

# The Weighted Euler Curve Transform for Shape and Image Analysis

Qitong Jiang  
The Ohio State University  
Department of Mathematics  
jiang.927@osu.edu

Sebastian Kurtek  
The Ohio State University  
Department of Statistics  
kurtek.1@stat.osu.edu

Tom Needham  
Florida State University  
Department of Mathematics  
tneedham@fsu.edu

## Abstract

*The Euler Curve Transform (ECT) of Turner et al. is a complete invariant of an embedded simplicial complex, which is amenable to statistical analysis. We generalize the ECT to provide a similarly convenient representation for weighted simplicial complexes, objects which arise naturally, for example, in certain medical imaging applications. We leverage work of Ghrist et al. on Euler integral calculus to prove that this invariant—dubbed the Weighted Euler Curve Transform (WECT)—is also complete. We explain how to transform a segmented region of interest in a grayscale image into a weighted simplicial complex and then into a WECT representation. This WECT representation is applied to study Glioblastoma Multiforme brain tumor shape and texture data. We show that the WECT representation is effective at clustering tumors based on qualitative shape and texture features and that this clustering correlates with patient survival time.*

## 1. Introduction

Tools from algebraic topology have become increasingly popular in shape analysis applications over the past several years. At an intuitive level, the topological perspective is appealing because algebraic topology is, at its core, designed to extract tractable algebraic invariants from complex shape data. The dominant technique in topological shape analysis is *persistent homology*, which summarizes multiscale topological features of a shape, where scale is measured relative to some *filtration function*. Roughly, for a continuous function  $f : X \rightarrow \mathbb{R}$  on a topological space  $X$  (satisfying certain tameness conditions), one computes the degree- $k$  homology of the sublevel sets  $f^{-1}((-\infty, r])$  and tracks “births” and “deaths” of homological features as the filtration value  $r$  is increased. This produces a summary statistic for the pair  $(X, f)$  called a *persistence diagram* (see standard references [19, 9]), which can be used as a proxy for  $X$  in shape analysis applications. This approach has been taken in several shape analysis tasks, with

shape data coming from cortical surfaces [13], brain artery systems [3], proteins [29] and leaf contours [37]. While the persistence diagram of a pair  $(X, f)$  provides a computationally tractable shape summary, the complex structure of the invariant means that it is difficult to incorporate into statistical models. A simpler invariant is the *Euler curve* of  $(X, f)$ ; this is an integer-valued function on  $\mathbb{R}$  whose value at  $r$  is the Euler characteristic (i.e., the alternating sum of ranks of the homology groups) of the sublevel set  $f^{-1}((-\infty, r])$ .

Given shape data, one must answer the question of which filtration function to apply in order to apply these topological methods. For a shape represented as a simplicial complex  $K$  embedded in a Euclidean space  $\mathbb{R}^d$ , recent work has advocated for using an ensemble of filtration functions given by the height function along directions sampled from the unit sphere  $S^{d-1}$  [41, 24, 20, 17, 4, 14, 21]. The collection of all persistence diagrams for these height filtrations is referred to as the persistent homology transform of  $K$ . Likewise, the collection of Euler curves for all filtration directions is called the Euler curve transform (ECT) for  $K$ . The ECT provides a particularly attractive shape representation, as its simplistic structure allows it to be easily incorporated into statistical models. This was the approach taken in [14], where the ECTs for Glioblastoma Multiforme (GBM) brain tumor shapes were used as covariates in a model for survival prediction.

In this paper, we consider a variant of the ECT, which we dub the *weighted Euler Characteristic Transform* (WECT). This object is defined for shape data consisting of an embedded simplicial complex  $K$  endowed with an extra weighting function  $g$ . The pair  $(K, g)$  is referred to as a *weighted simplicial complex*. The WECT invariant incorporates both the shape of  $K$  and the weighting function  $g$  into a topological summary. Our motivation for defining this summary also comes from analysis of brain tumor data, which is naturally given as a segmented grayscale image. The segmented shape is used to construct a simplicial complex  $K$  embedded in  $\mathbb{R}^2$ , and the grayscale pixel values inside the shape define the weight function  $g$ . While the WECT is a simple



















