Neyman - Pearson Classifier : NP classifier.

* Bayes classifier minimizes risk
  it minimizes weighted sum of all errors.

* In some cases we may not explicitly want to trade off one type of error with another.

* Type 1 error : False +ve

Results says you have Coronavirus, but you actually don't

* Type2 Error : False -ve

Test result says you do not have Coronavirus but you actually do have

* In some cases we need to minimize Type 2 under the constraint that type 1 error is below some threshold.

1) $\mathbb{P}[h_{NP}(x)=1 \mid x \in C_0] \leq \alpha$

2) $\mathbb{P}[h_{NP}(x)=0 \mid x \in C_1] \leq \mathbb{P}[h(x)=0 \mid x \in C_1]$

$\forall\ h \in \mathcal{H}$ s.t

$\mathbb{P}[h(x)=1 \mid x \in C_0] \leq \alpha$

* Type1 error of NP is bounded above by α
* Among all classifiers that satisfy this bound on type-1 error.

NP Classifier has least Type-2 errors.

Given the bound on Type1 error "α"

$$h_{NP}(x) = 1 \quad \text{if} \quad \frac{p_{x|y=1}}{p_{x|y=0}} > K$$

$$0 \quad \text{otherwise}$$

where k is s.t

$$\mathbb{P}\left[\frac{p_{x|y=1}}{p_{x|y=0}} \leq k \;\middle|\; x \in C_0 \right] = 1-\alpha$$

We now prove that this classifier that we have considered satisfies NP criterion.

The threshold "k" is choosen s.t.

$$\mathbb{P}\left[\frac{p_{x|y=1}}{p_{x|y=0}} \leq k \;\middle|\; x \in C_0\right] = 1-\alpha$$

Hence by construction we have

$$\mathbb{P}\left[\frac{p_{x|y=1}}{p_{x|y=0}} > k \;\middle|\; x \in C_0\right] = \alpha$$

$$\mathbb{P}[h_{NP}(x)=1 \mid x \in C_0] = \alpha$$

this satisfies the 1^{st} condition.

Now we need to show that, its Type-II error is less than other classifier satisfying the constraint on Type-I error

Let "h" be any classifier s.t

$$\mathbb{P}[h(x)=1 \mid x \in C_0] \leq \alpha$$

To complete the proof we need to show

$$\mathbb{P}[h_{NP}(x)=0 \mid x \in C_1] \leq \mathbb{P}[h(x)=0 \mid x \in C_1]$$

or equivalently

$$\mathbb{P}[h_{NP}(x)=1 \mid x \in C_1] \geq \mathbb{P}[h(x)=1 \mid x \in C_1]$$

Consider the Integral

$$I = \int_{\mathbb{R}^n} (h_{NP}(x) - h(x)) \left(p_{X|Y=1}(x) - k\, p_{X|Y=0}(x) \right) dx$$

$$= \int_{p_{X|Y=1} > k p_{X|Y=0}} (h_{NP}(x) - h(x)) \left(p_{X|Y=1}(x) - k\, p_{X|Y=0}(x) \right) dx \;+$$

$$\int_{p_{X|Y=1} \leq k p_{X|Y=0}} (h_{NP}(x) - h(x)) \left(p_{X|Y=1}(x) - k\, p_{X|Y=0}(x) \right) dx$$

We need to show that this integral is always non -ve

1^{st} Integral when $p_{X|Y=1}(x) > k\, p_{X|Y=0}(x)$

$h_{NP}(x) - h(x) =$

$1 - h(x) \geq 0$

$\Rightarrow$ 1^{st} Integral is non -ve

2^{nd} Integral $\quad p_{X|Y=1}(x) \leq k\, p_{X|Y=0}(x)$

$$h_{NP}(x) - h(x)$$

$$= 0 - h(x)$$

$$\leq 0$$

this will imply 2^{nd} Integral is non -ve.

$$\therefore \; I \geq 0$$

$$I = \int_{\mathbb{R}^n} \left(h_{NP}(x) - h(x)\right)\left(p_{X|Y=1}(x) - k\, p_{X|Y=0}(x)\right) dx$$

$$\geq 0$$

$$\left(\int h_{NP}(x)\, p_{X|Y=1}(x)\, dx - \int h(x)\, p_{X|Y=1}(x)\, dx\right) \geq$$

$$k\left[\int h_{NP}(x)\, p_{X|Y=0}(x)\, dx - \int h(x)\, p_{X|Y=0}(x)\, dx\right]$$

$$\int_{\mathbb{R}^n} h_{NP}(x)\, p_{X|Y=1}(x)\, dx = \mathbb{P}\left[h_{NP}(x) = 1 \mid x \in C_1\right]$$

$$\int_{\mathbb{R}^n} h(x)\, p_{X|Y=1}(x)\, dx = \mathbb{P}\left[h(x) = 1 \mid x \in C_1\right]$$

$$\int_{\mathbb{R}^n} h_{NP}(x)\, p_{x|y=0}(x)\, dx = \mathbb{P}[h_{NP}(x)=1 \mid x \in C_0]$$

$$\int_{\mathbb{R}^n} h(x)\, p_{x|y=0}(x)\, dx = \mathbb{P}[h(x)=1 \mid x \in C_0]$$

$\therefore$ The overall inequality

$$\mathbb{P}[h_{NP}(x)=1 \mid x \in C_1] - \mathbb{P}[h(x)=1 \mid x \in C_1] \geq$$

$$k\left[\mathbb{P}[h_{NP}(x)=1 \mid x \in C_0] - \mathbb{P}[h(x)=1 \mid x \in C_0]\right]$$

But for all "h" under consideration the RHS above is non -ve

$$\mathbb{P}[h_{NP}(x)=1 \mid x \in C_1] - \mathbb{P}[h(x)=1 \mid x \in C_1] \geq 0 \quad \blacksquare$$

* NP Classifier also needs the knowledge of class conditional densities.

Similar to Bayes Classifier, it also is based on the ration $\dfrac{p_{x|y=1}(x)}{p_{x|y=0}(x)}$

* In Bayes Classifier we say Class 1 if

$$\frac{p_{X|Y=1}}{p_{X|Y=0}} > \frac{p_{Y=0}}{p_{Y=1}} \frac{L(0,1)}{L(1,0)}$$

* In NP, the threshold "k" is set based on the allowed Type-1 error.