機器人學

導論

1. X-Y 平面 有三個自由度 → 2個移動、1個轉動
2. X-Y-Z 空間 有六個自由度 → 3個移動、3個轉動

移動

將剛體上的圓點連線到空間中的原點，獲得一向量用以表示向量於空間中狀態，若再將其向量微分便可獲得於其方向上之速度向量

轉動

將剛體(Body Frame)上的三個主軸寫成三個column vector，並藉由合併三個column vector獲得一旋轉矩陣 R (將Body Frame投影至World Frame A上)

p.s. 對於Rotation Matrix而言，其Inverse Matrix就是其Transpose Matrix

▼ orthogonal matrix

1. Always invertible: Inverse Matrix = Transpose Matrix
2. All columns length = 1
3. Mutually perpendicular (相互垂直)
4. 若為旋轉矩陣 det(R)=1, 若為鏡射矩陣 det(R)=-1

藉由上述3個條件，原本Rotation Matrix 有9個數字，但2、3置入6條件，所以只有3個獨立變數可以調整 ⇒ 空間中轉動擁有3個DOFs

$$= \begin{bmatrix} \hat{X}_B \cdot \hat{X}_A & \hat{Y}_B \cdot \hat{X}_A & \hat{Z}_B \cdot \hat{X}_A \\ \hat{X}_B \cdot \hat{Y}_A & \hat{Y}_B \cdot \hat{Y}_A & \hat{Z}_B \cdot \hat{Y}_A \\ \hat{X}_B \cdot \hat{Z}_A & \hat{Y}_B \cdot \hat{Z}_A & \hat{Z}_B \cdot \hat{Z}_A \end{bmatrix}$$

"direct cosines"

$$
{}^A_BR = \begin{bmatrix} | & | & | \\ {}^A\hat{X}_B & {}^A\hat{Y}_B & {}^A\hat{Z}_B \\ | & | & | \end{bmatrix}
$$

B relative to A

前後向量互換

$$
= \begin{bmatrix} \hat{X}_A \cdot \hat{X}_B & \hat{X}_A \cdot \hat{Y}_B & \hat{X}_A \cdot \hat{Z}_B \\ \hat{Y}_A \cdot \hat{X}_B & \hat{Y}_A \cdot \hat{Y}_B & \hat{Y}_A \cdot \hat{Z}_B \\ \hat{Z}_A \cdot \hat{X}_B & \hat{Z}_A \cdot \hat{Y}_B & \hat{Z}_A \cdot \hat{Z}_B \end{bmatrix} = \begin{bmatrix} - & {}^B\hat{X}_A{}^T & - \\ - & {}^B\hat{Y}_A{}^T & - \\ - & {}^B\hat{Z}_A{}^T & - \end{bmatrix}
$$

$$
= \begin{bmatrix} | & | & | \\ {}^B\hat{X}_A & {}^B\hat{Y}_A & {}^B\hat{Z}_A \\ | & | & | \end{bmatrix}^T = {}^B_AR^T
$$

▼ 描述物體的轉動狀態

Rx

$$
R_{\hat{X}_A}(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\theta & -sin\theta \\ 0 & sin\theta & cos\theta \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta & -s\theta \\ 0 & s\theta & c\theta \end{bmatrix}
$$

Ry

$$
R_{\hat{Y}_A}(\theta) = \begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix} = \begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}
$$

Rz

旋轉角度

$$R_{\hat{Z}_A}(\theta) = \begin{bmatrix} cos\theta & -sin\theta & 0 \\ sin\theta & cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

旋轉軸

▼ Rotation Matrix 的用法

- 描述一個Frame相對於另一個Frame的姿態

$$^{A}_{B}R = \begin{bmatrix} | & | & | \\ ^{A}\hat{X}_B & ^{A}\hat{Y}_B & ^{A}\hat{Z}_B \\ | & | & | \end{bmatrix}$$

- 將點由某一個Frame轉動姿態表達到另一個Frame的相對轉動姿態

$$^{A}P = ^{A}_{B}R\,^{B}P$$

- 將點於Frame中轉動

$$^{A}P' = R(\theta)^{A}P$$

Rotation Matrix 與 轉角

空間中的Rotation是3DOF，要如何將一般Rotation Matrix所表達的姿態拆成3次旋轉角度，以對應到3DOF?

注意事項

- Rotation is not commutable，所以多次旋轉的先後順序須明確定義
- 旋轉轉軸也須明確定義

拆解方式

- 對方向**固定不動**的軸旋轉: Fixed Angles
- 對**轉動物體的Frame**的轉軸方向旋轉: Euler Angles

Fixed Angles

先轉的軸放後面，後轉的軸放前面，如: Rxyz=Rz*Ry*Rx

$$ {}^{A}_{B}R_{XYZ}(\gamma,\beta,\alpha) = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix} $$

由R推算Fixed Angles

If $\beta \neq 90^\circ$

$$ \beta = Atan2(-r_{31}, \sqrt{{r_{11}}^2 + {r_{21}}^2}) $$
$$ \alpha = Atan2(r_{21}/c\beta, r_{11}/c\beta) $$
$$ \gamma = Atan2(r_{32}/c\beta, r_{33}/c\beta) $$

$-90^\circ \leq \beta \leq 90^\circ$

Single solution

If $\beta = 90^\circ$

$$ \alpha = 0^\circ $$
$$ \gamma = Atan2(r_{12}, r_{22}) $$

If $\beta = -90^\circ$

$$ \alpha = 0^\circ $$
$$ \gamma = -Atan2(r_{12}, r_{22}) $$

Euler Angles

先轉的放前面，後轉的放後面，如: Rxyz=Rx*Ry*Rz

對於以Euler Angles以ZYX為順序旋轉，相當於以Fixed Angles以XYZ為順序旋轉

$$ {}^{A}_{B}R_{Z'Y'Z'}(\alpha,\beta,\gamma) = \begin{bmatrix} c\alpha c\beta c\gamma - s\alpha s\gamma & -c\alpha c\beta s\gamma - s\alpha c\gamma & c\alpha s\beta \\ s\alpha c\beta c\gamma + c\alpha s\gamma & -s\alpha c\beta s\gamma + c\alpha c\gamma & s\alpha s\beta \\ -s\beta c\gamma & s\beta s\gamma & c\beta \end{bmatrix} $$

由R推算Euler Angles

If $\beta \neq 0°$

$$\beta = Atan2(\sqrt{r_{31}^{2} + r_{32}^{2}}, r_{33})$$
$$\alpha = Atan2(r_{23}/s\beta, r_{13}/s\beta)$$
$$\gamma = Atan2(r_{32}/s\beta, -r_{31}/s\beta)$$

If $\beta = 0°$

$$\alpha = 0°$$
$$\gamma = Atan2(-r_{12}, r_{11})$$

If $\beta = 180°$

$$\alpha = 0°$$
$$\gamma = Atan2(r_{12}, -r_{11})$$

小結

對於Euler Angles和Fixed Angles都有12(3*2*2)種表達方式(拆解方式)

補充常用拆解方式

- Angle-axis表達法: 藉由對一個unit vector旋轉theta角來表示(unit vector中有2DOF、轉角1DOF)
- Quaternion(四元素)表達法: 藉由複數的運算來加速對於矩陣運算的速度(可以減少對於Redundant數值的運算，Redundant是因為矩陣中9個元素只有3個是independent的，所以剩下6個是已經知道答案的東西)，雖然是四元素，但因為限制其為單位向量，所以仍為3DOF=4參數-1限制

$$q = \epsilon_4 + \epsilon_1\hat{\imath} + \epsilon_2\hat{\jmath} + \epsilon_3\hat{k}$$
$$= cos\frac{\theta}{2} + sin\frac{\theta}{2}(k_x\hat{\imath} + k_y\hat{\jmath} + k_z\hat{k})$$

note $\epsilon_1^2 + \epsilon_2^2 + \epsilon_3^2 + \epsilon_4^2 = 1$

4個參數+1個限制條件，也為3 DOFs

Mapping(移動+轉動)

藉由使用Homogeneous transformation matrix來達到將移動和轉動結合描述

$$
\begin{bmatrix} \underbrace{{}^{A}_{B}R}_{3\times 3} & \underbrace{{}^{A}P_{B\,org}}_{3\times 1} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}_{4\times 4}
= \begin{bmatrix} | & | & | & | \\ {}^{A}\hat{X}_B & {}^{A}\hat{Y}_B & {}^{A}\hat{Z}_B & {}^{A}P_{B\,org} \\ | & | & | & | \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Inverse Transformation Matrix

$$
{}^{A}_{B}T^{-1} = \begin{bmatrix} {}^{A}_{B}R^{T} & -{}^{A}_{B}R^{T}\,{}^{A}P_{B\,org} \\ 0 \quad 0 \quad 0 & 1 \end{bmatrix}
$$

順向運動學 Foward Kinematics

手臂幾何描述

- Joint
  - Every joint has 1 DoF
  - Every joint rotates to a certain axis
- Link
  - The rod (rigid body) that connects each joints
  - Numbered as: Link0: 地桿，不可移動, Link1: 和Link0相連，第一個可移動桿件

If we are going to describe multiple rods connecting each others, we need at least 4 variables

1. alpha: Link Twist, the angle between Axis i-1 and Axis i
2. a: Link Length, the smallest distance between Axis i-1 and Axis i
3. d: Link Offset, the distance between a i-1 and a i

4. theta: Joint Angle, the angle between a i-1 and a i

D-H Table

Determine XYZ on the rods:

- Z-axis: the rotation or moving direction of the axis
- X-axis: the direction of a (mentioned above), if Z i+1 and Z i is the same axis, we let X i be the axis perpendicular to both Z i+1 and Z i
- Y-axis: determined by Right-Hand Rule

Determine XYZ on link0 (地桿)

Revolute (旋轉) Joint:

> let XYZ as same as Axis1, normally we choose when theta1=0 (not necessary) ⇒ d1=0
>
> So when theta1=0, Frame1 is overlapped with Frame0

Prismatic (移動) Joint:

> let XYZ as same as Axis1, normally we choose when d1=0 (not necessary) ⇒ theta=0

Determine XYZ on the last link

Let X n be the same direction as X n-1 ⇒ a n = 0, alpha n = 0

Revolute (旋轉) Joint: normally we choose when theta1=0 (not necessary) ⇒ d1=0

> So when theta n=0, Last Frame (Frame n) = Frame n-1

Prismatic (移動) Joint: normally we choose when d1=0 (not necessary) ⇒ theta=0

α_{i-1}: 以$\hat{X}_{i-1}$方向看，$\hat{Z}_{i-1}$和$\hat{Z}_i$間的夾角

a_{i-1}: 沿著$\hat{X}_{i-1}$方向，$\hat{Z}_{i-1}$和$\hat{Z}_i$間的距離 $(a_i > 0)$

θ_i: 以$\hat{Z}_i$方向看，$\hat{X}_{i-1}$和$\hat{X}_i$間的夾角

d_i: 沿著$\hat{Z}_i$方向，$\hat{X}_{i-1}$和$\hat{X}_i$間的距離

(Craig)

a_i: the distance from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured along $\hat{X}_i$ $(a_i > 0)$

α_i: the angle from $\hat{Z}_i$ to $\hat{Z}_{i+1}$ measured about $\hat{X}_i$

d_i: the distance from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured along $\hat{Z}_i$

θ_i: the angle from $\hat{X}_{i-1}$ to $\hat{X}_i$ measured about $\hat{Z}_i$

(World)

Link Transformation

We can transform Axis i-1 to Axis i by:

1. Turn Z i-1 to Z i, by rotation: turn alpha i-1
2. Move Z i-1 to Z i, onto the same axis: move a i-1
3. Turn X i-1 to Xi, by rotation: turn theta i
4. Move i-1 Frame to i Frame, onto the same point: move d i

$$
{}^{i-1}_{i}T = T_{\hat{X}_{i-1}}(\alpha_{i-1})T_{\hat{X}_R}(a_{i-1})T_{\hat{Z}_Q}(\theta_i)T_{\hat{Z}_P}(d_i)
$$

$$
= \begin{bmatrix} c\theta_i & -s\theta_i & 0 & a_{i-1} \\ s\theta_i c\alpha_{i-1} & c\theta_i c\alpha_{i-1} & -s\alpha_{i-1} & -s\alpha_{i-1}d_i \\ s\theta_i s\alpha_{i-1} & c\theta_i s\alpha_{i-1} & c\alpha_{i-1} & c\alpha_{i-1}d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Transform from i to i-1(Craig)

$$
\begin{aligned}
{}^{i-1}_{i}T &= {}^{i-1}_{R}T\,{}^{R}_{Q}T\,{}^{Q}_{P}T\,{}^{P}_{i}T \\
&= T_{\hat{Z}_{i-1}}(\theta_i)T_{\hat{Z}_R}(d_i)T_{\hat{X}_Q}(a_i)T_{\hat{X}_P}(\alpha_i)
\end{aligned}
$$

$$
= \begin{bmatrix} c\theta_i & -s\theta_i c\alpha_i & s\theta_i s\alpha_i & a_i c\theta_i \\ s\theta_i & c\theta_i c\alpha_i & -c\theta_i s\alpha_i & a_i s\theta_i \\ 0 & s\alpha_i & c\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Transform from i to i-1(World)

逆向運動學

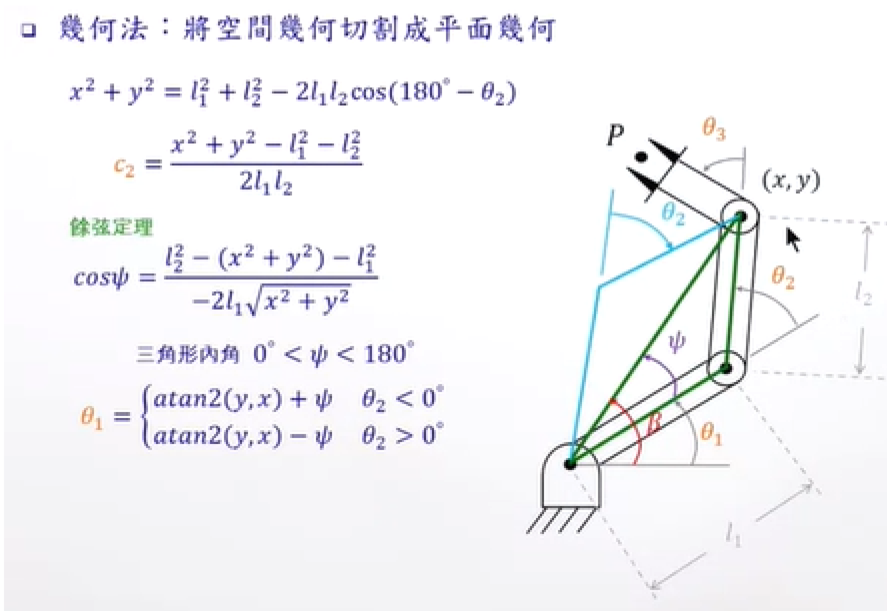
假設一手臂有六個自由度，而Transformation matrix中 9 個未知數，則因為3個向量皆為單位向量+3個向量倆倆垂直 = 3個自由度 + 移動矩陣的 3 個自由度 = 6個自由度 (符合假設)

- Reachable workspace: 手臂可以用一種以上姿態到達的位置
- Dexterous workspace: 手臂可以用任何姿態到達的位置
- Subspace: 手臂在定義頭尾的 T 所能達到的變動範圍

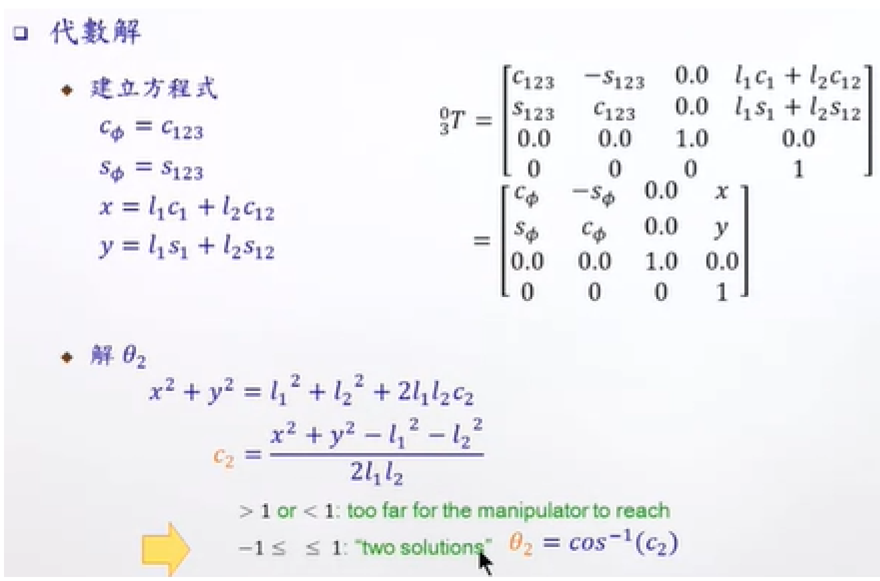
若後三個向量焦於一點則0-4P = 0-6P

多重解

1.幾和解


幾何法：將空間幾何切割成平面幾何
$x^2 + y^2 = l_1^2 + l_2^2 - 2l_1l_2\cos(180^\circ - \theta_2)$
$c_2 = \frac{x^2 + y^2 - l_1^2 - l_2^2}{2l_1l_2}$
餘弦定理
$cos\psi = \frac{l_2^2 - (x^2 + y^2) - l_1^2}{-2l_1\sqrt{x^2 + y^2}}$
三角形內角 $0^\circ < \psi < 180^\circ$
$\theta_1 = \begin{cases} atan2(y,x) + \psi & \theta_2 < 0^\circ \\ atan2(y,x) - \psi & \theta_2 > 0^\circ \end{cases}$
P
θ_3
(x,y)
θ_2
θ_2
l_2
ψ
β
θ_1
l_1

2.代數解


代數解
建立方程式
$c_\phi = c_{123}$
$s_\phi = s_{123}$
$x = l_1c_1 + l_2c_{12}$
$y = l_1s_1 + l_2s_{12}$
${}^0_3T = \begin{bmatrix} c_{123} & -s_{123} & 0.0 & l_1c_1 + l_2c_{12} \\ s_{123} & c_{123} & 0.0 & l_1s_1 + l_2s_{12} \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\phi & -s_\phi & 0.0 & x \\ s_\phi & c_\phi & 0.0 & y \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$
解 θ_2
$x^2 + y^2 = {l_1}^2 + {l_2}^2 + 2l_1l_2c_2$
$c_2 = \frac{x^2 + y^2 - {l_1}^2 - {l_2}^2}{2l_1l_2}$
> 1 or < 1: too far for the manipulator to reach
$-1 \le \quad \le 1$: "two solutions" $\theta_2 = cos^{-1}(c_2)$

- 將求得的 θ_2 帶入方程式

$$x = l_1 c_1 + l_2 c_{12} = (l_1 + l_2 c_2)c_1 + (-l_2 s_2)s_1 \triangleq k_1 c_1 - k_2 s_1$$

$$y = l_1 s_1 + l_2 s_{12} = (l_1 + l_2 c_2)s_1 + \quad (l_2 s_2)c_1 \triangleq k_1 s_1 + k_2 c_1$$

- 變數變換

define

$$r = +\sqrt{{k_1}^2 + {k_2}^2}$$

$$\gamma = Atan2(k_2, k_1)$$

then

$$k_1 = r\cos\gamma$$

$$k_2 = r\sin\gamma$$

And then

$$\frac{x}{r} = \cos\gamma\cos\theta_1 - \sin\gamma\sin\theta_1 = \cos(\gamma + \theta_1)$$

$$\frac{y}{r} = \cos\gamma\sin\theta_1 + \sin\gamma\cos\theta_1 = \sin(\gamma + \theta_1)$$

- 解 θ_1

$$\gamma + \theta_1 = Atan2\left(\frac{y}{r}, \frac{x}{r}\right) = Atan2(y, x)$$

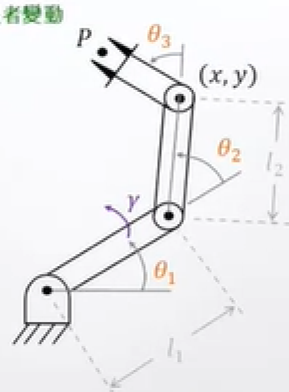
$$\Rightarrow \theta_1 = Atan2(y, x) - Atan2(k_2, k_1)$$

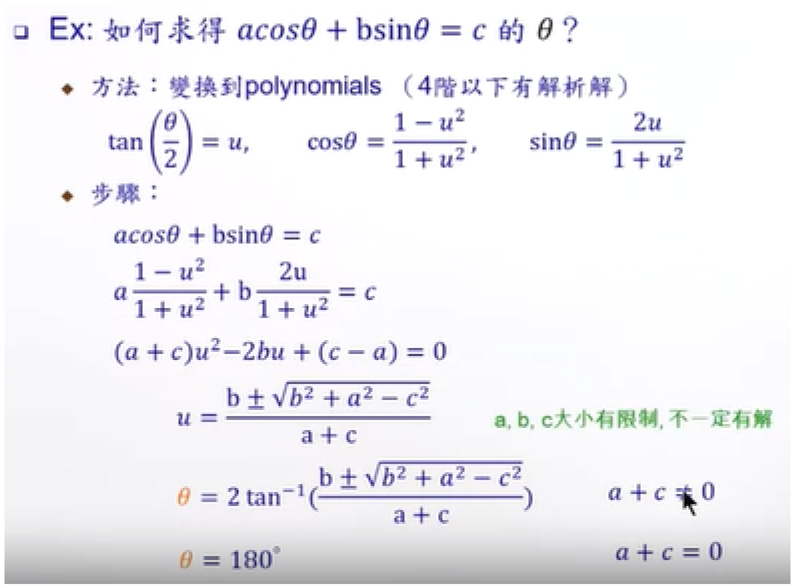
當θ_2選不同解，$c2$和$s2$變動，k_1和k_2變動，θ_1也跟者變動

- 解 θ_3

$$\theta_1 + \theta_2 + \theta_3 = Atan2(s_\phi, c_\phi) = \phi$$

$$\Rightarrow \theta_3 = \phi - \theta_1 - \theta_2$$



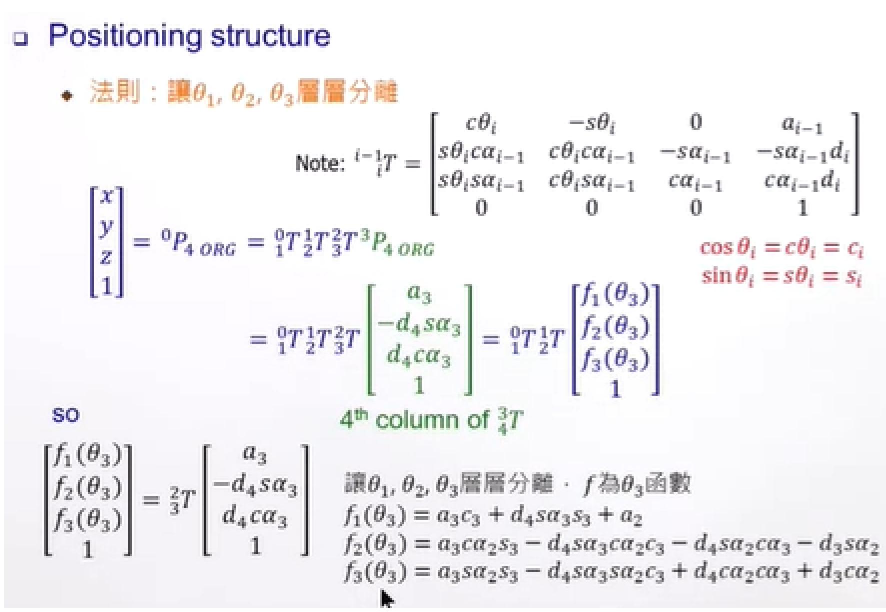


if b^2+a^2-c^2 < 0 ⇒ there is no solution

Pieper's Solution

若6-DOF manipulator具有三個連續的軸交在同一個點，則手臂有解析解

Theta 1,2,3 is for Positioning



只有C3跟S3是未知數，其他都是常數

- 下一步

$$ {}^{0}P_{4ORG} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = {}^{0}_{1}T\,{}^{1}_{2}T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = {}^{0}_{1}T \begin{bmatrix} g_1(\theta_2,\theta_3) \\ g_2(\theta_2,\theta_3) \\ g_3(\theta_2,\theta_3) \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 g_1 - s_1 g_2 \\ s_1 g_1 + c_1 g_2 \\ g_3 \\ 1 \end{bmatrix} $$

讓$\theta_1, \theta_2, \theta_3$層層分離， g為θ_2, θ_3函數

$$ g_1(\theta_2,\theta_3) = c_2 f_1 - s_2 f_2 + a_1 $$
$$ g_2(\theta_2,\theta_3) = s_2 c\alpha_1 f_1 + c_2 c\alpha_1 f_2 - s\alpha_1 f_3 - d_2 s\alpha_1 $$
$$ g_3(\theta_2,\theta_3) = s_2 s\alpha_1 f_1 + c_2 s\alpha_1 f_2 + c\alpha_1 f_3 + d_2 c\alpha_1 $$

$$ r = x^2 + y^2 + z^2 = g_1^2 + g_2^2 + g_3^2 $$
$$ = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 + 2a_1(c_2 f_1 - s_2 f_2) $$
$$ = (k_1 c_2 + k_2 s_2)2a_1 + k_3 $$

r僅為θ_2, θ_3函數

$$ k_1(\theta_3) = f_1 $$
$$ k_2(\theta_3) = -f_2 $$
$$ k_3(\theta_3) = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3 $$

- 此外

$$ z = g_3 = (k_1 s_2 - k_2 c_2)s\alpha_1 + k_4 $$

z僅為θ_2, θ_3函數

$$ k_1(\theta_3) = f_1 $$
$$ k_2(\theta_3) = -f_2 $$
$$ k_4(\theta_3) = f_3 c\alpha_1 + d_2 c\alpha_1 $$

- 整合 r 和 z 一起考量

$$ \begin{cases} r = (k_1 c_2 + k_2 s_2)2a_1 + k_3 \\ z = (k_1 s_2 - k_2 c_2)s\alpha_1 + k_4 \end{cases} $$

  - If $a_1 = 0$, $r = k_3(\theta_3) = f_1^2 + f_2^2 + f_3^2 + a_1^2 + d_2^2 + 2d_2 f_3$
  - If $s\alpha_1 = 0$, $z = k_4(\theta_3) = f_3 c\alpha_1 + d_2 c\alpha_1$
  - Else

$$ \frac{(r-k_3)^2}{4a_1^2} + \frac{(z-k_4)^2}{s^2\alpha_1} = k_1^2 + k_2^2 $$

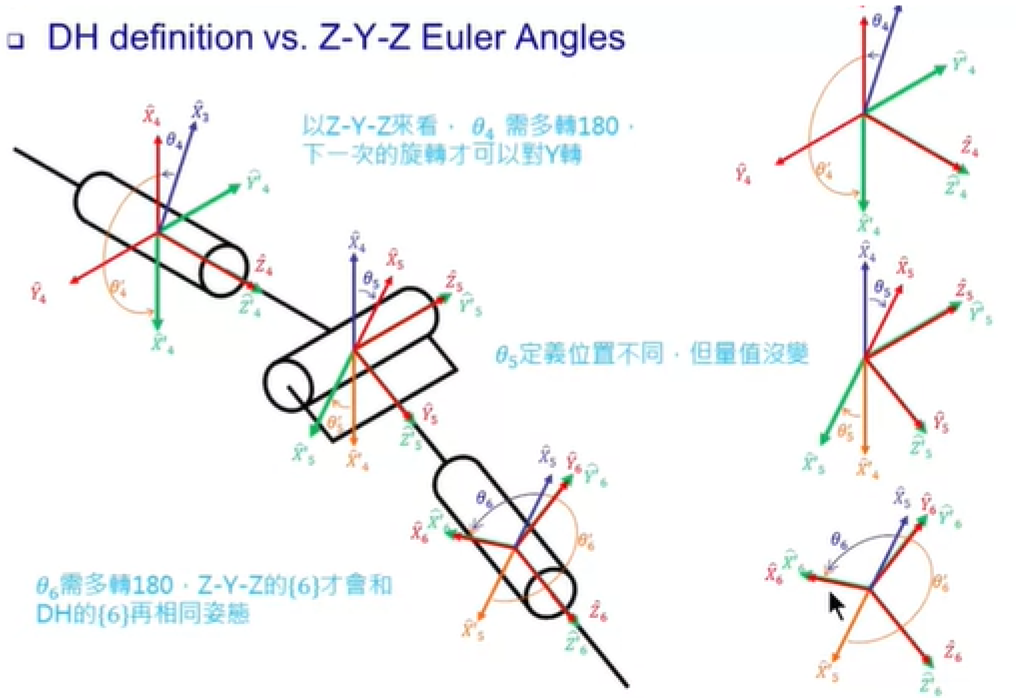
⇨ Solve θ_3 of all three cases by using "$u = \tan\left(\frac{\theta_3}{2}\right)$"

- 最後

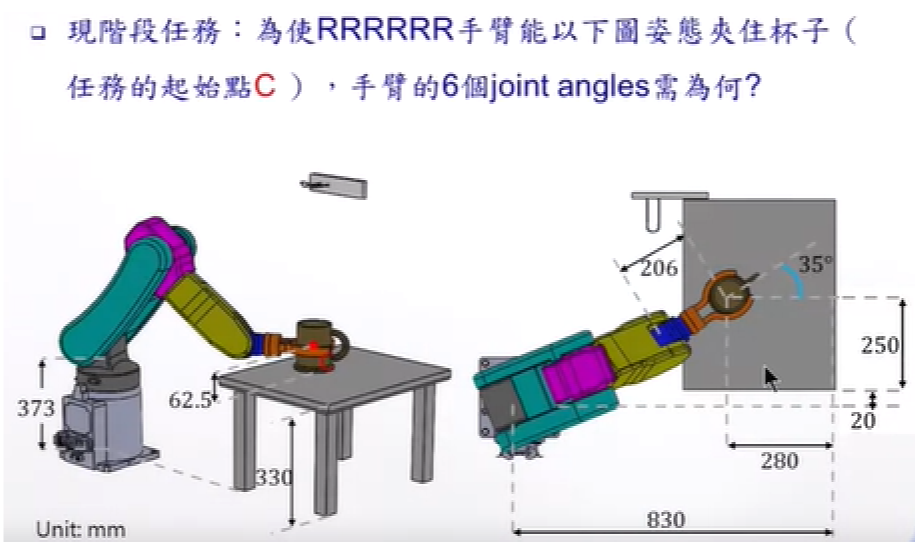
Using $r = (k_1 c_2 + k_2 s_2)2a_1 + k_3$ to solve θ_2

Using $x = c_1 g_1(\theta_2,\theta_3) - s_1 g_2(\theta_2,\theta_3)$ to solve θ_1

Theta 4,5,6 is for Orientation



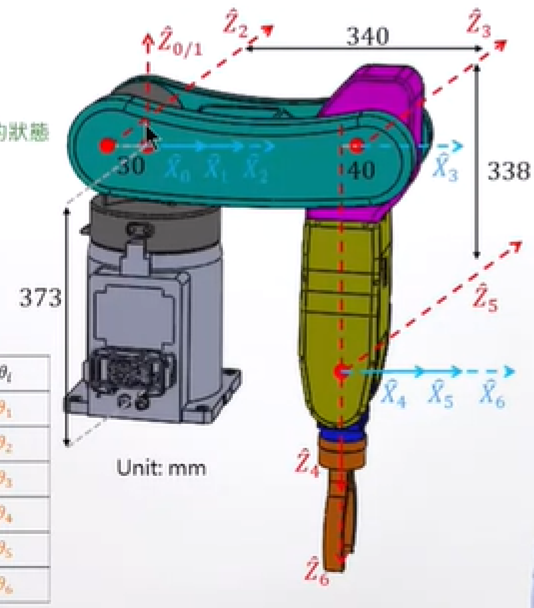
Example:



Solution

Step 1: 定義DH Table

圖中顯示各軸為0°的狀態

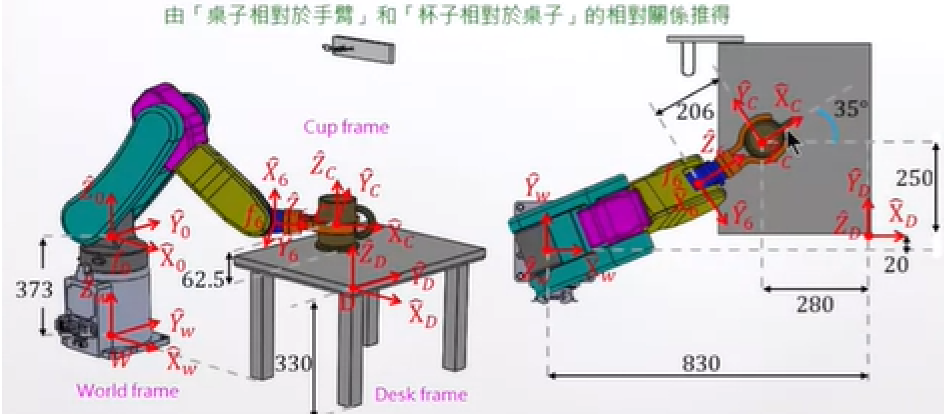


i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0°	0	0	θ_1
2	-90°	$a_1=-30$	0	θ_2
3	0°	$a_2=340$	0	θ_3
4	-90°	$a_3=-40$	$d_4=338$	θ_4
5	90°	0	0	θ_5
6	-90°	0	0	θ_6

Step 2: 找出${}^W_C T$，再進一步找出${}^0_6 T$

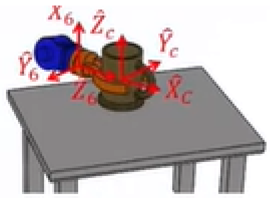
$$
{}^W_C T = {}^W_D T\, {}^D_C T = \begin{bmatrix} 1 & 0 & 0 & 830 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 330 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos 35^\circ & -\sin 35^\circ & 0 & -280 \\ \sin 35^\circ & \cos 35^\circ & 0 & 250 \\ 0 & 0 & 1 & 62.5 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

由「桌子相對於手臂」和「杯子相對於桌子」的相對關係推得



$${}^W_C T = {}^W_0 T\, {}^0_6 T\, {}^6_C T$$

$$= \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&373\\0&0&0&1\end{bmatrix} {}^0_6 T \begin{bmatrix}0&0&1&0\\0&-1&0&0\\1&0&0&206\\0&0&0&1\end{bmatrix}$$

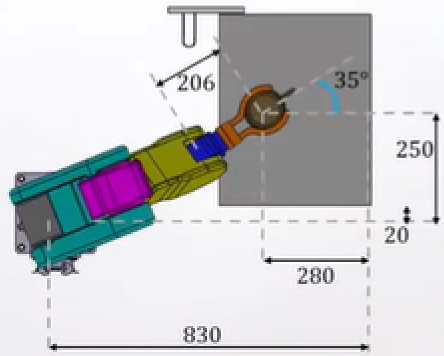


$${}^0_6 T = {}^W_0 T^{-1}\, {}^W_C T\, {}^6_C T^{-1}$$

$$= \begin{bmatrix}0&0.5736&0.8192&381.3\\0&-0.8192&0.5736&151.8\\1&0&0&19.5\\0&0&0&1\end{bmatrix}$$

$${}^0_6 R = \begin{bmatrix}0&0.5736&0.8192\\0&-0.8192&0.5736\\1&0&0\end{bmatrix}$$

$${}^0P_{6\,ORG} = \begin{bmatrix}381.3\\151.8\\19.5\end{bmatrix}$$



- Step 3: 找出$\theta_1 - \theta_6$
  - $\theta_1\ \theta_2\ \theta_3$角度求解

$$\begin{bmatrix}f_1(\theta_3)\\f_2(\theta_3)\\f_3(\theta_3)\\1\end{bmatrix} = {}^2_3 T\, {}^3P_{4\,ORG}$$

$$= \begin{bmatrix}c_3&-s_3&0&340\\s_3&c_3&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}-40\\338\\0\\1\end{bmatrix} = \begin{bmatrix}340-338s_3-40c_3\\338c_3-40s_3\\0\\1\end{bmatrix}$$

$$\begin{bmatrix}g_1(\theta_2,\theta_3)\\g_2(\theta_2,\theta_3)\\g_3(\theta_2,\theta_3)\\1\end{bmatrix} = {}^1_2 T \begin{bmatrix}f_1(\theta_3)\\f_2(\theta_3)\\f_3(\theta_3)\\1\end{bmatrix}$$

$$= \begin{bmatrix}c_2&-s_2&0&-30\\0&0&1&0\\-s_2&-c_2&0&0\\0&0&0&1\end{bmatrix}\begin{bmatrix}f_1(\theta_3)\\f_2(\theta_3)\\f_3(\theta_3)\\1\end{bmatrix} = \begin{bmatrix}340c_2-40c_{23}-338s_{23}-30\\0\\40s_{23}-338c_{23}-340s_2\\1\end{bmatrix}$$

$$r = (k_1c_2 + k_2s_2)2a_1 + k_3 = ||P||^2 = 1.68498214e5$$

$$z = (k_1s_2 - k_2c_2)s\alpha_1 + k_4 = 19.5$$

計算$\theta_1\ \theta_2\ \theta_3$角度

$$\frac{(r-k_3)^2}{4a_1^2} + \frac{(z-k_4)^2}{s^2\alpha_1} = k_1^2 + k_2^2 \qquad \Rightarrow solve\ \theta_3 = 2.5°$$

$$r = (k_1c_2 + k_2s_2)2a_1 + k_3 \qquad \Rightarrow solve\ \theta_2 = -52.2°$$

$$x = c_1g_1(\theta_2,\theta_3) - s_1g_2(\theta_2,\theta_3) \qquad \Rightarrow solve\ \theta_1 = 21.8°$$

1.68498214e5 修正 168813.18

- $\theta_4\ \theta_5\ \theta_6$ 角度求解

$$ {}^{0}_{3}R = \begin{bmatrix} 0.6006 & 0.7082 & -0.3710 \\ 0.24 & 0.2830 & 0.9286 \\ 0.7627 & -0.6468 & 0 \end{bmatrix} $$

$$ {}^{3}_{6}R = {}^{0}_{3}R^{-1}\,{}^{0}_{6}R = \begin{bmatrix} 0.7627 & 0.1477 & 0.6297 \\ -0.6468 & 0.1744 & 0.7424 \\ 0 & -0.9735 & 0.2286 \end{bmatrix} $$

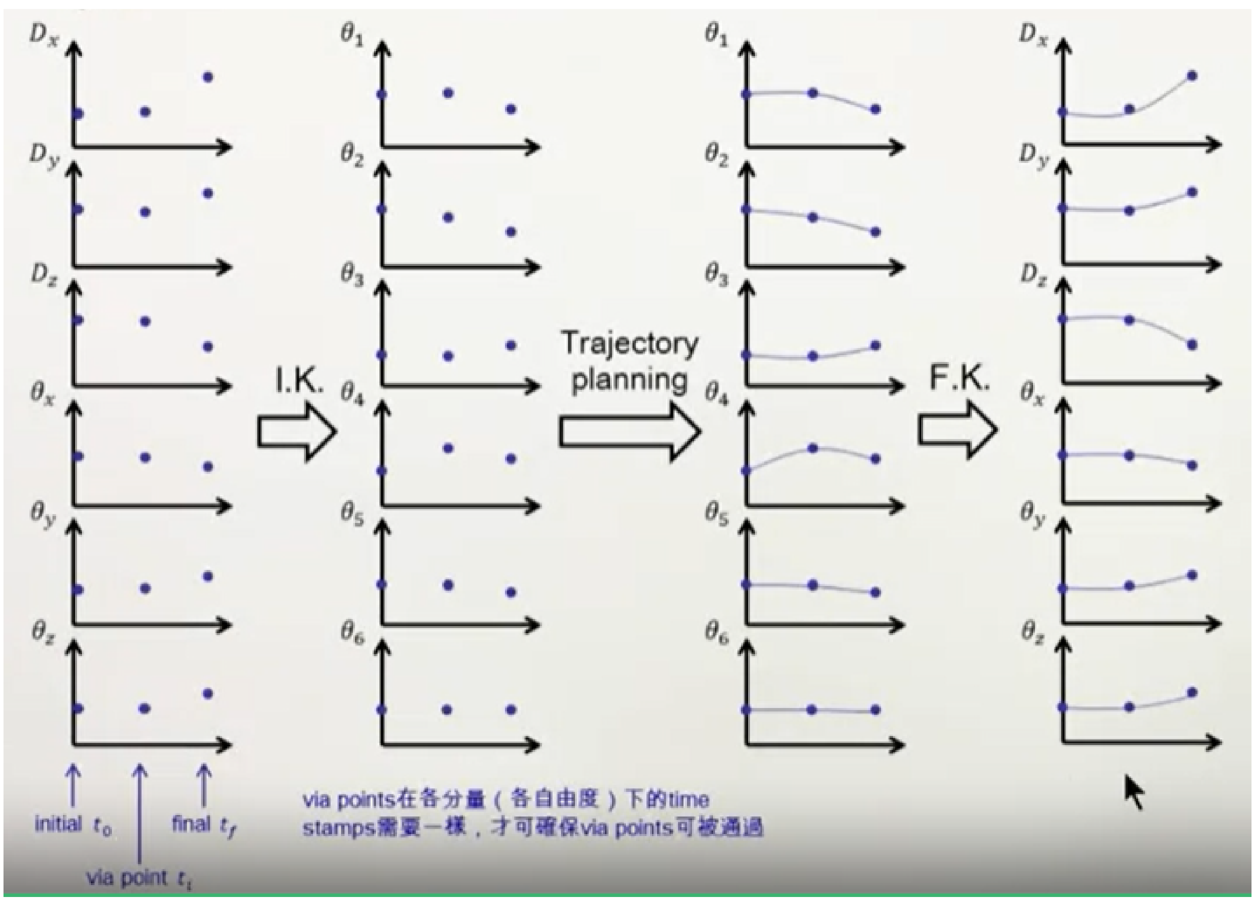
使用Z-Y-Z Euler angle求得剩下的joint angles

$\theta_4 = -20°$ $\theta_5 = -42°$ $\theta_6 = 15°$

Trajectory Planning

理想軌跡 (smooth path): continuous with continuous first derivative (速度，位移圓滑)

Joint Space



第一行圖: 上面三個Dx, Dy, Dz表示手臂要到達的位置、下面三個則為手臂一開始角度

三個點: 第一個點表示手臂初始位置、第二個點表示手臂中途位置、第三個點表示最終位置

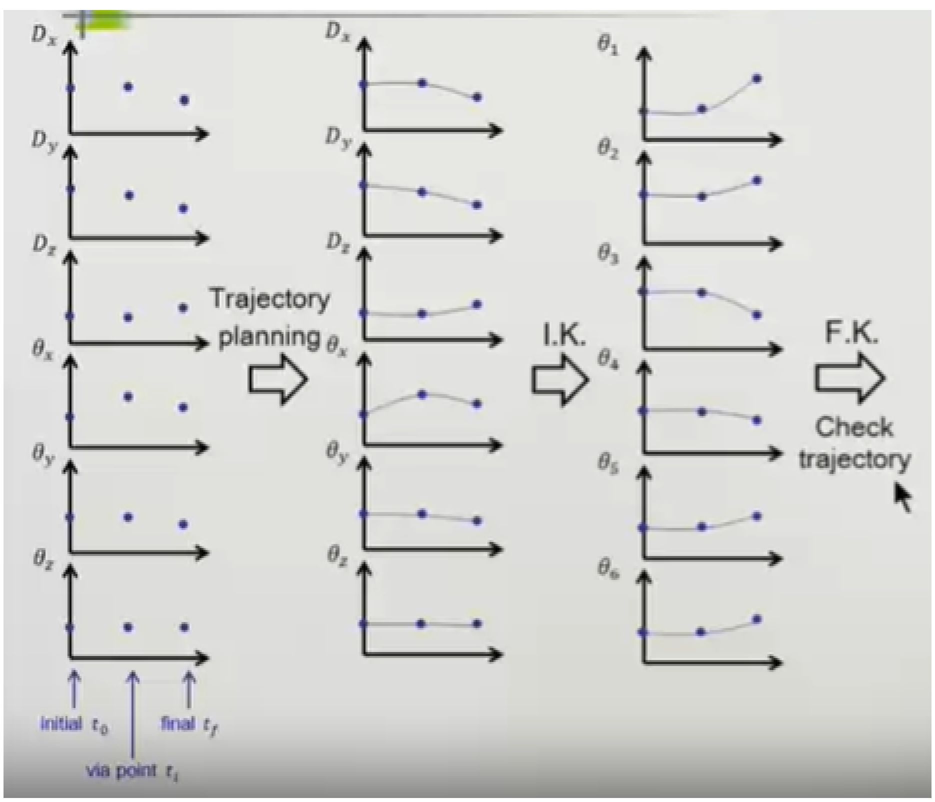
第二行圖: 找到這幾個點的Inverse Kinematics (joint angles)

第三行圖: 找到這幾個Joint Angles的Trajectory Planning

第四行圖: 找到Trajectory後，藉由Forward Kinematics觀察軌跡規劃情形

Cartesian-Space

1. 定義各坐標系相對關係(initial, via, final)
2. 對所有手臂末端點狀態規劃smooth trajectories
3. 將規劃好手臂末端點狀態的軌跡點轉換到Joint Space (檢查theta是否於短時間內有大量的移動)
4. 檢查Joint在Joint Space下軌跡的可行性 (運算附載高)



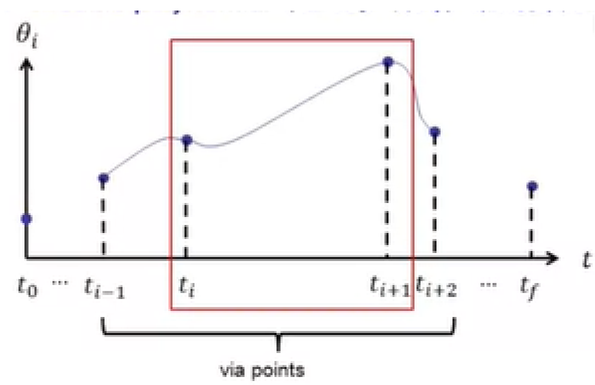
Methods

Cubic Polynomials

Principles:

1. Between each points, we connect them together than use different polynomials to plan.
2. The trajectory should be smooth (define 4 boundary conditions)
   a. At t(i) and t(i+1), they must be on the same point
   b. At t(i+1) and t(i+2), they must also be on the same point
   c. At t(i) and t(i+1), they must have the same velocity
   d. At t(i+1) and t(i+2), they must have the same velocity

By the principles above, we know we need to use a cubic polynomial to solve the problem (A cubic polynomial has 4 variables to adjust)



Solving the Cubic Polynomial:

General Solution: X(t) = A0 + A1*t + A2*t^2 + A3*t^3

At every part: Let t' = t - ti

Boundary Conditions:

X(t) @ t(i) = A0

X(t) @ t(i+1) = A0 + A1*(t(i+1) - t(i)) + A2*(t(i+1) - t(i))^2 + A3*(t(i+1) - t(i))^3

X'(t) @ t(i) = A1

X'(t) @ t(i+1) = A1 + 2*A2*(t(i+1) - t(i)) + 3*A3*(t(i+1) - t(i))^2

Solve it by polynomial or matrix operation

$$\begin{bmatrix}\theta_i\\ \theta_{i+1}\\ \dot{\theta}_i\\ \dot{\theta}_{i+1}\end{bmatrix}=\begin{bmatrix}1&0&0&0\\ 1&\Delta t&\Delta t^2&\Delta t^3\\ 0&1&0&0\\ 0&1&2\Delta t&3\Delta t^2\end{bmatrix}\begin{bmatrix}a_0\\ a_1\\ a_2\\ a_3\end{bmatrix}$$

$$\Theta = T_{4\times4}(\Delta t)\cdot A$$

$$\det(T_{4\times4}) = -\Delta t^4 \neq 0 \ as\ long\ as\ \Delta t \neq 0$$

所以

$$A = T_{4\times4}^{-1}\cdot\Theta$$

$$\begin{bmatrix}a_0\\ a_1\\ a_2\\ a_3\end{bmatrix}=\begin{bmatrix}1&0&0&0\\ 0&0&1&0\\ -\frac{3}{\Delta t^2}&\frac{3}{\Delta t^2}&-\frac{2}{\Delta t}&-\frac{1}{\Delta t}\\ \frac{2}{\Delta t^3}&-\frac{2}{\Delta t^2}&\frac{1}{\Delta t^2}&\frac{1}{\Delta t^2}\end{bmatrix}\begin{bmatrix}\theta_i\\ \theta_{i+1}\\ \dot{\theta}_i\\ \dot{\theta}_{i+1}\end{bmatrix}$$

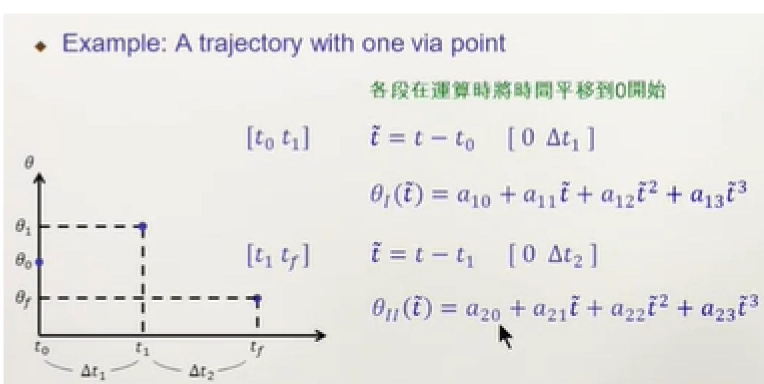
Since we have the solution above, how do we define the velocities?

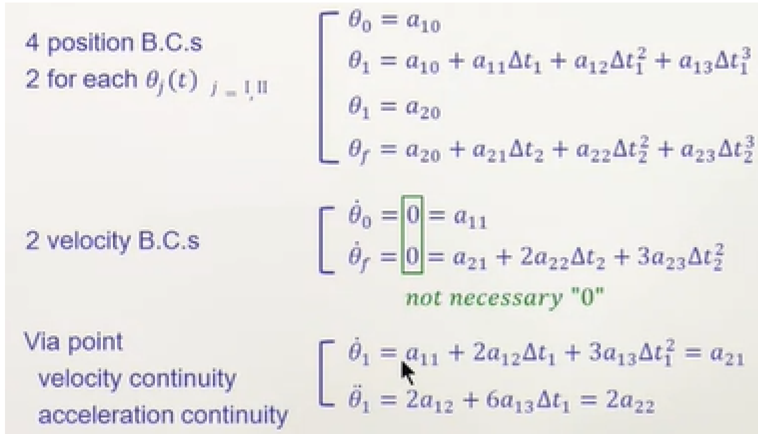
1. Just give it a value (Not Recommended, especially the trajectory falls near a singular point)
2. Generated automatically (If the velocity between two points are opposite sign, we prefer to make the velocity between 0, If they have the same sign, we prefer to make it the average)

By the two ways above, we can solve different parts of the cubic polynomial independently.

3. Let the acceleration to be SMOOTH also (But in this way, we have to solve different parts together)

EXAMPLE:





General Cubic Spline Function

Assume we have N+1 set points (1 initial, N-1 via, 1 final) ⇒ there will be N cubic functions

Si = Ai + Bi*x +Ci*x^2 + Di*x^3 ⇒ total of 4N unknown coefficients

Position conditions at both ends ⇒ 2N conditions

Velocity & Acceleration ⇒ 2(N-1) conditions

Ways to define the last 2 conditions:

1. Natural Cubic Spline (定義加速度): S’’(x1) = S’’(x(N+1)) = 0

   At the initial and the end of the trajectory, the arm doesn’t export force

2. Clamped Cubic Spline (定義速度): S’(x1) = u, S’(x(N+1)) = v

3. Periodic Cubic Spline (週期運動的連續性):

   If S(x1) = S(X(N+1)) ⇒ use S’(x1) = S’(x(N+1)), S’’(x1) = S’’(x(N+1))

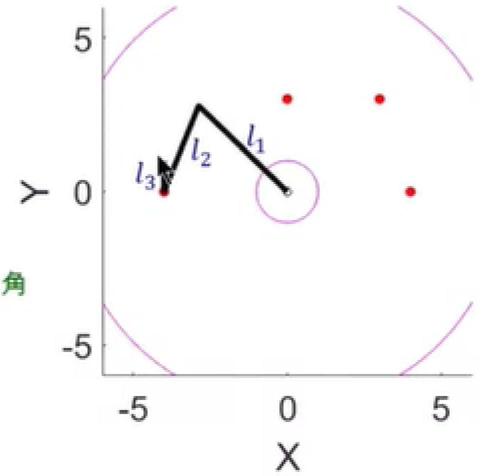
Example:

- 平面RRR手臂長度：$l_1 = 4, l_2 = 3$, and $l_3 = 1$

  下表定義 initial, via, via, 和 final points的位置

t	x	y	theta
0	-4	0	90
2	0	3	45
4	3	3	30
7	4	0	0

(X,Y) 定義在第二桿件的末端
Theta為第三桿件對X座標軸的夾角



- 方法一：以cubic polynomials在Cartesian-space下規劃軌跡

  1. 求出3個DOF (X, Y, θ)各自cubic polynomials的coefficients
     需通過4個點：每個DOF有3個cubic polynomials，共12個未知數

$$\Theta_{12\times1} = \underbrace{T_{12\times12}}_{\text{為}(\Delta t_1, \Delta t_2, \Delta t_3)\text{函數}} A_{12\times1}$$

$X/Y/\theta$

$$\begin{bmatrix} \theta_0 \\ \theta_1 \\ \theta_1 \\ \theta_2 \\ \theta_2 \\ \theta_f \\ \dot{\theta}_0 \\ \dot{\theta}_f \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \Delta t_1 & \Delta t_1^2 & \Delta t_1^3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \Delta t_2 & \Delta t_2^2 & \Delta t_2^3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \Delta t_3 & \Delta t_3^2 & \Delta t_3^3 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2\Delta t_3 & 3\Delta t_3^2 \\ 0 & 1 & 2\Delta t_1 & 3\Delta t_1^2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 6\Delta t_1 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 2\Delta t_2 & 3\Delta t_2^2 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 6\Delta t_2 & 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} a_{10} \\ a_{11} \\ a_{12} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ a_{31} \\ a_{32} \\ a_{33} \end{bmatrix}$$

The first 6 conditions are the condition for locations, 7/8 are the conditions for initial and final speed, the last 4 conditions are the continuous conditions(velocity & acceleration)for the via points.

By the example above, we have to solve X/Y/theta, so there are 36 unknown coefficients to solve

After finding each points' trajectory, we can use IK to find each joints' trajectory, and simulate it.

- 方法二：以cubic polynomials在Joint-space下規劃軌跡

  1. I.K.，求出initial、via、final points的Joint angles $(\theta_1, \theta_2, \theta_3)$
  2. 求出各$(\theta_1, \theta_2, \theta_3)$ cubic polynomials的coefficients

     需通過4個點：每個Joint angle有3個cubic polynomials，共12個未知數

$$\theta_{12\times3} = T_{12\times12}A_{12\times3}$$

6 position B.C.s

vel. & acl. continuity

$$\begin{array}{c} \begin{matrix} \theta_1 & \theta_2 & \theta_3 \end{matrix} \\ \begin{bmatrix} 2.3728 & 1.9552 & -2.7572 \\ 0.7297 & 2.3005 & -2.2449 \\ 0.7297 & 2.3005 & -2.2449 \\ 0.0426 & 1.8668 & -1.3858 \\ 0.0426 & 1.8668 & -1.3858 \\ -0.7688 & 1.9552 & -1.1864 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 & 4 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 9 & 27 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 6 & 27 \\ 0 & 1 & 4 & 12 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 12 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 4 & 12 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 12 & 0 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} \\ \\ \\ \\ A_{12\times3} \\ \\ \\ \\ \end{bmatrix}$$

3. Plan smooth trajectories for every DOF
4. Take each joints into FK, make sure the end effector works as planned

High-Order Polynomials

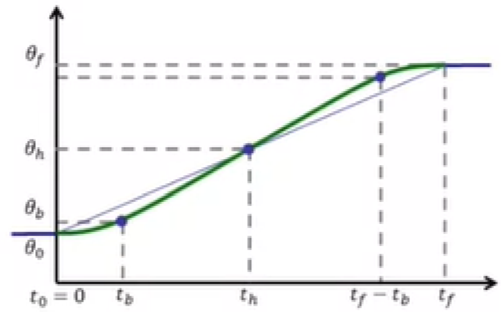
If we have to plan Location, Speed, Acceleration, then we have to use a Quintic polynomial(五次多項式) to do that, since it has 6 boundary conditions.

Linear Function with Parabolic Blends

For straight trajectories, or a trajectory that has multipule straight trajectories, causing the velocity uncontinuous.

- 規劃方式
  - linear（直線，一次多項式）
    - 等速

$$\dot{\theta} = \frac{\theta_h - \theta_b}{t_h - t_b} = \dot{\theta}_{t_b} \qquad \ldots①$$



$t_h = \frac{1}{2}t_f \quad \theta_h = \frac{\theta_f + \theta_0}{2}$

$assume\ \dot{\theta}_0 = \dot{\theta}_f = 0$

  - Parabolic（二次多項式）
    - 等加速度

$$\theta(t) = \theta_0 + \dot{\theta}_0 t + \frac{1}{2}\ddot{\theta}t^2$$

加速度

$$\dot{\theta}(t) = \dot{\theta}_0 + \ddot{\theta}t$$

$$\dot{\theta}(t_b) = \ddot{\theta}t_b \qquad \ldots②$$

- 交界處速度需要連續

②＝①

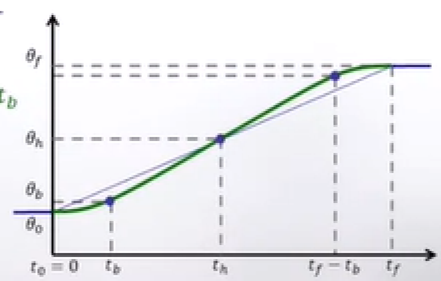
$$\ddot{\theta}t_b = \dot{\theta}_{t_b} = \frac{\theta_h - \theta_b}{t_h - t_b} = \frac{\frac{\theta_f + \theta_0}{2} - (\theta_0 + \frac{1}{2}\ddot{\theta}t_b^2)}{\frac{t_f}{2} - t_b} = \frac{\theta_f - \theta_0 - \ddot{\theta}t_b^2}{t_f - 2t_b}$$

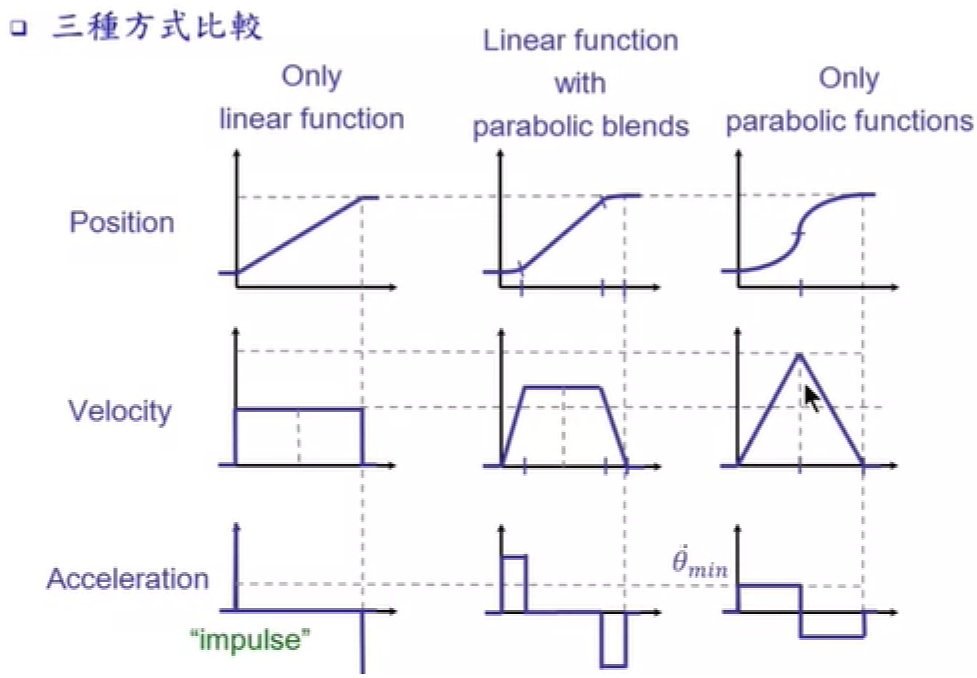
$$\dot{\theta}t_b^2 - \dot{\theta}t_f t_b + (\theta_f - \theta_0) = 0$$

$$\Rightarrow t_b = \frac{\ddot{\theta}t_f - \sqrt{\ddot{\theta}^2 t_f^2 - 4\ddot{\theta}(\theta_f - \theta_0)}}{2\ddot{\theta}}$$

判別式內需為正數或0，所得出t_b才為實數：$\ddot{\theta} \geq \frac{4(\theta_f - \theta_0)}{t_f^2}$

$$\dot{\theta}_{min} = \frac{4(\theta_f - \theta_0)}{t_f^2}$$

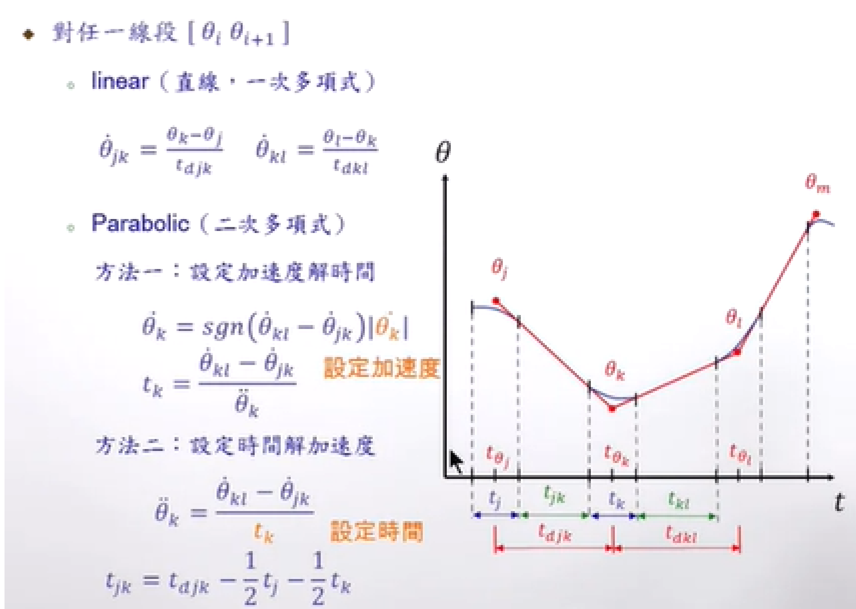




Tj: represents the parabolic blends for j

Tjk: represents the linear function for jk

We can see that, after blending with parabolic blends, we won’t reach the via points, so we can reach the via points by adding points in front and back of the via point, making the trajectory to pass the via point.



- 第一個線段
  - θ_1可視為整段軌跡起始點θ_0在時間上往後移（parabolic曲線段所需時間t_1的一半），以導入parabolic曲線段，讓速度由起始點開始可以連續

方法一：設定加速度解時間　設定加速度

$$\ddot{\theta}_1 = sgn(\theta_2 - \theta_1)|\ddot{\theta}_1|$$

$$\dot{\theta}_{12} = \frac{\theta_2-\theta_1}{t_{d12}-\frac{1}{2}t_1} = \ddot{\theta}_1 t_1$$

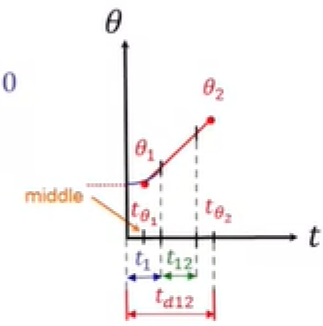
$$\frac{1}{2}\ddot{\theta}_1 t_1^2 - t_{d12}\ddot{\theta}_1 t_1 + (\theta_2 - \theta_1) = 0$$

$$t_1 = t_{d12} - \sqrt{{t_{d12}}^2 - \frac{2\,(\theta_2-\theta_1)}{\ddot{\theta}_1}}$$

方法二：設定時間解加速度

$$\ddot{\theta}_1 = \frac{\theta_2-\theta_1}{(t_{d12}-\frac{1}{2}t_1)}\frac{1}{t_1}$$

設定時間



- 最後一個線段
  - θ_n可視為整段軌跡起始點θ_f在時間上往前移（parabolic曲線段所需時間t_n的一半），以導入parabolic曲線段，讓速度由起始點開始可以連續

方法一：設定加速度解時間　設定加速度

$$\dot{\theta}_n = sgn(\theta_n - \theta_{n-1})|\dot{\theta}_n|$$

$$\dot{\theta}_{(n-1)n} = \frac{\theta_n-\theta_{n-1}}{t_{d(n-1)n}-\frac{1}{2}t_n} = \dot{\theta}_n(-t_n)$$

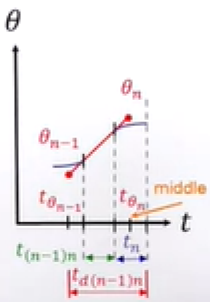
$$\frac{1}{2}\dot{\theta}_n t_n^2 - t_{d(n-1)n}\dot{\theta}_n t_n + (\theta_n - \theta_{n-1}) = 0$$

$$t_n = t_{d(n-1)n} - \sqrt{{t_{d(n-1)n}}^2 - \frac{2(\theta_n-\theta_{n-1})}{\dot{\theta}_n}}$$

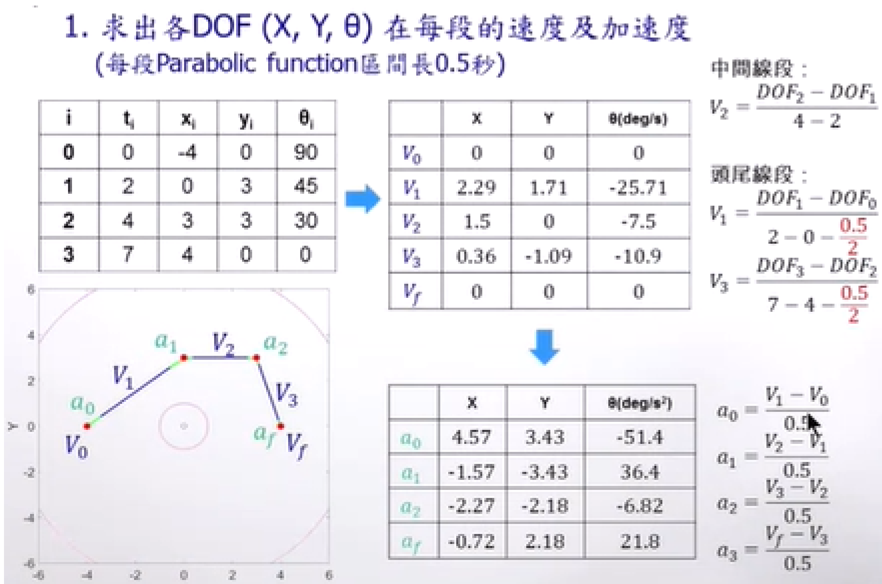
方法二：設定時間解加速度

$$\dot{\theta}_n = \frac{\theta_n-\theta_{n-1}}{(t_{d(n-1)n}-\frac{1}{2}t_n)}\frac{1}{-t_n}$$

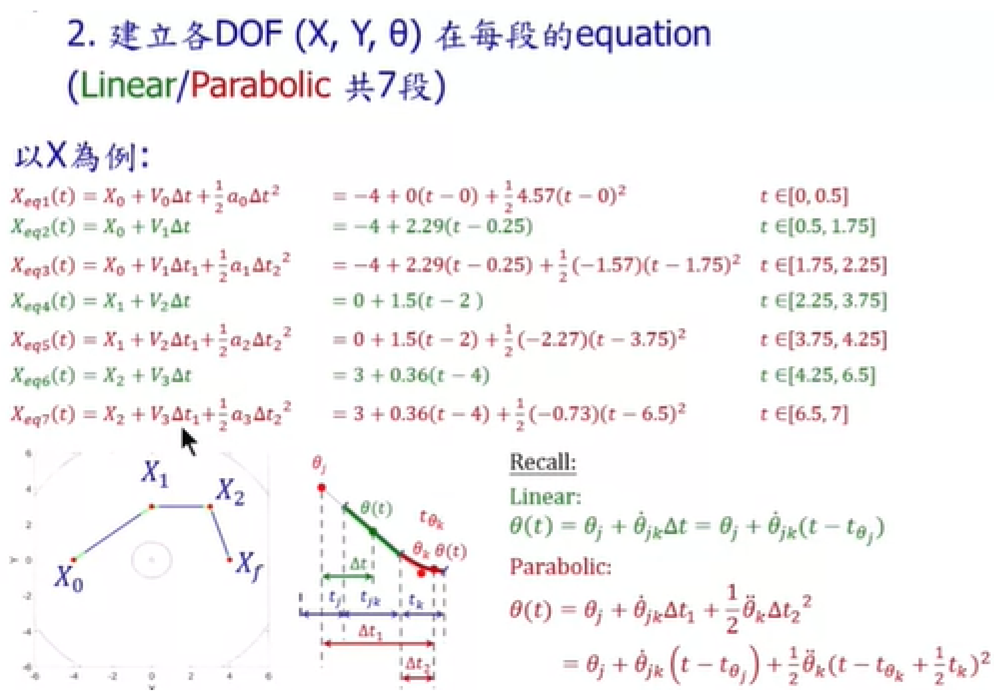
設定時間

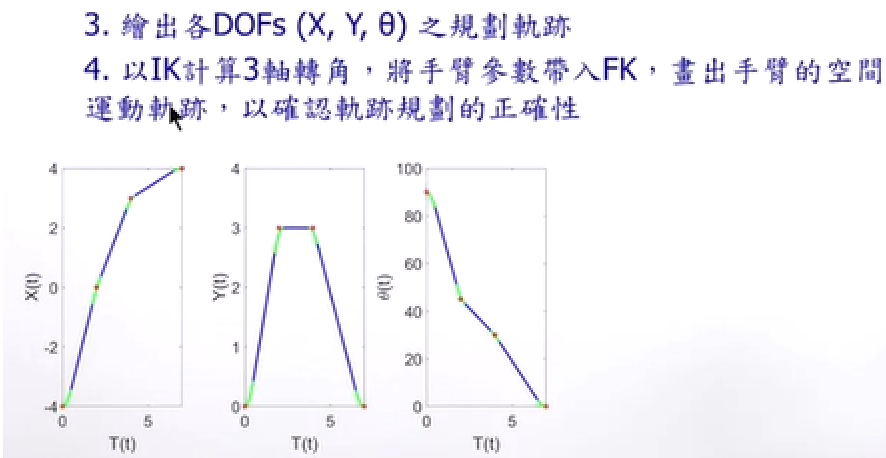


Example:

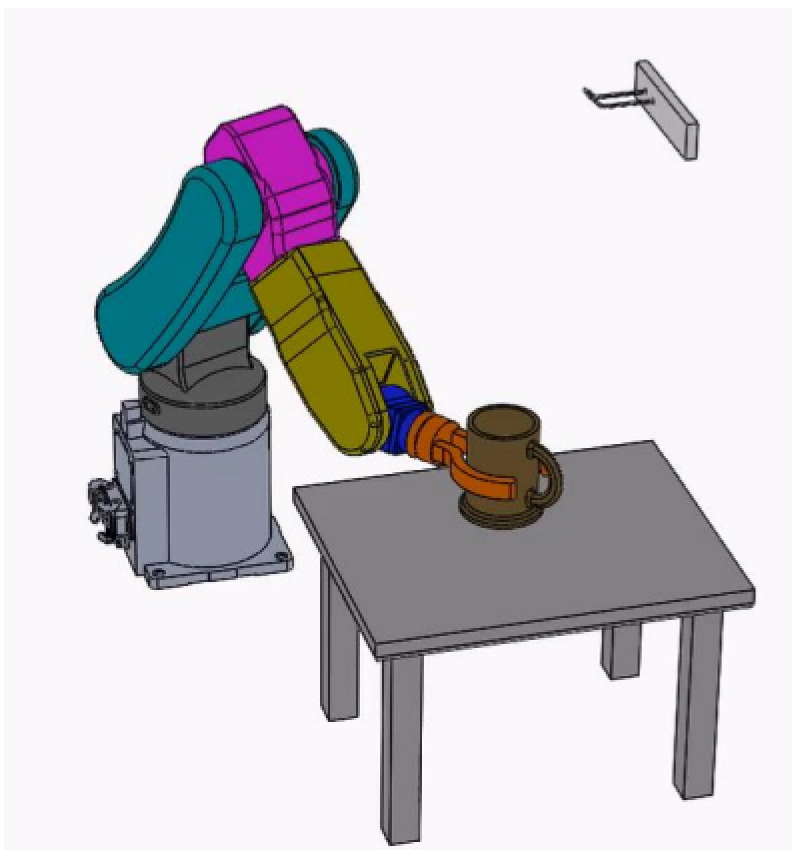


At the initial and the end, we'll have to move it to make sure the velocity would be smooth.





Real Examples



Mission

Step.1: Let the arm to grab the cup on the table, and move it to the hook.

Step.2: Build the DH table for the arm

Step.3: Use IK to calculate the 6 joint angles of the arm, so that the arm can grab the cup.

Step.4: Plan the trajectory, so that the arm can put the cup on the hook.

Add two support conditions:

1. Take the cup up vertically for a small distance.
2. Before putting the cup onto the hook, adjust the motion so that the cup can put onto the hook smoothly.

Condition definitions

Def.1: Clearly define each via point's location, time, posture

Def.2: Turn the def. above into a chart to ease trajectory planning.

	Time	X	Y	Z	Φ_x	Φ_y	Φ_z
P_0	0	550	270	19.5	0	0	35
P_1	2	550	270	79.5	0	0	35
P_2	6	330	372	367	0	-60	0
P_f	9	330	472	367	0	-60	0

Def.3: Find each point's Transformation matrix (The cup's center mass relevant to the base of the arm)

$$ {}^0_CT_0 = \begin{bmatrix} 0.8192 & -0.5736 & 0 & 550 \\ 0.5736 & 0.8192 & 0 & 270 \\ 0 & 0 & 1 & 19.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^0_CT_1 = \begin{bmatrix} 0.8192 & -0.5736 & 0 & 550 \\ 0.5736 & 0.8192 & 0 & 270 \\ 0 & 0 & 1 & 79.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

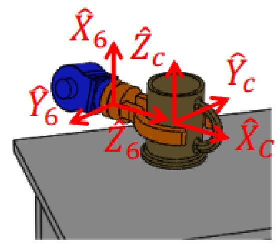
$$ {}^0_CT_2 = \begin{bmatrix} 0.5 & 0 & -0.866 & 330 \\ 0 & 1 & 0 & 372 \\ 0.866 & 0 & 0.5 & 367 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad {}^0_CT_f = \begin{bmatrix} 0.5 & 0 & -0.866 & 330 \\ 0 & 1 & 0 & 472 \\ 0.866 & 0 & 0.5 & 367 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Def.4: Find each point's transformation matrix (The tip of the end effector to the base of the arm)

$$ {}^{0}_{6}T = {}^{0}_{C}T\,{}^{6}_{C}T^{-1} $$

$$ = {}^{0}_{C}T \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 206 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} $$

Recall:



$$ {}^{0}_{6}T_0 = \begin{bmatrix} 0 & 0.5736 & 0.8192 & 381.3 \\ 0 & -0.8192 & 0.5736 & 151.8 \\ 1 & 0 & 0 & 19.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{0}_{6}T_1 = \begin{bmatrix} 0 & 0.5736 & 0.8192 & 381.3 \\ 0 & -0.8192 & 0.5736 & 151.8 \\ 1 & 0 & 0 & 79.5 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{0}_{6}T_2 = \begin{bmatrix} -0.866 & 0 & 0.5 & 227 \\ 0 & -1 & 0 & 372 \\ 0.5 & 0 & 0.866 & 188.6 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{0}_{6}T_f = \begin{bmatrix} -0.866 & 0 & 0.5 & 227 \\ 0 & -1 & 0 & 472 \\ 0.5 & 0 & 0.866 & 188.6 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

Def.5: From the transformation matrix above, we can find the location and posture of each point.

	Time	X	Y	Z	Φ_x	Φ_y	Φ_z
P_0	0	381.3	151.8	19.5	-145	-90	0
P_1	2	381.3	151.8	79.5	-145	-90	0
P_2	6	227	372	188.6	0	-30	180
P_f	9	227	472	188.6	0	-30	180

Solutions

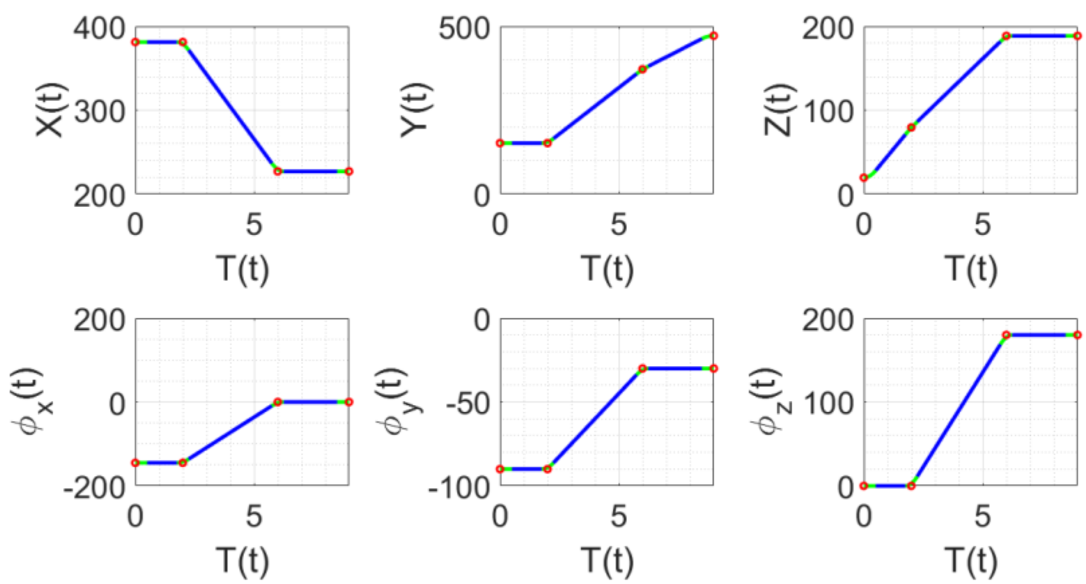
Sol.1: Linear function with parabolic blends (Cartesian-space)

Step.1: Find each trajectory's velocity and acceleration

deg/s	X	Y	Z	Φ_x	Φ_y	Φ_z
V_0	0	0	0	0	0	0
V_1	0	0	34.29	0	0	0
V_2	-38.56	55.04	27.27	36.25	15	45
V_3	0	36.36	0	0	0	0
V_f	0	0	0	0	0	0

deg/s²	X	Y	Z	Φ_x	Φ_y	Φ_z
a_0	0	0	68.57	0	0	0
a_1	-77.13	110.08	-14.02	72.5	30	90
a_2	77.13	-37.35	-54.55	-72.5	-30	-90
a_f	0	-72.73	0	0	0	0

Step.2: Draw each DoF’s trajectory

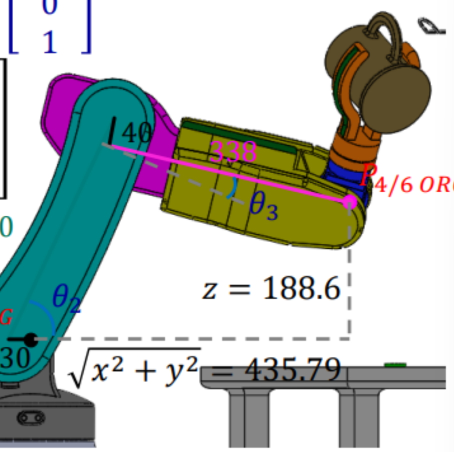


Step.3: Use IK to solve sach point’s joint angle, position and posture (P2)D

$$\begin{bmatrix} g_1(\theta_2,\theta_3) \\ g_2(\theta_2,\theta_3) \\ g_3(\theta_2,\theta_3) \\ 1 \end{bmatrix} = {}^{1}_{2}T \begin{bmatrix} f_1(\theta_3) \\ f_2(\theta_3) \\ f_3(\theta_3) \\ 1 \end{bmatrix} = {}^{1}_{2}T\,{}^{2}_{3}T\;{}^{3}P_{4\,ORG}$$

$$= \begin{bmatrix} c_2 & -s_2 & 0 & -30 \\ 0 & 0 & 1 & 0 \\ -s_2 & -c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_3 & -s_3 & 0 & 340 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -40 \\ 338 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 340c_2 - 40c_{23} - 338s_{23} - 30 \\ 0 \\ 40s_{23} - 338c_{23} - 340s_2 \\ 1 \end{bmatrix} = \begin{bmatrix} 435.79 \\ 0 \\ 188.6 \\ 1 \end{bmatrix}$$



$g_1(\theta_2,\theta_3) = 340c_2 - 40c_{23} - 338s_{23} - 30 = 435.79$

$g_1(\theta_2,\theta_3) = -40c_{23} - 338s_{23} + 340c_2 = 465.79$ - Eq1

$g_3(\theta_2,\theta_3) = +40s_{23} - 338c_{23} - 340s_2 = 188.6$ - Eq2

- $Eq1^2 + Eq2^2 \rightarrow Eq3$

$40^2 + 338^2 + 340^2 + 2(40)(340)(-c_3) + 2(338)(340)(-s_3) = 252530$

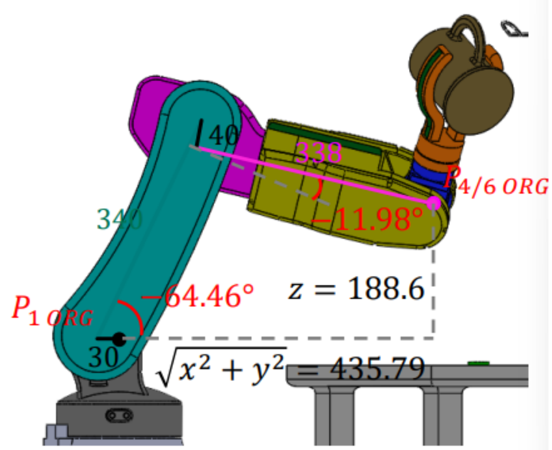
- 從Eq3解出第二/三軸轉角

$\theta_3 = -11.98°$ or $178.48°$

$\theta_2 = -64.46°$ or $20.37°$

- 第一軸轉角:

$\theta_1 = atan2(y,x) = 58.61°$

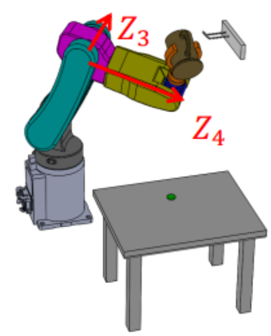


- 先求出${}^{0}_{3}R$:

$$ {}^{0}_{3}R = X(\alpha_0)Z(\theta_1)X(\alpha_1)Z(\theta_2)X(\alpha_2)Z(\theta_3) $$
$$ {}^{0}_{3}R = X(0°)Z(58.61°)\,X(-90°)Z(-64.46°)\,X(0°)Z(-11.98°) $$
$$ {}^{0}_{3}R = \begin{bmatrix} 0.1222 & 0.5064 & -0.8536 \\ 0.2003 & 0.8298 & 0.5209 \\ 0.9721 & -0.2346 & 0 \end{bmatrix} $$

- 為讓手臂姿態和ZYZ重合，需先做${}^{3}_{4}R$之中對X軸之旋轉：${}^{0}_{3''}R = {}^{0}_{3}R\,X(\alpha_3)$

$$ {}^{0}_{3''}R = {}^{0}_{3}R\,X(-90°) = \begin{bmatrix} 0.1222 & 0.8536 & 0.5064 \\ 0.2003 & -0.5209 & 0.8298 \\ 0.9721 & 0 & -0.2346 \end{bmatrix} $$
$$ {}^{3''}_{6}R = {}^{0}_{3''}R^{-1}\,{}^{0}_{6}R = \begin{bmatrix} 0.3802 & -0.2003 & 0.9030 \\ -0.7393 & 0.5209 & 0.4268 \\ -0.5558 & -0.8298 & 0.05 \end{bmatrix} $$



由${}^{3''}_{6}R$推算ZYZ Euler Angles

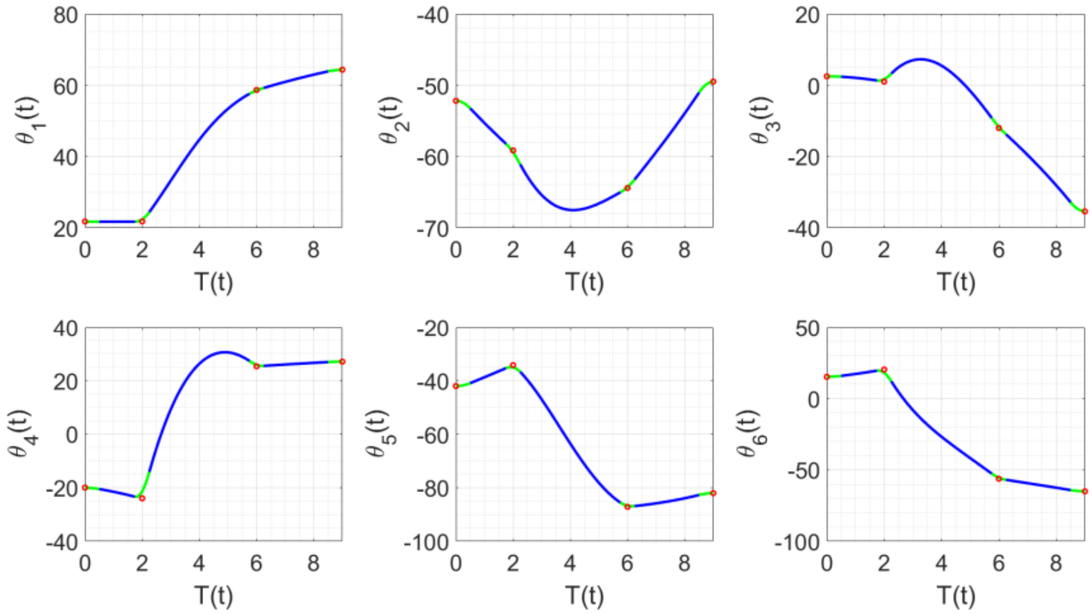
詳細方程式推導參見逆向運動學課程內容

$$ \beta = Atan2\left(\sqrt{{r_{31}}^2 + {r_{32}}^2}, r_{33}\right) = 87.13° \ or \ -87.13° $$
$$ \alpha = Atan2\left(\frac{r_{23}}{s\beta}, \frac{r_{13}}{s\beta}\right) = -154.70° \ or \ 25.30° $$
$$ \gamma = Atan2\left(\frac{r_{32}}{s\beta}, \frac{-r_{31}}{s\beta}\right) = 123.81° \ or \ -56.19° $$

- ZYZ的 (α,β,γ)和DH的$(\theta_4,\theta_5,\theta_6)$，在$(\theta_4,\theta_6)$有$+180°$的差異，需補回來

$$ \theta_4 = \alpha + 180° = 25.30° \ or \ -154.70° $$
$$ \theta_5 = \beta = -87.13° \ or \ 87.13° $$
$$ \theta_6 = \gamma + 180° = -56.19° \ or \ 123.81° $$

Step.4: Draw each via point’s joint angle to time’s trajectory



Step.5: Take each angles into FK, draw the cup's location and posture, to make sure the trajectory is correct.

$$\theta_1 = 58.61^\circ \ \theta_2 = -64.46^\circ \ \theta_3 = -11.98^\circ$$

$$\theta_4 = 25.30^\circ \ \theta_5 = -87.13^\circ \ \theta_6 = -56.19^\circ$$

根據DH table：

$${}^0_6T = {}^0_1T\,{}^1_2T\,{}^2_3T\,{}^3_4T\,{}^4_5T\,{}^5_6T$$

$$= \begin{bmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 & -30 \\ 0 & 0 & 1 & 0 \\ -\sin\theta_2 & -\cos\theta_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 & 340 \\ \sin\theta_3 & \cos\theta_3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \cos\theta_4 & -\sin\theta_4 & 0 & -40 \\ 0 & 0 & 1 & 338 \\ -\sin\theta_4 & -\cos\theta_4 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_5 & -\sin\theta_5 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ \sin\theta_5 & \cos\theta_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\theta_6 & -\sin\theta_6 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\theta_6 & -\cos\theta_6 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$= {}^0_6T_2$ (得出與設定4相同的transformation matrix)

Step.6: Use matlab to simulate trajectory.

Sol.2: Linear function with parabolic blends (Joint-space)

Step.1: Use IK to calculate 6 jooint angles

	X	Y	Z	Φ_x	Φ_y	Φ_z
P_0	381.3	151.8	19.5	-145	-90	0
P_1	381.3	151.8	79.5	-145	-90	0
P_2	227	372	188.6	0	-30	180
P_f	227	472	188.6	0	-30	180

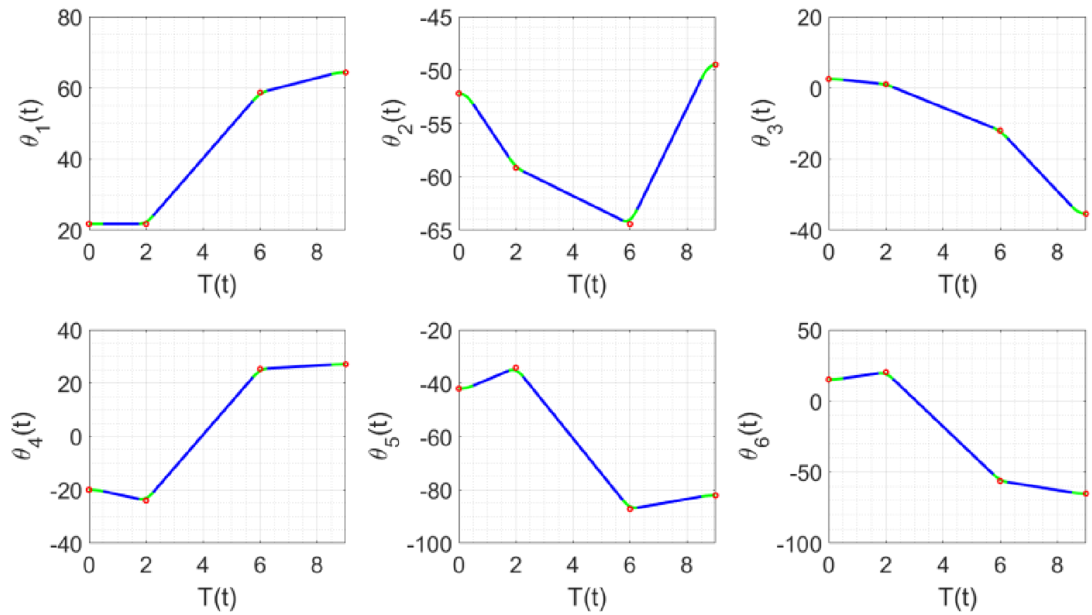
	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
P_0	22	-52	2.5	-20	-42	15
P_1	22	-59	1	-24	-34	20
P_2	59	-64	-12	25	-87	-56
P_f	64	-49	-35	27	-82	-65

Step.2: Calculate each joint’s velocity and acceleration

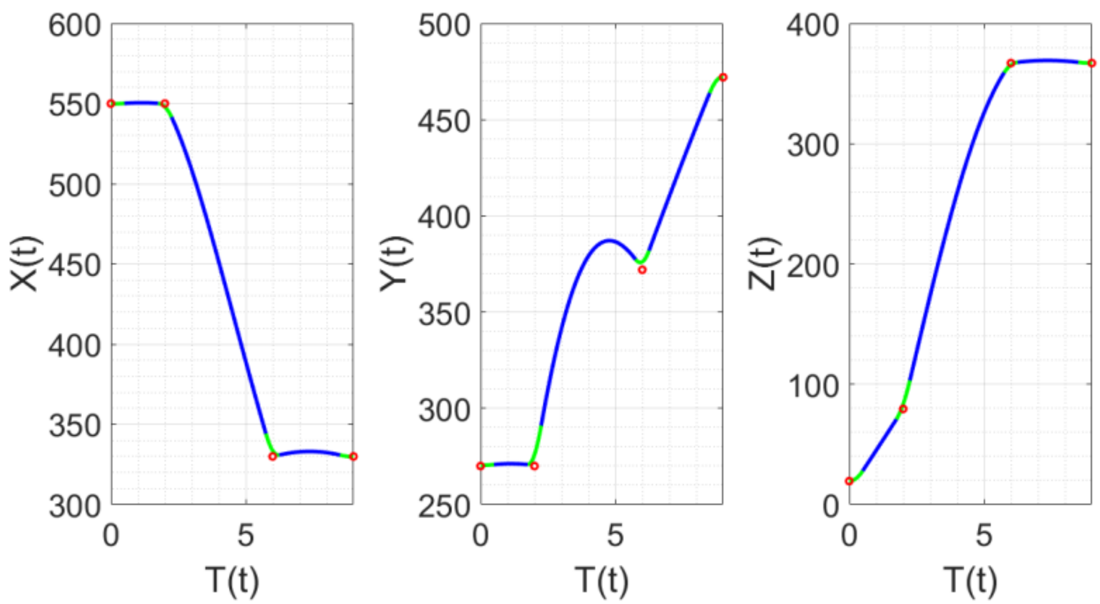
deg/s	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
V_0	0	0	0	0	0	0
V_1	0	-3.98	-0.85	-2.30	4.48	2.92
V_2	9.22	-1.32	-3.26	12.32	-13.23	-19.11
V_3	2.07	5.43	-8.50	0.64	1.85	-3.21
V_f	0	0	0	0	0	0

deg/s^2	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
a_0	0	-7.95	-1.70	-4.61	8.96	5.84
a_1	18.44	5.32	-4.81	29.25	-35.42	-44.06
a_2	-14.30	13.49	-10.49	-23.37	30.17	31.80
a_f	-4.14	-10.86	17.00	-1.28	-3.71	6.41

Step.3: Draw the trajectory of each joint (In order to check each joint will be smooth)



Step.4 Use FK to plot the trajectories, make sure the trajectories pass all via points



Step.5: Simulate the trajectory under the cup's frame