Scalar-Graviton Scattering

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Submitted in part fulfillment of the requirements for the degree of Bachelor of Science in
Department of Physics, Faculty of Science, Chulalongkorn University
Academic Year 2018

Project Title	Scalar-Graviton Scattering
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Department	Physics
Academic Year	2018

This project report is submitted to the Department of Physics, Faculty of Science, Chulalongkorn University, in partial fulfillment of the requirements for the degree of Bachelor of Science.

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Abstract

Scalar fields, despite being the simplest field to deal with, have applications in particle physics. At the most fundamental level, all particles interact with gravity via a currently hypothetical particle called a graviton. In this project, we focus on the graviton photoproduction process, $s + \gamma \to s + g$, which is a scalar-quantum electrodynamics process. The interaction terms are developed in analogy with electromagnetism. We study a technique that allows the amplitude of the cross section, which determines the rate of the reaction, to be expressed as product of two terms: a scalar-photon amplitude in scalar-QED and a Lorentz scalar that does not depend on the photon and graviton states. It allows cross-section to be straightforwardly calculated through scalar-QED and helicity formalism.

Acknowledgements

First, I would like to thank my advisor Dr.Pawin Ittisamai for guiding and teaching me basic knowledge. Thank for his effort that he tries to explain many difficult concepts in simple way. Moreover, his support and comment on this project.

I would like to thank my co-advisor Asst.Prof.Dr.Patipan Uttayarat, who gave me inspiration and basic concepts to this project. I would also like to thank Dr.Rujikorn Dhanawittayapol for his suggestion and teaching me throughout this project.

Last, but certainly not least, I would like to thank my family for giving me all the support and my friends for sharing a good time in the Department of Physics.

Contents

Conventions

- Einstein summation convention is used.
- The natural unit is used. $c = 1, \hbar = 1$
- For the Minkowski metric $\eta^{\mu\nu}$, we use η=diag(1,-1,-1,-1)

Chapter 1

Introduction

Classical physics can be used to explain phenomena in daily life but not microscopic phenomena such as blackbody radiation and the photoelectric effect. Thus, quantum theory plays a crucial role in explaining microscopic physics. Nevertheless, when we are interested in high-energy particles, quantum mechanics is no longer applicable because the number of particles is not conserved. So we have to combine special relativity and quantum mechanics together so as to explain high-energy subatomic particles. This is the origin of quantum field theories (QFT) that can be used to describe three fundamental forces, i.e., strong nuclear, weak nuclear and electromagnetic force. However, a theory constructed in this manner cannot explain gravity. We already have a theory for gravitation, the general theory of relativity. However, this theory works well only at the macroscopic level and has problems at the microscopic level. Many physicists believe that if general relativity and quantum could be combined together into a theory called quantum gravity, then we could explain gravitational force on the microscopic level. In quantum gravity, we believe that matter or particles interact with gravity through an exchange of a virtual particle called graviton. But until now there is no trace of this particle. Recently, LIGO has found gravitational wave[2] which led to many physicists hoping that we will have more knowledge on gravitons soon.

In QFT, we cannot observe all particles directly. Instead, we can measure a quantity called cross-section that can be obtained via theoretical predictions. Scalar-Quantum electrodynamics or SQED cross-section such as Scalar-Compton scattering is a standard exercise in many QFT textbooks since a simple form of a scalar field together with masslessness of a photon allows straightforward calculations that are fundamentals to processes in more complicated theories. Gravitons, like photons, are massless and gauge invariance, which, in this case, manifests via invariance under a general coordinate transformation (GCT). Thus, similar and straightforward calculations can be expected. Moreover, in cosmology, we have supporting evidence that dark matter could be composed of scalar particles[3][11]. This means that a scalar-graviton interaction has potential phenomenological applications.

The main purpose of this project is to study the graviton photoproduction $s+\gamma \to s+g$. The calculations can be done by using a technique that has been introduced in Holstein[1]. This technique suggested that the $s+\gamma \to s+g$ amplitude can be split into two parts where one of them is the Compton scattering amplitude. Calculations of the cross-section will follow using a much-used technique called helicity formalism. In the next chapter, we will give a brief review of quantum field theory and how to construct a spin-2 field that will lead us to a graviton. In chapter 3, we are going to find the interaction terms of gravitation in scalar-QED and evaluating the Compton scattering amplitude. Then, we are using the interaction terms and the amplitude to calculate the cross-section of the $s+\gamma \to s+g$ event. For the last chapter, we are going to discuss the result and its applications.

Chapter 2

Scalar Quantum Electrodynamics

As we know from a quantum mechanics course, the formalism describes only a conserved number of particles. We promote an observable to a Hermitian operator in a prescription called first quantization. However, when we consider a relativistic particle, a particle that has its own velocity near the speed of light, this particle can generate another particle so that quantum mechanic which describes a fixed number of particles will no longer be applicable. In this section, we will give a brief review of how to construct a theory that describes scalar particles and how they interact with an electromagnetic field through its quanta called photons. In this chapter, we are going to study some of the basics that are fundamentals to the main topic of this project. For brevity, we will refer the readers to the provided references for details. Sections 2.1-2.5 review basic classical field theoretical concepts. A massless spin-2 field is introduced in section 2.6. In sections 2.7-2.8 we review general quantum field theory prescriptions leading to a differential cross-section in terms of a scattering amplitude (the "M-matrix"). Then, we are going to discuss how to find the M-matrix in sections 2.9-2.13.

2.1 Notations

Before reviewing any theory, we will discuss about our notations first. In the three dimensional Euclidean space, we represent a three-vector in a bold letter or put an arrow on top of it as

$$\mathbf{A} = \vec{A} = (A_1, A_2, A_3). \tag{2.1}$$

We know that the inner product (or scalar product) of any two arbitrary three-vectors take the form

$$\mathbf{A} \cdot \mathbf{B} = \vec{A} \cdot \vec{B} = A_i B_i = \delta_{ij} A^i B^j = A_1 B_1 + A_2 B_2 + A_3 B_3. \tag{2.2}$$

However, quantum field theory in which we study incorporates special relativity and quantum mechanics. Thus, we work on the 4-dimensional space-time manifold. There are two kinds of 4-vectors, namely, for space-time coordinates

$$\begin{aligned} x^\mu &= (ct, \mathbf{x}) = (t, \mathbf{x}) = (t, \vec{x}) = (x^0, x^1, x^2, x^3), \\ x_\mu &= (ct, -\mathbf{x}) = (t, -\mathbf{x}) = (t, -\vec{x}) = (x^0, -x^1, -x^2, -x^3). \end{aligned} \tag{2.3}$$

We call x^μ and x_μ as contravariant and covariant vectors respectively. For an arbitrary 4-vector,

$$\begin{aligned} A^\mu &= (A^0, \mathbf{A}) = (A^0, \vec{A}), \\ A_\mu &= (A^0, -\mathbf{A}) = (A^0, -\vec{A}). \end{aligned} \tag{2.4}$$

A_μ and A^μ related to each other through the metric tensor η, known as the Minkowski metric, namely,

$$\begin{aligned} A^\mu &= \eta^{\mu\nu} A_\nu, \\ A_\mu &= \eta_{\mu\nu} A^\nu. \end{aligned} \tag{2.5}$$

Then we can define the inner product of two 4-vectors as

$$A^\mu B_\mu = A_\mu B^\mu = \eta_{\mu\nu} A^\mu B^\nu = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3, \tag{2.6}$$

where the inner product of 4-vectors is a Lorentz invariant scalar. A derivative in 4-dimensional space is called a 4-gradient. The 4-gradient is a four vector which we represent them as

$$\begin{aligned} \partial^\mu &= \frac{\partial}{\partial x_\mu} = \left(\frac{\partial}{\partial t}, -\nabla\right), \\ \partial_\mu &= \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right). \end{aligned} \tag{2.7}$$

From equation 2.7, we can construct an inner product which is a Lorentz invariant operator

$$\Box = \partial^2 = \partial^\mu \partial_\mu = \left(\frac{\partial^2}{\partial t^2}, -\nabla^2\right) \tag{2.8}$$

Another 4-vector that is useful is the energy-momentum four vectors which we can write as

$$\begin{aligned} p^\mu &= (E, \mathbf{p}) = (E, \vec{p}), \\ p_\mu &= (E, -\mathbf{p}) = (E, -\vec{p}), \end{aligned} \tag{2.9}$$

Thus, its Lorentz scalar takes the form

$$p^2 = p_\mu p^\mu = E^2 - \mathbf{p}^2 = m^2, \tag{2.10}$$

where m is the rest mass of the particle that relates to its energy by the Einstein relation

$$E^2 = \mathbf{p}^2 + m^2 = \vec{p}^2 + m^2. \tag{2.11}$$

This equation is the famous Einstein's equation which tells us about the relationship between mass and energy. As ones will see later in this report, we are always using this relation in QFT.

2.2 Klein-Gordon Equation

In quantum mechanics, when we use the Schrödinger equation, there is a problem that we do not treat time and spatial coordinates on an equal footing, namely,

$$i\frac{\partial \phi}{\partial t} = E\phi, \tag{2.12}$$

$$i\frac{\partial \phi}{\partial t} = \left(-\frac{\nabla^2}{2m} + V(x)\right)\phi, \tag{2.13}$$

To fix this problem, recall equation 2.11 and equation 2.12 which allow us to promote E to an operator. Since in quantum mechanics, p can be written in an operator form as $i\vec{\nabla}$, then we can write equation 2.12 as

$$\left(\frac{\partial^2}{\partial t^2}, -\nabla^2\right)\phi = \left(\frac{\partial^2}{\partial t^2}, -\frac{\partial^2}{\partial x^2}, -\frac{\partial^2}{\partial y^2}, -\frac{\partial^2}{\partial z^2}\right)\phi = (\Box + m^2)\phi = 0, \tag{2.14}$$

where equation 2.7 and equation 2.8 have been used. If we want to work on a charged particle, we must let the Klein-Gordon equation to have a complex form, namely

$$(\Box + m^2)\phi^* = 0, \tag{2.15}$$

where we only complex conjugated equation 2.14. Now we see that the derivative on space and time are on the same order. This equation is known as the Klein-Gordon equation. Its solution takes the form

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p e^{-ipx} + b_p^\dagger e^{ipx}), \tag{2.16}$$

where $\omega_p = \sqrt{\mathbf{p}^2 + m^2}$. Thus, by complex conjugation, we get

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p^\dagger e^{ipx} + b_p e^{-ipx}), \tag{2.17}$$

where $a_p^\dagger$ and $b_p^\dagger$ are both creation operators creating particles of opposite charges. a_p and b_p are both annihilation operators which annihilate particles that created from $a_p^\dagger$ and $b_p^\dagger$ respectively. Both ϕ and ϕ^* describe scalar particles with opposite charges. We are going to use these operators later in our scattering calculation.

2.3 Euler-Lagrange Equation and Action

In classical mechanics, to get an equation of motion, we only require the variation of the action to be zero, namely,

$$\delta S = 0, \tag{2.18}$$

where S is an action. This method is called the least action principle. We can say that the heart of classical mechanics is the least action principle. Likewise, the principle also applies in quantum field theory and other fields in physics. Field theory describes a system which is composed of a continuous set of degree of freedom instead of a discrete degree of freedom in classical mechanics. The continuous degrees of freedom suggests us to use a density function to interpret our system rather than a discrete one. This interpretation leads us to write a Lagrangian in the form

$$L = \int d^3x \mathcal{L}, \tag{2.19}$$

where $\mathcal{L}$ is called Lagrangian density. Thus the action can be written in term of Lagrangian density as

$$S = \int dt L = \int d^4x \mathcal{L}. \tag{2.20}$$

To get the Euler-Lagrange equation, which can be used to find an equation of motion, one can proceed it as follow. First, varying the action with respect to ϕ, we get

$$\begin{aligned} \delta S &= \int d^4x \delta\mathcal{L} \\ &= \int d^4x \Big[\frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta(\partial_\mu\phi)\Big]. \end{aligned} \tag{2.21}$$

Integrating by parts in the last term, we obtain

$$\begin{aligned}\delta S &= \int d^4x\Big[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi + \partial_\mu\Big\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\Big\}\Big]\\ &= \int d^4x\Big[\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\Big] + \int_\Omega da^\mu\Big\{\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\Big\},\end{aligned} \tag{2.22}$$

wherein the first line, for the first term, we have used Stokes' theorem; a^μ is a boundary area. We assume that our field vanishes on asymptotic boundaries which is true because if not our Lagrangian and its integration would have blown up and gave a non-physical interpretation. Thus the equation 2.22 becomes

$$\delta S = \int d^4x\Big[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big]. \tag{2.23}$$

Equating the above equation to zero one finds

$$\int d^4x\Big[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\Big]\delta\phi = 0 \tag{2.24}$$

since $\delta\phi$ is arbitrary which leads us to conclude that

$$\frac{\partial\mathcal{L}}{\partial\phi(x)} - \partial_\mu\frac{\partial\mathcal{L}}{\partial\partial_\mu\phi(x)} = 0. \tag{2.25}$$

This equation is the Euler-Lagrange equation. The Euler-Lagrange equation can be used to derive the equation such as the Klein-Gordon equation and the Dirac equation. In this project, we don't use this equation directly. However, there is the first step in equation 2.22 that use in deriving energy-momentum tensor in section 2.5.

2.4 Electromagnetic Interaction

In electromagnetic, there is one symmetry which is called gauge invariance. This gauge invariance allows us to choose a gauge that simplifies the calculations. When we consider electromagnetic with quantum theory, we usually describe it with A^μ which is composed of a scalar potential and vector potential, namely $A^\mu = (V, \vec{A})$. The gauge transformation takes the form

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu\alpha(x) \tag{2.26}$$

and

$$\phi(x) \to \phi(x)e^{-i\alpha(x)} \tag{2.27}$$

for A_μ and scalar field respectively, where e is just a conventional constant. In fact, e is a charge of a scalar particle. Now if we want to describe the electromagnetic and scalar field, in quantum theory, we must combine them in one Lagrangian that is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \partial^\mu\phi^*\partial_\mu\phi - m^2\phi\phi^*, \tag{2.28}$$

where $-\frac{1}{4}F_{\mu\nu}^2$ is the electromagnetic Lagrangian and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \tag{2.29}$$

However, by performing gauge transformation to equation 2.28, we find

$$\begin{aligned}
\mathcal{L}' =& -\frac{1}{4}F_{\mu\nu}^2 + \partial^\mu\phi^*(x)e^{i\alpha(x)}\partial_\mu\phi(x)e^{-i\alpha(x)} - m^2\phi^*(x)e^{i\alpha(x)}\phi(x)e^{-i\alpha(x)} \\
=& -\frac{1}{4}F_{\mu\nu}^2 + (e^{i\alpha}\partial^\mu\phi^* + i\phi^* e^{i\alpha}\partial_\mu\alpha)(e^{-i\alpha}\partial^\mu\phi - i\phi e^{-i\alpha}\partial_\mu\alpha) - m^2\phi^*\phi \\
=& -\frac{1}{4}F_{\mu\nu}^2 + \partial^\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi + i\phi^*\partial_\mu\alpha\partial^\mu\phi - i\phi\partial^\mu\phi^*\partial_\mu\alpha + \phi\phi^*\partial_\mu\alpha\partial_\mu\alpha \\
=&\mathcal{L} + i\phi^*\partial_\mu\alpha\partial^\mu\phi - i\phi\partial^\mu\phi^*\partial_\mu\alpha + \phi\phi^*\partial_\mu\alpha\partial_\mu\alpha
\end{aligned} \tag{2.30}$$

where in the first line $F_{\mu\nu}^2$ doesn't transform from gauge invariance. From equation 2.30, we see that Lagrangian in equation 2.28 is not gauge invariant. To fix this, we define a covariant derivative, instead of normal derivative, as

$$D_\mu = \partial_\mu + ieA_\mu. \tag{2.31}$$

By transforming $D_\mu\phi$, we find

$$\begin{aligned}
D'_\mu\phi' =&(\partial_\mu + ieA_\mu + \partial_\mu\alpha)\phi e^{-i\alpha} \\
=&e^{-i\alpha}\partial_\mu\phi - i\phi e^{-i\alpha}\partial_\mu\alpha + ieA_\mu\phi e^{-i\alpha} + \phi e^{-i\alpha}\partial_\mu\alpha \\
=&e^{-i\alpha}\partial_\mu\phi + ie^{-i\alpha}eA_\mu\phi \\
=&e^{-i\alpha}(\partial_\mu + ieA_\mu)\phi = e^{-i\alpha}D_\mu\phi.
\end{aligned} \tag{2.32}$$

In the same way for ϕ^*, we obtain

$$D'_\mu\phi'^* = e^{i\alpha}D_\mu\phi^*. \tag{2.33}$$

Using equation 2.32 and 2.33 instead of $\partial_\mu\phi$ and $\partial_\mu\phi^*$ in equation 2.28, we get

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + D^\mu\phi^* D_\mu\phi - m^2\phi\phi^* \tag{2.34}$$

which is now invariant under gauge transformation. By expanding equation 2.34 and comparing with the old Lagrangian, equation 2.28, we get the new Lagrangian, namely

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}^2 + \partial^\mu\phi^*\partial_\mu\phi - m^2\phi\phi^* - ieA_\mu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) - e^2A_\mu^2\phi^*\phi, \tag{2.35}$$

this Lagrangian comes with the extra terms which can be interpreted as an interaction between a scalar field and electromagnetic field. This prescription applies not only to electrodynamics but also to the other theory such as gravitation which is invariant under general coordinate transformation or GCT. This transformation implies the existence of energy-momentum tensor. We are going to discuss it in the next section.

2.5 Energy Momentum Tensor

Einstein has stated that, in special relativity, the laws of physics must be the same in all inertial frames of reference. Although we use general relativity in this project to describe gravitation, we can use the fact that every local point is almost flat space. So we can use special relativity in the local point, and thus the physics should be the same as other points. For instance, the physics at point x must be the same at other points y where x

and y related to each other by $y = x + \xi$. This postulate leads us to what we call General Coordinate Invariant. Let's consider a case of a scalar field. Then scalar field transforms as $\phi(x) \to \phi(x+\xi)$. For infinitesimal transformation, $\lim_{\xi\to 0} \xi$, Taylor's expansion gives

$$\begin{aligned} \phi(x+\xi) =& \phi(x) + \xi^\mu \partial_\mu \phi(x) + \cdots \\ \phi(x+\xi) \approx& \phi(x) + \xi^\mu \partial_\mu \phi(x) \\ \phi(x+\xi) - \phi(x) =& \xi^\mu \partial_\mu \phi \\ \delta\phi =& \xi^\mu \partial_\mu \phi. \end{aligned} \tag{2.36}$$

In the same way for $\mathcal{L}$,

$$\begin{aligned} \mathcal{L} &\to \mathcal{L} + \xi^\mu \partial_\mu \mathcal{L} \\ \delta\mathcal{L} &= \xi^\mu \partial_\mu \mathcal{L}. \end{aligned} \tag{2.37}$$

Since the Lagrangian is a function only of a ϕ and $\partial_\mu\phi$, by varying it and using equation 2.22, then get

$$\begin{aligned} \delta\mathcal{L} &= \sum_n \Big\{ \Big(\frac{\partial \mathcal{L}}{\partial \phi_n} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \Big) \delta\phi_n + \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \delta\phi_n \Big) \Big\} \\ \delta\mathcal{L} &= \sum_n \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \delta\phi_n \Big) \\ \xi^\nu \partial_\nu \mathcal{L} &= \sum_n \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \xi^\nu \partial_\nu \phi_n \Big) \\ 0 &= \sum_n \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \xi^\nu \partial_\nu \phi_n \Big) - \delta^\mu{}_\nu \xi^\nu \partial_\mu \mathcal{L} \\ 0 &= \partial_\mu \Big(\sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial_\nu \phi_n - \delta^\mu{}_\nu \mathcal{L} \Big), \end{aligned} \tag{2.38}$$

where we have used equations of motion in the second line, and equation 2.36 and 2.37 in the third line. We call

$$T^\mu{}_\nu = \sum_n \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_n)} \partial_\nu \phi_n - \delta^\mu{}_\nu \mathcal{L} \tag{2.39}$$

energy-momentum tensor and represent it as $T^\mu{}_\nu$. The $T^\mu{}_\nu$ is an on-shell conserved quantity, when the equation of motion is used, since $\partial_\mu T^\mu{}_\nu = 0$. The conservation tells us why we have energy and momentum conservation laws. It is true that $T^\mu{}_\nu$ implies the existence of GCT and vice verse. But another reason that we need $T^\mu{}_\nu$ is that an interaction term must be composed of $T^\mu{}_\nu$ to make our field, in this case, is a scalar field, interact with gravitation as we will see in section 3.2.

2.6 Massless Spin-2 Particle

Spin-2 particles can be described by the Fierz-Pauli action, see [8],that takes the form

$$S = \int d^4x \Big(-\frac{1}{2}\partial_\alpha h_{\mu\nu} \partial^\alpha h^{\mu\nu} + \partial_\mu h_{\nu\alpha} \partial^\nu h^{\mu\alpha} - \partial_\mu h^{\mu\nu} \partial_\nu h + \frac{1}{2}\partial_\alpha h \partial^\alpha h - \frac{1}{2} m^2 (h_{\mu\nu} h^{\mu\nu} - h^2) \Big), \tag{2.40}$$

where h is the trace of the $h^{\mu\nu}$ which is equal to $h^{\mu}{}_{\mu}$ and $h^{\mu\nu}$ is a symmetric tensor. If we take $m = 0$ for the action, it will be invariant under transformation of the form

$$h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu. \tag{2.41}$$

This transformation together with the massless spin-2 action leads to Einstien-Hilbert action, see [4]. This relation answers the question: why we believe that the action is the hope of understanding the quantization of gravity. We can vary the action for the massive case (we will take the mass to zero in the last step) differentiate them and take the trace. These three steps give us three equations:

$$(\Box - m^2)h_{\mu\nu} = 0, \tag{2.42}$$

$$\partial^\mu h_{\mu\nu} = 0 \tag{2.43}$$

and

$$h = 0. \tag{2.44}$$

The derivations of these equations can be found in [8] and [9]. Equation 2.42 takes a similar form to the Klein-Gordon equation. Thus, the solution of the equation must be

$$h_{\mu\nu}(x) = \int \frac{d^3p}{(2\pi)^3(2\omega_p)} \{h_{\mu\nu}(p)e^{ipx} + h^\dagger_{\mu\nu}(p)e^{-ipx}\}. \tag{2.45}$$

The general solution can be found by imposing the remaining two conditions, $\partial^\mu h_{\mu\nu} = 0$ and $h = 0$, to the solution, we get

$$h_{\mu\nu}(x) = \int \frac{d^3p}{(2\pi)^3(2\omega_p)} \sum_j \{a^j_p \epsilon^j_{\mu\nu}(p)e^{ipx} + a^{j\dagger}_p \epsilon^{j\dagger}_{\mu\nu}(p)e^{-ipx}\}. \tag{2.46}$$

where a^j_p, $a^{j\dagger}_p$ and $\epsilon^j_{\mu\nu}$ are an annihilation operator, a creation operator and orthogonal basis vector in j axis respectively. There are ten bases for the $h^{\mu\nu}$

$$\epsilon^0_{\mu\nu} = \frac{1}{2}\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \epsilon^1_{\mu\nu} = \frac{1}{2\sqrt{3}}\begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$\epsilon^2_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon^3_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\epsilon^4_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad \epsilon^5_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\epsilon^6_{\mu\nu} = \frac{1}{\sqrt{6}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}, \qquad \epsilon^7_{\mu\nu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\epsilon^8_{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \epsilon^9_{\mu\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

However, for the propagation along the z-axis, the massless particle requires only two polarizations, only two real polarization tensors remain which are $\epsilon^5_{\mu\nu}$ and $\epsilon^7_{\mu\nu}$. These are called physical modes, see [8] and [9]. Instead of working in this representation, let us make a combination of these two tensors as we did in photon case to change from X-Y polarization to circular polarization namely,

$$\epsilon^{\pm}_{\mu\nu} = \frac{1}{\sqrt{2}}\epsilon^5_{\mu\nu} \pm \frac{i}{\sqrt{2}}\epsilon^7_{\mu\nu}, \tag{2.47}$$

see [10]. Thus, we have

$$\epsilon^+_{\mu\nu} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.48}$$

and

$$\epsilon^-_{\mu\nu} = \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & -i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.49}$$

Recalling what we have in electrodynamics, for circular polarization, we have

$$\epsilon^+_{\mu} = -\frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ i \\ 0 \end{pmatrix} \tag{2.50}$$

and

$$\epsilon^-_{\mu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -i \\ 0 \end{pmatrix}. \tag{2.51}$$

From these two circular polarization we can see that

$$\epsilon^+_{\mu}\epsilon^+_{\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \epsilon^+_{\mu\nu} \tag{2.52}$$

and

$$\epsilon^-_{\mu}\epsilon^-_{\nu} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -i & 0 \\ 0 & -i & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \epsilon^-_{\mu\nu}. \tag{2.53}$$

Thus a graviton's polarization can be represented by a photon's polarization. Photon and graviton field are similar in terms of polarization and their mass. Only their spins are different. For a photon's field, is takes the form

$$A_\mu = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_p}} \{\epsilon_\mu^+ a_{p,+} e^{-ipx} + \epsilon_\mu^- a_{p,-}^\dagger e^{ipx}). \tag{2.54}$$

see [4]. Hence, for a graviton's field, we have

$$h_{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4(2\omega_p)} \{a_{p,+}\epsilon_\mu^+\epsilon_\mu^+ e^{ipx} + a_{p,-}^\dagger \epsilon_\mu^- \epsilon_\nu^- e^{-ipx}\}. \tag{2.55}$$

However, there are five gauge conditions leftover to be imposed. In this paper, we are using, see [10],

$$\partial_\mu h^{\mu\nu}(x) = 0 \quad \text{and} \quad h^\mu{}_\mu = 0. \tag{2.56}$$

This gauge condition will provide us an easier calculation in chapter 3. As ones will see in section 3.2, this field is a perturbative part of metric tensor g which describes a graviton. The graviton field $h_{\mu\nu}$ will play a crucial role in a calculation part of this project. In the next section, we will take one step closer to the calculation part. We are going to talk about how to calculate an amplitude of an event in QFT.

2.7 S-Matrix

Recalling what we have in quantum mechanics, to calculate a probability amplitude of the event that we have prepared at time t_i and let it evolve to time t_f then measure it. The probability is given by $|\langle f, t_f|i, t_i\rangle|^2$ where $|i, t_i\rangle$ is the initial state at time t_i and $|f, t_f\rangle$ is the final state at time t_f. However, when we label our state with time, this refers to the Schrodinger picture. We can convert this to the Heisenberg picture by leaving the state alone and pulling out all a time evolution operator. Thus it can be written as $|\langle f| S |i\rangle|^2$, where the S-matrix keeps all the information about how initial and final state interacts with each other. Let us consider the S-matrix element, $\langle f, t_f|i, t_i\rangle$. We will write the S-matrix in the perturbative form as $S = \mathbb{1} + i\mathcal{T}$ where $\mathcal{T}$ is known as transfer matrix. This perturbative form makes sense because the S-matrix must reduce to the identity matrix when there are no interactions. Since we already know that our theory must base on energy-momentum conservation. Thus it is helpful to factor a delta function out of the $\mathcal{T}$, namely

$$\mathcal{T} = (2\pi)^4\delta^4(\sum p)\mathcal{M}, \tag{2.57}$$

where $\delta^4(\sum p)$ is shorthand for $\delta(\sum p_f^\mu - \sum p_i^\mu)$ which $\sum p_i^\mu$ and $\sum p_f^\mu$ is total initial particles' momenta and total final particles' momenta respectively. Now, we can use the M-matrix, $\mathcal{M}$, to describes the interaction instead of the S-matrix since we already knew that the S-matrix is composed of the trivial part, $\mathbb{1}$, and interaction part $\mathcal{M}$. Thus using the M-matrix to describe the interaction is better than the S-matrix itself. In the case where interactions occur, we then have $|f\rangle \neq |i\rangle$, if not this will be a trivial one that is equal to $\mathbb{1}$. This leads to

$$\begin{aligned} |\langle f| S |i\rangle|^2 &= \left|\langle f| \mathbb{1} + (2\pi)^4\delta^4(\sum p)\mathcal{M} |i\rangle\right|^2 \\ &= \left|\langle f|i\rangle + \langle f| (2\pi)^4\delta^4(\sum p)\mathcal{M} |i\rangle\right|^2 \\ &= (2\pi)^8\delta^4(\sum p)\delta^4(\sum p')|\langle f| \mathcal{M} |i\rangle|^2 \\ &= (2\pi)^8\delta^4(\sum p)\delta^4(\sum p')|\mathcal{M}|^2, \end{aligned} \tag{2.58}$$

where $|\mathcal{M}| \equiv |\langle f|\, \mathcal{M}\, |i\rangle|$. This result suggests that the M-matrix is directly proportional to the S-matrix which represents the probability of the interaction. However, we can't directly measure the probability. Instead, the quantity that can be measured is a cross-section. In the next section, we will provide a formula that describes the relation between S-matrix and cross-section.

2.8 Cross-Section

In the early quantum mechanics, to calculate an interaction area, such as gold foil experiment, scientists introduced a cross-sectional area which represents the area of which particles can interact. The cross-sectional area, σ, is given by

$$\sigma = \frac{1}{T}\frac{1}{\Phi}N = \frac{N}{N_{inc}}A, \tag{2.59}$$

where T is the time of the experiment; Φ is the flux of incident particles; N is the number of scattered particles; N_{inc} is the number of incident particles; A is the area of the incident particle beam. This quantity has an area dimension. σ is useful since they are related to their interaction area which comes from interaction probability, N, times the incident area

Now we reviewed sufficient concepts about the cross-sectional area. Let us bring ourself to the quantum mechanical perspective. In quantum mechanics, the cross-section has more abstract meaning because we cannot interpret a particle with shape anymore and they have a probability for scattering. This cross-section has nothing to do with an area of a physical object. Although cross-section, σ, is an abstract quantity that represents the strength of an interaction, it can be thought of as an effective area for which the interaction can occur. Since quantum mechanics usually talk about probability, so it is natural to represent cross-section as

$$d\sigma = \frac{1}{T}\frac{1}{\Phi}dP, \tag{2.60}$$

where P is the probability for scattering. The probability is related to $\mathcal{M}$ matrix in the previous section. This relation will lead us to work with the $\mathcal{M}$ matrix instead of P. In a laboratory, it is impossible to place detectors in all direction. Instead, we can only measure the number of scattered particles in a certain solid angle, Ω, which detectors have been placed. So we must observe a differential cross-section in a laboratory and integrate it to get the cross-section if we want. The mathematical formula for the differential cross-section takes the form

$$d\sigma = \frac{1}{(2E_1)(2E_2)|\vec{v}_1 - \vec{v}_2|}|\mathcal{M}|^2 d\Pi_{\text{LIPS}}, \tag{2.61}$$

where

$$d\Pi_{\text{LIPS}} \equiv \prod_{\text{finalstate}} \frac{d^3p_j}{(2\pi)^3}\frac{1}{2E_j}(2\pi)^4\delta^4(\sum p) \tag{2.62}$$

is called the Lorentz-invariant phase space. Here, E_1, E_2, and $|\vec{v}_1 - \vec{v}_2|$ is energy for first particle, energy for second particle and relative velocity of particle respectively.

Now, consider $2 \to 2$ scattering in the center of mass frame, CM frame for short. We will use the result of this case in the next chapter. In the CM frame, for $2 \to 2$ scattering,

$$p_1 + p_2 \to p_3 + p_4, \tag{2.63}$$

we have $\vec{p_1} = -\vec{p_2}$ and $\vec{p_3} = -\vec{p_4}$ from the conservation of momentum and $E_1 + E_2 = E_3 + E_4$ from conservation of energy. Then

$$d\Pi_{\text{LIPS}} = (2\pi)^4\delta(E_1 + E_2 - E_3 - E_4)\delta^3(\vec{p_1} + \vec{p_2} - \vec{p_3} - \vec{p_4})\frac{d^3p_3}{(2\pi)^3}\frac{1}{2E_3}\frac{d^3p_4}{(2\pi)^3}\frac{1}{2E_4}, \tag{2.64}$$

for simplicity we have separated the delta function into two parts, energy and momentum. By applying this equation to equation 2.61 and integrating it with respect to omega, we find

$$\sigma = \frac{1}{64\pi^2 E_1 E_2(v_1 + v_2)}\int |\mathcal{M}|^2\delta(E_1 + E_2 - E_3 - E_4)\delta^3(\vec{p_1} + \vec{p_2} - \vec{p_3} - \vec{p_4})\frac{d^3p_3}{E_3}\frac{d^3p_4}{E_4}. \tag{2.65}$$

Since $\vec{p_1} + \vec{p_2} = 0$, by the changing Cartesian integration to the spherical one this leads to

$$\begin{aligned}\sigma &= \frac{1}{64\pi^2 E_1 E_2(v_1 + v_2)}\int |\mathcal{M}|^2\delta(E_1 + E_2 - E_3 - E_4)p_f^2 d\Omega\frac{dp_f}{E_3E_4}\\ &= \frac{1}{64\pi^2 E_1 E_2(v_1 + v_2)}\int \delta(\sqrt{s} - E_3 - E_4)p_f^2 d\Omega\frac{dp_f}{E_3E_4},\end{aligned} \tag{2.66}$$

where $|\vec{p_3}| = |\vec{p_4}| = p_f$ and $E_1 + E_2 = \sqrt{s}$. Write this equation in a new form as

$$\sigma = \frac{1}{64\pi^2 E_1 E_2(v_1 + v_2)}\int d\Omega dp_f|\mathcal{M}|^2\delta(f(p_f))g(p_f), \tag{2.67}$$

where

$$g(p_f) = \frac{p_f^2}{E_3E_4} = \frac{p_f^2}{\sqrt{(m_3^2 + p_f^2)(m_4^2 + p_f^2)}}, \tag{2.68}$$

and

$$f(p_f) = \sqrt{s} - \sqrt{m_3^2 + p_f^2} - \sqrt{m_4^2 + p_f^2}. \tag{2.69}$$

By using the relation

$$\delta(f(x)) = \left|\frac{df}{dx}\right|_a^{-1}\delta(x - a), \tag{2.70}$$

where $f(a) = 0$, we obtain

$$\int g(p_f)\delta(f(p_f))dp_f = \frac{1}{|df/dp_f|_a}\int g(p_f)\delta(p_f - a)dp_f = \frac{g(a)}{|df/dp_f|_a}, \tag{2.71}$$

where $a = p_f$. Since$f(p_f) = \sqrt{s} - \sqrt{m_3^2 + p_f^2} - \sqrt{m_4^2 + p_f^2}$, by differentiating f with respect to p_f, we get

$$\left|\frac{df}{dp_f}\right|_{a=p_f} = \left|\frac{p_f}{\sqrt{m_3^2 + p_f^2}} + \frac{p_f}{\sqrt{m_4^2 + p_f^2}}\right| = \left|\frac{p_f}{E_3} + \frac{p_f}{E_4}\right| = \left|p_f\frac{E_3 + E_4}{E_3E_4}\right|. \tag{2.72}$$

Substituting equation 2.71 and 2.72 into 2.67, we arrive at

$$\begin{aligned}\sigma &= \frac{1}{64\pi^2 E_1 E_2 (v_1+v_2)}\left|\frac{1}{p_f \frac{E_3+E_4}{E_3E_4}}\frac{p_f^2}{E_3E_4}\right|\int d\Omega|\mathcal{M}|^2 \\ &= \frac{1}{64\pi^2 E_1 E_2 (v_1+v_2)}\left|\frac{p_f}{E_3+E_4}\right|\int d\Omega|\mathcal{M}|^2 \\ &= \frac{1}{64\pi^2 E_1 E_2 (v_1+v_2)}\left|\frac{p_f}{\sqrt{s}}\right|\int d\Omega|\mathcal{M}|^2,\end{aligned} \tag{2.73}$$

where we have used $E_3+E_4 = E_1+E_2 = \sqrt{s}$. Next, considering the term $E_1E_2(v_1+v_2)$, we can rearrange it as follow

$$\begin{aligned}E_1E_2(v_1+v_2) &= E_1E_2(\frac{|\vec{p}_1|}{E_1}+\frac{|\vec{p}_2|}{E_2}) \\ &= E_1E_2(\frac{|\vec{p}_1|}{E_1}+\frac{|\vec{p}_1|}{E_2}) \\ &= |\vec{p}_1|(E_1+E_2) = |\vec{p}_1|\sqrt{s}.\end{aligned} \tag{2.74}$$

Applying this equation with equation 2.73, we then get

$$\begin{aligned}\sigma &= \frac{1}{64\pi^2|\vec{p}_1|\sqrt{s}}\left|\frac{p_f}{\sqrt{s}}\right|\int d\Omega|\mathcal{M}|^2 \\ &= \frac{1}{64\pi^2 s}\frac{|p_f|}{|\vec{p}_1|}\int d\Omega|\mathcal{M}|^2.\end{aligned} \tag{2.75}$$

It is worth noting here that s is a Lorentz invariant quantity which is represented by $s = (p_1+p_2)^2$, where p_1 and p_2 is first and second particle four-momentum respectively. In fact, there are other two quantities beside s: t and u. They are called Mandelstam variables. In the CM-frame, we can check that $s = (E_1+E_2)^2$ by

$$\begin{aligned}s = (p_1+p_2)^2 &= (E_1+E_2)^2 - (\vec{p}_1+\vec{p}_2)^2 \\ &= (E_1+E_2)^2 - (\vec{p}_1-\vec{p}_1)^2 \\ &= (E_1+E_2)^2.\end{aligned} \tag{2.76}$$

We will use these quantities again in chapter 3. Now, let us come back to the cross-section. From equation 2.75, the differential cross-section can be written as

$$d\sigma = \frac{1}{64\pi^2 s}\frac{|p_f|}{|\vec{p}_1|}d\Omega|\mathcal{M}|^2 \tag{2.77}$$

or

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|p_f|}{|\vec{p}_1|}|\mathcal{M}|^2. \tag{2.78}$$

Although we usually use $\frac{d\sigma}{d\Omega}$, a differential cross-section which measures in a small solid angle, to explain results, in calculation it is useful to write $\frac{d\sigma}{dt}$ instead since both σ and t are Lorentz invariant and can be used to transform to $\frac{d\sigma}{d\Omega}$ by chain rule. Since in the CM-frame we have

$$p_1^\mu = (E_1, 0, 0, |\vec{p_1}|) \tag{2.79}$$

and

$$p_3^\mu = (E_3, |\vec{p_3}|\sin\theta, 0, |\vec{p_3}|\cos\theta), \tag{2.80}$$

where θ is an angle between p_3 and p_1, we find that

$$\begin{aligned} t \equiv (p_1 - p_3)^2 &= p_1^2 + p_3^2 - 2p_1^\mu p_{3\mu} \\ &= m_1^2 + m^3 - 2p_1^\mu p_{3\mu} \\ &= m_1^2 + m^3 - E_1E_3 + 2|\vec{p_1}||\vec{p_3}|\cos\theta \end{aligned} \tag{2.81}$$

giving

$$dt = 2|\vec{p_1}||\vec{p_3}|d\cos\theta. \tag{2.82}$$

From calculus, we know that $d\Omega = d\cos\theta d\phi$ this leads to

$$d\Omega = d\cos\theta d\phi = \frac{dtd\phi}{2|\vec{p_1}||\vec{p_3}|}. \tag{2.83}$$

Combining with equation 2.77, we obtain

$$\begin{aligned} d\sigma &= \frac{1}{64\pi^2 s}\frac{|p_f|}{|\vec{p_1}|}\Big(\frac{dtd\phi}{2|\vec{p_1}||\vec{p_3}|}\Big)|\mathcal{M}|^2 \\ &= \frac{1}{128\pi^2 s|\vec{p_1}|^2}|\mathcal{M}|^2 dtd\phi \end{aligned} \tag{2.84}$$

By assuming that $|\mathcal{M}|^2$ doesn't depend on ϕ, integrating over ϕ this leads to

$$\begin{aligned} d\sigma &= \frac{1}{128\pi^2 s|\vec{p_1}|^2}|\mathcal{M}|^2 dt(2\pi) \\ &= \frac{1}{64\pi s|\vec{p_1}|^2}|\mathcal{M}|^2 dt, \end{aligned} \tag{2.85}$$

that is

$$\frac{d\sigma}{dt} = \frac{1}{64\pi s|\vec{p_1}|^2}|\mathcal{M}|^2. \tag{2.86}$$

For special case $m_1 = 0$, we can rearrange equation 2.76 to give

$$\begin{aligned} s &= (E_1 + E_2)^2 = E_1^2 + 2E_1E_2 + E_2^2 \\ s &= |\vec{p_1}|^2 + 2|\vec{p_1}|E_2 + |\vec{p_2}|^2 + m_2^2 \\ (s - m_2^2)^2 &= (2|\vec{p_1}|^2 + 2|\vec{p_1}|E_2)^2 \\ \frac{(s-m_2^2)^2}{4} &= |\vec{p_1}|^2(|\vec{p_1}| + E_2)^2 \\ \frac{(s-m_2^2)^2}{4} &= |\vec{p_1}|^2(E_1 + E_2)^2 = |\vec{p_1}|s \\ \frac{(s-m_2^2)^2}{4s} &= |\vec{p_1}|^2. \end{aligned} \tag{2.87}$$

Hence, for one massless particle, equation 2.86 reduces to

$$\frac{d\sigma}{dt} = \frac{1}{16\pi(s - m_2^2)^2}|\mathcal{M}|^2. \tag{2.88}$$

We will use this equation in section 3.1 and 3.4 where the incoming particle is a photon which is a massless particle. Of course in section 3.4, the outgoing particle is not a photon but rather a graviton. However, just like the photon, the graviton is massless, and hence equation 2.88 also holds for this situation.

2.9 Time-ordering Operator

To simplify the calculations, we can define the time-ordered operator which is denoted as $T\{...\}$. This operator indicates that all operator in the curly brackets should be placed so that later times are always to the left of earlier times. For example, $T\{a(t_1)b(t_2)\}$ where $a(t_1)$ and $b(t_2)$ are an operator at time t_1 and t_2, if t_2 more than t_1, the time-ordered operator gives $b(t_2)a(t_1)$. This time-ordered operator is another useful operator in QFT. It helps us write other equations in the simpler form. As ones will see, the time-ordered operator will play a central role in many sections from now on to simplify our equation and even make a calculation easier.

2.10 Propagators

The propagators are just a concept of the field that propagates from one point to another point. We use the terms propagator and propagate since the propagation of the field can be thought of as a wave that propagates through space. Let's begin with a simple example, the scalar propagation. Consider two space-time points x_1 and x_2 which the field will propagate between these points. The mathematical equation of the propagation can be written as $\langle 0| T\{\phi(x_1)\phi(x_2)\} |0\rangle$. Considering $\langle 0| \phi(x_1)\phi(x_2) |0\rangle$ by using the fact that

$$\phi(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega_k}} (a_k e^{-ikx} + a_k^\dagger e^{ikx}), \tag{2.89}$$

where $\omega_k = \sqrt{(\vec{k}^2 + m^2)}$. We get

$$\begin{aligned}\langle 0| \phi(x_1)\phi(x_2) |0\rangle &= \langle 0| \int \frac{d^3k_1}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{k_1}}} \int \frac{d^3k_2}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{k_2}}} a_{k_1} e^{-ik_1x_1} a_{k_2}^\dagger e^{ik_2x_2} |0\rangle \\ &= \langle 0| \int \frac{d^3k_1}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{k_1}}} \int \frac{d^3k_2}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{k_2}}} a_{k_1}^\dagger a_{k_2} e^{i(-k_1x_1+k_2x_2)} |0\rangle ,\end{aligned} \tag{2.90}$$

where the term $a_{k_1}^\dagger a_{k_2}^\dagger$, $a_{k_1}^\dagger a_{k_2}$ and $a_{k_1} a_{k_2}$ in the first line had been ignored since they equal to zero when they are sandwiched between ground states. From the fact that $\langle 0| a_{k_1} a_{k_2}^\dagger |0\rangle = (2\pi)^3 \delta^3(\vec{k}_1 - \vec{k}_2)$, we then have

$$\langle 0| \phi(x_1)\phi(x_2) |0\rangle = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} e^{ik(x_2-x_1)}, \tag{2.91}$$

where $k = k_1 = k_2$ from integrating the delta function. Now, we are ready for $\langle 0| T\{\phi(x_1)\phi(x_2)\} |0\rangle$. By using equation 3.5, we find

$$\begin{aligned}\langle 0| T\{\phi(x_1)\phi(x_2)\} |0\rangle &= \langle 0| \phi(x_1)\phi(x_2) |0\rangle \theta(t_1 - t_2) + \langle 0| \phi(x_2)\phi(x_1) |0\rangle \theta(t_2 - t_1) \\ &= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_k} [e^{ik(x_2-x_1)}\theta(t_1 - t_2) + e^{ik(x_1-x_2)}\theta(t_2 - t_1)].\end{aligned} \tag{2.92}$$

To simplify the above result introducing the mathematical identity[1]

$$e^{-i\omega_k t}\theta(t) + e^{i\omega_k t}\theta(-t) = \lim_{\epsilon\to 0} \frac{-2\omega_k}{2\pi i} \int_{-\infty}^{\infty} \frac{d\omega}{\omega^2 - \omega_k^2 + i\epsilon} e^{i\omega t}. \tag{2.93}$$

[1]For more information, see page 76 of reference [4]

The $i\epsilon$ has been put in to shift the poles out of the real axis. This trick makes an integration, which has poles on a real axis, easier by deforming the real integration to a contour integration. The appearance of the ϵ is acceptable because we are going to take the ϵ to zero at the end of the calculation. Combining equation 3.3 and 3.6, we get

$$\begin{aligned}\langle 0| T\{\phi(x_1)\phi(x_2)\} |0\rangle &= \lim_{\epsilon\to 0}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-i\vec{k}\cdot(\vec{x_1}-\vec{x_2})}[e^{-i\omega_k(t)}\theta(t)+e^{i\omega_k(t)}\theta(-t)] \\ &= \lim_{\epsilon\to 0}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}e^{-i\vec{k}\cdot(\vec{x_1}-\vec{x_2})}\Big(\frac{-2\omega_k}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2-\omega_k^2+i\epsilon}e^{i\omega(t_1-t_2)}\Big) \\ &= \lim_{\epsilon\to 0}\int \frac{d^4k}{(2\pi)^4}\frac{i}{\omega^2-\omega_k^2+i\epsilon}e^{ik\cdot(x_1-x_2)},\end{aligned} \tag{2.94}$$

where $t = t_1 - t_2$. Since we know that

$$\begin{aligned} k^2 &= \omega^2 - \vec{k}^2 \\ \omega^2 &= k^2 + \vec{k}^2, \end{aligned} \tag{2.95}$$

and

$$\omega_k^2 = \vec{k}^2 + m^2, \tag{2.96}$$

substituting this back to equation 2.94, we obtain

$$\begin{aligned} D_F(x_1,x_2) &= \langle 0| T\{\phi(x_1)\phi(x_2)\} |0\rangle \\ &= \lim_{\epsilon\to 0}\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{ik\cdot(x_1-x_2)} \\ &= \int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2+i\epsilon}e^{ik\cdot(x_1-x_2)}, \end{aligned} \tag{2.97}$$

which is called Feynman propagator. Notice that in the last line the limit has been abandoned for avoiding the unwieldy limit. Next, considering a more complicated example that involves a photon propagator. For the photon propagator, we have to deal with a vector rather than a scalar, lengthy calculations and multiple tricks will be needed. The result is presented without derivation

$$\langle 0| T\{A^\mu(x)A^\nu(y)\} |0\rangle = \int \frac{d^4k}{(2\pi)^4}e^{ik\cdot(x-y)}\frac{-ig^{\mu\nu}}{p^2+i\epsilon}, \tag{2.98}$$

where $g^{\mu\nu} = \eta^{\mu\nu}$ which is true for flat space, and p is a 4-momentum of a photon. Here we have used Feynman Gauge[2]. The propagator is an important part that is always appearing in Feynman diagrams. Unlike incoming and outgoing particles, although, a propagator is non-physical that is we can't directly observe it and appear in a calculation only, it is used to explain action at a distance in QFT by acting as an intermediate particle.

2.11 Dyson Series and Vacuum State

The ground state $|0\rangle$ that we discuss so far is the ground state in a free theory. However, when we are talking about an interaction theory. The ground state will be different from

[2]For outline of derivations, see page 128-130 of reference [3]

the free theory, and we represent it as $|\Omega\rangle$. Unlike a ground state in the free theory which doesn't allow anything to happen, the interaction theory allows interactions to occur in a ground state such as virtual particles which are always using as an example in the uncertainty principle. In this section, we will study how to write the interaction theory in the form of the free theory with which we know how to deal. Consider a scalar field, the Heisenberg equation of motion[5] provides $i\partial_t\phi(x) = [\phi, H]$ where H is the Hamiltonian in an interaction theory. In the Heisenberg picture we can separate the time evolution out of an operator, scalar field in this case, that is

$$\phi(\vec{x}, t) = S^\dagger(t, t_i)\phi(\vec{x})S(t, t_i), \tag{2.99}$$

where S is the time-evolution operator and t_i is an initial time. Note that we represent the time-evolution operator by S instead of U which is usually found in quantum mechanics since in quantum field theory time-operator is just the S-matrix. This time-evolution operator satisfies

$$i\partial_t S(t, t_i) = H(t)S(t, t_i). \tag{2.100}$$

The difference between free and interaction theory is simple; that is potential terms which cause particles to interact. So we can write the interaction Hamiltonian as

$$H(t) = H_0 + V(t), \tag{2.101}$$

where $H(t)$ is the interaction Hamiltonian, H_0 is the the free theory Hamiltonian, and $V(t)$ is a potential term. We need to write our field in the form of the free field which we know how to handle it. Since we know that t_i is the initial time, $\phi(\vec{x}, t_i) \equiv \phi(\vec{x})$ must be equal to $\phi_0(\vec{x})$. Thus, at an arbitrary time later, t, of the free field takes the form

$$\phi_0(\vec{x}, t) = e^{iH_0(t-t_i)}\phi(\vec{x})e^{-iH_0(t-t_i)}. \tag{2.102}$$

This is the relation of the Heisenberg picture and Schrodinger picture. The Schrodinger picture field, $\phi(\vec{x})$, is the picture that an operator does not change with time. The Heisenberg picture, in contrast, the field evolves with time and can be represented as $\phi(\vec{x}, t)$. By using equation 2.99, we can write the Heisenberg picture field in the form of the free field as

$$\begin{aligned}\phi(\vec{x}, t) &= S^\dagger(t, t_i)e^{-iH_0(t-t_i)}\phi_0(\vec{x})e^{iH_0(t-t_i)}S(t, t_i)\\ &= U^\dagger(t, t_i)\phi_0(\vec{x})U(t, t_i),\end{aligned} \tag{2.103}$$

where $U(t, t_i) = e^{iH_0(t-t_i)}S(t, t_i)$. To write an operator U in the form of a potential, differentiating $U(t, t_i)$ with respect to t and using equation 2.100: we find

$$\begin{aligned}i\partial_t U(t, t_i) &= i\partial_t e^{iH_0(t-t_i)}S(t, t_i)\\ &= i(\partial_t e^{iH_0(t-t_i)})S(t, t_i) + ie^{iH_0(t-t_i)}\partial_t S(t, t_i)\\ &= -e^{iH_0(t-t_i)}H_0 S(t, t_i) + e^{iH_0(t-t_i)}H(t)S(t, t_i)\\ &= -e^{iH_0(t-t_i)}H_0 S(t, t_i) + e^{iH_0(t-t_i)}H(t)e^{-iH_0(t-t_i)}e^{iH_0(t-t_i)}S(t, t_i)\\ &= e^{iH_0(t-t_i)}(-H_0 + H(t))e^{-iH_0(t-t_i)}e^{iH_0(t-t_i)}S(t, t_i)\\ &= e^{iH_0(t-t_i)}V(t)e^{-iH_0(t-t_i)}e^{iH_0(t-t_i)}S(t, t_i)\\ &= V_I(t)U(t, t_i),\end{aligned} \tag{2.104}$$

where $V_I(t) = e^{iH_0(t-t_i)}V(t)e^{-iH_0(t-t_i)}$. The solution for differential equation $i\partial_t U(t,t_i) = V_I(t)U(t,t_i)$ is

$$\begin{aligned} U(t,t_i) &= 1 - i\int_{t_i}^{t} dt' V_I(t') - \frac{1}{2}\int_{t_i}^{t} dt' \int_{t_i}^{t} dt'' T\{V_I(t')V_I(t'')\} + ... \\ &= T\Bigg\{ \exp\bigg[-i\int_{t_i}^{t} dt' V_I(t')\bigg]\Bigg\}, \end{aligned} \tag{2.105}$$

where time ordered operator appear because $V_I(t', t_i)$ and $V_I(t'', t_i)$ don't necessarily commute to each other. For instance,

$$\int_{t_i}^{t} dt' \int_{t_i}^{t'} dt'' T\{V_I(t')V_I(t'')\} = \int_{t_i}^{t} dt' \Big[\int_{t_i}^{t'} dt'' V_I(t')V_I(t'') + \int_{t'}^{t} dt'' V_I(t'')V_I(t')\Big]. \tag{2.106}$$

Equation 2.105 is known as Dyson series.

To apply the Dyson series with the S-matrix. First, define an interaction ground state $|\Omega\rangle$. This ground state is annihilated by $a_p(t)$ at an initial time which is long enough to be treated as a free theory. We normally set this initial time at $-\infty$. Assuming there is the reference time t_i that the interaction and free vacuum state are equal. We need to relate between those two vacuum states because we know how the free-field act on the free vacuum states. We can find the relation by evolving the vacuum state $|\Omega\rangle$ from $t = -\infty$ to the reference time t_i: in the Schrodinger picture, the state evolves as $S(t,t_i)|\Omega\rangle$. Thus $S(t,t_i)|\Omega\rangle$ is annihilated by $a_p(t_i)$ which is the annihilation operator for the free-vacuum state. As for free-vacuum operator, if we let the free-vacuum operator $|0\rangle$ evolve from $t = -\infty$ to t_i, for free-vacuum, the state evolves as $e^{iH_0(t-t_i)}|0\rangle$, is again annihilated by $a_p(t_i)$, namely $a_p(t_i)e^{iH_0(t-t_i)}|0\rangle = 0$. At time t_i, free and interacting state are both annihilated by $a_p(t_i)$, since at this time two pictures are taken to be equal. Thus, these two states must be different only by a numerical factor, assume that this factor is $\mathcal{N}$. So

$$\begin{aligned} \lim_{t\to-\infty} S(t,t_i)|\Omega\rangle &= \mathcal{N} \lim_{t\to-\infty} e^{iH_0(t-t_i)}|0\rangle \\ |\Omega\rangle &= \mathcal{N} \lim_{t\to\infty} S^\dagger(t,t_i)e^{iH_0(t-t_i)}|0\rangle \\ &= \mathcal{N} U_{i,-\infty}|0\rangle . \end{aligned} \tag{2.107}$$

Hermitian conjugating equation 2.107, we obtain

$$|\Omega\rangle = \mathcal{N}' \langle 0| U_{\infty,i} \; . \tag{2.108}$$

Next if an interaction occurs, to be precise if the field is not free instead it contains particles in the theory, then we can write this field as

$$\langle\Omega| T\{\phi(x_1)\phi(x_2)...\phi(x_n)\}|\Omega\rangle . \tag{2.109}$$

For simplicity, assume that $t_1.t_2 > ... > t_n$. Combining with equation 2.103, we get

$$\begin{aligned} &\langle\Omega| T\{\phi(x_1)\phi(x_2)...\phi(x_n)\}|\Omega\rangle = \langle\Omega|\phi(x_1)\phi(x_2)...\phi(x_n)|\Omega\rangle \\ &= \mathcal{N}\mathcal{N}' \langle 0| U_{\infty,0}U_{0,1}\phi(x_1)U_{1,0}U_{0,2}\phi(x_2)U_{2,0}...U_{0,n}\phi(x_n)U_{n,0}U_{0,-\infty}|0\rangle \\ &= \mathcal{N}\mathcal{N}' \langle 0| U_{\infty,1}\phi(x_1)U_{1,2}\phi(x_2)U_{2,3}...U_{n-1,n}\phi(x_n)U_{n,-\infty}|0\rangle \\ &= \mathcal{N}\mathcal{N}' \langle 0| T\{\phi(x_1)\phi(x_2)...\phi(x_n)U_{\infty,-\infty}\}|0\rangle . \end{aligned} \tag{2.110}$$

To remove the normalization factor $\mathcal{N}\mathcal{N}'$, recall that in free theory, we have $\langle 0|0\rangle = 1$. This condition must be true even in the interaction theory, that is $\langle\Omega|\Omega\rangle = 1$. From equation 2.107 and 2.108 together with $\langle\Omega|\Omega\rangle = 1$, these lead to

$$\begin{aligned} \mathcal{N}'\mathcal{N}\langle 0| U_{\infty,i}U_{i,-\infty}|0\rangle &= \langle\Omega|\Omega\rangle \\ \mathcal{N}\mathcal{N}' &= \frac{1}{\langle 0| U_{\infty,-\infty}|0\rangle}. \end{aligned} \tag{2.111}$$

Therefore

$$\begin{aligned} \langle\Omega| T\{\phi(x_1)\phi(x_2)...\phi(x_n)\}|\Omega\rangle &= \frac{\langle 0| T\{\phi(x_1)\phi(x_2)...\phi(x_n)U_{\infty,-\infty}\}|0\rangle}{\langle 0| U_{\infty,-\infty}|0\rangle} \\ = \frac{\langle 0| T\{\phi(x_1)\phi(x_2)...\phi(x_n)\exp\Big\{-i\int_{-\infty}^{\infty}dtV_I(t)\Big\}\}|0\rangle}{\langle 0| T\{\exp\Big\{-i\int_{-\infty}^{\infty}dtV_I(t)\Big\}\}|0\rangle}. \end{aligned} \tag{2.112}$$

We know that the free field transforms as

$$\phi(\vec{x},t_i) = e^{iH_0(t-t_i)}\phi_0(\vec{x})e^{-iH_0(t-t_i)}. \tag{2.113}$$

Since the potential is a function of fields, the potential must evolve in time like the free field. For instance, assume that $V(t_i) = \int d^3x\phi^n(\vec{x},t_i)$. At time $t = t_i$, we have

$$V(t_i) = \int d^3x\phi^n(\vec{x},t_i) = \int d^3x\phi_0^n(\vec{x},t_i) = \int d^3x\phi^n(\vec{x}). \tag{2.114}$$

Since the potential evolves like the free field,

$$\begin{aligned} V_I(t) &= \int d^3x\,\underbrace{e^{iH_0(t-t_i)}\phi_0(\vec{x},t_i)e^{-iH_0(t-t_i)}e^{iH_0(t-t_i)}\phi_0(\vec{x},t_i)e^{-iH_0(t-t_i)}...e^{iH_0(t-t_i)}\phi_0(\vec{x},t_i)e^{-iH_0(t-t_i)}}_{\text{n terms}} \\ &= \int d^3x\phi_0^n(\vec{x},t). \end{aligned} \tag{2.115}$$

The Lagrangian and the potential are related to each other by $V_I(t) = -\int d^3x\mathcal{L}_{int}$, where $\mathcal{L}_{int}$ is an interacting part of the Lagrangian. Substituting $V_I(t) = -\int d^3x\mathcal{L}_{int}$ in equation 2.112 leads to

$$\begin{aligned} \langle\Omega| T\{\phi(x_1)\phi(x_2)...\phi(x_n)\}|\Omega\rangle &= \frac{\langle 0| T\{\phi_0(x_1)\phi_0(x_2)...\phi_0(x_n)\exp\Big\{i\int_{-\infty}^{\infty}dt\int d^3x\mathcal{L}_{int}\Big\}\}|0\rangle}{\langle 0| T\{\exp\Big\{-i\int_{\infty}^{\infty}dt\int d^3x\mathcal{L}_{int}\Big\}\}|0\rangle} \\ &= \frac{\langle 0| T\{\phi_0(x_1)\phi_0(x_2)...\phi_0(x_n)e^{i\int d^4x\mathcal{L}_{int}}\}|0\rangle}{\langle 0| T\{e^{-i\int d^4x\mathcal{L}_{int}}\}|0\rangle}. \end{aligned} \tag{2.116}$$

We know how to calculate fields and states in the free field theory but not for the interaction field. However, this equation provides us the link between those two theories, free and interaction field. Thus, we can find the interaction field solutions through this relation. In the next two sections, we will improve this equation, see how it works and combine with a calculation.

2.12 Contractions and Bubbles

Now we familiarize ourselves with Feynman propagator and some notation in Dyson series, let's develop it so that can be used in a calculation. Recalling that the Feynman propagator $D_F(x_1, x_2)$ takes the form

$$D_F(x_1, x_2) = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\epsilon} e^{ik\cdot(x_1 - x_2)}, \tag{2.117}$$

In fact, the propagator itself is the green function of its free-equation of motion. So the Feynman propagator must satisfy

$$\Box_x D_{x1} = -i\delta_{x1}, \tag{2.118}$$

for simplicity we set $m = 0$, where $D_{xi} = D_{ix} = D(x, x_i)$ and $\delta_{xi} = \delta^4(x - x_i)$, see [4][7]. In the calculation, we will encounter the term such as $\langle\Omega|\, \phi^n \,|\Omega\rangle$. For illustration, let us calculate $\langle\Omega|\, \phi_1\phi_2 \,|\Omega\rangle$. Thus we can write $\langle\Omega|\, \phi_1\phi_2 \,|\Omega\rangle$ in the new form as

$$\begin{aligned} \langle \phi_1\phi_2 \rangle &= \int d^4x \delta_{x1} \langle \phi_x\phi_2 \rangle \\ &= i \int d^4x (\Box_x D_{x1}) \langle \phi_x\phi_2 \rangle \\ &= i \int d^4x D_{x1} (\Box_x \langle \phi_x\phi_2 \rangle). \end{aligned} \tag{2.119}$$

where $\langle \phi_1\phi_2 \rangle$ is shorthand for $\langle\Omega|\, \phi_1\phi_2 \,|\Omega\rangle$ and integrated by parts in the last step. To go any further, we need help from the Schwinger-Dyson equation which takes the from

$$(\Box_x + m^2) \langle \phi_x\phi_1...\phi_n \rangle = \left\langle \frac{d\mathcal{L}_{int}[\phi_x]}{d\phi_x} \phi_1...\phi_n \right\rangle - i \sum_i \delta^4(x - x_i) \langle \phi_1\phi...\phi_{i-1}\phi_{i+1}...\phi_n \rangle \,, \tag{2.120}$$

where $\phi_x = \phi(x)$ and $\phi_i = \phi(x_i)$, for prove see [4][6]. Applying the Schwinger-Dyson equation, then equation 2.119 becomes

$$\begin{aligned} \langle \phi_1\phi_2 \rangle &= i \int d^4x D_{x1} (\Box_x \langle \phi_x\phi_2 \rangle) \\ &= \int d^4x D_{x1}\delta_{x2} - i \int d^4x D_{x1} \left\langle \frac{d\mathcal{L}_{int}[\phi_x]}{d\phi_x} \phi_2 \right\rangle \\ &= D_{12} - i \int d^4x D_{x1} \left\langle \frac{d\mathcal{L}_{int}[\phi_x]}{d\phi_x} \phi_2 \right\rangle. \end{aligned} \tag{2.121}$$

To see more clearly about this process, assuming that $\mathcal{L}_{int} = \frac{\alpha}{3!}\phi^3$, then we get

$$\begin{aligned} \langle \phi_1\phi_2 \rangle &= D_{12} - i \int d^4x D_{x1} \left\langle \Big(\frac{d}{d\phi_x} \frac{\alpha}{3!} \phi_x^3 \Big) \phi_2 \right\rangle \\ &= D_{12} - i\frac{\alpha}{2} \int d^4x D_{x1} \langle \phi_x^2 \phi_2 \rangle \\ &= D_{12} - i\frac{\alpha}{2} \int d^4x \int d^4y \delta_{2y} D_{x1} \langle \phi_x^2 \phi_y \rangle \,. \end{aligned} \tag{2.122}$$

Again, we use $i\Box_y D_{y2} = \delta_{2y}$ and integration by part. We obtain

$$\begin{aligned}\langle\phi_1\phi_2\rangle &= D_{12} - i\frac{\alpha}{2}\int d^4x \int d^4y \delta_{2y} D_{x1}\left\langle \phi_x^2\phi_y\right\rangle \\ &= D_{12} + \frac{\alpha}{2}\int d^4x \int d^4y D_{x1}(\Box_y D_{y2})\left\langle \phi_x^2\phi_y\right\rangle \\ &= D_{12} - \frac{\alpha}{2}\int d^4x \int d^4y D_{x1}D_{y2}\Box_y\left\langle \phi_x^2\phi_y\right\rangle .\end{aligned} \tag{2.123}$$

Using Schwinger-Dyson, we arrive at

$$\begin{aligned}\langle\phi_1\phi_2\rangle &= D_{12} - \frac{\alpha^2}{4}\int d^4x \int d^4y D_{x1}D_{y2}\left\langle \phi_x^2\phi_y^2\right\rangle + 2i\frac{g}{2}\int d^4x \int d^4y D_{x_1}D_{2y}\delta_{xy}\left\langle \phi_x\right\rangle \\ &= D_{12} - \frac{\alpha^2}{4}\int d^4x \int d^4y D_{x1}D_{y2}\left\langle \phi_x^2\phi_y^2\right\rangle + ig\int d^4x D_{x_1}D_{2x}\left\langle \phi_x\right\rangle .\end{aligned} \tag{2.124}$$

Using the same trick for the last two, we get

$$\langle\phi_1\phi_2\rangle = D_{12} - g^2\int d^4x \int d^4y\Big(\frac{1}{2}D_{1x}D_{xy}^2D_{y2} + \frac{1}{4}D_{1x}D_{xx}D_{yy}D_{y2} + \frac{1}{2}D_{1x}D_{2x}D_{xy}D_{yy}\Big). \tag{2.125}$$

To obtain the result, we have to work on many steps in the Schwinger-Dyson equation and integrate by parts. However, there exist a trick which is called contractions. This method provides the same result but easier than to calculate the old one.

A contraction is a mathematical trick which says picking any two fields in the series and displacing them by a propagator. There is the theorem that is called Wick's theorem. This theorem allows us to contract all the field and convert them to propagators. Wick's theorem says

$$T\{\phi_1\phi_2...\phi_n\} =: \phi_1\phi_2...\phi_n + \text{all contractions} :, \tag{2.126}$$

where the notation : ... : is called the normal ordering. When the normal ordering takes place, all the annihilation operator will move to the right-hand side regardless of any commutation relation, for example,

$$: (a_1^\dagger + a_1)(a_2^\dagger + a_2) := a_1^\dagger a_2^\dagger + a_1^\dagger a_2 + a_2^\dagger a_1 + a_1 a_2. \tag{2.127}$$

Thus, we have

$$\langle 0| : \phi_0(x_1)\phi_0(x_2)...\phi_0(x_n) : |0\rangle , \tag{2.128}$$

where $n > 1$. For Wick contraction illustration, let's calculate

$$\begin{aligned}\langle 0| T\{\phi_1\phi_2\phi_3\phi_4\} |0\rangle &= \langle 0| : \phi_1\phi_2\phi_3\phi_4 : + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 |0\rangle \\ &= \langle 0| \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 + \phi_1\phi_2\phi_3\phi_4 |0\rangle ,\end{aligned} \tag{2.129}$$

where the lines above the field are present to indicate contractions. To use Wick contraction, coming back to the equation 2.116, Taylor expansion this equation together with using Wick contraction, we get

$$\begin{aligned}\langle\Omega| T\{\phi(x_1)\phi(x_2)...\phi(x_n)\} |\Omega\rangle &= \frac{\langle 0| T\{\phi_0(x_1)\phi_0(x_2)...\phi_0(x_n)e^{i\int d^4x\mathcal{L}_{int}}\} |0\rangle}{\langle 0| T\{e^{-i\int d^4x\mathcal{L}_{int}}\} |0\rangle} \\ &= \frac{\langle 0| T\{\phi_0(x_1)...\phi(x_n)\} |0\rangle + \int d^4x \langle 0| T\{\phi_0(x_1)...\phi(x_n)\mathcal{L}_{int}\} |0\rangle\} |0\rangle + ...}{\langle 0|0\rangle + \int d^4x \langle 0| T\{\mathcal{L}_{int}\} |0\rangle + \frac{1}{2}\int d^4x \int d^4y \langle 0| T\{\mathcal{L}_{int}^2\} |0\rangle + ...}\end{aligned} \tag{2.130}$$

For simplicity, assuming that $\mathcal{L}_{int} = \frac{\alpha}{3!}\phi^3(x)$ and consider$\langle\Omega|\, T\{\phi(x_1)\phi(x_2)\}$, therefore the denominator is

$$\begin{aligned}
&\langle 0|T\{\phi_0(x_1)\phi_0(x_2)e^{i\int d^4x\mathcal{L}_{int}}\}\,|0\rangle \\
&= \langle 0|\, T\{\phi_0(x_1)\phi_0(x_2)\}\,|0\rangle + \frac{i\alpha}{3!}\int d^4x\, \langle 0|\, T\{\phi_0(x_1)\phi_0(x_2)\phi_0^3(x)\}\,|0\rangle \\
&\quad + \frac{1}{2}\Big(\frac{i\alpha}{3!}\Big)^2 \int d^4x \int d^4y\, \langle 0|\, T\{\phi_0(x_1)\phi_0(x_2)\phi_0^3(x)\phi_0^3(y)\}\,|0\rangle + \mathcal{O}(\alpha^3).
\end{aligned} \tag{2.131}$$

The second term must vanish since it contains odd numbers of fields. The odd number of fields will leave one redundant field. This redundant field cannot make the bra and ket to have the same state, and thus they must vanish. The numerator takes the form

$$\begin{aligned}
&\langle 0|T\{\phi_0(x_1)\phi_0(x_2)e^{i\int d^4x\mathcal{L}_{int}}\}\,|0\rangle \\
&= \langle 0|\, T\{\phi_0(x_1)\phi_0(x_2)\}\,|0\rangle + \frac{1}{2}\Big(\frac{i\alpha}{3!}\Big)^2 \int d^4x \int d^4y\, \langle 0|\, T\{\phi_0(x_1)\phi_0(x_2)\phi_0^3(x)\phi_0^3(y)\}\,|0\rangle\,,
\end{aligned} \tag{2.132}$$

where the higher order terms $\mathcal{O}(\alpha^3)$ can be ignored. From the Wick contraction, the first term becomes

$$D(x_1, x_2) \equiv D_{12}, \tag{2.133}$$

and the Wick contractions for the last term consist of 8 ways:

1. There are nines ways of contraction for $\langle 0|\, T\{\phi_0(x_1)\phi_0(x_2)\phi_0(x)\phi_0(x)\phi_0(x)\phi_0(y)\phi_0(y)\phi_0(y)\}\,|0\rangle$, and they give $9D_{12}D_{xx}D_{xy}D_{yy}$.

2. There are six ways of contraction for $\langle 0|\, T\{\phi_0(x_1)\phi_0(x_2)\phi_0(x)\phi_0(x)\phi_0(x)\phi_0(y)\phi_0(y)\phi_0(y)\}\,|0\rangle$, and they give $6D_{12}D_{xy}^3$.

3. There are 18 ways of contraction for $\langle 0|\, T\{\phi_0(x_1)\phi_0(x_2)\phi_0(x)\phi_0(x)\phi_0(x)\phi_0(y)\phi_0(y)\phi_0(y)\}\,|0\rangle$, and they give $18D_{1x}D_{2x}D_{xy}D_{yy}$.

4. There are 9 ways of contraction for $\langle 0|\, T\{\phi_0(x_1)\phi_0(x_2)\phi_0(x)\phi_0(x)\phi_0(x)\phi_0(y)\phi_0(y)\phi_0(y)\}\,|0\rangle$, and they give $18D_{1x}D_{2y}D_{xx}D_{yy}$.

5. There are 18 ways of contraction for $\langle 0|\, T\{\phi_0(x_1)\phi_0(x_2)\phi_0(x)\phi_0(x)\phi_0(x)\phi_0(y)\phi_0(y)\phi_0(y)\}\,|0\rangle$, and they give $18D_{1x}D_{2y}D_{xy}^2$.

6. There are 18 ways of contraction for $\langle 0|\, T\{\phi_0(x_1)\phi_0(x_2)\phi_0(x)\phi_0(x)\phi_0(x)\phi_0(y)\phi_0(y)\phi_0(y)\}\,|0\rangle$, and they give $18D_{1y}D_{2y}D_{xx}D_{xy}$.

7. There are 9 ways of contraction for $\langle 0|\, T\{\phi_0(x_1)\phi_0(x_2)\phi_0(x)\phi_0(x)\phi_0(x)\phi_0(y)\phi_0(y)\phi_0(y)\}\,|0\rangle$, and they give $9D_{1y}D_{2x}D_{xx}D_{yy}$.

8. There are 18 ways of contraction for $\langle 0|\, T\{\phi_0(x_1)\phi_0(x_2)\phi_0(x)\phi_0(x)\phi_0(x)\phi_0(y)\phi_0(y)\phi_0(y)\}\,|0\rangle$, and they give $18D_{1y}D_{2x}D_{xy}^2$.

However, the fourth and the seventh contraction, the fifth and the eighth contraction, and the third and the sixth contraction are the same since we integrate over x and y. Hence, these contractions are symmetric under exchange between x and y. Therefore, the numerator becomes

$$\begin{aligned}&\langle 0|T\{\phi_0(x_1)\phi_0(x_2)e^{i\int d^4x\mathcal{L}_{int}}\}|0\rangle\\&=D_{12}-\alpha^2\int d^4x\int d^4y\Big(\frac{1}{2}D_{1x}D_{xy}^2D_{y2}+\frac{1}{2}D_{1x}D_{2x}D_{xy}D_{yy}+\frac{1}{4}D_{1x}D_{xx}D_{yy}D_{y2}\\&+\frac{1}{8}D_{12}D_{xx}D_{xy}D_{yy}+\frac{1}{12}D_{12}D_{xy}^3\Big)\end{aligned} \tag{2.134}$$

The same goes for the denominator. We get

$$\begin{aligned}\langle 0|\,T\{e^{i\int d^4x\mathcal{L}_{int}}\}\,|0\rangle&=\langle 0|0\rangle-\frac{1}{2}\Big(\frac{\alpha}{3!}\Big)^2\int d^4xd^4y\,\langle 0|\,T\{\phi_0^3(x)\phi_0^3(y)\}\,|0\rangle\\&=1-\alpha^2\int d^4x\int d^4y\Big(\frac{1}{8}D_{xx}D_{xy}D_{yy}+\frac{1}{12}D_{xy}^3\Big).\end{aligned} \tag{2.135}$$

where we have ignored the higher order term α^3 and the first order of α vanishes for the same reason as the numerator. Since we work on a perturbative theory, we can use the $\frac{1}{1-\alpha^2x}\approx 1+\alpha^2x$ approximation for the denominator. Thus, we obtain

$$\begin{aligned}&\langle\Omega|T\{\phi(x_1)\phi(x_2)...\phi(x_n)\}|\Omega\rangle\\&=\Big(D_{12}-\alpha^2\int d^4x\int d^4y\Big(\frac{1}{2}D_{1x}D_{xy}^2D_{y2}+\frac{1}{2}D_{1x}D_{2x}D_{xy}D_{yy}+\frac{1}{4}D_{1x}D_{xx}D_{yy}D_{y2}\\&+\frac{1}{8}D_{12}D_{xx}D_{xy}D_{yy}+\frac{1}{12}D_{12}D_{xy}^3\Big)\Big)\Big(1+\alpha^2\int d^4x\int d^4y\Big(\frac{1}{8}D_{xx}D_{xy}D_{yy}+\frac{1}{12}D_{xy}^3\Big)\Big).\end{aligned} \tag{2.136}$$

We see that the terms such as $D_{12}D_{xx}D_{xy}D_{yy}$, and $D_{12}D_{xy}^3$ are canceled out. We call these terms bubbles which are the terms that have nothing to do with the propagation from point x_1 to point x_2. That is, some of their points don't involve the propagation of the field. In conclusion, we have

$$\langle\Omega|\,T\{\phi(x_1)...\phi(x_n)\}\,|\Omega\rangle=\langle 0|\,T\{\phi_0(x_1)...\phi_0(x_n)e^{i\int d^4x\mathcal{L}_{int}}\}\,|0\rangle_{\text{no bubbles}}, \tag{2.137}$$

where "no bubbles" refers that every propagator must involve an external point and be the part of diagrams [4]. We can use this equation to draw diagrams, Feynman diagrams, which is easier to handle than using a brute force calculation. However, to get Feynman diagrams, we must must evaluate this equation first by expanding it with Taylor's series and then we can map the mathematical terms to pictures representation which is called Feynman rules. But in this project, we will calculate the amplitude from the start because we don't possess the Feynman rules for the $s+\gamma\to s+g$ yet. So Feynman rules and diagrams are not necessary and not emphasize here. So brute force is unavoidable. For the next section, we are discussing about how to extract the equation to M-matrix or amplitude.

2.13 M-matrix Calculation

Consider the term $-\frac{\alpha^2}{2}\int d^4x\int d^4yD_{1x}D_{xy}^2D_{y2}$. Since this term comes from Taylor's expansion of $\langle f|\,S\,|i\rangle=\langle\Omega|\,T\{\phi_1\phi_x^3\phi_y^3\phi_2\}\,|\Omega\rangle$, which is the S-matrix elements, thus it is a transfer matrix, see section 2.7. To evaluate the transfer matrix, substituting the propagator

$$D_{xy} = \frac{1}{(2\pi)^4}\int d^4p \frac{i}{p^2+i\epsilon} e^{ip(x-y)}, \tag{2.138}$$

where we are assuming that mass equal to zero, into the transfer matrix, we obtain

$$\begin{aligned}
\mathcal{T} =& -\frac{\alpha^2}{2}\int d^4x \int d^4y \int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4p_2}{(2\pi)^4}\int \frac{d^4p_3}{(2\pi)^4}\int \frac{d^4p_4}{(2\pi)^4} e^{ip_1(x_1-x)}e^{ip_2(y-x_2)} \\
&\times e^{ip_3(x-y)}e^{ip_4(x-y)} \frac{i}{p_1^2+i\epsilon}\frac{i}{p_2^2+i\epsilon}\frac{i}{p_3^2+i\epsilon}\frac{i}{p_4^2+i\epsilon} \\
=& -\frac{\alpha^2}{2}\int d^4x \int d^4y \int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4p_2}{(2\pi)^4}\int \frac{d^4p_3}{(2\pi)^4}\int \frac{d^4p_4}{(2\pi)^4} e^{i(-p_1+p_3+p_4)x}e^{i(p_2-p_3-p_4)y}e^{ip_1x_1-p_2x_2} \\
&\times \frac{i}{p_1^2+i\epsilon}\frac{i}{p_2^2+i\epsilon}\frac{i}{p_3^2+i\epsilon}\frac{i}{p_4^2+i\epsilon}.
\end{aligned} \tag{2.139}$$

Now, integrate overall x and y, these integrals produce two delta functions $(2\pi)^4\delta^4(-p_1+p_3+p_4)$ and $(2\pi)^4\delta^4(p_2-p_3-p_4)$ respectively. Next, we integrate over p_3 then the first delta function gives $p_3 = p_1 - p_4$ and the second becomes $(2\pi)^4\delta^4(p_1-p_2)$. We have

$$\begin{aligned}
\mathcal{T} =& -\frac{\alpha^2}{2}\int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4p_2}{(2\pi)^4}\int \frac{d^4p_4}{(2\pi)^4} e^{ip_1x_1-p_2x_2}(2\pi)^4\delta^4(p_1-p_2) \\
&\times \frac{i}{p_1^2+i\epsilon}\frac{i}{p_2^2+i\epsilon}\frac{i}{(p_1-p_4)^2+i\epsilon}\frac{i}{p_4^2+i\epsilon}.
\end{aligned} \tag{2.140}$$

We can convert equation 2.140 to the S-matrix by introducing the LSZ formula, see [4], which takes the form

$$\langle f|\, S\, |i\rangle = \Big[i\int d^4x_1 e^{-ip_1x_1}(\Box_1+m^2)\Big]...\Big[i\int d^4x_n e^{ip_nx_n}(\Box_n+m^2)\Big]\langle\Omega|\, T\{\phi_1...\phi_n\}\,|\Omega\rangle\,, \tag{2.141}$$

for this situation the LSZ formula becomes

$$\begin{aligned}
\langle f|\, S\, |i\rangle &= \Big[i\int d^4x_1 e^{-ip_ix_1}(\Box_1+m^2)\Big]\Big[i\int d^4x_2 e^{ip_fx_2}(\Box_2+m^2)\Big]\langle\Omega|\, T\{\phi_1\phi_2\}\,|\Omega\rangle \\
&= \Big[i\int d^4x_1 e^{-ip_ix_1}\Box_1\Big]\Big[i\int d^4x_2 e^{ip_fx_2}\Box_2\Big]\mathcal{T},
\end{aligned} \tag{2.142}$$

where the fields are massless and p_i^ν and p_f^ν are the initial and final 4-momentum respectively. Substituting equation 2.140 into the LSZ formula, we obtain

$$\langle f|\, S\, |i\rangle = -\Big[\int d^4x_1 e^{-ip_ix_1}(ip_1)^2\Big]\Big[\int d^4x_2 e^{ip_2x_2}(ip_2)^2\Big]\mathcal{T}. \tag{2.143}$$

Then integrating over x_1 and x_2 which again give two delta function $(2\pi)^4\delta^4(p_1-p_i)$ and $(2\pi)^4\delta^4(p_2-p_f)$ respectively. Rearrange the result by integrating over p_1 and p_2 then get

$$\begin{aligned}
\langle f|\, S\, |i\rangle =& \frac{\alpha^2}{2}\int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4p_2}{(2\pi)^4}\int \frac{d^4p_4}{(2\pi)^4}(2\pi)^{12}e^{ip_1x_1-p_2x_2}\delta^4(p_1-p_2)\delta^4(p_1-p_i)\delta^4(p_2-p_f) \\
&\times p_i^2p_f^2\frac{i}{p_1^2+i\epsilon}\frac{i}{p_2^2+i\epsilon}\frac{i}{(p_1-p_4)^2+i\epsilon}\frac{i}{p_4^2+i\epsilon} \\
=& \frac{\alpha^2}{2}\int \frac{d^4p_4}{(2\pi)^4}(2\pi)^4\delta^4(p_i-p_f)p_i^2p_f^2\frac{i}{p_i^2+i\epsilon}\frac{i}{p_f^2+i\epsilon}\frac{i}{(p_i-p_4)^2+i\epsilon}\frac{i}{p_4^2+i\epsilon},
\end{aligned} \tag{2.144}$$

since we will take the ϵ to zero in the end which implies that $(p_i+i\epsilon)^2 = p_i^2$ and $(p_f+i\epsilon)^2 = p_f^2$. Thus we arrive at

$$\langle f|\, S\, |i\rangle = -\frac{\alpha^2}{2}\int \frac{d^4p_4}{(2\pi)^4}(2\pi)^4\delta^4(p_i - p_f)\frac{i}{(p_i-p_4)^2+i\epsilon}\frac{i}{p_4^2+i\epsilon}, \tag{2.145}$$

We can see that only the propagators, the terms that don't connect with the external points, $\frac{i}{(p_i-p_4)^2+i\epsilon}$ and $\frac{i}{p_4^2+i\epsilon}$ for this case, are left. From S-matrix expansion,

$$\mathcal{S} = \mathbb{1} + (2\pi)^4\delta^4(\sum p_i)(i\mathcal{M}), \tag{2.146}$$

this implies

$$i\mathcal{M} = -\frac{\alpha^2}{2}\int \frac{d^4p_4}{(2\pi)^4}\frac{i}{(p_i-p_4)^2+i\epsilon}\frac{i}{p_4^2+i\epsilon} \tag{2.147}$$

or

$$i\mathcal{M} = -\frac{\alpha^2}{2}\frac{i}{(p_i-p_4)^2+i\epsilon}\frac{i}{p_4^2+i\epsilon}, \tag{2.148}$$

in momentum space, where the terms that appear in integral are only propagator terms. We have learned that only the internal line contributes to the propagator terms and we can ignore the integration since we are always dealing with the M-matrix in momentum space. The example on this section is provided to make the adoption in chapter 3 clearer. More or less, we are ready for our task, the scalar-graviton scattering. In the next chapter, we are going to discuss about this event which hoping that all the basics that have been introduced so far will help ones reach the answer along with many mathematics calculations.

Chapter 3

Scalar-Graviton Scattering

This chapter is the main part of the project: the developments leading to graviton photo-production by scalar-photon scattering (section 3.4). To get there, section 3.1 provides tools and formulas from Compton amplitude in scalar-QED. Then, section 3.2 and 3.3 tells us how to impose interaction terms between scalar and graviton, an essential ingredient for the main calculation in section 3.4.

3.1 Scalar-QED & Compton Amplitude

To understand how to calculate scattering amplitude. Let's start with a simple one, Scalar Quantum Electrodynamics (SQED). SQED is a simple example which will provide us with some basic calculation in QFT. Compton scattering, $e^- + \gamma \to e^- + \gamma$, is a simple illustration in QED that can apply to other phenomena. For simplicity, we will calculate Compton scattering in SQED instead of QED, since in QED, we have to deal with a spinorial object which is more complicated than a scalar. From now on, in this section, we will treat an electron as a scalar particle. The Lagrangian in SQED takes the form

$$\mathcal{L}_{SQED} = -\frac{1}{4}F_{\mu\nu}^2 + \partial_\mu\phi^*\partial^\mu\phi - ieA_\alpha(\phi\partial^\alpha\phi^* - \phi^*\partial^\alpha\phi) + e^2A_\alpha^2\phi^*\phi - m^2\phi^*\phi. \tag{3.1}$$

Now we consider Compton amplitude, namely

$$\begin{aligned}\langle\Omega|\,T\{\phi(x_1)...\phi(x_n)\}\,|\Omega\rangle =& 1 + i\int d^4xT\{\langle p_f;\epsilon_f,k_f|\,\mathcal{L}_{int}(x)\,|p_i;k_i,\epsilon_i\rangle\} \\ &- \frac{1}{2}\int d^4x\int d^4yT\{\langle p_f;\epsilon_f,k_f|\,\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)\,|p_i;k_i,\epsilon_i\rangle\},\end{aligned} \tag{3.2}$$

where $\mathcal{L}_{int} = -ieA_\alpha(\phi\partial^\alpha\phi^* - \phi^*\partial^\alpha\phi) + e^2A_\alpha^2\phi^*\phi$. Since in equation 3.1, the last term is a mass term and the first two terms are a photon and a scalar propagator respectively. Consider the second term in equation 3.2. Substituting $\mathcal{L}_{int}$ into the $i\int d^4xT\{\langle p_f;\epsilon_f,k_f|\,\mathcal{L}_{int}(x)\,|p_i;k_i,\epsilon_i\rangle\}$ term in equation 3.2, we find

$$\begin{aligned}& i\int d^4xT\{\langle p_f;\epsilon_f,k_f| - ieA_\alpha(\phi\partial^\alpha\phi^* - \phi^*\partial^\alpha\phi) + e^2A_\alpha^2\phi^*\phi\,|p_i;k_i,\epsilon_i\rangle\} \\ &= ie^2\int d^4xT\{\langle p_f;\epsilon_f,k_f|\,A_\alpha^2\phi^*\phi\,|p_i;k_i,\epsilon_i\rangle\},\end{aligned} \tag{3.3}$$

where in the first term in the first line is equal 0. Since the first term composes of odd A_α field. Computing the remaining term by help of Wick contraction, we have

$$\begin{aligned} & ie^2 \int d^4x T\{\langle p_f;\epsilon_f,k_f|\, A_\alpha^2\phi^*\phi\, |p_i;k_i,\epsilon_i\rangle\} \\ &= ie^2 \int d^4x T\{\langle 0|\, A(x_2)\phi_f(x_2)A_\alpha(x)A^\alpha(x)\phi^*(x)\phi(x)\phi_i^*(x_1)A(x_1)\,|0\rangle\}. \end{aligned} \tag{3.4}$$

The above equation can contract in 2 ways, namely

$$A(x_2)\phi_f(x_2)A_\alpha(x)A^\alpha(x)\phi^*(x)\phi(x)\phi_i^*(x_1)A(x_1) \tag{3.5}$$

and

$$A(x_2)\phi_f(x_2)A_\alpha(x)A^\alpha(x)\phi^*(x)\phi(x)\phi_i^*(x_1)A(x_1). \tag{3.6}$$

However, the above contractions are exactly the same. Thus we only multiply by two instead of calculating the two terms. Now we have

$$\begin{aligned} & ie^2 \int d^4x T\{\langle p_f;\epsilon_f,k_f|\, A_\alpha^2\phi^*\phi\, |p_i;k_i,\epsilon_i\rangle\} \\ &= 2ie^2 \int d^4x T\{\langle 0|\, A(x_2)\phi_f(x_2)A_\alpha(x)A^\alpha(x)\phi^*(x)\phi(x)\phi_i^*(x_1)A(x_1)\,|0\rangle\}. \end{aligned} \tag{3.7}$$

Scalar field doesn't provide any extra factor since it composes of only creation and annihilation operators, namely

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p e^{-ipx} + b_p^\dagger e^{ipx})$$

and

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p^\dagger e^{ipx} + b_p e^{-ipx}).$$

But for photon field, A_μ, it provides extra terms that is photon polarization and it takes the form

$$A_\alpha = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\sum_{i=1}^{2}(\epsilon^i{}_\alpha(k)a_i(k)e^{-ikx} + \epsilon^{i*}{}_\alpha(k)a^\dagger{}_i(k)e^{ikx}).$$

In the above equation, A_α composes of polarization, ϵ_α and ϵ_α^*, besides $e^{\pm ikx}$, annihilation and creation operators. Thus we will have the extra factors which are ϵ_α and ϵ_α^*. Using these facts in equation 3.10 together with Wick contraction, we obtain

$$ie^2 \int d^4x T\{\langle p_f;\epsilon_f,k_f|\, A_\alpha^2\phi^*\phi\, |p_i;k_i,\epsilon_i\rangle\} = 2ie^2\epsilon_i^\alpha\epsilon_{f\alpha}^*. \tag{3.8}$$

This term is called seagull. In equation 3.8, we have two polarizations, one for ϵ and the other one for ϵ^*. Since Lagrangian interaction has two photon-fields, apart from the fields that come from ket and bra. Each of them provides only ϵ or ϵ^* depending on it is a creation or annihilation operator respectively. Next we consider the second order term in equation

3.2. Writing the $\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)$ in the exact form as

$$\begin{aligned}\mathcal{L}_{int}(x)\mathcal{L}_{int}(y) =& -e^2 A_\alpha(x)(\phi(x)\partial^\alpha\phi^*(x) - \phi^*(x)\partial^\alpha\phi(x))A_\mu(y)(\phi(y)\partial^\mu\phi^*(y) - \phi^*(y)\partial^\mu\phi(y)).\\ =& -e^2 A_\alpha(x)A_\mu(y)[\phi(x)\partial^\alpha\phi^*(x)\phi(y)\partial^\mu\phi^*(y) - \phi^*(x)\partial^\alpha\phi(x)\phi(y)\partial^\mu\phi^*(y)\\ &-\phi(x)\partial^\alpha\phi^*(x)\phi^*(y)\partial^\mu\phi(y) + \phi^*(x)\partial^\alpha\phi(x)\phi^*(y)\partial^\mu\phi(y)]\\ =& -e^2 A_\alpha(x)A_\mu(y)[\phi(x)\partial^\alpha\phi^*(x)\phi(y)\partial^\mu\phi^*(y) - 2\phi^*(x)\partial^\alpha\phi(x)\phi(y)\partial^\mu\phi^*(y)\\ &+\phi^*(x)\partial^\alpha\phi(x)\phi^*(y)\partial^\mu\phi(y)],\end{aligned} \tag{3.9}$$

wherein the second line to the third line we have change dummy indices in the second and the third term in the bracket and use the fact that we have integral over x and y thus x and y are dummy variables. To keep equation 3.9 and 3.8 in the same order of e, in equation 3.9 we have ignored the terms that have a power index of e exceed 2. Substitute this back to the last term of equation 3.2 and by ignoring the integrations. We can do this since the integration only provides us with some delta function which can be handled by hand. Thus we get

$$\begin{aligned}&-\frac{1}{2}T\{\langle p_f;\epsilon_f,k_f|\,\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)\,|p_i;k_i,\epsilon_i\rangle\}\\ =&\frac{1}{2}e^2 T\{\langle p_f;\epsilon_f,k_f|\,A_\alpha(x)A_\mu(y)[\phi(x)\partial^\alpha\phi^*(x)\phi(y)\partial^\mu\phi^*(y) - 2\phi^*(x)\partial^\alpha\phi(x)\phi(y)\partial^\mu\phi^*(y)\\ &+\phi^*(x)\partial^\alpha\phi(x)\phi^*(y)\partial^\mu\phi(y)]\,|p_i;k_i,\epsilon_i\rangle\}\\ =&\frac{1}{2}e^2 T\{\langle\Omega|\,A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi(x)\partial^\alpha\phi^*(x)\phi(y)\partial^\mu\phi^*(y)\\ &-2A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi^*(x)\partial^\alpha\phi(x)\phi(y)\partial^\mu\phi^*(y)\\ &+A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi^*(x)\partial^\alpha\phi(x)\phi^*(y)\partial^\mu\phi(y)\,|\Omega\rangle\}.\end{aligned} \tag{3.10}$$

Now use Wick contraction in the above equation. For the first term, we can contract them as follow

$$A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi(x)\partial^\alpha\phi^*(x)\phi(y)\partial^\mu\phi^*(y), \tag{3.11}$$

$$A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi(x)\partial^\alpha\phi^*(x)\phi(y)\partial^\mu\phi^*(y), \tag{3.12}$$

$$A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi(x)\partial^\alpha\phi^*(x)\phi(y)\partial^\mu\phi^*(y), \tag{3.13}$$

and

$$A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi(x)\partial^\alpha\phi^*(x)\phi(y)\partial^\mu\phi^*(y). \tag{3.14}$$

Note here that

$$p' = p_i + k_i = p_f + k_f,$$

$$p' = p_i - k_f = p_f - k_i,$$

$$p' = p_i - k_f = p_f - k_i,$$

and

$$p' = p_i + k_i = p_f + k_f$$

are 4-momentum conservation that agree with equation 3.11-3.14 respectively. For the second term, we have

$$A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi^*(x)\partial^\alpha\phi(x)\phi(y)\partial^\mu\phi^*(y), \tag{3.15}$$

$$A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi^*(x)\partial^\alpha\phi(x)\phi(y)\partial^\mu\phi^*(y). \tag{3.16}$$

$$A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi^*(x)\partial^\alpha\phi(x)\phi(y)\partial^\mu\phi^*(y), \tag{3.17}$$

and

$$A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi^*(x)\partial^\alpha\phi(x)\phi(y)\partial^\mu\phi^*(y). \tag{3.18}$$

$$p' = p_i + k_i = p_f + k_f,$$

$$p' = p_i - k_f = p_f - k_i,$$

$$p' = p_i - k_f = p_f - k_i,$$

and

$$p' = p_i + k_i = p_f + k_f$$

are 4-momentum conservation that agree with equation 3.15-3.18 respectively. For the last term, we get

$$A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi^*(x)\partial^\alpha\phi(x)\phi^*(y)\partial^\mu\phi(y), \tag{3.19}$$

$$A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi^*(x)\partial^\alpha\phi(x)\phi^*(y)\partial^\mu\phi(y). \tag{3.20}$$

$$A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi^*(x)\partial^\alpha\phi(x)\phi^*(y)\partial^\mu\phi(y), \tag{3.21}$$

and

$$A(x_f)A(x_i)A_\alpha(x)A_\mu(y)\phi(x_f)\phi^*(x_i)\phi^*(x)\partial^\alpha\phi(x)\phi^*(y)\partial^\mu\phi(y). \tag{3.22}$$

Again,

$$p' = p_i + k_i = p_f + k_f,$$

$$p' = p_i - k_f = p_f - k_i,$$

$$p' = p_i - k_f = p_f - k_i,$$

and

$$p' = p_i + k_i = p_f + k_f$$

are 4-momentum conservation that agree with equation 3.19-3.22 respectively. Using all above contractions and momentum conservation, equation 3.10 becomes

$$
\begin{aligned}
&-\frac{1}{2}T\{\langle p_f;\epsilon_f,k_f|\,\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)\,|p_i;k_i,\epsilon_i\rangle\}\\
=&e^2\frac{i}{2(p'^2-m^2)}(\overbrace{\epsilon^*_{f\mu}\epsilon_{i\alpha}(ip'^{\alpha})(ip^{\mu}_f)+\epsilon_{i\mu}\epsilon^*_{f\alpha}(ip'^{\alpha})(ip^{\mu}_f)+\epsilon^*_{f\mu}\epsilon_{i\alpha}(ip'^{\mu})(ip^{\alpha}_f)+\epsilon_{i\mu}\epsilon^*_{f\alpha}(ip'^{\mu})(ip^{\alpha}_f)}^{\text{First term}}\\
&-2\Big\{\overbrace{\epsilon^*_{f\mu}\epsilon_{i\alpha}(-ip^{\alpha}_i)(ip^{\mu}_f)+\epsilon_{i\mu}\epsilon^*_{f\alpha}(-ip^{\alpha}_i)(ip^{\mu}_f)+\epsilon^*_{f\mu}\epsilon_{i\alpha}(-ip'^{\alpha})(ip'^{\mu})+\epsilon_{i\mu}\epsilon^*_{f\alpha}(-ip'^{\alpha})(ip'^{\mu})}^{\text{Second term}}\Big\}\\
&+\overbrace{\epsilon^*_{f\mu}\epsilon_{i\alpha}(-ip^{\alpha}_i)(-ip'^{\mu})+\epsilon_{i\mu}\epsilon^*_{f\alpha}(-ip^{\alpha}_i)(-ip'^{\mu})+\epsilon^*_{f\mu}\epsilon_{i\alpha}(-ip^{\mu}_i)(-ip'^{\alpha})+\epsilon_{i\mu}\epsilon^*_{f\alpha}(-ip^{\mu}_i)(-ip'^{\alpha})}^{\text{Last term}}),
\end{aligned}
\tag{3.23}
$$

where p', p_i and p_f is propagator, initial and final momentum respectively. Since every term consists of a scalar propagator, $\frac{i}{p'^2-m^2}$, thus factorizing out the scalar propagator in the second line is a good choice. We can calculate denominator of $\frac{i}{p'^2-m^2}$ as

$$(p_i-k_f)^2-m^2=\cancelto{m^2}{p_i^2}-2p_i\cdot k_f+\cancelto{0}{k_f^2}-m^2=-2p_i\cdot k_f=-2p_f\cdot k_i$$

and

$$(p_i+k_i)^2-m^2=\cancelto{m^2}{p_i^2}+2p_i\cdot k_i+\cancelto{0}{k_i^2}-m^2=2p_i\cdot k_i=2p_f\cdot k_f.$$

Substitute the above answers to equation 3.23, rearranging equation 3.23 and using the fact that $k_i\cdot\epsilon_i=0$ and $k_f\cdot\epsilon^*_f=0$, then get

$$
\begin{aligned}
&-\frac{1}{2}T\{\langle p_f;\epsilon_f,k_f|\,\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)\,|p_i;k_i,\epsilon_i\rangle\}\\
=&-\frac{ie^2}{2}(\frac{\epsilon^*_f\cdot p_f\epsilon_i\cdot p_i}{2p_i\cdot k_i}+\frac{\epsilon_i\cdot p_f\epsilon^*_f\cdot p_i}{-2p_i\cdot k_f}+\frac{\epsilon_i\cdot p_f\epsilon^*_f\cdot p_i}{-2p_i\cdot k_f}+\frac{\epsilon_i\cdot p_i\epsilon^*_f\cdot p_f}{2p_i\cdot k_i}\\
&+2\Big\{\frac{\epsilon^*_f\cdot p_f\epsilon_i\cdot p_i}{2p_i\cdot k_i}+\frac{\epsilon_i\cdot p_f\epsilon^*_f\cdot p_i}{-2p_i\cdot k_f}+\frac{\epsilon_i\cdot p_f\epsilon^*_f\cdot p_i}{-2p_i\cdot k_f}+\frac{\epsilon_i\cdot p_i\epsilon^*_f\cdot p_f}{2p_i\cdot k_i}\Big\}\\
&+\frac{\epsilon^*_f\cdot p_f\epsilon_i\cdot p_i}{2p_i\cdot k_i}+\frac{\epsilon_i\cdot p_f\epsilon^*_f\cdot p_i}{-2p_i\cdot k_f}+\frac{\epsilon_i\cdot p_f\epsilon^*_f\cdot p_i}{-2p_i\cdot k_f}+\frac{\epsilon_i\cdot p_i\epsilon^*_f\cdot p_f}{2p_i\cdot k_i})\\
=&-2ie^2(\frac{\epsilon^*_f\cdot p_f\epsilon_i\cdot p_i}{p_i\cdot k_i}-\frac{\epsilon_i\cdot p_f\epsilon^*_f\cdot p_i}{p_i\cdot k_f}),
\end{aligned}
\tag{3.24}
$$

where the denominator of each term in the second line comes from the propagator in front of parenthesis. Now summing up every terms together, the seagull and equation 3.24, we obtain

$$i\mathcal{M}=iA_{S-compton}=2ie^2\epsilon^{\alpha}_i\epsilon^*_{f\alpha}-2ie^2(\frac{\epsilon^*_f\cdot p_f\epsilon_i\cdot p_i}{p_i\cdot k_i}-\frac{\epsilon_i\cdot p_f\epsilon^*_f\cdot p_i}{p_i\cdot k_f}).\tag{3.25}$$

Thus, the amplitude for scalar Compton scattering is

$$A_{S-compton}=2e^2(\frac{\epsilon_i\cdot p_f\epsilon^*_f\cdot p_i}{p_i\cdot k_f}-\frac{\epsilon^*_f\cdot p_f\epsilon_i\cdot p_i}{p_i\cdot k_i}+\epsilon_i\cdot\epsilon^*_f).\tag{3.26}$$

This illustration provides a brief example of how to calculate an amplitude. The result of this example does not end here. As we shall see in the future, this result will play a crucial role in section 3.4. For the next section, we are going to extract the interaction term of the gravitation by calculating the perturbation of the Einstein-Hilbert action.

3.2 Gravitational Coupling to Matter

Before going into our main topic, and to preparing some basic, let's talk about gravitational coupling to matter. In general relativity, we have an important tensor called a metric tensor which is represented by $g_{\mu\nu}$. The metric tensor is like a root of many things in relativity since it can tell us how our spacetime curves, and use to find other quantities in relativity. However, in Quantum field theory, we consider everything on a flat space so this forces us to use Minkowski metric, $\eta_{\mu\nu}$. To consider how a matter in a flat space interacts with gravity, we need to perturb Minkowski metric to keep the flat space properties and can use quantum field theory to approach problems. Thus $g_{\mu\nu}$ takes the form

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{3.27}$$

where $h_{\mu\nu}$ is a perturbative field. Notice that if we had chosen $g^{\mu\nu}$ to take the form

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu}, \tag{3.28}$$

we would have found that

$$\begin{aligned} g_{\mu\alpha}g^{\alpha\nu} &= (\eta_{\mu\alpha} + \kappa h_{\mu\alpha})(\eta^{\alpha\nu} - \kappa h^{\alpha\nu}) \\ &= \eta_{\mu\alpha}\eta^{\alpha\nu} + \kappa h_{\mu\alpha}\eta^{\alpha\nu} - \kappa\eta_{\mu\alpha}h^{\alpha\nu} - \kappa^2 h_{\mu\alpha}h^{\alpha\nu} \\ &= \delta^{\nu}{}_{\mu} - \kappa^2 h_{\mu\alpha}h^{\alpha\nu}. \end{aligned} \tag{3.29}$$

Since we know that $g_{\mu\alpha}g^{\alpha\nu}$ should be equal $\delta^{\nu}{}_{\mu}$ but in the above equation appear the term $-\kappa^2 h_{\mu\alpha}h^{\alpha\nu}$ which it doesn't vanish. So, instead, we must choose $g^{\mu\nu}$ to take the form

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\alpha}h_{\alpha}{}^{\nu} \tag{3.30}$$

to cancel $-\kappa^2 h_{\mu\alpha}h^{\alpha\nu}$ out, namely,

$$\begin{aligned} g_{\mu\alpha}g^{\alpha\nu} &= (\eta_{\mu\alpha} + \kappa h_{\mu\alpha})(\eta^{\alpha\nu} - \kappa h^{\alpha\nu} + \kappa^2 h^{\alpha\beta}h_{\beta}{}^{\nu}) \\ &= \eta_{\mu\alpha}\eta^{\alpha\nu} - \kappa\eta_{\mu\alpha}h^{\alpha\nu} + \kappa^2\eta_{\mu\alpha}h^{\alpha\beta}h_{\beta}{}^{\nu} + \kappa h_{\mu\alpha}\eta^{\alpha\nu} - \kappa^2 h_{\mu\alpha}h^{\alpha\nu} + \kappa^3 h_{\mu\alpha}h^{\alpha\beta}h_{\beta}{}^{\nu} \\ &= \delta^{\nu}{}_{\mu} - \kappa^3 h_{\mu\alpha}h^{\alpha\beta}h_{\beta}{}^{\nu} \end{aligned} \tag{3.31}$$

From the above calculation, it seems that we should keep writing down the higher order in $g^{\mu\nu}$ for which the higher order in $g_{\mu\alpha}g^{\alpha\nu}$ will cancel. Since we are interested in a perturbative calculation, we use only a leading order to calculate. Then $g^{\mu\nu}$ and $g_{\mu\nu}$ take the form

$$\begin{aligned} g_{\mu\nu} &= \eta_{\mu\nu} + \kappa h_{\mu\nu}. \\ g^{\mu\nu} &= \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^{\mu\alpha}h_{\alpha}{}^{\nu}. \end{aligned} \tag{3.32}$$

When a matter interacts with gravity, we can write it action in general relativity as

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R + \int d^4x\sqrt{-g}\mathcal{L}_m,$$

where $g \equiv \det g$ and $\mathcal{L}_m$ means matter Lagrangian. Now we can ignore the first term in the action since it is only a dynamical term for space-time. We can find interaction terms between gravity and matter by expanding $g_{\mu\nu}$ in $\sqrt{-g}\mathcal{L}_m$ and keep working on the first order of expansion. First, let we work out $\sqrt{-g}$,

$$
\begin{aligned}
\sqrt{-g} \equiv \sqrt{-\det(g)} &= \sqrt{-\det(\eta+\kappa h)} = e^{\ln\left(\sqrt{-\det(\eta+\kappa h)}\right)} \\
&= e^{\frac{1}{2}\ln(-\det(\eta+\kappa h))} \qquad ;\text{from}\det(\eta+\kappa h) = \det(\eta)\det\left(1+\eta^{-1}\kappa h\right) \\
&= e^{\frac{1}{2}\ln[-\det(\eta)(\det\left(1+\kappa\eta^{-1}h\right))]} \\
&= e^{\ln\sqrt{-\det\eta}+\frac{1}{2}\ln\left(\det\left(1+\kappa\eta^{-1}h\right)\right)} \quad ;\text{Because } \det(\eta) = -1 \\
&= \sqrt{-\det\eta}\, e^{\frac{1}{2}\ln\left(\det\left(1+\kappa\eta^{-1}h\right)\right)} \quad ;\text{from } \ln(\det A) = \mathrm{Tr}(\ln A) \\
&= \sqrt{-\det\eta}\, e^{\frac{1}{2}\mathrm{Tr}\ln\left(1+\kappa\eta^{-1}h\right)} \quad ;\text{from } \ln(1+x) \approx x - \frac{x^2}{2} \\
&\approx e^{\frac{1}{2}\mathrm{Tr}\left\{(\kappa\eta^{-1}h)-\frac{(\kappa\eta^{-1}h)^2}{2}\right\}} \\
&\approx 1 + \frac{1}{2}\mathrm{Tr}\left(\kappa\eta^{-1}h - \frac{(\kappa\eta^{-1}h)^2}{2}\right) \quad ;\text{because } \eta^{-1} = \eta \\
&= 1 + \frac{1}{2}\mathrm{Tr}\left(\kappa\eta_{\mu\alpha}h^{\alpha\nu} - \frac{(\kappa\eta_{\mu\alpha}h^{\alpha\nu})^2}{2}\right) = 1 + \frac{1}{2}\kappa h^{\mu}{}_{\mu} - \frac{\kappa^2 h^{\mu\nu}h_{\mu\nu}}{4}
\end{aligned}
\tag{3.33}
$$

Assume that we work with Scalar particle then $\mathcal{L}_m = \partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi$, we get

$$
\begin{aligned}
\sqrt{-g}\mathcal{L}_m &= (1 + \frac{1}{2}\kappa h^{\alpha}{}_{\alpha} - \frac{\kappa^2 h^{\alpha\beta}h_{\alpha\beta}}{4})(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi) \\
&= (1 + \frac{1}{2}\kappa h^{\alpha}{}_{\alpha} - \frac{\kappa^2 h^{\alpha\beta}h_{\alpha\beta}}{4})(g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi) \\
&= (1 + \frac{1}{2}\kappa h^{\alpha}{}_{\alpha} - \frac{\kappa^2 h^{\alpha\beta}h_{\alpha\beta}}{4})((\eta^{\mu\nu} - \kappa h^{\mu\nu})\partial_\mu\phi^*\partial_\nu\phi - m^2\phi^*\phi) \\
&= (1 + \frac{1}{2}\kappa h^{\alpha}{}_{\alpha} - \frac{\kappa^2 h^{\alpha\beta}h_{\alpha\beta}}{4})(\underbrace{\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi}_{\text{Klein-Gordon } \mathcal{L}_m} - \kappa h^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi).
\end{aligned}
\tag{3.34}
$$

Since we are interested in the first order of the interaction then the above equation takes the form

$$
\begin{aligned}
\sqrt{-g}\mathcal{L}_m &= \mathcal{L}_m + \frac{1}{2}\kappa h^{\alpha}{}_{\alpha}(\partial_\mu\phi^*\partial^\mu\phi - m^2\phi^*\phi) - \kappa h^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi - \frac{\kappa^2 h^{\alpha\beta}h_{\alpha\beta}}{4}\mathcal{L}_m \\
&= \mathcal{L}_m + \frac{1}{2}\kappa h^{\alpha}{}_{\alpha}\mathcal{L}_m - \frac{\kappa^2 h^{\alpha\beta}h_{\alpha\beta}}{4}\mathcal{L}_m - \frac{\kappa}{2}h^{\mu\nu}(\partial_\mu\phi^*\partial_\nu\phi + \partial_\nu\phi^*\partial_\mu\phi) \\
&= \mathcal{L}_m - \frac{\kappa}{2}h_{\mu\nu}(-\eta^{\mu\nu}\mathcal{L}_m + \frac{\kappa h^{\mu\nu}}{2}\mathcal{L}_m + \partial^\mu\phi^*\partial^\nu\phi - \partial^\nu\phi^*\partial^\mu\phi) \\
&= \mathcal{L}_m - \frac{\kappa}{2}h_{\mu\nu}(-\eta^{\mu\nu}\mathcal{L}_m + \kappa h^{\mu\nu}\mathcal{L}_m - \frac{\kappa h^{\mu\nu}}{2}\mathcal{L}_m + \partial^\mu\phi^*\partial^\nu\phi - \partial^\nu\phi^*\partial^\mu\phi) \\
&= \mathcal{L}_m - \frac{\kappa}{2}h_{\mu\nu}(-\eta^{\mu\nu}\mathcal{L}_m + \partial^\mu\phi^*\partial^\nu\phi - \partial^\nu\phi^*\partial^\mu\phi) - \underbrace{\frac{\kappa^2 h^{\mu\nu}h_{\mu}{}^{\nu}}{4}\mathcal{L}_m}_{2^{nd}\text{ of }\kappa\approx 0} \\
&= \mathcal{L}_m - \frac{\kappa}{2}h_{\mu\nu}(-\eta^{\mu\nu}\mathcal{L}_m + \partial^\mu\phi^*\partial^\nu\phi - \partial^\nu\phi^*\partial^\mu\phi) \\
&= \mathcal{L}_m - \frac{\kappa}{2}h_{\mu\nu}T^{\mu\nu},
\end{aligned}
\tag{3.35}
$$

where the last term in the last line is a matter-gravitation coupling term. Noting that in general $T^{\mu\nu}$ in the above equation is $T^{\mu\nu}$ from matter which interacts with gravity. Although the calculation that shown above is a free scalar field which is a special case, however, the interaction term $-\frac{\kappa}{2}h_{\mu\nu}T^{\mu\nu}$ which we get here is a general one. To be precise, we can substitute the energy-momentum tensor of any theory in which we are interested. In the next section, we are finding the energy-momentum tensor for scalar-QED.

3.3 Energy Momentum Tensor for Scalar-QED

From Scalar-QED, we have $\mathcal{L}$ that takes the form

$$\mathcal{L}_{SQED} = -\frac{1}{4}F_{\mu\nu}^2 + \partial_\mu\phi^*\partial^\mu\phi - ieA_\alpha(\phi\partial^\alpha\phi^* - \phi^*\partial^\alpha\phi) + e^2A_\alpha^2\phi^*\phi - m^2\phi^*\phi. \tag{3.36}$$

We can find $T_{\mu\nu}$ from

$$T_{\mu\nu} = 2\frac{\delta\mathcal{L}_{SQED}}{\delta g^{\mu\nu}} - g_{\mu\nu}\mathcal{L}_{SQED}. \tag{3.37}$$

First, consider $\frac{\delta\mathcal{L}_{SQED}}{\delta g^{\mu\nu}}$ part, we can write $\mathcal{L}_{SQED}$ in the form

$$\begin{aligned}\mathcal{L}_{SQED} =& -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi \\ & - ieg^{\mu\nu}[A_\mu(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi)] + e^2g^{\mu\nu}A_\mu A_\nu\phi^*\phi - m^2\phi^*\phi.\end{aligned} \tag{3.38}$$

We should write the above equation in the form

$$\begin{aligned}\mathcal{L}_{SQED} =& -\frac{1}{4}g^{\mu\alpha}g^{\nu\beta}F_{\mu\nu}F_{\alpha\beta} + \frac{1}{2}g^{\mu\nu}\partial_\mu\phi^*\partial_\nu\phi + \frac{1}{2}g^{\mu\nu}\partial_\nu\phi^*\partial_\mu\phi \\ & - \frac{1}{2}ieg^{\mu\nu}[A_\mu(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi] - \frac{1}{2}ieg^{\mu\nu}[A_\nu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi] \\ & + \frac{1}{2}e^2g^{\mu\nu}A_\mu A_\nu\phi^*\phi + \frac{1}{2}e^2g^{\mu\nu}A_\nu A_\mu\phi^*\phi - m^2\phi^*\phi\end{aligned} \tag{3.39}$$

which is preserved the symmetric property of $T_{\mu\nu}$ after functional derivative with respect to $g^{\mu\nu}$. Then calculate the $T_{\mu\nu}$, we get

$$\begin{aligned}T_{\mu\nu} =& \overbrace{-F_{\mu\alpha}F^\alpha{}_\nu}^{\text{Electromagnetic}} + \overbrace{\partial_\mu\phi^*\partial_\nu\phi + \partial_\nu\phi^*\partial_\mu\phi}^{\text{Scalar}} - g_{\mu\nu}\mathcal{L}_{SQED} \\ & - \underbrace{ie[A_\mu(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi) + A_\nu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi)] + e^2A_\mu A_\nu\phi^*\phi + e^2A_\nu A_\mu\phi^*\phi}_{\text{Scalar-QED interaction}}\end{aligned} \tag{3.40}$$

We can substitute this term back into equation 3.35 which gives the gravitation scalar-QED interaction terms. Now, we can deal with the main calculation since we have already had the Lagrangian interaction. For the next section, we are going to use the Lagrangian interaction to obtain the amplitude for $s + \gamma \to s + g$ event where s, γ and g is a scalar particle, a photon, and a graviton respectively.

3.4 Graviton production via Photon-Scalar Scattering

The amplitude and interaction term alone doesn't tell us anything because we can't measure it. Unlike the amplitude, we can measure the cross-section in a laboratory and compare it with the one from our calculation. Thus, in this section, we are going to find the cross-section for the $s + \gamma \to s + g$ event. We can write interaction terms from both electrodynamics and gravity in the form

$$\mathcal{L}_{int} = \overbrace{-\frac{1}{2}\kappa h^{\mu\nu}T_{\mu\nu}}^{\text{gravity}} + \underbrace{eA_\alpha j^\alpha}_{\text{electromagnetic}} . \tag{3.41}$$

From Scalar-QED, we know that $eA_\alpha j^\alpha$ take the form

$$eA_\alpha j^\alpha = -ieA_\alpha(\phi\partial^\alpha\phi^* - \phi^*\partial^\alpha\phi) + e^2A_\alpha^2\phi^*\phi, \tag{3.42}$$

when we substitute $T_{\mu\nu}$ and $eA_\alpha j^\alpha$ into $\mathcal{L}_{int}$, we find

$$\begin{aligned}\mathcal{L}_{int} = &-\frac{1}{2}\kappa h^{\mu\nu}\{-F_{\mu\alpha}F^{\alpha}{}_{\nu} + \partial_\mu\phi^*\partial_\nu\phi + \partial_\nu\phi^*\partial_\mu\phi - ieA_\mu(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi) - ieA_\nu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi) \\ &- g_{\mu\nu}\mathcal{L}_{SQED}\} - ieA_\alpha(\phi\partial^\alpha\phi^* - \phi^*\partial^\alpha\phi) + e^2A_\alpha^2\phi^*\phi,\end{aligned} \tag{3.43}$$

We can extract Feynman rules and Feynman diagrams from

$$\begin{aligned}&\langle\Omega| T\{\phi(x_1)...\phi(x_n)\} |\Omega\rangle \\ &= \frac{\langle 0| T\{\phi_0(x_1)...\phi_0(x_n)e^{i\int d^4x\mathcal{L}_{int}[\phi_0]}\} |0\rangle}{\langle 0| T\{e^{i\int d^4x\mathcal{L}_{int}[\phi_0]}\} |0\rangle} \\ &= \langle 0| T\{\phi_0(x_1)...\phi_0(x_n)e^{i\int d^4x\mathcal{L}_{int}}\} |0\rangle_{\text{connected graph}} \\ &\approx 1 + i\langle 0| T\{\phi_0(x_1)...\phi_0(x_n)\int d^4x\mathcal{L}_{int}\} |0\rangle - \frac{1}{2}\langle 0| T\{\phi_0(x_1)...\phi_0(x_n)(\int d^4x\mathcal{L}_{int})^2\} |0\rangle + ... \\ &= 1 + i\langle 0| T\{\phi_0(x_1)...\phi_0(x_n)\int d^4x\mathcal{L}_{int}\} |0\rangle \\ &\quad -\frac{1}{2}\int d^4x\int d^4y\,\langle 0| T\{\phi_0(x_1)...\phi_0(x_n)\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)\} |0\rangle ,\end{aligned} \tag{3.44}$$

where Taylor's expansion is used in the third line. Since we are interested in photon scatter with scalar to produce a graviton which is $\phi\gamma \to \phi g$, we consider only in the first order of κ. Then expansion in the above equation can be written as

$$\begin{aligned}&\langle\Omega| T\{\phi(x_1)...\phi(x_n)\} |\Omega\rangle \\ &= 1 + i\int d^4x T\{\langle p_f;\epsilon_f\epsilon_f, k_f| \mathcal{L}_{int}(x) |p_i;k_i,\epsilon_i\rangle\} \\ &\quad -\frac{1}{2}\int d^4x\int d^4y T\{\langle p_f;\epsilon_f\epsilon_f, k_f| \mathcal{L}_{int}(x)\mathcal{L}_{int}(y) |p_i;k_i,\epsilon_i\rangle\},\end{aligned} \tag{3.45}$$

when $p_f, \epsilon_f\epsilon_f, k_f, \epsilon_i, k_i$ and p_i are momentum of scalar particle in final state, helicity of graviton in final state, 4-momentum of graviton in final state, helicity of photon in initial state, 4-momentum of photon in initial state and 4-momentum of scalar particle in initial state respectively. Note here that, since we are interested in the leading order term, we are going to consider only the terms which have the factor $e\kappa$, where we are ignoring the terms such as $e\kappa^2$, $e^2\kappa$, etc. Recalling the gauge conditions, we have $h^{\mu}{}_{\mu} = 0$ this makes $g_{\mu\nu}h^{\mu\nu} = 0$. Thus, the term $h^{\mu\nu}g_{\mu\nu}\mathcal{L}_{SQED}$ in $T^{\mu\nu}$, which occurs in $\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)$, can be ignored. First, we consider $h^{\mu\nu}F_{\mu\nu}F^{\alpha}eA_\alpha j^\alpha$ part from $\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)$. We call this term

$\gamma - pole$. Beware of the $\frac{1}{2}$ from this term since it comes from $(-\frac{1}{2}\kappa h^{\mu\nu}T_{\mu\nu} + eA_\alpha j^\alpha)^2 = (-\frac{1}{2}\kappa h^{\mu\nu}T_{\mu\nu})^2 - \kappa e h^{\mu\nu}T_{\mu\nu}A_\alpha j^\alpha + (eA_\alpha j^\alpha)^2$. The term takes the form

$$
\begin{aligned}
&T\{\langle p_f;\epsilon_f\epsilon_f,k_f|\,\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)\,|p_i;k_i,\epsilon_i\rangle_{\gamma-pole}\}\\
&= -T\{\langle p_f;\epsilon_f\epsilon_f,k_f|\,\kappa\,\overbrace{h^{\mu\nu}[F_{\mu\alpha}F^\alpha{}_\nu)]}^{\text{function of x}}\ \overbrace{eA_\alpha j^\alpha}^{\text{function of y}}\ |p_i;k_i,\epsilon_i\rangle\}\\
&= -e\kappa T\{\langle p_f;\epsilon_f\epsilon_f,k_f|\,h^{\mu\nu}[F_{\mu\alpha}F^\alpha{}_\nu]A_\alpha j^\alpha\,|p_i;k_i,\epsilon_i\rangle\}\\
&= -e\kappa T\{\langle p_f;\epsilon_f\epsilon_f,k_f|\,h^{\mu\nu}[\partial_\mu A_\alpha\partial^\alpha A_\nu - \partial_\alpha A_\mu\partial^\alpha A_\nu - \partial_\mu A_\alpha\partial_\nu A^\alpha + \partial_\alpha A_\mu\partial_\nu A^\alpha]A_\alpha j^\alpha\,|p_i;k_i,\epsilon_i\rangle\}.
\end{aligned}
\tag{3.46}
$$

Before going any further let us calculate $\partial_\mu A_\alpha$. We know that A_α has the form

$$
A_\alpha = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\sum_{i=1}^{2}(\epsilon^i{}_\alpha(k)a_i(k)e^{-ikx} + \epsilon^{i*}{}_\alpha(k)a^\dagger{}_i(k)e^{ikx}). \tag{3.47}
$$

Thus

$$
\partial_\mu A_\alpha = \int \frac{d^3k}{(2\pi)^3}\frac{1}{\sqrt{2\omega_k}}\sum_{i=1}^{2}(-ik_\mu\epsilon^i{}_\alpha(k)a_i(k)e^{-ikx} + ik_\mu\epsilon^{i*}{}_\alpha(k)a^\dagger{}_i(k)e^{ikx}). \tag{3.48}
$$

Substituting the above equation into $\langle p_f;\epsilon_f\epsilon_f,k_f|\,\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)\,|p_i;k_i,\epsilon_i\rangle$, we get

$$
\begin{aligned}
&T\{\langle p_f;\epsilon_f\epsilon_f,k_f|\,\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)\,|p_i;k_i,\epsilon_i\rangle_{\gamma-pole}\}\\
&= -e\kappa T\{\langle p_f;\epsilon_f\epsilon_f,k_f|\,h^{\mu\nu}[\partial_\mu A_\alpha\partial^\alpha A_\nu - \partial_\alpha A_\mu\partial^\alpha A_\nu - \partial_\mu A_\alpha\partial_\nu A^\alpha + \partial_\alpha A_\mu\partial_\nu A^\alpha]A_\gamma j^\gamma\,|p_i;k_i,\epsilon_i\rangle\}\\
&= -2e\kappa\ \overbrace{\epsilon_f^{\mu*}\epsilon_f^{\nu*}}^{\text{from } h^{\mu\nu}}\ \langle p_f|\,[(-ik_\mu\epsilon_\alpha + ik_\mu\epsilon^*_\alpha)(-ik^\alpha\epsilon_\nu + ik^\alpha\epsilon^*_\nu) - (-ik_\alpha\epsilon_\mu + ik_\alpha\epsilon^*_\mu)(-ik^\alpha\epsilon_\nu + ik^\alpha\epsilon^*_\nu)\\
&\quad - (-ik_\mu\epsilon_\alpha + ik_\mu\epsilon^*_\alpha)(-ik_\nu\epsilon^\alpha + ik_\nu\epsilon^{\alpha*}) + (-ik_\alpha\epsilon_\mu + ik_\alpha\epsilon^*_\mu)(-ik_\nu\epsilon^\alpha + ik_\nu\epsilon^{\alpha*})][A_\gamma j^\gamma]\,|p_i;k_i,\epsilon_i\rangle\,.
\end{aligned}
\tag{3.49}
$$

Noting that as a shorthand we use $\epsilon^*_{i,f}$ and $\epsilon_{i,f}$ instead of $\epsilon^*_{i,f}a^\dagger_{i,f}$ and $\epsilon_{i,f}a_{i,f}$ and leave $e^{\pm ikx}$ alone since it only gives energy and momentum conservation when we integrate over d^3k in which we already force energy-momentum to be conserved in a calculation. Here we choose only terms that have the form $\epsilon\epsilon^*$ otherwise the equation will vanish. In the third line, factor 2 comes from Wick contraction. To understand where this factor comes from, we will show the calculation only one term. We consider

$$
T\{\langle p_f;\epsilon_f\epsilon_f,k_f|\,\partial_\mu A_\alpha\partial^\alpha A_\nu\,|p_i;k_i,\epsilon_i\rangle\} = T\{\langle p_f;\epsilon_f\epsilon_f,k_f|\,\partial_\mu A_\alpha(x)\partial^\alpha A_\nu(x)A_\gamma A_i(x_1)\,|p_i\rangle\}
$$

where A_i is the photon field that has 4-momentum equal to k_i. The above equation consists of photon fields which can contract in the following way,

$$
\partial_\mu A_\alpha(x)\partial^\alpha A_\nu(x)A_\gamma A_i(x_1)
$$

and

$$
\partial_\mu A_\alpha(x)\partial^\alpha A_\nu(x)A_\gamma A_i(x_1).
$$

Now, let's continue our calculation,

$$
\begin{aligned}
&\langle p_f;\epsilon_f\epsilon_f,k_f|\,\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)\,|p_i;k_i,\epsilon_i\rangle_{\gamma-pole}\\
&= -2e\kappa\epsilon_f^{\mu*}\epsilon_f^{\nu*}\langle p_f|\,[(k_\mu\epsilon_\alpha k^{\alpha*}\epsilon^*_\nu + k^*_\mu\epsilon^*_\alpha k^\alpha\epsilon_\nu) - (k_\alpha\epsilon_\mu k^{\alpha*}\epsilon^*_\nu + k^*_\alpha\epsilon^*_\mu k^\alpha\epsilon_\nu) - (k_\mu\epsilon_\alpha k^*_\nu\epsilon^{\alpha*} + ik^*_\mu\epsilon^*_\alpha k_\nu\epsilon^\alpha)\\
&\quad + (k_\alpha\epsilon_\mu k^*_\nu\epsilon^{\alpha*} + k^*_\alpha\epsilon^*_\mu k_\nu\epsilon^\alpha)][A_\gamma j^\gamma]\,|p_i;k_i,\epsilon_i\rangle\,.
\end{aligned}
\tag{3.50}
$$

To avoid confusion, we put $*$ on k that multiplied by ϵ^* in the equation above to distinguish k that is multiplied by ϵ and ϵ^*. Because we use a scalar particle to scatter with a photon, then A_γ should be an annihilation operator that destroys a photon and becomes a photon propagator which j^γ is the scalar particle in SQED part. Thus ϵ^* in the above equation should be a creation operator of a photon which is a part of the propagator. We know that $A_\gamma j^\gamma = -iA_\gamma(\phi\partial^\gamma\phi^* - \phi^*\partial^\gamma\phi) + eA_\gamma^2\phi^*\phi$ and

$$\langle 0| T\{A_\gamma A_\nu\} |0\rangle = \frac{-ig_{\gamma\nu}}{k^2 + i\epsilon}. \tag{3.51}$$

To find $\langle p_f| j_\alpha |p_i\rangle$, we use the fact that

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p e^{-ipx} + b_p^\dagger e^{ipx}) \tag{3.52}$$

and

$$\phi^*(x) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}(a_p^\dagger e^{ipx} + b_p e^{-ipx}). \tag{3.53}$$

Since we are interested only the term that takes the form $a_p^\dagger a_p$ or $b_p^\dagger b_p$ (but don't forget to add time order, T), we get

$$\langle p_f| T\{j_\gamma\} |p_i\rangle = \langle p_f| T\{-i(\phi\partial^\gamma\phi^* - \phi^*\partial^\gamma\phi) + eA_\gamma\phi^*\phi\} |p_i\rangle \tag{3.54}$$

We are not interested in the term $eA_\gamma\phi^*\phi$ since it contains only one photon field which is vanished. Hence, we have

$$\begin{aligned}\langle p_f| T\{j^\gamma\} |p_i\rangle &= -i\langle p_f| T\{(\phi\partial^\gamma\phi^* - \phi^*\partial^\gamma\phi)\} |p_i\rangle \\ &= \langle p_f| T\{(a + b^\dagger)(p^\gamma a^\dagger - p^\gamma b) - (a^\dagger + b)(-p^\gamma a + p^\gamma b^\dagger)\} |p_i\rangle \\ &= \langle p_f| T\{(ap'^\gamma a'^\dagger - b^\dagger p'^\gamma b' + a^\dagger p'^\gamma a' - bp'^\gamma b'^\dagger)\} |p_i\rangle \\ &= (p_i^\gamma + p_f^\gamma)\end{aligned} \tag{3.55}$$

Now we match A_γ and ϵ^* in $\langle p_f; \epsilon_f\epsilon_f, k_f| \mathcal{L}_{int}(x)\mathcal{L}_{int}(y) |p_i; k_i, \epsilon_i\rangle$ to form a propagator because A_γ should be annihilated the photon in the propagator. We denote k^* by p' because k^* must be 4-momentum of the photon propagator since it comes from the exponential term of the annihilation part of the photon propagator. Thus we get

$$\begin{aligned}&\langle p_f; \epsilon_f\epsilon_f, k_f| \mathcal{L}_{int}(x)\mathcal{L}_{int}(y) |p_i; k_i, \epsilon_i\rangle_{\gamma-pole} \\ &= -2e\kappa(p_i^\gamma + p_f^\gamma)\Bigg((\epsilon_f^*\cdot p')(\epsilon_f^*\cdot\epsilon_i)\left(k_i^\mu\frac{-ig_{\mu\gamma}}{p'^2+i\epsilon}\right) + (k_i\cdot\epsilon_f^*)(\epsilon_i\cdot p')\left(\epsilon_f^{\nu*}\frac{-ig_{\nu\gamma}}{p'^2+i\epsilon}\right) \\ &\quad - (p'\cdot k_i)(\epsilon_f^*\cdot\epsilon_i)\left(\epsilon_f^{\mu*}\frac{-ig_{\mu\gamma}}{p'^2+i\epsilon}\right) - (k_i\cdot p')(\epsilon_f^*\cdot\epsilon_i)\left(\epsilon_{f_\nu}^*\frac{-ig_{\nu\gamma}}{p'^2+i\epsilon}\right) - (p'\cdot\epsilon_f^*)(k_i\cdot\epsilon_f^*)\left(\epsilon_i^\alpha\frac{-ig_{\alpha\gamma}}{p'^2+i\epsilon}\right) \\ &\quad - (k_i\cdot\epsilon_f^*)(p'\cdot\epsilon_f^*)\left(\epsilon_{i\alpha}\frac{-ig_\gamma^\alpha}{p'^2+i\epsilon}\right) + (p'\cdot\epsilon_i)(k_i\cdot\epsilon_f^*)\left(\epsilon_f^{\mu*}\frac{-ig_{\mu\gamma}}{p'^2+i\epsilon}\right) + (\epsilon_i\cdot\epsilon_f^*)(p'\cdot\epsilon_f^*)\left(k_{i\alpha}\frac{-ig_\gamma^\alpha}{p'^2+i\epsilon}\right)\Bigg) \\ &= \frac{4e\kappa}{p'^2+i\epsilon}\big(k_{i\gamma}(\epsilon_i\cdot\epsilon_f^*)(p'\cdot\epsilon_f^*) + \epsilon_{f_\gamma}^*(k_i\cdot\epsilon_f^*)(\epsilon_i\cdot p') - \epsilon_{f_\gamma}^*(p'\cdot k_i)(\epsilon_f^*\cdot\epsilon_i) - \epsilon_{i\gamma}(p'\cdot\epsilon_f^*)(k_i\cdot\epsilon_f^*)\big) \\ &\quad\times[i(p_i^\gamma + p_f^\gamma)],\end{aligned} \tag{3.56}$$

where we have use $g_{\mu\alpha}A^\alpha = A_\mu$ in the last line. From the conservation of 4-momentum, we know that $p' = k_i - k_f$, $k_{i,f}\cdot k_{i,f} = 0$ and $k_{i,f}\cdot\epsilon_{i,f} = 0$ since a photon is massless their polarization must be orthogonal to their momentum. Thus $p'^2 = -2k_i \cdot k_f$ since ϵ is just a conventional constant which we are going to take its limit to zero in the end. We get

$$\begin{aligned}
= &-\frac{2ie\kappa}{k_i \cdot k_f}[k_{i\gamma}(\epsilon_i \cdot \epsilon_f^*)(k_i \cdot \epsilon_f^*) - k_{i\gamma}(\epsilon_i \cdot \epsilon_f^*)\cancelto{0}{(k_f \cdot \epsilon_f^*)} \\
&+ \epsilon^*_{f_\gamma}(k_i \cdot \epsilon_f^*)\cancelto{0}{(\epsilon_i \cdot k_i)} - \epsilon^*_{f_\gamma}(k_i \cdot \epsilon_f^*)(\epsilon_i \cdot k_f) \\
&- \epsilon^*_{f_\gamma}\cancelto{0}{(k_i \cdot k_i)}(\epsilon_f^* \cdot \epsilon_i) + \epsilon^*_{f_\gamma}(k_f \cdot k_i)(\epsilon_f^* \cdot \epsilon_i) \\
&- \epsilon_{i\gamma}(k_i \cdot \epsilon_f^*)(k_i \cdot \epsilon_f^*) + \epsilon_{i\gamma}\cancelto{0}{(k_f \cdot \epsilon_f^*)}(k_i \cdot \epsilon_f^*)][p_i^\gamma + p_f^\gamma] \\
= &-\frac{2ie\kappa}{k_i \cdot k_f}[\epsilon^*_{\gamma f}(k_f \cdot k_i \epsilon_f^* \cdot \epsilon_i - \epsilon_f^* \cdot k_i \epsilon_i \cdot k_f) + \Big(k_i \cdot \epsilon_f^*\Big)\Big(k_{i\gamma}(\epsilon_i \cdot \epsilon_f^*) - \epsilon_{i\gamma}(k_i \cdot \epsilon_f^*)\Big)][p_i^\gamma + p_f^\gamma].
\end{aligned} \tag{3.57}$$

This is the amplitude for $\gamma - pole$. To be clear, we will write it again as

$$\begin{aligned}
A_{\gamma-pole} = &-\frac{1}{2i}\left\langle p_f; \epsilon_f\epsilon_f, k_f\right| \mathcal{L}_{int}(x)\mathcal{L}_{int}(y)\left|p_i; k_i, \epsilon_i\right\rangle_{\gamma-pole} \\
= &\frac{e\kappa}{2k_i \cdot k_f}[\epsilon_f^* \cdot (p_i + p_f)(k_f \cdot k_i \epsilon_f^* \cdot \epsilon_i - \epsilon_f^* \cdot k_i \epsilon_i \cdot k_f) \\
&+ \Big(k_i \cdot \epsilon_f^*\Big)\Big(k_i \cdot (p_i + p_f)(\epsilon_i \cdot \epsilon_f^*) - \epsilon_i \cdot (p_i + p_f)(k_i \cdot \epsilon_f^*)\Big)],
\end{aligned} \tag{3.58}$$

where i in the first line comes from $\mathcal{S} = 1 + i\delta^4(\Sigma p)\mathcal{M}$. We must divide our solution by i to get an amplitude since an amplitude is equal to $\mathcal{M}$. Next, we consider the first order of Taylor's series, equation 3.45, which takes the form $i\int d^4x \left\langle p_f; \epsilon_f\epsilon_f, k_f\right| \mathcal{L}_{int}(x)\left|p_i; k_i, \epsilon_i\right\rangle$. We know that our result doesn't vanish if and only if it is composed of even number of fields. Hence, we get only two terms for Lagrangian $\frac{1}{2}\kappa h^{\mu\nu} ieA_\mu(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi)$ and $\frac{1}{2}\kappa h^{\mu\nu} ieA_\nu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi)$. We will call this term seagull. We can calculate it as follow

$$\begin{aligned}
&i\int d^4x \left\langle p_f; \epsilon_f\epsilon_f, k_f\right| \mathcal{L}_{int}(x)\left|p_i; k_i, \epsilon_i\right\rangle_{seagull} \\
&= -\frac{e}{2}\kappa \left\langle p_f; \epsilon_f\epsilon_f, k_f\right| T\{h^{\mu\nu}A_\mu(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi) + h^{\mu\nu}A_\nu(\phi\partial_\mu\phi^* - \phi^*\partial_\mu\phi)\}\left|p_i; k_i, \epsilon_i\right\rangle \\
&= -e\kappa \left\langle p_f; \epsilon_f\epsilon_f, k_f\right| T\{h^{\mu\nu}A_\mu(\phi\partial_\nu\phi^* - \phi^*\partial_\nu\phi)\}\left|p_i; k_i, \epsilon_i\right\rangle \\
&= -e(\kappa\epsilon_f^{\mu*}\epsilon_f^{\nu*})(\epsilon_{\mu i})\left\langle p_f\right| T\{(a_p + b_p^\dagger)(ip_\nu a_p^\dagger - ip_\nu b_p) - (a_p^\dagger + b_p)(-ip_\nu a_p + ip_\nu b_p^\dagger)\}\left|p_i\right\rangle \\
&= -ie(\kappa\epsilon_f^{\mu*}\epsilon_f^{\nu*})(\epsilon_{\mu i})\left\langle p_f\right| T\{(a_p + b_p^\dagger)(p_\nu a_p^\dagger - p_\nu b_p) - (a_p^\dagger + b_p)(-p_\nu a_p + p_\nu b_p^\dagger)\}\left|p_i\right\rangle \\
&= -ie\kappa(\epsilon_f^{\mu*}\epsilon_f^{\nu*})(\epsilon_{\mu i})[p_{\nu i} + p_{\nu f}] \\
&= -ie\kappa(\epsilon_f^* \cdot \epsilon_i)(\epsilon_f^* \cdot (p_i + p_f)).
\end{aligned} \tag{3.59}$$

Now we have amplitude for seagull,

$$A_{seagull} = -e\kappa(\epsilon_f^* \cdot \epsilon_i)(\epsilon_f^* \cdot (p_i + p_f)). \tag{3.60}$$

Let us consider the term $(-\frac{1}{2}\kappa h^{\mu\nu}T_{\mu\nu} + eA_\alpha j^\alpha)^2 = (-\frac{1}{2}\kappa h^{\mu\nu}T_{\mu\nu})^2 - \kappa e h^{\mu\nu}T_{\mu\nu}A_\alpha j^\alpha + (eA_\alpha j^\alpha)^2$ again which is part of $\mathcal{L}^2_{int}$. We consider $-e\kappa A^\gamma j_\gamma h^{\mu\nu}(\partial_\mu\phi^*\partial_\nu\phi + \partial_\nu\phi^*\partial_\mu\phi)$. We call these terms A-Born and B-born terms which you will see at the moment this term will contribute to two different diagrams. For A-Born,

$$\begin{aligned}
&T\{\langle p_f;\epsilon_f\epsilon_f,k_f|\,\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)\,|p_i;k_i,\epsilon_i\rangle\}_{A-Born}\\
&=-e\kappa T\{\langle p_f;\epsilon_f\epsilon_f,k_f|\,A^\gamma(x)j_\gamma(x)h^{\mu\nu}(y)(\partial_\mu\phi^*(y)\partial_\nu\phi(y)+\partial_\nu\phi^*(y)\partial_\mu\phi(y))\,|p_i;k_i,\epsilon_i\rangle\}\\
&=-2e\kappa\epsilon_f^{*\mu}(y)\epsilon_f^{*\nu}(y)\epsilon_i^\gamma(x)T\{\langle p_f|\,j_\gamma(x)(\partial_\mu\phi^*(y)\partial_\nu\phi(y))\,|p_i\rangle\}\\
&=2ie\kappa\epsilon_f^{*\mu}(y)\epsilon_f^{*\nu}(y)\epsilon_i^\gamma(x)T\{\langle p_f|\,(\phi(x)\partial_\gamma\phi^*(x)-\phi^*(x)\partial_\gamma\phi(x))(\partial_\mu\phi^*(y)\partial_\nu\phi(y))\,|p_i\rangle\}\\
&=2ie\kappa\epsilon_f^{*\mu}(y)\epsilon_f^{*\nu}(y)\epsilon_i^\gamma(x)T\{\langle p_f|\,\phi(x)\partial_\gamma\phi^*(x)\partial_\mu\phi^*(y)\partial_\nu\phi(y)-\phi^*(x)\partial_\gamma\phi(x)\partial_\mu\phi^*(y)\partial_\nu\phi(y)\,|p_i\rangle\}\\
&=2ie\kappa\epsilon_f^{*\mu}(y)\epsilon_f^{*\nu}(y)\epsilon_i^\gamma(x)T\{\langle\Omega|\,\phi_f(x_2)[\phi(x)\partial_\gamma\phi^*(x)\partial_\mu\phi^*(y)\partial_\nu\phi(y)\\
&\quad-\phi^*(x)\partial_\gamma\phi(x)\partial_\mu\phi^*(y)\partial_\nu\phi(y)]\phi_i^*(x_1)\,|\Omega\rangle\}.
\end{aligned}\tag{3.61}$$

We can contract each term in 2 ways such that they will give us a connected graph. The first term can contract as

$$\phi_f(x_2)\phi(x)\partial_\gamma\phi^*(x)\partial_\mu\phi^*(y)\partial_\nu\phi(y)\phi_i^*(x_1)\tag{3.62}$$

and

$$\phi_f(x_2)\phi(x)\partial_\gamma\phi^*(x)\partial_\mu\phi^*(y)\partial_\nu\phi(y)\phi_i^*(x_1).\tag{3.63}$$

For the second term, they give contractions as

$$\phi_f(x_2)\phi^*(x)\partial_\gamma\phi(x)\partial_\mu\phi^*(y)\partial_\nu\phi(y)\phi_i^*(x_1)\tag{3.64}$$

and

$$\phi_f(x_2)\phi^*(x)\partial_\gamma\phi(x)\partial_\mu\phi^*(y)\partial_\nu\phi(y)\phi_i^*(x_1).\tag{3.65}$$

The contractions in 3.62 and 3.64 are similar since they have x_1 which propagate from x to y, and graviton is emitted at the y point. Thus we group 3.62 and 3.64 together and call this term A-Born. From contraction, we have scalar propagator and three terms for 4-momentum. Since each term has three derivatives, for A-Born, we have

$$\begin{aligned}
&T\{\langle p_f;\epsilon_f\epsilon_f,k_f|\,\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)\,|p_i;k_i,\epsilon_i\rangle\}_{A-Born}\\
&=2ie\kappa\epsilon_f^{*\mu}(y)\epsilon_f^{*\nu}(y)\epsilon_i^\gamma(x)\Big[(ip_{f\mu})(ip'_\gamma)(-ip'_\nu)\frac{i}{p'^2-m^2}-(ip_{f\mu})(-ip'_\nu)(-ip_{i\gamma})\frac{i}{p'^2-m^2}\Big]\\
&=-2ie\kappa\epsilon_f^{*\mu}(y)\epsilon_f^{*\nu}(y)\epsilon_i^\gamma(x)\frac{1}{p'^2-m^2}\Big[p_{f\mu}p'_\gamma p'_\nu+p_{f\mu}p'_\nu p_{i\gamma}\Big]\\
&=-2ie\kappa\frac{p_f\cdot\epsilon_f^*}{p'^2-m^2}\Big[p'\cdot\epsilon_i p'\cdot\epsilon_f^*+p'\cdot\epsilon_f^* p_i\cdot\epsilon_i\Big].
\end{aligned}\tag{3.66}$$

From the conservation of momentum, we have $p'=p_i+k_i=p_f+k_f$, and masslessness gives us the orthogonal of k and ϵ. Substituting this back to 3.67, we get

$$\begin{aligned}
&T\{\langle p_f;\epsilon_f\epsilon_f,k_f|\,\mathcal{L}_{int}(x)\mathcal{L}_{int}(y)\,|p_i;k_i,\epsilon_i\rangle\}_{A-Born}\\
&=-2ie\kappa\frac{p_f\cdot\epsilon_f^*}{p'^2-m^2}\Big[(p_i+k_i)\cdot\epsilon_i(p_f+k_f)\cdot\epsilon_f^*+(p_f+k_f)\cdot\epsilon_f^* p_i\cdot\epsilon_i\Big]\\
&=-2ie\kappa\frac{p_f\cdot\epsilon_f^*}{p'^2-m^2}\Big[p_i\cdot\epsilon_i p_f\cdot\epsilon_f^*+p_f\cdot\epsilon_f^* p_i\cdot\epsilon_i\Big]\\
&=-4ie\kappa\frac{(p_f\cdot\epsilon_f^*)^2(p_i\cdot\epsilon_i)}{p'^2-m^2}.
\end{aligned}\tag{3.67}$$

Here we can find $p'^2 - m^2$ from

$$p'^2 - m^2 = (p_i + k_i)^\mu (p_i + k_i)_\mu - m^2 = p_i^2 + 2p_i \cdot k_i + k_i^2 - m^2 = m^2 + 2p_i \cdot k_i + 0 - m^2 = 2p_i \cdot k_i.$$

Hence we get the A-Born amplitude,

$$\begin{aligned} A_{A-Born} &= -\frac{1}{2i} T\{\langle p_f; \epsilon_f \epsilon_f, k_f | \mathcal{L}_{int}(x) \mathcal{L}_{int}(y) | p_i; k_i, \epsilon_i \rangle\}_{A-Born} \\ &= e\kappa \frac{(p_f \cdot \epsilon_f^*)^2 (p_i \cdot \epsilon_i)}{p_i \cdot k_i}. \end{aligned} \tag{3.68}$$

Next, we group the remaining terms together, that is a contraction in 3.63 and 3.65. We call this B-Born. We obtain

$$\begin{aligned} &T\{\langle p_f; \epsilon_f \epsilon_f, k_f | \mathcal{L}_{int}(x) \mathcal{L}_{int}(y) | p_i; k_i, \epsilon_i \rangle\}_{B-Born} \\ &= 2ie\kappa \epsilon_f^{*\mu}(y) \epsilon_f^{*\nu}(y) \epsilon_i^\gamma(x) \frac{i}{p'^2 - m^2} \Big[(-ip_{i\nu})(ip'_\mu)(ip_{f\gamma}) - (-ip_{i\nu})(ip'_\mu)(-ip'_\gamma) \Big] \\ &= -2ie\kappa \frac{(p_i \cdot \epsilon_f^*)(p' \cdot \epsilon_f^*)}{p'^2 - m^2} \Big[p_f \cdot \epsilon_i + p' \cdot \epsilon_i \Big]. \end{aligned} \tag{3.69}$$

Now again, from the conservation of momentum, we use the fact that $p' = p_i - k_f = p_f - k_i$. Substituting p' back to equation 3.69, we get

$$\begin{aligned} &T\{\langle p_f; \epsilon_f \epsilon_f, k_f | \mathcal{L}_{int}(x) \mathcal{L}_{int}(y) | p_i; k_i, \epsilon_i \rangle\}_{B-Born} \\ &= -2ie\kappa_f \frac{(p_i \cdot \epsilon_f^*)(p_i - k_f) \cdot \epsilon_f^*}{p'^2 - m^2} \Big[p_f \cdot \epsilon_i + (p_f - k_i) \cdot \epsilon_i \Big] \\ &= -4ie\kappa \frac{(p_i \cdot \epsilon_f^*)^2 p_f \cdot \epsilon_i}{p'^2 - m^2}. \end{aligned} \tag{3.70}$$

Since $p'^2 - m^2$ can be calculated as follow

$$p'^2 - m^2 = (p_i - k_f)^\mu (p_i - k_f)_\mu - m^2 = p_i^2 - 2p_i \cdot k_f + k_f^2 - m^2 = m^2 - 2p_i \cdot k_f + 0 - m^2 = -2p_i \cdot k_f,$$

the B-Born amplitude takes the form

$$\begin{aligned} A_{B-Born} &= -\frac{1}{2i} T\{\langle p_f; \epsilon_f \epsilon_f, k_f | \mathcal{L}_{int}(x) \mathcal{L}_{int}(y) | p_i; k_i, \epsilon_i \rangle\}_{B-Born} \\ &= -e\kappa \frac{(p_i \cdot \epsilon_f^*)^2 p_f \cdot \epsilon_i}{p_i \cdot k_f}. \end{aligned} \tag{3.71}$$

Next, to get the final answer, we will sum over all each amplitude that we have calculated so far. The total amplitude is

$$\begin{aligned} A_{tot} =& A_{seagull} + A_{\gamma-pole} + A_{A-Born} + A_{B-Born} \\ =& -e\kappa(\epsilon_f^* \cdot \epsilon_i)(\epsilon_f^* \cdot (p_i + p_f)) + \frac{e\kappa}{2k_i \cdot k_f} \Big[\epsilon_f^* \cdot (p_i + p_f)(k_f \cdot k_i \epsilon_f^* \cdot \epsilon_i - \epsilon_f^* \cdot k_i \epsilon_i \cdot k_f) \\ &+ \Big(k_i \cdot \epsilon_f^* \Big) \Big(k_i \cdot (p_i + p_f)(\epsilon_i \cdot \epsilon_f^*) - \epsilon_i \cdot (p_i + p_f)(k_i \cdot \epsilon_f^*) \Big) \Big] + e\kappa \frac{(p_f \cdot \epsilon_f^*)^2 (p_i \cdot \epsilon_i)}{p_i \cdot k_i} \\ &- e\kappa \frac{(p_i \cdot \epsilon_f^*)^2 p_f \cdot \epsilon_i}{p_i \cdot k_f}. \end{aligned} \tag{3.72}$$

For simplicity, we will calculate the total amplitude in the CM frame. We will calculate some properties that we have in this frame. We can choose a coordinate such that the event after scattering line on a z-plane and after being on an x-z plane. Thus we have, 4-momentum before scattering, $k_i^\mu = (E_\gamma, 0, 0, k_i)$ and $p_i^\mu = (E_p, 0, 0-k_i)$ for photon and scalar respectively. After scattering, we have $p_f = (E'_p, -p\sin\theta, 0, -p\cos\theta)$ and $k_f = (E'_\gamma, p\sin\theta, 0, p\cos\theta)$. Photon has 2 polarization, left and right. To avoid confusion of convention, we will call + and - polarization instead of left and right. The polarizations before scattering can be written as

$$\epsilon_i^\pm = \frac{\mp}{\sqrt{2}}(\hat{x} \pm i\hat{y}) \tag{3.73}$$

and after scattering as

$$\epsilon_f^\pm = \frac{\mp}{\sqrt{2}}(\cos\theta\hat{x} \pm i\hat{y} - \sin\theta\hat{z}). \tag{3.74}$$

Thus we have

$$p_i \cdot \epsilon_f^{\pm *} = -k_i \cdot \epsilon_f^{\pm *} = \pm k_i \sin\theta \tag{3.75}$$

and

$$p_i \cdot \epsilon_i = p_f \cdot \epsilon_f^* = 0. \tag{3.76}$$

We can write equation 3.75 in a simple way as

$$p_i \cdot \epsilon_f^* = -k_i \cdot \epsilon_f^*. \tag{3.77}$$

Substituting the above equations to 3.72, we get

$$\begin{aligned}
A_{tot} =& - e\kappa(\epsilon_f^* \cdot \epsilon_i)(\epsilon_f^* \cdot p_i + \cancelto{0}{\epsilon_f^* \cdot p_f}) + \frac{e\kappa}{2k_i \cdot k_f}\Big[(\epsilon_f^* \cdot p_i + \cancelto{0}{\epsilon_f^* \cdot p_f})(k_f \cdot k_i \epsilon_f^* \cdot \epsilon_i - \epsilon_f^* \cdot k_i \epsilon_i \cdot k_f) \\
&+ \Big(k_i \cdot \epsilon_f^*\Big)\Big(k_i \cdot (p_i + p_f)(\epsilon_i \cdot \epsilon_f^*) - \epsilon_i \cdot (p_i + p_f)(k_i \cdot \epsilon_f^*)\Big)\Big] + e\kappa\frac{(p_f \cdot \epsilon_f^*)^2 \cancelto{0}{(p_i \cdot \epsilon_i)}}{p_i \cdot k_i} \\
&- e\kappa\frac{(p_i \cdot \epsilon_f^*)^2 p_f \cdot \epsilon_i}{p_i \cdot k_f} \\
=& - e\kappa(\epsilon_f^* \cdot \epsilon_i)(\epsilon_f^* \cdot p_i) + \frac{e\kappa}{2k_i \cdot k_f}\Big[(\epsilon_f^* \cdot p_i)(k_f \cdot k_i \epsilon_f^* \cdot \epsilon_i - \epsilon_f^* \cdot k_i \epsilon_i \cdot k_f) \\
&+ \Big(k_i \cdot \epsilon_f^*\Big)\Big(k_i \cdot (p_i + p_f)(\epsilon_i \cdot \epsilon_f^*) - \epsilon_i \cdot (p_i + p_f)(k_i \cdot \epsilon_f^*)\Big)\Big] - e\kappa\frac{(p_i \cdot \epsilon_f^*)^2 p_f \cdot \epsilon_i}{p_i \cdot k_f} \\
=& - e\kappa(\epsilon_f^* \cdot \epsilon_i)(\epsilon_f^* \cdot p_i) + \frac{e\kappa}{2}(\epsilon_f^* \cdot p_i)(\epsilon_f^* \cdot \epsilon_i) + \frac{e\kappa}{2k_i \cdot k_f}\Big[-\overbrace{(\epsilon_f^* \cdot p_i)}^{=-\epsilon_f^* \cdot k_i}(\epsilon_f^* \cdot k_i \epsilon_i \cdot k_f) \\
&+ \Big(k_i \cdot \epsilon_f^*\Big)\Big(k_i \cdot (p_i + p_f)(\epsilon_i \cdot \epsilon_f^*) - \epsilon_i \cdot (p_i + p_f)(k_i \cdot \epsilon_f^*)\Big)\Big] - e\kappa\frac{(p_i \cdot \epsilon_f^*)^2 p_f \cdot \epsilon_i}{p_i \cdot k_f} \\
=& - \frac{e\kappa}{2}(\epsilon_f^* \cdot p_i)(\epsilon_f^* \cdot \epsilon_i) + \frac{e\kappa}{2k_i \cdot k_f}\Big[(\epsilon_f^* \cdot k_i)^2 \epsilon_i \cdot (k_f - p_i - p_f) + (k_i \cdot \epsilon_f^*)k_i \cdot (p_i + p_f)(\epsilon_i \cdot \epsilon_f^*)\Big] \\
&- e\kappa\frac{(p_i \cdot \epsilon_f^*)^2 p_f \cdot \epsilon_i}{p_i \cdot k_f}.
\end{aligned} \tag{3.78}$$

Since the third term comes from $\gamma - pole$ and uses the fact that $k_f - p_i = k_i - p_f$, we can rearrange the amplitude as follow

$$
\begin{aligned}
A_{tot} =& -\frac{e\kappa}{2}(\epsilon_f^* \cdot p_i)(\epsilon_f^* \cdot \epsilon_i) + \frac{e\kappa}{2k_i \cdot k_f}\Big[(\epsilon_f^* \cdot k_i)^2(\cancelto{0}{\epsilon_i \cdot k_i} - 2\epsilon_i \cdot p_f) + (k_i \cdot \epsilon_f^*)k_i \cdot (p_i + p_f)(\epsilon_i \cdot \epsilon_f^*)\Big] \\
& - e\kappa\frac{(p_i \cdot \epsilon_f^*)^2 p_f \cdot \epsilon_i}{p_i \cdot k_f} \\
=& -\frac{e\kappa}{2}(\epsilon_f^* \cdot p_i)(\epsilon_f^* \cdot \epsilon_i) + \frac{e\kappa}{2k_i \cdot k_f}\Big[-2\overbrace{(\epsilon_f^* \cdot k_i)^2}^{=(\epsilon_f^* \cdot p_i)^2}(\epsilon_i \cdot p_f) + (k_i \cdot \epsilon_f^*)k_i \cdot (p_i + p_f)(\epsilon_i \cdot \epsilon_f^*)\Big] \\
& - e\kappa\frac{(p_i \cdot \epsilon_f^*)^2 p_f \cdot \epsilon_i}{p_i \cdot k_f} \\
=& -\frac{e\kappa}{2}(\epsilon_f^* \cdot p_i)(\epsilon_f^* \cdot \epsilon_i) - \frac{e\kappa}{k_i \cdot k_f}(\epsilon_f^* \cdot p_i)^2(\epsilon_i \cdot p_f) - e\kappa\frac{(p_i \cdot \epsilon_f^*)^2 p_f \cdot \epsilon_i}{p_i \cdot k_f} \\
& + \Big[\frac{e\kappa}{2k_i \cdot k_f}(k_i \cdot \epsilon_f^*)k_i \cdot (p_i + p_f)(\epsilon_i \cdot \epsilon_f^*)\Big] \\
=& -\frac{e\kappa}{2}(\epsilon_f^* \cdot p_i)(\epsilon_f^* \cdot \epsilon_i) - e\kappa(\epsilon_f^* \cdot p_i)^2(\epsilon_i \cdot p_f)(\frac{1}{k_i \cdot k_f} + \frac{1}{p_i \cdot k_f}) \\
& + \Big[\frac{e\kappa}{2k_i \cdot k_f}(k_i \cdot \epsilon_f^*)k_i \cdot (p_i + p_f)(\epsilon_i \cdot \epsilon_f^*)\Big].
\end{aligned}
\tag{3.79}
$$

Recall that $p_i + k_i = p_f + k_f$. Rearranging this equation, we get

$$p_i = k_f + p_f - k_i$$

$$p_i + p_f = k_f - k_i + 2p_f.$$

Contract the above equation with k_i and use the fact that $k_i^2 = 0$. Then we have

$$k_i \cdot (p_i + p_f) = k_i \cdot k_f + 2p_f \cdot k_i.$$

Substituting the above equation back to the last term of equation 3.79, then get

$$
\begin{aligned}
A_{tot} =& -\frac{e\kappa}{2}(\epsilon_f^* \cdot p_i)(\epsilon_f^* \cdot \epsilon_i) - e\kappa(\epsilon_f^* \cdot p_i)^2(\epsilon_i \cdot p_f)(\frac{1}{k_i \cdot k_f} + \frac{1}{p_i \cdot k_f}) \\
& + \frac{e\kappa}{2k_i \cdot k_f}\Big[k_i \cdot k_f \overbrace{(k_i \cdot \epsilon_f^*)}^{=-(\epsilon_f^* \cdot p_i)}(\epsilon_i \cdot \epsilon_f^*) + 2p_f \cdot k_i(k_i \cdot \epsilon_f^*)(\epsilon_i \cdot \epsilon_f^*)\Big] \\
=& -e\kappa\overbrace{(\epsilon_f^* \cdot p_i)}^{=-(\epsilon_f^* \cdot k_i)}(\epsilon_f^* \cdot \epsilon_i) - e\kappa\overbrace{(\epsilon_f^* \cdot p_i)^2}^{=(\epsilon_f^* \cdot k_i)^2}(\epsilon_i \cdot p_f)\frac{\overbrace{(k_i + p_i)}^{=p_f + k_f} \cdot k_f}{k_i \cdot k_f p_i \cdot k_f} \\
& + \frac{e\kappa}{k_i \cdot k_f}\Big[p_f \cdot k_i(k_i \cdot \epsilon_f^*)(\epsilon_i \cdot \epsilon_f^*)\Big] \\
=& e\kappa(\epsilon_f^* \cdot k_i)(\epsilon_f^* \cdot \epsilon_i) + \frac{e\kappa}{k_i \cdot k_f}\Big[p_f \cdot k_i(k_i \cdot \epsilon_f^*)(\epsilon_i \cdot \epsilon_f^*)\Big] \\
& - e\kappa(\epsilon_f^* \cdot k_i)^2(\epsilon_i \cdot p_f)\frac{p_f \cdot k_f}{k_i \cdot k_f p_i \cdot k_f} \\
=& e\kappa\frac{\epsilon_f^* \cdot k_i}{k_i \cdot k_f}\Big[(\epsilon_f^* \cdot \epsilon_i)(k_i \cdot k_f) + p_f \cdot k_i(\epsilon_i \cdot \epsilon_f^*) - (\epsilon_f^* \cdot k_i)(\epsilon_i \cdot p_f)\frac{p_f \cdot k_f}{p_i \cdot k_f}\Big]
\end{aligned}
\tag{3.80}
$$

$$
\begin{aligned}
A_{tot} =& e\kappa\frac{\epsilon_f^*\cdot k_i}{k_i\cdot k_f}\Big[k_i\cdot\underbrace{(k_f+p_f)}_{=k_i+p_i}(\epsilon_f^*\cdot\epsilon_i)-\underbrace{(\epsilon_f^*\cdot k_i)}_{=-(\epsilon_f^*\cdot p_i)}(\epsilon_i\cdot p_f)\frac{p_f\cdot k_f}{p_i\cdot k_f}\Big]\\
=& e\kappa\frac{\epsilon_f^*\cdot k_i}{k_i\cdot k_f}\Big[(k_i\cdot p_i)(\epsilon_f^*\cdot\epsilon_i)+(\epsilon_f^*\cdot p_i)(\epsilon_i\cdot p_f)\frac{p_f\cdot k_f}{p_i\cdot k_f}\Big],
\end{aligned}
$$

where we have used the fact that $k_f^2=0$ in the second line and $k_i^2=0$ in the fifth line. Now using the fact that $p_i+k_i=p_f+k_f$, square both sides of this equation, we find

$$
\begin{aligned}
p_i^2+2p_i\cdot k_i+\cancelto{0}{k_i^2} &= p_f^2+2p_f\cdot k_f+\cancelto{0}{k_f^2}\\
m^2+2p_i\cdot k_i &= m^2+2p_f\cdot k_f\\
p_i\cdot k_i &= p_f\cdot k_f.
\end{aligned}
$$

By substituting this back to equation 3.80, we arrive at

$$
\begin{aligned}
A_{tot} =& e\kappa\frac{\epsilon_f^*\cdot k_i}{k_i\cdot k_f}\Big[(p_f\cdot k_f)(\epsilon_f^*\cdot\epsilon_i)+(\epsilon_f^*\cdot p_i)(\epsilon_i\cdot p_f)\frac{p_f\cdot k_f}{p_i\cdot k_f}\Big]\\
=& e\kappa\frac{(\epsilon_f^*\cdot k_i)(p_f\cdot k_f)}{k_i\cdot k_f}\quad\underbrace{\Big[\epsilon_f^*\cdot\epsilon_i+\frac{(\epsilon_f^*\cdot p_i)(\epsilon_i\cdot p_f)}{p_i\cdot k_f}\Big]}_{\sim\text{S-Compton amplitude in the CM-frame}},
\end{aligned}
\tag{3.81}
$$

where "S-Compton amplitude in the CM-frame" means scalar-Compton scattering amplitude in the CM-frame. To evaluate this term, calculating S-Compton in the CM-frame and then using equation 3.76, then get

$$
\begin{aligned}
A_{S-compton} =& 2e^2\Big(\frac{\epsilon_i\cdot p_f\epsilon_f^*\cdot p_i}{p_i\cdot k_f}-\frac{\epsilon_f^*\cdot p_f\cancelto{0}{\epsilon_i\cdot p_i}}{p_i\cdot k_i}+\epsilon_i\cdot\epsilon_f^*\Big)\\
=& 2e^2\Big(\frac{\epsilon_i\cdot p_f\epsilon_f^*\cdot p_i}{p_i\cdot k_f}+\epsilon_i\cdot\epsilon_f^*\Big).
\end{aligned}
\tag{3.82}
$$

Instead of directly calculating equation 3.81, we can separate it into two parts namely,

$$
A_{tot}=\underbrace{\frac{\kappa}{2e}\frac{(\epsilon_f^*\cdot k_i)(p_f\cdot k_f)}{k_i\cdot k_f}}_{\text{H}}\cdot\underbrace{2e^2\Big[\epsilon_f^*\cdot\epsilon_i+\frac{(\epsilon_f^*\cdot p_i)(\epsilon_i\cdot p_f)}{p_i\cdot k_f}\Big]}_{\sim\text{S-Compton}}.
\tag{3.83}
$$

First, we consider the S-Compton like part. Recalling that we have two polarizations for photon those are + and -. There are four possible ways of photon polarization in Compton scattering, namely $++$, $+-$, $-+$ and $--$ each of which corresponds to the polarization that before and after interaction respectively. Thus we should sum all of those four possible amplitude. That is $A_{++}+A_{--}+A_{+-}+A_{-+}$. To calculate total amplitude, recalling that in the CM-frame we have

$$k_i^\mu=(E_\gamma,0,0,k_i), \tag{3.84}$$

$$p_i^\mu=(E_p,0,0-k_i), \tag{3.85}$$

$$p_f=(E_p',-p\sin\theta,0,-p\cos\theta), \tag{3.86}$$

$$k_f=(E_\gamma',p\sin\theta,0,p\cos\theta), \tag{3.87}$$

$$\epsilon_i^{\pm} = \frac{\mp}{\sqrt{2}}(\hat{x} \pm i\hat{y}) \tag{3.88}$$

and

$$\epsilon_f^{\pm} = \frac{\mp}{\sqrt{2}}(\cos\theta\hat{x} \pm i\hat{y} - \sin\theta\hat{z}), \tag{3.89}$$

where we have used p instead of $|p_i|$ or $|p_f|$ since in the CM-frame $|p_i| = |p_f| = p = E'_\gamma = E_\gamma$. Using the above equations lead to many useful properties that will be helpful in this calculation, namely

$$\epsilon_f^{\pm} \cdot p_f = -\epsilon_f^{*\pm} \cdot p_i = \mp\frac{p}{\sqrt{2}}\sin\theta, \tag{3.90}$$

$$\epsilon_f^{*\pm} \cdot \epsilon_i^{\pm} = -\frac{1}{2}(1+\cos\theta), \tag{3.91}$$

and

$$\epsilon_f^{*\mp} \cdot \epsilon_i^{\pm} = -\frac{1}{2}(1-\cos\theta) = \epsilon_f^{*\pm} \cdot \epsilon_i^{\mp} = \epsilon_f^{*\pm} \cdot \epsilon_i^{\mp}. \tag{3.92}$$

Now, the tools are ready to use in the calculation. Beginning with $A_{\pm\pm}$, by substituting equation 3.90 and 3.91 into 3.82, we then get

$$\begin{aligned} A_{\pm\pm} =& 2e^2(\frac{\epsilon_i^{\pm} \cdot p_f \epsilon_f^{*\pm} \cdot p_i}{p_i \cdot k_f} + \epsilon_i^{\pm} \cdot \epsilon_f^{*\pm}) \\ =& 2e^2(\frac{(\mp\frac{p}{\sqrt{2}}\sin\theta)(\pm\frac{p}{\sqrt{2}}\sin\theta)}{p_i \cdot k_f} - \frac{1}{2}(1+\cos\theta)) \\ =& -e^2(1+\cos\theta + \frac{p^2\sin^2\theta}{p_i \cdot k_f}), \end{aligned} \tag{3.93}$$

and $A_{\pm\mp}$ by substituting 3.90 and 3.92, then get

$$\begin{aligned} A_{\pm\mp} =& 2e^2(\frac{\epsilon_i^{\pm} \cdot p_f \epsilon_f^{*\mp} \cdot p_i}{p_i \cdot k_f} + \epsilon_i^{\pm} \cdot \epsilon_f^{*\mp}) \\ =& 2e^2(\frac{(\mp\frac{p}{\sqrt{2}}\sin\theta)(\mp\frac{p}{\sqrt{2}}\sin\theta)}{p_i \cdot k_f} - \frac{1}{2}(1-\cos\theta)) \\ =& -e^2(1-\cos\theta - \frac{p^2\sin^2\theta}{p_i \cdot k_f}). \end{aligned} \tag{3.94}$$

Before summing all the amplitudes together, let us introduce useful notations that is s, t and u which are equal to $(p_i+k_i)^2$, $(k_i-k_f)^2$ and $(p_i-k_f)^2$ respectively and $s+t+u=2m^2$. These notation are useful because they are frame independent. Thus writing the amplitude in these notation will be helpful. To transform the amplitude to the new notation, let's calculate some properties of this notation. Starting with

$$s = (p_i+k_i)^2 = m^2 + 2p_i \cdot k_i, \tag{3.95}$$

$$p_i = (E_p, 0, 0, -p), \tag{3.96}$$

$$E_p = \sqrt{p^2+m^2}, \tag{3.97}$$

$$E_\gamma = p = E'_\gamma \tag{3.98}$$

and

$$k_i = (E_\gamma, 0, 0, p), \tag{3.99}$$

by substituting equation 3.111-3.99 into 3.100, we get

$$\begin{aligned}
s =& m^2 + 2(p\sqrt{p^2+m^2} + p^2) \\
\frac{s-m^2}{2} =& p\sqrt{p^2+m^2} + p^2 \\
\frac{(s-m^2)^2}{4} =& p^2(p^2+m^2) + 2p^3\sqrt{p^2+m^2} + p^4 \\
\frac{(s-m^2)^2}{4} =& p^2\underbrace{(2p^2+m^2+2p^3\sqrt{p^2+m^2})}_{s} \\
p^2 =& \frac{(s-m^2)^2}{4s}.
\end{aligned} \tag{3.100}$$

For $t = (k_i - k_f)^2$, in addition of equation 3.99-3.111 and using k_f, we get

$$\begin{aligned}
t &= (k_i - k_f)^2 = -2k_i \cdot k_f \\
t &= -2(E_\gamma E'_\gamma - p^2\cos\theta) \\
\frac{t}{-2p^2} &= \frac{-2st}{(s-m^2)^2} = 1 - \cos\theta \\
\cos\theta &= \frac{(s-m^2)^2 + 2st}{(s-m^2)^2},
\end{aligned} \tag{3.101}$$

where equation 3.100 is used in the third line. Now for $u = (p_i - k_f)^2$, this leads us to

$$\begin{aligned}
u &= (p_i - k_f)^2 = m^2 - 2p_i \cdot k_f \\
p_i \cdot k_f &= -\frac{u-m^2}{2}.
\end{aligned} \tag{3.102}$$

Now substituting equation 3.100-3.102 into 3.93 together with the fact that $\sin^2\theta = 1 - \cos^2\theta$ and $s + t + u = 2m^2$ and rearranging it, we obtain

$$\begin{aligned}
A_{\pm\pm} =& -e^2\Big[1 + \cos\theta + \frac{p^2(1-\cos^2\theta)}{p_i \cdot k_f}\Big] \\
=& -e^2\Big[1 + \frac{(s-m^2)^2 + 2st}{(s-m^2)^2} + \frac{p^2(1 - (\frac{(s-m^2)^2+2st}{(s-m^2)^2})^2)}{-\frac{u-m^2}{2}}\Big] \\
=& -2e^2\Big[\frac{((s-m^2)^2 + st)(u-m^2) + t[(s-m^2)^2 + st]}{(s-m)^2(u-m^2)}\Big] \\
=& -2e^2\Big[\frac{((s-m^2)^2 + st)(\overbrace{u+t}^{=2m^2-s} - m^2)}{(s-m)^2(u-m^2)}\Big] \\
=& -2e^2\Big[\frac{((s-m^2)^2 + st)(m^2 - s)}{(s-m)^2(u-m^2)}\Big] \\
=& 2e^2\Big[\frac{(s-m^2)^2 + st}{(s-m)(u-m^2)}\Big].
\end{aligned} \tag{3.103}$$

The same goes for 3.94, we find

$$
\begin{aligned}
A_{\pm\mp} =& -e^2(1-\cos\theta - \frac{p^2(1-\cos^2\theta)}{p_i\cdot k_f}) \\
=& -e^2\Big[-\frac{2st}{(s-m^2)^2} - \frac{2t((s-m^2)^2+st)}{(s-m^2)^2(u-m^2)}\Big] \\
=&2e^2t\Big[\frac{(s-m^2)^2+st+s(u-m^2)}{(s-m^2)^2(u-m^2)}\Big] \\
=&2e^2t\Big[\frac{(s-m^2)^2+m^4+(st+su-m^2s)}{(s-m^2)^2(u-m^2)}\Big] \\
=&2e^2t\Big[\frac{(s-m^2)^2+m^4+m^2s-s^2}{(s-m^2)^2(u-m^2)}\Big] \\
=&2e^2\Big[\frac{-m^2t}{(s-m^2)(u-m^2)}\Big],
\end{aligned}
\tag{3.104}
$$

where we have used $s(t+u-m^2) = m^2s - s^2$ which comes from

$$t+u+s = 2m^2$$

$$t+u = 2m^2 - s$$

$$s(t+u-m^2) = s(2m^2-s-m^2)$$

$$s(t+u-m^2) = m^2s - s^2.$$

Before going any further, let us introduce a new variable $\frac{d\sigma}{dt}$. We know that σ(cross-section) and $t = (p_i - p_f)^2$ are frame independent. So $\frac{d\sigma}{dt}$ is one of good objects which comes with frame independent property. From section 2.8, for massless scatter with massive, we have

$$\frac{d\sigma}{dt} = \frac{1}{16\pi(s-m^2)^2}\sum|\mathcal{M}_{ij}|^2 = \frac{1}{16\pi(s-m^2)^2}\sum\frac{1}{2}|A_{ij}|^2, \tag{3.105}$$

where $\frac{1}{2}$ comes from the fact that an incoming photon has two polarizations. Thus we should average only 2 degrees of freedom. On the other hand if there are two incoming particles that each of them has two spin states, we must divided by 4 instead of 2, see [4] (page 232). To get $\sum|A_{ij}|^2$, we only sum $|A_{++}|^2$, $|A_{--}|^2$, $|A_{+-}|^2$ and $|A_{+-}|^2$. Thus from equation 3.103 and 3.104, squaring and summing them together, we obtain

$$
\begin{aligned}
\sum|A_{ij}|^2 =&|A_{++}|^2 + |A_{--}|^2 + |A_{+-}|^2 + |A_{+-}|^2 \\
=&2|A_{++}|^2 + 2|A_{+-}|^2 \\
=&2\Big[2e^2\frac{(s-m^2)^2+st}{(s-m^2)(u-m^2)}\Big]^2 + 2\Big[2e^2\frac{-m^2t}{(s-m^2)(u-m^2)}\Big]^2.
\end{aligned}
\tag{3.106}
$$

Since

$$
\begin{aligned}
(s-m^2)^2 + st =& s^2 - 2sm^2 + m^4 + st \\
=& m^4 + s(s-2m^2+t) \\
=& m^4 - us,
\end{aligned}
\tag{3.107}
$$

where in the third line we have used the fact that

$$s+t+u = 2m^2$$

$$-u = s + t - 2m^2,$$

substituting equation 3.107 into the first in equation 3.106, we get

$$\begin{aligned}\sum |A_{ij}|^2 =&2\Big[2e^2\frac{m^4 - us}{(s-m^2)(u-m^2)}\Big]^2 + 2\Big[2e^2\frac{-m^2t}{(s-m^2)(u-m^2)}\Big]^2\\ =&8e^4\frac{(m^4-us)^2+m^4t^2}{(s-m^2)^2(u-m^2)^2}.\end{aligned} \tag{3.108}$$

Substituting the above equation into equation 3.105, we get

$$\frac{d\sigma}{dt} = \frac{e^4}{4\pi}\frac{(m^4-us)^2+m^4t^2}{(s-m^2)^4(u-m^2)^2}. \tag{3.109}$$

Recall that $\frac{d\sigma}{dt}$ is frame independent. Thus $\frac{d\sigma}{dt}$ in the CM-frame is equal to $\frac{d\sigma}{dt}$ in a laboratory frame. So we have

$$\left(\frac{d\sigma}{dt}\right)_{\text{Lab frame}} = \frac{e^4}{4\pi}\frac{(m^4-us)^2+m^4t^2}{(s-m)^2(u-m^2)^2}. \tag{3.110}$$

In a laboratory frame, or lab frame for short, the electron is at rest, but the photon is moving. So in this frame, we choose the coordinate such that

$$p_i^\mu = (m, 0, 0, 0), \tag{3.111}$$

$$k_i^\mu = (\omega_i, \omega_i, 0, 0) \tag{3.112}$$

and

$$k_f^\mu = (\omega_f, \omega_f\cos\theta, 0, \omega_f\sin\theta) \tag{3.113}$$

for the electron and the photon respectively. Using equation 3.111-3.113, we can calculate each term in equation 3.109 as follow,
for $s - m^2$

$$\begin{aligned}s =&(p_i + k_i)^2\\ s =&p_i^2 + 2p_i\cdot k_i + k_i^2\\ s - m^2 =&2p_i\cdot k_i = 2m\omega_i,\end{aligned} \tag{3.114}$$

for $u - m^2$

$$\begin{aligned}u =&(p_i - k_f)^2\\ u =&p_i^2 - 2p_i\cdot k_f + k_f^2\\ u - m^2 =& -2m\omega_f,\end{aligned} \tag{3.115}$$

for t

$$\begin{aligned}t =&(k_i - k_f)^2\\ =&k_i^2 - 2k_i\cdot k_f + k_f^2 = -2k_i\cdot k_f\\ =& -2(\omega_i\omega_f - \omega_i\omega_f\cos\theta)\\ =& -4\omega_i\omega_f\sin^2\frac{\theta}{2},\end{aligned} \tag{3.116}$$

and $m^4 - su$

$$\begin{aligned}
(s-m^2)(u-m^2) =& -4m^2\omega_i\omega_f \\
su - m^2(s+u) + m^4 =& -4m^2\omega_i\omega_f \\
su - m^2(2m^2-t) + m^4 =& -4m^2\omega_i\omega_f \\
su - m^4 =& -4m^2\omega_i\omega_f(1-\sin^2\frac{\theta}{2}) \\
m^4 - su =& 4m^2\omega_i\omega_f\cos^2\frac{\theta}{2}.
\end{aligned} \tag{3.117}$$

Substituting equation 3.114-3.117 into 3.109, then get

$$\begin{aligned}
\frac{d\sigma}{dt}_{\text{S-Compton}} =& \frac{e^4}{4\pi}\frac{(m^4-us)^2+m^4t^2}{(s-m^2)^4(u-m^2)^2} \\
=& \frac{e^4}{4\pi}\frac{16m^4\omega_i^2\omega_f^2\cos^4\frac{\theta}{2}+16m^4\omega_i^2\omega_f^2\sin^4\frac{\theta}{2}}{(16m^4\omega_i^4)(4m^2\omega_f^2)} \\
=& \frac{e^4}{4\pi}\frac{1}{4m^2\omega_i^2}(\cos^4\frac{\theta}{2}+\sin^4\frac{\theta}{2}) \\
=& \frac{e^4}{32\pi m^2\omega_i^2}(1+\cos^2\theta),
\end{aligned} \tag{3.118}$$

where in the fourth line we use the trigonometric identity, namely $\cos^4 x + \sin^4 x = \frac{1}{2}(1 + \cos^2 2x)$. Actually, we want to find $\frac{d\sigma}{d\Omega}$ not $\frac{d\sigma}{dt}$. From chain rule, we have

$$\frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{d\sigma}{d\Omega}. \tag{3.119}$$

We can calculate $\frac{dt}{d\Omega}$ by using Compton's formula

$$\begin{aligned}
\frac{1}{\omega_f} - \frac{1}{\omega_i} =& \frac{1}{m}(1-\cos\theta) \\
\omega_f =& \frac{\omega_i}{1+\frac{\omega_i}{m}(1-\cos\theta)}.
\end{aligned} \tag{3.120}$$

Substituting the above equation back into 3.116 and differentiate it with respect to Ω, we obtain

$$\begin{aligned}
\frac{dt}{d\Omega}_{\text{S-Compton}} =& \frac{1}{2\pi}\frac{d}{d\cos\theta}\left(\frac{-2\omega_i^2(1-\cos\theta)}{1+\frac{\omega_i}{m}(1-\cos\theta)}\right) \\
=& \frac{1}{\pi}\frac{\omega_i^2}{(1+\frac{\omega_i}{m}(1-\cos\theta))^2} = \frac{\omega_f^2}{\pi},
\end{aligned} \tag{3.121}$$

where we use equation 3.120 in the second line. Substituting equation 3.121 and 3.118 into 3.119, we arrive at

$$\begin{aligned}
\frac{d\sigma}{d\Omega}_{\text{S-Compton}} =& \frac{\omega_f^2}{\pi}\frac{e^4}{32\pi m^2\omega_i^2}(1+\cos^2\theta) \\
=& \frac{\alpha^2}{2m^2}\frac{\omega_f^2}{\omega_i^2}(1+\cos^2\theta).
\end{aligned} \tag{3.122}$$

Now we get the cross-section for scalar Compton scattering, but we want scalar-graviton cross-section not Scalar Compton cross-section. Form equation 3.83, $\frac{d\sigma}{dt}$ takes the form

$$\begin{aligned}\frac{d\sigma}{dt} =&\frac{1}{32\pi(s-m^2)^2}\sum|A_{ij}|^2\\ =&\frac{1}{32\pi(s-m^2)^2}\sum\underbrace{\left|\frac{\kappa}{2e}\frac{(\epsilon_f^*\cdot k_i)(p_f\cdot k_f)}{k_i\cdot k_f}\right|^2}_{\mathrm{H}^2}\underbrace{\left|2e^2\Big[\epsilon_f^*\cdot\epsilon_i+\frac{(\epsilon_f^*\cdot p_i)(\epsilon_i\cdot p_f)}{p_i\cdot k_f}\Big]\right|^2}_{\sim\text{S-Compton}}\\ =&\underbrace{\left|\frac{\kappa}{2e}\frac{(\epsilon_f^*\cdot k_i)(p_f\cdot k_f)}{k_i\cdot k_f}\right|^2}_{\mathrm{H}^2}\frac{d\sigma}{dt}_{\text{S-Compton}}\end{aligned} \tag{3.123}$$

or

$$\frac{d\sigma}{d\Omega}=\underbrace{\left|\frac{\kappa}{2e}\frac{(\epsilon_f^*\cdot k_i)(p_f\cdot k_f)}{k_i\cdot k_f}\right|^2}_{\mathrm{H}^2}\frac{d\sigma}{d\Omega}_{\text{S-Compton}}. \tag{3.124}$$

However, H in the equation 3.83 is in the CM-frame. Thus let us come back to the CM-frame again. Recalling that H takes the form

$$|\mathrm{H}|=\frac{\kappa}{2e}\left|\frac{(\epsilon_f^*\cdot k_i)(p_f\cdot k_f)}{k_i\cdot k_f}\right|. \tag{3.125}$$

From equation 3.114 together with the fact that $p_i\cdot k_i=p_f\cdot k_f$, we then have

$$p_f\cdot k_f=\frac{s-m^2}{2}. \tag{3.126}$$

Substituting equation 3.84, 3.89, 3.100, 3.101 3.116 and 3.126 into equation 3.125 together with $\sin^2\theta=1-\cos^2\theta$, we obtain

$$\begin{aligned}|\mathrm{H}|^2=&\frac{\kappa^2}{4e^2}\left|\frac{p\sin\theta}{\sqrt{2}}\Big(\frac{s-m^2}{t}\Big)\right|^2\\ =&\frac{\kappa^2}{8e^2}\left|p^2\sin^2\theta\Big(\frac{s-m^2}{t}\Big)^2\right|\\ =&\frac{\kappa^2}{8e^2}\left|\Big(\frac{(s-m^2)^2}{4s}\Big)\Big(\frac{-4st(s-m^2)^2-4s^2t^2}{(s-m^2)^4}\Big)\Big(\frac{s-m^2}{t}\Big)^2\right|.\\ =&\frac{\kappa^2}{8e^2}\left|\frac{(s-m^2)^2+st}{t}\right|.\end{aligned} \tag{3.127}$$

Now, we can see that H^2 is frame independent since it composes of only s, t and m^2 which are Lorentz invariant. Let us come back to the lab frame again by using equation 3.114 and 3.116 which gives

$$\begin{aligned}|\mathrm{H}|^2=&\frac{\kappa^2}{8e^2}\left|\frac{4m^2\omega_i^2+(2m\omega_i+m^2)(-4\omega_i\omega_f\sin^2\frac{\theta}{2})}{-4\omega_i\omega_f\sin^2\frac{\theta}{2}}\right|\\ =&\frac{\kappa^2}{8e^2}\left|\frac{m^2\omega_i}{\omega_f\sin^2\frac{\theta}{2}}-2m\omega_i-m^2\right|.\end{aligned} \tag{3.128}$$

Next, using the Compton's formula in the second line of equation 3.128,

$$\frac{1}{\omega_f} = \frac{1 + 2\frac{\omega_i}{m}\sin^2\frac{\theta}{2}}{\omega_i},$$

we find

$$\begin{aligned} |\mathrm{H}|^2 =& \frac{\kappa^2}{8e^2}\left|\frac{m^2}{\sin^2\frac{\theta}{2}} + 2m\omega_i - 2m\omega_i - m^2\right| \\ =& \frac{\kappa^2}{8e^2}\left|\frac{m^2}{\sin^2\frac{\theta}{2}} - m^2\right| \\ =& \frac{m^2\kappa^2}{8e^2\sin^2\frac{\theta}{2}}\left|1 - \sin^2\frac{\theta}{2}\right| = \frac{m^2\kappa^2\cos^2\frac{\theta}{2}}{8e^2\sin^2\frac{\theta}{2}}. \end{aligned} \tag{3.129}$$

At this point, we obtain all of the ingredients for equation 3.124 namely equation 3.122 and 3.129. Thus, by substituting those two equations into equation 3.124, we get

$$\begin{aligned} \frac{d\sigma}{d\Omega} =& \mathrm{H}^2\frac{d\sigma}{d\Omega}_{\text{S-Compton}} \\ =& \frac{m^2\kappa^2\cos^2\frac{\theta}{2}}{8e^2\sin^2\frac{\theta}{2}} \cdot \frac{\alpha^2}{2m^2}\frac{\omega_f^2}{\omega_i^2}(1 + \cos^2\theta) \\ =& \frac{32\pi\mathrm{G}\cos^2\frac{\theta}{2}}{8e^2\sin^2\frac{\theta}{2}} \cdot \alpha^2\frac{\omega_f^2}{\omega_i^2}(\cos^4\frac{\theta}{2} + \sin^4\frac{\theta}{2}) \\ =& \mathrm{G}\alpha\frac{\omega_f^2}{\omega_i^2}\cos^2\frac{\theta}{2}(\cot^2\frac{\theta}{2}\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}), \end{aligned} \tag{3.130}$$

where we have used the facts that $\kappa^2 = 32\pi\mathrm{G}$ and $\cos^4 x + \sin^4 x = \frac{1}{2}(1 + \cos 2x)$ in the third line and $\alpha = \frac{e^2}{4\pi}$ in the last line.

Now we have finally arrived at the differential cross-section. In the next chapter, we are going to talk about its application and what we would get from extending the results from QED.

Chapter 4

Discussion & Applications

In the previous chapter, we have calculated so far is the amplitude for the $s+\gamma \to s+g$ event and its cross-section, but we haven't discussed its physical meanings yet. In this chapter, we will discuss our result and survey its applications.

4.1 Discussion

As ones see, our calculations are composed of many complex calculations. Unlike the Compton scattering, many indices and polarization vectors have appeared during the calculations which could be confusing. However, instead of directly calculating the amplitude, we can split our calculations into two parts where one of those parts is just a Compton scattering which is well-known in QED. To make calculating easier, we have to use the fact that cross-section is invariant under Lorentz transformation, and thus we can show that one of those split terms equal to the Compton scattering term in the CM-frame which its amplitude is already calculated in section 3.1. In calculating Compton cross-section, we have used the helicity method by summing over possible spin projected along a translation axis or helicity which is simple since we can easily count the possible photon helicity that is composed of + and - states. From this point, after putting the Compton cross-section and another term together, the calculation was easy and straightforward.

4.2 Applications

Dark matter, as its name would suggest, is the matter that can't be detected directly and doesn't interact with electromagnetism[11]. Even though we don't know what exactly the dark matter is, many physicists suggested that it would be composed of a scalar particle. Exploiting the assumption, we can get the cross-section of the interaction of dark matter and the graviton. Although, in this project, we have used the complex scalar field which interacts with a photon while the dark matter does not, we can adapt this method to the event with involve only the dark matter and graviton. By extracting Feynman diagrams from the calculation, ones can apply the Feynman diagrams to a similar calculation such as $g + s \to g + s$ event. However, as we have seen in the differential cross-section, we have the factor G and α which represent gravitational and electromagnetic interaction. If we instead consider only graviton, not graviton and photon interaction, we will get factor G^2 which will suppress the amplitude. This suppressed amplitude implies us why the graviton is hard to be detected.

Bibliography

[1] Holstein B.R., *Graviton Physics*. Am.J.Phys. 74,1002-1011, 2006.

[2] Abbott B.P., *et al.* (LIGO Scientific Collaboration and Virgo Collaboration), *Observation of Gravitational Waves from a Binary Neutron Star Inspiral*. Phys. Rev. Lett. 119, 161101, 2017.

[3] Juan, M., Tonatiuh, M., *A brief Review of the Scalar Field Dark Matter model*. J. Phys.: Conf. Ser. 378 012012, 2012.

[4] Matthew D.S., *Quantum Field Theory and the Standard Model*. Cambridge University, 2014.

[5] Dasgupta M., *AN INTRODUCTION TO QUANTUM FIELD THEORY*. University of Oxford, 2009.

[6] Srednicki M., *Quantum field theory*. Cambridge University Press, 2007.

[7] Hayes J. N., *The Propagators of Quantum Field Theories as Green's Function for Boundary Value Problems in Partial Differential Equations*. U.S. Naval Research Laboratory, 1964.

[8] Raimond A., *BRST Quantization of Massive Spin-2 Particles in the Stückelberg Formalism*. Würzburg, 2016.

[9] Hinterbichler K., *Theoretical Aspects of Massive Gravity*. Reviews of Modern Physics, Volume 84, 671-710, 2012.

[10] Grillo N., *Quantization of the Graviton Field, Characterization of the Physical Subspace and Unitarity in Causal Quantum Gravity*, hep-th/9911118.

[11] Roszkowski L., Sessolo E.M., Trojanowski S., *WIMP dark matter candidates and searches – current status and future prospects*, Rept.Prog.Phys. 81 (2018), arXiv:1707.06277.