ELE8088: Control & Estimation Theory **QUB, 2020/21**

Coursework Assignment

Deadlines: (Part I) 31 January 2022; (Part II) 28 March 2022

Released on: 25 October 2021

Instructions

Preparation and submission of solutions

(i) Typeset your solutions in LaTeX using the template provided on Canvas (read the instructions). Handwritten and scanned solutions, or solutions written in MS Word or other word processors will not be accepted.

(ii) Upload your solutions on Canvas. Note that you will be able to upload a single PDF file only (one for Part I and one for Part II).

(iii) If you have any questions or need a clarification or a hint, you have to post your question on Canvas as a discussion.

(iv) Some questions require the use of software; for example, you may need to write some Python code. Do not include your code in your solutions. Instead, upload it on `github.com` or `gist.github.com` and provide a link in your solutions. If you use a Google colab notebook, provide a link.

(v) If you have any exceptional circumstances (health problems, or other difficulties), please contact the advisor of studies, Dr D. McNeill, and submit an exceptional circumstances request.

(vi) Where necessary, include any relevant figures in your report. All figures must have an informative caption, correct axis labels and be legible.

(vii) Cite any sources you will use (you do not need to cite the lecture material). Follow the IEEE citation format.

(viii) Each team needs to upload a single PDF file, i.e., it suffices that one of the members of your team uploads your report.

(ix) The questions that start with the symbol require some Python programming.

Assessment and Feedback

(i) This coursework counts towards 25% of your total marks.

(ii) A detailed grading rubric is available on Canvas.

(iii) You solutions are graded for correctness (including justification) and style (e.g., quality of graphics and equations, whether you have captioned all your figures, whether your plots have labels and, if necessary, a legend).

(iv) The correctness of your approach counts more than the correctness of your result. Always provide justification in your solutions.

(v) The total marks of this assignment are 110/100. If you score higher than 100, the additional marks will contribute to your overall module mark (which cannot be higher than 100 marks).

(vi) You will receive detailed feedback for the first part of this coursework, but not a mark. You will receive your coursework mark after you submit both parts of the coursework.

Collaboration

(i) The team members are expected to contribute equally. All team members are expected to work on all problems.

(ii) You will need to submit a collaboration statement (400 words max.) with your reports. Provide *factual* data demonstrating the quality of your collaboration: include details on time/work management, communication, use of technologies (git) etc. Reflect on what you did well and what you can do better in your next collaborative project.

(iii) As soon as this coursework is released, have a meeting with your collaborator(s). Go through all questions and come up with a plan of actions.

(iv) Keep meeting minutes. After each meeting send a short email to each other with the list of actions you have agreed on.

(v) Use a source versioning system such as *git* to edit your code collaboratively — do not just exchange files over email. This is standard practice in the industry.

(vi) If there are any doubts regarding whether the team members contributed equally, you will be invited to an interview.

Deadlines

(i) Submit the first part (Control) of this coursework by **31 January 2022** at 16:00, UK time. The second part (Estimation) should be submitted by **28 March 2022** at 16:00, UK time.

(ii) There is plenty of time for you to work on this assignment, so do not leave this for the last week before the deadline. The deadlines are strict and no extension will be given.

(iii) There will be penalties for late submissions: 5% per day over the first five days and a mark of zero afterwards.

Plagiarism and Collusion

Queen's University Belfast has strict rules on plagiarism and collusion. By submitting your work you declare that you have read and understood the University regulations relating to academic offences, including collusion and plagiarism. Do not collude with other teams. Cite any resources you will use.

1 Part I: Control (55%)

1.1 Optimal control (18%)

(i) Prove that the function $g : \mathbb{R}^n \to \mathbb{R}$ given by $g(x) = \frac{1}{2}x^\intercal Qx + q^\intercal x + c$ with $Q \in \mathbb{S}^n_+$ is level-bounded if and only if $Q \in \mathbb{S}^n_{++}$.*

Next, consider the finite-horizon linear-quadratic optimal control problem

$$\mathbb{P}_N(x) : \operatorname*{minimise}_{\substack{u_0,u_1,\ldots,u_{N-1}\\x_0,x_1,\ldots,x_N}} \sum_{t=0}^{N-1} \tfrac{1}{2}\begin{bmatrix}x_t\\u_t\end{bmatrix}^\intercal \begin{bmatrix}Q & S\\S^\intercal & R\end{bmatrix}\begin{bmatrix}x_t\\u_t\end{bmatrix} + \tfrac{1}{2}x_N^\intercal P_f x_N, \tag{1a}$$

$$\text{subject to: } x_{t+1} = Ax_t + Bu_t,\ t \in \mathbb{N}_{[0,N-1]}, \tag{1b}$$

$$x_0 = x, \tag{1c}$$

where $\left[\begin{smallmatrix}Q & S\\S^\intercal & R\end{smallmatrix}\right] \succcurlyeq 0$, $Q \in \mathbb{S}^n_+$, $R \in \mathbb{S}^m_{++}$, $P_f \in \mathbb{S}^n_+$, and x is a given initial state.

(ii) Is Problem $\mathbb{P}_N$ convex? Does $\mathbb{P}_N$ have a minimiser?

(iii) Solve Problem $\mathbb{P}_N$ by eliminating the state sequence: determine the optimal sequence of control actions, the optimal sequence(s) of states and the optimal cost.

(iv) Solve Problem $\mathbb{P}_N$ by using the dynamic programming method.

Next, consider the following infinite-horizon optimal control problem

$$\mathbb{P}_\infty(x) : \operatorname*{minimise}_{(u_t)_{t\in\mathbb{N}},(x_t)_{t\in\mathbb{N}}} \sum_{t=0}^{\infty} \frac{1}{2}\begin{bmatrix}x_t\\u_t\end{bmatrix}^\intercal \begin{bmatrix}Q & S\\S^\intercal & R\end{bmatrix}\begin{bmatrix}x_t\\u_t\end{bmatrix}, \tag{2a}$$

$$\text{subject to: } x_{t+1} = Ax_t + Bu_t,\ t \in \mathbb{N}, \tag{2b}$$

$$x_0 = x, \tag{2c}$$

where $\left[\begin{smallmatrix}Q & S\\S^\intercal & R\end{smallmatrix}\right] \succcurlyeq 0$, $Q \in \mathbb{S}^n_+$, and $R \in \mathbb{S}^m_{++}$.

(v) Under what conditions do the dynamic programming iterates, $V_t^\star$, converge? Justify your answer.**

(vi) Prove that the optimal value of $\mathbb{P}_\infty(x)$, if it exists, is given by $V(x) = \frac{1}{2}x^\intercal Px$ where $P \in \mathbb{S}^n_{++}$ satisfies the algebraic equation

$$P = A^\intercal PA - (A^\intercal PB + S)(R + B^\intercal PB)^{-1}(B^\intercal PA + S^\intercal) + Q. \tag{3}$$

(vii) Let $A = \left[\begin{smallmatrix}1 & 0\\1 & 1\end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, $Q = I_2$, $R = 2$ and $S = \left[\begin{smallmatrix}-0.5\\0.5\end{smallmatrix}\right]$. The optimal value of $\mathbb{P}_N$ that you determined in question (iv) has the form $V_N^\star(x) = \frac{1}{2}x^\intercal P_N x$, where $P_N = P_N(P_0)$ depends on $P_0 = P_f$. Write a Python program that computes P_N (given N and P_f) and approximate $\lim_{N\to\infty} P_N(P_0)$ for different values of P_0. Verify that this limit satisfies Equation (3).

* Try first to prove this assuming that $q = 0$ and $c = 0$ and use the definition of level-boundedness. Next, use the result of Exercise 21 from Handout X1.

** In general, the sequence of functions $V_t^\star$ *may not converge*. Under certain assumptions, however, it converges. This is all discussed in Handout 6 (Section 6.3 in particular). However, note that this problem is different from that in Handout 6.

1.2 Linearisation theorem (16%)

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuously differentiable function with $f(0) = 0$. Suppose that there are constants $\beta > 0$ and $\eta > 0$ so that $\|Jf(x) - Jf(x')\| \leq \beta\|x - x'\|$, for all $x, x' \in \mathcal{B}_\eta$. Define $A = Jf(0)$.

(i) Prove that for all $x \in \mathcal{B}_\eta$, it is $\|f(x) - Ax\| \leq \frac{\beta}{2}\|x\|^2$.[*]

(ii) Prove that if $\rho(A) < 1$, then the dynamical system $x_{t+1} = f(x_t)$ is locally exponentially stable.[**]

(iii) Prove or disprove (by providing a counterexample) that if $\rho(A) = 1$, the system $x_{t+1} = f(x_t)$ is asymptotically stable.

1.3 Model predictive controller design (21%)

Consider the linear dynamical system

$$x_{t+1} = \begin{bmatrix} 1 & 0.7 \\ -0.1 & 1 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_t, \tag{4}$$

with $x_t \in \mathbb{R}^2$ and $u_t \in \mathbb{R}$. The system is subject to the state and input constraints

$$-\left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right] \leq x_t \leq \left[\begin{smallmatrix} 2 \\ 2 \end{smallmatrix}\right], \text{ and } -1 \leq u_t \leq 1, \tag{5}$$

and the stage cost is $\ell(x, u) = \|x\|_2^2 + u^2$.

(i) Design an MPC using the terminal cost $V_f(x) = 0$ and the terminal set $X_f = \{0\}$. Compute the sets of feasible states with a prediction horizon $N = 1, \ldots, 6$. Simulate the MPC-controlled dynamical system with $N = 10$ using one of the extreme points of X_N as the initial state, x_0.

(ii) Design an MPC by following the procedure outlined in Handout 10, Sections 10.2 and 10.3. Use a prediction horizon $N = 10$ and compute the set of feasible states, X_N. Simulate the MPC-controlled system starting from the extreme points of X_N.

(iii) Design an MPC controller using an ellipsoidal terminal set and prediction horizon $N = 10$. Provide simulation results starting from different initial states.

Now consider the nonlinear dynamical system[†]

$$x_{t+1} = \begin{bmatrix} 1 & 0.7 \\ -0.1 & 1 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_t + \frac{1}{20} \begin{bmatrix} x_t^\intercal x_t \\ \sin^2(x_{t,1}) \end{bmatrix}, \tag{6}$$

which is subject to the constraints given in (5).

(iv) Design a nonlinear model predictive controller using the methodology of Section 10.6 in Handout 10: determine the terminal cost function and the terminal set of constraints.

[*] Hint: Use the fundamental theorem of calculus, which you can find in Handout 2.

[**] You have to use Lyapunov's exponential stability theorem. Make sure you specify your Lyapunov function, prove that it satisfies *all* conditions of the theorem, and specify the invariant set X. In particular, you will have to prove that the Lyapunov function you will choose satisfies an appropriate Lyapunov decrease condition, a local upper bound and a global lower bound.

[†] We denote the two coordinates of $x_t \in \mathbb{R}^2$ by $x_{t,1}$ and $x_{t,2}$.

2 Part II: Estimation (55%)

2.1 Estimation (16%)

Suppose that in a certain country the weight, w (in kg), and height, h (in m), of the men of age between 35 and 45 years, are known to be jointly normally distributed with mean 75 kg and 1.72 m and variance-covariance matrix

$$\operatorname{Var}\left[\left[\begin{smallmatrix}w\\h\end{smallmatrix}\right] \middle| \text{man}\right] = \begin{bmatrix}27 & 0.25\\ 0.25 & 0.005\end{bmatrix}. \tag{7}$$

The weight and height of women of the same age group are also jointly normal, the mean weight is 70 kg, the mean height is 1.69 m and the variance-covariance matrix is

$$\operatorname{Var}\left[\left[\begin{smallmatrix}w\\h\end{smallmatrix}\right] \middle| \text{woman}\right] = \begin{bmatrix}21 & 0.22\\ 0.22 & 0.007\end{bmatrix}. \tag{8}$$

In that country and age group, 51 out of 100 people are women.*

(i) Design an *estimator*** of the height of a person given their weight. Is your estimator unbiased? What is the variance of your estimator? Use this estimator to estimate the height of a person with weight 78 kg.

(ii) Design an estimator of a person's weight given their height. What is the variance of your estimator? Use this estimator to estimate the weight of a person whose stature is 1.81 m. How will your results change if you know that the person is a man?

(iii) The *body mass index* is defined as BMI $= w/h^2$. A person is considered obese if their BMI is above 30. A person's height is 1.67 m. What is the probability that they are obese?

(iv) The *basal metabolic rate* (BMR) is the rate of energy expenditure of a person at rest and is measured in kcal/day. BMR can be calculated by

$$\text{BMR} = \begin{cases}10.5w + 0.0625h - 5a + 5, & \text{for men}\\ 10.2w + 0.0621h - 5a - 161, & \text{for women}\end{cases} \tag{9}$$

where a is the person's age in years. Suppose that a person's weight and height are uncorrelated with their age (in the aforementioned age group). A person's height is 1.83 m. Estimate their BMR.

2.2 Maximum likelihood estimation (24%)

We say that a nonnegative random variable X follows the exponential distribution with parameter $\lambda > 0$ if its probability density function (pdf) is

$$p_X(x) = \lambda e^{-\lambda x},\ x \geq 0.$$

* All of these values are arbitrary.

** An estimator is not the same as an estimate!

(i) Determine the expected value and variance of X.[†]

(ii) The file `exponential_data.csv` contains a list of 5,000 independent samples drawn from the random variable X. Determine the maximum likelihood estimate of λ from this data.

Next, let $z_1, \ldots, z_N$ be independent samples drawn from $\mathcal{N}(\mu, \sigma^2)$, where $\sigma^2 > 0$.

(iii) Prove that the maximum likelihood estimates of μ and σ^2 are given by

$$\widehat{\mu} = \frac{1}{N}\sum_{i=1}^{N} x_i, \tag{10a}$$

$$\widehat{\sigma^2} = \frac{1}{N}\sum_{i=1}^{N}(x_i - \hat{\mu}), \tag{10b}$$

respectively.

(iv) Determine the expected value and variance of $\widehat{\mu}$.

(v) Prove that the above estimator of σ^2 is biased.

(vi) The file `normal_data.csv` contains the values of $N = 1,000$ such independent samples. Determine $\widehat{\mu}$ and $\widehat{\sigma^2}$.

2.3 Kalman Filter (17%)

Consider a discrete-time control system of the form

$$x_{t+1} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} x_t + \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} u_t + \begin{bmatrix} 0.10 & 0.05 \\ 0.03 & 0.20 \end{bmatrix} w_t, \tag{11a}$$

$$y_t = \begin{bmatrix} 1 & 0 \end{bmatrix} x_t + \begin{bmatrix} -1 & 0.5 \end{bmatrix} w_t + v_t, \tag{11b}$$

with state variable $x_t \in \mathbb{R}^2$, input $u_t \in \mathbb{R}$, output $y_t \in \mathbb{R}$, while the dynamics and the output are disturbed by noise terms $w_t \in \mathbb{R}^2$ and $v_t \in \mathbb{R}$ which are time-independent zero-mean Gaussian random variables. The covariance matrix of w_t is $\Sigma_w = \left[\begin{smallmatrix} 1.1 & 0.2 \\ 0.2 & 1.5 \end{smallmatrix}\right]$ and the variance of v_t is $\sigma_v^2 = 0.84$. Moreover, $x_0, w_0, v_0, w_1, v_1, \ldots$ are mutually independent. The control signal u_t is known at time t, is independent of w_t and is determined using the measurements $y_0, y_1, \ldots, y_t$ (you do not need to know how u_t is computed). The initial state, x_0, is a Gaussian random variable, $x_0 \sim \mathcal{N}\left(\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 2 & 0 \\ 0 & 4 \end{smallmatrix}\right]\right)$.

(i) Derive the Kalman Filter equations[*] .

(ii) Prove that the covariance estimate $\Sigma_{t+1|t} = \mathrm{Var}[x_{t+1} \mid y_0, y_1, \ldots, y_t]$, converges to a matrix Σ_∞ as $t \to \infty$ and determine Σ_∞ (in Python).

(iii) Define the *state estimation error* as $e_t = \mathbb{E}[x_t - \hat{x}_{t|t}]$. Prove that e_t follows a linear dynamics of the form $e_{t+1} = F_t e_t$ and write a Python program that determines F_t given t (include F_0, F_1, F_{1000} and F_{1001} in your report).

[†] For the variance, you may use the law of the unconscious statistician.

[*] Determine the Kalman Filter equations for determining $\hat{x}_{t|t}$, $\Sigma_{t|t}$, $\hat{x}_{t+1|t}$ and $\Sigma_{t+1|t}$. Follow the same procedure as the one outlined in Handout 13.