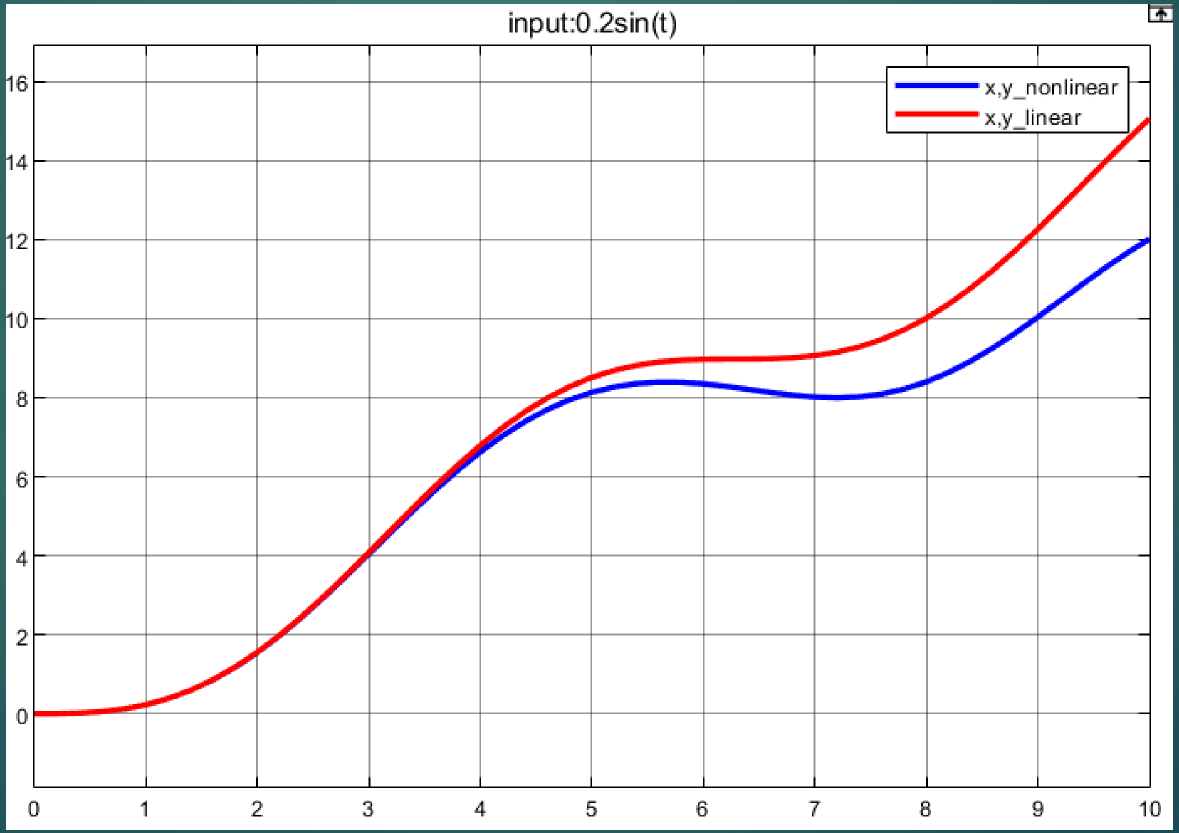
Simulation

- Linear vs Nonlinear (Open-loop)



Simulation

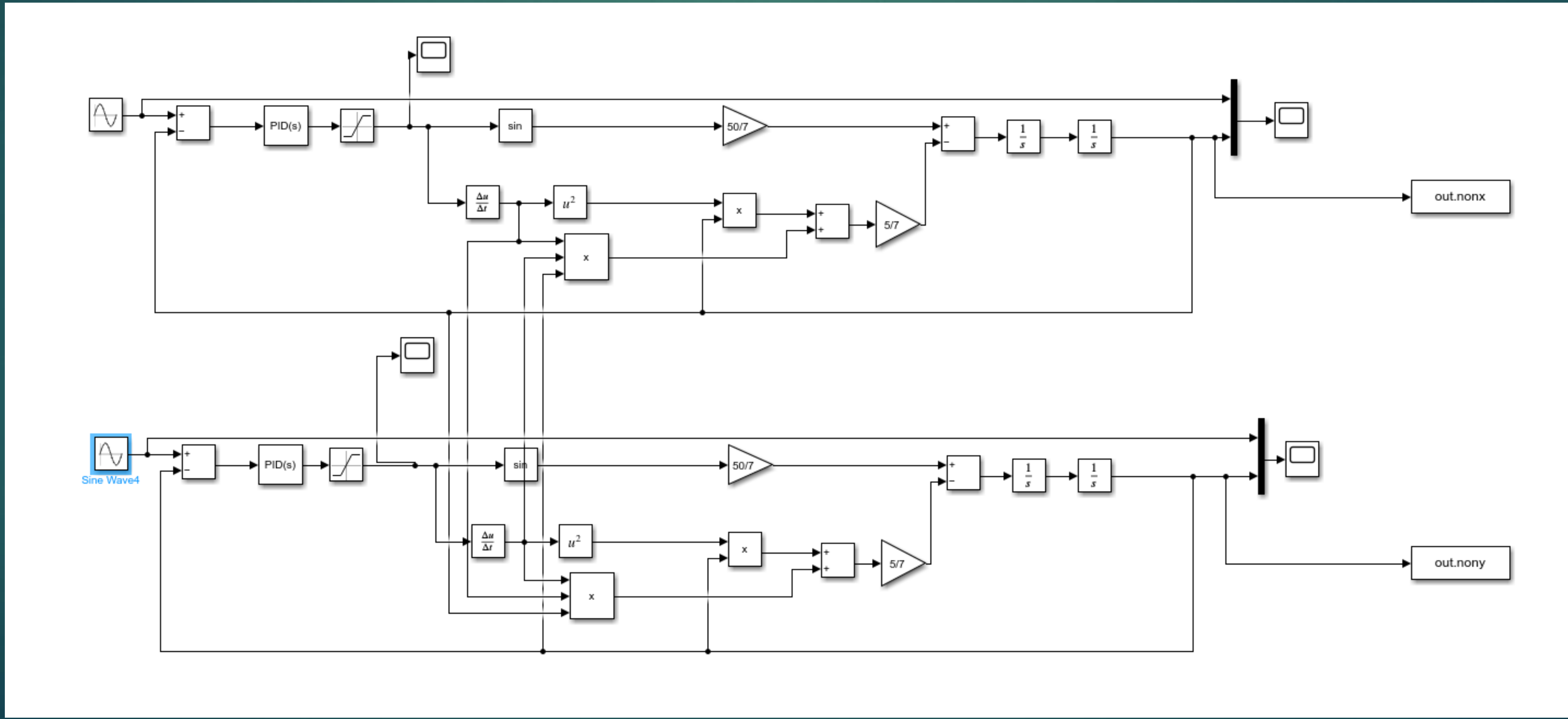
- Linear vs Nonlinear (Open-loop)



Simulation

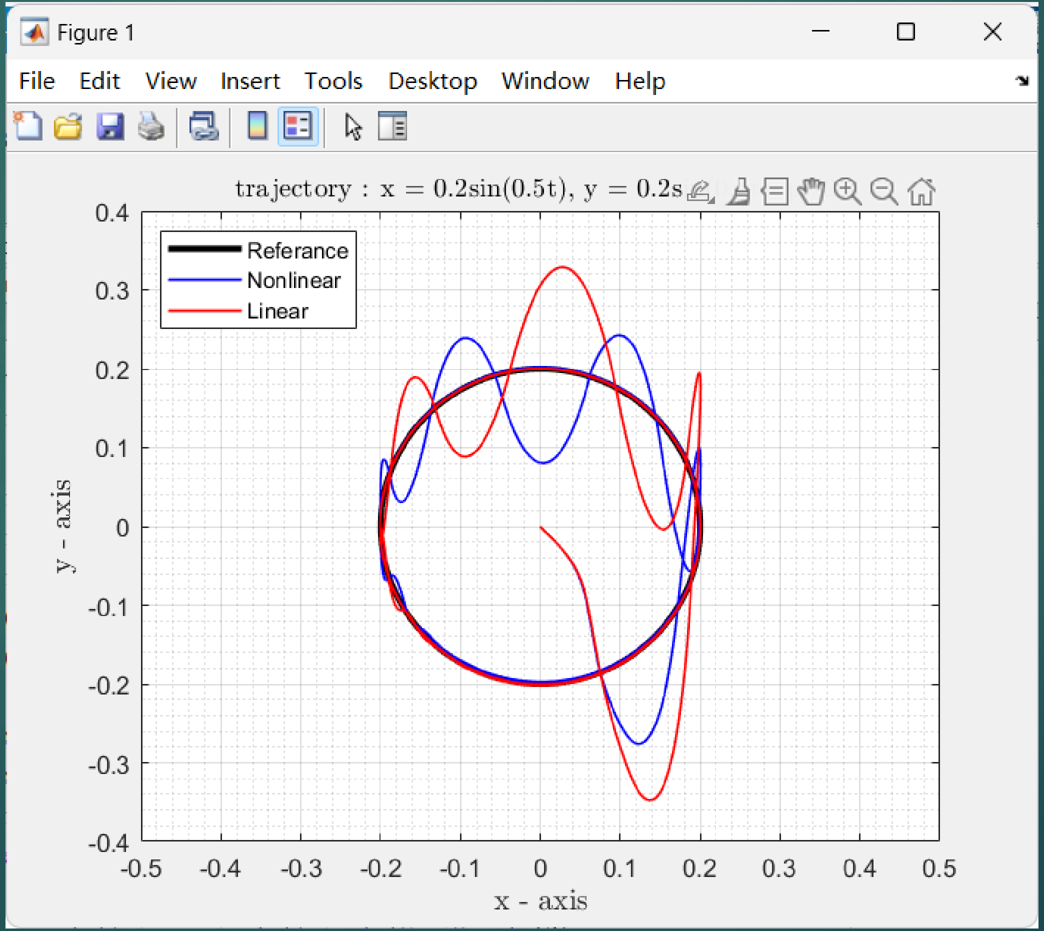
- Linear vs Nonlinear (Closed-loop with controller)

$K_P = 20, K_I = 8, K_D = 0.8$



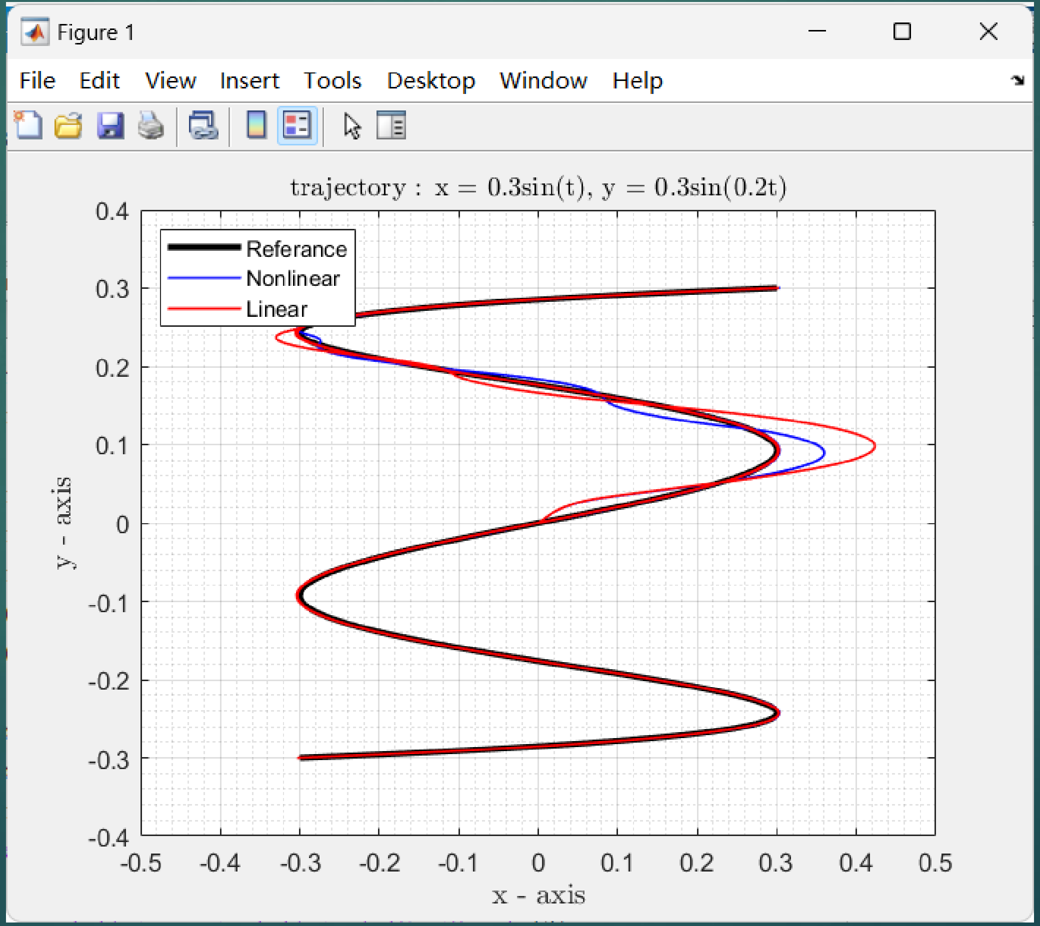
Simulation

- Trajectory



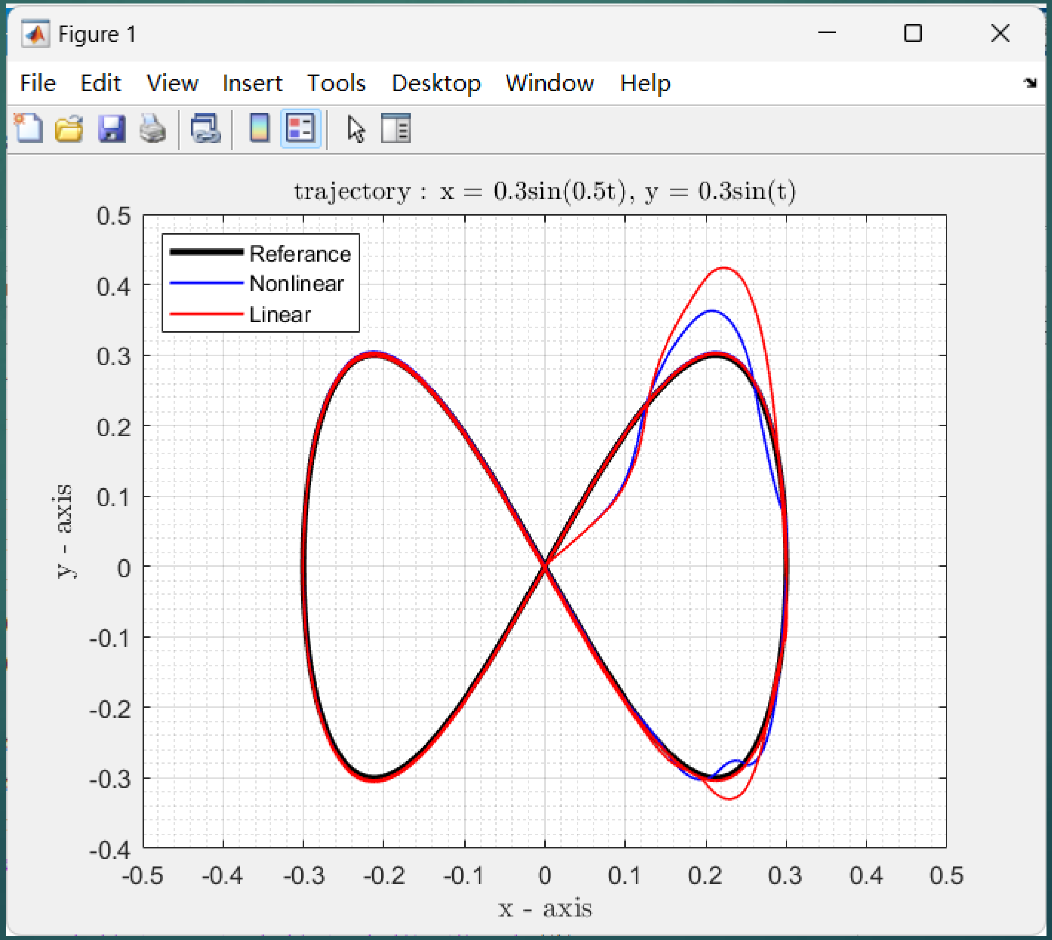
Simulation

- Trajectory



Simulation

- Trajectory



Reference

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Thank you!