ME 5224: Signals and Signals
Spring: 2023
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Homework 5 - Orthonormal Transformation

Spring 2023

Exercise 1. Orthogonality and Normalized Functions

The *inner product* of two functions over the interval $t \in (a, b)$ is defined as

$$\langle f(t), g(t)\rangle = \int_a^b f(t)g^*(t)dt$$

where g^* denotes the complex conjugate of $g(t)$.
The function $f(t)$ and $g(t)$ are called *orthogonal* if $\langle f(t), g(t)\rangle = 0$. A function $f(t)$ is said to be *normalized* if

$$\langle f(t), f(t)\rangle = \int_a^b |f(t)|^2 dt = 1$$

(a) Are the functions $sin(m\omega_0 t)$ and $sin(n\omega_0 t)$ orthogonal over the interval $t \in (0, T)$ where $T = 2\pi/\omega_0$ and $m \neq n$?

$x_1(t) = sin(m\omega_0 t) = \dfrac{e^{jm\omega_0 t} - e^{-jm\omega_0 t}}{2j}$, $x_2(t) = sin(n\omega_0 t) = \dfrac{e^{jn\omega_0 t} - e^{-jn\omega_0 t}}{2j}$

$$\begin{aligned}
\langle x_1(t), x_2(t)\rangle &= \int_0^{2\pi/\omega_0} (\frac{1}{2j})^2 (e^{jm\omega_0 t} - e^{-jm\omega_0 t})(e^{-jn\omega_0 t} - e^{jn\omega_0 t})dt \\
&= (\frac{1}{2j})^2 \int_0^{2\pi/\omega_0} e^{jt\omega_0(m-n)} - e^{jt\omega_0(m+n)} - e^{-jt\omega_0(m+n)} + e^{-jt\omega_0(m-n)} dt \\
&= -\frac{1}{2}\int_0^{2\pi/\omega_0} \frac{e^{jt\omega_0(m-n)} + e^{-jt\omega_0(m-n)}}{2} - \frac{e^{jt\omega_0(m+n)} + e^{-jt\omega_0(m+n)}}{2} dt \\
&= -\frac{1}{2}\int_0^{2\pi/\omega_0} cos(\omega_0(m-n)t) - cos(\omega_0(m+n)t)dt \\
&= -\frac{1}{2}\Big[\frac{sin(\omega_0(m-n)t)}{\omega_0(m-n)} - \frac{sin(\omega_0(m+n)t)}{\omega_0(m+n)}\Big]\Big|_0^{2\pi/\omega_0} = 0
\end{aligned}$$

Thus, they are orthogonal.

(b) Let $x(t)$ be an arbitrary real-valued signal, and let

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

and

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

be the odd and even parts of $x(t)$, respectively. Show that $x_o(t)$ and $x_e(t)$ are orthogonal over the interval $t \in (-T, T)$ for any T.

$$\langle x_o(t), x_e(t)\rangle = \int_{-T}^{T} x_o(t)x_e^*(t)dt = \int_{-T}^{T}\Big(\frac{x(t) - x(-t)}{2}\Big)\Big(\frac{x^*(t) + x^*(-t)}{2}\Big)dt$$

Since $x(t)$ is real-valued $\Longrightarrow x(t) = x^*(t) \Longrightarrow \displaystyle\int_{-T}^{T}\Big(\frac{x(t) - x(-t)}{2}\Big)\Big(\frac{x(t) + x(-t)}{2}\Big)dt$

$$\begin{aligned}
&= \frac{1}{4}\int_{-T}^{T} x^2(t) - x^2(-t)dt, \text{ let } A(t) = \int_{-T}^{T} x^2(t)dt,\ A(-t) = \int_{-T}^{T} x^2(-t)dt \\
&= \frac{1}{4}(A(t) - A(-t))\Big|_{-T}^{T} = \frac{1}{4}(A(T) - A(-T) - (A(T) - A(-T))) = 0
\end{aligned}$$

Thus, $x_o(t)$ and $x_e(t)$ are orthogonal over the interval $t \in (-T, T)$

(c) Find the value of a so that the functions $f(t) = t$ and $g(t) = t - at^2$ are orthogonal over the interval $t \in (0, 1)$. Determine the real-valued constants c_1 and c_2 so that $\frac{f(t)}{c_1}$ and $\frac{g(t)}{c_2}$ are both normalized.

First, solve $f(t)$ and $g(t)$ are orthogonal. $\Longrightarrow \displaystyle\int_0^1 t(t - at^2)dt = \frac{1}{3}t^3 - \frac{a}{4}t^4\Big|_0^1 = \frac{1}{3} - \frac{a}{4} = 0 \Longrightarrow a = \frac{4}{3}$

Then, normalize $f(t)$ and $g(t) \Longrightarrow \langle \frac{f(t)}{c_1}, \frac{f(t)}{c_1} \rangle = 1$, $\langle \frac{g(t)}{c_2}, \frac{g(t)}{c_2} \rangle = 1$

$$\langle \frac{f(t)}{c_1}, \frac{f(t)}{c_1} \rangle = \int_0^1 \Big(\frac{t}{c_1}\Big)^2 dt = \frac{t^3}{3c_1^2}\Big|_0^1 = \frac{1}{3c_1^2} = 1 \Longrightarrow c_1 = \sqrt{\frac{1}{3}}$$

$$\langle \frac{g(t)}{c_2}, \frac{g(t)}{c_2} \rangle = \int_0^1 \Big(\frac{t - \frac{4}{3}t^2}{c_2}\Big)^2 dt = \frac{1}{c_2^2}\int_0^1 t^2 - \frac{8}{3}t^3 + \frac{16}{9}t^4 dt = \frac{1}{c_2^2}\Big(\frac{1}{3}t^3 - \frac{2}{3}t^4 + \frac{16}{45}t^5\Big)\Big|_0^1$$

$$= \frac{1}{c_2^2}\Big(\frac{1}{3} - \frac{2}{3} + \frac{16}{45}\Big) = \frac{1}{c_2^2}\frac{15 - 30 + 16}{45} = 1 \Longrightarrow c_2 = \sqrt{\frac{1}{45}}$$

Exercise 2. Orthonormal Set of Functions

A set of functions $\{\phi_k(t)\}$, where k is any integer, is called and *orthonormal set* if (i) $\phi_k(t)$ and $\phi_m(t)$ are orthogonal for $k \neq m$ and (ii) all functions in $\{\phi_k(t)\}$ are normalized.
For each of the following problems, check if the given set of functions form an orthonormal set over the specified interval. If the functions are not normalized determine the constant C so that the set $\{\phi_k(t)/C\}$ is orthonormal.

(a) $\phi_k(t) = \dfrac{e^{jk\omega_0 t}}{\sqrt{T}}$ for $\omega_0 = \dfrac{2\pi}{T}$ and $t \in (0, T)$.

$$\langle \phi_k(t), \phi_m(t) \rangle = \int_0^T \frac{e^{jk\omega_0 t}}{\sqrt{T}} \frac{e^{-jm\omega_0 t}}{\sqrt{T}} dt = \frac{\omega_0}{2\pi}\int_0^T e^{jk\omega_0(k-m)t}dt = \frac{e^{j\omega_0(k-m)t}}{2\pi j(k-m)}\Big|_0^T = \frac{e^{j2\pi(k-m)} - 1}{2\pi j(k-m)} = 0$$

$\Longrightarrow$ The set is orthogonal.

$$\langle \phi_k(t), \phi_k(t) \rangle = \int_0^T \frac{e^{jk\omega_0 t}}{\sqrt{T}} \frac{e^{-jk\omega_0 t}}{\sqrt{T}} dt = \frac{1}{T}\int_0^T 1dt = 1 \Longrightarrow \phi_k(t) \text{ is orthonormal.}$$

$\because$ $\phi_k(t)$ and $\phi_m(t)$ are orthogonal for $k \neq m$, and all functions in $\{\phi_k(t)\}$ are normalized
$\therefore$ $\{\phi_k(t)\}$ is an orthonormal set.

(b) $\phi_k(t) = cos(k\omega_0 t)$ for $\omega_0 = \dfrac{2\pi}{T}$ and $t \in (0, T)$.

$$\langle \phi_k(t), \phi_m(t) \rangle = \int_0^T cos(k\omega_0 t)cos^*(m\omega_0 t)dt = \int_0^T \frac{e^{jk\omega_0 t} + e^{-jk\omega_0 t}}{2} \times \frac{e^{-jm\omega_0 t} + e^{jm\omega_0 t}}{2}dt$$

$$= \frac{1}{4}\int_0^T e^{j\omega_0 t(k-m)} + e^{-j\omega_0 t(k+m)} + e^{j\omega_0 t(k+m)} + e^{-j\omega_0 t(k-m)}dt$$

$$= \frac{1}{2}\int_0^T \frac{e^{j\omega_0 t(k-m)} + e^{-j\omega_0 t(k-m)}}{2} + \frac{e^{j\omega_0 t(k+m)} + e^{-j\omega_0 t(k+m)}}{2}dt$$

$$= \int_0^T cos(\omega_0 t(k-m)) + cos(\omega_0 t(k+m))dt$$

$$= \frac{sin(\omega_0 t(k-m))}{\omega_0(k-m)}\Big|_0^T + \frac{sin(\omega_0 t(k+m))}{\omega_0(k+m)}\Big|_0^T = 0$$

$$\langle \phi_k(t), \phi_k(t) \rangle = \int_0^T cos(k\omega_0 t)cos^*(k\omega_0 t)dt = \int_0^T \frac{e^{jk\omega_0 t} + e^{-jk\omega_0 t}}{2} \times \frac{e^{-jk\omega_0 t} + e^{jk\omega_0 t}}{2}dt$$

$$= \frac{1}{4}\int_0^T 1 + e^{-j2\omega_0 tk} + e^{j2\omega_0 tk} + 1dt = \frac{1}{2}\int_0^T \frac{e^{j2\omega_0 tk} + e^{-j2\omega_0 tk}}{2} + 1dt$$

$$= \frac{1}{2}\int_0^T cos(2k\omega_0 t) + 1dt = \frac{1}{2}\Big(\frac{sin(2k\omega_0 t)}{2k\omega_0} + t^2\Big)\Big|_0^T = \frac{T}{2}$$

$\because \langle \phi_k(t), \phi_k(t) \rangle \neq 1 \quad \therefore \dfrac{T}{2C^2} = 1 \Longrightarrow C = \sqrt{\dfrac{T}{2}} \Longrightarrow \{\phi_k(t)\}$ is not an orthonormal set.

Exercise 3. Function Expansion Using Orthonormal Functions

Given a complete orthonormal basis $\{\phi_k(t)\}_{k=-\infty}^{\infty}$ over the interval $t \in (a,b)$, then we can express a function $x(t)$ on the interval (a,b) as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) \tag{1}$$

Show that the coefficients, a_k, in the above expression can be determined using the formula

$$a_m = \int_a^b x(t)\phi_m^*(t)dt$$

(Hint: Multiple both sides of Equation 1 with $\phi_m^*(t)$ and integrate both sides.)

Since $\{\phi_k(t)\}$ is orthonormal over the interval $t \in (a,b)$

$$\Longrightarrow \int_b^a \phi_k(t)\phi_m^*(t)dt = \begin{cases} 1 & \text{, when } m = k \\ 0 & \text{, when } m \neq k \end{cases}$$

$$\int_b^a x(t)\phi_m^*(t)dt = \int_b^a \sum_{k=-\infty}^{\infty} a_k\phi_k(t)\phi_m^*(t)dt = \sum_{k=-\infty}^{\infty} a_k \int_b^a \phi_k(t)\phi_m^*(t)dt = a_m$$

Exercise 4. Fourier Series Expansion

As you have shown in Exercise 2, the set $\frac{e^{jk\omega_0 t}}{\sqrt{T}}$ is orthonormal over the interval $t \in (0,T)$. Using the result of Exercise 3 we see that we can expand a given function $x(t)$, which is periodic with period T using this set as

$$x(t) = \frac{1}{\sqrt{T}} \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

This representation is called the *orthonormal Fourier series representation* of $x(t)$.
Consider the periodic function with period $T = 2$

$$x(t) = \sum_{k=-\infty}^{\infty} (t-kT)\big[u(t-kT) - u(t-kT-2)\big]$$

(e.g. for $0 \leq t \leq 2$, $x(t) = t$). Compute the values of the coefficients, a_k, for all integers k.

$$\begin{aligned} a_k &= \frac{1}{\sqrt{2}}\int_0^2 te^{-jk\omega_0 t}dt, \text{ Let} \begin{cases} u = t & , du = dt \\ dv = e^{-jk\omega_0 t}dt & , v = -\frac{1}{jk\omega_0}e^{-jk\omega_0 t} \end{cases}, \ T\omega_0 = 2\pi \Longrightarrow \omega_0 = \pi \\ &= \frac{1}{\sqrt{2}}\Big(\frac{-t}{jk\omega_0}e^{-jk\omega_0 t}\Big|_0^2 - \int_0^2 -\frac{1}{jk\omega_0}e^{-jk\omega_0 t}dt\Big) = \frac{1}{\sqrt{2}}\Big(\frac{-2}{jk\pi}e^{-jk2\pi} + \frac{1}{k^2\pi^2}e^{-jk\pi t}\Big|_0^2\Big) \\ &= \frac{1}{\sqrt{2}}\Big(\frac{-2}{jk\pi}e^{-jk2\pi} + \frac{1}{k^2\pi^2}\Big(e^{-jk2\pi} - 1\Big)\Big) = \frac{-\sqrt{2}}{jk\pi} \quad \Big(e^{-jk2\pi} = cos(k2\pi) = 1\Big) \end{aligned}$$

Exercise 5. Parseval's Formula

For a signal expressed using equation(1) show that

$$\int_a^b |x(t)|^2 dt = \sum_{m=-\infty}^{\infty} |a_m|^2$$

This important result is known as the Parseval's formula. Note that the left side is the energy in x(t).

$x(t) = \sum_{k=-\infty}^{\infty} a_k\phi_k$, where $\{\phi_k(t)\}$ is an orthonormal set over $t \in (a,b)$

$$\int_a^b |x(t)|^2 dt = \int_a^b x(t)x^*(t)dt = \int_a^b \sum_{k=-\infty}^{\infty} a_k\phi_k(t) \sum_{k=-\infty}^{\infty} a_k\phi_k^*(t)dt = \sum_{k=-\infty}^{\infty} a_k^2 \int_a^b \phi_k(t)\phi_k^*(t)dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$