ME 5224: Signals and Signals
Spring: 2023
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Homework 1 - Signal Properties

Spring 2023

Exercise 1. Complex Numbers

Express the following complex number in the polar form z = A$e^{j\theta}$

(a) z = 1+j$\sqrt{3}$

$|\text{z}| = \sqrt{1+(\sqrt{3})^2} = 2$, $\quad \angle\text{z} = tan^{-1}(\frac{\sqrt{3}}{1}) = 60^\circ = \frac{\pi}{3}$, $\quad$ z = $2e^{j\frac{\pi}{3}}$

(b) z = a+jb for a>0

$|\text{z}| = \sqrt{a^2+b^2}$, $\quad \angle\text{z} = tan^{-1}(\frac{a}{b})$, $\quad$ z = $\sqrt{a^2+b^2}e^{j\theta} = \sqrt{a^2+b^2}e^{jtan^{-1}(\frac{a}{b})}$

(c) z = $(a+jb)^3$

Let z_0 = a+jb , z = z_0^3, by (b) $|z_0| = \sqrt{a^2+b^2}$, $\angle$z = $tan^{-1}(\frac{a}{b})$,

then z = $(a^2+b^2)^{\frac{3}{2}}e^{j3\theta} = (a^2+b^2)^{\frac{3}{2}}e^{j3tan^{-1}(\frac{a}{b})}$

(d) z = $\frac{e^{j\psi}}{e^{j\phi}}$

z = $e^{j\psi}e^{-j\phi} = e^{j(\psi-\phi)}$

(e) z = $\frac{e^{j\pi/3-1}}{1+j\sqrt{3}}$

By (a) , z = $\frac{\frac{1}{2}+j\frac{\sqrt{3}}{2}-1}{2(\frac{1}{2}+j\frac{\sqrt{3}}{2})} = \frac{\frac{-1}{2}+j\frac{\sqrt{3}}{2}}{2(\frac{1}{2}+j\frac{\sqrt{3}}{2})} = \frac{e^{j2\pi/3}}{2e^{j\pi/3}} = \frac{1}{2}e^{j\pi/3}$

Exercise 2. Complex Numbers

Express the following complex number in the *rectangular* form z = a + jb

(a) z = $e^{j\theta}$

z = cos(θ)+jsin(θ)

(b) z = $e^{a+j\theta}$

z = $e^ae^{j\theta} = e^a$(cos(θ)+jsin(θ))

(c) z = $\frac{e^{j\psi}}{e^{j\phi}}$

z = $e^{j(\psi-phi)}$ = cos($\psi-\phi$)+jsin($\psi-\phi$)

(d) z = $\frac{1}{c+jd}$

z = $\frac{1}{\sqrt{c^2+d^2}}\frac{1}{\frac{c}{\sqrt{c^2+d^2}}+\frac{jd}{\sqrt{c^2+d^2}}}$ = $\frac{1}{\sqrt{c^2+d^2}}$(cos(-θ)+jsin(-θ)) (cos(θ) = $\frac{c}{\sqrt{c^2+d^2}}$, sin(θ) = $\frac{d}{\sqrt{c^2+d^2}}$)

= $\frac{1}{\sqrt{c^2+d^2}}$(cos(θ)-jsin(θ))

(e) z = $\frac{e+jf}{c+jd}$

z = $\frac{\sqrt{e^2+f^2}}{\sqrt{c^2+d^2}}\frac{e^{j\phi}}{e^{j\theta}}$ = $\frac{\sqrt{e^2+f^2}}{\sqrt{c^2+d^2}}e^{j(\phi-\theta)}$

(cos(ϕ)=$\frac{e}{\sqrt{e^2+f^2}}$, sin(ϕ)=$\frac{f}{\sqrt{e^2+f^2}}$, cos(θ)=$\frac{c}{\sqrt{c^2+d^2}}$, sin(θ)=$\frac{d}{\sqrt{c^2+d^2}}$)

Exercise 3. Rectangular to Polar Conversion

Let A$e^{j\theta}$ = a+jb for a^2+b^2>0. Precisely specify a function θ = f(a, b) which is correct for all values of a and b results in a positive value of A.

$$\theta = \begin{cases} \arctan(b/a) & a > 0 \\ \pi + \arctan(b/a) & a < 0 \\ \pi/2 & a = 0, b > 0 \\ -\pi/2 & a = 0, b < 0 \end{cases}$$

Exercise 4. Transformations of Independent Variable

Let x(t)=sin(2πt)u(t+1/2)u(-t+1/2). Sketch and label carefully the following signals.

(a) x(t)

$$u(-t+\frac{1}{2}) = \begin{cases} 1 & ,t \le \frac{1}{2} \\ 0 & ,t > \frac{1}{2} \end{cases} \qquad u(t+\frac{1}{2}) = \begin{cases} 1 & ,t \ge \frac{-1}{2} \\ 0 & ,t < \frac{-1}{2} \end{cases}$$

$$\to u(-t+\frac{1}{2})u(t+\frac{1}{2}) = \begin{cases} 1 & ,\frac{-1}{2} < t < \frac{1}{2} \\ 0 & ,else \end{cases} \quad \to x(t) = \begin{cases} \sin(2\pi t) & ,\frac{-1}{2} < t < \frac{1}{2} \\ 0 & ,else \end{cases}$$

(b) x(t/2)

$$x(\frac{t}{2}) = \sin(\pi t)u(\frac{1}{2}(t+1))u(\frac{-1}{2}(t-1))$$

$$u(\frac{-1}{2}(t-1)) = \begin{cases} 1 & ,t \le 1 \\ 0 & ,t > 1 \end{cases} \qquad u(\frac{1}{2}(t+1)) = \begin{cases} 1 & ,t \ge -1 \\ 0 & ,t < -1 \end{cases}$$

$$\to u(\frac{1}{2}(t+1))u(\frac{-1}{2}(t-1)) = \begin{cases} 1 & ,-1 < t < 1 \\ 0 & ,else \end{cases} \quad \to x(t) = \begin{cases} \sin(\pi t) & ,-1 < t < 1 \\ 0 & ,else \end{cases}$$

(c) x(-t-1/2)

$$x(-t-\frac{1}{2}) = \sin(-2\pi t - \pi)u(-t)u(t+1)$$
$$= \begin{cases} \sin(-2\pi t - \pi) & ,-1 < t < 0 \\ 0 & ,else \end{cases}$$

Exercise 5. Signal Properties: Fundamental Periods

For each of the following continuous-time signals, determine if the signals are periodic, and specify their fundamental period.

(a) x(t) = bcos($2\pi ft+\theta$)

$$\text{The signal is periodic,} N_0 = \frac{2\pi}{2\pi f} = \frac{1}{f}$$

(b) x(t) = bcos(ωt+θ)

$$\text{The signal is periodic,} N_0 = \frac{2\pi}{\omega}$$

(c) x(t) = bcos($\omega_1 t+\theta$) + csin($\omega_2 t+\phi$) where $\omega_2 = 2\omega_1$

$$\text{Let } \omega_1 k_1 = \omega_2 k_2, \quad \frac{k_1}{k_2} = \frac{\omega_2}{\omega_1} = 2 \in \mathbf{Z} \rightarrow \text{The signal is periodic, with } N_0 = \frac{2\pi}{\omega_1}$$

(d) x(t) = bcos($\omega_1 t+\theta$)+csin($\omega_2 t+\theta$) where $\omega_2 = \sqrt{2}\omega_1$

$$\text{Let } \omega_1 k_1 = \omega_2 k_2, \quad \frac{k_1}{k_2} = \frac{\omega_2}{\omega_1} = \sqrt{2} \notin \mathbf{Z} \rightarrow \text{The signal is not periodic}$$

Exercise 6. Signal Properties: Fundamental Periods

For each of the following discrete-time signals, determine if the signals are periodic, and specify their fundamental period.

(a) x[k] = bcos[$2\pi fk+\theta$] where f=$\frac{1}{2}$

$$\text{The signal is periodic, } N_0 = \frac{2\pi N}{2\pi\frac{1}{2}} = 2N\text{, the minimum N that makes } N_0 \in Z \text{ is 1}$$

$$\rightarrow \text{the fundamental period is 2}$$

(b) x[k] = bcos[$2\pi fk+\theta$] where f=$\frac{1}{8}$

$$\text{The signal is periodic, } N_0 = \frac{2\pi N}{2\pi\frac{1}{8}} = 8N\text{, the minimum N that makes } N_0 \in Z \text{ is 1}$$

$$\rightarrow \text{the fundamental period is 8}$$

(c) x[k] = bcos[$2\pi fk+\theta$] where f=$\frac{2}{8}$

$$\text{The signal is periodic, } N_0 = \frac{2\pi N}{2\pi\frac{2}{8}} = 4N\text{, the minimum N that makes } N_0 \in Z \text{ is 1}$$

$$\rightarrow \text{the fundamental period is 4}$$

(d) x[k] = bcos[$2\pi fk+\theta$] where f=$\frac{7}{8}$

$$\text{The signal is periodic, } N_0 = \frac{2\pi N}{2\pi\frac{7}{8}} = \frac{8N}{7}\text{, the minimum N that makes } N_0 \in Z \text{ is 7}$$

$$\rightarrow \text{the fundamental period is 8}$$

(e) x[k] = bcos[$2\pi fk+\theta$] where f=$\frac{6}{11}$

$$\text{The signal is periodic, } N_0 = \frac{2\pi N}{2\pi\frac{6}{11}} = \frac{11N}{6}\text{, the minimum N that makes } N_0 \in Z \text{ is 6}$$

$$\rightarrow \text{the fundamental period is 11}$$

Exercise 7. Signal Properties: Energy and Power

Calculate the energy and power for the following signals.

(a) x(t) = e^{-t}u(t)

$$\text{Energy} = \lim_{T\to\infty}\int_{-T}^{T}|e^{-t}u(t)|^2dt = \lim_{T\to\infty}\int_{0}^{T}e^{-2t}dt = \lim_{T\to\infty} -\frac{1}{2}e^{-2t}\Big|_0^T = \tfrac{1}{2}$$

$$\text{Power} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|e^{-t}u(t)|^2dt = \lim_{T\to\infty}\frac{1}{2T}\times\frac{1}{2} = 0$$

(b) x(t) = $\cos(t)$

$$\text{Energy} = \lim_{T\to\infty}\int_{-T}^{T}|\cos(t)|^2\,dt = \lim_{T\to\infty}\int_{-T}^{T}\cos^2(t)dt = \lim_{T\to\infty}\int_{-T}^{T}\frac{1+\cos(2t)}{2}$$

$$= \lim_{T\to\infty}\frac{1}{2}t+\frac{1}{4}\sin(2t)\Big|_{-T}^{T} = \infty$$

$$\text{Power} = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|cos(t)|^2dt = \lim_{T\to\infty}\frac{1}{2T}\times T = \tfrac{1}{2}$$

(c) x[n] = $e^{j[\frac{\pi}{2}n+\frac{\pi}{4}]}$

$$\text{Energy} = \lim_{N\to\infty}\sum_{n=-N}^{N}|e^{j[\frac{\pi}{2}n+\frac{\pi}{4}]}|^2 = \lim_{N\to\infty}\sum_{n=-N}^{N}|e^{j2[\frac{\pi}{2}n+\frac{\pi}{4}]}| = \lim_{N\to\infty}\sum_{n=-N}^{N}\cos[\pi n+\frac{\pi}{2}]+j\sin[\pi n+\frac{\pi}{2}]$$

$$= \lim_{N\to\infty}\sum_{n=-N}^{N}-\sin[\pi n]+j\cos[\pi n] = \lim_{N\to\infty}\sum_{n=-N}^{N}\sqrt{\sin^2[\pi n]+\cos^2[\pi n]} = \lim_{N\to\infty}2N = \infty$$

$$\text{Power} = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|e^{j[\frac{\pi}{2}n+\frac{\pi}{4}]}|^2 = \lim_{N\to\infty}\frac{2N}{2N+1} = 1$$

(d) x[n] = $\cos[\frac{\pi}{4}$n]

$$\text{Energy} = \lim_{N\to\infty}\sum_{n=-N}^{N}|\cos[\frac{\pi}{4}n]|^2 = \lim_{N\to\infty}\sum_{n=-N}^{N}\cos^2[\frac{\pi}{2}n] = \lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1+\cos[\frac{\pi}{2}n]}{2}$$

$$= \lim_{N\to\infty}\frac{1}{2}\times 2N = \infty$$

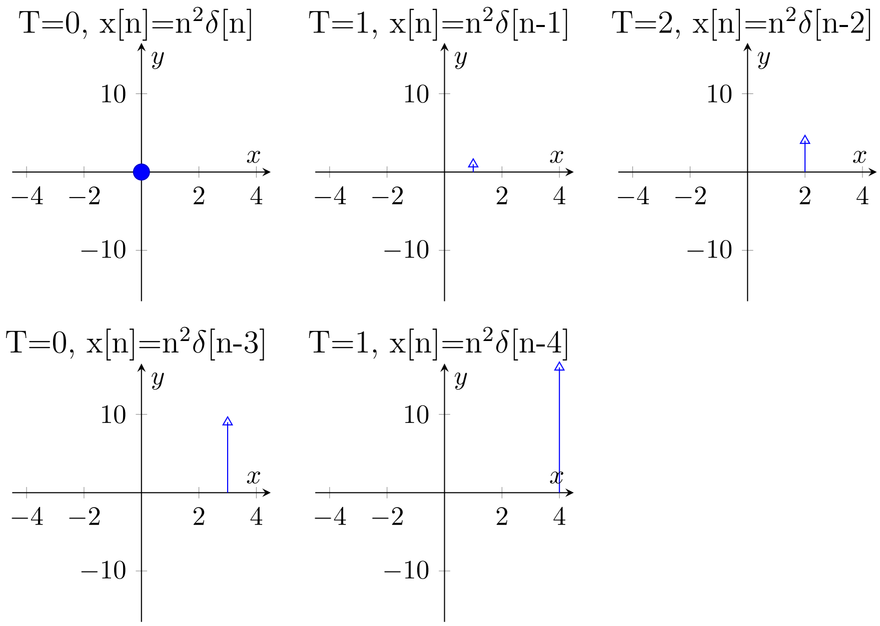
$$\text{Power} = \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}|cos[\frac{\pi}{4}n]|^2 = \lim_{N\to\infty}\frac{1}{2N+1}\times N = \frac{1}{2}$$

Exercise 8. Discrete-Time Impulse And Step Functions

(a) Calculate $\displaystyle\sum_{n=-\infty}^{\infty}n^2\delta(n-3)$

$$\delta(n-3) = \begin{cases}1 & ,n=3\\ 0 & ,else\end{cases} \quad\rightarrow\quad \sum_{n=-\infty}^{\infty}n^2\delta(n-3) = \begin{cases}9 & ,n=3\\ 0 & ,else\end{cases}$$

(b) Sketch the function x[n] $= \mathrm{n}^2\delta[n-T]$ for $T = 0,\ 1,\ \ldots,\ 4$



(c) Show that u[n] $= \displaystyle\sum_{k=-\infty}^{\infty}$ u[k]$\delta[n-k]$

Since $\delta[n-k]$ will equal 1 only when n-k=0, so by the summation from $-\infty$ to ∞, we will get u[n].

(d) Show that for all functions x[n], x[n]$= \displaystyle\sum_{k=-\infty}^{\infty}$ x[k]δ[n-k]

$$\sum_{k=-\infty}^{\infty} \mathrm{x[k]}\delta\mathrm{[n\text{-}k]} = \mathrm{x[k]}\Big|_{k=n}, \forall n \in Z = \mathrm{x[n]}$$