Problems on Probability

1) 4 Letters
Corresponding Address on 4 envelopes

Then the letters are put to 4 envelopes

Prob that exactly 3 letters are in correct envelops

Answer : zero.

2) A bag contains a Fair Coin & 2 headed Coin

Pick a Coin at random & toss it

it shows up head

→ what is the prob that its a fair coin.

→ You toss the Same Coin again & it shows tails

what is the prob that its a fair coin

Coin 1 : Fair

Coin 2 : Biased Coin.

head - H, tail - T

$$\mathbb{P}[\text{Fair} \mid H] = \frac{\mathbb{P}[H \mid \text{Fair}]\ \mathbb{P}[\text{Fair}]}{\mathbb{P}[H \mid \text{Fair}]\ \mathbb{P}[\text{Fair}] + \mathbb{P}[H \mid \text{Biased}]\ \mathbb{P}[\text{Biased}]}$$

$$= \frac{0.5 * 0.5}{0.5 * 0.5 + 1 * 0.5} = \frac{1}{3}$$

$$\mathbb{P}[\text{Fair} \mid H, T] = \frac{\mathbb{P}[H, T \mid \text{Fair}]\ \mathbb{P}[\text{Fair}]}{\mathbb{P}[H, T \mid \text{Fair}]\ \mathbb{P}[\text{Fair}] + \mathbb{P}[H, T \mid \text{Biased}]\ \mathbb{P}[\text{Biased}]}$$

$$= 1$$

3) X is a RV (CRV)

$$p_X(x) = K x(1-x) \quad , \quad 0 < x < 1$$

$k = ?$

$$\mathbb{P}_X : \mathbb{R} \to [0,1]$$

$$\mathbb{P}_X(x) = \int_{-\infty}^{x} p_X(x) \, dx$$

$$\mathbb{P}_X(\infty) = \int_{-\infty}^{\infty} p_X(x) \, dx$$

$$1 = \int_0^1 p_X(x) \, dx$$

$$= \int_0^1 K(1-x) \, x \, dx$$

$$= K \int_0^1 x - x^2 \, dx$$

$$= K \left[\frac{x^2}{2} \Big|_0^1 - \frac{x^3}{3} \Big|_0^1 \right]$$

$$= k \left[\frac{1}{2} - \frac{1}{3} \right]$$

$$1 = K * \frac{1}{6}$$

$$\Rightarrow \quad K = 6$$

4) $X \sim U[-1, 2]$

$$p_X(x) = \frac{1}{3} \qquad -1 \leq x \leq 2$$

$Y = |X|$

$\mathbb{P}[X \leq Y]$ & $\mathbb{P}[X < Y]$ $= ?$

for any real number x, $x \leq |x|$

Only when x is $-ve$: $x < |x|$

$$\mathbb{P}[X \leq Y] = \mathbb{P}[X \leq |X|] = 1$$

$$\mathbb{P}[X < Y] = \mathbb{P}[X < |X|]$$

$$= \mathbb{P}[X < 0]$$

$$= \int_{-\infty}^{0} p_X(x) \, dx$$

$$= \int_{-1}^{0} \frac{1}{3} \, dx = \frac{1}{3}$$

5)

$X \in \{1, 2, \dots, 20\}$

$$p_X(x) = \frac{1}{20} \quad \forall \ x \in X$$

$\mathbb{E} X$

$$\mathbb{E} X = \sum_x x \ p_X(x)$$

$$= \sum_{k=1}^{20} k \ \frac{1}{20}$$

$$= \frac{1}{20} \sum_{k=1}^{20} k$$

$$= \frac{1}{20} * \frac{20*21}{2}$$

$$= \frac{21}{2}$$

b) $X \sim U[0,1]$

$p_x(x) = 1 \qquad 0 < x \le 1$

$Y = (X+1)^2 \qquad$ find $\mathbb{E}Y$

LOTUS

$$\mathbb{E} f(X) = \int f(x)\, p_x(x)\, dx$$

$\to \mathbb{E}[X+Y]$

$= \mathbb{E}X + \mathbb{E}Y$

$\to \mathbb{E}C = C$

$\to \mathbb{E}aX = a\mathbb{E}X$

$Y = (X^2 + 2X + 1)$

$\mathbb{E}Y = \mathbb{E}[X^2 + 2X + 1]$

$= \mathbb{E}X^2 + 2\,\mathbb{E}X + \mathbb{E}1$

$$\mathbb{E}X = \int_0^1 x\, p_x(x)\, dx = \int_0^1 x\, dx = \frac{x^2}{2}\Big|_0^1 = \frac{1}{2}$$

$$\mathbb{E}X^2 = \int_0^1 x^2\, p_x(x)\, dx = \int_0^1 x^2\, dx = \frac{x^3}{3}\Big|_0^1 = \frac{1}{3}$$

$$\mathbb{E}Y = \frac{1}{3} + 2 * \frac{1}{2} + 1$$

$$= 7/3$$

7) Let X be a RV

$$Y = \frac{X - \mathbb{E}X}{\sqrt{Var(X)}}$$

what is the mean & Variance of Y

$\mathbb{E}X = \mu$

$Var(X) = \sigma^2 \Rightarrow \sqrt{Var(X)} = \sigma$

$$Y = \frac{X-\mu}{\sigma}$$

$$\mathbb{E}Y = \mathbb{E}\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}\mathbb{E}[X-\mu]$$

$$= \frac{1}{\sigma}(\mathbb{E}X - \mathbb{E}\mu) = \frac{1}{\sigma}(\mu-\mu)$$

$$= 0$$

$$Var\,Y = Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2}Var(X-\mu)$$

$\rightarrow Var(aX)$
$= a^2\, Var(X)$

$$= \frac{1}{\sigma^2}Var(X)$$

$\rightarrow Var(X+a)$

$$= \frac{1}{\sigma^2} * \sigma^2 = 1$$

$= Var\,X$

8) X, Y are iid Binary RV taking values 0 & 1 with prob $(1-p)$ & p.

$$Z = X + Y$$

Find the mass func of Z

$Z \in \{0, 1, 2\}$ $\qquad p_Z(0), \; p_Z(1), \; p_Z(2)$

$$p_Z(0) = \mathbb{P}[Z=0] = \mathbb{P}[X+Y=0] = \mathbb{P}[X=0, Y=0]$$

$$= \mathbb{P}[X=0] \; \mathbb{P}[Y=0]$$

$$= (1-p)(1-p) = (1-p)^2$$

$$p_Z(1) = \mathbb{P}[Z=1] = \mathbb{P}[X+Y=1] = \mathbb{P}[X=0, Y=1] + \mathbb{P}[X=1, Y=0]$$

$X=0$, $Y=1$ $\qquad$ $X=1$, $Y=0$

$$p_Z(2) = \mathbb{P}[Z=2] = \mathbb{P}[X+Y=2]$$

$$= \mathbb{P}[X=1, Y=1]$$

Q) We have a random number generator that generates random numbers Uniformly distributed over [0,1]

We generated 10 numbers from it.

what is the probability that the maximum of these numbers is less than 0.8

$x_1, x_2, \ldots, x_{10} \sim \text{iid } U[0,1]$

$M = \max(x_1, x_2, \ldots, x_{10})$

$M < 0.8 = \{x_1 < 0.8, x_2 < 0.8, \ldots, x_{10} < 0.8\}$

$\mathbb{P}[M < 0.8] = \left(\mathbb{P}[x_i < 0.8]\right)^{10}$

$= (0.8)^{10}$

10) Let X, Y be RV with mean 0 & Var 1

& Co-Var ρ, Find $\mathbb{E}[(X+Y)^2]$

$\mathbb{E}X = \mathbb{E}Y = 0$

$Var(X) = Var(Y) = 1$

$\rho_{XY} = \rho$

$$\mathbb{E}[(X+Y)^2]$$

$$= \mathbb{E}[X^2 + Y^2 + 2XY]$$

$$= \mathbb{E}X^2 + \mathbb{E}Y^2 + 2\,\mathbb{E}XY$$

$$= 1 + 1 + 2\rho$$

$$= 2(1+\rho)$$

$$Var(X) = \mathbb{E}X^2 - (\mathbb{E}X)^2$$

$$1 = \mathbb{E}X^2 - 0$$

$$\mathbb{E}X^2 = 1$$

$|||^{ly}$ $\mathbb{E}Y^2 = 1$

$$Cov(X,Y) = \mathbb{E}[XY] - \mathbb{E}X\,\mathbb{E}Y$$

$$\rho = \mathbb{E}[XY]$$

11) Let X, Y be independent RV with

$\mathbb{E}X = \mu_1$, $\mathbb{E}Y = \mu_2$

Let $Z = X + Y$, Find $\mathbb{E}[Z \mid Y]$

$$\mathbb{E}[Z \mid Y] = \mathbb{E}[(X+Y) \mid Y]$$

$$= \mathbb{E}[X \mid Y] + \mathbb{E}[Y \mid Y]$$

$$= \mu_1 + Y$$

$\mathbb{E}[X \mid Y] = \mathbb{E}X = \mu_1$ when X & Y are indpt

$\mathbb{E}[Y \mid Y] = Y$

Let X, Y iid standard Gaussian RV

What is the density of $3X + 4Y$.

$$X, Y \sim N(0,1)$$

$$\mathbb{E} Z = \mathbb{E}[3X + 4Y]$$

$$= 3\,\mathbb{E}X + 4\,\mathbb{E}Y$$

$$= 0$$

$$Var(Z) = Var(3X + 4Y)$$

$$= Var(3X) + Var(4Y) + Cov(3X, 4Y)$$

$$= Var(3X) + Var(4Y)$$

$$= 3^2\,Var(X) + 4^2\,Var(Y)$$

$$= 3^2 + 4^2 = 25$$

$$Z \sim N(0, 25)$$

$$p_Z(z) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(z-\mu)^2}{2\sigma^2}\right\}$$

$$= \frac{1}{5\sqrt{2\pi}} \exp\left\{-\frac{z^2}{50}\right\}$$