ME 5224: Signals and Signals
Spring: 2023
Instructor: Dr. I-Fan Lin

Homework 3 - DT and CT Impulse Response of LTI Systems

Spring 2023

Exercise 1. Determine the Impulse Response of DT LTI Systems

For each of the following systems with input x and output y, **i**) prove that the system is linear, **ii**) prove that the system is time-invariant, **iii**) compute the system's impulse response. Simplify your answer as much as possible.

(a) $y_n = \sum_{k=0}^{\infty} b_k x_{n-k}$

i) Prove that the system is linear

Let $y_1[n] = \sum_{k=0}^{\infty} b[k]x_1[n-k]$, $y_2[n] = \sum_{k=0}^{\infty} b[k]x_2[n-k]$, $x_3[n] = ax_1[n] + bx_2[n]$

$\Longrightarrow y_3[n] = \sum_{k=0}^{\infty} b[k]x_3[n-k] = \sum_{k=0}^{\infty} b[k](ax_1[n-k]+bx_2[n-k]) = a\sum_{k=0}^{\infty} b[k]x_1[n-k]+b\sum_{k=0}^{\infty} b[k]x_2[n-k]$

$\Longrightarrow y_3[n] = ay_1[n] + by_2[n] \Longrightarrow$ The system is **LINEAR**.

ii) Prove that the system is time-invariant

Let $y_1[n]$ be the output of the shifted input $x[n-n_0]$, and $y_2[n]$ be the shifted output.

$$\begin{cases} y_1[n] = \sum_{n=0}^{\infty} b[k]x[n-k-n_0] \\ y_2[n] = y[n-n_0] = \sum_{n=0}^{\infty} b[k]x[n-k-n_0] \end{cases}$$

$\because y_1[n] = y_2[n] \Longrightarrow$ The system is **TIME-INVARIANT**.

iii) Compute the system's impulse response. (x[n]=$\delta[n]$)

$y[n] = \sum_{k=0}^{\infty} b[k]\delta[n-k] = \begin{cases} b[n] & ,n \geq 0 \\ 0 & ,n < 0 \end{cases} = b[n]u[n]$

(b) $y_n = \frac{1}{3}(x_n - \frac{1}{2}(x_{n-1} + x_{n+1}))$

i) Prove that the system is linear

Let $y_1[n] = \dfrac{1}{3}(x_1[n] - \dfrac{1}{2}(x_1[n-1] + x_1[n+1])), y_2[n] = \dfrac{1}{3}(x_2[n] - \dfrac{1}{2}(x_2[n-1] + x_2[n+1])),$

$x_3[n] = ax_1[n] + bx_2[n]$

$$\begin{aligned}
\Longrightarrow y_3[n] &= \frac{1}{3}(x_3[n] - \frac{1}{2}(x_3[n-1] + x_3[n+1])) \\
&= \frac{1}{3}((ax_1[n] + bx_2[n]) - \frac{1}{2}((ax_1[n-1] + bx_1[n-1]) + (ax_1[n+1] + bx_2[n+1]))) \\
&= a(\frac{1}{3}(x_1[n] - \frac{1}{2}(x_1[n-1] + x_1[n+1]))) + b(\frac{1}{3}(x_2[n] - \frac{1}{2}(x_2[n-1] + x_2[n+1]))) \\
&= ay_1[n] + by_2[n]
\end{aligned}$$

$\Longrightarrow$ The system is **LINEAR**.

ii) Prove that the system is time-invariant
Let $y_1[n]$ be the output of the shifted input $x[n-n_0]$, and $y_2[n]$ be the shifted output.

$$\begin{cases} y_1[n] = \frac{1}{3}(x[n-n_0] - \frac{1}{2}(x[n-n_0-1] + x[n-n_0+1])) \\ y_2[n] = y[n-n_0] = \frac{1}{3}(x[n-n_0] - \frac{1}{2}(x[n-n_0-1] + x[n-n_0+1])) \end{cases}$$

$\because y_1[n] = y_2[n] \implies$ The system is **TIME-INVARIANT**.

iii) Compute the system's impulse response. (x[n]=$\delta[n]$)

$$y[n] = \frac{1}{3}(\delta[n] - \frac{1}{2}(\delta[n-1] + \delta[n+1])) = \begin{cases} -1/6 & , n = -1 \\ 1/3 & , n = 0 \\ -1/6 & , n = 1 \end{cases}$$

(c) $y_n = \frac{1}{2}y_{n-1} + x_n$

i) Prove that the system is linear
Let $y_1[n] = \frac{1}{2}y_1[n-1] + x_1[n]$, $y_2[n] = \frac{1}{2}y_2[n-1] + x_2[n]$, $x_3[n] = ax_1[n] + bx_2[n]$
Assume $y'[n] = y[n] - \frac{1}{2}y[n-1] \implies y'_1[n] = x_1[n]$, $y'_2[n] = x_2[n]$
$\implies y_3[n] = \frac{1}{2}y_3[n-1] + x_3[n] \implies y'_3[n] = x_3[n] = ax_1[n] + bx_2[n] = ay'_1[n] + by'_2[n]$
$\implies$ The system is **LINEAR**.

ii) Prove that the system is time-invariant
Let $y_1[n]$ be the output of the shifted input $x[n-n_0]$, and $y_2[n]$ be the shifted output.

$$\begin{cases} y_1[n] = \frac{1}{2}y[n-1] + x[n-n_0] \\ y_2[n] = \frac{1}{2}y[n-1] + x[n-n_0] \end{cases}$$

$\because y_1[n] = y_2[n] \implies$ The system is **TIME-INVARIANT**.

iii) Compute the system's impulse response. (x[n]=$\delta[n]$)

$$y[n] = \frac{1}{2}y[n-1] + \delta[n] = \begin{cases} 1 & , n = 0 \\ 1/2 & , n = 1 \\ 1/4 & , n = 2 \\ \vdots \end{cases} = (\tfrac{1}{2})^n u[n]$$

Exercise 2. DT Impulse Response

Consider the discrete-time LTI system described by the equation

$$y_n = x_n - 3x_{n-1} + 2x_{n-2}$$

(a) Compute the impulse response of the system.
$y[n] = \delta[n] - 3\delta[n-1] + 2\delta[n-2]$

(b) Express the system in the form $y_n = x_n * h_n$
$y[n] = (\delta[n] - 3\delta[n-1] + 2\delta[n-2]) * x[n]$

(c) Find the output when the input is given by $x_n = u_n$

$$y[n] = u[n] - 3u[n-1] + 2u[n+1] = \begin{cases} 1 & , n = 0 \\ -2 & , n = 1 \\ 0 & , else \end{cases}$$

(d) Find the output when the input is given by $x_n = 1$
$y[n] = 1 - 3 + 2 = 0$

Exercise 3. Determine the Impulse Response of CT LTI Systems

For each of the following systems with input x and output y, **i**) prove that the system is linear, **ii**) prove that the system is time-invariant, **iii**) compute the system's impulse response. Simplify your answer as much as possible.

(a) $y(t) = \int_{-\infty}^{\infty} r(\tau - t)x(\tau)d\tau$

i) Prove that the system is linear

Let $y_1(t) = \int_{-\infty}^{\infty} r(\tau - t)x_1(\tau)d\tau$, $y_2(t) = \int_{-\infty}^{\infty} r(\tau - t)x_2(\tau)d\tau$, $x_3(t) = ax_1(t) + bx_2(t)$

$$\begin{aligned}\implies y_3(t) &= \int_{-\infty}^{\infty} r(\tau - t)x_3(t)d\tau = \int_{-\infty}^{\infty} r(\tau - t)(ax_1(t) + bx_2(t))d\tau \\ &= a\int_{-\infty}^{\infty} r(\tau - t)x_1(t)d\tau + b\int_{-\infty}^{\infty} r(\tau - t)x_2(t)d\tau = ay_1(t) + by_2(t)\end{aligned}$$

$\implies$ The system is **LINEAR**.

ii) Prove that the system is time-invariant

Let $y_1(t)$ be the output of the shifted input $x(t - t_0)$, and $y_2(t)$ be the shifted output.

$$\begin{cases} y_1(t) = \int_{-\infty}^{\infty} r(\tau - t)x(\tau - t_0)d\tau \\ y_2(t) = y(t - t_0) = \int_{-\infty}^{\infty} r(\tau - (t - t_0))x(\tau)d\tau = \int_{-\infty}^{\infty} r(\tau - t)x(\tau - t_0)d\tau \end{cases}$$

$\because y_1(t) = y_2(t) \implies$ The system is **TIME-INVARIANT**.

iii) Compute the system's impulse response

$y(t) = \int_{-\infty}^{\infty} r(\tau - t)\delta(\tau)d\tau$

$\because \delta(t) = \begin{cases} 1 & \text{, t = 0} \\ 0 & \text{, else} \end{cases} \implies r(\tau - t)\delta(\tau) = r(-t) \implies y(t) = r(-t)$

(b) $y(t) = x(t) + 2x(t + 1) + 3x(t - 1)$

i) Prove that the system is linear

Let $y_1(t) = x_1(t) + 2x_1(t + 1) + 3x_1(t - 1)$, $y_2(t) = x_2(t) + 2x_2(t + 1) + 3x_2(t - 1)$

$x_3(t) = ax_1(t) + bx_2(t)$

$$\begin{aligned}\implies y_3(t) &= x_3(t) + 2x_3(t + 1) + 3x_3(t - 1) \\ &= (ax_1(t) + bx_2(t)) + 2(ax_1(t + 1) + bx_2(t + 1)) + 3(ax_1(t - 1) + bx_2(t - 1)) \\ &= a(x_1(t) + 2x_1(t + 1) + 3x_1(t - 1)) + b(x_2(t) + 2x_2(t + 1) + 3x_2(t - 1)) \\ &= ay_1(t) + by_2(t)\end{aligned}$$

$\implies$ The system is **LINEAR**.

ii) Prove that the system is time-invariant

Let $y_1(t)$ be the output of the shifted input $x(t - t_0)$, and $y_2(t)$ be the shifted output.

$$\begin{cases} y_1(t) = x(t - t_0) + 2x(t + 1 - t_0) + 3x(t - 1 - t_0) \\ y_2(t) = y(t - t_0) = x(t - t_0) + 2x(t + 1 - t_0) + 3x(t - 1 - t_0) \end{cases}$$

$\because y_1(t) = y_2(t) \implies$ The system is **TIME-INVARIANT**.

iii) Compute the system's impulse response

$$y(t) = \delta(t) + 2\delta(t + 1) + 3\delta(t - 1) = \begin{cases} 2 & \text{, t = -1} \\ 1 & \text{, t = 0} \\ 3 & \text{, t = 1} \end{cases}$$

(c) $\dfrac{dy(t)}{dt} = -x(t) \quad \Longrightarrow \quad y(t) = -\int_{-\infty}^{t} x(\tau)d\tau$

i) Prove that the system is linear

Let $y_1(t) = -\int_{-\infty}^{t} x_1(\tau)d\tau, y_2(t) = -\int_{-\infty}^{t} x_2(\tau)d\tau, x_3(t) = ax_1(t) + bx_2(t)$

$$\Longrightarrow y_3(t) = -\int_{-\infty}^{t} x_3(\tau)d\tau = y(t) = -\int_{-\infty}^{t} ax_1(\tau) + bx_2(\tau)d\tau = -a\int_{-\infty}^{t} x_1(\tau)d\tau - b\int_{-\infty}^{t} x_2(\tau)d\tau$$
$$= -ay_1(t) - by_2(t)$$

$\Longrightarrow$ The system is **LINEAR**.

ii) Prove that the system is time-invariant

Let $y_1(t)$ be the output of the shifted input $x(t - t_0)$, and $y_2(t)$ be the shifted output.

$$\begin{cases} y_1(t) = -\int_{-\infty}^{t} x(\tau - t_0)d\tau \\ y_2(t) = y(t - t_0) = -\int_{-\infty - t_0}^{t - t_0} x(\tau)d\tau = \int_{-\infty}^{t} x(\tau - t_0)d\tau \end{cases}$$

$\because y_1(t) = y_2(t) \Longrightarrow$ The system is **TIME-INVARIANT**.

iii) Compute the system's impulse response

$$y(t) = -\int_{-\infty}^{t} \delta(\tau)d\tau \quad \because \delta(t) = \begin{cases} 1 & \text{, t = 0} \\ 0 & \text{, else} \end{cases} \Longrightarrow y(t) = \begin{cases} -1 & \text{, t} \geq 0 \\ 0 & \text{, t} < 0 \end{cases} = -u(t)$$

Exercise 4. CT Convolution

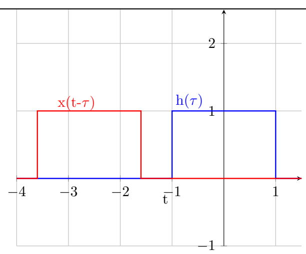
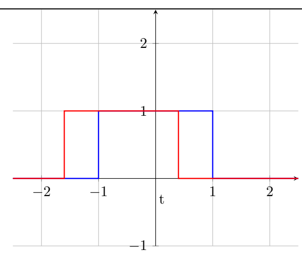
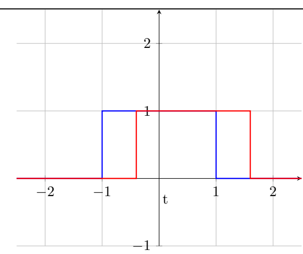
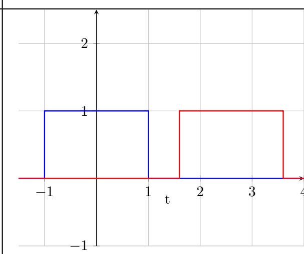
Find the outputs of the following LTI systems with the following inputs.

(a) Impulse response of $h(t) = u(t + 1) - u(t - 1)$; input of $x(t) = u(t) - u(t - 2)$

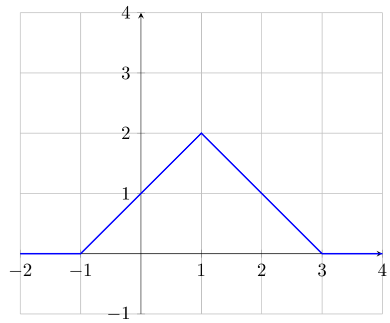
$$y(t) = h(t)*x(t) = (u(t+1)-u(t-1))*(u(t)-u(t-2)) = \int_{-\infty}^{\infty} (u(\tau+1)-u(\tau-1))(u(t-\tau)-u(t-\tau-2))d\tau$$

$$u(t + 1) - u(t - 1) = \begin{cases} 1 & , -1 < \text{t} < 1 \\ 0 & \text{, else} \end{cases} \quad , x(t) = u(t) - u(t - 2) = \begin{cases} 1 & , 0 < \text{t} < 2 \\ 0 & \text{, else} \end{cases}$$

Flip x(τ), and shift by an arbitrary value of t to get x($t - \tau$), then find the regions of τ-overlap.

Region I. No-overlap	Region II. Partially-overlap	Region III. Partially-overlap	Region IV. No-overlap
t < -1	-1 < t < 1	1 < t < 2	t > 2
			
$y(t) = 0$	$y(t) = \int_{-1}^{t} 1d\tau = t + 1$	$y(t) = \int_{t-2}^{1} 1d\tau = 3 - t$	$y(t) = 0$

$$y(t) = \begin{cases} \text{t+1} & , -1 < \text{t} < 1 \\ \text{3-t} & , 1 < \text{t} < 3 \\ 0 & \text{, else} \end{cases} \Longrightarrow$$

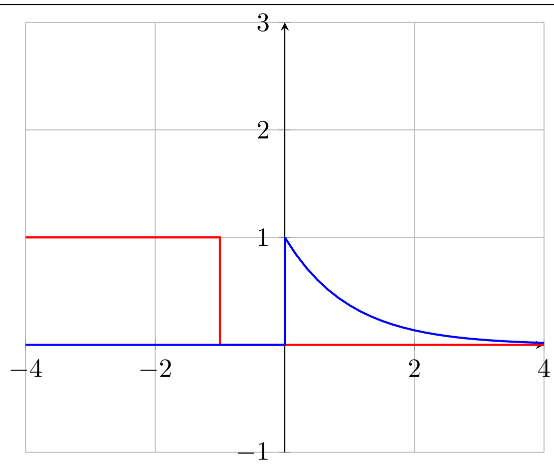
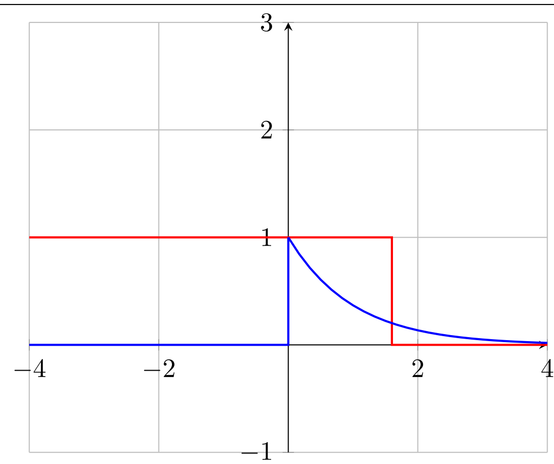


(b) Impulse response of $h(t) = e^{-at}u(t)$; input of $x(t) = u(t)$ for a > 0

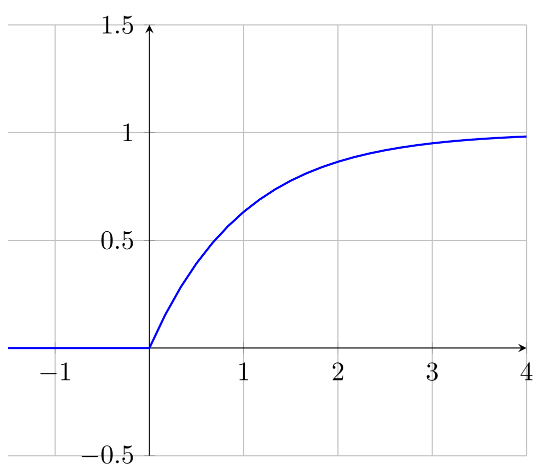
$$y(t) = h(t) * x(t) = (e^{-at}u(t)) * u(t) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)u(t-\tau)d\tau$$

$$h(t) = e^{-at}u(t) = \begin{cases} e^{-at} & ,\text{t} \geq 0 \\ 0 & ,\text{else} \end{cases} \quad , x(t) = u(t) = \begin{cases} 1 & ,\text{t} \geq 0 \\ 0 & ,\text{else} \end{cases}$$

Flip x(τ), and shift by an arbitrary value of t to get x($t-\tau$), then find the regions of τ-overlap.

Region I. No-overlap	Region II. Partially-overlap
t < 0	t ≥ 0
	
$y(t) = 0$	$y(t) = \int_0^t e^{-a\tau}d\tau = -\frac{1}{a}(e^{-at} - 1)u(t)$

$$y(t) = -\frac{1}{a}(e^{-at} - 1)u(t) \Longrightarrow (a = 1 \quad \text{below})$$

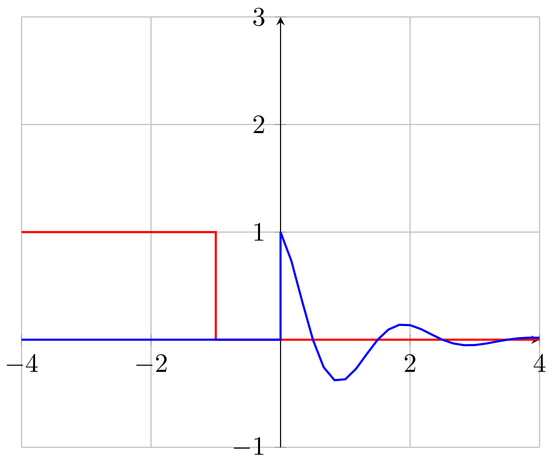
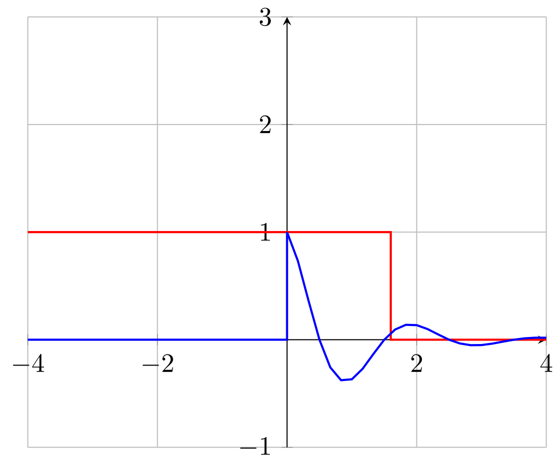


(c) Impulse response of $h(t) = e^{-at}cos(\omega t)u(t)$; input of $x(t) = u(t)$ for $a > 0$ and $\omega \in \mathbb{R}$

$$y(t) = h(t) * x(t) = (e^{-at}cos(\omega t)u(t)) * u(t) = \int_{-\infty}^{\infty} e^{-a\tau}cos(\omega\tau)u(\tau)u(t-\tau)d\tau$$

$$h(t) = cos(\omega t)u(t) = \begin{cases} cos(\omega t) & ,\text{t} \geq 0 \\ 0 & ,\text{else} \end{cases} \quad , x(t) = u(t) = \begin{cases} 1 & ,\text{t} \geq 0 \\ 0 & ,\text{else} \end{cases}$$

Flip x(τ), and shift by an arbitrary value of t to get x($t-\tau$), then find the regions of τ-overlap.

Region I. No-overlap	Region II. Partially-overlap
t < 0	t ≥ 0
	
$y(t) = 0$	$y(t) = \int_0^t e^{-a\tau}cos(\omega\tau)d\tau$

$$y(t) = \int_0^t e^{-a\tau}cos(\omega\tau)d\tau = \frac{1}{2}\int_0^t e^{-a\tau}(e^{j\omega\tau} + e^{-j\omega\tau})d\tau = \frac{1}{2}\Big[\frac{1}{-a+j\omega}e^{\tau(j\omega-a)} + \frac{1}{-a-j\omega}e^{-\tau(j\omega+a)}\Big]\Big|_0^t$$

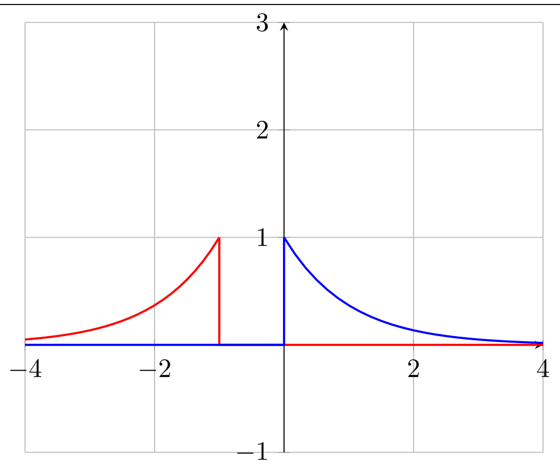
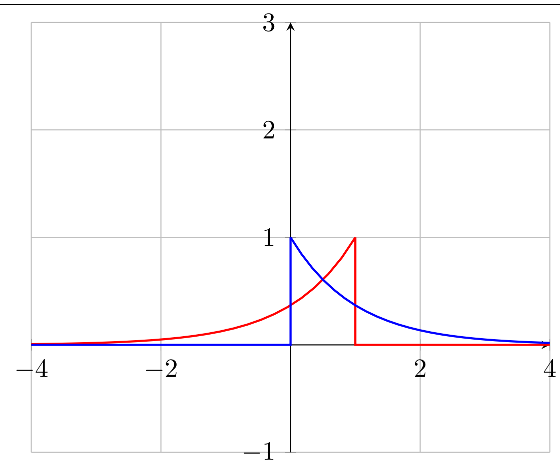
$$= \frac{1}{2}\Big[\frac{-a-j\omega-a+j\omega}{-ja\omega+\omega^2+a^2+ja\omega}e^{-a\tau}\Big(e^{j\omega\tau}+e^{-j\omega\tau}\Big)\Big]_0^t = \frac{2a}{a^2+\omega^2}\Big(1-e^{-at}cos(\omega t)\Big)u(t)$$

(d) Impulse response of $h(t) = e^{-at}u(t)$; input of $x(t) = e^{-bt}u(t)$ for a $\neq$ b > 0

$$y(t) = h(t) * x(t) = (e^{-at}u(t)) * (e^{-bt}u(t)) = \int_{-\infty}^{\infty} e^{-a\tau}u(\tau)e^{-b(t-\tau)}u(t-\tau)d\tau$$

$$h(t) = e^{-at}u(t) = \begin{cases} e^{-at} & ,\text{t} \geq 0 \\ 0 & ,\text{else} \end{cases} \quad , x(t) = e^{-bt}u(t) = \begin{cases} e^{-bt} & ,\text{t} \geq 0 \\ 0 & ,\text{else} \end{cases}$$

Flip x(τ), and shift by an arbitrary value of t to get x($t-\tau$), then find the regions of τ-overlap.

Region I. No-overlap	Region II. Partially-overlap
t < 0	t ≥ 0
	
$y(t) = 0$	$y(t) = \int_0^t e^{-a\tau}e^{-b(t-\tau)}d\tau$

$$y(t) = \int_0^t e^{-a\tau}e^{-b(t-\tau)}d\tau = e^{-bt}\int_0^t e^{(-a+b)\tau}d\tau = \frac{e^{-bt}}{-a+b}(e^{(-a+b)t}-1) = \frac{1}{a-b}(e^{-bt}-e^{-at})u(t)$$

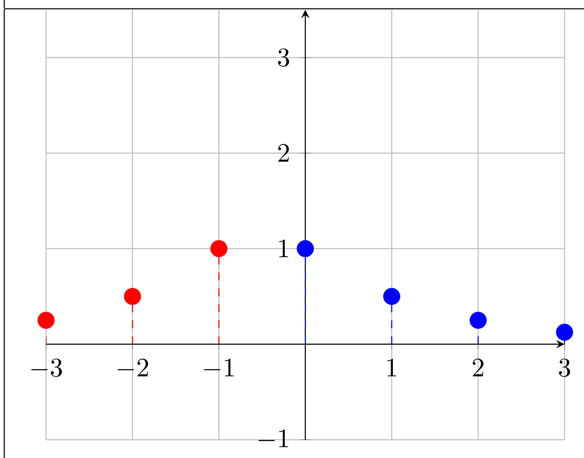
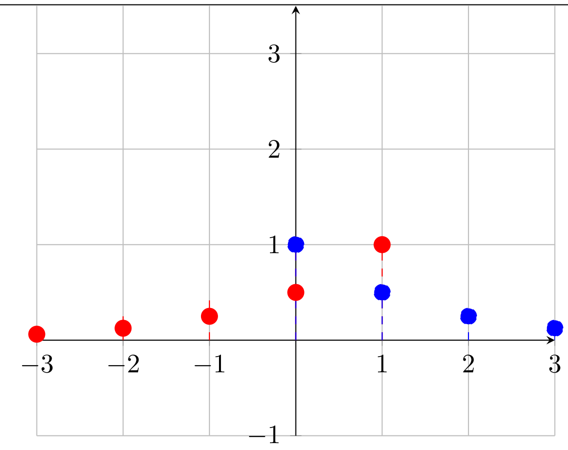
Exercise 5. DT Convolution

Calculate the output of a LTI system with impulse response $h[n]$, input $x[n]$, and output $y[n]$

(a) $h[n] = a^n u[n]$ and $x[n] = b^n u[n]$ where $a \neq b$

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} a^k u[k] b^{n-k} u[n-k]$$

Flip $x[k]$, and shift by an arbitrary value of n to get $x[n-k]$, then find the regions of k-overlap.

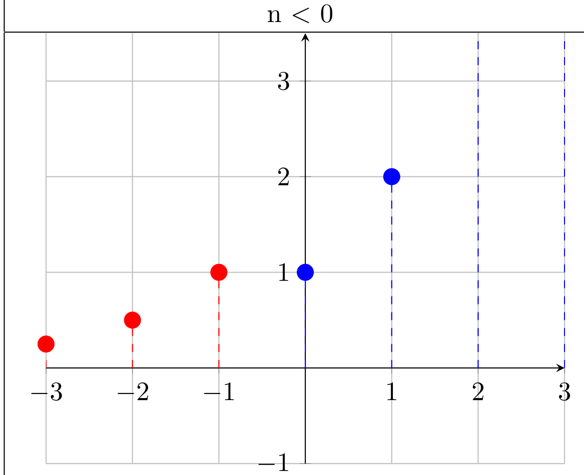
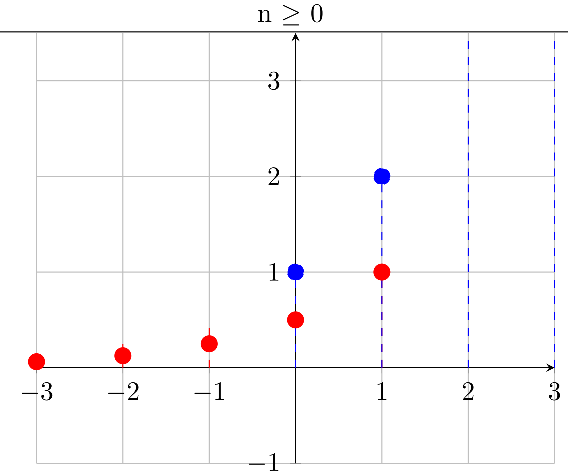
Region I. No-overlap	Region II. Partially-overlap
n $<$ 0	n $\geq$ 0
	
$y[n] = 0$	$y[n] = \sum_{k=0}^{n} a^k b^{n-k}$

$$y[n] = \sum_{k=0}^{n} a^k b^{n-k} = b^n \sum_{k=0}^{n}\left(\frac{a}{b}\right)^k = b^n \frac{1-(a/b)^{n+1}}{1-a/b} = \frac{b^{n+1}-a^{n+1}}{b-a}u[n]$$

(b) $h[n] = a^n u[n]$ and $x[n] = a^n u[n]$

$$y[n] = h[n] * x[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=-\infty}^{\infty} a^k u[k] a^{n-k} u[n-k]$$

Flip $x[k]$, and shift by an arbitrary value of n to get $x[n-k]$, then find the regions of k-overlap.

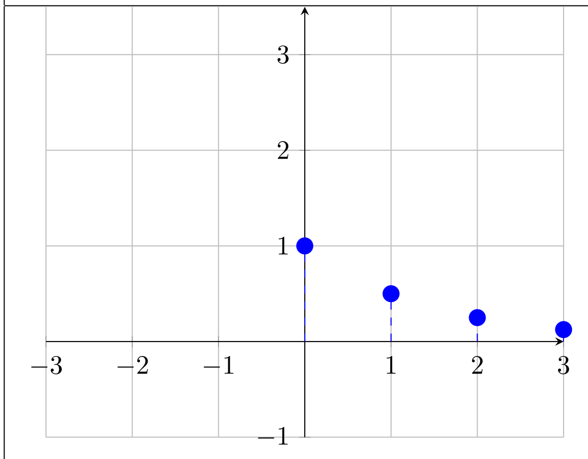
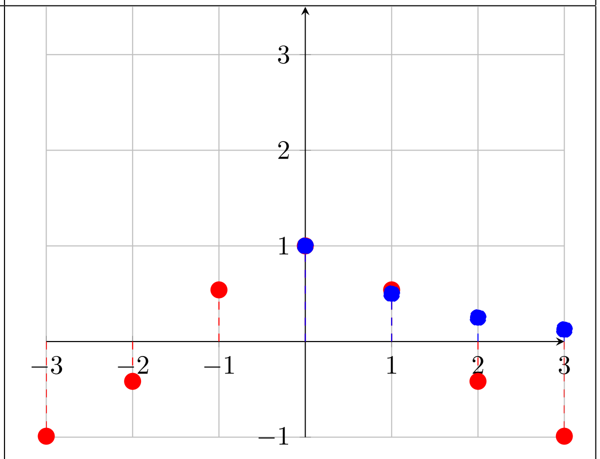
Region I. No-overlap	Region II. Partially-overlap
n $<$ 0	n $\geq$ 0
	
$y[n] = 0$	$y[n] = \sum_{k=0}^{n} a^k a^{n-k}$

$$y[n]=\sum_{k=0}^{n}a^k a^{n-k}=\sum_{k=0}^{n}a^n=(n+1)a^n u[n]$$

(c) $h[n]=a^n u[n]$ and $x[n]=cos[\omega n]$ where $|a|<1$

$$y[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]=\sum_{k=-\infty}^{\infty}a^k u[k]cos[\omega(n-k)]$$

Flip $x[k]$, and shift by an arbitrary value of n to get $x[n-k]$, then find the regions of k-overlap.

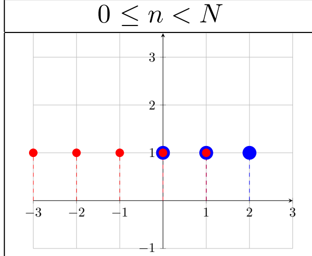
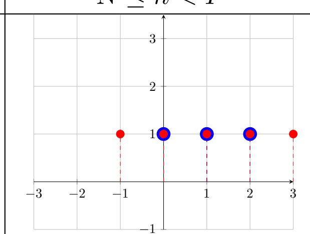
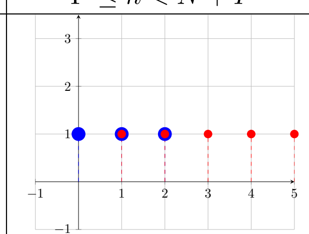
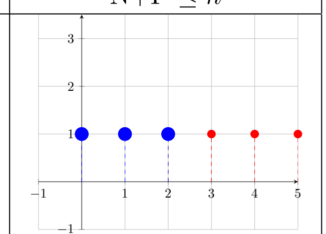
Region I. No-overlap	Region II. Partially-overlap
n <0	n ≥ 0
	
$y[n]=0$	$y[n]=\sum_{k=0}^{n}a^k cos[\omega(n-k)]$

$$\begin{aligned}\text{y}[n]=&\sum_{k=0}^{n}a^k cos[\omega(n-k)]=\frac{1}{2}\sum_{k=0}^{n}a^k(e^{j\omega(n-k)}+e^{-j\omega(n-k)})\\=&\frac{1}{2}\sum_{k=0}^{n}(ae^{-j\omega})^k e^{j\omega n}+\frac{1}{2}\sum_{k=0}^{n}(ae^{-j\omega})^k e^{-j\omega n}=\frac{1}{2}\frac{e^{j\omega n}}{1-ae^{-j\omega}}+\frac{1}{2}\frac{e^{j\omega n}}{1-ae^{j\omega}}\\=&\frac{e^{j\omega n}-ae^{j\omega(n+1)}+e^{-j\omega n}-ae^{-j\omega(n+1)}}{2(1-ae^{-j\omega})(1-ae^{j\omega})}\end{aligned}$$

(d) $h[n]=u[n]-u[n-N]$ and $x[n]=u[n]-u[n-P]$ for P $>$ N

$$y[n]=h[n]*x[n]=\sum_{k=-\infty}^{\infty}h[k]x[n-k]=\sum_{k=-\infty}^{\infty}(u[k]-u[k-N])(u[n-k]-u[n-k-P])$$

Flip $x[k]$, and shift by an arbitrary value of n to get $x[n-k]$, then find the regions of k-overlap.

Partially-overlap	Fully-overlap	Partially-overlap	No-overlap
$0\leq n<N$	N $\leq n<P$	P $\leq n<N+P$	N+P $\leq n$
			
$y[n]=n+1$	$y[n]=N$	$y[n]=N-(n-P)+1$	$y[n]=0$

$$y[n]=\begin{cases}n+1 & ,0\leq n<N\\ N & ,N\leq n<P\\ N+P-n+1 & ,P\leq n<N+P\\ 0 & ,N+P\leq n\end{cases}$$