**ME 5224: Signals and Signals**
**Spring: 2023**
**Instructor: Dr. I-Fan Lin**

# Homework 4 - LTI System Properties

Spring 2023

## Exercise 1. Properties of Convolution

(a) Consider a CT LTI system $y(t) = x(t) * h(t)$. Show the input $\frac{dx(t)}{dt}$ results in the output $\frac{dy(t)}{dt}$

Let $y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \Longrightarrow \dfrac{dy(t)}{dt} = \dfrac{d}{dt}\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)\dfrac{d}{dt}x(t-\tau)d\tau = h(t) * \dfrac{dx(t)}{dt}$

(b) Consider a DT LTI system $y_n = x_n * h_n$. Prove that $\sum_{n=-\infty}^{\infty} y_n = \Big(\sum_{n=-\infty}^{\infty} x_n\Big)\Big(\sum_{n=-\infty}^{\infty} h_n\Big)$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \Longrightarrow \sum_{n=-\infty}^{\infty} y[n] = \sum_{n=-\infty}^{\infty}\sum_{k=-\infty}^{\infty} x[n]h[n-k] = \sum_{k=-\infty}^{\infty}\Big(x[k]\sum_{n=-\infty}^{\infty} h[n-k]\Big)$$

$$= \Big(\sum_{n=-\infty}^{\infty} x_n\Big)\Big(\sum_{n=-\infty}^{\infty} h_n\Big)$$

(c) Consider a CT LTI system $y(t) = x(t) * h(t)$. Prove that if $x(t)$ is periodic with period $T$, then $y(t)$ is also periodic with period $T$.

$x(t)$ is periodic with period $T \Longrightarrow x(t) = x(t+kT), k \in \mathbb{Z}$

$y(t+kT) = x(t+kT) * h(t) = x(t) * h(t) = y(t) \Longrightarrow y(t)$ is periodic with period $T$.

## Exercise 2. Properties of Convolution

Let $x_n$ be a signal which is nonzero only in the interval $0 \le n < M$ and $h_n$ be a signal which is nonzero only in the interval $0 \le n < N$.

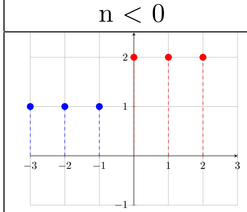
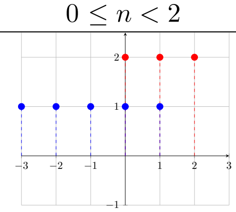
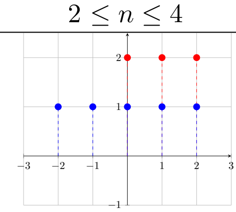
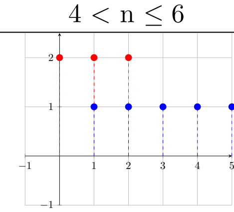
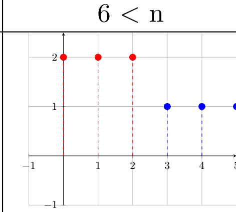
(a) Determine the interval $L_1 \le n \le L_2$ over which $y_n = x_n * h_n$ is nonzero. Express $L_1$ and $L_2$ in terms of $M$ and $N$.

Let $x[n] = u[n] - u[n-M]$, $h[n] = u[n] - u[n-N]$

$$\Longrightarrow y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} (u[k] - u[k-N])(u[n-k] - u[n-k-M])$$

$\Longrightarrow$Overlap will occur only when $n \ge 0$ and $n - N + 1 \le M - 1 \Longrightarrow 0 \le n \le M + N - 2$

(b) Verify the result in the previous part by analytically computing the convolution of signals $x[n] = u[n] - u[n-5]$ and $h[n] = 2(u[n] - u[n-3])$

| No-overlap | Partially-overlap | Fully-overlap | Partially-overlap | No-overlap |
|---|---|---|---|---|
| n $<$ 0 | $0 \le n < 2$ | $2 \le n \le 4$ | $4 <$ n $\le 6$ | $6 <$ n |
|  |  |  |  |  |
| $y[n] = 0$ | $y[n] = \sum_{k=0}^{n} 2$ | $y[n] = \sum_{k=2}^{n} 6$ | $y[n] = \sum_{k=4}^{n} 2$ | $y[n] = 0$ |

$$y[n] = \begin{cases} 2 & ,0 \le n < 2 \\ 6 & ,2 \le n \le 4 \\ 2 & ,4 < n \le 6 \\ 0 & ,\text{else} \end{cases} \Longrightarrow y[n] \neq 0, \text{ when } 0 \le n \le 6, \text{ 6=5+3-2}$$

(c) Verify the result in the previous part by analytically computing the convolution of signals $x[n] = u[n] - u[n-5]$ and $h[n] = 2(u[n] - u[n-2])$

$$y[n] = x[n] * h[n] = x[n] * 2(\delta[n] + \delta[n-1]) = 2x[n] + 2x[n-1]$$
$$= 2(u[n] - u[n-5]) + 2(u[n-1] - u[n-6])$$
$$\Longrightarrow y[n] \neq 0 \text{ for } 0 \le \text{n} \le 5 = 5 + 2 - 2$$

## Exercise 3. Properties of LTI Systems

Prove the following properties

(a) The commutative property of DT convolution, that is, $x_n * y_n = y_n * x_n$

$$x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[k]y[n-k] \Longrightarrow \text{Let } l = n - k \Longrightarrow \sum_{l=-\infty}^{\infty} x[n-l]y[l] = y[n] * x[n]$$

(b) The associative property of DT convolution, that is, $(x_n * y_n) * z_n = x_n * (y_n * z_n)$

Let $w[n] = x[n] * y[n]$, $g[n] = y[n] * z[n]$

$$(x[n] * y[n]) * z[n] = w[n] * z[n] = \sum_{l=-\infty}^{\infty} w[l] * z[n-l] = \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]y[l-k]z[n-l]$$
$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{l=-\infty}^{\infty} y[l-k]z[n-l], \text{ Let } m = n - l,\ l = n - m$$
$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} y[n-k-m]z[m] = \sum_{k=-\infty}^{\infty} x[k]g[n-k]$$
$$= x[n] * g[n] = x[n] * (y[n] * z[n])$$

(c) The distributive property of DT convolution, that is $x[n] * (y[n] + z[n]) = x[n] * y[n] + x[n] * z[n]$

$$x[n] * (y[n] + z[n]) = \sum_{k=-\infty}^{\infty} x[k](y[n-k] + z[n-k]) = \sum_{k=-\infty}^{\infty} x[k]y[n-k] + \sum_{k=-\infty}^{\infty} x[k]z[n-k]$$
$$= x[n] * y[n] + x[n] * z[n]$$

(d) Let $h[n]$ be the impulse response of a DT system. Then the system is causal if and only if h[n]=0 for n <0.

Suppose $h[n_0] \neq 0$, when $n_0 < 0$, $y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] + h[n_0]x[n-n_0]$

$\Longrightarrow x[n-n_0]$ is a future value of $x[n] \Longrightarrow$ The system is causal $\iff h[n] = 0, \quad \forall n < 0$

## Exercise 4. Causal and Stable LTI Systems

For the following discrete-time and continuous-time LTI systems, determine whether each system is causal and/or stable. Justify your answers.

(a) $h[n] = \left(\frac{1}{2}\right)^n u[-n]$

$h[-1] = 2u[1] \neq 0 \qquad \Longrightarrow$ non-causal

$$\sum_{n=-\infty}^{\infty} \left|h[n]\right| = \sum_{n=-\infty}^{0} \left(\frac{1}{2}\right)^n = \sum_{n=0}^{\infty} 2^n = \infty \qquad \Longrightarrow \text{non-stable}$$

(b) $h[n] = \left(-\frac{1}{2}\right)^n u[n] + (1.01)^n u[n-1]$

$h[n] = 0,\ \forall n < 0 \qquad \Longrightarrow$ causal

$$\sum_{n=-\infty}^{\infty} \left|h[n]\right| = \sum_{n=1}^{\infty} \left(-\frac{1}{2}\right)^n + (1.01)^n + 1 = \infty \qquad \Longrightarrow \text{non-stable}$$

(c) $h(t) = e^{2t}u(-1-t)$
$h(-1) = e^{-2} \neq 0 \qquad \Longrightarrow$ non-causal
$\int_{-\infty}^{\infty} \Big|h(t)\Big| dt = \int_{-\infty}^{\infty} e^{2t} = \infty \qquad \Longrightarrow$ non-stable

(d) $h(t) = te^{-t}u(t)$
$h(t) = 0,\ \forall t < 0 \qquad \Longrightarrow$ causal
$\int_{-\infty}^{\infty} \Big|h(t)\Big| dt = \int_{0}^{\infty} te^{-t}dt,\ \begin{cases} u = t & ,du = dt \\ dv = e^{-t}dt & ,v = -e^{-t} \end{cases} = -te^{-t}\Big|_0^{\infty} + \int_{-\infty}^{\infty} e^{-t}dt = 0 + 1 = 1 \qquad \Longrightarrow$ stable

## Exercise 5. LTI Differential Equations

Determine the impulse response for the following system under the assumption that the system is initially at rest.

$$\frac{dy(t)}{dt} = -ay(t) + x(t)$$

The system can be rewritten into $h'(t) = -ah(t) + \delta(t)$. First we consider the natural response of the system $h_n(t)$, which is the solution of $h'(t) + ah(t) = 0 \Longrightarrow h_n(t) = ce^{-at}$. Then we consider the particular response of the system $h_p(t)$, which is the solution of $h'(t) + ah(t) = \delta(t) \Longrightarrow h_p(t) = \frac{1}{a}\delta(t)$
$\because h(t) = h_n(t) + h_p(t) = ce^{-at} + \frac{1}{a}\delta(t)$, and the system is initially at rest $h(0) = 0$
$\therefore c = -\frac{1}{a} \Longrightarrow h(t) = \frac{1}{a}(-e^{-at} + \delta(t))$

## Exercise 6. DT Differential Equations

Consider the DT LTI system described by the equation

$$y[n] = \frac{1}{2}y[n-1] + x[n]$$

where $\lim_{n\to-\infty} y[n] = 0$

(a) Compute the impulse response of the system.
input $x[n] = \delta[n]$, we have $y[n] = \frac{1}{2}y[n-1] + \delta[n]$, $\lim_{n\to-\infty} y[n] = 0 \Longrightarrow y[n] = 0,\ \forall n < 0$
$y[0] = 0 + 1 = 1$, $y[1] = \frac{1}{2} \times 1 + 0 = \frac{1}{2}$, $y[2] = \frac{1}{4} \ldots y[n] = (\frac{1}{2})^n \quad \therefore h[n] = (\frac{1}{2})^n u[n]$

(b) Express the system in the form $y[n] = x[n] * h[n]$.
$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} (\frac{1}{2})^k x[n-k]$$

(c) Find the output when the input is given by $x[n] = u[n]$.
$$y[n] = \sum_{k=0}^{\infty} (\frac{1}{2})^k u[n-k] = \sum_{k=0}^{n} (\frac{1}{2})^k u[n] = u[n]\frac{1 - (\frac{1}{2})^{n+1}}{1 - \frac{1}{2}} = u[n](2 - (\frac{1}{2})^n)$$

(d) Find the output when the input is given by $x[n] = 1$.
$$y[n] = \sum_{k=0}^{\infty} (\frac{1}{2})^k = \frac{1}{1 - \frac{1}{2}} = 2$$

## Exercise 7. System Response to a Complex Exponential Input

For the following continuous-time and discrete-time systems with the given input and output, determine whether the system is definitely *not* LTI.

(a) $S_1[e^{j7t}] = te^{j7t}$
$y(t) = te^{j7t}$, $x(t) = e^{j7t}$, $y(t) = h(t) * x(t) \Longrightarrow h(t) = t \Longrightarrow$ **Time-variant**.

(b) $S_2[e^{j7t}] = e^{j7(t-2)}$
$y(t) = e^{j7(t-2)}$, $x(t) = e^{j7t}$, $y(t) = h(t) * x(t) \Longrightarrow h(t) = e^{-j14} \Longrightarrow$ **Time-invariant**.

(c) $S_3[e^{j7t}] = sin(7t)$
$y(t) = sin(7t) = \dfrac{e^{j7t} - e^{-j7t}}{2j}$, $x(t) = e^{j7t}$, $y(t) = h(t) * x(t) \Longrightarrow h(t) = \dfrac{1 - e^{-j14t}}{2j} \Longrightarrow$ **Time-variant**.

(d) $S_4[e^{j\pi n/4}] = e^{j\pi n/4}u[n]$
$y[n] = e^{j\pi n/4}u[n]$, $x[n] = e^{j\pi n/4}$, $y[n] = h[n] * x[n] \Longrightarrow h[n] = u[n] \Longrightarrow$ **Time-variant**.

(e) $S_5[e^{j\pi n/4}] = e^{j3\pi n/4}$
$y[n] = e^{j3\pi n/4}$, $x[n] = e^{j\pi n/4}$, $y[n] = h[n] * x[n] \Longrightarrow h[n] = e^{j\pi n/2} \Longrightarrow$ **Time-variant**.

(f) $S_6[e^{j\pi n/4}] = 2e^{3\pi/4}e^{j\pi n/4}$
$y[n] = 2e^{3\pi/4}e^{j\pi n/4}$, $x[n] = e^{j\pi n/4}$, $y[n] = h[n] * x[n] \Longrightarrow h[n] = 2e^{3\pi/4} \Longrightarrow$ **Time-invariant**.

## Exercise 8. System Response to a Complex Exponential Input

Let $y[n] = S[x[n]]$ be a LTI system with discrete-time input $x[n]$, discrete-time output $y[n]$, and impulse response $h[n]$.

(a) Write an explicit expression for the output in terms of the input and the impulse response.

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k]$$

(b) If the input to the system is $x[n] = e^{j\omega n}$, then show that the output must have the form

$$y[n] = C[\omega]e^{j\omega n}$$

where $C[\omega]$ is a complex value that is a function of $\omega$. Also, calculate an explicit expression for $C[\omega]$ in terms of the inputs response.

$$x[n] = e^{j\omega n},\ y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega(n-k)} = \Big(\sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}\Big)e^{j\omega n}$$

$$\Longrightarrow C[\omega] = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k}$$

(c) Show that if $h[n]$ is real valued, then for all $\omega \in \mathbb{R}$

$$C[-\omega] = C^*[\omega]$$

or equivalently that if $C[\omega] = A[\omega]e^{j\theta}$, then

$$\theta[\omega] = -\theta[\omega]$$

$$C[\omega] = \sum_{k=-\infty}^{\infty} h[k]e^{-j\omega k} \Longrightarrow C^*[\omega] = \sum_{k=-\infty}^{\infty} h^*[k]e^{j\omega k} \quad \because h[n] \text{ is real valued} \Longrightarrow h^*[k] = h[k]$$

$$\Longrightarrow C^*[\omega] = \sum_{k=-\infty}^{\infty} h[k]e^{j\omega k} = C[-\omega]$$

(d) Use the result of part (c) above to compute the output $y[n]$ when $x[n] = cos[\omega n]$.

$$x[n] = cos[\omega n] = \frac{e^{j\omega n} + e^{-j\omega n}}{2},\ y[n] = h[n] * x[n] = \tfrac{1}{2}\Big(\sum_{k=-\infty}^{\infty} h[k]\frac{e^{j\omega[n-k]} + e^{-j\omega[n-k]}}{2}\Big)$$

By using (c) $\Longrightarrow y[n] = \frac{1}{2}C[\omega]e^{j\omega n} + \frac{1}{2}C[-\omega]e^{-j\omega n}$

(e) Use the result of part (c) above to compute the output $y[n]$ when $x[n] = Bcos[\omega n + \phi]$

$$x[n] = Bcos[\omega n+\phi] = \frac{B}{2}\Big(e^{j(\omega n+\phi)} + e^{-j(\omega n+\phi)}\Big),\ y[n] = \frac{B}{2}\Big(\sum_{k=-\infty}^{\infty} h[k]\Big(e^{j(\omega[n-k]+\phi)} + e^{-j(\omega[n-k]+\phi)}\Big)\Big)$$

By using (c) $\Longrightarrow y[n] = \dfrac{B}{2}e^{j\phi}C[\omega]e^{j\omega n} + \dfrac{B}{2}e^{-j\phi}C[-\omega]e^{-j\omega n}$

(f) Use the result of part (c) above to compute the output $y[n]$ when $x[n] = sin[\omega n]$

$$x[n] = sin[\omega n] = \frac{e^{j\omega n} - e^{-j\omega n}}{2j},\ y[n] = \sum_{k=-\infty}^{\infty} h[k]\frac{e^{j\omega[n-k]} - e^{-j\omega[n-k]}}{2j}$$

By using (c) $\Longrightarrow y[n] = \dfrac{1}{2j}C[\omega]e^{j\omega n} - \dfrac{1}{2j}C[-\omega]e^{-j\omega n}$

(g) Use the result of part (c) above to compute the output $y[n]$ when $x[n] = Bsin[\omega n + \phi]$

$$x[n] = Bsin[\omega n + \phi] = \frac{B}{2j}\left(e^{j(\omega n+\phi)} - e^{-j(\omega n+\phi)}\right),\ y[n] = \sum_{k=-\infty}^{\infty} h[k]\frac{B}{2j}\left(e^{j(\omega n+\phi)} - e^{-j(\omega n+\phi)}\right)$$

By using (c) $\Longrightarrow y[n] = \frac{B}{2j}e^{j\phi}C[\omega]e^{j\omega n} + \frac{B}{2j}e^{-j\phi}C[-\omega]e^{-j\omega n}$