Local and Stochastic Volatility Models: An Investigation into the Pricing of Exotic Equity Options

A dissertation submitted to the Faculty of Science, University of the Witwatersrand, Johannesburg, South Africa, in fulfillment of the requirements of the degree of Master of Science.

Abstract

The assumption of constant volatility as an input parameter into the Black-Scholes option pricing formula is deemed primitive and highly erroneous when one considers the terminal distribution of the log-returns of the underlying process. To account for the 'fat tails' of the distribution, we consider both local and stochastic volatility option pricing models. Each class of models, the former being a special case of the latter, gives rise to a parametrization of the skew, which may or may not reflect the correct dynamics of the skew. We investigate a select few from each class and derive the results presented in the corresponding papers. We select one from each class, namely the implied trinomial tree (Derman, Kani & Chriss 1996) and the SABR model (Hagan, Kumar, Lesniewski & Woodward 2002), and calibrate to the implied skew for SAFEX futures. We also obtain prices for both vanilla and exotic equity index options and compare the two approaches.

Lisa Majmin

September 29, 2005

I declare that this is my own, unaided work. It is being submitted for the Degree of Master of Science to the University of the Witwatersrand, Johannesburg. It has not been submitted before for any degree or examination to any other University.

(Signature)

(Date)

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I owe my deepest gratitude to my parents for their unconditional support and kindness.

Contents

Chapter 1

Introduction

Since the derivation of an arbitrage-free and risk-neutral closed-form solution to European option pricing (Black & Scholes 1973), a number of advancements and modifications to the original modelling techniques have been suggested. These attempt to account for certain behavioural patterns displayed by the underlying (equity index in our case) which are contrary to the assumptions that have been made in the original lognormal one-factor model. The original model is Markovian in nature and consists of a deterministic drift term (which is the continuously compounded risk free rate in the risk-neutral world) and a term that accounts for random or volatile behaviour. In pricing European options that have a terminal payoff dependent on the underlying, the assumptions that are made pertain to continuous trading, transaction costs, borrowing and lending and the returns distribution of the underlying. At maturity of the option, and throughout the option life, it is assumed that the terminal distribution of the underlying is lognormal with a constant standard deviation (volatility). The focus of this thesis is to examine two classes of models that have been proposed to account for the leptokurtotic terminal distribution of the underlying, alternatively the non-constant volatility feature.

The first class, local volatility models, are deterministic in nature and can be calibrated using all available market data (European options, current spot and risk free rate etc.). They are deemed arbitrage-free and self-consistent yet produce volatility surfaces which, because they are static in nature, do not display the correct dynamics of the implied volatility skew from which they are derived (this will be seen in Chapter 5). This can be explained by considering the analogy between local volatility surfaces and forward rate curves. Given that the arbitrage-free short rate for some time henceforth is given by the current expected value of the forward rate for that time, so too can the local volatility function be seen as the arbitrage-free expected value of the instantaneous volatility when the underlying is at a particular level at a particular time henceforth. Forward rates are generally not realised, and to use such a model could be considered naive, and would possibly result in losses resulting from inaccuracies in hedge ratios. Nevertheless they are still quite ubiquitous as their implementation is a fairly straightforward task. A unique local volatility surface is constructed using the traded vanilla options. The surface can then be used to price and hedge path-dependent (exotic) options on the underlying. The models retain market completeness, as all input options can be replicated. One can ultimately reach the conclusion that although local volatility models provide a mechanism to extract the local volatility function, they do not provide any reasonable progress in terms of skew-modelling. They also lead to errors resulting from interpolation and extrapolation. These models give rise to a non-parametric surface but fail to explain the existence of the volatility smile (skew).

Chapter 2 provides a fairly detailed introduction to this class of models. Chapter 3 describes the construction of arbitrage-free binomial trees of spot prices and associated probabilities, local volatilities and Arrow-Debreu prices (to be defined). In addition the procedure given in (Derman & Kani 1994), further refinements given in (Barle & Cakici 1995) and (Brandt & Wu 2002) are also discussed. Chapter 4 extends this notion and allows for further flexibility in the trinomial scheme developed in (Derman, Kani & Chriss 1996). In Chapter 5, the result presented in (Dupire 1994), which enables the local volatility to be determined from European option prices, is derived and further extended to allow for the determination of such data from implied volatility. The chapter culminates in the extensive analysis of the dynamics of a local volatility model. The implied Black volatility is derived using perturbation techniques from (Hagan & Woodward 1998). The final result, which is discussed in (Hagan et al. 2002), reveals the flawed dynamics of these models. It is shown that as the current forward level moves, the implied skew for a particular maturity moves in the opposite direction, contrary to known (empirical) behaviour. The local volatility surfaces inferred from vanilla market data are oftentimes unintuitive and lack any reasonable explanation for observed trends. Therefore, in terms of pricing and hedging options, local volatility models lack robustness and will be inaccurate for such tasks.

The inability of such models to accurately price exotics and hedge vanilla options necessitates the further advancement and modification of the lognormal model. Thus, we next consider stochastic volatility models. Most often, these models are chosen for their tractability as well as their pricing and hedging ability. The calibration of the parameters (usually constant) of each of the models is once again performed using the traded vanilla options. Calibration, pricing and hedging is a model-dependent procedure. These models are two-factor and Markovian in nature. The standard Brownian motions may or may not be correlated, depending on the specification of the model. In accordance with risk-neutral valuation, a hedge portfolio is constructed to replicate the option value throughout its life, which results in a partial differential equation for the option value, dependent of the underlying and its volatility or variance process. Calibration of these models is usually performed via a numerical minimization scheme using the market vanilla options. Although both local and stochastic volatility models agree on the vanilla inputs, they generally disagree on the pricing of the exotics i.e. the dynamics of the skew.

This class of models has often been deemed as incomplete as we cannot create a hedge portfolio using the underlying and risk free asset alone. In general, the procedure of creating a hedge is performed using options as well as the above-mentioned assets; this completes the market. In Chapter 6, we derive the partial differential equation and discuss incompleteness with reference to the market price of volatility risk which arises from the change of measure in risk-neutral valuation. Chapter 7 briefly discusses the lognormal stochastic volatility model given in (Hull & White 1987). Chapter 8 reviews the model given in (Heston 1993). A fairly detailed analysis of the Fourier transform technique for option pricing is also provided. The last model we consider is the SABR model in (Hagan et al. 2002), which is derived and explained in Chapter 9. This model is particularly attractive in that it provides closed-form solutions to both vanilla options and their implied volatilities. The authors also assert that it predicts the correct dynamics of the skew. The PDEs satisfied by contingent claims in the two-factor models given in the Chapters 7, Chapter 8 and Chapter 9 above, are solved via numerical approximation such as Monte Carlo simulation (antithetic variates technique and hybrid quasi-Monte Carlo), singular perturbation techniques as in (Hagan et al. 2002) or other mathematical methods which include the Fourier transform as in (Heston 1993).

Chapter 10 deals with the calibration and pricing of various vanilla and exotic options. For the local volatility case, we use the trinomial tree described in Chapter 4 and for the stochastic volatility model, we use that described in Chapter 9. All Excel VBA modules and dlls that are provided are briefly described and results are presented. Other VBA code is provided to generate binomial implied tress, described in Chapter 3 and Monte Carlo simulations of the Hull-White lognormal model and Heston's Ornstein-Uhlenbeck model. A full description is provided in this chapter.

Chapter 2

Local Volatility Models: Implied Binomial and Trinomial Trees

In the Black-Scholes framework (Black & Scholes 1973), the stock price evolves lognormally according to the stochastic differential equation

$$\frac{dS}{S} = \mu dt + \sigma dZ \tag{2.1}$$

where μ is the expected continuously compounded rate of return, σ is the volatility of the stock price, and dZ is a standard Brownian motion with mean zero and variance dt. Both μ and σ are assumed constant. The left hand side of (2.1) is the return provided by the stock in a period dt. Black, Scholes and Merton (Black & Scholes 1973), (Merton 1973) use no-arbitrage arguments, with the assumption a of constant riskfree rate, in the valuation of European derivatives dependent on the stock which follows (2.1). Forming a portfolio that consists of the derivative, and a variable but quantifiable amount of stock, that ensures the portfolio is riskless over an infinitesimal time period dt, they argue that the portfolio should earn the riskless rate. The resulting partial differential equation, which governs derivatives dependent on the underlying traded asset, is then solved with the parameter σ being the only input that is not readily available.

In a discrete-time framework such as (Cox, Ross & Rubinstein 1979) binomial implementation of (2.1), it can also be argued that at each time step, an equivalent portfolio of the stock and riskless asset must replicate the derivative at each node to prevent any arbitrage opportunities. The risk-neutral evolution of the stock is constructed with constant logarithmic stock price spacing, which corresponds to a constant volatility over the entire life of the option. It can be shown that a necessary and sufficient condition for arbitrage-free pricing in a complete market is the existence and uniqueness of an equivalent martingale measure π. The measure is used to price derivatives as the discounted expected value of the payoff at maturity. Under this measure, the stock price and all European contingent claims dependent on it, normalized by the riskfree asset, are martingales.

Vanilla options are generally quoted in terms of implied volatility Σ. This is the constant volatility which, upon substitution into a Black type pricing formula (Black-Scholes, SAFEX Black, Black), will equate the model price to the market price. Use of Σ does not imply belief in Geometric Brownian Motion at that Σ, rather that the formula returns the required price. The Black formulae are increasing functions of volatility, which means that a unique implied volatility per option can always be found.

Since the 1987 crash, it became clear that equity index options with lower (higher) strikes have higher (lower) volatilities. So, out-the-money puts trade at a higher implied volatility than out-the-money calls. By the work of (Breeden & Litzenberger 1978), this can be interpreted as a non-lognormal distribution for the underlying. Thus the relationship between volatility, strike and time to maturity of European options generates an implied volatility surface $\Sigma(S,t)$ that is contrary to the assumption of constant volatility. Following from this, another surface $\sigma(S,t)$, called the local volatility surface, can be created. This is the surface which records the standard deviations of returns given a stock price of S at a time t. In a classical discrete time framework, the volatility, both implied and local, is the same throughout the tree. At first blush there exists a different tree for every different implied volatility that is quoted. Rather, what is required is a tree that can be used simultaneously for all options.

There is an analogy of the relationship that exists between the yield-to-maturity and the forward rates of a discount instrument, and the implied volatility and the local volatilities of an option (Derman, Kani & Zou 1996). The implied volatility of a European option, which is that implied constant future local volatility, equates the Black-Scholes price with the market price. Similarly, the yield-to-maturity of a bond is the implied constant forward capitalization rate that equates the present value of the coupon and principal payments to the current market price. As one would price a non-input bond by obtaining the forward curve from the current yield curve and use these rates to discount the coupons, so too can one use the implied volatility surface of standard European options to deduce future local volatilities for the valuation of exotic options. This does not mean that local volatilities necessarily predict future realised volatility accurately, just as forward rates are also seldom realised. By going long/short relevant bonds, forward rates can be locked in. Analogously, future local volatilities can also be locked in by using options.

Local volatility models are completely deterministic since all information required for the calibration is available. The market smile, which refers to the relationship between the volatility, strike and time-to-maturity of the option, is used as an input to deduce the volatility as a function of the stock price and time $\sigma(S,t)$. A variation in Σ implies a variation in σ with S and t. The model proposes that it is possible to extract the entire surface $\sigma(S,t)$ from standard European option prices. So current options prices uniquely determine the local forward volatilities in the tree.

The idea behind local volatility models is that one can use the discrete set of highly liquid European options for calibration purposes with the intention of valuing and hedging exotic options. At each node, the volatility to the next time period can be calculated and this is then the local volatility. The volatility becomes time- and state-dependent.

Chapter 3

The Derman and Kani Implied Binomial Tree

The first implied recombining binomial tree was developed in (Rubinstein 1994). It is backward inductive and uses the actively traded European options, that mature simultaneously, as inputs. Consequently, it can only be used for valuing other exotic options that expire at the same time as the European options. This model served as a predecessor to other more complex and useful models. In this chapter, the model proposed in (Derman & Kani 1994) will be explored. The inputs are actively traded European options that have various strikes and maturities. This will enable a much wider range of over-the-counter options to be valued and hedged.

The process described by Derman and Kani is forward inductive, creating a binomial tree with uniformly spaced time steps. The root of the tree starts at $t = 0$ with the current spot price, and future time steps are built using all observable data. At each step, the transition probabilities and prices of the underlying must be determined. The range of available European option prices, in addition to theoretical forward prices, are used, since this will ensure the tree is in agreement with the markets' expectation. The resulting tree is then risk-neutral in nature. There is one additional degree of freedom that is solved by a centring condition used in (Cox et al. 1979).

In general, $S_{n,i}$ is the spot price at node (n, i) where $n \geq 0$ is the time step and $0 \leq i \leq n$ is the state. The spot price is $S_{0,0}$. Assuming all information has been calculated up to time step n, at time step $n + 1$ there are $2n + 3$ unknown parameters: $n + 2$ stock prices at nodes $(n + 1, i)$ for $0 \leq i \leq n + 1$ and $n + 1$ risk-neutral transition probabilities $p_{n,i}$ from node (n, i) to node $(n + 1, i + 1)$.

There are $2n + 2$ known quantities at t_{n+1}:

1. $n + 1$ theoretical forward prices $f_{n,i} = S_{n,i}e^{r\Delta t}$, which is the forward price for time $n + 1$ at time n, given that we are at node (n, i). As usual in equity option pricing, the risk free rate is constant throughout the tree.

2. $n + 1$ European option prices with valuation date today, maturity $T = t_{n+1}$ and strikes $S_{n,i}$ for

$0 \leq i \leq n$. These will generally be obtained by interpolation of the implied volatility obtained from the market that corresponds to the strike.

The final degree of freedom is assigned to the centring condition.

Using the risk-neutrality of the implied tree, the expected value, one period later, of the stock price at any node, is its known forward price. Thus

$$f_{n,i} = p_{n,i} S_{n+1,i+1} + (1 - p_{n,i}) S_{n+1,i} \tag{3.1}$$

Hence

$$f_{n,i} - S_{n+1,i} = p_{n,i} (S_{n+1,i+1} - S_{n+1,i})$$

and

$$p_{n,i} = \frac{f_{n,i} - S_{n+1,i}}{(S_{n+1,i+1} - S_{n+1,i})} \tag{3.2}$$

Note that this is an exact generalization of the constant volatility equation

$$\begin{aligned} \pi &= \frac{e^{r\Delta t} - u}{u - d} \\ &= \frac{Se^{r\Delta t} - Su}{Su - Sd} \end{aligned}$$

The option prices are to be interpolated from the market values. They refer to $n+1$ independent options expiring at t_{n+1} with strike levels $S_{n,i}$ and spot $S_{0,0}$. At this strike level, $S_{n,i}$ splits the up and down nodes at t_{n+1}. The node $S_{n+1,i+1}$ ($S_{n+1,i}$) and all those above (below) contribute to the value of a call (put) option. Although the condition is not explicitly checked, the node is chosen according to the inequality $S_{n,i} \leq S_{n+1,i+1} \leq S_{n,i+1}$.

Using (3.1), the $n+1$ option equations and the centring condition of the tree, the stock prices at t_{n+1}, $S_{n+1,i}$ and transition probabilities $p_{n,i}$ for $0 \leq i \leq n$ can then be determined.

3.1 Arrow-Debreu Prices

The implied tree makes use of Arrow-Debreu prices. $\lambda_{n,i}$ is the price today of a security that pays unity at period n, state i and zero elsewhere. Thus it is computed by forward induction as the sum over all paths, from the root of the tree to node (n,i), of the product of the risklessly-discounted transition probabilities at each node in each path leading to node (n,i). The Arrow-Debreu prices for the step $n+1$, $\lambda_{n+1,i}$ are given by

$$\lambda_{0,0} = 1$$

$$e^{r\Delta t}\lambda_{n+1,i} = \begin{cases} p_{n,n}\lambda_{n,n} & \text{for} \quad i = n+1 \\ p_{n,i-1}\lambda_{n,i-1} + (1 - p_{n,i})\lambda_{n,i} & \text{for} \quad 1 \leq i \leq n \\ (1 - p_{n,0})\lambda_{n,0} & \text{for} \quad i = 0 \end{cases} \tag{3.3}$$

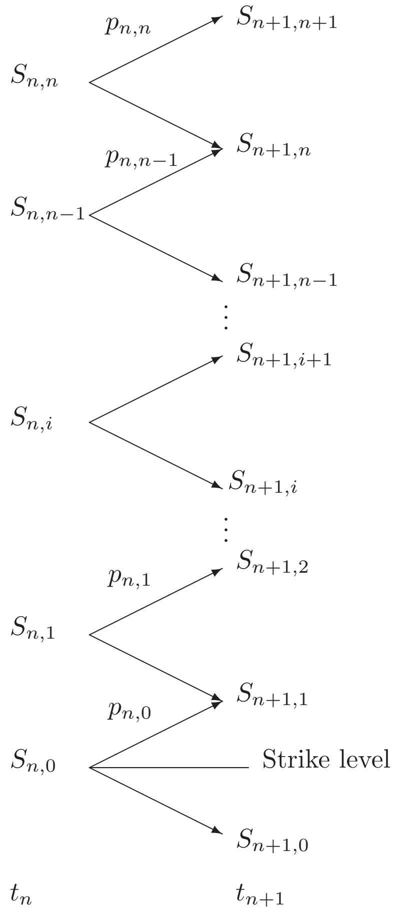

Figure 3.1: Constructing $S_{n+1,i}$, $0 \leq i \leq n$ at t_{n+1} from $S_{n,i}$ at t_n

Let $C\left(S_{n,i}, t_{n+1}\right)$ and $P\left(S_{n,i}, t_{n+1}\right)$ denote the known (possibly interpolated) market values of European call and put prices respectively, with strike $S_{n,i}$ and maturity t_{n+1}. The value in a binomial context that assumes constant volatility, with strike K and maturity t_{n+1} is given as

$$C\left(K, t_{n+1}\right) = e^{-r(n+1)\Delta t} \sum_{j=0}^{n+1} \binom{n+1}{j} \pi^j \left(1-\pi\right)^{n+1-j} \max\left(S_{n+1,j} - K, 0\right)$$

and

$$P\left(K, t_{n+1}\right) = e^{-r(n+1)\Delta t} \sum_{j=0}^{n+1} \binom{n+1}{j} \pi^j \left(1-\pi\right)^{n+1-j} \max\left(K - S_{n+1,j}, 0\right)$$

where π is the risk-neutral probability of an upward movement throughout the tree.

Analogously, in the case of transition probabilities (the probability of an upward or downward movement from t_n to t_{n+1}) that change throughout the tree

$$C\left(K, t_{n+1}\right) = \sum_{j=0}^{n+1} \lambda_{n+1,j} \max\left(S_{n+1,j} - K, 0\right) \tag{3.4}$$

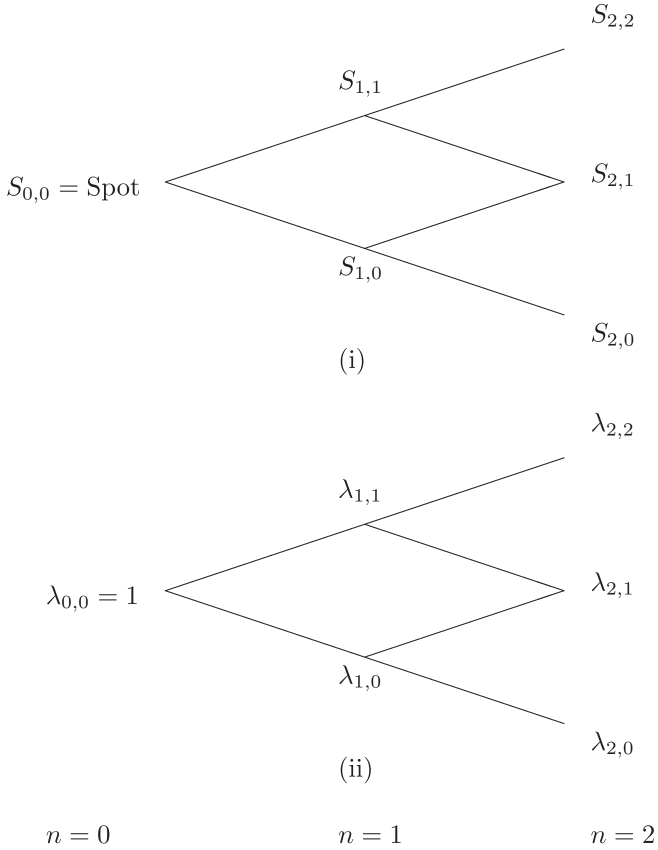

Figure 3.2: Binomial Tree of (i) Stock Prices and (ii) Arrow-Debreu Prices

and

$$P\left(K, t_{n+1}\right) = \sum_{j=0}^{n+1} \lambda_{n+1,j} \max\left(K - S_{n+1,j}, 0\right) \tag{3.5}$$

3.2 Upper Tree

Consider the portion of the tree that extends from the centre upwards. The European call prices, $C\left(S_{n,i}, t_{n+1}\right)$, will be required for the evaluation of the stock prices. When the strike is taken to be $S_{n,i}$, it is only necessary to consider the nodes from $S_{n+1,i+1}$ upwards. The interpolated implied volatility relating to the required strike is then used, for consistency, in the Cox-Ross-Rubinstein binomial tree. For accuracy, the Black-Scholes formula can also be used. This will be dealt with in §3.7.

Using (3.3)

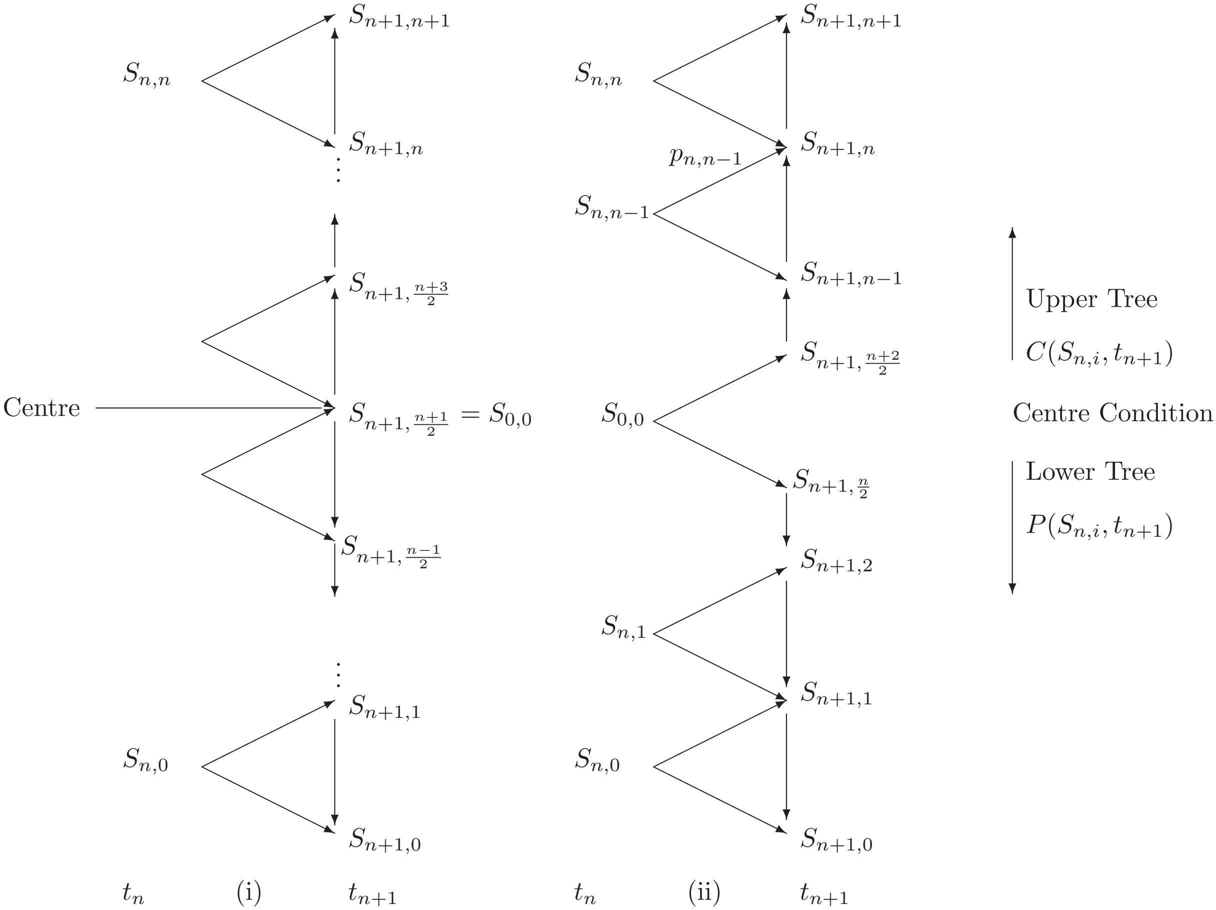

Figure 3.3: Inductive Procedure for $S_{n+1,i}$, $0 \leq i \leq n$ when n is (i) odd and (ii) even

$$
\begin{aligned}
& C\left(S_{n,i}, t_{n+1}\right) \\
& = \sum_{j=i+1}^{n+1} \lambda_{n+1,j}\left(S_{n+1,j} - K\right) \\
& = e^{-r\Delta t} \sum_{j=i+1}^{n} \left(\lambda_{n,j-1} p_{n,j-1} + \lambda_{n,j}\left(1 - p_{n,j}\right)\right)\left(S_{n+1,j} - S_{n,i}\right) \\
& + e^{-r\Delta t} \lambda_{n,n} p_{n,n}\left(S_{n+1,n+1} - S_{n,i}\right)
\end{aligned}
$$

Expanding (3.4) and using (3.3), the call price can be simplified as follows:

$$
\begin{aligned}
&e^{r\Delta t}C\left(S_{n,i},t_{n+1}\right)\\
&=p_{n,n}\lambda_{n,n}\left(S_{n+1,n+1}-S_{n,i}\right)+\sum_{j=i+1}^{n}\left(\left(1-p_{n,j}\right)\lambda_{n,j}+p_{n,j-1}\lambda_{n,j-1}\right)\left(S_{n+1,j}-S_{n,i}\right)\\
&=\sum_{j=i+1}^{n}p_{n,j}\lambda_{n,j}\left(\left(S_{n+1,j+1}-S_{n,i}\right)-\left(S_{n+1,j}-S_{n,i}\right)\right)\\
&+\sum_{j=i+1}^{n}\lambda_{n,j}\left(S_{n+1,j}-S_{n,i}\right)+p_{n,i}\lambda_{n,i}\left(S_{n+1,i+1}-S_{n,i}\right)\\
&=\sum_{j=i+1}^{n}p_{n,j}\lambda_{n,j}\left(S_{n+1,j+1}-S_{n+1,j}\right)+\sum_{j=i+1}^{n}\lambda_{n,j}\left(S_{n+1,j}-S_{n,i}\right)\\
&+p_{n,i}\lambda_{n,i}\left(S_{n+1,i+1}-S_{n,i}\right)
\end{aligned}
$$

Using (3.1), the price then becomes

$$
\begin{aligned}
&e^{r\Delta t}C\left(S_{n,i},t_{n+1}\right)\\
&=\sum_{j=i+1}^{n}\lambda_{n,j}\left(f_{n,j}-S_{n+1,j}\right)+\sum_{j=i+1}^{n}\lambda_{n,j}\left(S_{n+1,j}-S_{n,i}\right)+p_{n,i}\lambda_{n,i}\left(S_{n+1,i+1}-S_{n,i}\right)
\end{aligned}
$$

Thus

$$
e^{r\Delta t}C\left(S_{n,i},t_{n+1}\right)=p_{n,i}\lambda_{n,i}\left(S_{n+1,i+1}-S_{n,i}\right)+\sum_{j=i+1}^{n}\lambda_{n,j}\left(f_{n,j}-S_{n,i}\right) \tag{3.6}
$$

Using (3.6) and (3.2), the stock prices $S_{n+1,i+1}$ can be found in terms of $S_{n+1,i}$:

$$
\begin{aligned}
&S_{n+1,i+1}\left[e^{r\Delta t}C\left(S_{n,i},t_{n+1}\right)-\sum_{j=i+1}^{n}\lambda_{n,j}\left(f_{n,j}-S_{n,i}\right)-\lambda_{n,i}\left(f_{n,i}-S_{n+1,i}\right)\right]\\
=\quad &S_{n+1,i}\left[e^{r\Delta t}C\left(S_{n,i},t_{n+1}\right)-\sum_{j=i+1}^{n}\lambda_{n,j}\left(f_{n,j}-S_{n,i}\right)-\lambda_{n,i}\left(f_{n,i}-S_{n+1,i}\right)\right]
\end{aligned}
$$

Finding the stock prices $S_{n+1,i+1}$ in terms of $S_{n+1,i}$., the upper node formula is

$$
S_{n+1,i+1}\quad=\quad\frac{S_{n+1,i}\left[e^{r\Delta t}C\left(S_{n,i},t_{n+1}\right)-\Sigma_i\right]-\lambda_{n,i}S_{n,i}\left(f_{n,i}-S_{n+1,i}\right)}{e^{r\Delta t}C\left(S_{n,i},t_{n+1}\right)-\Sigma_i-\lambda_{n,i}\left(f_{n,i}-S_{n+1,i}\right)} \tag{3.7}
$$

where Σ_i refers to

$$
\sum_{j=i+1}^{n}\lambda_{n,j}\left(f_{n,j}-S_{n,i}\right)
$$

All that is required is to know one initial $S_{n+1,i}$. This is to obtained from the centring condition that is discussed below.

3.3 Centre of the Tree

At each time step, the starting point is the centre of the tree. There are two cases to consider.

3.3.1 Odd Number of Nodes

If n is odd, then the number of nodes $(n+2)$ at t_{n+1} is odd. Select the central node $S_{n+1,\frac{n+1}{2}}$, to be the spot today, $S_{0,0}$. The remainder of the upper part of the tree can be found using (3.7). The transition probabilities can be found using (3.2).

3.3.2 Even number of nodes

If n is even and there are an even number of nodes at t_{n+1}, then set the average of the logarithm of the two central nodes equal the logarithm of today's spot. So for $i=\frac{n}{2}$, $\ln S_{0,0}=\frac{1}{2}\left(\ln S_{n+1,\frac{n}{2}+1}+\ln S_{n+1,\frac{n}{2}}\right)$.

The centring condition implies that $S_{n+1,\frac{n}{2}}=S_{n,i}^{2}/S_{n+1,\frac{n}{2}+1}$, where $S_{n,i}=S_{0,0}$. Using this condition in (3.7), for $i=\frac{n}{2}$

$$\begin{aligned}
&S_{n+1,i+1}\left[e^{r\Delta t}C\left(S_{n,i},t_{n+1}\right)-\Sigma_i\right]-S_{n+1,i+1}\lambda_{n,i}\left(f_{n,i}-S_{n+1,i}\right)\\
&=S_{n+1,i}\left[e^{r\Delta t}C\left(S_{n,i},t_{n+1}\right)-\Sigma_i\right]-\lambda_{n,i}S_{n,i}\left(f_{n,i}-S_{n+1,i}\right)
\end{aligned}$$

Using (3.6) and (3.1) the above becomes

$$\begin{aligned}
&S_{n+1,i+1}\left[e^{r\Delta t}C\left(S_{n,i},t_{n+1}\right)-\Sigma_i\right]-S_{n+1,i+1}\lambda_{n,i}f_{n,i}+S_{n+1,i+1}\lambda_{n,i}S_{n+1,i}\\
&=S_{n+1,i}p_{n,i}\lambda_{n,i}\left(S_{n+1,i+1}-S_{n,i}\right)-\lambda_{n,i}S_{n,i}\left[p_{n,i}S_{n+1,i+1}+\left(1-p_{n,i}\right)S_{n+1,i}\right]\\
&+\lambda_{n,i}S_{n,i}S_{n+1,i}
\end{aligned}$$

since $\left[e^{r\Delta t}C\left(S_{n,i},t_{n+1}\right)-\Sigma_i\right]=p_{n,i}\lambda_{n,i}\left(S_{n+1,i+1}-S_{n,i}\right)$

Upon simplification

$$S_{n+1,\frac{n}{2}+1} = \frac{S_{0,0}\left[e^{r\Delta t}C\left(S_{0,0},t_{n+1}\right)+\lambda_{n,i}S_{0,0}-\Sigma_i\right]}{\lambda_{n,i}f_{n,i}-e^{r\Delta t}C\left(S_{0,0},t_{n+1}\right)+\Sigma_i} \tag{3.8}$$

After this initial node is calculated, all nodes above it for $\frac{n}{2}+2\leq i\leq n+1$ can then be calculated using (3.6).

3.3.3 Lower Tree

In a manner analogous to the upper part of the tree, the lower part is determined using the interpolated European put market prices, $P\left(S_{n,i},t_{n+1}\right)$.

When the strike is taken to be $S_{n,i}$, it is only necessary to consider the nodes from $S_{n+1,i}$ downwards.

Using (3.3) and expanding (3.5)

$$
\begin{aligned}
& P\left(S_{n,i}, t_{n+1}\right) \\
& = \sum_{j=0}^{i} \lambda_{n+1,j}\left(K - S_{n+1,j}\right) \\
& = e^{-r\Delta t} \sum_{j=1}^{i} \left[\lambda_{n,j-1} p_{n,j-1} + \lambda_{n,j}\left(1 - p_{n,j}\right)\right]\left(S_{n,i} - S_{n+1,j}\right) \\
& + e^{-r\Delta t} \lambda_{n,0}\left(1 - p_{n,0}\right)\left(S_{n,i} - S_{n+1,0}\right)
\end{aligned}
$$

The put price can be simplified as follows:

$$
\begin{aligned}
& e^{r\Delta t} P\left(S_{n,i}, t_{n+1}\right) \\
& = \sum_{j=0}^{i-1} \lambda_{n,j} p_{n,j}\left(S_{n,i} - S_{n+1,j+1}\right) + \sum_{j=0}^{i} \lambda_{n,j}\left(1 - p_{n,j}\right)\left(S_{n,i} - S_{n+1,j}\right) \\
& = \sum_{j=0}^{i-1} \lambda_{n,j} p_{n,j}\left[\left(S_{n,i} - S_{n+1,j+1}\right) - \left(S_{n,i} - S_{n+1,j}\right)\right] + \sum_{j=0}^{i-1} \lambda_{n,j}\left(S_{n,i} - S_{n+1,j}\right) \\
& + \lambda_{n,i}\left(1 - p_{n,i}\right)\left(S_{n,i} - S_{n+1,i}\right) \\
& = \sum_{j=0}^{i-1} \lambda_{n,j} p_{n,j}\left(S_{n+1,j} - S_{n+1,j+1}\right) + \sum_{j=0}^{i-1} \lambda_{n,j}\left(S_{n,i} - S_{n+1,j}\right) \\
& + \lambda_{n,i}\left(1 - p_{n,i}\right)\left(S_{n,i} - S_{n+1,i}\right)
\end{aligned}
$$

Using (3.1), the price is then

$$
\begin{aligned}
& e^{r\Delta t} P\left(S_{n,i}, t_{n+1}\right) \\
& = \sum_{j=0}^{i-1} \lambda_{n,j}\left(S_{n+1,j} - f_{n,j}\right) + \sum_{j=0}^{i-1} \lambda_{n,j}\left(S_{n,i} - S_{n+1,j}\right) + \lambda_{n,i}\left(1 - p_{n,i}\right)\left(S_{n,i} - S_{n+1,i}\right) \\
& = \sum_{j=0}^{i-1} \lambda_{n,j}\left(S_{n,i} - f_{n,j}\right) + \lambda_{n,i}\left(1 - p_{n,i}\right)\left(S_{n,i} - S_{n+1,i}\right)
\end{aligned}
$$

Thus

$$
e^{r\Delta t} P\left(S_{n,i}, t_{n+1}\right) = \lambda_{n,i}\left(1 - p_{n,i}\right)\left(S_{n,i} - S_{n+1,i}\right) + \sum_{j=0}^{i-1} \lambda_{n,j}\left(S_{n,i} - f_{n,j}\right) \tag{3.9}
$$

Using (3.2), an expression for the lower nodes in terms of the higher ones can be found according to

$$
\begin{aligned}
& e^{r\Delta t} P\left(S_{n,i}, t_{n+1}\right) \\
& = \lambda_{n,i}\left[1 - \left(\frac{f_{n,i} - S_{n+1,i}}{S_{n+1,i+1} - S_{n+1,i}}\right)\right]\left(S_{n,i} - S_{n+1,i}\right) + \sum_{j=0}^{i-1} \lambda_{n,j}\left(S_{n,i} - f_{n,j}\right)
\end{aligned}
$$

Upon simplification

$$
\begin{aligned}
&\left(S_{n+1,i+1} - S_{n+1,i}\right) e^{r\Delta t} P\left(S_{n,i}, t_{n+1}\right) \\
&= \lambda_{n,i}\left(S_{n+1,i+1} - f_{n,i}\right) + \left(S_{n+1,i+1} - S_{n+1,i}\right) \sum_{j=0}^{i-1} \lambda_{n,j}\left(S_{n,i} - f_{n,j}\right)
\end{aligned}
$$

Solving for $S_{n+1,i}$

$$
S_{n+1,i} = \frac{S_{n+1,i+1}\left[e^{r\Delta t} P\left(S_{n,i}, t_{n+1}\right) - \Sigma_i\right] + \lambda_{n,i} S_{n,i}\left(f_{n,i} - S_{n+1,i+1}\right)}{e^{r\Delta t} P\left(S_{n,i}, t_{n+1}\right) - \Sigma_i + \lambda_{n,i}\left(f_{n,i} - S_{n+1,i+1}\right)} \tag{3.10}
$$

where Σ_i refers to

$$
\sum_{j=0}^{i-1} \lambda_{n,j}\left(S_{n,i} - f_{n,j}\right)
$$

3.4 Transition Probabilities

Throughout the tree, the transition probabilities $p_{n,i}$ must satisfy $0 < p_{n,i} < 1$. This is to prevent any arbitrage opportunities: if $p_{n,i} > 1$, then $S_{n+1,i+1}$ will fall below $f_{n,i}$ and similarly, if $p_{n,i} < 0$, $S_{n+1,i}$ will be higher than $f_{n,i}$. This leads to the requirement that throughout the tree, $f_{n,i} \leq S_{n+1,i+1} \leq f_{n,i+1}$. If there is a violation of this inequality at node $S_{n+1,i+1}$, choose the stock price that ensures the logarithmic spacing between this node and the adjacent node is the same as that between corresponding nodes at the previous time step. For $i < n$,

$$
\ln \frac{S_{n+1,i+1}}{S_{n+1,i}} = \ln \frac{S_{n,i+1}}{S_{n,i}}
$$

For $i = n$, if $f_{n,n} \geq S_{n+1,n+1}$, then

$$
\ln \frac{S_{n+1,n+1}}{S_{n+1,n}} = \ln \frac{S_{n,n}}{S_{n,n-1}}
$$

So the above conditions can be written as

$$
\begin{aligned}
S_{n+1,i+1} &= S_{n+1,i} \frac{S_{n,i+1}}{S_{n,i}} \\
S_{n+1,n+1} &= S_{n+1,n} \frac{S_{n,n}}{S_{n,n-1}}
\end{aligned}
$$

3.5 Local Volatility

As usual, we denote the expectation and variance by $\mathbb{E}[\cdot]$ and $\mathbb{V}[\cdot]$ respectively. To calculate the local volatilities, the binomial nature of the tree with the log-returns are used. If $\ln X$ evolves to $\ln Y$ with probability p and to $\ln Z$ with probability $(1-p)$, then

$$
\mathbb{E}[\ln X] = p \ln Y + (1-p) \ln Z
$$

and

$$
\begin{aligned}
&\mathbb{V}[\ln X] \\
&= \mathbb{E}\left[(\ln X)^2\right] - \mathbb{E}[\ln X]^2 \\
&= p(\ln Y)^2 + (1-p)(\ln Z)^2 - [p\ln Y + (1-p)\ln Z]^2 \\
&= p(\ln Y)^2 + (1-p)(\ln Z)^2 - p^2(\ln Y)^2 - 2p(1-p)\ln Y \ln Z - (1-p)^2(\ln Z)^2 \\
&= (\ln Y)^2 p(1-p) + (\ln Z)^2 p(1-p) - 2p(1-p)\ln Y \ln Z \\
&= p(1-p)\left[\ln\frac{Y}{Z}\right]^2
\end{aligned}
$$

The local volatility $\sigma_{n,i}$ is calculated as the annualized standard deviation of the log-returns at the node (n,i). The general case in a binomial context is to consider the movement in the tree from $\ln S_{n,i}$ to $\ln S_{n+1,i+1}$ with probability $p_{n,i}$ and to $\ln S_{n+1,i}$ with probability $(1-p_{n,i})$. It is clear that $\mathbb{E}\left[\ln \frac{S_{n+1,i}}{S_{n,i}}\right]$ differs from $\mathbb{E}[\ln S_{n+1,i}]$, for $0 \leq i \leq n$, by the constant value of $\ln S_{n,i}$. It is also the case that $\mathbb{V}\left[\ln \frac{S_{n+1,i}}{S_{n,i}}\right]$ and $\mathbb{V}[\ln S_{n+1,i}]$ are equal. This result is invoked for simplification of the calculations. Therefore, since the volatility is generally taken to be per annum but the period of interest is over Δt, for $0 \leq i \leq n$

$$
\begin{aligned}
\sigma^2_{n,i} &= \frac{1}{\Delta t} p_{n,i}(1-p_{n,i})\left[\ln\left(\frac{S_{n+1,i+1}}{S_{n+1,i}}\right)\right]^2 \\
\sigma_{n,i} &= \frac{1}{\sqrt{\Delta t}}\sqrt{p_{n,i}}\sqrt{1-p_{n,i}}\ln\left(\frac{S_{n+1,i+1}}{S_{n,i}}\right) \qquad (3.11)
\end{aligned}
$$

3.6 Computational Algorithm

Implementation of the Derman-Kani procedure is performed by taking the input, which is the implied volatility of European options of certain strikes and maturities (generally taken at equally spaced intervals in time), and producing a risk-neutral binomial tree that describes the evolution of the underlying from $t = 0$ until expiry of the final maturity of the given option inputs. The time steps in the tree will be equal to the expiry dates of the input options.

3.6.1 Input Data

The following data is standard input:

1. Valuation date (taken to be $t = 0$)
2. Spot on valuation date
3. Expiry date (last maturity date of European options)

4. Risk-free rate

5. Implied volatilities relevant to each strike at each time step

3.6.2 Algorithm

1. Taking the valuation date as the root of the tree (corresponding to $n = 0$), the levels are built up by starting at the centre. Depending on what level is being built, the first requirement is to determine whether n, corresponding to the current time step t_n, is even or odd.

   * If $n = 0$ Mod 2, the next level to be built t_{n+1} will have an even number of nodes. $S_{n+1,\frac{n}{2}} = S_{0,0}^2/S_{n+1,\frac{n}{2}+1}$ and (3.8) are used to determine the two central nodes.
   * Else for n being odd, the number of nodes at t_{n+1} is odd and $S_{n+1,\frac{n+1}{2}} = S_{0,0}$

2. The remainder of the upper nodes, provided $n > 0$, are then calculated using (3.7). In order to calculate the call option prices with strike $S_{n,i}$ for $i > \frac{n}{2}+1$ if n even or $i > \frac{n+1}{2}$ if n odd, the input data is recalled. Linear interpolation is performed on the implied volatility of the strikes to find the volatility that is required to price the options. The necessary interpolation is performed on the implied volatilities to obtain σ. Once this value has been deduced from the discrete set of data at the expiration t_{n+1}, the Cox-Ross-Rubinstein binomial model is used to price the call option. This is done to be consistent with the binomial framework; there

$$C\left(S_{n,i}, t_{n+1}\right) = e^{-r(n+1)\Delta t} \sum_{j=0}^{n+1} \binom{n+1}{j} p^j \left(1-p\right)^{n+1-j} \max\left(S_{0,0} u^j d^{n+1-j} - S_{n,i}, 0\right)$$

   where p is the probability of an upward movement. The multiplicative up factor, u is calculated by $u = e^{\sigma\sqrt{\Delta t}}$ and $1/u = d$. It is necessary to calculate the above summation for $C\left(S_{n,i}, t_{n+1}\right)$ using a loop. Considering the values for j such that $S_{0,0}u^j d^{n-j} - S_{n,i} \geq 0$, j is solved for.

   Use a loop to calculate $C\left(S_{n,i}, t_{n+1}\right)$. Consider values of j such that $S_{0,0}u^j d^{n-j} - S_{n,i} \geq 0$

$$\begin{aligned} u^j d^{n-j} &\geq \frac{S_{n,i}}{S_{0,0}} \\ \ln u^{2j-n} &\geq \ln \frac{S_{n,i}}{S_{0,0}} \\ 2j - n &\geq \frac{\ln \frac{S_{n,i}}{S_{0,0}}}{\ln u} \end{aligned}$$

   So,

$$j \geq \left\lceil \frac{1}{2} \frac{\ln \frac{S_{n,i}}{S_{0,0}}}{\ln u} + \frac{n}{2} + \frac{1}{2} \right\rceil \equiv \alpha \tag{3.12}$$

where $[\cdot]$ denotes rounding.

Using a loop that begins at node $n+1$ and steps down to node α, the binomial coefficient at j for each $n+1 \geq j \geq \alpha$ is determined using the so-called 'in-out' recursion relation

$$\begin{aligned}\binom{n}{n} &= 1 \\ \binom{n}{j-1} &= \binom{n}{j}\frac{j}{(n-j)}\end{aligned}$$

Once the option price is obtained, use (3.6) to find the remainder of the nodes in the upper part of the tree.

The no-arbitrage condition, $f_{n,i} \leq S_{n+1,i+1} \leq f_{n,i+1}$, must be checked as each node value is calculated. If the above inequality is violated, then

* For $i < n$:

$$S_{n+1,i+1} = S_{n+1,i}\frac{S_{n,i+1}}{S_{n,i}}$$

* For $i = n$:

$$S_{n+1,n+1} = S_{n+1,n}\frac{S_{n,n}}{S_{n,n-1}}$$

The central and upper part of the tree can be fully determined from the stated procedure.

3. The inductive procedure for the lower part of the tree is initiated from the central portion of the tree, and then steps downwards until the entire set of nodes have been determined. Provided $n > 0$, the first node in this portion of the tree will be calculated using either (3.9) if n is odd or $S_{n+1,\frac{n}{2}} = S_{0,0}^2/S_{n+1,\frac{n}{2}+1}$ otherwise. The remainder of the nodes will all be determined using (3.9).

   The procedure is the same as that for the upper part of the tree. The differences arise in the calculation of the option prices, which are put options in this case.

   The put option prices with maturity t_{n+1} and strike $S_{n,i}$ for $i < \frac{n}{2}+1$ if n even or $i < \frac{n+1}{2}$ if n odd must be returned from the input data. The binomial put option price is given by

$$P\left(S_{n,i}, t_{n+1}\right) = e^{-r(n+1)\Delta t}\sum_{j=0}^{n+1}\binom{n+1}{j}p^j\left(1-p\right)^{n+1-j}\max\left(S_{n,i} - S_{0,0}u^j d^{n+1-j}, 0\right)$$

   where p is the probability of an upward movement. The same values are attributed to u and d.

   While performing a loop to calculate $P\left(S_{n,i}, t_{n+1}\right)$, it is only relevant to consider the values for j such that $S_{0,0}u^j d^{n-j} - S_{n,i} \leq 0$. From (3.12), it is clear that $\alpha - 1 \geq j \geq 0$ since it is the remainder of the nodes that contribute to the put price. The summation loop begins at $j = 0$ and continues upwards to node $\alpha - 1$. The binomial coefficient is determined using the 'in-out' recursion relation

$$\begin{aligned}\binom{n}{0} &= 1 \\ \binom{n}{j+1} &= \binom{n}{j}\frac{n-j}{j+1}\end{aligned}$$

The no-arbitrage condition, $f_{n,i} \leq S_{n+1,i+1} \leq f_{n,i+1}$, must also be checked as each node value is calculated. If the above inequality is violated, then

* For $i > 0$:

$$S_{n+1,i} = S_{n+1,i+1} \frac{S_{n,i}}{S_{n,i+1}}$$

* For $i = 0$:

$$S_{n+1,0} = S_{n+1,1} \frac{S_{n,0}}{S_{n,1}}$$

The entire tree, at t_{n+1} is then fully determined. The n transition probabilities and Arrow-Debreu prices can be calculated using (3.2) and (3.3) respectively. The tree of local volatilities are also determined using (3.11).

3.7 Barle and Cakici Algorithm Modifications

A number of adjustments in (Barle & Cakici 1995) have been suggested to the above procedure. Considering the Cox-Ross-Rubinstein binomial tree, there seems to be a higher chance of obtaining negative probabilities with high interest rates and constant local volatility. To solve this, the time steps can be made smaller or a tree of forward prices can be built which can then be translated back to the prices of the underlying. High interest rates seem to pose a similar problem in the construction of the implied tree. The changes to algorithm are described below.

1. Use the Black-Scholes option pricing formula to calculate the prices of the European options, as it is computationally faster and converges far better than the C-R-R formula. Moreover, this is more sensible since the market volatilities are Black-Scholes volatilities, not Cox-Ross-Rubinstein volatilities.

2. Since negative transition probabilities are to be excluded, it is shown below that the price $S_{n+1,i+1}$ is confined to the interval

$$f_{n,i} \leq S_{n+1,i+1} \leq f_{n,i+1} \tag{3.13}$$

Instead of (3.13), Derman and Kani assume $S_{n,i} \leq S_{n+1,i+1} \leq S_{n,i+1}$. Another difference is that strikes are the forward not the spot levels.

Consider the upper portion of the tree:

For the interpolated European call option prices, the strike K is chosen to be $f_{n,i}$ to be consistent with (3.13). Substituting $f_{n,i}$ into the (3.4), we get to be consistent with the above inequality:

$$C(f_{n,i}, t_{n+1}) = \sum_{j=0}^{n+1} \lambda_{n+1,j} \max(S_{n+1,j} - f_{n,i}, 0) \tag{3.14}$$

Once again, only the nodes from $S_{n+1,i+1}$ need to be considered. Using (3.3)

$$C(f_{n,i}, t_{n+1}) = e^{-r\Delta t} \sum_{j=i+1}^{n} [\lambda_{n,j-1} p_{n,j-1} + \lambda_{n,j}(1 - p_{n,j})] (S_{n+1,j} - f_{n,i})$$
$$+ e^{-r\Delta t} \lambda_{n,n} p_{n,n} (S_{n+1,n+1} - f_{n,i})$$

Expanding and using (3.1), the call price can be simplified as follows:

$$
\begin{aligned}
& e^{r\Delta t} C\left(f_{n,i}, t_{n+1}\right) \\
& = \sum_{j=i+1}^{n} \left[\left(1-p_{n,j}\right)\lambda_{n,j} + p_{n,j-1}\lambda_{n,j-1}\right]\left(S_{n+1,j} - f_{n,i}\right) \\
& + p_{n,n}\lambda_{n,n}\left(S_{n+1,n+1} - f_{n,i}\right) \\
& = \sum_{j=i+1}^{n} p_{n,j}\lambda_{n,j}\left[\left(S_{n+1,j+1} - f_{n,i}\right) - \left(S_{n+1,j} - f_{n,i}\right)\right] \\
& + \sum_{j=i+1}^{n} \lambda_{n,j}\left(S_{n+1,j} - f_{n,i}\right) + p_{n,i}\lambda_{n,i}\left(S_{n+1,i+1} - f_{n,i}\right) \\
& = \sum_{j=i+1}^{n} \lambda_{n,j}\left(\left[p_{n,j}S_{n+1,j+1} + \left(1-p_{n,j}\right)S_{n+1,j}\right] - f_{n,i}\right) \\
& + p_{n,i}\lambda_{n,i}\left(S_{n+1,i+1} - f_{n,i}\right)
\end{aligned}
$$

Using the risk-neutral equation for the price of a forward at t_n with expiry t_{n+1}

$$
f_{n,i} = p_{n,i}S_{n+1,i+1} + \left(1-p_{n,i}\right)S_{n+1,i}
$$

The price then becomes

$$
e^{r\Delta t} C\left(f_{n,i}, t_{n+1}\right) = \sum_{j=i+1}^{n} \lambda_{n,j}\left(f_{n,j} - f_{n,i}\right) + p_{n,i}\lambda_{n,i}\left(S_{n+1,i+1} - f_{n,i}\right) \tag{3.15}
$$

Now define

$$
\Delta_i^C = e^{r\Delta t} C\left(f_{n,i}, t_{n+1}\right) - \sum_{j=i+1}^{n} \lambda_{n,j}\left(f_{n,j} - f_{n,i}\right)
$$

So, (3.15) is reduced to

$$
\Delta_i^C = p_{n,i}\lambda_{n,i}\left(S_{n+1,i+1} - f_{n,i}\right) \tag{3.16}
$$

which is a known quantity. Using the risk-neutrality of the implied tree and substituting (3.2) into (3.15), the following recursion formula is obtained for the stock prices in the upper portion of the tree:

$$
\begin{aligned}
& S_{n+1,i+1}\left[e^{r\Delta t} C\left(f_{n,i}, t_{n+1}\right) - \sum_{j=i+1}^{n} \lambda_{n,j}\left(f_{n,j} - f_{n,i}\right) - \lambda_{n,i}\left(f_{n,i} - S_{n+1,i}\right)\right] \\
& = S_{n+1,i}\left[e^{r\Delta t} C\left(f_{n,i}, t_{n+1}\right) - \sum_{j=i+1}^{n} \lambda_{n,j}\left(f_{n,j} - f_{n,i}\right)\right] + \lambda_{n,i} f_{n,i}\left(S_{n+1,i} - f_{n,i}\right)
\end{aligned}
$$

Upon simplification

$$S_{n+1,i+1} = \frac{S_{n+1,i}\Delta_i^C + \lambda_{n,i} f_{n,i} \left(S_{n+1,i} - f_{n,i}\right)}{\Delta_i^C - \lambda_{n,i} \left(f_{n,i} - S_{n+1,i}\right)} \tag{3.17}$$

Consider the lower portion of the tree:

The same reasoning applies to the interpolated European put option prices. The strike is now taken to be $f_{n,i}$, it is only necessary to consider the nodes from $S_{n+1,i}$ downwards. The put option price with strike $f_{n,i}$ and maturity t_{n+1} is given as

$$P\left(f_{n,i}, t_{n+1}\right) = \sum_{i=0}^{n+1} \lambda_{n+1,i} \max\left(f_{n,i} - S_{n+1,i}, 0\right) \tag{3.18}$$

Using (3.3) and expanding (3.18)

$$\begin{aligned} P\left(f_{n,i}, t_{n+1}\right) = {} & e^{-r\Delta t} \sum_{j=1}^{i} \left[\lambda_{n,j-1} p_{n,j-1} + \lambda_{n,j} \left(1 - p_{n,j}\right)\right] \left(f_{n,i} - S_{n+1,j}\right) \\ & + e^{-r\Delta t} \lambda_{n,0} \left(1 - p_{n,0}\right) \left(f_{n,i} - S_{n+1,0}\right) \end{aligned}$$

The put price can be simplified as follows:

$$\begin{aligned} & e^{r\Delta t} P\left(f_{n,i}, t_{n+1}\right) \\ & = \sum_{j=0}^{i-1} \lambda_{n,j} p_{n,j} \left(f_{n,i} - S_{n+1,j+1}\right) + \sum_{j=0}^{i} \lambda_{n,j} \left(1 - p_{n,j}\right) \left(f_{n,i} - S_{n+1,j}\right) \\ & = \sum_{j=0}^{i-1} \lambda_{n,j} \left[p_{n,j} \left(f_{n,i} - S_{n+1,j+1}\right) + \left(1 - p_{n,j}\right) \left(f_{n,i} - S_{n+1,j}\right)\right] \\ & + \lambda_{n,i} \left(1 - p_{n,i}\right) \left(f_{n,i} - S_{n+1,i}\right) \\ & = \sum_{j=0}^{i-1} \lambda_{n,j} \left(f_{n,i} - \left[p_{n,j} S_{n+1,j+1} + \left(1 - p_{n,j}\right) S_{n+1,j}\right]\right) \\ & + \lambda_{n,i} \left(1 - p_{n,i}\right) \left(f_{n,i} - S_{n+1,i}\right) \end{aligned}$$

Using (3.1), the price is then

$$\begin{aligned} & e^{r\Delta t} P\left(S_{n,i}, t_{n+1}\right) \\ & = \sum_{j=0}^{i-1} \lambda_{n,j} \left(f_{n,i} - f_{n,j}\right) + \lambda_{n,i} \left(1 - p_{n,i}\right) \left(f_{n,i} - S_{n+1,i}\right) \end{aligned}$$

Now define

$$\Delta_i^P = e^{r\Delta t} P\left(f_{n,i}, t_{n+1}\right) - \sum_{j=0}^{i-1} \lambda_{n,j} \left(f_{n,i} - f_{n,j}\right)$$

So, the put option price is reduced to

$$\Delta_i^P = \lambda_{n,i}\left(1-p_{n,i}\right)\left(f_{n,i}-S_{n+1,i}\right) \tag{3.19}$$

The following recursion formula is obtained for the lower nodes in terms of the higher ones:

$$\Delta_i^P = \lambda_{n,i}\frac{S_{n+1,i+1}-f_{n,i}}{S_{n+1,i+1}-S_{n+1,i}}\left(f_{n,i}-S_{n+1,i}\right)$$

$$\begin{aligned}&S_{n+1,i}\left[\lambda_{n,i}\left(S_{n+1,i+1}-f_{n,i}\right)-\Delta_i^P\right]\\&=\lambda_{n,i}f_{n,i}\left(S_{n+1,i+1}-f_{n,i}\right)-\Delta_i^P S_{n+1,i+1}\end{aligned}$$

Thus,

$$S_{n+1,i} = \frac{\lambda_{n,i}f_{n,i}\left(S_{n+1,i+1}-f_{n,i}\right)-\Delta_i^P S_{n+1,i+1}}{\lambda_{n,i}f_{n,i}\left(S_{n+1,i+1}-f_{n,i}\right)-\Delta_i^P} \tag{3.20}$$

3. Centre of the Tree

Instead of the centring condition given by Derman and Kani, it seems more reasonable to allow the underlying to follow the most likely movement - exponential increase at the risk free rate. So, instead of having the spine of the tree remain as $S_{0,0}$, it bends along with the capitalization implied by the risk free rate. So for n odd,

$$S_{n+1,\frac{n+1}{2}} = S_{0,0}e^{r(n+1)\Delta t}$$

If n is even, for $i=\frac{n}{2}$

$$f_{n,i} = \frac{1}{2}\left[\ln S_{n+1,i}+\ln S_{n+1,i+1}\right]$$

So

$$S_{n+1,i}S_{n+1,i+1} = f_{n,i}^2$$

The forward price, rather than the stock price from previous time step, is used to take into account the exponential growth rate at the risk free rate.

Substituting this into (3.17) and solving for the lower node, $S_{n+1,i}$, where $i=\frac{n}{2}$ first,

$$\begin{aligned}S_{n+1,i+1}\left[\Delta_i^C-\lambda_{n,i}f_{n,i}+\lambda_{n,i}S_{n+1,i}\right] &= \Delta_i^C S_{n+1,i}+\lambda_{n,i}f_{n,i}S_{n+1,i}-\lambda_{n,i}f_{n,i}^2\\ S_{n+1,i+1}\Delta_i^C-\lambda_{n,i}f_{n,i}S_{n+1,i+1}+\lambda_{n,i}f_{n,i}^2 &= \Delta_i^C S_{n+1,i}+\lambda_{n,i}f_{n,i}S_{n+1,i}-\lambda_{n,i}f_{n,i}^2\end{aligned}$$

Since $\Delta_i^C = p_{n,i}\lambda_{n,i}\left(S_{n+1,i+1} - f_{n,i}\right)$

$$\begin{aligned}
&S_{n+1,i+1}\left[p_{n,i}\lambda_{n,i}\left(S_{n+1,i+1} - f_{n,i}\right) - \lambda_{n,i}f_{n,i}\right] + \lambda_{n,i}f_{n,i}^2 \\
&= \Delta_i^C S_{n+1,i} + \lambda_{n,i}f_{n,i}S_{n+1,i} - \lambda_{n,i}f_{n,i}^2
\end{aligned}$$

Using (3.2) and

$$S_{n+1,i+1} - f_{n,i} = \left(1 - p_{n,i}\right)\left(S_{n+1,i+1} - S_{n+1,i}\right)$$

Upon simplification

$$\begin{aligned}
&S_{n+1,i+1}\frac{f_{n,i} - S_{n+1,i}}{\left(S_{n+1,i+1} - S_{n+1,i}\right)}\lambda_{n,i}\left(1 - p_{n,i}\right)\left(S_{n+1,i+1} - S_{n+1,i}\right) \\
&- S_{n+1,i+1}\lambda_{n,i}f_{n,i} + \lambda_{n,i}f_{n,i}^2 \\
&= \Delta_i^C S_{n+1,i} + \lambda_{n,i}f_{n,i}S_{n+1,i} - \lambda_{n,i}f_{n,i}^2
\end{aligned}$$

So

$$\begin{aligned}
&\lambda_{n,i}f_{n,i}^2 - \lambda_{n,i}S_{n+1,i+1}\left[p_{n,i}S_{n+1,i} - S_{n+1,i} - p_{n,i}f_{n,i}\right] \\
&= \Delta_i^C S_{n+1,i} + \lambda_{n,i}f_{n,i}S_{n+1,i} - \lambda_{n,i}f_{n,i}^2
\end{aligned}$$

Using the centring condition $S_{n+1,i}S_{n+1,i+1} = f_{n,i}^2$

$$\begin{aligned}
\lambda_{n,i}S_{n+1,i+1}p_{n,i}f_{n,i} - \lambda_{n,i}p_{n,i}f_{n,i}^2 &= \Delta_i^C S_{n+1,i} + \lambda_{n,i}f_{n,i}S_{n+1,i} - \lambda_{n,i}f_{n,i}^2 \\
\lambda_{n,i}p_{n,i}f_{n,i}\left(S_{n+1,i+1} - f_{n,i}\right) &= \Delta_i^C S_{n+1,i} + \lambda_{n,i}f_{n,i}S_{n+1,i} - \lambda_{n,i}f_{n,i}^2 \\
\lambda_{n,i}f_{n,i}^2 - f_{n,i}\Delta_i^C &= \Delta_i^C S_{n+1,i} + \lambda_{n,i}f_{n,i}S_{n+1,i}
\end{aligned}$$

The last line follows from a substitution of Δ_i^C. The node just below the centre, $S_{n,i}$ for $i = \frac{n}{2}$, can be solved for according to

$$S_{n+1,i} = \frac{f_{n,i}\left(\lambda_{n,i}f_{n,i} - \Delta_i^C\right)}{\lambda_{n,i}f_{n,i} + \Delta_i^C}$$

So, if the number of nodes is either even or odd, the centring condition gives rise to the remainder of the nodes of the tree.

4. Negative Transition Probabilities

In (Derman & Kani 1994), the problem of obtaining transition probabilities that indicated an arbitrage opportunity was dealt with by maintaining the logarithmic spacing between adjacent nodes equal to that of the previous level. Yet, this may still be violating the inequality $f_{n,i} \leq S_{n+1,i+1} \leq f_{n,i+1}$. To avoid this, a choice of any point between $f_{n,i}$ and $f_{n,i+1}$ is sufficient. Simply choose the average of the two forwards.

3.7.1 Non-Constant Time Intervals and a Dividend Yield

If it is the case that the input data (option expiry times) is not equally spaced, the resulting binomial tree should display such a feature. The original Derman-Kani algorithm will not be able to allow for direct modification, as the option prices used to determine the tree of spot prices are calculated using a binomial tree approach. One would have to perform interpolation to obtain the required data at equally spaced dates.

A dividend yield can easily be accounted for by slightly modifying the theoretical forward prices (and European option prices) calculated. At node (n, i), the forward price with a dividend yield q is given by:

$$f_{n,i} = S_{n,i}e^{(r-q)\Delta t}.$$

In the Barle & Cakici algorithm, the additional inputs required are all the options' expiries. The above procedure is modified by replacing the constant Δt with the relevant time interval. Given N option expiry times and a total time period of T, for non-constant time intervals, we have that

$$T = \sum_{i=1}^{N} \Delta t_i,$$

where $\Delta t_i = t_i - t_{i-1}$. Therefore, the forward price at node (n, i) is given by

$$f_{n,i} = S_{n,i}e^{(r-q)\Delta t_{i+1}}.$$

3.8 Discrete Dividends and a Term Structure of Interest Rates

(Brandt & Wu 2002) suggest two further modifications to the original algorithm to incorporate discrete dividends and to allow for a non-constant interest rate. The centring condition and the strikes of the European options are those suggested in (Barle & Cakici 1995) as this ensures the phenomenon of negative probabilities associated with the nodes is eliminated from the middle section of the tree. Thus, the economically interesting region of the tree is unaffected.

Once again, the N nodes of the tree are equally spaced Δt apart, where $\Delta t = \frac{T}{N}$, T being the final maturity. The construction of the tree is identical to that proposed by Derman and Kani. Assuming all information has been evaluated up to time step t_n, that is:

- $S_{n,i}$
- $\lambda_{n,i}$ are known for nodes (n, i), $0 \leq i \leq n$

Consider the upper portion of the tree:

For each $S_{n,i}$, the movement is to $S_{n+1,i+1}$ with probability $p_{n,i}$ and to $S_{n+1,i}$ with probability $1 - p_{n,i}$, for $\frac{n+1}{2} \leq i \leq n+1$ if n is odd, or $\frac{n}{2} + 1 \leq i \leq n+1$ if n is even. Assume $S_{n+1,i}$ is known and as before, $f_{n,i}$ denotes the price at node (n, i) of a forward contract with maturity date t_{n+1}.

Solve for $S_{n+1,i+1}$ as follows:

- Risk-neutrality of the tree implies:

$$f_{n,i} = p_{n,i}S_{n+1,i+1} + (1 - p_{n,i})S_{n+1,i}$$

So,

$$p_{n,i} = \frac{f_{n,i} - S_{n+1,i}}{S_{n+1,i+1} - S_{n+1,i}}$$

- The theoretical forward price with discrete dividends is:

$$f_{n,i} = S_{n,i}e^{r_{n+1}\Delta t} - D_{n+1} \tag{3.21}$$

where r_{n+1} denotes the interest rate applicable between t_n and t_{n+1} and D_{n+1} is the discrete dividend with ex-dividend date t_{n+1}. If the dividends are paid in-between nodes, the tree is adjusted by paying the forward value of the dividends at the nodes following the ex-dividend dates.

Let $c_i(K, t_{n+1})$ denote the price at node (n, i) of a 'one step ahead' European call option that matures at t_{n+1}. Setting the strike as $f_{n,i}$,

$$c_i(f_{n,i}, t_{n+1}) = e^{-r_{n+1}\Delta t}p_{n,i}(S_{n+1,i+1} - f_{n,i}) \tag{3.22}$$

Substituting in for $p_{n,i}$:

$$c_i(f_{n,i}, t_{n+1}) = e^{-r_{n+1}\Delta t}(S_{n+1,i+1} - f_{n,i})\frac{f_{n,i} - S_{n+1,i}}{S_{n+1,i+1} - S_{n+1,i}}$$

$$c_i(f_{n,i}, t_{n+1})\,(S_{n+1,i+1} - S_{n+1,i}) = e^{-r_{n+1}\Delta t}(S_{n+1,i+1} - f_{n,i})$$

$$S_{n+1,i+1}\left(c_i(f_{n,i}, t_{n+1}) - e^{-r_{n+1}\Delta t}(f_{n,i} - S_{n+1,i})\right)$$
$$= S_{n+1,i}c_i(f_{n,i}, t_{n+1}) - e^{-r_{n+1}\Delta t}f_{n,i}(f_{n,i} - S_{n+1,i})$$

Solving for $S_{n+1,i+1}$ in terms of $S_{n+1,i}$:

$$S_{n+1,i+1} = \frac{S_{n+1,i}c_i(f_{n,i}, t_{n+1}) + e^{-r_{n+1}\Delta t}f_{n,i}(S_{n+1,i} - f_{n,i})}{c_i(f_{n,i}, t_{n+1}) + e^{-r_{n+1}\Delta t}(S_{n+1,i} - f_{n,i})} \tag{3.23}$$

Similarly, for the lower portion of the tree:

Let $p_i(K, t_{n+1})$ denote the price at node (n, i) of a European put option that matures at t_{n+1}. Setting the strike as $f_{n,i}$,

$$p_i(f_{n,i}, t_{n+1}) = e^{-r_{n+1}\Delta t}(1 - p_{n,i})(f_{n,i} - S_{n+1,i}) \tag{3.24}$$

Substituting in for $1 - p_{n,i}$ and solving for $S_{n+1,i}$ in terms of $S_{n+1,i+1}$:

$$S_{n+1,i} = \frac{S_{n+1,i+1}p_i(f_{n,i}, t_{n+1}) + e^{-r_{n+1}\Delta t}f_{n,i}(f_{n,i}) - S_{n+1,i+1}}{p_i(f_{n,i}, t_{n+1}) + e^{-r_{n+1}\Delta t}(f_{n,i} - S_{n+1,i+1})} \tag{3.25}$$

where $0 \leq i \leq \frac{n+1}{2} - 1$ if n is odd and $0 \leq i \leq \frac{n}{2} - 1$ if n is even.

Consider the centre of the tree. The conditions pertaining to even and odd nodes are as described in §3.7.

(i) For n odd, $S_{n+1,\frac{n+1}{2}} = f_{n,i}$.

(ii) If n is even, then $S_{n+1,i}S_{n+1,i+1} = f_{n,i}^2$ for $i = \frac{n}{2}$. Using this condition in (3.23), the $S_{n+1,i+1}$ can be solved for:

$$\begin{aligned} &S_{n+1,i+1}\left[c_i(f_{n,i}, t_{n+1}) + e^{-r_{n+1}\Delta t}(S_{n+1,i} - f_{n,i})\right] \\ &= S_{n+1,i}c_i(f_{n,i}, t_{n+1}) + e^{-r_{n+1}\Delta t} f_{n,i}(S_{n+1,i} - f_{n,i}) \end{aligned}$$

Multiplying through by $e^{r_{n+1}\Delta t}$:

$$\begin{aligned} &S_{n+1,i+1}e^{r_{n+1}\Delta t}c_i(f_{n,i}, t_{n+1}) + S_{n+1,i+1}(S_{n+1,i} - f_{n,i}) \\ &= S_{n+1,i}e^{r_{n+1}\Delta t}c_i(f_{n,i}, t_{n+1}) + f_{n,i}(S_{n+1,i} - f_{n,i}) \end{aligned}$$

Using (3.22) and $f_{n,i} - p_{n,i}S_{n+1,i+1} = (1 - p_{n,i})S_{n+1,i}$:

$$\begin{aligned} &S_{n+1,i+1}p_{n,i}(S_{n+1,i+1} - f_{n,i}) + S_{n+1,i+1}S_{n+1,i} - S_{n+1,i+1}f_{n,i} \\ &= S_{n+1,i}p_{n,i}(S_{n+1,i+1} - f_{n,i}) + f_{n,i}(S_{n+1,i} - f_{n,i}) \end{aligned}$$

Upon rearrangement,

$$\begin{aligned} &S_{n+1,i+1}S_{n+1,i} - S_{n+1,i}p_{n,i}(S_{n+1,i+1} - f_{n,i}) - f_{n,i}(S_{n+1,i} - f_{n,i}) \\ &= S_{n+1,i+1}f_{n,i} - S_{n+1,i+1}p_{n,i}(S_{n+1,i+1} - f_{n,i}) \\ &f_{n,i}^2 - p_{n,i}f_{n,i}^2 + p_{n,i}S_{n+1,i}f_{n,i} - f_{n,i}S_{n+1,i} + f_{n,i}^2 \\ &= S_{n+1,i+1}f_{n,i} - S_{n+1,i+1}p_{n,i}(S_{n+1,i+1} - f_{n,i}) \\ &f_{n,i}^2(1 - p_{n,i}) + f_{n,i}^2 - f_{n,i}(f_{n,i} - p_{n,i}S_{n+1,i+1}) \\ &= S_{n+1,i+1}f_{n,i} - S_{n+1,i+1}p_{n,i}(S_{n+1,i+1} - f_{n,i}) \end{aligned}$$

Therefore,

$$f_{n,i}^2 + p_{n,i}f_{n,i}(S_{n+1,i+1} - f_{n,i}) \quad = \quad S_{n+1,i+1}f_{n,i} - S_{n+1,i+1}p_{n,i}(S_{n+1,i+1} - f_{n,i})$$

Using (3.22) and solving for $S_{n+1,i+1}$,

$$S_{n+1,i+1}\left(f_{n,i} - e^{r_{n+1}\Delta t}c_i(f_{n,i}, t_{n+1})\right) \quad = \quad f_{n,i}\left(f_{n,i} + e^{r_{n+1}\Delta t}c_i(f_{n,i}, t_{n+1})\right)$$

$$S_{n+1,i+1} \quad = \quad \frac{f_{n,i}\left(f_{n,i} + e^{r_{n+1}\Delta t}c_i(f_{n,i}, t_{n+1})\right)}{f_{n,i} - e^{r_{n+1}\Delta t}c_i(f_{n,i}, t_{n+1})}$$

So, $S_{n+1,i}$ is then given by

$$\begin{aligned} S_{n+1,i} &= f_{n,i}^2 / S_{n+1,i+1} \\ &= \frac{f_{n,i}\left(f_{n,i} - e^{r_{n+1}\Delta t}c_i(f_{n,i}, t_{n+1})\right)}{f_{n,i} + e^{r_{n+1}\Delta t}c_i(f_{n,i}, t_{n+1})} \end{aligned}$$

In practice, these one-step ahead European option prices, $c_i(f_{n,i}, t_{n+1})$ and $p_i(f_{n,i}, t_{n+1})$ are unknown but can be inferred from the observed call and put option prices at t_0. For strike K, we have

$$c_i(K, t_{n+1}) = e^{-r_{n+1}\Delta t}[p_{n,i}(S_{n+1,i+1} - K)^+ + (1 - p_{n,i})(S_{n+1,i} - K)^+]$$

and

$$p_i(K, t_{n+1}) = e^{-r_{n+1}\Delta t}[p_{n,i}(K - S_{n+1,i+1})^+ + (1 - p_{n,i})(K - S_{n+1,i})^+]$$

Since $S_{n+1,k} \leq K = f_{n,k} \leq S_{n+1,i+1}$, we have for all $S_{n+1,i+1} > K$:

$$c_i(K, t_{n+1}) = e^{-r_{n+1}\Delta t}[p_{n,i}(S_{n+1,i+1} - K) + (1 - p_{n,i})(S_{n+1,i} - K)]$$

and for all $S_{n+1,i+1} < K$:

$$p_i(K, t_{n+1}) = e^{-r_{n+1}\Delta t}[p_{n,i}(K - S_{n+1,i+1}) + (1 - p_{n,i})(K - S_{n+1,i})]$$

By equating the risk-neutral forward equation and (3.21),

$$S_{n,i}e^{r_{n+1}\Delta t} - D_{n+1} = p_{n,i}S_{n+1,i+1} + (1 - p_{n,i})S_{n+1,i}$$

Substituting this into the above equations for $c_i(K, t_{n+1})$ and $p_i(K, t_{n+1})$:

$$\begin{aligned} c_i(K, t_{n+1}) &= S_{n,i} - e^{-r_{n+1}\Delta t}K - e^{-r_{n+1}\Delta t}D_{n+1} \\ p_i(K, t_{n+1}) &= e^{-r_{n+1}\Delta t}K - S_{n,i} + e^{-r_{n+1}\Delta t}D_{n+1} \end{aligned}$$

Rewriting the call pricing equation in terms of the one-step ahead options,

$$\begin{aligned} &C(f_{n,k}, t_{n+1}) \\ &= \sum_{i=k}^{n} \lambda_{n,i}c_i(f_{n,k}, t_{n+1}) \\ &= \lambda_{n,k}c_k(f_{n,k}, t_{n+1}) + \sum_{i=k+1}^{n} \lambda_{n,i}[S_{n,i} - e^{-r_{n+1}\Delta t}f_{n,k} - e^{-r_{n+1}\Delta t}D_{n+1}] \end{aligned}$$

Similarly,

$$\begin{aligned} &P(f_{n,k}, t_{n+1}) \\ &= \sum_{i=0}^{k} \lambda_{n,i}p_i(f_{n,k}, t_{n+1}) \\ &= \lambda_{n,k}p_k(f_{n,k}, t_{n+1}) + \sum_{i=0}^{k-1} \lambda_{n,i}[e^{-r_{n+1}\Delta t}f_{n,k} - S_{n,i} + e^{-r_{n+1}\Delta t}D_{n+1}] \end{aligned}$$

Thus, the one-step ahead option prices can be solved for as

$$c_k(f_{n,k}, t_{n+1}) = \frac{C(f_{n,k}, t_{n+1}) - \sum_{i=k+1}^{n} \lambda_{n,i}[S_{n,i} - e^{-r_{n+1}\Delta t}f_{n,k} - e^{-r_{n+1}\Delta t}D_{n+1}]}{\lambda_{n,k}} \tag{3.26}$$

and

$$p_i(f_{n,k}, t_{n+1}) = \frac{P(f_{n,k}, t_{n+1}) - \sum_{i=0}^{k-1} \lambda_{n,i}[e^{-r_{n+1}\Delta t}f_{n,k} - S_{n,i} + e^{-r_{n+1}\Delta t}D_{n+1}]}{\lambda_{n,k}} \tag{3.27}$$

The changes that have been made affect the Arrow-Debreu prices as follows:

$$\lambda_{0,0} = 1$$
$$\lambda_{n+1,i} = \begin{cases} e^{-r_{n+1}\Delta t} p_{n,n}\lambda_{n,n} & \text{for} \quad i = n+1 \\ e^{-r_{n+1}\Delta t}[p_{n,i-1}\lambda_{n,i-1} + (1-p_{n,i})\,\lambda_{n,i}] & \text{for} \quad 1 \leq i \leq n \\ e^{-r_{n+1}\Delta t}\,(1-p_{n,0})\,\lambda_{n,0} & \text{for} \quad i = 0 \end{cases} \tag{3.28}$$

Chapter 4

Implied Trinomial Tree of Derman, Kani and Chriss

4.1 Introduction

The construction of implied binomial trees extends the Black-Scholes theory by making it consistent with the observed smile, the result being an implied evolution for the underlying in equilibrium. Yet, these trees have just enough parameters to be constructed: the tree will be unique up to the specified choice for the centre of the tree. A unique tree may be disadvantageous in the sense that no-arbitrage conditions may easily be violated, or an implausible distribution may be obtained. Since not all market prices that are required for the calculation of transition probabilities are available (there are only a discrete set of traded options which are used for interpolation), it may be more reasonable to have an implied tree that may not match every option price, but rather produce a more plausible distribution.

In order to achieve this, the construction of an implied trinomial tree is suggested in (Derman, Kani & Chriss 1996). These trees are more flexible as they have additional degrees of freedom for parameterization. This allows the pre-specification of the state space which corresponds to choosing the stock price at each node in advance. This can be advantageous if selected judiciously.

The stock price process is assumed to follow the stochastic differential equation:

$$\frac{dS}{S} = \mu(t)\,dt + \sigma(S,t)\,dZ \tag{4.1}$$

where $\mu(t)$ is the expected rate of return at time t, $\sigma(S,t)$ is the local volatility function and dZ a standard Wiener process of mean 0 and variance dt. Since all uncertainty in the volatility is derived from the stock price, options can still be hedged using stock alone so the valuation is preference-free. The form of the local volatility function needs to be determined from the market prices of traded options. This determines the future evolution of the underlying, and all options can be priced ensuring that the model is consistent with prices of liquid options.

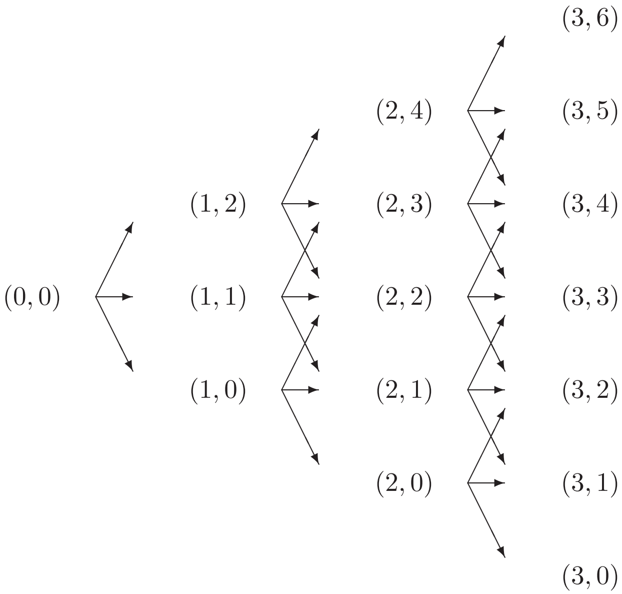

Figure 4.1: Nodes of the trinomial tree

The expected value of the stock price one time step ahead is the forward price,

$$f_{n,i} \quad = \quad S_{n,i}e^{r\Delta t} \tag{4.2}$$

where r is the risk free rate and $0 \leq i \leq 2n$ refers to the state at time step t_n.

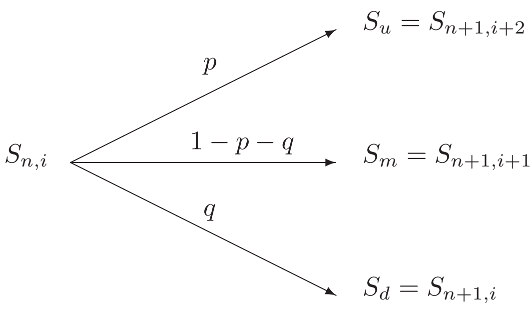

Figure 4.2: A single time step move in a trinomial tree with associated risk-neutral probabilities

Since the tree is risk-neutral in nature, for $0 \leq i \leq 2n$

$$f_{n,i} = p_{n,i} S_{n+1,i+2} + (1 - p_{n,i} - q_{n,i}) S_{n+1,i+1} + q_{n,i} S_{n+1,i} \tag{4.3}$$

The following distributional properties are required for the determination of the local volatility at each node: if $X \sim \Phi(\mu_x, \sigma_x)$, then $Z = e^X$ is lognormal with (Kwok 1998, §1.3.1)

$$\mathbb{E}[Z] = \exp\left(\mu_x + \tfrac{\sigma_x^2}{2}\right) \tag{4.4}$$

and

$$\mathbb{V}[Z] = \exp\left(2\mu_x + \sigma_x^2\right)\left[\exp\left(\sigma_x^2\right) - 1\right] \tag{4.5}$$

In the risk-neutral environment,

$$\ln \frac{S(t + \Delta t)}{S(t)} \sim \Phi\left(\left(r - \tfrac{\sigma^2}{2}\right)\Delta t, \sigma\sqrt{\Delta t}\right)$$

where σ^2 is the annualized variance rate of the lognormal process. Using (4.4) and (4.5),

$$\mathbb{E}\left[\frac{S(t + \Delta t)}{S(t)}\right] = e^{r\Delta t}$$

and

$$\mathbb{V}\left[\frac{S(t + \Delta t)}{S(t)}\right] = e^{2r\Delta t}\left(e^{\sigma^2\Delta t} - 1\right)$$

So, the following results hold for the expected value and variance of the stock price at $t + \Delta t$:

$$\mathbb{E}[S(t + \Delta t)] = S(t) e^{r\Delta t} \tag{4.6}$$

$$\mathbb{V}[S(t + \Delta t)] = \left(S(t)e^{r\Delta t}\right)^2 \left(e^{\sigma^2\Delta t} - 1\right) \tag{4.7}$$

If σ is the stock price volatility and $\mathbb{E}[S(t_{n+1})|S(t_n) = S_{n,i}] = f_{n,i}$, then

$$\begin{aligned}\mathbb{V}[S(t_{n+1})] &= \mathbb{E}\left[(S(t_{n+1}) - f_{n,i})^2\right] \\ &= p_i (S_{n+1,i+2} - f_{n,i})^2 + q_i (S_{n+1,i} - f_{n,i})^2 + (1 - p_i - q_i)(S_{n+1,i+1} - f_{n,i})^2\end{aligned}$$

Using (4.6) and (4.7),

$$\begin{aligned}f_{n,i}^2\left(e^{\sigma^2\Delta t} - 1\right) &= p_{n,i}(S_{n+1,i+2} - f_{n,i})^2 + (1 - p_{n,i} - q_{n,i})(S_{n+1,i+1} - f_{n,i})^2 \\ &\quad + q_{n,i}(S_{n+1,i} - f_{n,i})^2 \\ &= f_{n,i}^2\sigma^2\Delta t + O(\Delta t)^2 \end{aligned} \tag{4.8}$$

where $O(\Delta t)^2$ represents higher order terms in Δt.

(4.8) will be required for the calculation of the local volatility. Clearly, the truncated Taylor expansion indicates a level of inaccuracy.

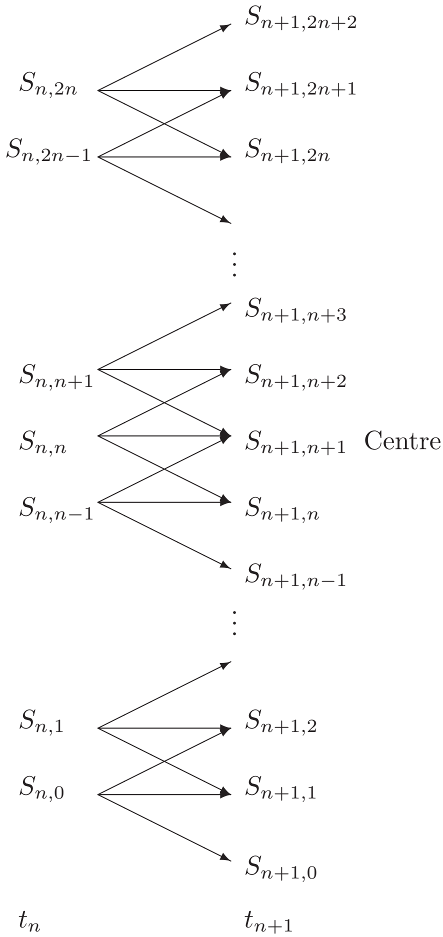

Figure 4.3: The recombining trinomial tree

The actively traded European put and call prices will be used to determine the second parameter. When volatilities are not constant, a judicious choice of the state space in an attempt to solve for the transition probabilities will eliminate the third unknown parameter. Once a a trinomial tree of spot prices has been constructed, we use the theoretical forwards and relevant European option prices to calculate these probabilities.

Again, the trinomial tree makes use of the Arrow-Debreu prices (compare §3.1). Recall that $\lambda_{n,i}$ is the price today of a security that pays unity at period n, state i and zero elsewhere. This time, $\lambda_{n+1,i}$ at time step t_{n+1} and state $0 \leq i \leq 2n+2$ is given by:

$$\lambda_{0,0} = 1$$

$$e^{r\Delta t}\lambda_{n+1,i} = \begin{cases} p_{n,2n}\lambda_{n,2n} & \text{for} \quad i = 2n+2 \\ p_{n,2n-1}\lambda_{n,2n-1} + (1 - p_{n,2n} - q_{n,2n})\,\lambda_{n,2n} & \text{for} \quad i = 2n+1 \\ p_{n,i-2}\lambda_{n,i-2} + (1 - p_{n,i-1} - q_{n,i-1})\,\lambda_{n,i-1} + q_{n,i}\lambda_{n,i} & \text{for} \quad 2 \leq i \leq 2n \\ (1 - p_{n,0} - q_{n,0})\,\lambda_{n,0} + q_{n,1}\lambda_{n,1} & \text{for} \quad i = 1 \\ q_{n,0}\lambda_{n,0} & \text{for} \quad i = 0 \end{cases} \tag{4.9}$$

4.2 Constructing the State Space

The choice of a trinomial scheme provides an additional degree of freedom which allows us significant freedom in choosing the state space. Depending on the relationship between implied volatility, strike and time to expiration, the choice of state space may vary from being regular to being skewed. Uniform mesh sizes are generally adequate when the implied volatility varies quite slowly. If it varies significantly with strike or time to maturity, it may be necessary to choose a node spacing that changes accordingly and is skewed. Negative transition probabilities can be avoided by selecting node spacing that incorporates the skew evident in the market prices at each maturity.

Our strategy will be to first generate a regular trinomial lattice, assuming interest rates and dividend yields are zero. This translates into a constant time spacing Δt and logarithmic mesh spacing ΔS. Then we modify Δt or ΔS at different time and stock levels to capture the basic term- and skew-structures of local volatility in the market.

In certain cases, it may not be possible to avoid negative probabilities, even if the forward value at a particular node lies between $S_{n+1,i}$ and $S_{n+1,i+2}$. In such cases, the option price that produces the negative probability can be overwritten. The implied tree will not fit all option price data but will necessarily fit the forward prices and hence provide transition probabilities that are in the correct range.

4.2.1 Term Structure Adjustments

First consider the case when there is a significant term structure of implied volatility but no skew structure. The local volatility is a function of time, $\sigma(t)$. For some constant c, rubber time $\tilde{t}$ is implicitly defined by

$$t \quad = \quad c\int_0^{\tilde{t}} \sigma^2(u)du, \tag{4.10}$$

where $\sigma(u)$ is the instantaneous (local) volatility at time u. There is no skew structure.

If, for example, we have that $\sigma^2(u) = a + bu$, where a and b are positive constants, then

$$\begin{aligned} t &= c \int_0^{\tilde{t}} (a + bu)\, du \\ &= c \left[au + \tfrac{1}{2} bu^2 \right]_0^{\tilde{t}} \\ &= ca\tilde{t} + \tfrac{1}{2} cb\tilde{t}^2 \\ \Rightarrow \tilde{t} &= \frac{-ca + \sqrt{c^2a^2 + 2cbt}}{cb} \end{aligned}$$

Alternatively, if

$$\sigma(u) = \begin{cases} \sigma_1 & \text{if } u \leq u_1 \\ \sigma_2 & \text{if } u > u_1 \end{cases}$$

as in raw interpolation of yield curves. Then $t = c\tilde{t}\sigma_1^2$ if $\tilde{t} \leq u_1$. So,

$$\tilde{t} = \frac{t}{c\sigma_1^2}.$$

If $\tilde{t} = u_1$, then $t = cu_1\sigma_1^2$. Lastly, if $\tilde{t} > u_1$, then

$$\begin{aligned} t &= c \left(\int_0^{u_1} \sigma^2(u) du + \int_{u_1}^{\tilde{t}} \sigma^2(u) du \right) \\ &= cu_1\sigma_1^2 + \left(\tilde{t} - u_1\right) \sigma_2^2 \\ \Rightarrow \tilde{t} &= \frac{t - cu_1\sigma_1^2}{\sigma_2^2} + u_1. \end{aligned}$$

Using rubber time as opposed to standard time transforms the evolution process into a constant volatility process. This can be shown by defining a new stock price variable $\tilde{S}$ by $\tilde{S}(t) := S(\tilde{t})$ and a new Brownian motion $\tilde{Z}$ by

$$\tilde{Z}(t) = \tilde{Z}\left(c \int_0^{\tilde{t}} \sigma^2(u) du \right) := \sqrt{c} \int_0^{\tilde{t}} \sigma(u) dZ(u), \tag{4.11}$$

for some constant c; we will make a convenient choice later. So,

$$d\tilde{Z}(t) = \sqrt{c}\sigma(\tilde{t}) dZ(\tilde{t}) \tag{4.12}$$

We need to verify that $\tilde{Z}(t)$ is indeed a Brownian motion.

Suppose we have a probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. Recall the definition of Brownian motion.

Definition 1 *(Rogers & Williams 2000) A real-valued stochastic process* $\{W_t : t \in \mathbb{R}^+\}$ *is a Brownian motion if it has the properties*

(i) $W_0 = 0, \; \forall\, \omega;$

(ii) $t \longmapsto W_t(\omega)$ *is a continuous function of* $t \in \mathbb{R}^+$, $\forall\ \omega$;

(iii) *For every* t, $h \geq 0$, $W_{t+h} - W_t$ *is independent of* $\{W_u : 0 \leq u \leq t\}$, *and has a Gaussian distribution with mean 0 and variance h.*

In (4.12), it is clear $\tilde{t}$ exists and is unique.

Clearly (i) and (ii) are satisfied. For the distributional properties:

$$\mathbb{E}\left[\tilde{Z}(t)\right] = \mathbb{E}\left[\sqrt{c}\int_0^{\tilde{t}} \sigma(u)dZ(u)\right] = 0 \tag{4.13}$$

This is a result of the martingale property of the Itô integral.

$$\begin{aligned}\mathbb{V}\left[\tilde{Z}(t)\right] &= \mathbb{V}\left[\sqrt{c}\int_0^{\tilde{t}} \sigma(u)dZ(u)\right] \\ &= \mathbb{E}\left[c\int_0^{\tilde{t}} \sigma^2(u)du\right] \\ &= \mathbb{E}\left[t\right] = t.\end{aligned}$$

The second line follows from the Itô isometry (Oksendal 2004, §3.1.5).

For the independent increments: let $0 \leq \tau_1 \leq \tau_2 \leq \tau_3 \leq \tau_4$ and consider

$$\begin{aligned}&\mathbb{E}\left[\left(\tilde{Z}(\tau_4) - \tilde{Z}(\tau_3)\right)\left(\tilde{Z}(\tau_2) - \tilde{Z}(\tau_1)\right)\right] \\ &= \mathbb{E}\left[\tilde{Z}(\tau_4)\tilde{Z}(\tau_2) - \tilde{Z}(\tau_3)\tilde{Z}(\tau_2) + \tilde{Z}(\tau_3)\tilde{Z}(\tau_1) - \tilde{Z}(\tau_4)\tilde{Z}(\tau_1)\right] \\ &= \mathbb{E}\left[\tilde{Z}(\tau_4)\tilde{Z}(\tau_2)\right] - \mathbb{E}\left[\tilde{Z}(\tau_3)\tilde{Z}(\tau_2)\right] + \mathbb{E}\left[\tilde{Z}(\tau_3)\tilde{Z}(\tau_1)\right] - \mathbb{E}\left[\tilde{Z}(\tau_4)\tilde{Z}(\tau_1)\right] \\ &= \mathbb{E}\left[c\int_0^{\tilde{\tau_4}} \sigma(u)dZ(u)\int_0^{\tilde{\tau_2}} \sigma(v)dZ(v)\right] - \mathbb{E}\left[c\int_0^{\tilde{\tau_3}} \sigma(u)dZ(u)\int_0^{\tilde{\tau_2}} \sigma(v)dZ(v)\right] \\ &+ \mathbb{E}\left[c\int_0^{\tilde{\tau_3}} \sigma(u)dZ(u)\int_0^{\tilde{\tau_1}} \sigma(v)dZ(v)\right] - \mathbb{E}\left[c\int_0^{\tilde{\tau_4}} \sigma(u)dZ(u)\int_0^{\tilde{\tau_1}} \sigma(v)dZ(v)\right] \\ &= \mathbb{E}\left[c\int_0^{\tilde{\tau_4}}\int_0^{\tilde{\tau_2}} \sigma(u)dZ(u)\sigma(v)dZ(v)\right] - \mathbb{E}\left[c\int_0^{\tilde{\tau_3}}\int_0^{\tilde{\tau_2}} \sigma(u)dZ(u)\sigma(v)dZ(v)\right] \\ &+ \mathbb{E}\left[c\int_0^{\tilde{\tau_3}}\int_0^{\tilde{\tau_1}} \sigma(u)dZ(u)\sigma(v)dZ(v)\right] - \mathbb{E}\left[c\int_0^{\tilde{\tau_4}}\int_0^{\tilde{\tau_1}} \sigma(u)dZ(u)\sigma(v)dZ(v)\right] \\ &= 0\end{aligned}$$

This is as a result of the linearity of $\mathbb{E}\left[\cdot\right]$, Fubini's Theorem (Rogers & Williams 2000, §II.12) and the fact that $Z(t)$ is a standard Brownian motion so

$$\mathbb{E}\left[Z(u)Z(v)\right] \quad = \quad 0$$

Using the definition of scaled time and (4.11),

$$\begin{aligned}\frac{d\tilde{S}(t)}{\tilde{S}(t)} &= \frac{d\left(S(\tilde{t})\right)}{S(\tilde{t})} \\ &= \ldots + \sigma(\tilde{t})d\left(Z(\tilde{t})\right) \\ &= \frac{1}{\sqrt{c}}d\tilde{Z}(t),\end{aligned}$$

by (4.12). Hence, the new stock price variable has a constant volatility of $\frac{1}{\sqrt{c}}$.

Now c is chosen to ensure that the rescaled and standard times coincide at a fixed future time (usually the last maturity of the input data), that is we want $T = \tilde{t}(T)$. Thus using (4.10)

$$c = T \Bigg/ \int_0^T \sigma^2(u)du \tag{4.14}$$

In the trinomial tree with N known equally-spaced time points $0 = t_0,\ t_1,\ \ldots,\ t_N = T$, the requirement is to find the unknown scaled time points $0 = \tilde{t}_0,\ \tilde{t}_1,\ \ldots,\ \tilde{t}_N = T$ such that $\sigma(\tilde{t}_k)^2\Delta\tilde{t}_k$ is a constant for all times t_k. This ensures the tree will recombine. (Derman, Kani & Chriss 1996) show that this can be done by solving for $1 \leq k \leq N$:

$$\tilde{t}_k = \frac{T\sum_{i=1}^{k}\frac{1}{\sigma^2(\tilde{t}_i)}}{\sum_{i=1}^{N}\frac{1}{\sigma^2(\tilde{t}_i)}} \tag{4.15}$$

The formula for the term structure (4.15) is implicit and hence quite difficult to implement. We now derive an alternative iterative scheme which will enable all scaled times to be determined explicitly.

The notation for the remainder of this section for variance is as follows:

- $\sigma_I^2(t)$: Implied Black-Scholes variance for an option with maturity t.
- $\sigma_f^2(t)$: Forward variance which will be defined below.
- $\sigma_l^2(t)$: Local variance as a function of time.

It will be required that for $1 \leq k \leq N$

$$\hat{c} \equiv \int_{\tilde{t}_{k-1}}^{\tilde{t}_k} \sigma_l^2(\tilde{t}(s))ds$$

is independent of t, for some new constant $\hat{c}$; again, we will choose this in due course.

Since there is no strike structure, the local variance reduces to the forward Black-Scholes implied variance (Gatheral 2004, §2.4).

So,

$$\int_{\tilde{t}_{k-1}}^{\tilde{t}_k} \sigma_f^2(\tilde{t}(s))ds = \hat{c} \tag{4.16}$$

where $\hat{c}$ is a constant.

The implied forward variance at time 0 between t and $t + \Delta t$ is given by

$$\sigma_f^2(0; t, t + \Delta t) = \frac{\sigma_I^2(t + \Delta t)(t + \Delta t) - \sigma_I^2(t)t}{\Delta t}$$

Taking the limit as $\Delta t \to 0$

$$\begin{aligned}
\sigma_f^2(t) &= \frac{d}{ds}\sigma_I^2(s)s\bigg|_{s=t} \\
&= \frac{d}{dt}\sigma_I^2(t)t \qquad (4.17)
\end{aligned}$$

Then

$$\begin{aligned}
N\hat{c} = \sum_{k=1}^{N} \int_{\tilde{t}_{k-1}}^{\tilde{t}_k} \sigma_f^2\left(\tilde{t}(s)\right) ds &= \int_0^T \sigma_f^2\left(\tilde{t}\right) d\tilde{t} \\
&= \int_0^T \sigma_f^2(t) dt \\
&= \int_0^T \left(\frac{d}{dt}\sigma_I^2(t)t\right) dt \\
&= \sigma_I^2(T)T.
\end{aligned}$$

But,

$$\begin{aligned}
\int_0^T \sigma_f^2(t)dt &= \int_0^T \sigma_f^2(\tilde{t})d\tilde{t} \\
&= \int_0^T \frac{1}{c} dt \\
&= \frac{T}{c}.
\end{aligned}$$

So we have that

$$\begin{aligned}
\hat{c} &= \sigma_I^2(T) \cdot \frac{T}{N}, \\
c &= \frac{1}{\sigma_I^2(T)}.
\end{aligned}$$

Given $\sigma_I(t_1)$ and $\sigma_I(t_2)$ at maturities t_1 and t_2 respectively, it is the case that $\sigma_I^2(t_1)t_1 < \sigma_I^2(t_2)t_2$. This must be true to ensure the forward ATM implied volatility between t_1 and t_2 is always positive. Performing linear interpolation on $\sigma_I(t)$ or $\sigma_I^2(t)$ does not always give rise to positive forward volatilities. The problem is analogous to that of yield curve interpolation where the interpolation method must be carefully chosen to ensure that forward rates cannot be negative (Hagan & West 2005). So, $\sigma_f^2(t) \leftrightarrow f(t)$ and $\sigma_I^2(t) \leftrightarrow r(t)$, where $r(t)$ is the risk free yield-to-maturity of a discount instrument maturing at time t and $f(t)$ is the instantaneous forward rate. The relationship between $r(t)$ and $f(t)$ is

$$f(t) = \frac{d}{dt}r(t)t$$

which generalizes to (4.17). So,

$$r(t) = \frac{1}{t}\int_0^t f(s)ds$$

and

$$\sigma_I^2(t) = \frac{1}{t}\int_0^t \sigma_f^2(s)ds \tag{4.18}$$

Given two points, $r(t_{j-1})$ and $r(t_j)$, the discrete forward rate that is applicable between t_{j-1} and t_j, $f_{d,j}$ is given by

$$f_{d,j} = \frac{r(t_j)t_j - r(t_{j-1})t_{j-1}}{t_j - t_{j-1}}$$

and the discrete forward implied volatility applicable between t_{j-1} and t_j, $\sigma_{d,f;j}$, is given by

$$\sigma_{d,f;j} = \sqrt{\frac{\sigma_I^2(t_j)t_j - \sigma_I^2(t_{j-1})t_{j-1}}{t_j - t_{j-1}}} \tag{4.19}$$

Two satisfactory methods prescribed for yield curve interpolation are raw and monotone convex interpolation (Hagan & West 2005). The raw interpolation method is selected here, as it is by definition that method which has piecewise constant forward curves, which enables us to find closed-form solutions for the ATM volatility. The interpolation function for $\sigma_I^2(t)$ turns out to be $\sigma_I^2(t) = B + \frac{C}{t}$ where B and C are to be derived below. Given $\sigma_I(t_1)$ and $\sigma_I(t_2)$ (two endpoints), the discrete forward implied volatility $\sigma_{d,f;2}$ is calculated using (4.19). The implied volatility at any time $t \in [t_1, t_2)$ can be found using (4.18):

$$\begin{aligned}
\sigma_I^2(t)t &= \int_0^t \sigma_f^2(s)ds \\
&= \int_0^{t_1} \sigma_f^2(s)ds + \int_{t_1}^t \sigma_f^2(s)ds \\
&= \sigma_I^2(t_1)t_1 + (t - t_1)\sigma_{d,f;2}^2
\end{aligned}$$

since the interpolation method is raw. Therefore,

$$\begin{aligned}
&\sigma_I(t) \\
&= \sqrt{\sigma_I^2(t_1)\frac{t_1}{t} + \frac{(t-t_1)}{t}\sigma_{d,f;2}^2} \\
&= \sqrt{\sigma_I^2(t_1)\frac{t_1}{t} + \frac{(t-t_1)}{t}\left(\frac{\sigma_I^2(t_2)t_2 - \sigma_I^2(t_1)t_1}{t_2 - t_1}\right)} \\
&= \sqrt{\frac{\sigma_I^2(t_1)t_1t_2 - t_1^2\sigma_I^2(t_1) + t\left(\sigma_I^2(t_2)t_2 - \sigma_I^2(t_1)t_1\right) - \sigma_I^2(t_2)t_1t_2 - \sigma_I^2(t_1)t_1^2}{t\left(t_2 - t_1\right)}} \\
&= \sqrt{\frac{\sigma_I^2(t_2)t_2 - \sigma_I^2(t_1)t_1}{t_2 - t_1} + \frac{t_1t_2\left(\sigma_I^2(t_1) - \sigma_I^2(t_2)\right)}{t\left(t_2 - t_1\right)}} \\
&:= \sqrt{B + \frac{C}{t}}
\end{aligned} \tag{4.20}$$

Thus, the implied ATM volatilities for all times between any two points can be found using this method of interpolation. This method guarantees that the ATM forward volatilities are positive.

4.2.2 Skew Structure Adjustments

If there is significant skew structure in the implied volatility, but no term structure, we assume the local volatility is a function of the underlying, $\sigma(S)$. Define a scaled stock price $\tilde{S}$ by

$$\tilde{S} \quad = \quad S_0 \exp\left[c\int_{S_0}^{S}\frac{1}{x\sigma(x)}dx\right] \tag{4.21}$$

for some constant c. So,

$$\ln\tilde{S} \quad = \quad \ln S_0 + c\int_{S_0}^{S}\frac{1}{x\sigma(x)}dx$$

The scaled stock price has constant volatility c. To see this, consider the stochastic equation

$$\frac{dS}{S} = \sigma(S)dZ$$

Use Itô's Lemma (Björk 2004, §3.5)

Theorem 1 *Let X be the process given by*

$$dX(t) \quad = \quad a(X,t)dt + b(X,t)dW_t$$

where a(X,t) and b(X,t) are adapted processes, and let f be a $C^{1,2}$-function. Define the process Z by $Z(t) = f(t, X(t))$. Then Z has a stochastic differential given by

$$df\left(X(t),t\right) \quad = \quad \left(\frac{\partial f}{\partial t} + a(X,t)\frac{\partial f}{\partial x} + \frac{1}{2}b(X,t)^2\frac{\partial^2 f}{\partial x^2}\right)dt + b(X,t)\frac{\partial f}{\partial x}dW_t \tag{4.22}$$

Let

$$f(x,t) = c\int_{S_0}^{x}\frac{1}{y\sigma(y)}dy$$

Then

$$f_x = \frac{c}{x\sigma(x)}$$
$$f_{xx} = -c\left(\frac{\sigma(x) + x\sigma^{'}(x)}{\left(x\sigma(x)\right)^2}\right)$$
$$f_t = 0$$

Thus

$$d\ln\tilde{S} = \left(0 + 0\cdot\frac{c}{x\sigma(x)} - \tfrac{1}{2}S^2\sigma^2c\left(\frac{\sigma + S\sigma^{'}}{\left(S\sigma\right)^2}\right)\right)dt + S\sigma\frac{c}{S\sigma}dZ$$
$$= -\frac{c}{2}\left(\sigma + \sigma^{'}S\right)dt + cdZ$$

This indicates that there is an induced drift from the relationship between local volatility and the level of the underlying. Through a choice of a small enough time step, the drift can be accommodated for in a trinomial environment, starting from a constant volatility state space. The constant c can be chosen to be the ATM local volatility, $\sigma(S_{n,n})$.

Let $\tilde{S}_k$ and $\tilde{S}_{k+1}$ denote two scaled stock prices at times t_k and t_{k+1} respectively. We start off in continuous time and derive a discretization of (4.21) for the upper portion of the tree. Since

$$\tilde{S}_k = S_0 \exp\left[c \int_{S_0}^{S_k} \frac{1}{x\sigma(x)} dx\right],$$

and

$$\tilde{S}_{k+1} = S_0 \exp\left[c \int_{S_0}^{S_{k+1}} \frac{1}{x\sigma(x)} dx\right].$$

So, we get that

$$\begin{aligned}
\tilde{S}_{k+1} &= \tilde{S}_k \exp\left[c \int_{S_0}^{S_{k+1}} \frac{1}{x\sigma(x)} dx - c\int_{S_0}^{S_k} \frac{1}{x\sigma(x)} dx\right] \\
&= \tilde{S}_k \exp\left[c \int_{S_k}^{S_{k+1}} \frac{1}{x\sigma(x)} dx\right] \\
&\approx \tilde{S}_k \exp\left[\frac{c}{\sigma(S_k)} \int_{S_k}^{S_{k+1}} \frac{1}{x} dx\right] \\
&= \tilde{S}_k \exp\left[\frac{c}{\sigma(S_k)} \ln \frac{S_{k+1}}{S_k}\right]
\end{aligned}$$

For nodes above the central node, $n < k \leq 2n$, we have

$$\begin{aligned}
\frac{\sigma(S_k)}{c} \ln \frac{\tilde{S}_{k+1}}{\tilde{S}_k} &= \ln \frac{S_{k+1}}{S_k} \\
\Rightarrow S_{k+1} &= S_k \exp\left[\frac{\sigma(S_k)}{c} \ln \frac{\tilde{S}_{k+1}}{\tilde{S}_k}\right].
\end{aligned} \tag{4.23}$$

while for nodes below the central node, $0 \leq k < n$, we have that

$$\begin{aligned}
-\frac{\sigma(S_k)}{c} \ln \frac{\tilde{S}_{k+1}}{\tilde{S}_k} &= \ln \frac{S_k}{S_{k+1}} \\
\Rightarrow S_k &= S_{k+1} \exp\left[-\frac{\sigma(S_k)}{c} \ln \frac{\tilde{S}_{k+1}}{\tilde{S}_k}\right].
\end{aligned} \tag{4.24}$$

When the volatility is constant and equal to c, the trees with S and $\tilde{S}$ spacing will coincide. Suppose the implied skew is determined by the Taylor series expansion up to the linear term: $\Sigma = \Sigma_0 + b(K - K_0)$, where Σ_0 represents the ATM implied volatility, b is the slope which can be interpreted as the percentage per point increase/decrease in the implied volatility with decrease/increase in strike K, and K_0 is the current index level. To determine the local volatility, σ, at level K in the vicinity of K_0, there is the relation (Derman 1999)

$$\sigma = \Sigma_0 + 2b(K - K_0) \tag{4.25}$$

4.2.3 Term and Skew Structure

If the local volatility is a separable function (product) of strike and term structure, scaling is performed on time and stock price independently. The result is a state space that has a term structure with constant skew structure superimposed on it. In general, this may work anyway, even if there is not outright separability.

4.3 Solving for the Transition Probabilities

Let $C(K, t_{n+1})$ and $P(K, t_{n+1})$ denote the price of call and put option prices, with maturity t_{n+1} and strike K, that are available from market data respectively. As in the binomial implied tree, only a discrete set of data points that relate price/implied volatility to strike are available. Consequently, interpolation must be performed on the implied volatility to obtain the value corresponding to the required strike. This volatility is used to price the relevant option using a constant volatility trinomial tree. This constant volatility can be selected to be the average of the at-the-money forward volatilities (strike is equal to the forward/futures value for that maturity). A superior method for pricing is to use the Black-Scholes option pricing formula.

At time step t_{n+1}, the unknowns to be determined are the transition probabilities $p_{n,i}$ and $q_{n,i}$ for $0 \leq i \leq 2n$. There are $2n+3$ stock prices that are chosen, in addition to the $2n+1$ theoretical forward prices $f_{n,i}$, using (4.2), and $2n+1$ European option prices that are to be determined from market data.

- The risk-neutral forward equation equation, (4.3), relates the stock prices at each node to transition probabilities.
- For the upper portion of the tree, the European call options, with strike K, maturity t_{n+1} and spot $S_{0,0}$ being valued at $t = 0$ are considered. The price can be written as:

$$C(K, t_{n+1}) = \sum_{j=0}^{2n+2} \lambda_{n+1,j} \max(S_{n+1,j} - K, 0)$$

Expanding this using (4.9)

$$\begin{aligned}
&e^{r\Delta t} C(K, t_{n+1}) \\
&= p_{n,2n}\lambda_{n,2n} \max(S_{n+1,2n+2} - K, 0) \\
&+ (p_{n,2n-1}\lambda_{n,2n-1} + (1 - p_{n,2n} - q_{n,2n})\lambda_{n,2n}) \max(S_{n+1,2n+1} - K, 0) \\
&+ \sum_{j=2}^{2n} (p_{n,j-2}\lambda_{n,j-2} + (1 - p_{n,j-1} - q_{n,j-1})\lambda_{n,j-1} + q_{n,j}\lambda_{n,j}) \max(S_{n+1,j} - K, 0) \\
&+ ((1 - p_{n,0} - q_{n,0})\lambda_{n,0} + q_{n,1}\lambda_{n,1}) \max(S_{n+1,1} - K, 0) \\
&+ q_{n,0}\lambda_{n,0} \max(S_{n+1,0} - K, 0)
\end{aligned}$$

Since a strike of $S_{n+1,i+1}$ for $n+1 \leq i \leq 2n$ is to be selected, the above can be rewritten as

$$\begin{aligned}
& e^{r\Delta t} C\left(S_{n+1,i+1}, t_{n+1}\right) \\
&= p_{n,2n}\lambda_{n,2n}\left(S_{n+1,2n+2} - S_{n+1,i+1}\right) \\
&+ \left(p_{n,2n-1}\lambda_{n,2n-1} + \left(1-p_{n,2n}-q_{n,2n}\right)\lambda_{n,2n}\right)\left(S_{n+1,2n+1} - S_{n+1,i+1}\right) \\
&+ \sum_{j=i+2}^{2n}\left(p_{n,j-2}\lambda_{n,j-2} + \left(1-p_{n,j-1}-q_{n,j-1}\right)\lambda_{n,j-1} + q_{n,j}\lambda_{n,j}\right)\left(S_{n+1,j} - S_{n+1,i+1}\right)
\end{aligned}$$

Using (4.3), the price can then be simplified

$$\begin{aligned}
& e^{r\Delta t} C\left(S_{n+1,i+1}, t_{n+1}\right) \\
&= \sum_{j=i+1}^{2n}\lambda_{n,j}\left(\left(p_{n,j}S_{n+1,j+2} + \left(1-p_{n,j}-q_{n,j}\right)S_{n+1,j+1} + q_{n,j}S_{n+1,j}\right) - S_{n+1,i+1}\right) \\
&+ p_{n,i}\lambda_{n,i}\left(S_{n+1,i+2} - S_{n+1,i+1}\right) \\
&= \sum_{j=i+1}^{2n}\lambda_{n,j}\left(f_{n,j} - S_{n+1,i+1}\right) + p_{n,i}\lambda_{n,i}\left(S_{n+1,i+2} - S_{n+1,i+1}\right) \qquad (4.26)
\end{aligned}$$

Since the state space is chosen, the forwards and option prices known, the unknown is then $p_{n,i}$ which can be solve for according to

$$p_{n,i} = \frac{e^{r\Delta t} C\left(S_{n+1,i+1}, t_{n+1}\right) - \sum_{j=i+1}^{2n}\lambda_{n,j}\left(f_{n,j} - S_{n+1,i+1}\right)}{\lambda_{n,i}\left(S_{n+1,i+2} - S_{n+1,i+1}\right)} \qquad (4.27)$$

The probability $q_{n,i}$ can be solved for by using (4.3)

$$q_{n,i} = \frac{f_{n,i} - p_{n,i}\left(S_{n+1,i+2} - S_{n+1,i+1}\right) - S_{n+1,i+1}}{S_{n+1,i} - S_{n+1,i+1}} \qquad (4.28)$$

- Considering the lower portion of the tree from the central node ($S_{n,n}$ at t_n) downwards. The European put option prices with strike K, maturity t_{n+1}, spot $S_{0,0}$ being valued at $t = 0$, are required. The price can be written as:

$$P\left(K, t_{n+1}\right) = \sum_{j=0}^{2n+2}\lambda_{n+1,j}\max\left(K - S_{n+1,j}, 0\right)$$

Using (4.9)

$$\begin{aligned}
& e^{r\Delta t} P\left(K, t_{n+1}\right) \\
&= p_{n,2n}\lambda_{n,2n}\max\left(K - S_{n+1,2n+2}, 0\right) \\
&+ \left(p_{n,2n-1}\lambda_{n,2n-1} + \left(1-p_{n,2n}-q_{n,2n}\right)\right)\max\left(K - S_{n+1,2n+1}, 0\right) \\
&+ \sum_{j=2}^{2n}\left(p_{n,j-2}\lambda_{n,j-2} + \left(1-p_{n,j-1}-q_{n,j-1}\right)\lambda_{n,j-1} + q_{n,j}\lambda_{n,j}\right)\max\left(K - S_{n+1,j}, 0\right) \\
&+ \left(\left(1-p_{n,0}-q_{n,0}\right)\lambda_{n,0} + q_{n,1}\lambda_{n,1}\right)\max\left(K - S_{n+1,1}, 0\right) \\
&+ q_{n,0}\lambda_{n,0}\max\left(K - S_{n+1,0}, 0\right)
\end{aligned}$$

Since a strike of $S_{n+1,i+1}$ for $0 \leq i \leq n$ is to be selected, the above can be rewritten as

$$
\begin{aligned}
& e^{r\Delta t} P\left(S_{n+1,i+1}, t_{n+1}\right) \\
& = \sum_{j=2}^{i+1} \left(p_{n,j-2}\lambda_{n,j-2} + \left(1 - p_{n,j-1} - q_{n,j-1}\right)\lambda_{n,j-1} + q_{n,j}\lambda_{n,j}\right)\left(S_{n+1,i+1} - S_{n+1,j}\right) \\
& + \left(\left(1 - p_{n,0} - q_{n,0}\right)\lambda_{n,0} + q_{n,1}\lambda_{n,1}\right)\left(S_{n+1,i+1} - S_{n+1,1}\right) \\
& + q_{n,0}\lambda_{n,0}\left(S_{n+1,i+1} - S_{n+1,0}\right)
\end{aligned}
$$

Using (4.3),

$$
\begin{aligned}
& e^{r\Delta t} P\left(S_{n+1,i+1}, t_{n+1}\right) \\
& = \sum_{j=0}^{i-1} \lambda_{n,j}\left(S_{n+1,i+1} - \left(p_{n,j}S_{n+1,j+2} + \left(1 - p_{n,j} - q_{n,j}\right)S_{n+1,j+1} + q_{n,j}S_{n+1,j}\right)\right) \\
& + q_{n,i}\lambda_{n,i}\left(S_{n+1,i+1} - S_{n+1,i}\right) \\
& = \sum_{j=0}^{i-1} \lambda_{n,j}\left(S_{n+1,i+1} - f_{n,j}\right) + q_{n,i}\lambda_{n,i}\left(S_{n+1,i+1} - S_{n+1,i}\right) \qquad (4.29)
\end{aligned}
$$

The only unknown in this equation is $q_{n,i}$. Solving for this

$$
q_{n,i} = \frac{e^{r\Delta t} P\left(S_{n+1,i+1}, t_{n+1}\right) - \sum_{j=0}^{i-1} \lambda_{n,j}\left(S_{n+1,i+1} - f_{n,j}\right)}{\lambda_{n,i}\left(S_{n+1,i+1} - S_{n+1,i}\right)} \qquad (4.30)
$$

Using (4.3), $p_{n,i}$ can be solved

$$
p_{n,i} = \frac{f_{n,i} + q_{n,i}\left(S_{n+1,i+1} - S_{n+1,i}\right) - S_{n+1,i+1}}{S_{n+1,i+2} - S_{n+1,i+1}} \qquad (4.31)
$$

4.4 Negative Transition Probabilities

If a reasonable choice of state space is chosen, negative probabilities are usually avoided. However, there are two instances when this may occur:

1. If $f_{n,i}$ lies out of the required range between $S_{n+1,i}$ and $S_{n+1,i+2}$, a riskless arbitrage may exist.
   - If $f_{n,i} > S_{n+1,i+2}$, then from (4.3) it is clear that either $q_{n,i} < 0$ or $p_{n,i} > 1$
   - If $f_{n,i} < S_{n+1,i}$, then either $q_{n,i} > 1$ or $p_{n,i} < 0$
2. Negative probabilities may arise as a result of the magnitude of local volatility obtained from the implied tree. If a very high (low) value is obtained for the call option price $C\left(S_{n+1,i+1}, t_{n+1}\right)$ in (4.27), then a very high (low) value will be obtained for the local volatility $\sigma_{n,i}$. Since the state space has already been calculated, the extreme values of local volatility may not correspond to probabilities that are between 0 and 1. To avoid this, the option price is overwritten and another is selected that ensures the forward condition, mentioned above, is maintained at each node.

There are numerous ways to select probabilities between 0 and 1 which ensure the forward condition, $S_{n+1,i} < f_{n,i} < S_{n+1,i+2}$, holds at every tree node for $0 \leq i \leq 2n$. For example, select the middle transition probability to be 0 and set the up and down probabilities to $p_{n,i} = (f_{n,i} - S_{n+1,i}) / (S_{n+1,i+2} - S_{n+1,i})$ and $q_{n,i} = 1 - p_{n,i}$.

Alternatively, if $S_{n+1,i+1} < f_{n,i} < S_{n+1,i+2}$ occurs then set

$$p_{n,i} = \frac{1}{2}\left[\frac{f_{n,i} - S_{n+1,i+1}}{S_{n+1,i+2} - S_{n+1,i+1}} + \frac{f_{n,i} - S_{n+1,i}}{S_{n+1,i+2} - S_{n+1,i}}\right]$$

and

$$q_{n,i} = \frac{1}{2}\left[\frac{S_{n+1,i+2} - f_{n,i}}{S_{n+1,i+2} - S_{n+1,i}}\right]$$

and if $S_{n+1,i} < f_{n,i} < S_{n+1,i+1}$ occurs, set

$$p_{n,i} = \frac{1}{2}\left[\frac{f_{n,i} - S_{n,i}}{S_{n+1,i+2} - S_{n+1,i}}\right]$$

and

$$q_{n,i} = \frac{1}{2}\left[\frac{S_{n+1,i+2} - f_{n,i}}{S_{n+1,i+2} - S_{n+1,i}} + \frac{S_{n+1,i+1} - f_{n,i}}{S_{n+1,i+1} - S_{n+1,i}}\right]$$

In the case that there may be a significant term or skew structure, the mesh of stock prices that is constructed should not violate the forward price condition. If this occurs, a new state space should be selected.

4.5 Local Volatility

To determine the local volatility, $\sigma_{n,i}$ at time step n and node (n,i) where $0 \leq i \leq 2n$, use equations (4.3) and (4.8). So,

$$\begin{aligned}&\sigma_{n,i}\\ = &\sqrt{\frac{p_{n,i}\left(S_{n+1,i+2} - f_{n,i}\right)^2 + q_{n,i}\left(S_{n+1,i} - f_{n,i}\right)^2 + \left(1 - p_{n,i} - q_{n,i}\right)\left(S_{n+1,i+1} - f_{n,i}\right)^2}{f_{n,i}^2 \Delta t}}\end{aligned} \tag{4.32}$$

4.6 Computational Algorithm

4.6.1 Input Data

The input data, which is identical to the case of the implied binomial tree in Chapter 3, is required for the specification of the state space.

1. Valuation date (taken to be $t = 0$)

2. Spot on valuation date

3. Expiry dates of all European options

4. Futures (or forward) level corresponding to the valuation date; if this is not provided, the algorithm uses the theoretical forward rates[1]

5. Risk-free rate

6. Dividend yield

7. Implied volatilities for various strikes relevant at each time step t_j for $1 \leq j \leq \hat{N}$ where $t_{\hat{N}} = T$

8. The number of time steps in the implied tree N - this does not necessarily have to agree with $\hat{N}$.

9. Specification of the nature of the input:

   (I) No extreme term or skew structure, normal trinomial state space.

   (II) Term Structure: State space with unequal time steps.

   (III) Skew Structure: Require ATM implied volatility, slope of the linear function to construct a state space with nodal spacing that varies vertically.

   (IV) Both: A skewed state space is first constructed, followed by a term structure.

4.6.2 Constructing the required state space

Case I: Normal State Space

In this case, there is no extreme term or skew structure associated with the local volatility function. An underlying constant-volatility trinomial tree will be appropriate for the determination of the transition probabilities and Arrow-Debreu prices. A single time step of the trinomial tree is constructed from the combination of two steps of the binomial Cox-Ross-Rubinstein tree.

Consider the Cox-Ross-Rubinstein constant volatility trinomial tree[2]:

---

[1] We will be pricing European and some path-dependent options on the ALSI40 equity index. Most relevant information is provided by implied skew data (from the South African Futures Exchange, SAFEX, or a dealer) on the futures contracts on this index that trade. It is also convenient for interpolation purposes as the ATM implied volatility can be found using raw interpolation on the relative strikes, this is X/F, where X is the strike and F is the ATM futures level.

[2] Since two steps of a binomial C-R-R tree equates to one step of the C-R-R trinomial tree, then $Su^2 = SU$ and $Sd^2 = SD$, where $u = e^{\sigma\sqrt{\frac{\Delta t}{2}}}$ is the upward movement in a binomial tree of time step $\frac{\Delta t}{2}$ and $d = e^{-\sigma\sqrt{\frac{\Delta t}{2}}}$ the downward movement. When one time step in the trinomial tree is Δt, then $U = \left(e^{\sigma\sqrt{\Delta t/2}}\right)^2 = e^{\sigma\sqrt{2\Delta t}}$, and similarly $D = e^{-\sigma\sqrt{2\Delta t}}$. Given that the risk-neutral probability of an upward movement in the binomial tree over a time step $\frac{\Delta t}{2}$ is given by

$$\begin{aligned}\pi &= \frac{e^{r\frac{\Delta t}{2}} - d}{u - d} \\ &= \frac{e^{r\frac{\Delta t}{2}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}}{e^{\sigma\sqrt{\frac{\Delta t}{2}}} - e^{-\sigma\sqrt{\frac{\Delta t}{2}}}},\end{aligned}$$

then in the trinomial tree of time step Δ, the probabilities associated with the up, down and middle movements are given by $p_U = p = \pi^2$, $p_D = q = (1-\pi)^2$ and $p_M = 1 - p_U - p_D$.

- $S_u = Se^{\sigma\sqrt{2\Delta t}}$
- $S_m = S$
- $S_d = Se^{-\sigma\sqrt{2\Delta t}}$
- $p = \left(\frac{e^{r\Delta t/2} - e^{-\sigma\sqrt{\Delta t/2}}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2$
- $q = \left(\frac{e^{\sigma\sqrt{\Delta t/2}} - e^{r\Delta t/2}}{e^{\sigma\sqrt{\Delta t/2}} - e^{-\sigma\sqrt{\Delta t/2}}}\right)^2$

where S is the spot price at the current time step, p and q represent the transition probabilities of an up and down movement respectively, σ is the constant volatility of an ATM option and S_u, S_m and S_d are the spot prices at the following time step. The movement is shown explicitly in Figure 4.2. The time steps $\Delta t = $ (Expiry Date – Valuation Date)/(N $\cdot$ 365).

The probabilities, as stated above, will not be used. The required transition probabilities, $p_{n,i}$, $q_{n,i}$ and $1 - p_{n,i} - q_{n,i}$, are what is required in the implied tree approach. The state space is a platform that can be selected using the additional degree of freedom. This is a result of selecting a trinomial as opposed to a binomial tree.

Consider the upper portion of the tree (from $S_{n+1,n+2}$ to $S_{n+1,2n+2}$) at $t = \tilde{t}_{n+1}$: (4.27) and (4.28) are used to calculate $p_{n,i}$ and $q_{n,i}$ for $n+1 \leq i \leq 2n$. For this part of the tree, the European call option prices are required. Since there is a term structure of implied volatility and the constructed state space has time intervals which will not always coincide with the input dates, it will be necessary to perform linear interpolation in the vertical direction (on the strikes) and raw interpolation in the horizontal direction (on the implied volatilities at a date that is not an input). See Figure 4.4. This is done to obtain the implied volatility at a non-input strike for an option expiring at a non-input date. To ensure that forward rates are always positive, the raw interpolation method is required.

Refer to Figure 4.4. To calculate the implied volatility, σ_I, for strike X and maturity $\tilde{t}_{n+1}$, linear interpolation is first performed on the implied volatilities σ_1 and σ_2 (which relates to strikes X_1 and X_2) at maturity t_k to obtain σ_{12}. The next linear interpolation is performed on σ_3 and σ_4 at maturity t_{k+1} to obtain σ_{34}. These implied volatilities relate to the strike X at maturities t_k and t_{k+1}.

Consider the calculation of σ_{12} and σ_{34}:

$$\sigma_{12} = \frac{X - X_1}{X_2 - X_1}\sigma_2 + \frac{X_2 - X}{X_2 - X_1}\sigma_1$$
$$\sigma_{34} = \frac{X - X_3}{X_4 - X_3}\sigma_4 + \frac{X_4 - X}{X_4 - X_3}\sigma_3$$

Then, σ_I at $\tilde{t}_{n+1}$ is calculated using by (4.20):

$$\sigma_I = \sqrt{\frac{t_k t_{k+1}\left(\sigma_{12}^2(t_k) - \sigma_{34}^2(t_{k+1})\right)}{\tilde{t}_{n+1}\left(t_{k+1} - t_k\right)} + \frac{\sigma_{34}^2(t_{k+1})t_{k+1} - \sigma_{12}^2(t_k)t_k}{t_{k+1} - t_k}}.$$

Once the implied volatility has been evaluated, either a constant volatility trinomial tree or the Black-Scholes formula can be used to price the option.

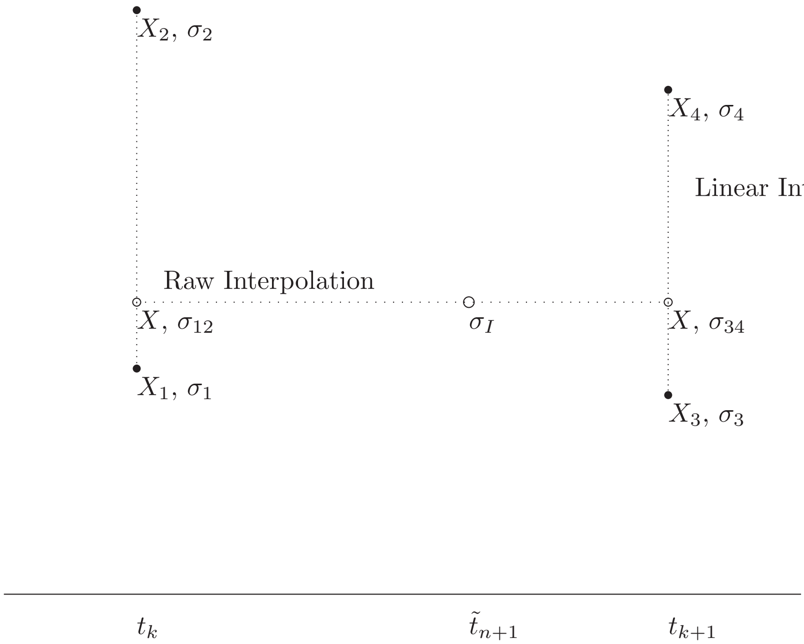

Figure 4.4: Interpolating implied volatility, σ_I, for strike X at scaled time $\tilde{t}_{n+1}$

The lower portion of the tree (from the central node at $\tilde{t}_n$ downwards) is analogous to the upper portion. To determine the transition probabilities $p_{n,i}$ and $q_{n,i}$ for $0 \leq i \leq n$, equations (4.31) and (4.30) can be used respectively. For this portion of the tree, the European put option prices are required and interpolation is performed on the implied volatility and maturity.

Case II: Term Structure

For construction of the state space, incorporating a term structure of implied volatility is done by applying the results of §4.2.1. The implied ATM volatility (the strike of the option is the futures or forward price) is the only input required for construction of the state space, as it is assumed that there is no strike structure. The resulting implied trinomial tree will have unequal time steps. So, $\sigma^2(\tilde{t}_k)\Delta\tilde{t}_k$ is to be a constant for $1 \leq k \leq N$, where $\tilde{t}_k$ refers to the scaled time and $\sigma^2(\tilde{t}_k)$ refers to the local variance over $\Delta\tilde{t}_k = \tilde{t}_k - \tilde{t}_{k-1}$. The procedure is described as follows:

- Calculate or read the relative strikes X/F at each of the input dates t_j for $1 \leq j \leq \hat{N}$.
- Linearly interpolate X to find the implied ATM volatility at that time - require the implied volatility that corresponds to the value $X/F = 1$, where F is the forward/ futures level.
- Perform raw interpolation on the implied ATM volatilities between each time to obtain the forward implied ATM volatilities. These will be constant between any two input dates as a result of the raw interpolation method used.

Once all the forward implied ATM volatilities have been found, use (4.16) to solve for the scaled times by induction. The necessary calculations are simplified due to the forward implied ATM volatilities being constant.

Suppose the known scaled time, $\tilde{t}_{k-1}$, falls between t_{j-1} and t_j and $\sigma_{d,f;j}$ is the constant forward implied ATM volatility between times t_{j-1} and t_j (4.5). In order to determine $\tilde{t}_k$, the induction requires a 'Do While' loop and a variable, dlocalintegral, that is reset to 0 once each scaled time has been found or until (4.16) is satisfied. The search for $\tilde{t}_k$ begins by testing whether the area given by $(t_j - \tilde{t}_{k-1})\sigma_{d,f;j}$ is greater than or smaller than $\hat{c}$. If it is greater than $\hat{c}$, then $\tilde{t}_k < t_j$. If not, the variable, dlocalintergral, starting at 0, is incremented by this area and the search continues by checking whether dlocalintergral $+ (t_{j+1} - t_j)\sigma_{d,f;j+1}$ is greater then or less than $\hat{c}$. So, dlocalintegral is incremented by the discrete amounts until such time it is equivalent to $\hat{c}$. The discrete amounts are the areas given in general by $\sigma^2_{d,f;j}\Delta t$. The loop is only terminated if an increment to dlocalintegral results in a value that is greater than $\hat{c}$. The procedure to determine $\tilde{t}_k$ can be summerized as follows:

dlocalintegral $= 0$

Do While dlocalintegral $< \hat{c}$

- dlocalintegral $=$ dlocalintegral $+ \sigma^2_{d,f;j}(t_j - \tilde{t}_{k-1})$ This brings $\tilde{t}_k$ up to t_j, j must be incremented to $j+1$.
- Once again, the condition for dlocalintegral is checked.

  If dlocalintegral $> \hat{c}$, then $\tilde{t}_k < t_j$ and

$$\tilde{t}_k = \frac{\hat{c} - \text{dlocalintegral}}{\sigma^2_{d,f;j}} + \tilde{t}_{k-1} \tag{4.33}$$

  If dlocalintegral $< \hat{c}$, then dlocalintegral is incremented by $\sigma^2_{d,f;j+1}(t_{j+1} - t_j)$. In this case, the 'while' loop continues until $\tilde{t}_k$ falls between t_{m-1} and t_m for $1 \le m \le \hat{N}$ and use (4.33) to determine $\tilde{t}_k$.

Once the scaled times have been solved for, they can be used in the calculation of the transition probabilities. The state space (stock price mesh) is constructed using a constant volatility recombining trinomial tree.

Since there is no strike structure, the implied volatilities of the ATM options for $\tilde{t}_k$ will be used to price the options, either using the Black-Scholes formula or the trinomial tree constructed with the unequal time steps. These volatilities will be interpolated between input dates using raw interpolation to ensure $\sigma^2_I(t)t > 0$.

Case III: Skew Structure

We now apply the results of §4.2.2. In the case that the local volatility function is of the form $\sigma(S)$, the state space is constructed to accommodate a linear relationship between implied volatility and strike. We assume that there is no term structure. The input requirement is

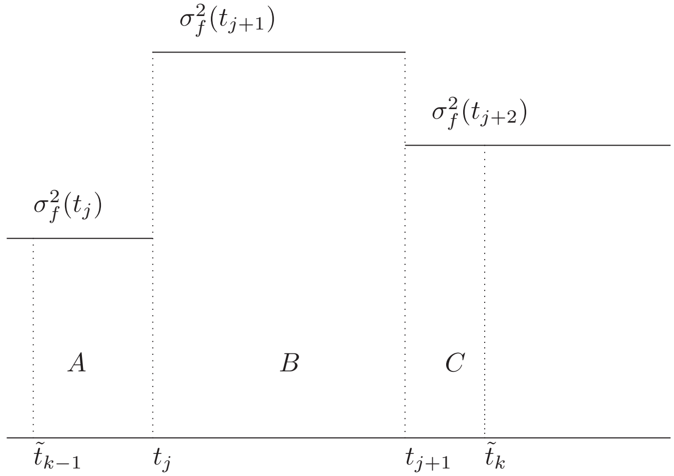

Figure 4.5: $\tilde{t}_k$ is calculated inductively by ensuring integrals equate to $\hat{c}$, $A + B + C = \hat{c}$

1. The ATM implied volatility Σ_0. Since it is assumed that there is no term structure, the value for Σ_0 can be taken as an average of all the ATM implied volatilities.

2. The slope of the function (b): this is generally the percentage point increase in ATM implied volatility per point decrease in the strike of the option. As previously mentioned, this is the Taylor series expansion.

The procedure is to first construct the nodal prices, $S_{j,i}$, at time step j for $0 \leq i \leq 2j$. These are then to be adjusted using (4.23), (4.24) and (4.25).

Once the trinomial tree (state space) has been completed, the requirement is to determine the transition probabilities $p_{n,i}$ and $q_{n,i}$ for $0 \leq i \leq 2n$ as well as the Arrow-Debreu prices $\lambda_{n+1,i}$ for $0 \leq i \leq 2n+2$.

Since the scaled stock price is assumed to have a constant volatility, the input to create the tree will be some constant volatility, Σ_{ATM} which will be adjusted by using the linear relationship (4.25). We are interested in relative changes, not absolute therefore, the futures level at each input will be interpolated to find the futures at the time indicated by the node tree. Once we have this value, to obtain the local volatility, (4.25) becomes:

$$\sigma = \Sigma_0 + 2b\left(\frac{K}{F_{ATM}} - 1\right).$$

This state space is then used to find all the transition probabilities and Arrow-Debreu prices using (4.9).

Case IV Both

The trinomial scheme can easily be constructed to accommodate both a strike and term structure. We begin by constructing a skewed state space as was described in the third case above. This is then stretched in time according to the second case above. This is the simplest and most tractable method to obtain a surface which depicts the observed volatility phenomena.

4.6.3 Non-Constant Time Intervals and a Dividend Yield

If it is the case that the input data (option expiry times) is not equally spaced, the resulting trinomial tree should display such a feature. The original Derman-Kani-Chriss algorithm can be altered to allow for such a modification in the case where the option prices are calculated using Black-Scholes, not a trinomial tree.

This is done in exactly the same manner as in Chapter 3, §3.7.1.

Chapter 5

Characterization of Local Volatility and the Dynamics of the Smile

5.1 Introduction

A risk-neutral diffusion process for the evolution of the underlying is proposed in (Dupire 1994):

$$\frac{dS}{S} = r(t)dt + \sigma(S,t)dW, \tag{5.1}$$

where $r(t)$ is the expected instantaneous stock price return and $\sigma(S,t)$ is the local volatility function. $W(t)$ is standard Brownian motion. Here the spot follows a one dimensional diffusion process, and so the model is complete (it allows for arbitrage pricing and hedging). Option prices can be calculated by discounting an expectation with respect to a risk-neutral probability, under which the discounted spot has no drift, but retains the same diffusion coefficient. In the case of European options, the expectation is taken over terminal values of the spot, while path-dependent options are priced as discounted expected values of the terminal payoff over all paths. Knowledge of the prices of path-dependent options is equivalent to knowledge of the full risk-neutral diffusion, while knowing the European option prices only amounts to knowledge of the spot distribution at the various option expiry times. The full diffusions contain more information than the conditional laws, as distinct diffusions may generate identical conditional laws. One attempts to choose the local volatility function $\sigma(S,t)$ so as to have the model replicate the prices of European options (for various strikes and maturities) seen trading in the market. The more maturities we have, the closer we are to knowledge of the full risk-neutral diffusion.

5.2 Kolmogorov Equations

Before examining the local volatility function, it is necessary to derive the forward and backward Kolmogorov equations.

We start by developing the theory and intuition behind the backward equation. Let $\mathcal{F}_t^W = \sigma\left(W_s : s \leq t\right)$

be the sigma-algebra generated by Brownian motion $\{W_t : t \geq 0\}$, where, s, $t \geq 0$.

Let X be the unique solution that satisfies the integral equation (Björk 2004, §5.1):

$$X_t = x_0 + \int_0^t \mu(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s$$

subject to the existence of a constant k, such that for all x, y and t, the following hold:

$$\begin{aligned} ||\mu(t,x) - \mu(t,y)|| &\leq k||x-y||, \\ ||\sigma(t,x) - \sigma(t,y)|| &\leq k||x-y||, \\ ||\mu(t,x)|| + ||\sigma(t,y)|| &\leq k(1 + ||x||), \end{aligned}$$

The solution has the following properties:

1. X is $\mathcal{F}_t^W$-adapted.
2. X has continuous trajectories.
3. X is a Markov process.

Let

$$u(y,t) = \mathbb{E}_{X(t)=y}\left[\Phi(X(T))\right] \tag{5.2}$$

be the expected value of a terminal condition at a time $T > t$, given that $X(t) = y$.

For any function $u(X, s)$, apply Itô's Lemma (Björk 2004, §4.10)

$$\begin{aligned} du(X(s), s) &= u_x dX + \tfrac{1}{2}u_{xx}(dX)^2 + u_s ds \\ &= \left(u_s + \mu u_x + \tfrac{1}{2}\sigma^2 u_{xx}\right) ds + \sigma u_x dW_s \end{aligned}$$

Thus,

$$u(X(T), T) - u(X(t), t) = \int_t^T \left(u_s + \mu u_x + \tfrac{1}{2}\sigma^2 u_{xx}\right) ds + \int_t^T \sigma u_x dW_s$$

Taking the expected value,

$$\mathbb{E}_{X(t)=y}\left[\Phi(X(T))\right] - u(X(t), t) = \int_t^T \mathbb{E}_t\left[u_s + \mu u_x + \tfrac{1}{2}\sigma^2 u_{xx}\right] ds$$

$$\Rightarrow u_s + \mu u_x + \tfrac{1}{2}\sigma^2 u_{xx} = 0 \tag{5.3}$$

This follows from the fact that Brownian motion is a martingale and (5.2). Thus, for all $t < T$, $u(x,t)$ satisfies (5.3) subject to $u(x,T) = \Phi(T)$. The above PDE (5.3) and boundary condition is the Cauchy problem. The above result is the Feynman-Kač stochastic representation formula, conditional on the process $\sigma(X(s), s)\frac{\partial u}{\partial x}$ being in $\mathcal{L}^2$ (Björk 2004, §5.5).

In the multi-dimensional case ($1 \leq i \leq n$), we have the vector-valued SDE

$$dX_i = \mu_i(X,t)dt + \sum_j \sigma_{ij} dW_j, \tag{5.4}$$

where each W_j is an independent Brownian motion for $1 \leq j \leq m$. Given the following:

- A (column-vector valued) function $\mu : \mathbb{R}_+ \times \mathbb{R}^n$.
- A function $C : \mathbb{R}_+ \times \mathbb{R}^n \to M(n,n)$, which can be written as $C(t,x) = \sigma(t,x)\sigma^T(t,x)$, for some function $\sigma : \mathbb{R}_+ \times \mathbb{R}^n \to M(n,d)$.
- A real valued function $\Phi : \mathbb{R}^n \to \mathbb{R}$.

Then for $t < T$

$$u(y,t) = \mathbb{E}_{X(t)=y}\left[\Phi(X(T))\right]$$

solves

$$u_t + \mathcal{A}u = 0,$$

with $u(x,T) = \Phi(x)$. $\mathcal{A}$ is the Itô operator defined for any function $g(t,x)$ with $g \in C^2(\mathbb{R}^n)$ as

$$\left(\mathcal{A}g\right)(x) = \sum_{i=1}^{n} \mu_i(t,x)\frac{\partial g}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} C_{i,j}(t,x)\frac{\partial^2 g}{\partial x_i \partial x_j} \tag{5.5}$$

and is the infinitesimal generator of the n-dimensional SDE (5.4). (By letting $f(X,t) = X$ and applying a multi-dimensional Itô formula, the original SDE is recovered.) Using the above results, consider the boundary value problem:

$$\begin{aligned} \left(\frac{\partial u}{\partial t} + \mathcal{A}u\right)(t,y) &= 0, & (s,y) &\in (0,T) \times \mathbb{R}^n, \\ u(T,y) &= \mathbf{I}_B(y), & y &\in \mathbb{R}^n \end{aligned}$$

where $\mathbf{I}_B$ is the indicator function of the set B. Since

$$u(t,y) = \mathbb{E}_{\mathbf{I}_B(X(T))}\left[X(t) = y\right] = \mathbb{P}\left[X(T) \in B \,|X(t) = y\right]$$

we have that X is the solution of the n-dimensional SDE (5.4). Turning this argument around, we obtain the following proposition:

Proposition 1 ***Kolmogorov backward equation***
(Björk 2004, §5.10) Let X be a solution to (5.4). Then the transition probabilities $P(t,y;T,B) = \mathbb{P}\left[X(T) \in B \,|X(t) = y\right]$ are given as the solution to the equation

$$\begin{aligned} \left(\frac{\partial P}{\partial t} + \mathcal{A}P\right)(t,y;T,B) &= 0, & (t,y) &\in (0,T) \times \mathbb{R}^n, \\ P(t,y;T,B) &= \boldsymbol{I}_B(y). \end{aligned}$$

By writing $\lambda(B)$ for the Lebesgue measure of the set B, the transition density of the process X is given by (Etheridge 2002, §4.8)

$$p(t,y;T,x) := \lim_{\lambda(B)\to 0} \frac{1}{\lambda(B)}\mathbb{P}\left[X(T) \in B \,|X(t) = y\right]. \tag{5.6}$$

This can be thought of as the probability that $X(T) = x$, given that $X(t) = y$ where $t < T$. Since X is a continuous-time Markov process, it has a well-defined transition density function. p must satisfy the Chapman-Kolmogorov equation which can be written as (Rogers & Williams 2000, §III.1)

$$p(t, y; T, x) = \int_{\mathbb{R}^n} p(t, y; t_1, x_1)p(t_1, x_1; T, x)dx_1$$

for any t_1 satisfying $t < t_1 < T$. Thus, the Chapman-Kolmogorov equation calculates the probability of $p(t, y; T, x)$ by integrating over all the probabilities of getting from (t, y) to (T, x) via (t_1, x_1), for all intermediate positions x_1.

There is a corresponding result for transition density functions:

Proposition 2 *(Björk 2004, §5.10) Let X be a solution to (5.4). Assume that the measure $P(t, y; T, dx)$ has a density $p(t, y; T, x)dx$. Then we have*

$$\begin{aligned} \left(\frac{\partial p}{\partial t} + \mathcal{A}p\right)(t, y; T, x) = 0, &\quad (s, y) \in (0, T) \times \mathbb{R}^n, \\ p(t, y; T, x) \to \delta_x, &\quad \textit{as } t \to T. \end{aligned}$$

The differential operator is working on the backward variables (t, y).

In order to present a similar proposition regarding the forward Kolmogorov equation (Fokker-Planck equation), define the adjoint Itô operator by

$$\left(\mathcal{A}^* g\right)(t, x) = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left[g\mu_i(t, x)\right] + \tfrac{1}{2}\sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}\left[C_{i,j}(t, x)g\right] \tag{5.7}$$

for any function $g(t, x)$ with $g \in C^2(\mathbb{R}^n)$. Fix two points in time, t and T, where $t < T$. Let time s be such that $t < s < T$.

Proposition 3 ***Kolmogorov forward equation***
(Björk 2004, §5.12) Let X be a solution to (5.4) with a density $p(t, y; T, x)dx$. Then p will satisfy the forward Kolmogorov equation

$$\begin{aligned} \frac{\partial}{\partial T}p(t, y; T, x) = \mathcal{A}^* p(t, y; T, x), &\quad (T, x) \in (0, \infty) \times \mathbb{R}^n, \\ p(t, y; T, x) \to \delta_y, &\quad \textit{as } T \downarrow t. \end{aligned}$$

The forward Kolmogorov describes the probability distribution by solving an initial-value problem, while the backward Kolmogorov equation describes the expected final payoff by solving a final-value problem.

5.3 Relationship between Prices and Distributions

In this section, we will derive the expression for local volatility that appears in (Dupire 1994) using the approach presented in (Derman & Kani 1994). Let $\phi(S_T, S_0, T)$ denote the risk-neutral probability density function associated with (5.1). It is defined as the probability that the stock price reaches S_T

at time T having the initial value S_0 at time 0. It satisfies both the backward and forward Kolmogorov equations with boundary condition $\phi(S_0, S_T, 0) = \delta(S_T - S_0)$, where $\delta(x)$ is the Dirac delta function. These equations are parabolic partial differential equations. The backward equation involves derivatives with respect to current state and time while the forward equation involves derivatives with respect to future state and time. The backward Kolmogorov equation requires terminal conditions and is solved for, backwards in time:

$$\frac{\partial \phi}{\partial T} = \frac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 \phi}{\partial S_0^2} + r S_0 \frac{\partial \phi}{\partial S_0} \tag{5.8}$$ [1]

The forward Kolmogorov (Fokker-Planck) equation requires initial conditions and is solved for $T > 0$:

$$\frac{\partial \phi}{\partial T} = \frac{1}{2}\frac{\partial^2}{\partial S_T^2}\left(\sigma^2 S_T^2 \phi\right) - r\frac{\partial}{\partial S_T}\left(S_T \phi\right) \tag{5.9}$$ [2]

Let $Z(0,t)$ be the discount function given by

$$Z(0,t) = \exp\left(-\int_0^t r(s)\,ds\right) \tag{5.10}$$

Now we briefly review the main results of (Breeden & Litzenberger 1978). The collection of European call option prices, $C(S_0, K, T)$, with current spot S_0, maturity T and of different strikes K, yields the risk-neutral density function ϕ through the relationship:

$$C(S_0, K, T) = Z(0,T)\int_K^\infty \phi(S_0, S_T, T)(S_T - K)\,dS_T \tag{5.11}$$

We now use the following well-known formula, the Leibnitz rule, for differentiation of a definite integral with respect to a parameter (Abramowitz & Stegun 1974):

$$\frac{d}{da}\int_{\psi(a)}^{\phi(a)} f(x,a)\,dx = f(\phi(a),a)\frac{d\phi(a)}{da} - f(\psi(a),a)\frac{d\psi(a)}{da} + \int_{\psi(a)}^{\phi(a)} \frac{d}{da} f(x,a)\,dx$$

[1] If we consider the diffusion process of (5.1), Proposition (2) can be applied. For all $t < T$, we have that

$$\frac{\partial \phi}{\partial t} + \mathcal{A}\phi = 0$$

$$\frac{\partial \phi}{\partial t} + r S_0 \frac{\partial \phi}{\partial S_0} + \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 \phi}{\partial {S_0}^2} = 0$$

$$\Rightarrow r S_0 \frac{\partial \phi}{\partial S_0} + \tfrac{1}{2}\sigma^2 S_0^2 \frac{\partial^2 \phi}{\partial {S_0}^2} = \frac{\partial \phi}{\partial T},$$

where the Itô operator is defined by (5.5).

[2] Using Proposition (3) and the definition of the adjoint Itô operator (5.7), we have that for all $T > t$ (as the forward equation relies on initial conditions and describes behaviour forward in time),

$$\frac{\partial \phi}{\partial T} - \mathcal{A}^*\phi = 0$$

$$\frac{\partial \phi}{\partial T} - \left(-r\frac{\partial}{\partial S_T}(S_T\phi) + \frac{1}{2}\frac{\partial^2}{\partial S_T^2}(\sigma^2 S_T^2 \phi)\right) = 0$$

$$\Rightarrow \frac{1}{2}\frac{\partial^2}{\partial S_T^2}\left(\sigma^2 S_T^2 \phi\right) - r\frac{\partial}{\partial S_T}(S_T\phi) = \frac{\partial \phi}{\partial T}$$

Differentiating (5.11) once with respect to strike:

$$
\begin{aligned}
&\frac{\partial}{\partial K} C\left(S_0, K, T\right) \\
&= \frac{\partial}{\partial K}\left(Z(0,T) \lim_{x \to \infty} \int_K^x \phi\left(S_0, S_T, T\right)\left(S_T - K\right) dS_T\right) \\
&= Z(0,T) \lim_{x \to \infty}\left(\phi\left(S_0, x, T\right)(x-K) \frac{dx}{dK} - \phi\left(S_0, K, T\right)(K-K) \frac{dK}{dK} + \int_K^x \phi\left(S_0, S_T, T\right)(-1) dS_T\right) \\
&= -Z(0,T) \int_K^{\infty} \phi\left(S_0, S_T, T\right) dS_T \qquad (5.12)
\end{aligned}
$$

In this case, $a = K$, $\phi(a)$ and $\psi(a)$ are constants, and $f(x,a) = \phi\left(S_0, S_T, T\right)\left(S_T - K\right)$. The first and second term in the third line above are zero. Differentiating again with respect to strike:

$$
\begin{aligned}
&\frac{\partial^2}{\partial K^2} C\left(S_0, K, T\right) \\
&= -Z(0,T) \lim_{x \to \infty}\left(\phi\left(S_0, x, T\right) \frac{dx}{dK} - \phi\left(S_0, K, T\right) \frac{dK}{dK} + \int_K^x \frac{d}{dK}\left(\phi\left(S_0, K, T\right)\right) dS_T\right) \\
&= Z(0,T) \phi\left(S_0, K, T\right) \qquad (5.13)
\end{aligned}
$$

as in (Breeden & Litzenberger 1978). In practice, there are only a discrete set of of option prices and the continuum is completed using interpolation. The right hand side of the above equation is an Arrow-Debreu price: it is the price of a security that has a a payoff of $\delta\left(S_T - K\right)$. It can be constructed using butterflies[3].

Multiplying (5.9) by $Z(0,T)\left(S_T - K\right)$ and integrating with respect to S_T yields

$$
Z(0,T) \int_K^{\infty} \frac{\partial \phi}{\partial T}\left(S_T - K\right) dS_T = Z(0,T) \int_K^{\infty}\left(\frac{1}{2} \frac{\partial^2}{\partial S_T^2}\left(\sigma^2 S_T^2 \phi\right) - r \frac{\partial}{\partial S_T}\left(S_T \phi\right)\right)\left(S_T - K\right) dS_T \qquad (5.14)
$$

Consider the first term on the right hand side of (5.14). We use integration by parts:

$$
\int_a^{\infty} f'(S_T) g(S_T) dS_T = f(S_T) g(S_T)\Big|_a^{\infty} - \int_a^{\infty} f(S_T) g'(S_T) dS_T
$$

We use this with $f'(S_T) = \frac{\partial^2}{\partial S_T^2}\left(\sigma^2 S_T^2 \phi\right)$ and $g(S_T) = \left(S_T - K\right)$. So,

$$
\begin{aligned}
&Z(0,T) \int_K^{\infty} \frac{1}{2} \frac{\partial^2}{\partial S_T^2}\left(\sigma^2 S_T^2 \phi\right)\left(S_T - K\right) dS_T \\
&= \frac{Z(0,T)}{2} \lim_{x \to \infty}\left(\frac{\partial}{\partial S_T}\left(\sigma^2 S_T^2 \phi\right)\left(S_T - K\right)\Big|_K^x - \int_K^x \frac{\partial}{\partial S_T}\left(\sigma^2 S_T^2 \phi\right) dS_T\right) \\
&= \frac{Z(0,T)}{2} \lim_{x \to \infty}\left(\frac{\partial}{\partial S_T}\left(\sigma^2 S_T^2 \phi\right)\left(S_T - K\right)\Big|_K^x - \sigma^2 S_T^2 \phi\Big|_K^x\right) \\
&= Z(0,T) \frac{1}{2} \sigma^2 K^2 \phi\left(S_T, K, T\right) \\
&= \frac{1}{2} \sigma^2 K^2 \frac{\partial^2}{\partial K^2} C\left(S_0, K, T\right) \qquad (5.15)
\end{aligned}
$$

[3] A butterfly payoff can be constructed by going long n (European) call options, strike K_1, going short $2n$ call options, strike K_2 and going long n call options with strike K_3, such that $K_2 - K_1 = \frac{1}{n} = K_3 - K_2$. The infinitesimal width results as $n \to \infty$.

We have used the fact that $\lim_{x\to\infty} \sigma^2 x^n \phi(S_0, x, T) \to 0$ for $n = 2$, 3, and (5.13)[4].

Now consider the second term on the right hand side of (5.14). Again perform integration by parts with $f'(S_T) = \frac{\partial}{\partial S_T}(S_T\phi)$ and $g(S_T) = (S_T - K)$. We get

$$\begin{aligned}
&- Z(0,T) \int_K^\infty r \frac{\partial}{\partial S_T}(S_T\phi)(S_T - K)\, dS_T \\
&= -rZ(0,T) \lim_{x\to\infty} \left((S_T\phi)(S_T - K) \Big|_K^x - \int_K^x (S_T\phi)\, dS_T \right) \\
&= rZ(0,T) \int_K^\infty ((S_T - K) + K)\, \phi dS_T \\
&= r\left[C(S_0, K, T) - K \frac{\partial}{\partial K} C(S_0, K, T) \right]
\end{aligned} \tag{5.16}$$

In the second line above, $\lim_{x\to\infty} (x\phi)(x - K) \to 0$ as was shown in the Lemma (1). In the last line, (5.11) and (5.12) are substituted in.

[4]Consider the second term. Since implied volatilities/prices for strikes at zero or ∞ do no exist, one could for example assume that:

(i) For all strikes K satisfying $0 \le K < K_l$, where K_l is the lowest quoted strike, we have $\Sigma(K,T) = \Sigma(K_l,T)$.

(ii) For all strikes K satisfying $K_h \le K < \infty$, where K_h is the highest quoted strike, we have $\Sigma(K,T) = \Sigma(K_h,T)$.

Obtaining the risk-neutral probability density function (pdf) from such a skew, using the results of (Breeden & Litzenberger 1978), implies that at the above two extremes, a lognormal density function (with a constant volatility) is assumed. If $y = \ln x$ $(x > 0)$ is normally distributed with mean $-\infty < \mu < \infty$ and standard deviation $\sigma \ge 0$ (at each extreme, mean and variance will differ), then x is lognormally distributed with pdf

$$\frac{1}{x\sigma\sqrt{2\pi}} \exp\left(-\tfrac{1}{2} \left(\frac{\ln x - \mu}{\sigma} \right)^2 \right)$$

We now prove the following lemma:

Lemma 1 *For $n \in \mathbb{N}$, $\lim_{x\to\infty} x^n \phi(S_0, x, T) = 0$:*

$$\begin{aligned}
&\lim_{x\to\infty} x^n \phi(S_0, x, T) \\
&= \lim_{x\to\infty} x^n \exp\left(-\tfrac{1}{2} \left(\frac{\ln x - \mu}{\sigma} \right)^2 \right) \\
&= \frac{1}{\sigma\sqrt{2\pi}} \lim_{x\to\infty} x^{n-1} \exp\left(-\tfrac{1}{2} \left(\frac{\ln x - \mu}{\sigma} \right)^2 \right)
\end{aligned}$$

Now, let $w = \ln x$. Then, $e^w = x$ and the limit becomes

$$\begin{aligned}
&\frac{1}{\sigma\sqrt{2\pi}} \lim_{w\to\infty} e^{w(n-1)} \exp\left(-\tfrac{1}{2} \left(\frac{w - \mu}{\sigma} \right)^2 \right) \\
&= \frac{1}{\sigma\sqrt{2\pi}} \lim_{w\to\infty} \exp\left(\frac{w(n-1)\sigma^2 - \frac{1}{2}(w-\mu)^2}{\sigma^2} \right) \\
&= 0
\end{aligned}$$

since

$$\lim_{w\to\infty} \left(w(n-1)\sigma^2 - \tfrac{1}{2}(w-\mu)^2 \right) = -\infty.$$

In order to simplify the term on the left hand side of (5.14), we note that

$$
\begin{aligned}
\frac{\partial C}{\partial T} &= \frac{\partial}{\partial T}\left(Z(0,T)\int_K^\infty \phi(S_T-K)dS_T\right) \\
&= Z(0,T)\int_K^\infty \frac{\partial \phi}{\partial T}\left(S_T-K\right)dS_T - rC\left(S_0,K,T\right)
\end{aligned}
$$

Therefore,

$$
Z(0,T)\int_K^\infty \frac{\partial \phi}{\partial T}\left(S_T-K\right)dS_T = \frac{\partial C}{\partial T} + rC\left(S_0,K,T\right) \tag{5.17}
$$

Substituting (5.15), (5.16) and (5.17) into (5.14), we get

$$
\frac{\partial C}{\partial T} + rC\left(S_0,K,T\right) = \frac{1}{2}\sigma^2 K^2 \frac{\partial^2}{\partial K^2}C\left(S_0,K,T\right) + r\left[C\left(S_0,K,T\right) - K\frac{\partial}{\partial K}C\left(S_0,K,T\right)\right]
$$

Therefore,

$$
\frac{1}{2}\sigma^2\left(K,T\right)K^2\frac{\partial^2 C}{\partial K^2} - rK\frac{\partial C}{\partial K} - \frac{\partial C}{\partial T} = 0
$$

Solving for the $\sigma\left(K,T\right)$

$$
\sigma\left(K,T\right) = \sqrt{\frac{\frac{\partial C}{\partial T} + rK\frac{\partial C}{\partial^2 K}}{\frac{1}{2}K^2\frac{\partial^2 C}{\partial K^2}}} \tag{5.18}
$$

which is the result proven in (Dupire 1994).

Using the following results, (5.18) can be viewed as the definition for local volatility in (5.1):

1. If (5.1) holds, then the distribution function $\phi\left(S_0,K,T\right)$ completely determines European option prices $C\left(S_0,K,T\right)$ for all strikes K and maturities T.

2. Conversely, call prices completely determine the distribution function using (5.13).

3. The local volatility function $\sigma(S,t)$, for future stock prices S at times t, can be determined from (5.18).

The stock price diffusion process can be entirely determined from knowledge of the stock price distribution function.

5.4 Local Volatility in terms of Implied volatility

In this section, we will need distinguish between partial differentiation of the form

$$
\lim_{\varepsilon\to 0}\frac{f(x(t),t+\varepsilon)-f(x(t),t)}{\varepsilon},
$$

which we denote $\frac{\partial f}{\partial t}$, and partial differentiation of the form

$$
\lim_{\varepsilon\to 0}\frac{f(x(t+\varepsilon),t+\varepsilon)-f(x(t),t)}{\varepsilon},
$$

which we denote $\frac{df}{dt}$. Suppose $f = f(x,t)$ and $x = x(t)$. So, f is a function of t alone. Of course

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} \tag{5.19}$$

This convention will be followed even if f is not a function of t alone e.g. if $f = f(x,t,s)$, $x = x(t)$, then

$$\lim_{\varepsilon \to 0} \frac{f(x(t+\varepsilon), t+\varepsilon, s) - f(x(t), t, s)}{\varepsilon}$$

will be denoted $\frac{df}{dt}$ in order to distinguish from

$$\lim_{\varepsilon \to 0} \frac{f(x(t), t+\varepsilon, s) - f(x(t), t, s)}{\varepsilon}$$

which is $\frac{\partial f}{\partial t}$. Of course (5.19) holds again, even though $\frac{df}{dt}$ is a function of both t and s.

(5.18) for local volatility can be extended to include a dividend yield q and is given by (Wilmott 2000, §25.6):

$$\sigma(K,T) = \sqrt{\frac{\frac{dC}{dT} + (r-q)K\frac{dC}{dK} + qC}{\frac{1}{2}K^2\frac{d^2C}{dK^2}}}, \tag{5.20}$$

By noting that the implied volatility Σ is a function of strike and expiry i.e. $\Sigma = \Sigma(K,T)$, the above partial derivatives can be obtained with respect to Σ i.e. the local volatility function can be expressed as a function of implied volatility rather than of option prices.

1. Consider $\frac{dC}{dT}$:

$$\frac{dC}{dT} = \frac{\partial C}{\partial T} + \frac{\partial C}{\partial \Sigma}\frac{\partial \Sigma}{\partial T}$$

Now,

$$C(S,K,T,\Sigma,r,q) = Se^{-q(T-t)}\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2)$$

where t and S are the current time and stock price, T the expiry of the option with strike K, r and q are the interest rate and dividend yield respectively and

$$d_{1,2} = \frac{\ln\frac{S}{K} + \left(r - q \pm \frac{1}{2}\Sigma^2\right)(T-t)}{\Sigma\sqrt{T-t}}.$$

Therefore

$$\frac{dC}{dT} = -qSe^{-q(T-t)}\Phi(d_1) + Se^{-q(T-t)}\Phi'(d_1)\frac{dd_1}{dT} + rKe^{-r(T-t)}\Phi(d_2) - Ke^{-r(T-t)}\Phi'(d_2)\frac{dd_2}{dT} \tag{5.21}$$

Looking at $\frac{dd_1}{dT}$ and $\frac{dd_2}{dT}$:

$$\begin{aligned}
&\frac{dd_1}{dT} \\
&= \frac{\partial d_1}{\partial T} + \frac{\partial d_1}{\partial \Sigma}\frac{\partial \Sigma}{\partial T} \\
&= \frac{\left(\Sigma\frac{\partial\Sigma}{\partial T}(T-t) + \left(r-q+\frac{1}{2}\Sigma^2\right)\right)\Sigma\sqrt{T-t} - \left(\ln\frac{S}{K} + \left(r-q+\frac{1}{2}\Sigma^2\right)(T-t)\right)\left(\frac{\partial\Sigma}{\partial T}\sqrt{T-t} + \frac{\Sigma}{2\sqrt{T-t}}\right)}{\Sigma^2(T-t)} \\
&:= \frac{\mathcal{X}}{\mathcal{Y}}
\end{aligned}$$

which is obtained from the quotient rule. Multiplying out the numerator, $\mathcal{X}$:

$$\begin{aligned}
&\mathcal{X} \\
&= \Sigma^2 \frac{\partial \Sigma}{\partial T}(T-t)\sqrt{T-t} + \left(r-q+\tfrac{1}{2}\Sigma^2\right)\Sigma\sqrt{T-t} - \frac{1}{2}\Sigma \ln\frac{S}{K}\frac{1}{\sqrt{T-t}} \\
&- \frac{1}{2}\Sigma\left(r-q+\frac{1}{2}\Sigma^2\right)\sqrt{T-t} - \frac{\partial \Sigma}{\partial T}\sqrt{T-t}\ln\frac{S}{K} - \frac{\partial \Sigma}{\partial T}(T-t)\sqrt{T-t}\left(r-q+\frac{1}{2}\Sigma^2\right)
\end{aligned}$$

First dividing through by $\mathcal{Y}$ then simplifying, we get

$$\begin{aligned}
\frac{dd_1}{dT} &= \frac{\partial \Sigma}{\partial T}\sqrt{T-t} + \frac{1}{2}\left(r-q+\frac{1}{2}\Sigma^2\right)\frac{1}{\Sigma\sqrt{T-t}} - \frac{1}{2}\ln\frac{S}{K}\frac{1}{\Sigma(T-t)\sqrt{T-t}} \\
&- \frac{\partial \Sigma}{\partial T}\left(\ln\frac{S}{K} + \left(r-q+\frac{1}{2}\Sigma^2\right)(T-t)\right)\frac{1}{\Sigma^2\sqrt{T-t}} \\
&= \frac{\partial \Sigma}{\partial T}\left(\sqrt{T-t} - \frac{d_1}{\Sigma}\right) - \frac{1}{2}\frac{d_1}{T-t} + \frac{r-q+\frac{1}{2}\Sigma^2}{\Sigma\sqrt{T-t}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{dd_2}{dT} &= \frac{\partial d_1}{\partial T} - \frac{\partial \Sigma}{\partial T}\sqrt{T-t} - \frac{1}{2}\frac{\Sigma}{\sqrt{T-t}} \\
&= \frac{\partial \Sigma}{\partial T}\left(\sqrt{T-t} - \frac{d_1}{\Sigma}\right) - \frac{1}{2}\frac{d_1}{T-t} + \frac{r-q+\frac{1}{2}\Sigma^2}{\Sigma\sqrt{T-t}} - \frac{\partial \Sigma}{\partial T}\sqrt{T-t} - \frac{1}{2}\frac{\Sigma}{\sqrt{T-t}} \\
&= \frac{r-q+\frac{1}{2}\Sigma^2}{\Sigma\sqrt{T-t}} - \frac{1}{2}\frac{\Sigma}{\sqrt{T-t}} - \frac{1}{2}\frac{d_1}{T-t} - \frac{\partial \Sigma}{\partial T}\frac{d_1}{\Sigma}.
\end{aligned}$$

Substituting into (5.21) and using the well-known fact that

$$Se^{-q(T-t)}\Phi'(d_1) = Ke^{-r(T-t)}\Phi'(d_2) \tag{5.22}$$

$$\begin{aligned}
\frac{dC}{dT} &= -qSe^{-q(T-t)}\Phi(d_1) + rKe^{-r(T-t)}\Phi(d_2) \\
&+ Se^{-q(T-t)}\Phi'(d_1)\left(\frac{\partial \Sigma}{\partial T}\left(\sqrt{T-t} - \frac{d_1}{\Sigma}\right) - \frac{1}{2}\frac{d_1}{T-t} + \frac{r-q+\frac{1}{2}\Sigma^2}{\Sigma\sqrt{T-t}}\right) \\
&- Ke^{-r(T-t)}\Phi'(d_2)\left(\frac{r-q+\frac{1}{2}\Sigma^2}{\Sigma\sqrt{T-t}} - \frac{1}{2}\frac{\Sigma}{\sqrt{T-t}} - \frac{1}{2}\frac{d_1}{T-t} - \frac{\partial \Sigma}{\partial T}\frac{d_1}{\Sigma}\right) \\
&= -qSe^{-q(T-t)}\Phi(d_1) + rKe^{-r(T-t)}\Phi(d_2) \\
&+ Se^{-q(T-t)}\Phi'(d_1)\left(\frac{\partial \Sigma}{\partial T}\sqrt{T-t} + \frac{1}{2}\frac{\Sigma}{\sqrt{T-t}}\right)
\end{aligned} \tag{5.23}$$

2. The next Greek is called dual delta and is found along similar lines. See (Hakala & Wystup 2002, §1.8.2) for example. We have that

$$\frac{dC}{dK} = Se^{-q(T-t)}\Phi'(d_1)\left(\frac{\partial \Sigma}{\partial K}\sqrt{T-t}\right) - e^{-r(T-t)}\Phi(d_2) \tag{5.24}$$

3. Lastly, we require the Greek, dual gamma (Hakala & Wystup 2002, §1.8.2). This is

$$\frac{d^2C}{dK^2} = Se^{-q(T-t)}\Phi'(d_1)\left(\left(\frac{\partial \Sigma}{\partial K}\right)^2 d_1\sqrt{T-t}\left(\frac{d_1}{\Sigma}-\sqrt{T-t}\right)+2\frac{\partial \Sigma}{\partial K}\frac{d_1}{\Sigma K}+\frac{1}{K^2\Sigma\sqrt{T-t}}+\frac{\partial^2\Sigma}{\partial K^2}\sqrt{T-t}\right) \tag{5.25}$$

Considering the numerator in (5.20) first, and dividing through by the $\frac{1}{2}$ from the denominator:

$$\begin{aligned}
&2\frac{dC}{dT}+2\left(r-q\right)K\frac{dC}{dK}+2qC\\
&=-2qSe^{-q(T-t)}\Phi(d_1)+2rKe^{-r(T-t)}\Phi(d_2)\\
&+2Se^{-q(T-t)}\Phi'(d_1)\left(\frac{\partial\Sigma}{\partial T}\sqrt{T-t}+\frac{1}{2}\frac{\Sigma}{\sqrt{T-t}}\right)\\
&+2\left(r-q\right)K\left(Se^{-q(T-t)}\Phi'(d_1)\left(\frac{\partial\Sigma}{\partial K}\sqrt{T-t}\right)-e^{-r(T-t)}\Phi(d_2)\right)\\
&+2q\left(Se^{-q(T-t)}\Phi(d_1)-Ke^{-r(T-t)}\Phi(d_2)\right)\\
&=2Se^{-q(T-t)}\Phi'(d_1)\left(\frac{\partial\Sigma}{\partial T}\sqrt{T-t}+\frac{1}{2}\frac{\Sigma}{\sqrt{T-t}}+\left(r-q\right)K\sqrt{T-t}\frac{\partial\Sigma}{\partial K}\right)\\
&+2rKe^{-r(T-t)}\Phi(d_2)-2\left(r-q\right)Ke^{-r(T-t)}\Phi(d_2)-2qKe^{-r(T-t)}\Phi(d_2)\\
&=Se^{-q(T-t)}\Phi'(d_1)\left(2\frac{\partial\Sigma}{\partial T}\sqrt{T-t}+\frac{\Sigma}{\sqrt{T-t}}+2\left(r-q\right)K\sqrt{T-t}\frac{\partial\Sigma}{\partial K}\right)\\
&:=\mathcal{N}
\end{aligned}$$

Now, expanding the denominator:

$$\begin{aligned}
K^2\frac{d^2C}{dK^2} &= K^2Se^{-q(T-t)}\Phi'(d_1)\left(\left(\frac{\partial \Sigma}{\partial K}\right)^2 d_1\sqrt{T-t}\left(\frac{d_1}{\Sigma}-\sqrt{T-t}\right)\right.\\
&\left.+2\frac{\partial \Sigma}{\partial K}\frac{d_1}{\Sigma K}+\frac{1}{K^2\Sigma\sqrt{T-t}}+\frac{\partial^2\Sigma}{\partial K^2}\sqrt{T-t}\right)\\
&:=\mathcal{D}
\end{aligned}$$

Multiplying $\mathcal{N}$ and $\mathcal{D}$ by $\frac{\Sigma\sqrt{T-t}}{Se^{-q(T-t)}\Phi'(d_1)}$ to get $\mathcal{N}'$ and $\mathcal{D}'$ respectively,

$$\mathcal{N}'=2\Sigma\frac{\partial\Sigma}{\partial T}\left(T-t\right)+\Sigma^2+2\Sigma\left(r-q\right)K\left(T-t\right)\frac{\partial\Sigma}{\partial K}$$

and

$$\begin{aligned}
\mathcal{D}' &= K^2\left(\frac{\partial\Sigma}{\partial K}\right)^2 d_1^2\left(T-t\right)-\Sigma\left(T-t\right)K^2\sqrt{T-t}d_1\left(\frac{\partial\Sigma}{\partial K}\right)^2\\
&+2K\frac{\partial\Sigma}{\partial K}d_1\sqrt{T-t}+1+\Sigma\left(T-t\right)K^2\frac{\partial^2\Sigma}{\partial K^2}\\
&=\left(1+Kd_1\frac{\partial\Sigma}{\partial K}\sqrt{T-t}\right)^2+K^2\left(T-t\right)\Sigma\left(\frac{\partial^2\Sigma}{\partial K^2}-d_1\left(\frac{\partial\Sigma}{\partial K}\right)^2\sqrt{T-t}\right)
\end{aligned}$$

So, $\sigma(K,T)$ is given by

$$\begin{aligned}\sigma(K,T) &= \sqrt{\frac{\mathcal{N}'}{\mathcal{D}'}} \\ &= \sqrt{\frac{2\Sigma\frac{\partial\Sigma}{\partial T}(T-t)+\Sigma^2+2\Sigma(r-q)K(T-t)\frac{\partial\Sigma}{\partial K}}{\left(1+Kd_1\frac{\partial\Sigma}{\partial K}\sqrt{T-t}\right)^2+K^2(T-t)\Sigma\left(\frac{\partial^2\Sigma}{\partial K^2}-d_1\left(\frac{\partial\Sigma}{\partial K}\right)^2\sqrt{T-t}\right)}}\end{aligned} \tag{5.26}$$

The local volatility surface can be used to price exotic options which are not generally traded, to be consistent with traded instruments. The surface generated from the traded instruments, at time t, is the market's view of future volatility. In order to reduce any exposure to model error, hedging should be performed using the same surface (Wilmott 2000, §25.13).

5.5 Dynamics of the Volatility Surface

This section will ultimately reveal the flawed nature of local volatility models, displaying their inability to correctly predict the dynamics of the skew.

5.5.1 The Forward Measure

Let T be the maturity of a zero coupon bond $Z(t,T)$, for $t<T$, which will be used as a numèraire for the valuation of derivatives.

5.5.2 Local Volatility Model

Given the following SDE which describes the dynamics of the forward price of an asset under the forward measure (Hagan & Woodward 1998),

$$dF(t) = {}^5\alpha(t)A(F)dW(t), \quad F(0)=f_0 \tag{5.27}$$

where $\alpha(t)$ is a function of time but not state, $A(F)$ is a function of state but not time and f_0 the time $t=0$ forward price for settlement date t_{set}. The European call and put option prices, with expiry t_{ex} and strike K, are given by the expected values

$$V_{\text{call}}(0,f_0) = Z(0,t_{\text{set}})\mathbb{E}\left[(F(t_{\text{ex}})-K)^+\,|F(0)=f_0\right] \tag{5.28}$$

$$V_{\text{put}}(0,f_0) = Z(0,t_{\text{set}})\mathbb{E}\left[(K-F(t_{\text{ex}}))^+\,|F(0)=f_0\right] \tag{5.29}$$

under the forward measure.[6]

[5] It is desirable that local volatility surfaces can be separated as was noted in (Derman, Kani & Chriss 1996). A separable local volatility surface allows the algorithmic generation of a trinomial tree of local volatility, that matches the observed and modelled implied skew well. This was seen in Chapter 4.

[6] The expiry date and the settlement date need not be equal.

For example, in Black's model, $\alpha(t) = \sigma_B$ is the quoted volatility of an option and $A(F) = F$. In this case, the price of a European call and put option at time $t = 0$, with expiry t_{ex} and strike K, are:

$$\begin{aligned}
V_{\text{call}}(0, f_0) &= Z(0, t_{\text{set}}) \left(f_0 \Phi(d_1) - K\Phi(d_2) \right) \\
V_{\text{put}}(0, f_0) &= Z(0, t_{\text{set}}) \left(K\Phi(-d_2) + f_0 \Phi(-d_1) \right),
\end{aligned}$$

with $Z(0, t_{\text{set}})$ the time 0 discount factor to the settlement date, t_{set} and

$$d_{1,2} = \frac{\ln \frac{f_0}{K} \pm \frac{1}{2}\sigma_B^2 t_{\text{ex}}}{\sigma_B \sqrt{t_{\text{ex}}}}.$$

In order to solve (5.28) and (5.29) in more general cases, singular perturbation techniques can be applied to obtain either the actual prices or more conveniently, the implied volatility for Black's model. We will first review the method of solving differential equations using perturbation techniques.

5.5.3 Perturbation Techniques

Gauge Functions and Ordering

If the limit of a positive function $f(\varepsilon)$, as $\varepsilon \geq 0$ tends to zero exists, then there are three possibilities (Nayfeh 1981, §1.3):

1. $f(\varepsilon) \to 0$
2. $f(\varepsilon) \to A$
3. $f(\varepsilon) \to \infty$

as $\varepsilon \to 0$, where $0 < A < \infty$.

To determine the rate at which a function tends to zero (or infinity), a set of comparison functions (the rates at which these functions tend to zero or infinity are known) are used. These are termed *gauge functions*. The simplest being the powers of ε,

$$1, \varepsilon, \varepsilon^2, \varepsilon^3, \ldots$$

with

$$1 > \varepsilon > \varepsilon^2 > \varepsilon^3 > \ldots$$

for small ε, and inverse powers of ε,

$$\varepsilon^{-1}, \varepsilon^{-2}, \varepsilon^{-3}, \ldots$$

with

$$\varepsilon^{-1} < \varepsilon^{-2} < \varepsilon^{-3} < \ldots$$

Other gauge functions are: $\exp(1/\varepsilon)$, $\exp(-1/\varepsilon)$, $\ln(1/\varepsilon)$, $[\ln(1/\varepsilon)]^{-1}$, etc. These are required to supplement the powers of ε as they tend to zero faster than any power of ε. The set of gauge functions establish the rate at which a function tends to zero or infinity. This is done by comparing the rate at which the function tends to zero or infinity with the rate that the gauge functions tend to zero or infinity respectively.

The order symbols O and o can be classified according to the following:

If the function $f(\varepsilon)$ tends to zero at the same rate that the function $g(\varepsilon)$ tends to zero, we have that

$$f(\varepsilon) = O(g(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0,$$

if

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = P \quad 0 < |P| < \infty$$

where $g(\varepsilon)$ is the gauge function.

If it is not possible to determine the rate at which a function tends to its limit, but sufficient to determine the rate with respect to a given gauge function (whether it is faster or slower than the gauge function), then we have the following:

$$f(\varepsilon) = o(g(\varepsilon)) \quad \text{as} \quad \varepsilon \to 0,$$

if

$$\lim_{\varepsilon \to 0} \frac{f(\varepsilon)}{g(\varepsilon)} = 0$$

Asymptotic Sequences and Expansions

Definition 2 *An expansion of $f(x, \varepsilon)$ of the form*

$$f(x, \varepsilon) = f_0(x) + \varepsilon f_1(x) + \varepsilon^2 f_2(x) + \ldots + \varepsilon^n f_n(x) + O(\varepsilon^{n+1}),$$

as $\varepsilon \to 0$, is a regular or straightforward perturbation expansion, with ε as perturbation parameter.

The function $f(x)$ is a solution to an algebraic, integral or differential equation. $f(x, \varepsilon)$ is the perturbed solution with $\varepsilon^j f_j(x)$ as the jth order solution. The regular expansion is performed up to order 2 or 3 then substituted into the original equation for $f(x)$. The solution is then obtained by equating like powers of ε and solving for each.

If it is not possible to find an asymptotic expansion (to be defined below) for a given function, it will be necessary to use a general sequence of functions.

Definition 3 ***Asymptotic Sequence***
(Cole & Kevorkian 1981, §1.2) Consider a sequence of functions of ε, $\{\phi_n(\varepsilon)\}$ for $n = 1, 2, \ldots$. Such a sequence is asymptotic if

$$\phi_{n+1}(\varepsilon) = o(\phi_n(\varepsilon)) \quad \textit{as } \varepsilon \to \varepsilon_0$$

for each n. If the sequence is infinite and $\phi_{n+1} = o(\phi_n)$ uniformly in n, the sequence is said to be uniform in n.

Definition 4 ***Asymptotic Expansion***

(Cole & Kevorkian 1981, §1.2) A sum of terms of the form $\sum_{n=1}^{N} a_n(x)\phi_n(\varepsilon)$ *is called an asymptotic expansion of the function* $f(x,\varepsilon)$ *to* N *terms (*N *may be infinite) as* $\varepsilon \to \varepsilon_0$ *with respect to the sequence* $\{\phi_n(\varepsilon)\}$ *if*

$$f(x,\varepsilon) - \sum_{n=1}^{M} a_n(x)\phi_n(\varepsilon) = o(\phi_M) \quad \text{as } \varepsilon \to \varepsilon_0$$

for each $M = 1, 2, \ldots, N$*. Or equivalently,*

$$f(x,\varepsilon) - \sum_{n=1}^{M-1} a_n(x)\phi_n(\varepsilon) = O(\phi_M) \quad \text{as } \varepsilon \to \varepsilon_0$$

for each $M = 2, \ldots, N$*.*

If the order relations hold uniformly in the domain, then the expansion becomes uniformly valid in the domain. Given a function $f(x,\varepsilon)$ and an asymptotic sequence $\{\phi_n(\varepsilon)\}$, each of the $a_n(x)$ can be uniquely calculated using the above definition. Thus,

$$\begin{aligned}
a_1(x) &= \lim_{\varepsilon\to\varepsilon_0} \frac{f(x,\varepsilon)}{\phi_1(\varepsilon)} \\
a_2(x) &= \lim_{\varepsilon\to\varepsilon_0} \frac{f(x,\varepsilon) - a_1(x)\phi_1(\varepsilon)}{\phi_2(\varepsilon)} \\
a_k(x) &= \lim_{\varepsilon\to\varepsilon_0} \frac{f(x,\varepsilon) - \sum_{n=1}^{k-1} a_n(x)\phi_n(\varepsilon)}{\phi_k(\varepsilon)}
\end{aligned}$$

Solutions for Algebraic, Integral or Differential Equations

Let $f(x,\varepsilon)$ be a solution to an algebraic, integral or differential equation, with x the independent variable and ε a small parameter. If the equation cannot be solved for arbitrary ε, the solution can be represented as an asymptotic expansion of the parameter. This is called a *parameter perturbation.* In the straightforward expansion given by Definition 2, the term $\varepsilon^{n+1}f(x)^{n+1}$ should be a small correction to the term $\varepsilon^n f(x)^n$. Consequently, this type of expansion breaks down when $\varepsilon^{n+1}f_{n+1} = O(\varepsilon^n f_n)$, where $n = 0, 1,$ or 2. If the expansion, using a finite number of terms does not represent the solution for all values of x, then the expansion is non-uniformly valid for all x. This then leads to *singular perturbation problems.* These are the rule, as opposed to the exception (regular perturbation problems).

5.5.4 Solving for Option Prices and Implied Volatility

Let $V(t,f)$ be the date t value of a European call option, with strike K and expiry and settlement dates t_{ex} and t_{set} respectively. As before, $F(t)$ is the forward price process which follows equation (5.27). Under the forward measure we have

$$V(t,f) = Z(t,t_{\text{set}})\mathbb{E}\left[(F(t_{\text{ex}}) - K)^+ \,|F(t) = f\right]. \tag{5.30}$$

Let

$$Q(t,f) := \mathbb{E}\left[(F(t_{\text{ex}}) - K)^+ \,|F(t) = f\right] \tag{5.31}$$

be the expected payoff of the option. The expectation is over the probability distribution generated by $F(t)$. $Q(t,T)$ must satisfy the backward Kolmogorov equation in §5.2:

$$Q_t + \frac{1}{2}\alpha^2(t)A^2(f)Q_{ff} = 0, \quad t < t_{\text{ex}}, \tag{5.32}$$

subject to the terminal condition

$$Q(t_{\text{ex}}, f) = (f-K)^+. \tag{5.33}$$

We begin by selecting an appropriate perturbation parameter, $\varepsilon \equiv A(K) \ll 1$, and scale equations (5.32) and (5.33) by defining

$$\psi := \psi(t) = \int_t^{t_{\text{ex}}} \alpha^2(s)ds, \tag{5.34}$$

$$x := x(f) = \frac{1}{\varepsilon}(f-K), \tag{5.35}$$

$$\tilde{Q}(\psi, x) = \frac{1}{\varepsilon}Q(t,f). \tag{5.36}$$

Note that in Black's model, $\alpha(t) = \sigma_B$ and $A(K) = K$. Thus it might not be the case that $A(K) \ll 1$ in equity markets, while it would be in the interest rate market. This problem is easily resolved by some normalization procedure that can be applied to $A(K)$ with the inverse procedure applied to $\alpha(t)$. We will see a similar strategy in Chapter 9.

In terms of the variables x, ψ and $\tilde{Q}$, we have

$$\begin{aligned}
\frac{\partial Q}{\partial t} &= \varepsilon \frac{\partial \tilde{Q}}{\partial \psi}\frac{\partial \psi}{\partial t} \\
&= -\varepsilon \frac{\partial \tilde{Q}}{\partial \psi}\alpha^2(t), \\
\frac{\partial Q}{\partial f} &= \varepsilon \frac{\partial \tilde{Q}}{\partial x}\frac{\partial x}{\partial f} \\
&= \varepsilon \frac{\partial \tilde{Q}}{\partial x}\frac{1}{\varepsilon} = \frac{\partial \tilde{Q}}{\partial x}, \\
\frac{\partial^2 Q}{\partial f^2} &= \frac{\partial^2 \tilde{Q}}{\partial x^2}\frac{1}{\varepsilon}
\end{aligned}$$

For $\psi > 0$, (5.32) then becomes

$$\begin{aligned}
-\varepsilon \tilde{Q}_\psi \alpha^2(t) + \frac{1}{2}\alpha^2(t)A^2(f)\tilde{Q}_{xx}\frac{1}{\varepsilon} &= 0 \\
\Rightarrow \tilde{Q}_\psi - \frac{1}{2}\frac{A^2(K+\varepsilon x)}{A^2(K)}\tilde{Q}_{xx} &= 0,
\end{aligned} \tag{5.37}$$

since $f = K + \varepsilon x$, and (5.33) transforms to

$$\tilde{Q} = x^+, \quad \psi = 0. \tag{5.38}$$

Therefore, in terms of $\tilde{Q}(\psi, x)$ the option value is given by

$$V(t,f) = Z(t, t_{\text{set}})A(K)\tilde{Q}\left(\psi(t), \frac{f-K}{A(K)}\right).$$

By using a Taylor expansion of $A(K+\varepsilon x)$,

$$
\begin{aligned}
A(K+\varepsilon x) &= \sum_{j=0}^{\infty} \frac{A^j(K)}{j!}\varepsilon^j x^j \\
&= A(K)\sum_{j=0}^{\infty} \frac{\varepsilon^j x^j}{j!}\frac{A^j(K)}{A(K)} \\
\Rightarrow \frac{A(K+\varepsilon x)}{A(K)} &= \sum_{j=0}^{\infty} \frac{\nu_j}{j!}\varepsilon^j x^j,
\end{aligned}
$$

where

$$\nu_j = \frac{A^{(j)}(K)}{A(K)}, \quad \text{for } j = 1, 2, \ldots$$

All expansions will be done up to order ε^2. Therefore,

$$
\begin{aligned}
\frac{A^2(K+\varepsilon x)}{A^2(K)} &= \left(\sum_{j=0}^{\infty} \frac{\nu_j}{j!}\varepsilon^j x^j\right)^2 \\
&= \sum_{j=0}^{\infty}\left(\frac{\nu_j}{j!}\varepsilon^j x^j\right)^2 + 2\sum_{j=0}^{\infty}\sum_{k=0}^{j-1}\frac{\nu_j}{j!}\varepsilon^j x^j \frac{\nu_k}{k!}\varepsilon^k x^k \\
&= 1 + \nu_1^2\varepsilon^2 x^2 + 2\nu_1\varepsilon x + 2\frac{\nu_2}{2}\varepsilon^2 x^2\left(1+\nu_1\varepsilon x\right) + \ldots \\
&= 1 + 2\nu_1\varepsilon x + \left(\nu_1^2+\nu_2\right)\varepsilon^2 x^2 + \ldots
\end{aligned}
$$

Substituting this into (5.37):

$$\tilde{Q}_\psi - \frac{1}{2}\left(1 + 2\nu_1\varepsilon x + \left(\nu_1^2+\nu_2\right)\varepsilon^2 x^2 + \ldots\right)\tilde{Q}_{xx} = 0.$$

Therefore, for $\psi > 0$, up to ε^2,

$$\tilde{Q}_\psi - \frac{1}{2}\tilde{Q}_{xx} = \nu_1\varepsilon x\tilde{Q}_{xx} + \frac{1}{2}\varepsilon^2\left(\nu_1^2+\nu_2\right)x^2\tilde{Q}_{xx} + \ldots \tag{5.39}$$

subject to

$$\tilde{Q} = x^+, \quad \text{at } \psi = 0.$$

In order to solve for $\tilde{Q}$, we perform a regular perturbation expansion (with ε as the expansion parameter) according to Definition 2:

$$\tilde{Q} = {}^7\tilde{Q}^0 + \varepsilon\tilde{Q}^1 + \varepsilon^2\tilde{Q}^2 + \ldots$$

Substituting this expansion into (5.39), we get

$$
\begin{aligned}
&\tilde{Q}^0_\psi - \frac{1}{2}\tilde{Q}^0_{xx} + \varepsilon\tilde{Q}^1_\psi - \frac{1}{2}\varepsilon\tilde{Q}^1_{xx} + \varepsilon^2\tilde{Q}^2_\psi - \frac{1}{2}\varepsilon^2\tilde{Q}^2_{xx} + \ldots \\
&= \nu_1\varepsilon x\left(\tilde{Q}^0_{xx} + \varepsilon\tilde{Q}^1_{xx} + \varepsilon^2\tilde{Q}^2_{xx} + \ldots\right) + \frac{1}{2}\varepsilon^2\left(\nu_1^2+\nu_2\right)x^2\left(\tilde{Q}^0_{xx} + \varepsilon\tilde{Q}^1_{xx} + \varepsilon^2\tilde{Q}^2_{xx} + \ldots\right) \\
&= \nu_1\varepsilon x\tilde{Q}^0_{xx} + \nu_1\varepsilon^2 x\tilde{Q}^1_{xx} + \frac{1}{2}\varepsilon^2\left(\nu_1^2+\nu_2\right)x^2\tilde{Q}^0_{xx} + \ldots
\end{aligned}
$$

Equating like powers of ε, the following hierarchy of PDEs result:

[7] The powers of the function $\tilde{Q}$ refer to the order of the solution. This notation is used here and in Chapter 9, as the use of subscripts will be for partial differentiation.

1. At order 0, we have

$$\tilde{Q}^0_\psi - \frac{1}{2}\tilde{Q}^0_{xx} = 0, \qquad \text{for } \psi > 0$$
$$\tilde{Q}^0 = x^+ \qquad \text{at } \psi = 0, \tag{5.40}$$

which is basically the heat equation.

2. At order ε, we have

$$\tilde{Q}^1_\psi - \frac{1}{2}\tilde{Q}^1_{xx} = \nu_1 x\tilde{Q}^0_{xx}, \qquad \text{for } \psi > 0$$
$$\tilde{Q}^1 = 0 \qquad \text{at } \psi = 0. \tag{5.41}$$

3. At order ε^2, we have

$$\tilde{Q}^2_\psi - \frac{1}{2}\tilde{Q}^2_{xx} = \nu_1 x\tilde{Q}^1_{xx} + \frac{1}{2}\left(\nu_1^2 + \nu_2\right)x^2\tilde{Q}^0_{xx}, \qquad \text{for } \psi > 0$$
$$\tilde{Q}^2 = 0 \qquad \text{at } \psi = 0. \tag{5.42}$$

We begin by solving for $\tilde{Q}^0$. The solution can be obtained using the convolution of the heat kernel with the initial condition. This method is used in Chapter 9 to solve a similar PDE. Here, we apply the Laplace Transform Method.

Definition 5 *(James 1999, §2.2) We define the Laplace transform of a function $f(x)$ by the expression*

$$\mathcal{L}\{f(x)\} = \int_0^\infty e^{-\phi x} f(x)dx = F(\phi), \tag{5.43}$$

where ϕ is a complex variable and $e^{-\phi x}$ is called the kernel of the transformation

Clearly, the Laplace transform of the function $f(x)$ exists if and only if the improper integral converges for some values of ϕ. To establish the sufficient conditions on $f(x)$ to ensure the transform exists, we define the following:

Definition 6 ***Exponential Order***
(James 1999, §2.2.3) A function $f(x)$ is said to be of exponential order as $x \to \infty$ if there exists a real number μ and positive constants M and N such that

$$|f(x)| < Me^{\mu x}$$

for all $x > N$.

The choice of μ is not unique. Thus, let the greatest lower bound μ_c of the set of possible values of μ be the **abscissa of convergence** of the function $f(x)$. The following theorem provides sufficient conditions for ensuring the existence of the Laplace transform of a function. They are not necessary conditions and are restrictive. There exist functions with infinite discontinuities that possess Laplace transforms.

Theorem 2 ***Existence of Laplace transform***
(James 1999, §2.2.3) If the function $f(x)$ is piecewise-continuous on $[0,\infty]$ and is of exponential order, with abscissa of convergence μ_c, then its Laplace transform exists, with region of convergence $\Re(\phi) > \mu_c$ in the ϕ domain; that is

$$\mathcal{L}\{f(x)\} = \int_0^\infty e^{-\phi x} f(x)dx := F(\phi), \quad \Re(\phi) > \mu_c$$

The inverse transform $f(x)$ of the function $F(\phi)$, written $\mathcal{L}^{-1}\{F(\phi)\}$, where $F(\phi) = \mathcal{L}\{f(x)\}$ can generally be found in a table of transforms (Abramowitz & Stegun 1974, §29).

Let $\phi = \phi_r + i\phi_i$ be the integration variable. We begin by taking the Laplace transform of the term $\tilde{Q}^0_\psi$ from (5.40) with $\Re(\phi) > 0$, and use integration by parts to solve it. The integration is performed with respect to ψ.

$$\begin{aligned}
\int_0^\infty \frac{\partial \tilde{Q}^0(\psi,x)}{\partial \psi} e^{-\phi\psi} d\psi &= \lim_{\alpha\to\infty} \left(\tilde{Q}^0(\psi,x)e^{-\phi\psi}\Big|_0^\alpha + \phi \int_0^\alpha \tilde{Q}^0(\psi,x)e^{-\phi\psi}d\psi \right) \\
&= \lim_{\alpha\to\infty} \left(\tilde{Q}^0(\psi,x)e^{-(\phi_r+i\phi_i)\psi}\Big|_0^\alpha + \phi \int_0^\alpha \tilde{Q}^0(\psi,x)e^{-(\phi_r+i\phi_i)\psi}d\psi \right) \\
&= (0 - x^+) + \phi\mathcal{L}\left\{\tilde{Q}^0(\psi,x)\right\} \\
&= \begin{cases} -x + \phi\widetilde{Q^0(\phi,x)} & x > 0 \\ \phi\widetilde{Q^0(\phi,x)} & x < 0 \end{cases}
\end{aligned}$$

where we have defined $\widetilde{Q^0(\phi,x)} = \mathcal{L}\left\{\tilde{Q}^0(\phi,x)\right\}$. The requirement that $\Re(\phi) > 0$ ensures that $\lim_{\alpha\to\infty} \tilde{Q}^0(\alpha,x)e^{-\phi\alpha}$ is zero.

Let $\widetilde{Q^0_{xx}(\phi,x)}$ denote the Laplace transform of $\tilde{Q}^0_{xx}$. Since the differentiation is with respect to x, we treat it as a ordinary differentiation and treat ϕ as a constant. Therefore, the ODE to be solved is:

$$\frac{d^2\widetilde{Q^0(\phi,x)}}{dx^2} - 2\phi\widetilde{Q^0(\phi,x)} = \begin{cases} -2x & x > 0 \\ 0 & x < 0 \end{cases}$$

For $x > 0$, the solution is of the form:

$$\widetilde{Q^0(\phi,x)} = \text{CF} + \text{PI},$$

where the CF is the complimentary function and PI, the particular integral. The CF satisfies the homogenous ODE

$$\frac{d^2\widetilde{Q^0(\phi,x)}}{dx^2} - 2\phi\widetilde{Q^0(\phi,x)} = 0$$

This is a second order ODE with constant coefficients. The solution is of the form

$$\widetilde{Q^0(\phi,x)} = A(\phi)e^{\sqrt{2\phi}x} + B(\phi)e^{-\sqrt{2\phi}x}$$

where $A(\phi)$ and $B(\phi)$ are constants of integration which are functions of ϕ. For the PI, choose a solution of the form $\widetilde{Q^0_{PI}(\phi,x)} = ax^2 + bx + c$, where a, b and c are constants. Noting that $\frac{d^2\widetilde{Q^0_{PI}(\phi,x)}}{dx^2} = 2a$ and substituting this into the non-homogenous ODE, we get

$$2a - 2\phi\left(ax^2 + bx + c\right) = -2x.$$

Equating coefficients, we get

$$
\begin{aligned}
-2\phi b &= -2, \\
2a - 2\phi c &= 0, \\
-2\phi a &= 0.
\end{aligned}
$$

Thus, $\widetilde{Q_{PI}^0(\phi, x)} = \frac{x}{\phi}$. Therefore,

$$
\widetilde{Q^0(\phi, x)} = \frac{x}{\phi} + A(\phi)e^{\sqrt{2\phi}x} + B(\phi)e^{-\sqrt{2\phi}x} \tag{5.44}
$$

In the case that $x < 0$, the solution is

$$
\widetilde{Q^0(\phi, x)} = C(\phi)e^{\sqrt{2\phi}x} + D(\phi)e^{-\sqrt{2\phi}x} \tag{5.45}
$$

In order to avoid solutions which grow exponentially at $\pm\infty$, we set $A(\phi) = D(\phi) = 0$. Consequently, the solution will be bounded and unique.

To determine the function values $B(\phi)$ and $C(\phi)$, we equate $\widetilde{Q^0(\phi, x)}$ and $\frac{d\widetilde{Q^0(\phi,x)}}{dx}$ at $x = 0$. This can be interpreted as saying that the rate of change of the expected value of the payoff, with respect to the independent variable, in both the transformed and untransformed space, is equivalent to the expected value of the payoff at $x = 0$.

$$
\widetilde{Q^0(\phi, 0)} = \begin{cases} B(\phi) & x > 0 \\ C(\phi) & x < 0 \end{cases} \tag{5.46}
$$

$$
\frac{d\widetilde{Q^0(\phi, x)}}{dx} = \begin{cases} \frac{1}{\phi} - \sqrt{2\phi}B(\phi)e^{-\sqrt{2\phi}x} & x > 0 \\ \sqrt{2\phi}C(\phi)e^{\sqrt{2\phi}x} & x < 0 \end{cases} \tag{5.47}
$$

(5.46) requires $B(\phi) = C(\phi)$. Using this in (5.47) and setting $x = 0$, we get that

$$
\begin{aligned}
\frac{1}{\phi} - \sqrt{2\phi}B(\phi) &= \sqrt{2\phi}B(\phi) \\
\Rightarrow B(\phi) &= \frac{1}{\phi}\frac{1}{2\sqrt{2\phi}} \\
&= \frac{1}{(2\phi)^{3/2}}.
\end{aligned}
$$

Thus, we have that

$$
\widetilde{Q^0(\phi, x)} = \left\{ \begin{array}{ll} \frac{x}{\phi} & x > 0 \\ 0 & x < 0 \end{array} \right\} + \frac{1}{(2\phi)^{3/2}} e^{-\sqrt{2\phi}|x|}.
$$

Using a table of Laplace transform inversion formulae (Abramowitz & Stegun 1974, §29), we have that

$$
\tilde{Q}^0(\psi, x) = x\Phi\left(\frac{x}{\sqrt{\psi}}\right) + \sqrt{\frac{\psi}{2\pi}} e^{-x^2/2\psi}, \tag{5.48}
$$

where $\Phi(\cdot)$ is the cumulative normal density function.

We now note two ways of generating further solutions to PDEs given an initial one. Suppose $F(\psi, x)$ is a previously determined solution to the PDE $F_\psi = \frac{1}{2}F_{xx}$. Firstly, differentiating F any number of times with respect to x and/or ψ results in another solution (with a different boundary condition). Secondly, suppose we are trying to solve the PDE:

$$u_\psi(\psi, x) - \frac{1}{2}u_x(\psi, x) = m\psi^j F(\psi, x).$$

We claim $u(\psi, x) = m\frac{\psi^{j+1}}{j+1}F(\psi, x)$ is a solution, of course it has initial condition $u(0, x) = 0$. To see this, we calculate

$$\begin{aligned}
&u_\psi(\psi, x) - \frac{1}{2}u_{xx}(\psi, x) \\
&= \frac{\partial}{\partial\psi}\left(m\frac{\psi^{j+1}}{j+1}F(\psi, x)\right) - \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(m\frac{\psi^{j+1}}{j+1}F(\psi, x)\right) \\
&= m\left(\psi^j F(\psi, x) + \frac{\psi^{j+1}}{j+1}F_\psi(\psi, x)\right) - \frac{m}{2}\frac{\psi^{j+1}}{j+1}F_{xx}(\psi, x) \\
&= m\psi^j F(\psi, x).
\end{aligned} \tag{5.49}$$

Let $G(\psi, x) = \tilde{Q}_0$. G is a solution to the PDE (5.40); we will use the above tricks to find solutions for $\tilde{Q}_1$ and $\tilde{Q}_2$. In preparation for this, we calculate some of the partial derivatives of G with respect to ψ and x.

Using the product rule of differentiation,

$$\begin{aligned}
G_x &= \Phi\left(\frac{x}{\sqrt{\psi}}\right) + x\frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi} - x\frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi} = \Phi\left(\frac{x}{\sqrt{\psi}}\right) \\
G_{xx} &= \frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi} \\
G_\psi &= x\left(\frac{-1}{\sqrt{2\pi}}e^{-x^2/2\psi}\frac{x}{2\psi\sqrt{\psi}}\right) + \frac{1}{2}\frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{\psi}}e^{-x^2/2\psi} + \sqrt{\frac{\psi}{2\pi}}e^{-x^2/2\psi}\frac{x^2}{2\psi^2} \\
&= e^{-x^2/2\psi}\left(\frac{-x^2}{2\psi\sqrt{2\pi\psi}} + \frac{1}{2\sqrt{2\pi\psi}} + \frac{x^2}{2\psi\sqrt{2\pi\psi}}\right) \\
&= \frac{1}{2}\frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi} \\
G_{\psi\psi} &= \frac{-1}{2\sqrt{2\pi}}\frac{1}{2}\frac{1}{\psi\sqrt{\psi}}e^{-x^2/2\psi} + \frac{1}{2}\frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi}\frac{x^2}{2\psi^2} \\
&= \frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi}\left(\frac{x^2-\psi}{4\psi^2}\right) \\
G_{\psi x} &= \frac{1}{2}\frac{-x}{\psi\sqrt{2\pi\psi}}e^{-x^2/2\psi} \\
G_{\psi\psi\psi} &= \frac{-1}{\sqrt{2\pi}}\frac{1}{2}\frac{1}{\psi\sqrt{\psi}}e^{-x^2/2\psi}\left(\frac{x^2-\psi}{4\psi^2}\right) + \frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi}\frac{x^2}{2\psi^2}\left(\frac{x^2-\psi}{4\psi^2}\right) + \frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi}\left(\frac{-2x^2}{4\psi^3} + \frac{1}{4\psi^2}\right) \\
&= \frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi}\left(\frac{x^2-\psi}{4\psi^2}\right)\left(\frac{x^2}{2\psi^2} - \frac{1}{2\psi}\right) + \frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi}\left(\frac{\psi-2x^2}{4\psi^3}\right) \\
&= \frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi}\left(\frac{\psi^2-\psi x^2+x^4-\psi x^2+2\psi^2-4\psi x^2}{8\psi^4}\right) \\
&= \frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi}\left(\frac{3\psi^2-6\psi x^2+x^4}{8\psi^4}\right)
\end{aligned}$$

Using the partial derivatives and property (5.49), at $O(\varepsilon)$, we are solving for $\tilde{Q}^1$. For $\psi > 0$, we have

$$\tilde{Q}^1_\psi - \frac{1}{2}\tilde{Q}^1_{xx} = \nu_1 x G_{xx} = -2\nu_1\psi G_{\psi x},$$

with $\tilde{Q}^1 = 0$ at $\psi = 0$. Therefore,

$$\begin{aligned}\tilde{Q}^1 &= -2\nu_1\frac{\psi^2}{2}G_{\psi x} = -\nu_1\psi^2 G_{\psi x}\\ &= \nu_1\psi x G_\psi \end{aligned} \tag{5.50}$$

At $O(\varepsilon^2)$, we have for $\psi > 0$:

$$\tilde{Q}^2_\psi - \frac{1}{2}\tilde{Q}^2_{xx} = \nu_1 x\tilde{Q}^1_{xx} + \frac{1}{2}\left(\nu_1^2 + \nu_2\right)x^2 G_{xx} \tag{5.51}$$

In order to simplify this, we begin by finding $\tilde{Q}^1_{xx}$.

$$\begin{aligned}\tilde{Q}^1_{xx} &= \frac{\partial^2}{\partial x^2}\nu_1\psi x G_\psi\\ &= \nu_1\psi\frac{\partial^2}{\partial x^2}\left(x\frac{1}{2\sqrt{2\pi\psi}}e^{-x^2/2\psi}\right)\\ &= \frac{\nu_1\psi}{2\sqrt{2\pi\psi}}\frac{\partial^2}{\partial x^2}\left(xe^{-x^2/2\psi}\right)\\ &= \frac{\nu_1\psi}{2\sqrt{2\pi\psi}}\frac{\partial}{\partial x}\left(e^{-x^2/2\psi}\left(1-\frac{x^2}{\psi}\right)\right)\\ &= \frac{\nu_1\psi}{2\sqrt{2\pi\psi}}\left(e^{-x^2/2\psi}\frac{-x}{\psi}\left(1-\frac{x^2}{\psi}\right) - e^{-x^2/2\psi}\frac{2x}{\psi}\right)\\ &= \frac{\nu_1\psi}{2\sqrt{2\pi\psi}}e^{-x^2/2\psi}\left(\frac{x^3}{\psi^2}-\frac{x}{\psi}-\frac{2x}{\psi}\right)\\ &= \frac{\nu_1\psi}{\sqrt{2\pi\psi}}e^{-x^2/2\psi}\left(\frac{x^3-3\psi x}{2\psi}\right)\end{aligned}$$

Substituting this into (5.51), we get

$$\begin{aligned}\tilde{Q}^2_\psi - \frac{1}{2}\tilde{Q}^2_{xx} &= \nu_1 x\frac{\nu_1\psi}{\sqrt{2\pi\psi}}e^{-x^2/2\psi}\left(\frac{x^3-3\psi x}{2\psi}\right) + \frac{1}{2}\left(\nu_1^2+\nu_2\right)x^2\frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi}\\ &= \frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi}\left(\nu_1^2\left(\frac{x^4-3\psi x^2}{2\psi}+\frac{x^2}{2}\right)+\frac{\nu_2}{2}x^2\right)\\ &= \frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi}\left(\nu_1^2\left(\frac{x^4-2\psi x^2}{2\psi}\right)+\frac{\nu_2}{2}x^2\right)\end{aligned}$$

A suitable linear expression using the partial derivatives G_ψ, $G_{\psi\psi}$ and $G_{\psi\psi\psi}$ is to be constructed to equate to the coefficient of ν_1^2. Let $\frac{1}{\sqrt{2\pi\psi}}e^{-x^2/2\psi} \equiv C$. $G_{\psi\psi\psi}$ is the only expression which contains a term with x^4. Since we require $C\frac{x^4}{2\psi}$, we need to take the multiple $4\psi^3 G_{\psi\psi\psi}$. This results in

$$C4\psi^3\left(\frac{3\psi^2-6\psi x^2+x^4}{8\psi^4}\right) = C\left(\frac{x^4}{2\psi}-3x^2+\frac{3}{2}\psi\right)$$

The next term we require is $-Cx^2$. Currently having $-3Cx^2$, we add to the above expression $8\psi^2 G_{\psi\psi} = C(2x^2-2\psi)$ which gives $C\left(\frac{x^4}{2\psi}-x^2-\frac{\psi}{2}\right)$. The final step involves removing the term $-\frac{C}{2}\psi$. This is achieved by adding the term $\psi G_\psi = \frac{C}{2}\psi$. Therefore, we have that

$$\frac{1}{\sqrt{2\pi\psi}} e^{-x^2/2\psi} \nu_1^2 \left(\frac{x^4 - 2\psi x^2}{2\psi} \right) = 4\psi^3 G_{\psi\psi\psi} + 8\psi^2 G_{\psi\psi} + \psi G_\psi \tag{5.52}$$

The same procedure follows for the coefficient of ν_2, $\frac{C}{2}x^2$. The partial derivatives to be used will be G_ψ and $G_{\psi\psi}$ since the highest power of x in the coefficient is x^2. Since we require $\frac{C}{2}x^2$, begin with the multiple $2\psi^2 G_{\psi\psi} = \frac{C}{2}(x^2 - \psi)$. To remove the term $-\frac{C}{2}\psi$, add $\psi G_\psi = \frac{C}{2}\psi$. Therefore,

$$\frac{1}{\sqrt{2\pi\psi}} e^{-x^2/2\psi} \frac{\nu_2}{2} x^2 = 2\psi^2 G_{\psi\psi} + \psi G_\psi \tag{5.53}$$

Using property (5.49), (5.52) and (5.53) are integrated, enabling us to solve for $\tilde{Q}^2$:

$$\begin{aligned} 4\psi^3 G_{\psi\psi\psi} + 8\psi^2 G_{\psi\psi} + \psi G_\psi &= \psi^4 G_{\psi\psi\psi} + \frac{8}{3}\psi^3 G_{\psi\psi} + \frac{\psi^2}{2} G_\psi, \\ 2\psi^2 G_{\psi\psi} + G_\psi &= \frac{2}{3}\psi^3 G_{\psi\psi} + \frac{1}{2}\psi^2 G_\psi. \end{aligned}$$

So, the solution $\tilde{Q}^2$ is:

$$\tilde{Q}^2 = \nu_1^2 \left(\psi^4 G_{\psi\psi\psi} + \frac{8}{3}\psi^3 G_{\psi\psi} + \frac{\psi^2}{2} G_\psi \right) + \nu_2 \left(\frac{2}{3}\psi^3 G_{\psi\psi} + \frac{1}{2}\psi^2 G_\psi \right) \tag{5.54}$$

We then write this in a concise format. Looking at the coefficient of ν_1^2, we have that

$$\begin{aligned} \psi^4 G_{\psi\psi\psi} + \frac{8}{3}\psi^3 G_{\psi\psi} + \frac{\psi^2}{2} G_\psi &= \psi^4 C \left(\frac{3\psi^2 - 6\psi x^2 + x^4}{8\psi^4} \right) + \frac{8}{3}\psi^3 C \left(\frac{x^2 - \psi}{4\psi^2} \right) + \frac{\psi^2}{2}\frac{1}{2}C \\ &= \left(\frac{x^4}{8} - \frac{3}{4}\psi x^2 + \frac{3}{8}\psi^2 + \frac{2}{3}\psi x^2 - \frac{2}{3}\psi^2 + \frac{1}{4}\psi^2 \right) C \\ &= \left(\frac{x^4}{8} - \frac{1}{12}\psi x^2 - \frac{1}{24}\psi^2 \right) C. \end{aligned}$$

Expressing the above line in terms of $G_{\psi\psi}$ and G_ψ, we get that

$$\begin{aligned} \frac{1}{2}\psi^2 x^2 G_{\psi\psi} + \frac{1}{12}\psi x^2 G_\psi - \frac{1}{12}\psi^2 G_\psi &= \left(\frac{x^4}{8} - \frac{1}{8}\psi x^2 + \frac{1}{24}\psi^2 x^2 - \frac{1}{24}\psi^2 \right) C \\ &= \left(\frac{x^4}{8} - \frac{1}{12}\psi x^2 - \frac{1}{24}\psi^2 \right) C. \end{aligned} \tag{5.55}$$

The coefficient of ν_2 is then expanded:

$$\begin{aligned} \frac{2}{3}\psi^3 G_{\psi\psi} + \frac{1}{2}\psi^2 G_\psi &= \frac{2}{3}\psi^3 C \left(\frac{x^2 - \psi}{4\psi^2} \right) + \psi^2 \frac{C}{4} \\ &= \left(\frac{1}{6}\psi x^2 + \frac{1}{12}\psi^2 \right) C. \end{aligned}$$

Rewriting the above line in terms of G_ψ, we get

$$\begin{aligned} \left(\frac{1}{6}\psi x^2 + \frac{1}{12}\psi^2 \right) C &= \frac{1}{12} \left(2\psi x^2 + \psi^2 \right) C \\ &= \frac{\psi}{6} \left(2x^2 + \psi \right) G_\psi. \end{aligned} \tag{5.56}$$

Therefore, substituting (5.55) and (5.56) into (5.54), we have

$$\tilde{Q}^2 = \frac{1}{2}\nu_1^2\psi^2x^2G_{\psi\psi} + \frac{1}{12}\nu_1^2\left(x^2-\psi\right)\psi G_\psi + \frac{1}{6}\nu_2\left(2x^2+\psi\right)\psi G_\psi. \tag{5.57}$$

The solution of $\tilde{Q}$ up to $O(\varepsilon^2)$ is then given by substituting (5.48), (5.50) and (5.57) into the regular perturbation expansion:

$$\begin{aligned}\tilde{Q} &= \tilde{Q}^0 + \varepsilon\tilde{Q}^1 + \varepsilon^2\tilde{Q}^2 + \dots \\ &= G + \varepsilon\nu_1\psi x G_\psi + \frac{1}{2}\varepsilon^2\nu_1^2\psi^2x^2G_{\psi\psi} + \varepsilon^2\left(\frac{4\nu_2+\nu_1^2}{12}x^2 + \frac{2\nu_2-\nu_1^2}{12}\psi\right)\psi G_\psi \end{aligned}\tag{5.58}$$

Define

$$\tilde{\psi} := \psi\left(1 + \varepsilon\nu_1 x + \varepsilon^2\left(\frac{4\nu_2+\nu_1^2}{12}x^2 + \frac{2\nu_2-\nu_1^2}{12}\psi\right) + \dots\right).$$

Then $\tilde{Q}(\psi,x)$ can be re-written as $\tilde{Q}(\psi,x) = G(\tilde{\psi},x)$. This is true up to $O(\varepsilon^2)$. Given the definition of $\tilde{\psi}$, we can expand $G(\tilde{\psi},x)$ around ψ:

$$\begin{aligned}G(\tilde{\psi},x) &= G(\psi,x) + \psi\left(\varepsilon\nu_1 x + \varepsilon^2\left(\frac{4\nu_2+\nu_1^2}{12}x^2 + \frac{2\nu_2-\nu_1^2}{12}\psi\right) + \dots\right)G_\psi(\psi,x) \\ &\quad + \frac{\psi^2}{2}\left(\varepsilon\nu_1 x + \varepsilon^2\left(\frac{4\nu_2+\nu_1^2}{12}x^2 + \frac{2\nu_2-\nu_1^2}{12}\psi\right) + \dots\right)^2 G_{\psi\psi}(\psi,x) + \dots \\ &= G(\psi,x) + \varepsilon\nu_1\psi x G_\psi(\psi,x) + \varepsilon^2\left(\frac{4\nu_2+\nu_1^2}{12}x^2 + \frac{2\nu_2-\nu_1^2}{12}\psi\right)\psi G_\psi(\psi,x) + \frac{1}{2}\varepsilon^2\nu_1^2\psi^2x^2G_{\psi\psi}(\psi,x) + \dots \\ &= \tilde{Q}(\psi,x)\end{aligned}$$

by (5.58).

Recall the European call option value at time t is given by

$$V(t,f) = Z(t,t_{\text{set}})\varepsilon\tilde{Q}(\psi,x) = Z(t,t_{\text{set}})\varepsilon G(\tilde{\psi},x) \overset{8}{=} Z(t,t_{\text{set}})G(\varepsilon^2\tilde{\psi},\varepsilon x) = Z(t,t_{\text{set}})G\left(A^2(K)\tilde{\psi}, f-K\right),$$

which is directly from the definition of x and ε. By defining

$$\begin{aligned}\psi^* := A^2(K)\tilde{\psi} &= A^2(K)\psi\left(1 + \varepsilon\nu_1 x + \varepsilon^2\left(\frac{4\nu_2+\nu_1^2}{12}x^2 + \frac{2\nu_2-\nu_1^2}{12}\psi\right) + \dots\right) \\ &= A^2(K)\psi\left(1 + \nu_1(f-K) + \frac{4\nu_2+\nu_1^2}{12}(f-K)^2 + \frac{2\nu_2-\nu_1^2}{12}A^2(K)\psi + \dots\right),\end{aligned}$$

[8] By directly substituting $G(\varepsilon^2\tilde{\psi},\varepsilon x)$ into (5.48), we can show that $\varepsilon G(\tilde{\psi},x) = G(\varepsilon^2\tilde{\psi},\varepsilon x)$. We have that

$$\varepsilon G(\tilde{\psi},x) = \varepsilon\left(x\Phi\left(\frac{x}{\sqrt{\tilde{\psi}}}\right) + \sqrt{\frac{\tilde{\psi}}{2\pi}}e^{-x^2/2\tilde{\psi}}\right).$$

Then,

$$\begin{aligned}G(\varepsilon^2\tilde{\psi},\varepsilon x) &= \varepsilon x\Phi\left(\frac{\varepsilon x}{\sqrt{\tilde{\psi}\varepsilon^2}}\right) + \sqrt{\frac{\tilde{\psi}\varepsilon^2}{2\pi}}e^{-\varepsilon^2x^2/2\tilde{\psi}\varepsilon^2} \\ &= \varepsilon x\Phi\left(\frac{x}{\sqrt{\tilde{\psi}}}\right) + \varepsilon\sqrt{\frac{\tilde{\psi}}{2\pi}}e^{-x^2/2\tilde{\psi}}.\end{aligned}$$

we find the option value is given by

$$V(t,f) = Z(t,t_{\text{set}})G\left(\psi^*, f-K\right). \tag{5.59}$$

To obtain the equivalent Black implied volatility, we take $f-K$ to be $O(\varepsilon)$ and $A^2(K)\psi$ as $O(\varepsilon^2)$ and use the Maclaurin expansion for small x of $\sqrt{1+x} = 1+\frac{1}{2}x - \frac{1}{8}x^2+\ldots$. Up to order ε^2,

$$\begin{aligned}
\sqrt{\psi^*} &= A(K)\sqrt{\psi}\left(1+\frac{1}{2}\nu_1\left(f-K\right)+\frac{4\nu_2+\nu_1^2}{24}\left(f-K\right)^2+\frac{2\nu_2-\nu_1^2}{24}A^2(K)\psi-\frac{1}{8}\nu_1^2(f-K)^2+\ldots\right)\\
&= A(K)\sqrt{\psi}\left(1+\frac{1}{2}\nu_1\left(f-K\right)+\frac{2\nu_2-\nu_1^2}{12}\left(f-K\right)^2+\frac{2\nu_2-\nu_1^2}{24}A^2(K)\psi+\ldots\right).
\end{aligned} \tag{5.60}$$

Given that $\nu_1 = \frac{A'(K)}{A(K)}$, the first two terms of the expansion for $\sqrt{\psi^*}$ are

$$A(K)\sqrt{\psi}+A(K)\sqrt{\psi}\frac{1}{2}\frac{A'(K)}{A(K)}\left(f-K\right) = \sqrt{\psi}\left(A(K)+\frac{1}{2}A'(K)\left(f-K\right)\right). \tag{5.61}$$

Define $f_{\text{av}} := \frac{1}{2}(f+K)$. By noting that $K = f_{\text{av}} - \frac{1}{2}(f-K)$, we can expand A around f_{av}:

$$\begin{aligned}
A(K) &= A\left(f_{\text{av}}-\tfrac{1}{2}(f-K)\right) = A(f_{\text{av}}) - \frac{1}{2}(f-K)A'(f_{\text{av}}) + \frac{1}{2}\frac{(f-K)^2}{4}A''(f_{\text{av}})+\ldots\\
A'(K) &= A'(f_{\text{av}}) - \frac{1}{2}(f-K)A''(f_{\text{av}}) + \frac{1}{2}\frac{(f-K)^2}{4}A'''(f_{\text{av}})+\ldots
\end{aligned}$$

Then, substituting this into the right hand side of (5.61), we get

$$\begin{aligned}
&\sqrt{\psi}\left(A(K)+\frac{1}{2}A'(K)\left(f-K\right)\right)\\
&= \sqrt{\psi}\Bigg(A(f_{\text{av}}) - \frac{1}{2}(f-K)A'(f_{\text{av}}) + \frac{1}{8}(f-K)^2A''(f_{\text{av}})\\
&+\frac{1}{2}\left(f-K\right)\left(A'(f_{\text{av}}) - \frac{1}{2}(f-K)A''(f_{\text{av}}) + \frac{1}{8}(f-K)^2A'''(f_{\text{av}})\right)+\ldots\Bigg)\\
&= \sqrt{\psi}A(f_{\text{av}})\left(1-\frac{1}{8}\left(f-K\right)^2\frac{A''(f_{\text{av}})}{A(f_{\text{av}})}\right)+O(\varepsilon^3)\\
&= \sqrt{\psi}A(f_{\text{av}})\left(1-\frac{1}{8}\gamma_2\left(f-K\right)^2\right)+O(\varepsilon^3)
\end{aligned}$$

where, for $k = 1,2,\ldots$,

$$\gamma_k = \frac{A^{(k)}(f_{\text{av}})}{A(f_{\text{av}})}.$$

Since we are expanding to $O\left(\varepsilon^2\right)$, the ν_k in the last two terms of (5.60) can be changed to γ_k without affecting the computation since they are $O\left(\varepsilon^2\right)$ and are both multiplied by $A(f_{\text{av}})\sqrt{\psi}$ which is $O\left(\varepsilon\right)$. [9]

[9] By noting that $K = f_{\text{av}} - \frac{1}{2}(f-K)$, we expand the following:

$$\begin{aligned}
A(K) &= A(f_{\text{av}}) - \frac{1}{2}(f-K)A'(f_{\text{av}}) + \frac{1}{8}(f-K)^2A''(f_{\text{av}})+\ldots\\
A'(K) &= A'(f_{\text{av}}) - \frac{1}{2}(f-K)A''(f_{\text{av}}) + \frac{1}{8}(f-K)^2A'''(f_{\text{av}})+\ldots\\
A''(K) &= A''(f_{\text{av}}) - \frac{1}{2}(f-K)A'''(f_{\text{av}}) + \frac{1}{8}(f-K)^2A''''(f_{\text{av}})+\ldots
\end{aligned}$$

Upon substitution into (5.60),

$$\begin{aligned}\sqrt{\psi^*} &= A(f_{\text{av}})\sqrt{\psi}\left(1 - \frac{1}{8}\gamma_2\,(f-K)^2 + \frac{2\gamma_2-\gamma_1^2}{12}\,(f-K)^2 + \frac{2\gamma_2-\gamma_1^2}{24}A^2(f_{\text{av}})\psi + \ldots\right)\\ &= A(f_{\text{av}})\sqrt{\psi}\left(1 + \frac{\gamma_2-2\gamma_1^2}{24}\,(f-K)^2 + \frac{2\gamma_2-\gamma_1^2}{24}A^2(f_{\text{av}})\psi + \ldots\right)\end{aligned} \tag{5.62}$$

In order to obtain the implied volatility, it is necessary to consider the special case where we start with Black's model: $dF(t) = \sigma_B F(t)dW$; so $\alpha(t) = \sigma_B$ and $A(F) = F$. The value of the option price $V(t,f)$ is then given by

$$V(t,f) = Z(t,t_{\text{set}})G\left(\psi_B, f-K\right),$$

where $\sqrt{\psi_B}$ is obtained by substituting σ_B and F into the expression for $\sqrt{\psi^*}$, (5.62). We have that

Looking at the remaining four terms of (5.60), we use the above expansions to verify that the substitution of γ_k in place of ν_k can be done using the fact that $(f-K)^2$ and $A^2(f_{\text{av}})$ are $O(\varepsilon^2)$, and $A(f_{\text{av}})$ is $O(\varepsilon)$:

1.

$$\begin{aligned}\frac{1}{6}A(K)\sqrt{\psi}\nu_2(f-K)^2 &= \frac{\sqrt{\psi}}{6}A''(K)(f-K)^2\\ &= \frac{\sqrt{\psi}}{6}\left(A''(f_{\text{av}}) - \frac{1}{2}(f-K)A'''(f_{\text{av}}) + \frac{1}{8}(f-K)^2A''''(f_{\text{av}}) + \ldots\right)(f-K)^2\\ &= O(\varepsilon^3)\end{aligned}$$

2.

$$\begin{aligned}\frac{-1}{12}A(K)\sqrt{\psi}\nu_2^2(f-K)^2 &= \frac{-\sqrt{\psi}}{12}\frac{(A'(K))^2}{A(K)}(f-K)^2\\ &= \frac{-\sqrt{\psi}}{12}\left(A''(f_{\text{av}}) - \frac{1}{2}(f-K)A'''(f_{\text{av}}) + \frac{1}{8}(f-K)^2A''''(f_{\text{av}}) + \ldots\right)^2\cdot\\ &\quad\left(A(f_{\text{av}}) - \frac{1}{2}(f-K)A'(f_{\text{av}}) + \frac{1}{8}(f-K)^2A''(f_{\text{av}}) + \ldots\right)^{-1}(f-K)^2\\ &= \frac{-\sqrt{\psi}}{12}\left((A''(f_{\text{av}}))^2 + \ldots\right)\left(A(f_{\text{av}}) + \frac{1}{2}(f-K)A'(f_{\text{av}}) + \ldots\right)(f-K)^2\\ &= O(\varepsilon^3)\end{aligned}$$

3.

$$\begin{aligned}\frac{1}{12}A^3(K)\psi^{3/2}\nu_2 &= \frac{\psi^{3/2}}{12}A^2(K)A''(K)\\ &= \frac{\psi^{3/2}}{12}\left(A(f_{\text{av}}) - \frac{1}{2}(f-K)A'(f_{\text{av}}) + \frac{1}{8}(f-K)^2A''(f_{\text{av}}) + \ldots\right)^2\cdot\\ &\quad\left(A''(f_{\text{av}}) - \frac{1}{2}(f-K)A'''(f_{\text{av}}) + \frac{1}{8}(f-K)^2A''''(f_{\text{av}}) + \ldots\right)\\ &= O(\varepsilon^3)\end{aligned}$$

4.

$$\begin{aligned}\frac{-1}{24}A^3(K)\psi^{3/2}\nu_1^2 &= \frac{-\psi^{3/2}}{24}A(K)A'(K)\\ &= \frac{-\psi^{3/2}}{24}\left(A(f_{\text{av}}) - \frac{1}{2}(f-K)A'(f_{\text{av}}) + \frac{1}{8}(f-K)^2A''(f_{\text{av}}) + \ldots\right)\cdot\\ &\quad\left(A'(f_{\text{av}}) - \frac{1}{2}(f-K)A''(f_{\text{av}}) + \frac{1}{8}(f-K)^2A'''(f_{\text{av}}) + \ldots\right)\\ &= O(\varepsilon^3)\end{aligned}$$

$A(f_{\text{av}}) = f_{\text{av}}$ and $\sqrt{\psi} = \sigma_B\sqrt{t_{\text{ex}} - t}$. We also note that $\gamma_1 = \frac{1}{f_{\text{av}}}$ and $\gamma_k = 0$ for all $k > 1$. This yields

$$\sqrt{\psi_B} = f_{\text{av}}\sigma_B\sqrt{t_{\text{ex}} - t}\left(1 - \frac{(f-K)^2}{12f_{\text{av}}^2} - \frac{\sigma_B^2\,(t_{\text{ex}} - t)}{24} + \ldots\right)$$

Since $G(\psi_B, f - K)$ is an increasing function of ψ_B, the Black price will coincide with the correct price if and only if

$$\sqrt{\psi_B} = \sqrt{\psi^*}.$$

Doing this yields the expression for implied volatility:

$$\begin{aligned}
&f_{\text{av}}\sigma_B\sqrt{t_{\text{ex}} - t}\left(1 - \frac{(f-K)^2}{12f_{\text{av}}^2} - \frac{\sigma_B^2\,(t_{\text{ex}} - t)}{24} + \ldots\right)\\
&= A(f_{\text{av}})\sqrt{\psi}\left(1 + \frac{\gamma_2 - 2\gamma_1^2}{24}(f-K)^2 + \frac{2\gamma_2 - \gamma_1^2}{24}A^2(f_{\text{av}})\psi + \ldots\right).
\end{aligned} \tag{5.63}$$

Now

$$\left(1 - \frac{(f-K)^2}{12f_{\text{av}}^2} - \frac{\sigma_B^2\,(t_{\text{ex}} - t)}{24} + \ldots\right)\left(1 + \frac{(f-K)^2}{12f_{\text{av}}^2} + \frac{\sigma_B^2\,(t_{\text{ex}} - t)}{24}\right) = 1 + O(\epsilon^3)$$

(It is a difference of squares.) Hence, by multiplying both sides of (5.63) by

$$\left(1 + \frac{(f-K)^2}{12f_{\text{av}}^2} + \frac{\sigma_B^2\,(t_{\text{ex}} - t)}{24}\right),$$

and dividing by $f_{\text{av}}\sqrt{t_{\text{ex}} - t}$, we get

$$\begin{aligned}
&\sigma_B\\
&= \frac{\sqrt{\psi}}{\sqrt{t_{\text{ex}} - t}}\frac{A(f_{\text{av}})}{f_{\text{av}}}\left[1 + \frac{\gamma_2 - 2\gamma_1^2}{24}(f-K)^2 + \frac{2\gamma_2 - \gamma_1^2}{24}A^2(f_{\text{av}})\psi + \ldots\right]\left[1 + \frac{(f-K)^2}{12f_{\text{av}}^2} + \frac{\sigma_B^2\,(t_{\text{ex}} - t)}{24} + \ldots\right]\\
&= \frac{\sqrt{\psi}}{\sqrt{t_{\text{ex}} - t}}\frac{A(f_{\text{av}})}{f_{\text{av}}}\left[1 + \left(\gamma_2 - 2\gamma_1^2 + \frac{2}{f_{\text{av}}^2}\right)\frac{(f-K)^2}{24} + \frac{2\gamma_2 - \gamma_1^2}{24}A^2(f_{\text{av}})\psi + \frac{\sigma_B^2\,(t_{\text{ex}} - t)}{24} + \ldots\right].
\end{aligned}$$

As a separate calculation, note that

$$\begin{aligned}
\sigma_B &= \frac{\sqrt{\psi}}{\sqrt{t_{\text{ex}} - t}}\frac{A(f_{\text{av}})}{f_{\text{av}}}\left[1 + \left(\gamma_2 - 2\gamma_1^2 + \frac{2}{f_{\text{av}}^2}\right)\frac{(f-K)^2}{24} + \frac{2\gamma_2 - \gamma_1^2}{24}A^2(f_{\text{av}})\psi + \frac{\sigma_B^2\,(t_{\text{ex}} - t)}{24} + \ldots\right]\\
&= \frac{\sqrt{\psi}}{\sqrt{t_{\text{ex}} - t}}\frac{A(f_{\text{av}})}{f_{\text{av}}}[1 + O(\epsilon^2)].
\end{aligned}$$

So squaring both sides we get

$$\sigma_B^2 = \frac{\psi}{t_{\text{ex}} - t}\frac{A^2(f_{\text{av}})}{f_{\text{av}}^2}[1 + O(\epsilon^2)],$$

so

$$\begin{aligned}
\sigma_B^2(t_{\text{ex}} - t) &= \psi\frac{A^2(f_{\text{av}})}{f_{\text{av}}^2}[1 + O(\epsilon^2)]\\
&= \psi\frac{A^2(f_{\text{av}})}{f_{\text{av}}^2} + O(\epsilon^4)
\end{aligned}$$

since $\psi \frac{A^2(f_{\text{av}})}{f_{\text{av}}^2}$ is itself $O(\epsilon^2)$. Now, returning to the expression for σ_B, we have at time $t = 0$, $f = f_0$

$$
\begin{aligned}
&\sigma_B \\
&= \frac{\sqrt{\psi}}{\sqrt{t_{\text{ex}} - t}} \frac{A(f_{\text{av}})}{f_{\text{av}}} \left[1 + \left(\gamma_2 - 2\gamma_1^2 + \frac{2}{f_{\text{av}}^2} \right) \frac{(f-K)^2}{24} + \left(2\gamma_2 - \gamma_1^2 + \frac{1}{f_{\text{av}}^2} \right) \frac{A^2(f_{\text{av}})\psi}{24} + \ldots \right] \\
&= a \frac{A(f_{\text{av}})}{f_{\text{av}}} \left[1 + \left(\gamma_2 - 2\gamma_1^2 + \frac{2}{f_{\text{av}}^2} \right) \frac{(f-K)^2}{24} + \left(2\gamma_2 - \gamma_1^2 + \frac{1}{f_{\text{av}}^2} \right) \frac{a^2 A^2(f_{\text{av}})(t_{\text{ex}} - t)}{24} + \ldots \right], \qquad (5.64)
\end{aligned}
$$

where

$$
\begin{aligned}
a^2 &= \frac{1}{t_{\text{ex}} - t} \psi \\
&= \frac{1}{t_{\text{ex}} - t} \int_t^{t_{\text{ex}}} \alpha^2(s) ds \qquad (5.65)
\end{aligned}
$$

The equivalent implied volatility is given by (5.64), and is then used in Black's formula to price call and put options. Although the formula is not exact, (Hagan & Woodward 1998) suggest its accuracy is comparable to that of a tree or PDE approach.

5.5.5 Incorrect Local Volatility Dynamics

Since we are trying to establish the dynamics of local volatility models, we consider the SDE, equation (2.7) in (Hagan et al. 2002):

$$dF = \sigma_{\text{loc}}(F) F dW, \quad F(0) = f_0. \qquad (5.66)$$

Here, $\sigma_{\text{loc}}(F)$ is the local volatility which is a function of the forward price only. Black's implied volatility, as a function of strike K and the $t = 0$ forward price f_0, is then given by equation (2.8) in (Hagan et al. 2002):

$$\sigma_B(K, f_0) = {}^{10}\sigma_{\text{loc}}(f_{\text{av}}) \left(1 + \frac{1}{24} \frac{\sigma''_{\text{loc}}(f_{\text{av}})}{\sigma_{\text{loc}}(f_{\text{av}})} (f_0 - K)^2 + \ldots \right).$$

Let f_0 be the $t = 0$ forward price and the implied volatility for a given strike K be $\sigma_B^0(K)$. To first order,

$$\sigma_B^0(K) = \sigma_{\text{loc}} \left(\tfrac{1}{2} (f_0 + K) \right)$$

By translating $K = 2F - f_0$, we have that

$$\sigma_B^0(2K - f_0) = \sigma_{\text{loc}}(K).$$

[10] Given (5.66) to describe the dynamics of the forward price, it is clear that in (5.27), $\alpha(t) = 1$ (therefore $a = 1$) and $A(f_{\text{av}}) = \sigma_{\text{loc}}(f_{\text{av}}) f_{\text{av}}$. So, substituting theses values into (5.64), we get

$$
\begin{aligned}
\sigma_B(K, f) &= 1 \cdot \frac{\sigma_{\text{loc}}(f_{\text{av}}) f_{\text{av}}}{f_{\text{av}}} \left(1 + \frac{1}{24} \frac{\sigma''_{\text{loc}}(f_{\text{av}})}{\sigma_{\text{loc}}(f_{\text{av}})} (f - K)^2 + \ldots \right) \\
&= \sigma_{\text{loc}}(f_{\text{av}}) \left(1 + \frac{1}{24} \frac{\sigma''_{\text{loc}}(f_{\text{av}})}{\sigma_{\text{loc}}(f_{\text{av}})} (f - K)^2 + \ldots \right).
\end{aligned}
$$

Suppose the underlying forward value moves from f_0 at time $t = 0$ to f_1 at time $t = 1$,

$$\sigma^0_{\text{loc}}(K) = \sigma^0_B(2K - f_0) = {}^{11}\sigma^1_{\text{loc}}(K),$$

where $\sigma^0_{\text{loc}}(K)$ and $\sigma^1_{\text{loc}}(K)$ represents the local volatilities at times $t = 0$ and $t = 1$, as a function of the strike K, respectively. Then for an option with strike X, the implied volatility predicted by the model at time $t = 0$ is given by:

$$\begin{aligned}
\sigma^1_{\text{loc}}(K) &= \sigma^1_B(2K - f_1) \\
\Rightarrow \sigma^0_B(2K - f_0) &= \sigma^1_B(2K - f_1), \\
\Rightarrow \sigma^0_B(X + f_1 - f_0) &= \sigma^1_B(X + f_1 - f_0) = \sigma^1_B(X).
\end{aligned}$$

So if f has moved left/right by $f_1 - f_0$, then σ_B moves to the right/left. This is contrary to known observations and so shows that local volatility models are severely compromised.

Therefore, the dynamics are incorrect. A consequence of this is that the delta hedge value, Δ, will also be incorrect. Consider Black's formula for a European call option:

$$\begin{aligned}
V_{\text{call}}(0, f_0) &= {}^{12}Z(0, t_{\text{set}})\left(f_0\Phi(d_1) - K\Phi(d_2)\right) \\
&= BS(f_0, K, \sigma_B(f_0, K), t_{\text{ex}}).
\end{aligned}$$

Then,

$$\Delta := \frac{\partial V_{\text{call}}}{\partial f_0} = \frac{\partial BS}{\partial f_0} + \frac{\partial BS}{\partial \sigma_B}\frac{\partial \sigma_B}{\partial f_0}.$$

The second term is the local volatility model's correction to the delta risk which is Black's vega risk multiplied by $\frac{\partial \sigma_B}{\partial f_0}$. Since the predicted dynamics are in the opposite direction to what is observed, we can conclude that the sign of this term should be opposite to that calculated. (Hagan et al. 2002) asserts that the Black model yields more accurate hedges than local volatility models.

[11] The local volatility, being state but not time dependent, is the same for each strike K at $t = 0$ and $t = 1$.

[12] In international markets, options on futures are not fully margined and hence, the buyer will pay a premium upfront. Consequently, the pricing formula is the Standard Black (Black 1976) formula for vanilla options. In South Africa, options are fully margined (no premium upfront) and the pricing formula differs in that there is no discount function. i.e. the price of call and put options are in (West 2005*b*, Chapter 10):

$$\begin{aligned}
V_{\text{call}} &= f_0\Phi\left(d_1\right) - K\Phi\left(d_2\right), \\
V_{\text{put}} &= K\Phi\left(-d_2\right) - f_0\Phi\left(-d_1\right), \\
d_{1,2} &= \frac{\ln\frac{f_0}{K} \pm \sigma^2\psi_{\text{ex}}}{\sigma\sqrt{\psi_{\text{ex}}}},
\end{aligned}$$

where the current futures level is f_0, the strike K, the volatility is σ and time to maturity, $\psi_{\text{ex}} = t_{\text{ex}} - t$. Options are American, but there are no profitable early exercise opportunities.

Chapter 6

Stochastic Volatility Models

6.1 Introduction

In the preceding chapters, the focus was on deterministic, non-parametric models that enabled the local volatility to be determined. Local volatility is, at a future market level and time, (within each model) the volatility the index must have to ensure current market prices are fair. These models enable the determination of Arrow-Debreu prices, which are required for the pricing and hedging of path-dependent or exotic options. However, the models provide results that are contrary to observed phenomena. This then leads to portfolios that do not contain the correct hedges, and mispricing of options. This will be discussed in greater detail in Chapter 9.

Another approach to the determination of the future volatility, is rather that of stochastic volatility (parametric) models.

In the classical Black-Scholes framework, the first fundamental theorem of mathematical finance states that the pricing model is arbitrage free if and only if there exists an equivalent martingale measure (EMM) (Björk 2004, §10.9). Under this measure, the traded assets, normalized by the numèraire (risk-free asset in general), are martingales. Furthermore, if the market model is complete (every contingent claim is attainable), the second fundamental theorem provides the result that the EMM will be unique. Alternatively viewed, every contingent claim can be perfectly hedged with the traded asset and risk-free asset alone. In an arbitrage-free complete market, the prices of contingent claims can be given as their discounted expected values under the unique EMM. This means that the discounted value of a contingent claim is given by the initial cost of setting up the replicating strategy and the gains from trading. It is assumed that all trading strategies are self-financing and admissible (i.e. the value of the replicating portfolio is bounded below by zero). The martingale representation theorem is required when constructing this portfolio.

Finding the EMM can be interpreted as applying Girsanov's theorem and defining a martingale (Radon-Nokodym) process, often referred to as the stochastic Dolèans exponential (Bingham & Kiesel 2004, §5.10.3), that changes the drift of the discounted stock price process to zero. So, under the EMM, it becomes a martingale and through Itô's Lemma, any sufficiently smooth function of this process (price process of the simple contingent claim) will also be a martingale. Using this result, the Feynman-Kač theorem is then applied to this function to obtain the above-mentioned result, the arbitrage-free price of the claim.

Stochastic volatility models are generally two-factor models where the volatility, as well as the stock price, are modelled using diffusion processes driven by Brownian motion(s). Since an additional source of randomness is added to the model, without volatility being a traded asset, the market model becomes incomplete. The introduction of variance swaps into the market will complete the market (Hagan et al. 2002). Incompleteness translates into non-uniqueness of the EMM. Consequently, when Girsanov's theorem is applied, there is no unique function (*market price of volatility risk*, $\hat{\lambda}$), that will ensure the uniqueness of the EMM. There are a number of different measures which can been used. Each corresponds to different choices for $\hat{\lambda}$. In general, one can choose to maximize utility or minimize risk. In complete markets, all derivatives dependent on the underlying process will have the same value for $\hat{\lambda}$. Thus, the market determines the value for $\hat{\lambda}$. If $\hat{\lambda} = 0$, then the market is said to be risk-neutral. This corresponds to the Minimal Martingale measure. The determination of this parameter is a calibration issue to be dealt with in subsequent chapters.

Another consequence of incomplete markets is that a perfect hedge cannot be created with traded asset and risk-free asset alone. Some common processes for the volatility, denoted $\sigma_t = f(Y_t)$ are:

- Lognormal (Hull & White 1987),
$$dY_t = aY_t dt + bY_t d\hat{Z}_t$$
- Ornstein-Uhlenbeck (Fouque, Papanicolaou & Sircar 2000),
$$dY_t = \alpha(m - Y_t)dt + \beta Y_t d\hat{Z}_t$$
- Cox-Ingersoll-Ross (Heston 1993),
$$dY_t = \kappa(m' - Y_t)dt + v\sqrt{Y_t} d\hat{Z}_t$$

where the Brownian motion $(\hat{Z}_t)$ that is driving the volatility process is correlated with the Brownian motion (W_t) that is driving the asset price equation:

$$dX_t = \mu X_t dt + \sigma_t X_t dW_t.$$

The Brownian motions are generally assumed to have instantaneous correlation $\rho \in$ [-1,1]. This is defined as

$$d\langle W, \hat{Z}\rangle_t \quad = \quad \rho dt,$$

where $\langle W, \hat{Z}\rangle_t$ is defined as the covariation of W_t and $\hat{Z}_t$. For the process X_t defined above, $\langle X\rangle_t = \int_0^t \sigma_s^2 ds$, is the quadratic variation of the martingale part. $\hat{Z}_t$, by using the Choleski decomposition, can also be written as

$$\hat{Z}_t \quad = \quad \rho W_t + \sqrt{1-\rho^2} Z_t$$

where (Z_t) is a standard Brownian motion independent of (W_t).

6.2 Derivative Pricing

In this section, the pricing PDE will be derived according to the method described in (Fouque et al. 2000). Denote the underlying probability space $(\Omega, \mathcal{F}, P)$, where $\Omega = C\big([0,\infty) : \mathbb{R}^2\big)$, the space of all continuous trajectories $(W_t(\omega), Z_t(\omega)) = \omega(t)$ in $\mathbb{R}^2$. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is generated by the Brownian motions and satisfies the usual conditions i.e. $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$, and the filtration is right continuous. Using no-arbitrage arguments, the pricing function, $P(t, X_t, Y_t)$, of a European derivative will be shown to satisfy a PDE. The pricing PDE will be derived assuming volatility is a function of a mean-reverting Ornstein-Uhlenbeck (OU) process. The following equations define the processes of the traded asset and volatility, X_t and $f(Y_t)$ respectively:

$$\begin{aligned} dX_t &= \mu X_t dt + \sigma_t X_t dW_t \\ \sigma_t &= f(Y_t) \\ dY_t &= \alpha(m - Y_t)dt + \beta Y_t d\hat{Z}_t \end{aligned}$$

The pricing function is determined by constructing a hedged portfolio of assets. Since there is additional Brownian motion, it is not sufficient to hedge with the underlying asset alone. One requires another option with a different expiration date to hedge the volatility risk. The argument is completely familiar from the elementary theory of interest rate derivatives.

Let $P^1(t,x,y)$ denote the price of a European derivative with maturity T_1 and payoff function $h(X_{T_1})$. The requirement is to find hedge amounts (a_t, b_t, c_t) such that

$$P^1(T_1, X_{T_1}, Y_{T_1}) = a_{T_1} X_{T_1} + b_{T_1}\beta_{T_1} + c_{T_1} P^2(T_1, X_{T_1}, Y_{T_1}) \tag{6.1}$$

where $\beta_t = e^{rt}$, where r is the short-term interest rate and $P^2(t, X_t, Y_t)$ is the price of a European derivative with the same payoff function h but with a maturity T_2 where $T_2 > T_1 > t$. The above equation equates the terminal payoff of the first derivative with the hedged portfolio. This portfolio must also satisfy the self-financing condition ($\forall t < T_1$):

$$dP^1(t, X_t, Y_t) = a_t dX_t + b_t r e^{rt} dt + c_t dP^2(t, X_t, Y_t) \tag{6.2}$$

No arbitrage implies that $\forall t < T_1$, the following must always hold:

$$P^1(t, X_t, Y_t) = a_t X_t + b_t e^{rt} + c_t P^2(t, X_t, Y_t) \tag{6.3}$$

A multi-dimensional version of Itô's Lemma is required in order to evaluate the infinitesimal change in the derivatives with respect to time and the two spacial variables.

Proposition 4 ***Itô's Formula***
(Björk 2004, §4.8) Take a vector Wiener process $W = (W_1, \ldots, W_n)$ with correlation matrix $\underline{\rho}$ as given, and assume that the vector process $X = (X_1, \ldots, X_n)^T$ has a stochastic integral. Then the following hold:

- *For any $C^{1,2}$ function f, the stochastic differential of the process $f(t, X(t))$ is given by*

$$df(t, X(t)) = \frac{\partial f}{\partial t} dt + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} dX_i + \frac{1}{2} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j} dX_i dX_j,$$

with

$$\begin{cases} (dt)^2 = 0, & \\ dt \cdot dW_i = 0, & i = 1, \ldots, n. \\ dW_i \cdot dW_j = \rho_{ij} dt. & \end{cases}$$

- *If $k = n$ and dX has the structure*

$$dX_i = \mu_i dt + \sigma_i dW_i, \qquad i = 1, \ldots, n. \tag{6.4}$$

where μ_i and σ_i are scalar processes for $i = 1, \ldots, n$, then

$$df = \left\{ \frac{\partial f}{\partial t} + \sum_{i=1}^{n} \mu_i \frac{\partial f}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} \sigma_i \sigma_j \rho_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j} \right\} dt + \sum_{i=1}^{n} \sigma_i \frac{\partial f}{\partial x_i} dW_i$$

Using a two-dimensional version of Itô's formula:

$$dg(t, X_t, Y_t) = \frac{\partial g}{\partial t} dt + \frac{\partial g}{\partial x} dX_t + \frac{\partial g}{\partial y} dY_t + \frac{1}{2} \left(\frac{\partial^2 g}{\partial x^2} d\langle X \rangle_t + 2 \frac{\partial^2 g}{\partial xy} d\langle X, Y \rangle_t + \frac{\partial^2 g}{\partial y^2} d\langle Y \rangle_t \right)$$

In general, the quadratic covariation is given by $d\langle X, Y \rangle_t = \sigma_X(t, X_t) \sigma_Y(t, Y_t) dt$.

Applying this to (6.2) yields

$$\begin{aligned} &\left(\frac{\partial P^1}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2 P^1}{\partial x^2} + \rho \beta x f(y) \frac{\partial^2 P^1}{\partial xy} + \frac{1}{2} \beta^2 \frac{\partial^2 P^1}{\partial y^2} \right) dt + \frac{\partial P^1}{\partial x} dX_t + \frac{\partial P^1}{\partial y} dY_t \\ &= \left(c_t \left(\frac{\partial P^2}{\partial t} + \frac{1}{2} f(y)^2 x^2 \frac{\partial^2 P^2}{\partial x^2} + \rho \beta x f(y) \frac{\partial^2 P^2}{\partial xy} + \frac{1}{2} \beta^2 \frac{\partial^2 P^2}{\partial y^2} \right) + b_t r e^{rt} \right) dt \\ &+ \left(a_t + c_t \frac{\partial P^2}{\partial x} \right) dX_t + c_t \frac{\partial P^2}{\partial y} dY_t \end{aligned}$$

Define the following differential operator:

$$D_1 = \frac{1}{2} f(y)^2 x^2 \frac{\partial^2}{\partial x^2} + \rho \beta x f(y) \frac{\partial^2}{\partial xy} + \frac{1}{2} \beta^2 \frac{\partial^2}{\partial y^2}$$

The above can be simplified as

$$\begin{aligned} &\left(\frac{\partial P^1}{\partial t} + D_1 P^1 \right) dt + \frac{\partial P^1}{\partial x} dX_t + \frac{\partial P^1}{\partial y} dY_t \\ &= \left(c_t \left(\frac{\partial}{\partial t} + D_1 \right) P^2 + b_t r e^{rt} \right) dt + \left(a_t + c_t \frac{\partial P^2}{\partial x} \right) dX_t + c_t \frac{\partial P^2}{\partial y} dY_t \end{aligned} \tag{6.5}$$

Equating $d\hat{Z}_t$ and dW_t terms is equivalent to equating dY_t and dX_t terms respectively. So, solving for a_t and c_t:

$$\begin{aligned} c_t \frac{\partial P^2}{\partial y} dY_t &= \frac{\partial P^1}{\partial y} dY_t \\ \Rightarrow c_t &= \frac{\partial P^1 / \partial y}{\partial P^2 / \partial y} \end{aligned}$$

as expected (analogously to the hedge arguments in interest rate models).

$$\begin{aligned} \frac{\partial P^1}{\partial x} dX_t &= \left(a_t + c_t \frac{\partial P^2}{\partial x} \right) dX_t \\ \Rightarrow a_t &= \frac{\partial P^1}{\partial x} - \left(\frac{\partial P^1 / \partial y}{\partial P^2 / \partial y} \right) \frac{\partial P^2}{\partial x} \end{aligned}$$

Substituting into (6.3) to solve for b_t,

$$\begin{aligned} b_t &= e^{-rt}\left(P^1 - a_t X_t - c_t P^2\right) \\ &= e^{-rt}\left[P^1 - \left(\frac{\partial P^1}{\partial x} - \left(\frac{\partial P^1/\partial y}{\partial P^2/\partial y}\right)\frac{\partial P^2}{\partial x}\right)X_t - \left(\frac{\partial P^1/\partial y}{\partial P^2/\partial y}\right)P^2\right] \end{aligned}$$

Define the Black-Scholes partial differential operator by

$$\mathcal{L}_{BS}(f(y)) = \frac{\partial}{\partial t} + \frac{1}{2}f(y)^2 x^2 \frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right) \tag{6.6}$$

(Under the assumption of constant volatility, the price of a European derivative $P(t,x)$ satisfies $\mathcal{L}_{BS}P = 0$ where $f(y) = \sigma$ and satisfies the terminal condition $P(T,x) = h(x)$). Equating the dt terms in (6.5),

$$\left(\frac{\partial}{\partial t} + D_1\right)P^1 = c_t\left(\frac{\partial}{\partial t} + D_1\right)P^2 + b_t r e^{rt}$$

Substituting in for a_t, b_t and c_t:

$$\begin{aligned} &\left(\frac{\partial}{\partial t} + D_1\right)P^1 \\ &= \frac{\partial P^1/\partial y}{\partial P^2/\partial y}\left(\frac{\partial}{\partial t} + D_1\right)P^2 + r\left(P^1 - \left(\frac{\partial P^1}{\partial x} - \left(\frac{\partial P^1/\partial y}{\partial P^2/\partial y}\right)\frac{\partial P^2}{\partial x}\right)x - \left(\frac{\partial P^1/\partial y}{\partial P^2/\partial y}\right)P^2\right) \end{aligned}$$

Gathering terms in P^1 and P^2:

$$\frac{1}{\partial P^1/\partial y}\left[\left(\frac{\partial}{\partial t} + D_1\right)P^1 + rx\frac{\partial P^1}{\partial x} - rP^1\right] = \frac{1}{\partial P^2/\partial y}\left[\left(\frac{\partial}{\partial t} + D_1\right)P^2 + rx\frac{\partial P^2}{\partial x} - rP^2\right]$$

Thus,

$$\frac{1}{\partial P^1/\partial y}D_2 P^1(t,x,y) = \frac{1}{\partial P^2/\partial y}D_2 P^2(t,x,y) \tag{6.7}$$

where D_2 is a partial differential operator defined as

$$D_2 = \frac{\partial}{\partial t} + D_1 + r\left(x\frac{\partial}{\partial x} - \cdot\right)$$

which is $\mathcal{L}_{BS}(f(y))$ plus second-order terms from the additional diffusion process.

In equation (6.7), the left-hand side is dependent on T_1 and independent of T_2 while the opposite result holds for the right-hand side. Consequently, both must equal a function that does not depend on expiry. Both sides can only be functions of the independent variables. For reasons to follow, this function is denoted by

$$\alpha(m-y) - \beta\left(\frac{(\mu - r)}{f(y)}\rho + \hat{\lambda}(t,x,y)\sqrt{1-\rho^2}\right)$$

where $\hat{\lambda}(t,x,y)$ is an arbitrary function. For the general case:

$$dY_t = \mu_Y(t,Y_t)dt + \sigma_Y(t,Y_t)d\hat{Z}_t,$$

this function is written as

$$\mu_Y(t,y) - \sigma_Y(t,y)\left(\frac{(\mu - r)}{f(y)}\rho + \hat{\lambda}(t,x,y)\sqrt{1-\rho^2}\right)$$

So, the pricing function $P(t,x,y)$, with terminal condition $P(T,x,y) = h(x)$ must satisfy the PDE:

$$\begin{aligned}&\frac{\partial P}{\partial t} + \frac{1}{2}f(y)^2x^2\frac{\partial^2 P}{\partial x^2} + \rho\beta x f(y)\frac{\partial^2 P}{\partial xy} + \frac{1}{2}\beta^2\frac{\partial^2 P}{\partial y^2}\\ &+ r\left(x\frac{\partial P}{\partial x} - P\right) + (\alpha(m-y) - \beta\Lambda(t,x,y))\frac{\partial P}{\partial y} = 0\end{aligned} \tag{6.8}$$

where

$$\Lambda(t,x,y) = \left(\frac{(\mu - r)}{f(y)}\rho + \hat{\lambda}(t,x,y)\sqrt{1-\rho^2}\right) \tag{6.9}$$

The above equation can be grouped according to differential operators:

$$\underbrace{\frac{\partial}{\partial t} + \frac{1}{2}f(y)^2x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right)}_{\mathcal{L}_{BS}(f(y))} + \underbrace{\rho\beta x f(y)\frac{\partial^2}{\partial xy}}_{\text{correlation}} + \underbrace{\frac{1}{2}\beta^2\frac{\partial^2}{\partial y^2} + \alpha(m-y)\frac{\partial}{\partial y}}_{\mathcal{L}_{OU}} + \underbrace{\beta\Lambda\frac{\partial}{\partial y}}_{\text{premium}}$$

The first term is as defined above, the second is due to the correlation of the two processes, the third is the differential operator acting on a OU-diffusion. The general definition of the operator is given by

$$\mathcal{L} = \frac{1}{2}\sigma^2(t,x)\frac{\partial^2}{\partial x^2} + \mu(t,x)\frac{\partial}{\partial x}$$

where $\sigma(t,x)$ and $\mu(t,x)$ are the coefficients of Brownian motion and dt of the diffusion respectively. The fourth term results from the market price of the volatility risk. To see what effect this has on $P(t,x,y)$, apply Itô's Lemma:

$$\begin{aligned}&dP(t,X_t,Y_t)\\ &= \frac{\partial P}{\partial t}dt + \frac{\partial P}{\partial x}(\mu x dt + f(y)x dW_t) + \frac{\partial P}{\partial y}\left(\alpha(m-y)dt + \beta\left(\rho dW_t + \sqrt{1-\rho^2}dZ_t\right)\right)\\ &+ \frac{1}{2}\left(f(y)^2x^2\frac{\partial^2 P}{\partial x^2} + 2\rho\beta x f(y)\frac{\partial^2 P}{\partial xy} + \beta^2\frac{\partial^2 P}{\partial y^2}\right)dt\\ &= \left(\frac{\partial P}{\partial t} + \mu x\frac{\partial P}{\partial x} + \alpha(m-y)\frac{\partial P}{\partial y} + \frac{1}{2}\left(f(y)^2x^2\frac{\partial^2 P}{\partial x^2} + 2\rho\beta x f(y)\frac{\partial^2 P}{\partial xy} + \beta^2\frac{\partial^2 P}{\partial y^2}\right)\right)dt\\ &+ \left(f(y)x\frac{\partial P}{\partial x} + \beta\rho\right)dW_t + \beta\sqrt{1-\rho^2}dZ_t\end{aligned} \tag{6.10}$$

From (6.8), it is clear that

$$\begin{aligned}&\frac{\partial P}{\partial t} + \frac{1}{2}f(y)^2x^2\frac{\partial^2 P}{\partial x^2} + \rho\beta x f(y)\frac{\partial^2 P}{\partial xy} + \frac{1}{2}\beta^2\frac{\partial^2 P}{\partial y^2} =\\ &rP - rx\frac{\partial P}{\partial x} - \alpha(m-y)\frac{\partial P}{\partial y} + \beta\Lambda(t,x,y)\frac{\partial P}{\partial y}\end{aligned}$$

Substituting this into (6.10):

$$
\begin{aligned}
&dP(t, X_t, Y_t) \\
&= \left(\mu x \frac{\partial P}{\partial x} + \alpha(m-y)\frac{\partial P}{\partial y} + rP - rx\frac{\partial P}{\partial x} - \alpha(m-y)\frac{\partial P}{\partial y} + \beta\Lambda(t,x,y)\frac{\partial P}{\partial y}\right) dt \\
&+ \left(f(y)x\frac{\partial P}{\partial x} + \beta\rho\right) dW_t + \beta\sqrt{1-\rho^2}dZ_t \\
&= \left(\mu x\frac{\partial P}{\partial x} + rP - rx\frac{\partial P}{\partial x} + \beta\left(\frac{(\mu - r)}{f(y)}\rho + \hat{\lambda}(t,x,y)\sqrt{1-\rho^2}\right)\frac{\partial P}{\partial y}\right) dt \\
&+ \left(f(y)x\frac{\partial P}{\partial x} + \beta\rho\right) dW_t + \beta\sqrt{1-\rho^2}dZ_t \\
&= \left(\frac{(\mu - r)}{f(y)}\left(xf(y)\frac{\partial P}{\partial x} + \beta\rho\frac{\partial P}{\partial y}\right) + rP + \hat{\lambda}(t,x,y)\beta\sqrt{1-\rho^2}\frac{\partial P}{\partial y}\right) dt \\
&+ \left(f(y)x\frac{\partial P}{\partial x} + \beta\rho\right) dW_t + \beta\sqrt{1-\rho^2}dZ_t
\end{aligned}
$$

Looking at the dt term, it is clear that as the volatility risk β increases, the rate of return of the option increases by $\hat{\lambda}$ multiplied by that infinitesimal amount, in addition to that amount due to the market price of risk associated with the underlying, $\frac{\mu - r}{f(y)}$. Note in this case $f(y) = \sigma_t$.

6.3 Arbitrage Pricing

6.3.1 Equivalent Martingale Measure

Consider the probability triple $(\Omega, P, \mathcal{F}_T)$ over the finite time interval $[0, T]$ and a market model that consists of a risky asset price process $S(t)$ and a risk free asset $B(t)$ for $t \geq 0$.

Definition 7 *(Björk 2004, §10.2) A probability measure Q on $\mathcal{F}_T$ is called an* ***equivalent martingale measure*** *for the above market model, with numèraire $B(t)$ on $[0, T]$, if it has the following properties:*

- *Q is equivalent to P on $\mathcal{F}_T$*
- *The relative price process $S(t)/B(t)$ is a martingale under Q on $[0, T]$*

Let $V^h(t)$ represent the value of a portfolio h at time t.

Definition 8 *(Björk 2004, §7.2) An* ***arbitrage*** *possibility on a financial market is a self-financed portfolio h such that*

$$
\begin{aligned}
V^h(0) &= 0, \\
P(V^h(T) \geq 0) &= 1, \\
P(V^h(T) > 0) &> 0
\end{aligned}
$$

We say the market is ***arbitrage free*** *if there are no arbitrage possibilities*

Theorem 3 ***The First Fundamental Theorem***
(Björk 2004, §10.14) The model is arbitrage free if and only if there exists an equivalent martingale measure Q

So, the existence of an equivalent martingale measure Q ensures that the no arbitrage condition prevails. Q may not necessarily be unique. Assuming the interest rate associated with the risk free asset is constant, then the following theorem is applicable to the relative price process of any contingent claim, denoted $\Pi(S(t), t)$, dependent upon the same underlying source of randomness:

Theorem 4 ***Risk Neutral Valuation***
(Björk 2004, §10.19)

$$\frac{\Pi(t, S(t))}{B(t)} = \mathbb{E}^Q\left[\frac{\Pi(T, S(T))}{B(T)}\right]$$

where Q is a (not necessarily unique) equivalent martingale measure.

Theorem 5 ***The Second Fundamental Theorem***
Assume the market is arbitrage free. Then the market is complete if and only if the equivalent martingale measure is unique.

A unique EMM (equivalent martingale measure) is the requirement to ensure the market is complete. This can be interpreted as the ability to replicate or hedge any derivative uniquely. In an incomplete market, the requirement of no arbitrage is not sufficient to to price a derivative uniquely. There may be several EMMs which price the derivative in a way that is consistent with no arbitrage. Consequently, a number of different prices may be consistent with the no-arbitrage condition. The next theorem relates completeness and no arbitrage to the sources of randomness that are present in the market.

Theorem 6 *(Björk 2004, §8.3) Let M denote the number of underlying* ***traded*** *assets in the model excluding the risk free asset, and let R denote the number of random sources. Generically we then have the following relations:*

1. *The model is arbitrage free iff $M < R$.*
2. *The model is complete iff $M \geq R$.*
3. *The model is complete and arbitrage free iff $M = R$.*

6.3.2 Martingale Representation Theorem

Alternatively viewed, in a complete and arbitrage-free environment, all contingent claims can be replicated uniquely. The following two theorems are required in order to determine the replicating portfolio. Since the price process of the contingent claim is a martingale under the unique EMM, the exact form for the hedge (amounts of each asset to be held in the multi-dimensional case) can be determined.

Theorem 7 ***Representation of Wiener Functionals***
(Björk 2004, §11.1) Let W be a d-dimensional Wiener process, and let X be a stochastic variable such that

- $X \in \mathcal{F}_T^W$
- $\mathbb{E}\left[|X|\right] < \infty$

Then there exist uniquely determined $\mathcal{F}_T^W$-adapted processes $h_1,\ h_2,\ \ldots,\ h_d$, such that X has the representation

$$X = \mathbb{E}\left[X\right] + \sum_{i=1}^{d} \int_0^d h_i(s) dW_i(s)$$

This result leads to the martingale representation theorem.

Theorem 8 ***Martingale Representation Theorem***
(Björk 2004, §11.2) Let W be a d-dimensional Wiener process, and assume the filtration $\underline{\mathcal{F}}$ is defined as

$$\mathcal{F}_t = \mathcal{F}_t^W \quad t \in [0, T]$$

Let M be any $\mathcal{F}_t$-adapted martingale. Then there exist uniquely determined $\mathcal{F}_t$-adapted processes $h_1,\ h_2,\ \ldots,\ h_d$ such that M has the representation

$$M(t) = M(0) + \sum_{i=1}^{d} \int_0^d h_i(s) dW_i(s)$$

This result is an existence result and does not give details of the process of h. For the description of h, consider an n-dimensional process X with the dynamics:

$$dX(t) \quad = \quad \mu(t)dt + \sigma(t)dW(t)$$

where μ and σ are adapted processes taking values in $\mathbb{R}^n$. If we assume $M(t) = f(t, X_t)$ for some deterministic smooth function $f(t, x)$. Applying Itô's formula yields the following:

$$df(t, X(t)) = \left(\frac{\partial f}{\partial t} + \mathcal{A}f\right) dt + \left[(\nabla_x) f\right] \sigma(t) dW(t)$$

where $\mathcal{A}$, a partial differential operator, is the Itô operator defined for any function $g(t, x)$ with $g \in C^2(\mathbb{R}^n)$ as

$$\left(\mathcal{A}g\right)(t, x) = \sum_{i=1}^{n} \mu_i(t, x) \frac{\partial g}{\partial x_i} + \frac{1}{2} \sum_{i,j=1}^{n} C_{i,j}(t, x) \frac{\partial^2 g}{\partial x_i \partial x_j} \tag{6.11}$$

where

$$C(t, x) = \sigma(t, x) \sigma^T(t, x)$$

and ∇_x (nabla or grad) is defined for $g \in C^1(\mathbb{R}^n)$ as

$$\nabla_x g = \left[\frac{\partial g}{\partial x_1}, \cdots, \frac{\partial g}{\partial x_n}\right]$$

Now, since $f(t, X(t))$ is assumed to be a martingale, the drift is zero and

$$\begin{aligned} df(t, X(t)) &= [(\nabla_x) f]\, \sigma(t) dW(t) \\ &= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \sigma^i(t) dW^i(t) \end{aligned}$$

So, the integrand h has an explicit description:

$$h_i(t) = \frac{\partial f}{\partial x_i} \sigma^i(t), \quad i = 1,\ 2,\ \ldots,\ d.$$

In order to ensure that the process followed by the discounted asset, and consequently any simple contingent claim dependent on it, is a martingale under the unique EMM, Girsanov's theorem is required to ensure the drift of the discounted original process vanishes. Alternatively, the drift of the stock price, under risk-neutral conditions, becomes the risk free rate.

So, the transformation occurs from the measure P to the EMM Q and a P-Wiener process W^P can then be expressed as

$$dW_t^P = \varphi_t dt + dW_t^Q,$$

where W^Q is a Q-Wiener process. The process φ_t must satisfy the Novikov condition to be defined below. To change to the risk-neutral measure Q on $\mathcal{F}_T$, choose a non-negative random variable $L_T \in \mathcal{F}_T$ and define Q by

$$\tfrac{dQ}{dP} = L_T, \quad \text{on } \mathcal{F}_T$$

This can be restated in the following theorem:

Theorem 9 ***The Girsanov Theorem***

(Björk 2004, §11.3) Let W^P be a d-dimensional P-Wiener process on $(\Omega, \mathcal{F}, P, \underline{\mathcal{F}})$ and let φ be any d-dimensional adapted column vector process. Choose a fixed time T and define the process L on $[0, T]$ by

$$\begin{aligned} dL_t &= \varphi_t^Q L_t dW_t^P, \\ L_0 &= 1, \end{aligned}$$

i.e.

$$L_t = \exp\left(\int_0^t \varphi_s^Q dW_s^P - \frac{1}{2}\int_0^t ||\varphi_s^Q||^2 ds\right).$$

Assume that

$$\mathbb{E}^P[L_T] = 1,$$

and define the new probability measure Q on $\mathcal{F}_T$ by

$$\frac{dQ}{dP} = L_T, \quad \textit{on } \mathcal{F}_T$$

Then

$$dW_t^P = \varphi_t dt + dW_t^Q,$$

where W^Q is a Q-Wiener process.

In the above theorem, the explicit form of L is given by

$$L_t = \exp\left(\sum_{i=1}^{d} \int_0^t \varphi_i(s) dW_i^P(s) - \frac{1}{2} \int_0^t \sum_{i=1}^{d} \varphi_i^2(s) ds\right)$$

This process is often referred to as the Dolèans exponential. It is in fact the market price of risk associated with the traded asset that all contingent claims are dependent on. Risk-neutral valuation determines the price of these derivatives.

Lemma 2 ***The Novikov Condition***
(Björk 2004, §11.5) Assume that the process φ is such that

$$\mathbb{E}^P\left[\exp\left(\frac{1}{2}\int_0^T ||\varphi_t||^2 dt\right)\right] < \infty.$$

Then L is a martingale and in particular $\mathbb{E}^P[L_T] = 1$.

So, when the stock price and the risk free asset satisfy the following:

$$dS_t = \mu S_t dt + \sigma S_t dW_t^P$$
$$dB_t = r B_t dt$$

Define L by

$$dL_t = \varphi_t L_t dW_t^P$$

and setting $dQ = L_T dP$ on $\mathcal{F}_T$, applying Girsanov's theorem, the dynamics of S under the measure Q become

$$dS_t = (\mu + \sigma \varphi_t) S_t dt + \sigma S_t dW_t^Q$$

For Q to be an EMM, the instantaneous return of the **traded** asset S must be the risk free rate. So, to determine the process φ_t, we have

$$\mu + \sigma \varphi_t = r$$

So, φ_t is the market price of risk, λ, associated with the traded asset. There is a one-to-one correspondence between the EMM Q and λ. Under this measure, it can be shown that the pricing function $F(t, x)$ satisfies the following partial differential equation and boundary condition:

$$\frac{\partial F}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} + rx\frac{\partial F}{\partial x} - rF = 0,$$
$$F(T,x) = \Phi(x)$$

The Feynman-Kač representation formula is required to give the solution to the above system:

Proposition 5 ***Feynman-Kač***
(Björk 2004, §5.6) Assume that F is a solution to the boundary value problem

$$\begin{aligned}\frac{\partial F}{\partial t}(t,x) + \frac{1}{2}\sigma^2(t,x)\frac{\partial^2 F}{\partial x^2}(t,x) + \mu(t,x)\frac{\partial F}{\partial x}(t,x) - rF(t,x) &= 0,\\ F(T,x) &= \Phi(x)\end{aligned}$$

Assume furthermore that the process $\sigma(s,X_s)\frac{\partial F}{\partial x}(s,X_s)$ is in $\mathcal{L}^2$, where X is defined below. Then F has the representation

$$F(t,x) = e^{-r(T-t)}\mathbb{E}_{t,x}\left[\Phi(X_T)\right]$$

where X satisfies the SDE

$$\begin{aligned}dX_s &= \mu(s,X_s)ds + \sigma(s,X_s)dW_s,\\ X_t &= x\end{aligned}$$

Thus, after the EMM has been found, the solution to the pricing function is obtained using the above proposition. Equivalently, the pricing of the derivative under the unique EMM Q is accomplished using the following risk-neutral valuation proposition (in the one dimensional case):

Proposition 6 ***Risk-Neutral Valuation***
(Björk 2004, §15.2) Assuming the absence of arbitrage, the pricing function $F(t,x)$ of the claim with maturity T, $\Phi(X(T))$, is given by the formula

$$F(t,x) = e^{-r(T-t)}\mathbb{E}^{Q}_{t,x}\left[\Phi(X(T))\right],$$

where the dynamics of X under the EMM Q are given by

$$dX(t) = \left\{\mu(t,X(t)) - \lambda(t,X(t))\sigma(t,X(t))\right\}dt + \sigma(t,X(t))dW(t)$$

Here, W is a Q-Wiener process and the subscripts t and x indicate that $X(t) = x$. $\lambda(t,X(t))$ is the market price of risk process associated with $X(t)$ at time t, $\mu(t,X(t))$ and $\sigma(t,X(t))$ are the instantaneous expected return and volatility respectively. $\mathbb{E}^{Q}_{t,x}\left[\cdot\right]$ is the expectation at time t with starting value $X(t) = x$ with respect to the EMM Q.

Proposition 7 *(Björk 2004, §15.4) The martingale measure Q is characterized by any of the following equivalent facts:*

- *The local mean rate of return of any derivative process $\Pi(t)$ equals the short rate of interest, i.e. the $\Pi(t)$-dynamics have the following structural form under Q*

$$d\Pi(t) = r\Pi(t)dt + \sigma_\Pi(t)dW(t)$$

where W is a Q-Wiener process, and σ_Π is the same under Q as under P.

- *With Π as above, the process $\Pi(t)/B(t)$ is a Q-martingale, i.e. it has a zero drift term.*

In the standard Black-Scholes environment, the market model is arbitrage free. Since it is also complete, all claims can be perfectly hedged. The following lemma shows that hedging is equivalent to the existence of a stochastic integral representation of the normalized claim, $X/S_0(t)$ for $t \in [0, T]$. Here, $S_0(t)$ is a suitable numèraire asset and $\tilde{S}_i(t)$ refers to the normalized stock price process for $0 \leq i \leq N$ at time t.

Lemma 3 *(Björk 2004, §10.15) Consider a T-clam X. Fix a martingale measure Q and assume that the normalized claim, $X/S_0(t)$, is integrable. If the Q-martingale M, defined by*

$$M(t) = \mathbb{E}^Q\left[\frac{X}{S_0(T)}|\mathcal{F}_t\right], \tag{6.12}$$

admits an integral representation of the form

$$M(t) = x + \sum_{i=1}^{N}\int_0^t h_i(s)d\tilde{S}_i(s), \tag{6.13}$$

then X can be hedged using $S_i(t)$ for $0 \leq i \leq N$.
Furthermore, the replicating portfolio for $(h_1, h_2, \ldots, h_N)$ is given by (6.13) and h_0 is given by $h_0 = M(t) - \sum_{i=1}^{N} h_i(t)\tilde{S}_i(t)$.

Considering the case when there is a risk free asset and one traded asset, the above theorems are required to determine the hedge portfolio. Using the martingale representation theorem, there exists a process $g(t)$ such that $M(t)$ (as given by (6.12)) satisfies

$$dM(t) = g(t)dW(t)$$

where W is a Q-Wiener process. Under the EMM Q, the normalized stock price process, $\tilde{S}(t)$, is a martingale and therefore satisfies

$$d\tilde{S}(t) = \tilde{S}(t)\sigma dW(t)$$

Therefore,

$$dW(t) = \frac{1}{\sigma\tilde{S}}d\tilde{S}(t)$$

It is also clear from the lemma above that the model is complete if there exists a process $h_1(t)$ such that

$$dM(t) = h_1(t)dW(t)$$

Thus, using these two equations for the martingale $M(t)$, the hedge process $h_1(t)$ is found to be

$$h_1(t) = \frac{g(t)}{\sigma\tilde{S}(t)}$$

and

$$h_0 = M(t) - h_1(t)\tilde{S}(t)$$

So, in the ordinary Black-Scholes scenario where the market model is arbitrage-free and complete, the existence and uniqueness of a replicating portfolio suffices through the application of the martingale representation theorem and Girsanov's theorem.

6.3.3 Incomplete Markets

In an arbitrage-free market, there exists a market price of risk process which is common to the underlying and to all derivatives dependent on it. In a complete market, the price of any derivative can be uniquely determined by the requirement of absence of arbitrage and the uniqueness of the EMM. This means that the derivative can equally well be replaced by its replicating portfolio. The stochastic volatility model is that of an incomplete market since volatility is usually not a traded asset. There is another source of randomness in the model. The no-arbitrage requirement no longer ensures a unique price for the derivative will be obtained. This can be interpreted as the existence of several possible EMMs and associated market prices of risk. The price of the derivative is determined by two factors:

- The derivative must be priced in such a way that arbitrage is avoided. All derivatives must be priced by the same EMM. This property is established in the following proposition:

  Proposition 8 *(Björk 2004, §15.1) Assume the market for derivatives is arbitrage free. Then there exists a universal process $\lambda(t)$ such that, with probability 1, and for all t, we have*

  $$\frac{\mu_F - r}{\sigma_F} = \lambda(t)$$

  regardless of the specific choice of the derivative F.

- In an incomplete market, aggregate supply and demand has an effect on the price. This must be taken into consideration when selecting a particular EMM or equivalently, a market price of risk (since there is a one-to-one correspondence).

Essentially, the pricing procedure of Proposition 6 is still applicable. What remains to be determined is the market price of volatility risk, $\hat{\lambda}$. These class of models are parametric in nature. The most obvious way of determining the required parameters is to use existing markets prices. As soon as the parameter values are such that the models price derivatives consistently, the models can then be used to price and hedge more exotic options.

The calibration will be dealt with in more detail on a case by case basis in the chapters that follow.

Chapter 7

Hull-White Model

7.1 Introduction

This chapter will address the stochastic volatility model presented in (Hull & White 1987). It is a two-factor model in which the variance follows a lognormal stochastic process.

7.2 The Two Factor Model

Consider a derivative $f(S_t, V, t)$ at time t, where the underlying asset has price S_t and instantaneous variance $V = \sigma_t^2$, which obey the following stochastic processes:

$$dS = \phi S dt + \sigma S dW^1 \tag{7.1}$$

$$dV = \mu V dt + \xi V dW^2 \tag{7.2}$$

$$dW^1 dW^2 = \rho dt \tag{7.3}$$

where $\phi = \phi(S, \sigma, t)$, $\mu = \mu(\sigma, t)$ and $\xi = \xi(\sigma, t)$. dW^1 and dW^2 are standard Brownian motions with correlation ρ. Assume the risk free rate r is constant.

From Chapter 6, the equation satisfied by a derivative that is dependent on the underlying and its volatility or variance is given by:

$$\frac{\partial f}{\partial t} + \frac{1}{2} V S^2 \frac{\partial^2 f}{\partial S^2} + \rho\sigma\xi S V \frac{\partial^2 f}{\partial V \partial S} + \frac{1}{2}\xi^2 V \frac{\partial^2 f}{\partial V^2} + rS\frac{\partial f}{\partial S} + (\mu V - \hat{\lambda}\xi) V \frac{\partial f}{\partial V} - rf = 0,$$

where in the function $(\mu V - \hat{\lambda}\xi)$, $\hat{\lambda}(S, V, t)$ is the market price of volatility risk. The higher the value of $\hat{\lambda}$, the more adverse investors are to take on volatility risk. We assume for simplicity that it has the effect of being incorporated in the parameters and we redefine μ in the above equation.

Thus, any derivative $f(S, \sigma^2, t)$ dependent on the underlying asset and associated variance, satisfies the partial differential equation (PDE) and terminal condition $\Phi(S_T)$ at $t = T$:

$$\begin{aligned}\frac{\partial f}{\partial t} + \frac{1}{2}VS^2\frac{\partial^2 f}{\partial S^2} + \rho\sigma\xi SV\frac{\partial^2 f}{\partial V \partial S} + \frac{1}{2}\xi^2 V\frac{\partial^2 f}{\partial V^2} + rS\frac{\partial f}{\partial S} + \mu V\frac{\partial f}{\partial V} - rf =& 0\\ f(S_T, \sigma_T^2, T) =& \Phi(S_T)\end{aligned} \tag{7.4}$$

The risk-neutral valuation from Chapter 6 Proposition 6, is applied to the pricing of the derivative f (in the one dimensional case): Therefore, the price of the option can be expressed as:

$$f(S_t, \sigma_t^2, t) = e^{-r(T-t)} \int f(S_T, \sigma_T^2, T)p(S_T \left| S_t, \sigma_t^2 \right.)dS_T, \tag{7.5}$$

where
T is the maturity of the option;
S_t is the asset price at time t;
σ_t is the volatility at time t;
$p(S_T \left| S_t, \sigma_t^2 \right.)$ is the conditional pdf of S_T in a risk-neutral world given S_t and σ_t^2;
$f(S_T, \sigma_T^2, T)$ is the payoff of the option.

7.3 Pricing Under Zero Correlation

Hull and White initially use the simplifying assumption that the Brownian motions are uncorrelated and that the instantaneous expected return and volatility of the variance V are independent of S. In doing so, they obtain the option price in terms of an expansion using the moments of the mean variance, $\bar{V}$, as it is not possible to obtain an analytic form for the distribution.

Define $\bar{V}$ as the mean variance over the life of the option:

$$\bar{V} = \frac{1}{T-t}\int_t^T \sigma^2(\tau)d\tau \tag{7.6}$$

Consider the bivariate normal density function, $f : \mathbb{R}^2 \to \mathbb{R}$, given by

$$\begin{aligned}&f(x, y)\\ &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{x-\mu_1}{\sigma_1}\right)^2 - 2\rho\left(\frac{x-\mu_1}{\sigma_1}\right)\left(\frac{y-\mu_2}{\sigma_2}\right) + \left(\frac{y-\mu_2}{\sigma_2}\right)^2\right)\right]\end{aligned} \tag{7.7}$$

When $\rho = 0$, $f(x, y) = f_X(x)f_Y(y)$ where $f(x, y)$ is the bivariate normal density and

$$\begin{aligned}f_X(x) &= \frac{1}{\sqrt{2\pi}\sigma_1}\exp\left[-\frac{1}{2}\left(\frac{x-\mu_1}{\sigma_1}\right)^2\right]\\ f_Y(y) &= \frac{1}{\sqrt{2\pi}\sigma_2}\exp\left[-\frac{1}{2}\left(\frac{x-\mu_2}{\sigma_2}\right)^2\right]\end{aligned}$$

In order to calculate a conditional expectation, we require the following:

$$\begin{aligned}f_{Y|X}(y|x) &= \frac{f(x, y)}{\int_{-\infty}^{\infty} f(x, y)dy}\\ &= \frac{f(x, y)}{f_X(x)}\end{aligned}$$

for any x such that $f_X(x) > 0$. Using the fact that for any three related random variables x, y and z, the conditional density functions are related by convolution by

$$p(x|y) = \int g(x|z)h(z|y)dz$$

equation (7.5) can be simplified.

The distribution of S_T can be written as

$$p(S_T|\sigma_t^2, S_t) = \int g(S_T|S_t, \bar{V})h(\bar{V}|S_t, \sigma_t^2)d\bar{V}$$

Substituting this into (7.5) yields

$$\begin{aligned} f(S_t, \sigma_t^2, t) &= e^{-r(T-t)} \int\int f(S_T)g(S_T|S_t, \bar{V})h(\bar{V}|S_t, \sigma_t^2)dS_T d\bar{V} \\ &= \int \left(e^{-r(T-t)} \int f(S_T)g(S_T|S_t, \bar{V})dS_T \right) h(\bar{V}|S_t, \sigma_t^2)d\bar{V} \end{aligned}$$

where it is established that the inner integral is the Black-Scholes option price on a security with mean variance $\bar{V}$. Any path that V follows over the life of the option, whether stochastic or not, still has the same mean variance, $\bar{V}$. Thus, the lognormal distribution depends on the risk free rate, the initial stock price, term and $\bar{V}$. So, any path followed by V still leads to the same $\bar{V}$ and ultimately, the same terminal lognormal distribution.

So, it is clear that when S and V are instantaneously uncorrelated, the distribution of $\ln \frac{S_T}{S_t}$, conditional upon $\bar{V}$ is normal with mean $r(T-t) - \frac{1}{2}\bar{V}(T-t)$ and variance $\bar{V}(T-t)$. This is equivalent to Black-Scholes with time-dependent volatility: the constant volatility parameter σ is replaced by $\bar{\sigma} = \sqrt{\bar{V}}$.

To see this, consider the filtration generated by the stock price and the volatility:

$$\mathcal{F}_t = \sigma(S_u, V_u; 0 \le u \le t), \quad t \in [0, T]$$

Since the Brownian motions of the above processes are independent,

$$\mathcal{F}_t = \sigma(S_u; 0 \le u \le t) \vee \sigma(V_u; 0 \le u \le t), \quad t \in [0, T]$$

where $\mathcal{F} \vee \mathcal{G}$ is the smallest sigma-algebra containing all sets of $\mathcal{F}$ and $\mathcal{G}$. Denote the sigma-algebra generated by V from $t = 0$ to $t = T$ as $\sigma(V) = \sigma(V_u; 0 \le u \le T)$ and denote the payoff function as $f(S_T)$. So,

$$\mathcal{F}_t \subset \mathcal{F}_t \vee \sigma(V_u; t \le u \le T) = \sigma(V) \vee \sigma(S_u; 0 \le u \le t), \quad t \in [0, T]$$

Considering the arbitrage-free price at time-t and using iterated expectations, as well as the Markov property of the stock price,

$$\begin{aligned} \Pi^{\hat{\lambda}}_{f(S_T)}(t) &= e^{-r(T-t)}\mathbb{E}^{\hat{\lambda}}\left[f(S_T)\,|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}\mathbb{E}^{\hat{\lambda}}\left[\mathbb{E}^{\hat{\lambda}}\left[f(S_T)\,|\sigma(V) \vee \sigma(S_u; 0 \le u \le t)\right]|\mathcal{F}_t\right] \\ &= e^{-r(T-t)}\mathbb{E}^{\hat{\lambda}}\left[\mathbb{E}^{\hat{\lambda}}\left[f(S_T)\,|\sigma(V) \vee \sigma(S_t)\right]|\mathcal{F}_t\right] \end{aligned}$$

where the superscript stresses the fact that the EMM of choice is dependent on the parameter $\hat{\lambda}$. The inner expectation is the Black-Scholes computation with time-dependent volatility. In this case, the solution to the risk-neutral diffusion of the stock price

$$dS = rSdt + \sigma(t)SdW^1$$

is given by

$$S_T = S_0 \exp\left(\left(r - \frac{\bar{V}}{2}\right)T + \int_0^T \sigma(\tau)dW_\tau^1\right)$$

Conditional on S_0, the distribution of $\ln(S_T/S_0)$ is given by

$$\ln\left(\frac{S_T}{S_0}\right) \sim \Phi\left(\left(r - \frac{\bar{V}}{2}\right)T; \bar{V}T\right)$$

Therefore, the price of a derivative $\Pi_{f(S_T)}(t)$ with time-dependent volatility is given by the Black-Scholes price

$$C(S_t, r, \bar{V}, T, K) = S_t\Phi(d_1) - e^{-r(T-t)}K\Phi(d_2),$$

where

$$d_{1,2} = \frac{\ln\frac{S_t}{K} + \left(r \pm \frac{\bar{V}}{2}\right)(T-t)}{\sqrt{\bar{V}\,(T-t)}}.$$

Therefore, given the assumption of zero correlation, the price of the contingent claim at time t is given by

$$\Pi_{f(S_T)}^{\hat{\lambda}}(t) = e^{-r(T-t)}\mathbb{E}^{\hat{\lambda}}\left[\Pi_{f(S_T)}^{BS}(t)\,|\mathcal{F}_t\right]$$

An interesting result due to (Renault & Touzi 1996) gives the result that with $\rho = 0$, the implied curve from from any volatility process, is a smile. The full statement is as follows: *In a stochastic volatility model where $(\sigma_t)_{t\geq 0}$ and $(W_t)_{t\geq 0}$ are independent, suppose the risk premium process is a function of Y_t and t but not of X_t: $\hat{\lambda}_t = \hat{\lambda}(t, Y_t)$. Then, provided $\overline{\sigma^2}$ to be defined below, is an L^2 random variable, the implied volatility curve $I(K)$ for fixed t, x, T is a smile - that is, it is locally convex around the minimum $K_{min} = xe^{r(T-t)}$, which is the forward price of the stock.* Here, T, K and x are the maturity, strike and current stock price respectively. To retain generality,

$$\overline{\sigma^2} = \frac{1}{T-t}\int_t^T f(Y_s)^2\, ds$$

This result provides a sense of robustness to the general class of volatility models.

7.4 Pricing Under Non-Zero Correlation

Since the assumption of zero correlation between the underlying and its variance is empirically incorrect, a numerical procedure will be required to solve the PDE with $\rho \neq 0$. Consider the bivariate normal distribution function (7.7). The distributional properties of $\ln \frac{S_T}{S_t}$ conditional upon V depends upon the path followed by V, not just $\bar{V}$.

The first approach to be considered is to use the Antithetic Variable Technique (suggested by Hull and White) in a pure Monte Carlo simulation. An alternative Monte Carlo Simulation technique, Quasi-Monte Carlo, will also be used to generate future stock prices to price vanilla and exotic options.

7.4.1 Monte Carlo Simulation: Antithetic Variates Approach

Given the stochastic differential equations for the stock price and the volatility, Itô's Lemma (Björk 2004, §3.5) can be used to solve for both S and V at time T (the maturity of the option being priced). Given

$$\begin{aligned} dS &= \phi S dt + \sigma S dW^1 \\ dV &= \mu V dt + \xi V dW^2 \\ dW^1 dW^2 &= \rho dt \end{aligned}$$

as before, where $\sigma^2 = V$. Applying Itô's lemma can be shown to yield the following solutions:

$$\begin{aligned} S_T &= S_0 \exp\left(\left(r - \frac{V}{2}\right) T + W_T^1\right) \\ V_T &= V_0 \exp\left(\left(\mu - \frac{\xi^2}{2}\right) T + W_T^2\right) \end{aligned}$$

The standard approach to simulate S and V for all $t \in [0, T]$ is to discretize the time interval into n equally spaced smaller intervals Δt apart, where $\Delta t = T/n$. Since $W_t^1, W_t^2 \sim \Phi\left(0, \sqrt{t}\right)$, for $1 \leq i \leq n$:

$$S_i = S_{i-1} \exp\left(\left(r - \frac{V_{i-1}}{2}\right) \Delta t + u_i \sqrt{V_{i-1} \Delta t}\right) \tag{7.8}$$

$$V_i = V_{i-1} \exp\left(\left(\mu - \frac{\xi^2}{2}\right) \Delta t + \rho \xi u_i \sqrt{\Delta t} + \xi \sqrt{1 - \rho^2} v_i \sqrt{\Delta t}\right) \tag{7.9}$$

where $u_i, v_i \sim \Phi(0,1)$ are independent and ρ is the correlation between the stock price and the volatility. The antithetic variates technique increases the efficiency of Monte Carlo simulation by reducing the variance of the simulation estimates (Glasserman 2004). This particular technique introduces negative dependence between pairs of replications. In a simulation driven by standard normal random variables, antithetic variates can be implemented by pairing a sequence of i.i.d. (independent and identically distributed) $\Phi(0,1)$ variables with the negation of the sequence. For a distribution F symmetric about the origin, $F^{-1}(1-u)$ and $F^{-1}(u)$ have the same magnitudes but different signs, u being a uniform random variable over [0,1]. Essentially, a pair of simulations of the Brownian path is run using the original sequence and its reflection, resulting in a lower variance.

So, generating sequences u_i and v_i for $1 \leq i \leq n$, form the sequences $-u_i$ and $-v_i$ that are required in (7.8) and (7.9). Since the price of a call option with strike K and maturity T is to be estimated, the value of

$$e^{-rT}(S_n - K)^+$$

is to be determined as follows:

	u_i	$-u_i$
v_i	C_1	C_2
$-v_i$	C_3	C_4

for $1 \leq i \leq n$. Thus, the price of the option can be found from the average of the above estimates i.e. $\frac{1}{4}(C_1 + C_2 + C_3 + C_4)$. The parameters μ, ξ, ρ, σ_0 (and possibly $\hat{\lambda}$ if this is incorporated into the model) are usually determined from at-the-money European option price data. This process is referred to as *cross-sectional fitting*. Denote the set of unknown parameters as Υ. In order to determine Υ, a least-squares fit to the observed call option prices $C^{OBS}(K,T)$ for all strikes K and expiration dates T in some set κ and we solve

$$\min_{\Upsilon} \Sigma_{(K,T)\in\kappa} \left(C(K,T;\Upsilon) - C^{OBS}(K,T)\right)^2$$

7.4.2 Hybrid Quasi-Monte Carlo Simulation

Application of low-discrepancy sequences (LDS) to the generation of sample points for Monte Carlo sampling leads to quasi-Monte Carlo approaches. These methods have been found to be successful in high-dimensional integral problems that arise in computational finance (Cheng & Druzdzel 1986). Discrepancy is a measure of the non-uniformity of a sequence of points in the hypercube $[0,1]^d$, where d is the dimension of the problem. Given that the measure of star discrepancy is:

$$D_N^*(x_1,\ x_2,\ \ldots,\ x_N) = \sup_{0\leq v_j<1, j=1,\ \ldots,\ d} \left| \frac{1}{N} \sum_{i=1}^{N} \prod_{j=1}^{d} 1_{0\leq x_i^j < v_j} - \prod_{j=1}^{d} v_j \right|$$

where, for every subset E of $[0,1]^d$ of the form $[0,v_1) \times \ldots \times [0,v_d)$, we divide the number of points x_k in E by N and take the absolute difference between this quotient and the volume of E. The maximum distance is the star discrepancy D_N^*. A sequence of points in $[0,1]^d$ is a LDS if for any $N > 1$

$$D_N^*(x_1,\ x_2,\ \ldots,\ x_N) \leq c(d) \cdot \frac{(\ln N)^d}{N}$$

where $c(d)$ is a constant which depends upon the dimension d. By having the fraction of points within any subset E of $[0,1]^d$ of the form $[0,v_1)\times\ldots\times[0,v_d)$ be as close as possible to its volume, the LDSs will spread uniformly over $[0,1]^d$. The traditional Monte Carlo using (pseudo) random numbers, has a convergence rate of only $O\left(\frac{1}{\sqrt{N}}\right)$, which is independent of d and depends only of the number of simulations N. Quasi-Monte Carlo rate of convergence can be much faster with errors approaching size of $O\left(\frac{1}{N}\right)$ in optimal cases. The theoretic upper bound rate of convergence (or maximum error) for the multi-dimensional LDSs is of $O\left(\frac{(\ln N)^d}{N}\right)$ (Dias 2004).

Given a two-factor option pricing model, we require two matrices of $\Phi(0,1)$ numbers of size $(N \times M)$, where N is the number of sample paths and M is the number of points along each path (usually the number of days to expiry). Effectively, the dimension is $2 \times N \times M$ which is very large. In such cases, it becomes difficult to construct quasi-Monte Carlo point sets with meaningful equidistribution properties (Owen 1998).

We will use a method based on a Latin-Hypercube technique called *stratified sampling without replacement* (Vose 2000). It uses Monte Carlo methods to extend quasi-Monte Carlo methods to higher dimensional problems, thus creating a hybrid quasi-Monte Carlo technique. The procedure is as follows:

1. Construct a one-dimensional LDS (Faure, Halton or Sobol) of length N.

2. Use $U(0,1)$ random variates to permute the original sequence $2 \times M$ times to construct the columns of the matrices.

3. To obtain $\Phi(0,1)$ random variates, use Moro's inversion formula (Jackson & Staunton 2001, §12.3).

We will generate a Faure sequence in the above procedure (Faure 1982). Let p be the first prime number such that $p \geq d$ and p^m is the upper bound of the sample size. Let $c_{ij} = \binom{i}{j} \bmod p$, $0 \leq j \leq i \leq m$. The base p representation for $n = 0, 1, \ldots$ is

$$n = \sum_{i=0}^{m-1} a_i(n) p^i$$

where $a_i(n) \in [0, p)$ are integers. The first coordinate of the point x_n is given by

$$x_n^1 = \sum_{j=0}^{m-1} a_j(n) p^{-j-1}$$

The other coordinates are given by

$$\begin{cases} \bar{a_j}(n) = \sum_{l=j}^{m-1} c_{lj} a_l(n) \bmod p, & j \in \{0, 1, \ldots, m-1\}, \\ a_j(n) = \bar{a_j}(n), & j \in \{0, 1, \ldots, m-1\}, \\ x_n^i = \sum_{j=0}^{m-1} a_j(n) p^{-j-1}, & \end{cases}$$

in order of $i = 2, \ldots, d$. In our case, $p = 2$ and the VBA algorithm to generate such a sequence can be found in (Jackson & Staunton 2001, §12.3). To construct the columns of each of the matrices, we use the algorithm presented by (Dias 2004) to randomly permute the original sequence.

In order to obtain $\Phi(0,1)$ numbers without damaging the low-discrepancy properties (uniformity and order), we will use Moro's inversion formula presented in (Moro February 1995). Moro presented an algorithm that used the (Beasley & Springer 1977) algorithm for the central part of the Normal distribution and modelled the tails using truncated Chebyschev series. It divides the domain for $u \sim U[0,1]$ into two regions:

1. The central region of the distribution, $0.08 < u \leq 0.92$, is modelled as in Beasley and Springer;

2. The tails of the distribution, $u \leq 0.08$ or $u > 0.92$, are modelled with Chebyschev series.

The tail performance is important for out-the-money option problems since the option's exercise occurs at the more extreme values, emphasizing the weight of the tail towards the option's value.

Chapter 8

The Heston Model

8.1 Introduction

The two-factor model proposed in (Heston 1993) uses a solution technique based on Fourier transforms (characteristic functions). It allows for arbitrary correlation between the volatility and the returns process. The volatility follows an Ornstein-Uhlenbeck (OU) process. We will briefly review this process before proceeding.

8.2 The Mean Reverting Ornstein-Uhlenbeck Process

The mean reverting OU process is an Itô process with a a linear pull-back term in the drift. It is defined as a solution to

$$dY_t = \alpha\left(m - Y_t\right)dt + \beta dZ_t \tag{8.1}$$

where $(Z_t)_{t\geq 0}$ is a Brownian motion, α is the rate of mean reversion and m is the long-run mean of Y. To obtain a solution to this, first consider $e^{\alpha t}Y_t$. Taking the differential,

$$d\left(e^{\alpha t}Y_t\right) = \alpha e^{\alpha t}Y_t + e^{\alpha t}dY_t,$$

Therefore,

$$e^{\alpha t}dY_t = d\left(e^{\alpha t}Y_t\right) - \alpha e^{\alpha t}Y_t \tag{8.2}$$

Multiplying both sides of (8.1) by $e^{\alpha t}$, we get

$$e^{\alpha t}dY_t = e^{\alpha t}\alpha\left(m - Y_t\right)dt + e^{\alpha t}\beta dZ_t \tag{8.3}$$

Together with (8.2) implies

$$d\left(e^{\alpha t}Y_t\right) = \alpha m e^{\alpha t}dt + e^{\alpha t}\beta dZ_t$$

Solving for this given that $Y_0 = y$:

$$\begin{aligned}
e^{\alpha t} Y_t &= y + \int_0^t \alpha m e^{\alpha s} ds + \int_0^t \beta e^{\alpha s} dZ_s \\
\Rightarrow Y_t &= e^{-\alpha t} y + \int_0^t \alpha m e^{-\alpha(t-s)} ds + \int_0^t \beta e^{-\alpha(t-s)} dZ_s \\
&= e^{-\alpha t} y + m\left(1 - e^{-\alpha t}\right) + \int_0^t \beta e^{-\alpha(t-s)} dZ_s \\
&= m + e^{-\alpha t}(y - m) + \int_0^t \beta e^{-\alpha(t-s)} dZ_s
\end{aligned}$$

To consider the distributional properties, it is clear that the integral $\int_0^t \beta e^{-\alpha(t-s)} dZ_s$ has mean zero and variance $\mathbb{E}\left[\left(\int_0^t \beta e^{-\alpha(t-s)} dZ_s\right)^2\right]$. By the Itô Isometry (Oksendal 2004, §3.1.5),

$$\mathbb{E}\left[\left(\int_0^t \beta e^{-\alpha(t-s)} dZ_s\right)^2\right] = \int_0^t \left(\beta e^{-\alpha(t-s)}\right)^2 ds = \int_0^t e^{-2\alpha(t-s)} \beta^2 ds = \frac{\beta^2}{2\alpha}\left(1 - e^{-2\alpha t}\right).$$

Hence, $Y_t \sim N\left(m + e^{-\alpha t}(y - m), \frac{\beta^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)\right)$.

8.3 Stochastic Volatility Model

The following SDEs are assumed to model the processes of spot asset S_t and the volatility $\sqrt{V_t}$:

$$\begin{aligned}
dS_t &= \mu S dt + \sqrt{V_t} S dW_t^1 && (8.4) \\
d\sqrt{V_t} &= -\beta\sqrt{V_t} dt + \delta dW_t^2 && (8.5) \\
dW_t^1 dW_t^1 &= \rho dt && (8.6)
\end{aligned}$$

Using Itô's Lemma, let $g(t, x) = x^2$ to determine the process followed by the variance, V_t.

$$\begin{aligned}
g_t &= 0, \\
g_x &= 2x, \\
g_{xx} &= 2
\end{aligned}$$

So,

$$\begin{aligned}
dg(t, x) &= g_t dt + g_x dX + \frac{1}{2} g_{xx} (dX)^2 \\
\Rightarrow dV_t &= 2\sqrt{V_t}\left(-\beta\sqrt{V_t} dt + \delta dW_t^2\right) + \delta^2 dt \\
&= -2\beta V_t dt + 2\delta\sqrt{V_t} dW_t^2 + \delta^2 dt \\
&= \left(\delta^2 - 2\beta V_t\right) dt + 2\delta\sqrt{V_t} dW_t^2
\end{aligned}$$

Rewriting this as a square-root process:

$$dV_t = \kappa(\theta - V_t) dt + \sigma\sqrt{V_t} dW_t^2 \qquad (8.7)$$

Assume a constant interest rate r. Therefore, the price at time t of a discount bond that mature at time $t+\tau$ is given by

$$Z(t, t+\tau) = e^{-r\tau}$$

As standard arbitrage arguments have already shown in Chapter 6, the value of any contingent claim $P(S, V, t)$ must satisfy the following PDE:

$$\begin{aligned}\frac{\partial P}{\partial t} + \frac{1}{2} V S^2 \frac{\partial^2 P}{\partial S^2} + \rho\sigma S V \frac{\partial^2 P}{\partial S \partial V} + \frac{1}{2}\sigma^2 V \frac{\partial^2 P}{\partial V^2} + r\left(S\frac{\partial P}{\partial S} - P\right) \\ + \left(\kappa\left(\theta - V\right) - \hat{\lambda}(S, V, t)\right)\frac{\partial P}{\partial V} = 0\end{aligned}$$

As before, $\hat{\lambda}(S, V, t)$ is the market price of volatility risk which is independent of the particular contingent claim. This parameter can be obtained from an existing price and used to price all other claims. The model selects a functional form of $\hat{\lambda}(S, V, t) = \hat{\lambda} V$. So, the PDE a European call option, $C(S, V, t)$, with strike K and maturity T satisfies is:

$$\begin{aligned}\frac{\partial C}{\partial t} + \frac{1}{2} V S^2 \frac{\partial^2 C}{\partial S^2} + \rho\sigma S V \frac{\partial^2 C}{\partial S \partial V} + \frac{1}{2}\sigma^2 V \frac{\partial^2 C}{\partial V^2} + r\left(S\frac{\partial C}{\partial S} - C\right) \\ + \left(\kappa\left(\theta - V\right) - \hat{\lambda} V\right)\frac{\partial C}{\partial V} = 0\end{aligned} \tag{8.8}$$

subject to the following boundary conditions:

$$\begin{aligned}C(S, V, T) &= (S_T - K)^+, \\ C(0, V, T) &= 0, \\ \frac{\partial C}{\partial S}(\infty, V, t) &= 1, \\ rS\frac{\partial C}{\partial S}(S, 0, t) + \kappa\theta\frac{\partial C}{\partial V}(S, 0, t) - rC(S, 0, t) + \frac{\partial C}{\partial t}(S, 0, t) &= 0, \\ C(S, \infty, t) &= S.\end{aligned}$$

By analogy with Black-Scholes, a solution of the form

$$C(S, V, t) = SP^1 - Z(t, T)KP^2 \tag{8.9}$$

is proposed. The first term is the present value of the spot upon optimal exercise and the second term is the present value of the strike payment. Both P^1 and P^2 must satisfy (8.8). Let $x = \ln S$. Then, as has been previously shown:

$$dx_t = \left(\mu - \tfrac{V}{2}\right)dt + \sqrt{V}dW_t^1 \tag{8.10}$$

(8.8) can be rewritten in terms of x as:

$$\begin{aligned}\frac{\partial C}{\partial t} + \frac{1}{2} V \frac{\partial^2 C}{\partial x^2} + \rho\sigma V \frac{\partial^2 C}{\partial x \partial V} + \frac{1}{2}\sigma^2 V \frac{\partial^2 C}{\partial V^2} + \left(r - \tfrac{V}{2}\right)\frac{\partial C}{\partial x} - rC \\ + \left(\kappa\left(\theta - V\right) - \hat{\lambda} V\right)\frac{\partial C}{\partial V} = 0\end{aligned} \tag{8.11}$$

and the solution (8.9) becomes

$$C(x,V,t) = e^x P^1 - Z(t,T) K P^2 \tag{8.12}$$

Finding all the required partial derivatives of $C(x,V,t)$:

$$\begin{aligned}
C_x &= e^x \left(P^1 + P^1_x\right) - ZKP^2_x, \\
C_{xx} &= e^x \left(P^1 + 2P^1_x + P^1_{xx}\right) - ZKP^2_{xx}, \\
C_V &= e^x P^1_V - ZKP^2_V, \\
C_{VV} &= e^x P^1_{VV} - ZKP^2_{VV}, \\
C_{Vx} &= e^x \left(P^1_V + P^1_{Vx}\right) - ZKP^2_{Vx}, \\
C_t &= e^x P^1_t - ZKP^2_t - rZKP^2
\end{aligned}$$

where $Z = Z(t,T) = e^{-r(T-t)}$.

Substituting into (8.11), we get

$$\begin{aligned}
e^x P^1_t - ZKP^2_t - rZKP^2 + \frac{1}{2}V\left(e^x\left(P^1 + 2P^1_x + P^1_{xx}\right) - ZKP^2_{xx}\right) \\
+\rho\sigma V\left(e^x\left(P^1_V + P^1_{Vx}\right) - ZKP^2_{Vx}\right) + \frac{1}{2}\sigma^2 V\left(e^x P^1_{VV} - ZKP^2_{VV}\right) \\
+\left(r - \tfrac{V}{2}\right)\left(e^x\left(P^1 + P^1_x\right) - ZKP^2_x\right) - r\left(e^x P^1 - ZKP^2\right) \\
+\left(\kappa\left(\theta - V\right) - \hat{\lambda}V\right)\left(e^x P^1_V - ZKP^2_V\right) = 0
\end{aligned}$$

Gathering terms in P^1 and P^2:

$$\begin{aligned}
e^x\left[P^1_t + \left(r + \tfrac{V}{2}\right)P^1_x + \left(\kappa\theta - \left(\kappa + \hat{\lambda} - \rho\sigma\right)V\right)P^1_V + \frac{1}{2}VP^1_{xx} + \rho\sigma VP^1_{Vx} + \frac{1}{2}\sigma^2 VP^1_{VV}\right] \\
-ZK\left[P^2_t + \left(r - \tfrac{V}{2}\right)P^2_x + \left(\kappa\left(\theta - V\right) - \hat{\lambda}V\right)P^2_V + \frac{1}{2}VP^2_{xx} + \rho\sigma VP^2_{Vx} + \frac{1}{2}\sigma^2 VP^2_{VV}\right] = 0
\end{aligned}$$

So, the proposed solutions, P^1 and P^2 must satisfy the PDEs:

$$\begin{aligned}
\frac{1}{2}V\frac{\partial^2 P^j}{\partial x^2} + \rho\sigma V\frac{\partial^2 P^j}{\partial x\partial V} + \frac{1}{2}\sigma^2 V\frac{\partial^2 P^j}{\partial V^2} + \left(r + u_j V\right)\frac{\partial P^j}{\partial x} \\
+\left(a - b_j V\right)\frac{\partial P^j}{\partial V} + \frac{\partial P^j}{\partial t} = 0,
\end{aligned} \tag{8.13}$$

for $j = 1,2$, where $u_1 = \frac{1}{2}$, $u_2 = -\frac{1}{2}$, $a = \kappa\theta$, $b_1 = \kappa + \hat{\lambda} - \rho\sigma$ and $b_2 = \kappa + \hat{\lambda}$, subject to the terminal condition:

$$P^j\left(x,V,T\right) = 1_{\{x \geq \ln K\}}. \tag{8.14}$$

Thus, for $0 < t < T$

$$P^j(x,V,t) = \mathbb{E}^Q_t\left[1\left|x \geq \ln K\right.\right]. \tag{8.15}$$

The above PDE (8.13) is the Fokker-Plank equation (Kolmogorov forward equation). Equation (8.14) can be seen as the conditional probability that the option expires in the money.

8.4 Solution Technique: Fourier Transform

Define the Fourier Transform and inverse transform of a function $f(x)$, respectively as follows (James 1999):

Definition 9

$$\mathcal{F}\left\{f(x)\right\} = \int_{-\infty}^{\infty} f(x)e^{-i\phi x}dx = F(\phi) \tag{8.16}$$

$$\mathcal{F}^{-1}\left\{F(\phi)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\phi)e^{i\phi x}d\phi = f(x) \tag{8.17}$$

where $f(x)$ is absolutely integrable for all $t \in \mathbb{R}$ and has at most a finite number of maxima and minima, and a finite number of discontinuities in any finite interval, $i = \sqrt{-1}$ and ϕ is the transform variable.[1] In the following analysis, it is assumed that the Fourier transform variable $\phi = \phi_r + i\phi_i$, where ϕ_r and ϕ_i are real, is complex in nature. This is done to ensure the existence of the transformation of the payoff function. In this instance, the *generalized* Fourier inversion formula is required:

$$\mathcal{F}^{-1}\left\{F(\phi)\right\} = \frac{1}{2\pi}\int_{i\phi_i-\infty}^{i\phi_i+\infty} F(\phi)e^{i\phi x}d\phi \tag{8.18}$$

In typical option payoffs, the Fourier transform of the payoff will only exist if ϕ_i is restricted to a strip of regularity, $\alpha < \phi_i < \beta$. Since the payoff depends only on the underlying, a one-dimensional transform is required which will enable the two-dimensional PDE to be solved by means of one-dimensional integration.

8.4.1 The Direct Application of the Fourier Technique: Standard Black-Scholes Model

To gain insight into the effectiveness of the Fourier transform and inversion formulae, a brief outline of the process will be described below. This facilitates the reasoning behind application of such a technique. We will consider the standard Black-Scholes pricing model, where the underlying S follows the log-normal diffusion process with constant volatility σ and risk free rate r. Let the log-price process be represented by x. So,

$$\begin{aligned} dS &= rSdt + \sigma SdZ \\ \Rightarrow dx &= \left(r - \frac{\sigma^2}{2}\right)dt + \sigma dZ \end{aligned}$$

[1] This definition may vary in terms of:

1. The signs of the exponential being reversed. i.e.

$$\mathcal{F}\left\{f(x)\right\} = \int_{-\infty}^{\infty} f(x)e^{i\phi x}dx = F(\phi), \qquad \mathcal{F}^{-1}\left\{F(\phi)\right\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\phi)e^{-i\phi x}d\phi = f(x)$$

2. The constant $\frac{1}{2\pi}$ may be split symmetrically as $\frac{1}{\sqrt{2\pi}}$ in front of (8.16) and (8.17):

$$\mathcal{F}\left\{f(x)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{\pm i\phi x}dx = F(\phi), \qquad \mathcal{F}^{-1}\left\{F(\phi)\right\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(\phi)e^{\mp i\phi x}d\phi = f(x)$$

The pricing function of a standard European call option on the underlying, $C(x,t)$, is a solution to the Black-Scholes PDE with boundary condition:

$$C_t + \left(r - \frac{\sigma^2}{2}\right) C_x + \frac{\sigma^2}{2} C_{xx} - rC = 0 \tag{8.19}$$

$$C(x,T) = (e^{x_T} - K)^+ \tag{8.20}$$

Here, $0 \leq t \leq T$, T being the expiry and K, the strike.

Since the payoff depends only on the underlying, a one-dimensional transform is required. Using a variation of Definition (9), the transform and inverse transform of the pricing function are:

$$\widetilde{C}(\phi,t) = \int_{-\infty}^{\infty} e^{i\phi x} C(x,t) dx \tag{8.21}$$

$$C(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\phi x} \widetilde{C}(\phi,t) d\phi \tag{8.22}$$

Substituting (8.22) into (8.19), we get

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\phi x} \left(\widetilde{C_t} + \left(r - \frac{\sigma^2}{2}\right) \widetilde{C_x} + \frac{\sigma^2}{2} \widetilde{C_{xx}} - r\widetilde{C}\right) d\phi = 0$$

$$\Rightarrow \widetilde{C_t} + \left(r - \frac{\sigma^2}{2}\right) \widetilde{C_x} + \frac{\sigma^2}{2} \widetilde{C_{xx}} - r\widetilde{C} = 0 \tag{8.23}$$

In the above PDE, the notation $\widetilde{C_x} = \widetilde{\frac{\partial C}{\partial x}}$, refers to the Fourier transform of the partial derivative. The requirement is to evaluate the Fourier transform of the partial derivatives: C_x, C_{xx}. Since the Fourier transform is performed in the variable x for fixed t, the transform of the partial derivative in t will remain a partial derivative in t (more precisely, the Fourier transform of $\frac{\partial C}{\partial t}$, $\widetilde{\frac{\partial C}{\partial t}}$ is $\frac{\partial \widetilde{C}}{\partial t}$).

Using integration by parts and recalling that $\phi = \phi_r + i\phi_i$,

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{i\phi x} \frac{\partial C}{\partial x} dx &= \lim_{\alpha\to\infty} e^{i\phi x} C(x,t)\Big|_{-\alpha}^{\alpha} + i\phi \int_{\infty}^{\infty} e^{i\phi x} C(x,t) dx \\
&= \lim_{\alpha\to\infty} e^{i(\phi_r + i\phi_i)x} C(x,t)\Big|_{-\alpha}^{\alpha} + i\phi \int_{\infty}^{\infty} e^{i\phi x} C(x,t) dx \\
&= \lim_{\alpha\to\infty} e^{i\phi_r x} e^{-\phi_i x} C(x,t)\Big|_{-\alpha}^{\alpha} + i\phi \int_{\infty}^{\infty} e^{i\phi x} C(x,t) dx \\
&= i\phi\widetilde{C}
\end{aligned}$$

In the above limit, as $x \to \pm\infty$, the term $e^{i\phi_r x}$ behaves as an oscillating constant. For the limit to exist, we require $\phi_i > 0$ (this proves to be consistent with the required strip of regularity of the option payoff to be demonstrated below). We also have that the option price, $C(x,t)$, tends to zero when the asset's price tends to zero (or the logarithm of the asset's price tends to $-\infty$) and $C(x,t)$ tends to the asset's price when this tends to ∞ (or the logarithm of the asset's price tends to ∞). Consequently, the limit goes to zero.

For the second transform, using integration by parts and the fact that $\frac{\partial C}{\partial x}$ tends to ∞ when the asset's price tends to ∞ and zero when the asset's price tends to zero:

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{i\phi x} \frac{\partial^2 C}{\partial x^2} dx &= \lim_{\alpha \to \infty} e^{i\phi x} \frac{\partial C}{\partial x}\bigg|_{-\alpha}^{\alpha} + i\phi \int_{-\infty}^{\infty} e^{i\phi x} \frac{\partial C}{\partial x} dx \\
&= (i\phi)^2 \widetilde{C} \\
&= -\phi^2 \widetilde{C}
\end{aligned}$$

The above limit tends to zero for the same reasons as in the first case. We require $\phi_i > 0$.

We now substitute the above two transformations into (8.23) and get

$$\widetilde{C}_t + \left[\left(r - \tfrac{\sigma^2}{2}\right) i\phi - \tfrac{\sigma^2}{2}\phi^2 - r\right] \widetilde{C} = 0$$

The above equation is an ODE (variable separable) in t for every fixed ϕ. Thus,

$$\frac{d\widetilde{C}}{dt} = \left[r - \left(r - \tfrac{\sigma^2}{2}\right) i\phi + \tfrac{\sigma^2}{2}\phi^2\right] \widetilde{C}$$

Solving this

$$\begin{aligned}
\frac{d\widetilde{C}}{\widetilde{C}} &= \left[r - \left(r - \tfrac{\sigma^2}{2}\right) i\phi + \tfrac{\sigma^2}{2}\phi^2\right] dt \\
\Rightarrow \widetilde{C}(\phi, t) &= \widetilde{C}(\phi, T) \exp\left[\left(-r + \left(r - \tfrac{\sigma^2}{2}\right) i\phi - \tfrac{\sigma^2}{2}\phi^2\right)(T - t)\right]
\end{aligned}$$

where $\widetilde{C}(\phi, t)$ and $\widetilde{C}(\phi, T)$ is the Fourier transform of the option price and payoff respectively. First finding $\widetilde{C}(\phi, T)$ and recalling that $\phi = \phi_r + i\phi_i$:

$$\begin{aligned}
\widetilde{C}(\phi, T) &= \int_{-\infty}^{\infty} e^{i\phi x} (e^x - K)^+ dx \\
&= \int_{\ln K}^{\infty} e^{i\phi x} e^x dx - K \int_{\ln K}^{\infty} e^{i\phi x} dx \\
&= \frac{1}{1 + i\phi} \lim_{\alpha \to \infty} e^{x(1+i\phi)}\bigg|_{\ln K}^{\alpha} - \frac{K}{i\phi} \lim_{\alpha \to \infty} e^{i\phi x}\big|_{\ln K}^{\alpha}
\end{aligned}$$

The upper limit above does not exist unless $\phi_i > 1$ in the first case and unless $\phi_i > 0$ in the second. Applying this restriction ($\phi_i > 1$),

$$\begin{aligned}
\widetilde{C}(\phi, T) &= \frac{1}{1 + i\phi}\left(0 - K^{1+i\phi}\right) - \frac{K}{i\phi}\left(0 - K^{i\phi}\right) \\
&= \frac{-i\phi K^{1+i\phi} + K^{1+i\phi} + i\phi K^{1+i\phi}}{i\phi\,(1 + i\phi)} \\
&= -\frac{K^{1+i\phi}}{\phi^2 - i\phi} \qquad (8.24)
\end{aligned}$$

The final step is to invert $\widetilde{C}(\phi, t)$, the option price time t. We require the generalized Fourier transform since ϕ is complex with $\phi_i > 1$:

$$\begin{aligned}
C(x, t) &= \frac{1}{2\pi} \int_{i\phi_i - \infty}^{i\phi_i + \infty} e^{-i\phi x} \widetilde{C}(\phi, T) \exp\left[\left(-r + \left(r - \tfrac{\sigma^2}{2}\right) i\phi - \tfrac{\sigma^2}{2}\phi^2\right)(T - t)\right] d\phi \\
&= \frac{e^{-r(T-t)}}{2\pi} \int_{i\phi_i - \infty}^{i\phi_i + \infty} -\frac{K^{1+i\phi}}{\phi^2 - i\phi} \exp\left[-i\phi x + \left(r - \tfrac{\sigma^2}{2}\right) i\phi\,(T - t) - \tfrac{\sigma^2}{2}\phi^2\,(T - t)\right] d\phi
\end{aligned}$$

Simplifying the transformed payoff using partial fractions, for a and b constants, we get

$$\begin{aligned}
\frac{-K^{1+i\phi}}{\phi(\phi-i)} &= K^{1+i\phi}\left[\frac{a}{\phi}+\frac{b}{\phi-i}\right] \\
-1 &= (a+b)\,\phi - ia \\
\Rightarrow a &= -1, \\
\text{and } b &= i
\end{aligned}$$

Therefore,

$$\begin{aligned}
C(x,t) &= \frac{Ke^{-r(T-t)}}{2\pi}\int_{i\phi_i-\infty}^{i\phi_i+\infty}\frac{iK^{i\phi}}{\phi-i}\exp\left[-i\phi x+\left(r-\frac{\sigma^2}{2}\right)i\phi\,(T-t)-\frac{\sigma^2}{2}\phi^2\,(T-t)\right]d\phi \\
&\quad-\frac{Ke^{-r(T-t)}}{2\pi}\int_{i\phi_i-\infty}^{i\phi_i+\infty}\frac{iK^{i\phi}}{\phi}\exp\left[-i\phi x+\left(r-\frac{\sigma^2}{2}\right)i\phi\,(T-t)-\frac{\sigma^2}{2}\phi^2\,(T-t)\right]d\phi \\
&= \frac{Ke^{-r(T-t)}}{2\pi}\left(\Re(I_1)+\Re(I_2)\right)
\end{aligned}$$

The remainder of the calculation involves calculating the real part of the integrals as shown above. A contour integral is required. The solution to the stochastic volatility model will be provided in detail in section (8.4.3). The following pricing definition and theorems summarize the above results for general pricing procedures.

Definition 10 *(Grimmet & Stirzaker 2001) The characteristic function of X is the function $\phi:\mathbb{R}\longrightarrow\mathbb{C}$ defined by*

$$\psi(\phi)=\mathbb{E}\left[e^{i\phi X}\right],\quad i=\sqrt{-1}$$

So, given the risk-neutral density and distribution function $p_T(x)$ and $P_T(x)$ respectively, of the diffusion process $x(T)$, we have

$$\begin{aligned}
\psi_T(\phi) &= \int_{-\infty}^{\infty} e^{i\phi x}\,dP_T(x) \\
&= \int_{-\infty}^{\infty} e^{i\phi x}\,p_T(x)dx
\end{aligned}$$

The characteristic function always exists and the defining integral converges absolutely. It is also uniformly continuous in ϕ. From the above definition, $\psi(\phi)$, it can be seen that $\psi(\phi)$ and $\psi(-\phi)$ are conjugate quantities (Kendall 1945, §4.3):

$$\Re(\psi(\phi)) = \frac{1}{2}\left(\psi(\phi)+\psi(-\phi)\right) \tag{8.25}$$

$$\Im(\psi(\phi)) = \frac{1}{2i}\left(\psi(\phi)-\psi(-\phi)\right) \tag{8.26}$$

Another interesting result is given by the *Inversion Theorem*:

$$P_T(x)-P_T(0)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\psi_T(\phi)\frac{1-e^{-ix\phi}}{i\phi}d\phi$$

Characteristic functions, $\psi(\phi)$, are continuous in ϕ and defined in every finite ϕ interval. It is also the case that $\psi(0) = 1$.

Let the function $f(x)$ represent a European option payoff that depends only the value of the underlying at maturity of the option and $x(T) = \ln S(T)$.

Theorem 10 ***The Characteristic Formula***
(Sepp 2003) We assume that $x(T)$ has the analytic characteristic function $\psi_T(\phi)$ with the strip of regularity $S_\phi = \{\phi : \alpha < \phi_i < \beta\}$. Next we assume that $e^{-\phi_i x} f(x) \in L^1(\mathbb{R})$ where ϕ_i is located in the payoff strip S_f with transform $F(\phi)$, $\phi_i \in S_f$.

Then, if $S_F = S_f \cap S_\phi$ is not empty, the option value is given by

$$f(x(t)) = \frac{e^{-r(T-t)}}{2\pi} \int_{i\phi_i-\infty}^{i\phi_i+\infty} \psi_T(-\phi) F(\phi) d\phi \tag{8.27}$$

where $\phi \in S_F = S_f \cap S_\phi$.

The derivation of the result is from risk-neutral pricing. Using the definition of the characteristic function, the generalized Fourier inversion formula and Fubini's theorem,

$$\begin{aligned} f(x(t)) &= \mathbb{E}^Q \left[e^{-r(T-t)} f(x(T)) \right] \\ &= e^{-r(T-t)} \mathbb{E}^Q \left[\frac{1}{2\pi} \int_{i\phi_i-\infty}^{i\phi_i+\infty} e^{-i\phi x(T)} F(\phi) d\phi \right] \\ &= \frac{e^{-r(T-t)}}{2\pi} \int_{i\phi_i-\infty}^{i\phi_i+\infty} \mathbb{E}^Q \left[e^{-i\phi x(T)} \right] F(\phi) d\phi \\ &= \frac{e^{-r(T-t)}}{2\pi} \int_{i\phi_i-\infty}^{i\phi_i+\infty} \psi_T(-\phi) F(\phi) d\phi \end{aligned}$$

The whole integrand exists if $\phi \in S_F$.

8.4.2 Application of the Characteristic Function

Analysis follows that of (Attari 2004). In this section and the remainder of the chapter, the Fourier transform of Definition (9) will be used. Considering the case of the standard Black-Scholes pricing model or possibly, a variation thereof (which may include stochastic volatility). Let $S_T = S_t e^{r(T-t)+x}$, where where S follows a log-normal diffusion equation with constant risk free rate r and x accounts for the volatility process. The price of a European call option, $C(t, S_t; T, K)$, with strike K and maturity T, can be expressed as

$$\begin{aligned} C(t, S_t; T, K) &= e^{-r(T-t)} \mathbb{E}_t^Q \left[(S_T - K)^+ \right] \\ &= e^{-r(T-t)} \mathbb{E}_t^Q \left[S_T \,| S_T > K \right] - e^{-r(T-t)} \mathbb{E}_t^Q \left[K \,| S_T > K \right] \\ &= S_t \mathbb{E}_t^Q \left[e^{x(t,T)} \,| x(t,T) \geq n \right] - e^{-r(T-t)} \mathbb{E}_t^Q K \left[1 \,| x(t,T) \geq n \right] \\ &= S_t \int_n^\infty e^{x(t,T)} p(x(t,T)) dx(t,T) - e^{-r(T-t)} K \int_n^\infty p(x(t,T)) dx(t,T) \\ &= S_t P^1 - e^{-r(T-t)} K P^2 \end{aligned}$$

where $p(x)$ is the risk-neutral density associated with x and we define the following:

$$n = \ln\left(\frac{e^{-r(T-t)}K}{S_t}\right)$$
$$P^1 = \int_n^\infty e^{x(t,T)}p(x(t,T))dx(t,T)$$
$$P^2 = \int_n^\infty p(x(t,T))dx(t,T)$$

It is well-known that $0 \leq P^1 \leq 1$ and since $e^x p(x) \geq 0$ for all x, the product can be considered to be a probability density function. From the above analysis, it is clear that the choice of solution (8.9) is justified. Similarly, the expressions for P^j, $j = 1, 2$ above, are analogous to that in the stochastic volatility case, (8.15).

If we let $\psi_1(\phi)$ and $\psi_2(\phi)$ be the characteristic functions of the density functions $e^x p(x)$ and $p(x)$ respectively, then we have

$$\psi_1(\phi) = \int_{-\infty}^\infty e^{i\phi x}e^x p(x)dx,$$
$$\psi_2(\phi) = \int_{-\infty}^\infty e^{i\phi x}p(x)dx \tag{8.28}$$

Substituting into P^j for $j = 1, 2$:

$$P^j = \int_n^\infty \left(\frac{1}{2\pi}\int_{-\infty}^\infty \psi_j(\phi)e^{-i\phi x}d\phi\right)dx$$
$$= \frac{1}{2\pi}\int_{-\infty}^\infty \psi_j(\phi)\left(\int_n^\infty e^{-i\phi x}dx\right)d\phi$$

Since the integration variable ϕ must be complex, the solutions, P^j are given by the generalized Fourier transform:

$$P^j = \frac{1}{2\pi}\int_{i\phi_i-\infty}^{i\phi_i+\infty} \psi_j(\phi)\left(\int_n^\infty e^{-i\phi x}dx\right)d\phi \tag{8.29}$$

Here, the integration variable, ϕ is in the complex plane. ϕ_i is required to lie within certain bounds on the imaginary axis (strip of regularity). In the second line, the order of integration has been changed using Fubini's theorem (Rogers & Williams 2000, §II.12). As a solution, we require the real part of (8.29).

In many financial models, the existence of an explicit formula for the characteristic function exists. Given these functions, the prices of a wide range of options, dependent on the underlying, can be computed. These prices are Fourier-inversion integrals which are numerically evaluated.

8.4.3 Solution to the Stochastic Volatility Process

Given that V_t and x_t follow the risk-neutral diffusions

$$dx_t = (r + u_j V)\,dt + \sqrt{V}dW_t^1,$$
$$dV_t = (a - b_j V)\,dt + \sigma\sqrt{V_t}dW_t^2$$

Consider a twice-differetiable function $f(x,V,t)$ that is given by:

$$f(x,V,t)=\mathbb{E}\left[g\left(x(T),V(T)\right)|x(t),V(t)\right] \tag{8.30}$$

for some function $g\left(x(T),V(T)\right)$ where $T>t$. Using Itô's Lemma,

$$\begin{aligned}df=&\left(\frac{1}{2}V\frac{\partial^2 f}{\partial x^2}+\rho\sigma V\frac{\partial^2 f}{\partial x\partial V}+\frac{1}{2}\sigma^2 V\frac{\partial^2 f}{\partial V^2}+\left(r+u_jV\right)\frac{\partial f}{\partial x}+\left(a-b_j\right)V\frac{\partial f}{\partial V}+\frac{\partial f}{\partial t}\right)dt\\&+\left(r+u_jV\right)\frac{\partial f}{\partial x}dW^1+\left(a-b_jV\right)\frac{\partial f}{\partial V}dW^2\end{aligned}$$

By iterated expectations, f must be a martingale:

$$\begin{aligned}\mathbb{E}\left[f(x,V,t)\,|\mathcal{F}_u\right]&=\mathbb{E}\left[\mathbb{E}\left[g(x(T),V(T))\,|\mathcal{F}_t\right]|\mathcal{F}_u\right]\\&=\mathbb{E}\left[g(x(T),V(T))\,|\mathcal{F}_u\right]\end{aligned}$$

for $0\leq u\leq t$ with $(\mathcal{F}_t)_{t\geq 0}$ the filtration that contains all information generated by the correlated Brownian motions. So it is clear that f, a conditional expectation, is a martingale and therefore

$$\mathbb{E}\left[df\right]=0$$

So, from the above equation, the dt term must be zero. This yields the forward Kolmogorov equation:

$$\frac{1}{2}V\frac{\partial^2 f}{\partial x^2}+\rho\sigma V\frac{\partial^2 f}{\partial x\partial V}+\frac{1}{2}\sigma^2 V\frac{\partial^2 f}{\partial V^2}+\left(r+u_jV\right)\frac{\partial f}{\partial x}+\left(a-b_jV\right)\frac{\partial f}{\partial V}+\frac{\partial f}{\partial t}=0 \tag{8.31}$$

From (8.30), we have the terminal condition:

$$f(x,V,T)=g(x,V)$$

If we let

$$g(x,V,\phi)=e^{i\phi x} \tag{8.32}$$

then, the solution is the characteristic function (as can be seen from Definition (10)). Thus, for $j=1,2$,

$$f_j(\phi):=f_j(x,V,t;\phi)=\psi_j(\phi)$$

from (8.28).

Using the fact that the Fourier transform and the solutions, P^j for $j=1,2$, satisfy the Kolmogorov forward equation, we can solve explicitly for $f_j(\phi)$ and invert them to obtain the required probabilities $P^j(x,V,t)$, subject to the terminal condition from (8.30)

$$f_j(\phi)=e^{i\phi x} \tag{8.33}$$

We are required to find the Fourier transform of the terminal condition as given by (8.14), as done in

(8.29) is

$$
\begin{aligned}
\widetilde{P}_j(\phi) &:= \widetilde{P}_j(x,V,T;\phi) \\
&= \int_{-\infty}^{\infty} e^{-i\phi x} P^j(x,V,T)dx \\
&= \int_{-\infty}^{\infty} e^{-i\phi x} 1_{\{x\geq \ln K\}}dx \\
&= \lim_{\alpha\to\infty} \int_{\ln K}^{\alpha} e^{-i\phi x}dx \\
&= \frac{-1}{i\phi} \lim_{\alpha\to\infty} e^{-i\phi x}\Big|_{\ln K}^{\alpha} \\
&= \frac{-1}{i\phi} \left[\lim_{\alpha\to\infty} e^{-i(\phi_r+i\phi_i)\alpha} - e^{-i(\phi_r+i\phi_i)\ln K}\right] \\
&= \frac{1}{i\phi} e^{-i\phi \ln K} \\
&= \frac{1}{i\phi} K^{-i\phi}
\end{aligned}
\tag{8.34}
$$

Since ϕ is in the complex plane, the above limit only exists if the imaginary part, $\phi_i < 0$.

The Kolmogorov forward equation is a mixed partial differential equation, which is linear with variable coefficients. To solve it, a solution of the following form is suggested:

$$f_j(\phi) = \exp\left(C(\tau;\phi) + D(\tau;\phi)V + i\phi x\right)$$

where $\tau = T - t$. Calculating all partial derivatives that are required:

$$
\begin{aligned}
f_x &= i\phi f, \\
f_{xx} &= -\phi^2 f, \\
f_V &= Df, \\
f_{VV} &= D^2 f, \\
f_{xV} &= iD\phi f, \\
f_t &= (C_t + VD_t)\, f
\end{aligned}
$$

Substituting this into (8.31) and dividing through by f,

$$-\frac{1}{2}V\phi^2 + i\rho\sigma VD\phi + \frac{1}{2}\sigma^2 VD^2 + (r + u_j V)\, i\phi + (a - b_j V)\, D + (C_t + VD_t) = 0$$

Gathering terms in V:

$$V\left(-\frac{1}{2}\phi^2 + i\rho\sigma D\phi + \frac{1}{2}\sigma^2 D^2 - b_j D + iu_j\phi + D_t\right) + (ir\phi + aD + C_t) = 0$$

So, from the above it is clear that the following two ordinary differential equations (ODEs) are obtained

$$
\begin{aligned}
-\frac{1}{2}\phi^2 + i\rho\sigma D\phi + \frac{1}{2}\sigma^2 D^2 - b_j D + iu_j\phi + \frac{dD}{dt} &= 0, \\
ir\phi + aD + \frac{dC}{dt} &= 0
\end{aligned}
$$

subject to

$$C(0) = 0, \quad D(0) = 0$$

Since $dt = -d\tau$,

$$\frac{1}{2}\sigma^2 D^2 + D\left(i\rho\sigma\phi - b_j\right) + \left(iu_j\phi - \frac{1}{2}\phi^2\right) = \frac{dD}{d\tau},$$
$$ir\phi + aD = \frac{dC}{d\tau}$$

Solving for D first:

$$\frac{dD}{\frac{1}{2}\sigma^2 D^2 + D\left(i\rho\sigma\phi - b_j\right) + \left(iu_j\phi - \frac{1}{2}\phi^2\right)} = d\tau$$

Factorizing the denominator in D,

$$\begin{aligned} D &= \frac{-\left(i\rho\sigma\phi - b_j\right) \pm \sqrt{\left(i\rho\sigma\phi - b_j\right)^2 - 4\frac{1}{2}\sigma^2\left(iu_ji\phi - \frac{1}{2}\phi^2\right)}}{\sigma^2} \\ &= \frac{\left(b_j - i\rho\sigma\phi\right) \pm \sqrt{\left(i\rho\sigma\phi - b_j\right)^2 - \sigma^2\left(2iu_j\phi - \phi^2\right)}}{\sigma^2} \end{aligned}$$

Define the following:

$$\begin{aligned} \hat{f} &= \sqrt{\left(i\rho\sigma\phi - b_j\right)^2 - \sigma^2\left(2iu_j\phi - \phi^2\right)} \\ X^+ &= \frac{\left(b_j - i\rho\sigma\phi\right) + \hat{f}}{\sigma^2} \\ X^- &= \frac{\left(b_j - i\rho\sigma\phi\right) - \hat{f}}{\sigma^2} \\ \hat{g} &= \frac{X^+}{X^-} \end{aligned}$$

So,

$$\frac{dD}{\left(D - X^+\right)\left(D - X^-\right)} = \frac{1}{2}\sigma^2 d\tau$$

Separating out into partial fractions:

$$\frac{A}{D - X^+} + \frac{B}{D - X^-} = \frac{1}{\left(D - X^+\right)\left(D - X^-\right)}$$

Equating the numerators:

$$A\left(D - X^-\right) + B\left(D - X^+\right) = 1$$

Multiplying out and equating terms in powers of D, we get:

$$\begin{aligned} (A + B)D - \left(AX^- + BX^+\right) &= 1 \\ \Rightarrow A &= -B \\ \text{and } A\left(X^- - X^+\right) &= -1 \\ \Rightarrow A &= \frac{1}{X^+ - X^-} \end{aligned}$$

Now,

$$\begin{aligned} X^+ - X^- &= \frac{(b_j - i\rho\sigma\phi) + \hat{f}}{\sigma^2} - \frac{(b_j - i\rho\sigma\phi) - \hat{f}}{\sigma^2} \\ &= \frac{2\hat{f}}{\sigma^2} \end{aligned}$$

Therefore,

$$\begin{aligned} A &= \frac{\sigma^2}{2\hat{f}} \\ B &= \frac{-\sigma^2}{2\hat{f}} \end{aligned}$$

Now, the ODE is of the form:

$$\begin{aligned} \frac{\sigma^2 dD}{2\hat{f}\left(D - X^+\right)} - \frac{\sigma^2 dD}{2\hat{f}\left(D - X^-\right)} &= \frac{1}{2}\sigma^2 d\tau \\ \Rightarrow \frac{dD}{D - X^+} - \frac{dD}{D - X^-} &= \hat{f} d\tau \end{aligned}$$

Integrating both sides, we get

$$\ln\left(D - X^+\right) - \ln\left(D - X^-\right) = \hat{f}\tau + k$$

for some constant k. Since $D(0) = 0$,

$$\begin{aligned} k &= \ln\left(-X^+\right) - \ln\left(-X^-\right) \\ &= \ln\frac{X^+}{X^-} \end{aligned}$$

$$\Rightarrow \ln\frac{D - X^+}{D - X^-} = \hat{f}\tau + \ln\hat{g}$$

Exponentiating both sides then substituting in for g in the right hand side:

$$\begin{aligned} \frac{D - X^+}{D - X^-} &= \hat{g}e^{\hat{f}\tau} \\ \Rightarrow D\left(1 - \hat{g}e^{\hat{f}\tau}\right) &= X^+ - \frac{X^+}{X^-}e^{\hat{f}\tau}X^- \\ \Rightarrow D(\tau;\phi) &= X^+\frac{1 - e^{\hat{f}\tau}}{1 - \hat{g}e^{\hat{f}\tau}} \end{aligned} \tag{8.35}$$

Now, it is possible to solve for C. Looking at

$$ir\phi + aD + \frac{dC}{dt} = 0$$

and noting that $dt = -d\tau$, we solve the ODE:

$$\frac{dC}{d\tau} = ir\phi + aD \tag{8.36}$$

subject to $C(0) = 0$. Therefore,

$$\frac{dC}{d\tau} = ir\phi + aX^{+}\frac{1-e^{\hat{f}\tau}}{1-\hat{g}e^{\hat{f}\tau}}$$

$$\Rightarrow dC = ir\phi d\tau + aX^{+}\frac{1-e^{\hat{f}\tau}}{1-\hat{g}e^{\hat{f}\tau}}d\tau$$

Consider the integral

$$\frac{1-e^{\hat{f}\tau}}{1-\hat{g}e^{\hat{f}\tau}}d\tau$$

To integrate the above expression, a simple substitution and the method of partial fractions is used. Let $x = e^{\hat{f}\tau}$. Then, $dx = \hat{f}xd\tau$ and the integral is equal to

$$\frac{1-x}{\hat{f}x\left(1-\hat{g}x\right)}dx$$

Separating into partial fractions,

$$\frac{A}{x} + \frac{B}{1-\hat{g}x} = \frac{1-x}{x\left(1-\hat{g}x\right)}$$

$$\Rightarrow A\left(1-\hat{g}x\right) + Bx = 1-x$$

$$\Rightarrow A = 1$$

$$\text{and } B - \hat{g} = -1$$

$$\Rightarrow B = \hat{g} - 1$$

Upon integration and then substituting in for x, we get

$$\begin{aligned}
\frac{1}{\hat{f}}\int \frac{1-x}{x\left(1-\hat{g}x\right)}dx &= \frac{1}{\hat{f}}\int\left(\frac{1}{x} + \frac{\hat{g}-1}{1-\hat{g}x}\right)dx \\
&= \frac{1}{\hat{f}}\left(\ln x - \left(\frac{\hat{g}-1}{\hat{g}}\right)\ln\left(1-\hat{g}x\right) + l\right) \\
&= \tau - \frac{\hat{g}-1}{\hat{g}\hat{f}}\ln\left(1-\hat{g}e^{\hat{f}\tau}\right) + \hat{\ }
\end{aligned}$$

for some constants l and $\hat{l}$. Therefore, the ODE (8.36) is integrated to give:

$$C = ir\phi\tau + aX^{+}\left(\tau - \frac{\hat{g}-1}{\hat{g}\hat{f}}\ln\left(1-\hat{g}e^{\hat{f}\tau}\right)\right) + m$$

where m is a constant of integration. Substituting in $C(0) = 0$, we solve for m.

$$0 = -aX^{+}\frac{\hat{g}-1}{\hat{g}\hat{f}}\ln\left(1-\hat{g}\right) + m$$

$$\Rightarrow m = aX^{+}\frac{\hat{g}-1}{\hat{g}\hat{f}}\ln\left(1-\hat{g}\right)$$

Therefore,

$$\begin{aligned}
C &= ir\phi\tau + aX^{+}\left(\tau - \frac{\hat{g}-1}{\hat{g}\hat{f}}\ln\left(1-\hat{g}e^{\hat{f}\tau}\right)\right) + aX^{+}\frac{\hat{g}-1}{\hat{g}\hat{f}}\ln\left(1-\hat{g}\right) \\
&= ir\phi\tau + aX^{+}\left(\tau - \frac{\hat{g}-1}{\hat{g}\hat{f}}\ln\left(\frac{1-\hat{g}e^{\hat{f}\tau}}{1-\hat{g}}\right)\right)
\end{aligned}$$

Simplifying the term $X^+\frac{\hat{g}-1}{\hat{g}\hat{f}}$, we get

$$\begin{aligned} X^+\frac{\hat{g}-1}{\hat{g}\hat{f}} &= \frac{X^+}{X^-}\left(\frac{X^+-X^-}{\hat{g}\hat{f}}\right) \\ &= \frac{2\hat{f}}{\sigma^2\hat{f}} \\ &= \frac{2}{\sigma^2} \end{aligned}$$

So,

$$C(\tau;\phi) = ir\phi\tau + a\left(X^+\tau - \frac{2}{\sigma^2}\ln\left(\frac{1-\hat{g}e^{\hat{f}\tau}}{1-\hat{g}}\right)\right) \tag{8.37}$$

We can now calculate the required $P^j(x,V,t)$ for $j=1,2$ as in (8.29).

$$\begin{aligned} P^j(x,V,t) &= \frac{1}{2\pi}\int_{i\phi_i-\infty}^{i\phi_i+\infty} f_j(\phi)\frac{K^{-i\phi}}{i\phi}d\phi \\ &= \frac{-1}{2\pi}\int_{i\phi_i-\infty}^{i\phi_i+\infty} f_j(\phi)\frac{iK^{-i\phi}}{\phi}d\phi \end{aligned} \tag{8.38}$$

We only require the real part of this integral, $\Re\left(P^j(x,V,t)\right)$, which will be evaluated using a contour integral. In doing so, it is necessary to ensure $\phi_i < 0$. The following definitions and theorems from real and complex analysis are required.

Definition 11 *(Rudin 1987, §10.1) Let Ω denote an open set. Suppose f is a complex function defined in Ω. If $z_0 \in \Omega$ and if*

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}$$

exists, we denote this limit by $f'(z_0)$ and call it the derivative of f at z_0. If $f'(z_0)$ exists for every $z_0 \in \Omega$, then f is holomorphic (or analytic) in Ω.

Definition 12 *(Rudin 1987, §10.8) If X is a topological space, a curve in X is a continuous mapping γ of a compact interval $[\alpha,\beta] \subset \mathbb{R}^1$ into X; here $\alpha < \beta$. We call $[\alpha,\beta]$ the parameter interval of γ and denote the range of γ by γ^*. Thus, γ is a mapping, and γ^* is the set of all points $\gamma(t)$, for $\alpha \le t \le \beta$.*

If $\gamma(\alpha)$ coincides with $\gamma(\beta)$, then γ is a closed curve.

A path is a piecewise continuously differentiable curve in the plane. A closed path is a closed curve which is also a path.

Suppose γ is a path, and f is a continuous function on γ^*. The integral of f over γ is defined as an integral over the parameter interval $[\alpha,\beta]$ of γ:

$$\int_\gamma f(z)dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)dt. \tag{8.39}$$

Theorem 11 ***Cauchy's Theorem***

(Rudin 1987, §10.12) Suppose $F \in H(\Omega)$, the space of analytic functions, and F' is continuous in Ω. Then

$$\int_\gamma F'(z)dz = 0 \tag{8.40}$$

for every closed path γ in Ω.

Theorem 12 ***Lebesgue Dominated Convergence Theorem***

(Goldberg 1976, §11.8B) Let $\{f_n\}_{n=1}^\infty$ be a sequence of functions in $\mathcal{L}[a,b]$ such that

$$\lim_{n\to\infty} f_n(x) = f(x) \quad \textit{almost everywhere} \quad (a \le x \le b).$$

Suppose there exists $g \in \mathcal{L}[a,b]$ such that

$$|f_n(x)| \le g(x) \quad \textit{almost everywhere} \quad (a \le x \le b).$$

Then $f \in \mathcal{L}[a,b]$ and

$$\lim_{n\to\infty} \int_a^b f_n \quad = \textstyle\int_a^b f.$$

Define

$$\Re\left(\frac{-1}{2\pi}\int_{i\phi_i-\infty}^{i\phi_i+\infty} f_j(\phi)\frac{iK^{-i\phi}}{\phi}d\phi\right) := \Re\left(\frac{-1}{2\pi}I\right)$$

and let $F(\phi) = f_j(\phi)\frac{iK^{-i\phi}}{\phi}$. In order to evaluate the integral I for $\phi_i < 0$, we use a contour integral given by six parametric curves:

1. $\Gamma_1 : \phi = k,\ k \in (m,n)$, for $m,n > 0 \in \mathbb{R}$;
2. $\Gamma_2 : \phi = n + ip,\ p \in (0, \phi_i)$;
3. $\Gamma_3 : \phi = k + i\phi_i,\ k \in (n, -n)$;
4. $\Gamma_4 : \phi = -n + ip,\ p \in (\phi_i, 0)$;
5. $\Gamma_5 : \phi = k,\ k \in (-n, -m)$;
6. $\Gamma_6 : \phi = me^{i\theta},\ \theta \in (-\pi, 0)$;

Note, there is a pole (singularity) at $\phi = 0$, therefore we require Γ_6.

Since the integrand is analytic on this contour, Cauchy's theorem implies that

$$\sum_{j=1}^{6}\int_{\Gamma_j} F(\phi)d\phi = 0$$

To find $\Re(I)$, the integrals must be evaluated, taking limits as $m \to 0$, $n \to \infty$.

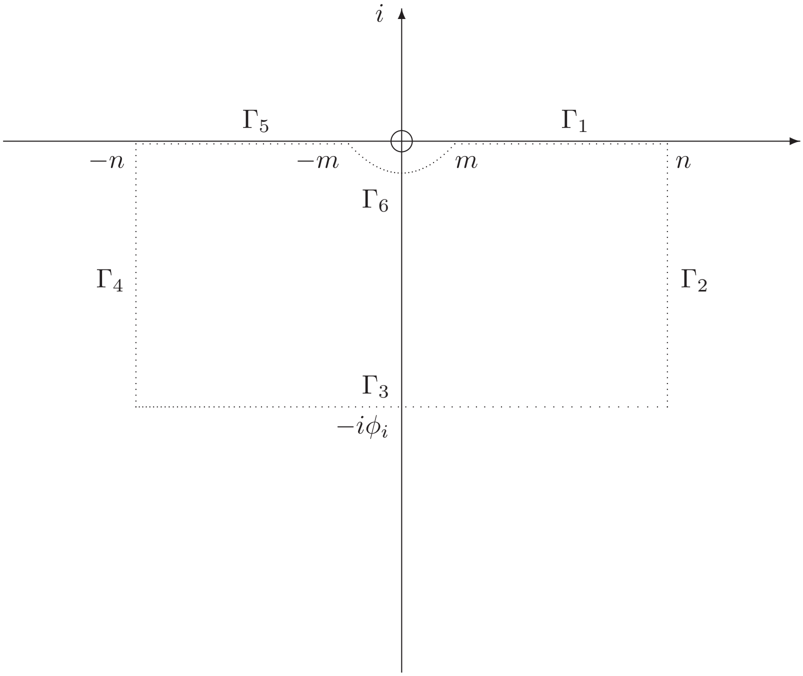

Figure 8.1: Contour integral for the evaluation of I

Consider

$$\begin{aligned}
&\int_{\Gamma_1} F(\phi)d\phi + \int_{\Gamma_5} F(\phi)d\phi \\
&= \lim_{m\to 0, n\to\infty} \int_m^n f_j(t)\frac{iK^{-it}}{t}dt + \int_{-n}^{-m} f_j(t)\frac{iK^{-it}}{t}dt \\
&= \int_{-\infty}^{\infty} f_j(t)\frac{iK^{-it}}{t}dt \qquad (8.41)
\end{aligned}$$

Evaluating $\int_{\Gamma_2} F(\phi)d\phi$. A parameterization of ϕ is required, therefore equation (8.39) is used. Here, $\gamma(t) = n + it$ for $t \in [0, \phi_i]$. We have

$$\begin{aligned}
\int_{\Gamma_2} F(\phi)d\phi &= \lim_{n\to\infty} \int_0^{\phi_i} f_j(n+it)\frac{iK^{-i(n+it)}}{n+it}(-i)dt \\
&= \lim_{n\to\infty} \int_0^{\phi_i} f_j(n+it)\frac{K^{t-in}}{n-it}dt \\
&= \int_0^{\phi_i} \lim_{n\to\infty}\left(f_j(n+it)\frac{K^{t-in}}{n+it}\right)dt
\end{aligned}$$

Looking at the individual terms as $n \to \infty$: $f_j(n+it)$ behaves like an oscillating constant, K^{t-in} behaves like a constant and $\frac{1}{n+it} \to \infty$. Therefore, as a result of Lebesgue's dominated convergence theorem (12), the entire integrand tends to zero as n tends to ∞. So,

$$\int_{\Gamma_2} F(\phi)d\phi = 0 \qquad (8.42)$$

Similarly, with the parameterization $\gamma(t) = -n + i\phi_i - it$ for $t \in [0, \phi_i]$.

$$\begin{aligned}
\int_{\Gamma_4} F(\phi)d\phi &= \lim_{n\to\infty} \int_0^{\phi_i} f_j(-n + i\phi_i - it)\frac{iK^{-i(-n+i\phi_i-it)}}{-n + i\phi_i - it}(-i)dt \\
&= \lim_{n\to\infty} \int_0^{\phi_i} f_j(-(n - i(t - \phi_i)))\frac{K^{in-(t-\phi_i)}}{-n - i(t - \phi_i)}dt \\
&= \int_0^{\phi_i} \lim_{n\to\infty} \left(f_j(-(n + i(\phi_i - t)))\frac{K^{in-(t-\phi_i)}}{n + i(\phi_i - t)} \right) dt
\end{aligned}$$

Thus,

$$\int_{\Gamma_4} F(\phi)d\phi = 0 \tag{8.43}$$

Next, consider the integral along Γ_3. The parameterization is $\gamma(t) = t + i\phi_i$ for $t \in [n, -n]$. We have

$$\begin{aligned}
\int_{\Gamma_3} F(\phi)d\phi &= \lim_{n\to\infty} \int_n^{-n} f_j(t + i\phi_i)\frac{iK^{-i(t+i\phi_i)}}{t + i\phi_i}dt \\
&= -i \lim_{n\to\infty} \int_{-n}^{n} f_j(t + i\phi_i)\frac{K^{\phi_i-it}}{t + i\phi_i}dt \\
&= i \lim_{n\to\infty} \int_n^{-n} f_j(-t + i\phi_i)\frac{K^{\phi_i+it}}{-t + i\phi_i}dt \\
&= i \lim_{n\to\infty} \int_n^{-n} f_j(-\overline{(t + i\phi_i)})\overline{\left(\frac{K^{\phi_i-it}}{-(t + i\phi_i)}\right)}dt \\
&= \overline{\left(i \lim_{n\to\infty} \int_n^{-n} f_j(t + i\phi_i)\frac{K^{\phi_i-it}}{-(t + i\phi_i)}dt \right)} \\
&= \overline{\int_{\Gamma_3} F(\phi)d\phi}
\end{aligned}$$

The negative sign going from line (1) to line (2) results from reversing the path of integration, going from line (2) to line (3), a change of variable from t to $-t$ was made. Line (4) to line (5) is a result of a conjugate symmetry that applies to Fourier transforms. For a real-valued integrable function f, we have (Lewis 2000)

$$F(\phi) := \int_{-\infty}^{\infty} e^{-i\phi x} f(x)dx = \overline{F(-\overline{\phi})}$$

The final results indicates that the real part of this contour integral is in fact the whole integral.

Lastly, consider $\int_{\Gamma_6} F(\phi)d\phi$.

$$\begin{aligned}
\int_{\Gamma_6} F(\phi)d\phi &= \lim_{m\to 0} \int_{-\pi}^{0} f_j(me^{i\theta})\frac{iK^{-ime^{i\theta}}}{me^{i\theta}}mie^{i\theta}d\theta \\
&= -\lim_{m\to 0} \int_{-\pi}^{0} f_j(me^{i\theta})K^{-ime^{i\theta}}d\theta \\
&= -\int_{-\pi}^{0} \lim_{m\to 0} \left(f_j(me^{i\theta})K^{-ime^{i\theta}} \right) d\theta
\end{aligned}$$

Since $f_j(\phi)$ is the Fourier transform of the (risk-neutral) probability density function, we have that

$\lim_{m\to 0} f_j(me^{i\theta})$ tends to 1. Therefore,

$$\int_{\Gamma_6} F(\phi)d\phi = -\int_{-\pi}^{0} d\theta$$
$$= -\pi \tag{8.44}$$

Putting everything together using (8.41), (8.42), (8.43) and (8.44), we find

$$\begin{aligned}
\Re\left(\frac{-1}{2\pi}\int_{i\phi_i-\infty}^{i\phi_i+\infty} f_j(\phi)\frac{iK^{-i\phi}}{\phi}d\phi\right) &= \frac{-1}{2\pi}\left(-\lim_{n\to\infty}\int_{\Gamma_3} F(\phi)d\phi\right)\\
&= \frac{-1}{2\pi}\left(\lim_{m\to 0, n\to\infty}\sum_{j=1, j\neq 3}^{6}\int_{\Gamma_j}\Re\left(F(\phi)\right)d\phi\right)\\
&= \frac{-1}{2\pi}\left(-\pi + \int_{-\infty}^{\infty}\Re\left(f_j(\phi)\frac{iK^{-i\phi}}{\phi}\right)d\phi\right)\\
&= \frac{1}{2} - \frac{1}{2\pi}\int_{-\infty}^{\infty}\Re\left(f_j(\phi)\frac{iK^{-i\phi}}{\phi}\right)d\phi\\
&= \frac{1}{2} + \frac{1}{2\pi}\int_{-\infty}^{\infty}\Re\left(f_j(\phi)\frac{K^{-i\phi}}{i\phi}\right)d\phi
\end{aligned}$$

By writing $\Re\left(f_j(\phi)\frac{K^{-i\phi}}{i\phi}\right)$ as $\Re\left(f_j(\phi)\frac{e^{-i\phi\ln K}}{i\phi}\right)$, we can further simplify the integral:

$$\begin{aligned}
\int_{-\infty}^{\infty}\Re\left(f_j(\phi)\frac{e^{-i\phi\ln K}}{i\phi}\right)d\phi &= \int_{-\infty}^{\infty}\Re\left(\frac{f_j(\phi)}{i\phi}\left(\cos\left(\phi\ln K\right) - i\sin\left(\phi\ln K\right)\right)\right)d\phi\\
&= \int_{-\infty}^{\infty}\Re\left(\frac{1}{i\phi}\left(\Re(f_j) + i\Im(f_j)\right)\left(\cos\left(\phi\ln K\right) - i\sin\left(\phi\ln K\right)\right)\right)d\phi\\
&= \int_{-\infty}^{\infty}\Re\left(\frac{1}{\phi}\left(-i\Re(f_j) + \Im(f_j)\right)\left(\cos\left(\phi\ln K\right) - i\sin\left(\phi\ln K\right)\right)\right)d\phi\\
&= \int_{-\infty}^{\infty}\frac{1}{\phi}\left(-\Re(f_j)\sin\left(\phi\ln K\right) + \Im(f_j)\cos\left(\phi\ln K\right)\right)d\phi
\end{aligned}$$

$\frac{1}{\phi}$ is an odd function, $\cos\left(\phi\ln K\right)$ and $\sin\left(\phi\ln K\right)$ are even and odd functions respectively, and using (8.26), we have that

$$\begin{aligned}
\Re(f_j(-\phi)) &= \frac{1}{2}\left(f_j(-\phi) + f_j(\phi)\right) = \Re(f_j(\phi)),\\
\Im(f_j(-\phi)) &= \frac{1}{2i}\left(f_j(-\phi) - f_j(\phi)\right) = -\Im(f_j(-\phi))
\end{aligned}$$

showing that $\Re(f_j(-\phi))$ is an even function and $\Im(f_j(-\phi))$, odd.

Thus, both terms in the integral above are even functions. Therefore,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\Re\left(f_j(\phi)\frac{e^{-i\phi\ln K}}{i\phi}\right)d\phi = \frac{1}{\pi}\int_{0}^{\infty}\Re\left(f_j(\phi)\frac{e^{-i\phi\ln K}}{i\phi}\right)d\phi$$

So, $P^j(x, V, t)$ for $j = 1, 2$ can be numerically evaluated as:

$$P^j(x, V, t) = \frac{1}{2} + \frac{1}{\pi}\int_{0}^{\infty}\Re\left(f_j(\phi)\frac{e^{-i\phi\ln K}}{i\phi}\right)d\phi \tag{8.45}$$

8.5 Computational Procedures

Given that the solution to the European call pricing problem is given by (8.12) where, P^1 and P^2 can be found using (8.45), we will proceed to implement a hybrid quasi-Monte Carlo scheme using an Euler-Maruyama (EM) discretization method. A second approach, which involves the numerical evaluation of the probabilities, will be to perform Gauss-Legendre integration.

8.5.1 Quasi-Monte Carlo Simulation

The simulation technique employed here will be identical to that of §7.4.2. The risk-neutral diffusion equations associated with the stock price process and the variance will be discretized using the Euler-Maruyama method (Higham 2001).

We have

$$\begin{aligned} dS_t &= rSdt + \sqrt{V_t}SdW_t^1 \\ dV_t &= \kappa^*\left(\theta^* - V_t\right)dt + \sigma\sqrt{V_t}dW_t^2 \end{aligned}$$

where $\kappa^* = \kappa + \hat{\lambda}$ and $\theta^* = \frac{\kappa\theta}{\kappa+\hat{\lambda}}$ which arises from assuming that $\hat{\lambda}(S,V,t) = \hat{\lambda}V$ as in (8.8). The Brownian motions Z^1 and Z^2 have correlation ρ.

A scalar autonomous SDE written can be written in integral form as:

$$X(t) = X_0 + \int_0^t f(X(s))ds + \int_0^t g(X(s))dW(s), \quad 0 \le t \le T.$$

where the second integral is with respect to Brownian motion and X_0 is the initial condition. This can be rewritten as:

$$dX(t) = f(X(t))dt + g(X(t))dW(t), \quad X(0) = X_0 \quad 0 \le t \le T$$

Discretizing the interval $[0,T]$ in to N equally spaces intervals $\Delta t = \frac{T}{N}$, then for $j = 1, \ \ldots, \ N$,

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})\left(W(\tau_j) - W(\tau_{j-1})\right)$$

where $\tau_j = j\Delta t$. Using this for the risk-neutral underlying and variance processes above, we have

$$S_j = S_{j-1} + (r-q)S_{j-1}\Delta t + \sqrt{V_{j-1}}S_{j-1}u_j\sqrt{\Delta t} \tag{8.46}$$

$$V_j = V_{j-1} + \kappa^*(\theta^* - V_{j-1})\Delta t + \sigma\sqrt{V_{j-1}}\left(\rho u_j + \sqrt{1-\rho^2}v_j\right)\sqrt{\Delta t} \tag{8.47}$$

Here, u_j and v_j are the quasi-random $\Phi(0,1)$ numbers which are generated as in chapter (7.1). The Choleski decomposition relates the two independent Brownian motions (u_j and v_j).

Another important consideration is the variance process may become negative. In this instance, it is necessary either to reflect the value by taking the absolute value, or to absorb it by setting it to zero.

8.5.2 Gauss-Legendre Integration

The simplest form of Gaussian Integration is based on the use of an optimally chosen polynomial to approximate an integrand $f(t)$ over the interval $[-1,1]$ (Burden & Faires 1997).

It can be shown that the best estimate of the integral is then:

$$\int_{-1}^{1} f(t)dt = \sum_{k=1}^{n} c_k f(t_k)$$

where t_i are designated evaluation points (abscissae), and c_i are the weights of that point in the sum. These values are found in tables in (Abramowitz & Stegun 1974, §25.4). A simple linear transformation can be made that will translate any interval $[\alpha, \beta]$ to $[-1, 1]$:

$$x = \frac{2t - \alpha - \beta}{\beta - \alpha}$$

such that the integral becomes

$$\int_{\alpha}^{\beta} f(t)dt = \int_{-1}^{1} \frac{\beta - \alpha}{2} f\left(\frac{\beta - \alpha}{2} x + \frac{\alpha + \beta}{2}\right) dx$$

The simplest form uses a uniform weighting over the interval, and the particular points at which to evaluate $f(t)$ are the roots of a particular class of polynomials, the Legendre polynomials, over the interval. These polynomials are orthogonal on $[-1, 1]$, with respect to the weight function $w(t) \equiv 1$, and for each n, the polynomial P_n has n distinct zeros which lie in $(-1, 1)$. This leads to the result that for any polynomial $P(t)$ of degree $(2n - 1)$, then

$$\int_{-1}^{1} P(t)dt = \sum_{k=1}^{n} c_k P(t_k).$$

where

$$c_k = \int_{-1}^{1} \prod_{j=1, j \neq k}^{n} \frac{t - t_j}{t_k - t_j} dt$$

and $t_0, t_1, \ldots, t_n$ are the zeros of the nth Legendre polynomial.

These points are not evenly spaced and increasing the degree (number of chosen points to evaluate the function) of integration improves convergence. Rather than using higher degrees of quadrature, one increases the number subintervals (each may have some given degree of quadrature).

This method of integration has one significant further advantage in many situations. In the evaluation of an integral on an interval, it is not necessary to evaluate the function at the endpoints which proves valuable when evaluating various improper integrals, such as those with infinite limits.

We are required to evaluate the integral:

$$\int_{0}^{\infty} \Re\left(f_j(\phi) \frac{e^{-i\phi \ln K}}{i\phi}\right) d\phi$$

The degree n is chosen to be 16. If the maturity of the option is longer than a year, the default setting will be to select a total length of 512, dividing it into subintervals of length 8 (i.e. 64 pieces), to be evaluated as a separate integral. So, the integral can be rewritten as

$$\begin{aligned}\sum_{m=1}^{64} \int_{p_{m-1}}^{p_m} \Re\left(f_j(\phi) \frac{e^{-i\phi \ln K}}{i\phi}\right) d\phi &= \sum_{m=1}^{64} \int_{-1}^{1} \frac{p_m - p_{m-1}}{2} \Re\left(f_j(\widetilde{\phi}) \frac{e^{-i\widetilde{\phi} \ln K}}{i\widetilde{\phi}}\right) d\hat{\phi} \\ &= \sum_{m=1}^{64} \frac{p_m - p_{m-1}}{2} \sum_{k=1}^{16} c_k \Re\left(f_j(\hat{t}_k) \frac{e^{-i\hat{t}_k \ln K}}{i\hat{t}_k}\right) \qquad (8.48)\end{aligned}$$

where $p_m - p_{m-1} = 8$ and $p_m = 8m$. Here,

$$\widetilde{\phi} = \frac{p_m - p_{m-1}}{2}\hat{\phi} + \frac{p_{m-1} + p_m}{2},$$

$$\hat{\phi} = \frac{2\phi - p_{m-1} - p_m}{p_m - p_{m-1}}$$

and

$$\hat{t}_k = \frac{p_m - p_{m-1}}{2}t_k + \frac{p_{m-1} + p_m}{2}.$$

As before, the weights c_k and abscissae t_k, for $k = 1, \ldots, 16$, can be found in a standard table.

The case where we consider shorter dated options can be divided in to two scenarios (the subinterval length of 8 is maintained in both cases):

1. If the strike is within ε of the (forward) ATM range;
2. If the option is either ITM or OTM.

ε can be selected according to desired efficiency and accuracy. In the first case above, the number of pieces is increased which increases the total length the integral is evaluated over. If the second case above arises, then the number of pieces is selected to be 64 or more, depending on a suitable criterion.

Once the number of pieces has been calculated, the abscissae t_k and weights c_k are substituted into (8.48).

This procedure is computationally quicker and more accurate than the quasi-Monte Carlo simulation above (Schöbel, R and Zhu, J 1999).

The European call price, $C(x, V, t)$, at time t is given by:

$$\begin{aligned}
C(x,V,t) &= e^x P^1 - Z(t,T)KP^2 \\
&= e^x\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left(f_1(\phi)\frac{e^{-i\phi\ln K}}{i\phi}\right)d\phi\right) - Z(t,T)K\left(\frac{1}{2} + \frac{1}{\pi}\int_0^\infty \Re\left(f_2(\phi)\frac{e^{-i\phi\ln K}}{i\phi}\right)d\phi\right) \\
&= e^x\left(\frac{1}{2} + \frac{1}{\pi}\sum_{m=1}^{64}\frac{p_m - p_{m-1}}{2}\sum_{k=1}^{16} c_k \Re\left(f_1(\hat{t}_k)\frac{e^{-i\hat{t}_k \ln K}}{i\hat{t}_k}\right)\right) \\
&\quad - KP(t,T)\left(\frac{1}{2} + \frac{1}{\pi}\sum_{m=1}^{64}\frac{p_m - p_{m-1}}{2}\sum_{k=1}^{16} c_k \Re\left(f_2(\hat{t}_k)\frac{e^{-i\hat{t}_k \ln K}}{i\hat{t}_k}\right)\right)
\end{aligned}$$

Chapter 9

SABR Model

9.1 Introduction

The SABR (stochastic-$\alpha\beta\rho$ model) model is a two-factor stochastic volatility model, which allows for correlation between the underlying and its volatility. Under the forward measure, the following SDEs model the forward $\hat{F}$, and volatility $\hat{\alpha}$, processes:

$$d\hat{F} = \hat{\alpha} C(\hat{F}) dW_1, \tag{9.1}$$

$$d\hat{\alpha} = \nu \hat{\alpha} dW_2, \tag{9.2}$$

$$dW_1 dW_2 = \rho dt, \tag{9.3}$$

where $C(\hat{F})$[1] is the diffusion coefficient, $\hat{F}(0) = f$ and $\hat{\alpha}(0) = \alpha$. $\hat{\alpha}$ is a 'volatility-like' parameter, ν is the volatility of volatility (volvol) and as usual ρ is the correlation between $\hat{F}$ and $\hat{\alpha}$. Clearly, the forward price is a martingale under the forward measure that will be used for the analysis below.

9.2 Black Volatilities of Vanilla Options Priced with the SABR Model

We shall spend a considerable amount of effort deriving the price of vanilla options as a function of α, ν, ρ and $C(\cdot)$; in what is considered to be the typical case $C(\hat{F}) = \hat{F}^{\beta}$, and so the price of vanilla options, and hence the skew, becomes a closed form function of α, ν, ρ and β. We then have a calibration mechanism: these parameters are chosen so that this skew most closely matches that in the market. The ease with which this is done makes the model very tractable. Even when the market skew cannot reliably be observed, methods such as that of (West 2005*a*) are available.

[1] In order for the perturbation expansion to be reasonably accurate and work well, $C(\hat{F})$ should be a smooth function which does not get too close to zero.

Let $V(t, f, \alpha)$ be the value of a European call option (on a forward contract) at date t with strike K, where $\hat{F}(t) = f$ and $\hat{\alpha}(t) = \alpha$. The option has exercise date t_{ex} and settlement date t_{set}. Then, the undiscounted value of the option is given by

$$V(t, f, \alpha) = \mathbb{E}\left[\left((\hat{F}(t_{\text{ex}}) - K\right)^+ \middle| \hat{F}(t) = f, \hat{\alpha}(t) = \alpha\right] \tag{9.4}$$

$$= \int_{-\infty}^{\infty} \int_{K}^{\infty} (F - K) p(t, f, \alpha; t_{\text{ex}}, F, A) dF dA, \tag{9.5}$$

where $p(t, f, \alpha; t_{\text{ex}}, F, A)$ is the probability density function defined by

$$p(t, f, \alpha; t_{\text{ex}}, F, A) dF dA = \mathbb{P}\left[F < \hat{F}(t_{\text{ex}}) < F + dF, A < \hat{\alpha}(t_{\text{ex}}) < A + dA \middle| \hat{F}(t) = f, \hat{\alpha}(t) = \alpha\right]. \tag{9.6}$$

The analysis is to be carried out on a small volatility expansion. Therefore, we begin by replacing $\hat{\alpha} \to \varepsilon\hat{\alpha}$ and $\nu \to \varepsilon\nu$ in (9.1) and (9.2)[2].

From Chapter 5, Proposition 3, we know that for $T > t$, p follows the forward Kolmogorov equation:

$$p_T = \tfrac{1}{2}\varepsilon^2 A^2 \left(C^2(F)p\right)_{FF} + \varepsilon^2 \rho\nu \left(A^2 C(F) p\right)_{FA} + \tfrac{1}{2}\varepsilon^2\nu^2 \left(A^2 p\right)_{AA} \tag{9.7}$$

with

$$p = \delta(F - f)\delta(A - \alpha),$$

at $T = t$. Here, subscripts refer to partial derivatives. Therefore,

$$p(t, f, \alpha; t_{\text{ex}}, F, A) = \delta(F - f)\delta(A - \alpha) + \int_t^{t_{\text{ex}}} p_T(t, f, \alpha; T, F, A) dT.$$

Using this, the undiscounted call option price can be re-written as

$$\begin{aligned}
&V(t, f, \alpha) \\
&= \int_{-\infty}^{\infty} \int_{K}^{\infty} (F - K)\delta(F - f)\delta(A - \alpha) dF dA + \int_{-\infty}^{\infty} \int_{K}^{\infty} \int_t^{t_{\text{ex}}} (F - K) p_T(t, f, \alpha; T, F, A) dT dF dA \\
&= (f - K)^+ + \int_{-\infty}^{\infty} \int_{K}^{\infty} \int_t^{t_{\text{ex}}} (F - K) p_T(t, f, \alpha; T, F, A) dT dF dA, \qquad (9.8)
\end{aligned}$$

where we have used the following property of the Dirac delta function:

$$\int_{-\infty}^{\infty} f(x)\delta(x - \alpha) dx = f(\alpha).$$

[2] The quantities being perturbed are the volatility and the volvol. In such instances, the perturbation transformations are of the form:

$$\hat{\alpha} \to \varepsilon\hat{\alpha}$$

$$\nu \to \varepsilon\nu.$$

In the method of Matched Asymptotic Expansions, we are required to find the distinguished limit which is that power of ε that balances the derivatives. In this case, we balance the terms p_T and $\frac{1}{2}\varepsilon^2 A^2 \left(C^2(F)p\right)_{FF}$ in (9.7). This means choosing suitable transformation variables (to be discussed in §9.2.1) such that we obtain an equation of the form $p_T = p_{zz} + \ldots$.

Substituting (9.7) for p_T into (9.8) and by integrating out A from the terms $\varepsilon^2\rho\nu\left(A^2C(F)p\right)_{FA}$ and $\frac{1}{2}\varepsilon^2\nu^2\left(A^2p\right)_{AA}$, we have

$$
\begin{aligned}
&V(t,f,\alpha)\\
&= (f-K)^+ + \int_{-\infty}^{\infty}\int_K^{\infty}\int_t^{t_{\mathrm{ex}}}(F-K)\Bigg[\frac{1}{2}\varepsilon^2A^2\left(C^2(F)p\right)_{FF}\\
&+\varepsilon^2\rho\nu\left(A^2C(F)p\right)_{FA}+\frac{1}{2}\varepsilon^2\nu^2\left(A^2p\right)_{AA}\Bigg]dTdFdA\\
&= (f-K)^+ + \frac{1}{2}\varepsilon^2\int_{-\infty}^{\infty}\int_K^{\infty}\int_t^{t_{\mathrm{ex}}}(F-K)A^2\left(C^2(F)p\right)_{FF}dTdFdA.
\end{aligned}
$$

This is to exclude the cases of $A\to\pm\infty$[3]. By switching the order of integration and then integrating by parts with respect to F, the undiscounted price then becomes

$$
\begin{aligned}
V(t,f,\alpha) &= (f-K)^+ + \frac{1}{2}\varepsilon^2\int_t^{t_{\mathrm{ex}}}\int_{-\infty}^{\infty}\int_K^{\infty}A^2(F-K)\left(C^2(F)p\right)_{FF}dFdAdT\\
&= (f-K)^+ + \frac{1}{2}\varepsilon^2\int_t^{t_{\mathrm{ex}}}\int_{-\infty}^{\infty}A^2\lim_{x\to\infty}\left(\left.(F-K)\left(C^2(F)p\right)_F\right|_K^x-\int_K^x\left(C^2(F)p\right)_F dF\right)dAdT\\
&= (f-K)^+ + \frac{1}{2}\varepsilon^2\int_t^{t_{\mathrm{ex}}}\int_{-\infty}^{\infty}A^2\lim_{x\to\infty}\left(\left.(F-K)\left(C^2(F)p\right)_F\right|_K^x-\left.\left(C^2(F)p\right)\right|_K^x\right)dAdT\\
&= (f-K)^+ + \frac{1}{2}\varepsilon^2\int_t^{t_{\mathrm{ex}}}\int_{-\infty}^{\infty}A^2\left(0-(0-C^2(K)p(t,f,\alpha;T,K,A))^4\right)dAdT\\
&= (f-K)^+ + \frac{1}{2}\varepsilon^2C^2(K)\int_t^{t_{\mathrm{ex}}}\int_{-\infty}^{\infty}A^2p(t,f,\alpha;T,K,A)dAdT
\end{aligned}
$$

Simplifying this further, we define

$$
P(t,f,\alpha;T,K) := \int_{-\infty}^{\infty}A^2p(t,f,\alpha;T,K,A)dA. \tag{9.9}
$$

---

[3]We have that

$$
\begin{aligned}
\int_{-\infty}^{\infty}\int_K^{\infty}\int_t^{t_{\mathrm{ex}}}(F-K)\varepsilon^2\rho\nu\left(A^2C(F)p\right)_{FA}dTdFdA &= \int_K^{\infty}\int_t^{t_{\mathrm{ex}}}(F-K)\varepsilon^2\rho\nu\left(\int_{-\infty}^{\infty}\left(A^2C(F)p\right)_{FA}dA\right)dTdF\\
&= \int_K^{\infty}\int_t^{t_{\mathrm{ex}}}(F-K)\varepsilon^2\rho\nu\left(\lim_{x\to\infty}\left.\left(A^2C(F)p\right)_F\right|_{-x}^x\right)dTdF\\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
\int_K^{\infty}\int_t^{t_{\mathrm{ex}}}\frac{1}{2}(F-K)\varepsilon^2\nu^2\left(\int_{-\infty}^{\infty}\left(A^2p\right)_{AA}dA\right)dTdF &= \int_K^{\infty}\int_t^{t_{\mathrm{ex}}}\frac{1}{2}(F-K)\varepsilon^2\nu^2\left(\lim_{x\to\infty}\left.\left(A^2p\right)_A\right|_{-x}^x\right)dTdF\\
&= 0.
\end{aligned}
$$

[4]The cases when $F\to\infty$ are not included as $\lim_{x\to\infty}p(t,f,\alpha;T,x,A)\to 0$. Thus,

$$
\begin{aligned}
\lim_{x\to\infty}\left.(F-K)\left(C^2(F)p\right)_F\right|_K^x &= \lim_{x\to\infty}\left((x-K)\left(C^2(x)p(t,f,\alpha;T,x,A)\right)_F\right)-0\\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
-\lim_{x\to\infty}\left.\left(C^2(F)p\right)\right|_K^x &= -\lim_{x\to\infty}C^2(x)p(t,f,\alpha;T,x,A)+C^2(K)p(t,f,\alpha;T,K,A)\\
&= C^2(K)p(t,f,\alpha;T,K,A).
\end{aligned}
$$

From Chapter 5, Proposition 2, for $t < T$, P satisfies the backward Kolmogorov equation:

$$P_t + \frac{1}{2}\varepsilon^2\alpha^2 C^2(f)P_{ff} + \varepsilon^2\rho\nu\alpha^2 C(f)P_{f\alpha} + \frac{1}{2}\varepsilon^2\nu^2\alpha^2 P_{\alpha\alpha} = 0,$$

with

$$P = \alpha^2\delta(f-K),$$

for $t = T$. Since P depends on $T-t$, not t or T separately, let

$$\begin{aligned} \tau &= T - t, \\ \Rightarrow d\tau &= dT \end{aligned}$$

If we let $\tau_{\text{ex}} = t_{\text{ex}} - t$, then

$$V(t,f,\alpha) = (f-K)^+ + \frac{1}{2}\varepsilon^2 C^2(K)\int_0^{\tau_{\text{ex}}} P(\tau,f,\alpha;K)d\tau$$

since P depends on the initial condition at $\tau = 0$ as opposed to the terminal condition at $t = T$. Thus, for $\tau > 0$, P solves

$$P_\tau = \frac{1}{2}\varepsilon^2\alpha^2 C^2(f)P_{ff} + \varepsilon^2\rho\nu\alpha^2 C(f)P_{f\alpha} + \frac{1}{2}\varepsilon^2\nu^2\alpha^2 P_{\alpha\alpha}, \tag{9.10}$$

subject to the initial condition ($\tau = 0$):

$$P = \alpha^2\delta(f-K). \tag{9.11}$$

P is Gaussian probability density function[5] which is dependent on $C(f)$, the diffusion coefficient. Since it is asserted in (Hagan et al. 2002) that small changes in the exponent will result in large changes in the density, the exponent will be expanded. The near-identity transform method is applied by transforming the dependent and independent variables order by order into a canonical problem, instead of using a straightforward (regular) expansion to solve the problem order by order.

9.2.1 Multiple Scales Technique

Multiple scales techniques are often applied to ODEs involving the viscous damping of harmonic oscillators and boundary layer problems in fluid mechanics. We briefly review the technique here given in (Nayfeh

[5] As can be seen in (Etheridge 2002, §3.1), the transition density function for a Brouwnian motion with zero mean and variance t given by

$$p(0,x;t,y) = \frac{1}{\sqrt{2\pi t}}\exp\left(-\frac{(x-y)^2}{2t}\right),$$

is the solution of a diffusion process satisfies the backward Kolmogorov equation (from Proposition 2) in §5.2. Then P, which satisfies (9.10) with initial condition (9.11), has solution

$$\begin{aligned} P &= \frac{\alpha^2}{\sqrt{2\pi\varepsilon^2\alpha^2 C^2(K)\tau}}\exp\left(-\frac{(f-K)^2}{2\varepsilon^2\alpha^2 C^2(K)\tau}\right) \\ &= \frac{\alpha}{\sqrt{2\pi\varepsilon^2 C^2(K)\tau}}\exp\left(-\frac{(f-K)^2}{2\varepsilon^2\alpha^2 C^2(K)\tau}\right). \end{aligned} \tag{9.12}$$

1981, §8.1.3). Let $u(t;\varepsilon)$ be a solution to an ODE, where the expansion involves multiple independent variables (scales). So, u is a function of multiple scales, instead of a single variable. So, we have

$$u(t,\varepsilon) = \hat{u}(t, \varepsilon t, \varepsilon^2 t, \ldots; \varepsilon)$$

We define the different time scales T_i, for $i \geq 0$ as

$$T_i = \varepsilon^i t.$$

Clearly, T_{n+1} is slower than T_n since $\varepsilon^{n+1} t < \varepsilon^n t$. Therefore,

$$u(t,\varepsilon) = \hat{u}(T_0, T_1, T_2, \ldots; \varepsilon)$$

The order of the expansion is related to the number of scales. If we require an expansion valid for times as large as $O(\frac{1}{\varepsilon^M})$ (M is a positive integer), then we are required to introduce $M+1$ independent variables, $T_0,\ T_1,\ \ldots,\ T_M$.

To obtain the solution to the ODE, the following procedure is to be followed:

1. Begin by expanding $\hat{u}$ in powers of ε:

$$\begin{aligned} u(t;\varepsilon) &= \hat{u}(T_0,\ T_1,\ \ldots,\ T_M; \varepsilon) \\ &= u_0(T_0,\ T_1,\ \ldots,\ T_M) + \varepsilon u_1(T_0,\ T_1,\ \ldots,\ T_M) + \ldots + \varepsilon^M u_M(T_0,\ T_1,\ \ldots,\ T_M) + O\left(\varepsilon T_M\right) \end{aligned} \tag{9.13}$$

   The expansion breaks down when $\varepsilon T_M = O(1)$, i.e. $t = O\left(\frac{1}{\varepsilon^{M+1}}\right)$.

2. Rewrite $\frac{d}{dt}$ and $\frac{d^2}{dt^2}$ in terms of $\frac{\partial}{\partial T_i}$, for $0 \leq i \leq M$. We have

$$\frac{d}{dt} = \sum_{i=0}^{M} \frac{\partial T_i}{\partial t} \frac{\partial}{\partial T_i} = \sum_{i=0}^{M} \varepsilon^i \frac{\partial}{\partial T_i}. \tag{9.14}$$

   Also,

$$\frac{d^2}{dt^2} = \left(\sum_{i=0}^{M} \varepsilon^i \frac{\partial}{\partial T_i} \right)^2 = \sum_{i=0}^{M} \left(\varepsilon^i \frac{\partial}{\partial T_i} \right)^2 + 2 \sum_{i=0}^{M} \sum_{j=0}^{i-1} \varepsilon^i \varepsilon^j \frac{\partial}{\partial T_i T_j}. \tag{9.15}$$

3. Substitute (9.13), (9.14) and/or (9.15) into the ODE and equate like powers of ε. A system of $M+1$ PDEs is obtained in the variables $T_0,\ T_1,\ \ldots,\ T_M$ for $u_0,\ u_1,\ \ldots,\ u_M$. As seen in §5.5.4, the PDEs are solved by first integrating with respect to T_0. In doing so, arbitrary functions of $T_1,\ T_2,\ \ldots,\ T_M$ are obtained by insisting that for $T_0,\ T_1,\ \ldots,\ T_M,\ \frac{u_i}{u_{i-1}} < \infty$.

9.2.2 Near-identity Transform Method: Option Price Expansion

To solve for P, the near-identity transform method is applied order by order up to $O(\varepsilon^2)$, to transform the problem into a simple canonical problem.

We begin by performing a transformation up to leading order:

In expanding the exponent of P in (9.12), it is shown that if $(f-K)$ is $O(\varepsilon)$, then

$$\frac{(f-K)^2}{2\varepsilon^2\alpha^2C^2(K)\tau}(1+\ldots) =^{6} \frac{1}{2\tau}\left(\frac{1}{\varepsilon\alpha}\int_K^f \frac{dp}{C(p)}\right)^2(1+\ldots)$$

We are now going to transform the independent variables, f and α. This is similar to the process discussed in §9.2.1, where a new set of independent variables is introduced. Since we are expanding an exponential (regular perturbation expansions fail), we are required to re-scale $(f-K)$ to be $O(\varepsilon)$. This is closely linked with the method of Matched Asymptotic Expansions, when finding the distinguished limit to balance the inner and outer expansions in a boundary layer problem. In this case, by selecting a new set of variables judiciously, we can balance the partial derivatives and obtain a solution consisting of a leading order term which describes the physical situation well.

So, define

$$z := \frac{1}{\varepsilon\alpha}\int_K^f \frac{dp}{C(p)}, \tag{9.16}$$

$$B(\varepsilon\alpha z) := C(f), \tag{9.17}$$

and change variables from f to z but leave α unchanged. The new independent variables, z and α are $O(1)$ and we will be looking at the limit as $\varepsilon \to 0$, with z fixed.

So, $P = e^{-\frac{z^2}{2}}$, and we have the following transformation in partial derivatives:

$$\frac{\partial}{\partial f} =^{7} \frac{1}{\varepsilon\alpha B(\varepsilon\alpha z)}\frac{\partial}{\partial z}, \tag{9.18}$$

$$\frac{\partial}{\partial \alpha} = \frac{\partial}{\partial \alpha} - \frac{z}{\alpha}\frac{\partial}{\partial z}. \tag{9.19}$$

[6]Using a Taylor series expansion, we can show that up to $O(\varepsilon)$,

$$\frac{(f-K)^2}{C^2(K)} = \left(\int_K^f \frac{dp}{C(p)}\right)^2,$$

if $(f-K)$ is $O(\varepsilon)$. For a function $h(f)$, we have:

$$h(f) = h(K) + h'(K)(f-K).$$

Therefore, when $h(f) = \int_K^f \frac{dp}{C(p)}$, the expansion becomes

$$\left(\int_K^f \frac{dp}{C(p)}\right)^2 = \left(\int_K^K \frac{dp}{C(p)} + \frac{(f-K)}{C(K)}\right)^2 = \frac{(f-K)^2}{C^2(K)}.$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial f^2} &= \frac{\partial}{\partial f}\left(\frac{1}{\varepsilon\alpha B\left(\varepsilon\alpha z\right)}\frac{\partial}{\partial z}\right)\\
&= \frac{\partial}{\partial f}\left(\frac{1}{\varepsilon\alpha B\left(\varepsilon\alpha z\right)}\right)\frac{\partial}{\partial z} + \frac{1}{\varepsilon\alpha B\left(\varepsilon\alpha z\right)}\frac{\partial}{\partial f}\frac{\partial}{\partial z}\\
&= \left(\frac{1}{\varepsilon\alpha B\left(\varepsilon\alpha z\right)}\frac{\partial}{\partial z}\right)\left(\frac{1}{\varepsilon\alpha B\left(\varepsilon\alpha z\right)}\right)\frac{\partial}{\partial z} + \frac{1}{\varepsilon\alpha B\left(\varepsilon\alpha z\right)}\left(\frac{1}{\varepsilon\alpha B\left(\varepsilon\alpha z\right)}\frac{\partial}{\partial z}\right)\frac{\partial}{\partial z}\\
&= \frac{1}{\varepsilon\alpha B\left(\varepsilon\alpha z\right)}\frac{-1}{\varepsilon\alpha}\frac{B'\left(\varepsilon\alpha z\right)}{B^2\left(\varepsilon\alpha z\right)}\varepsilon\alpha\frac{\partial}{\partial z} + \frac{1}{\varepsilon^2\alpha^2 B^2\left(\varepsilon\alpha z\right)}\frac{\partial^2}{\partial z^2}\\
&= \frac{1}{\varepsilon^2\alpha^2 B^2\left(\varepsilon\alpha z\right)}\left(\frac{\partial^2}{\partial z^2} - \varepsilon\alpha\frac{B'\left(\varepsilon\alpha z\right)}{B\left(\varepsilon\alpha z\right)}\frac{\partial}{\partial z}\right),\\
\frac{\partial^2}{\partial f\,\partial\alpha} &= \frac{\partial}{\partial f}\left(\frac{\partial}{\partial\alpha} - \frac{z}{\alpha}\frac{\partial}{\partial z}\right)\\
&= \left(\frac{1}{\varepsilon\alpha B\left(\varepsilon\alpha z\right)}\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial\alpha} - \frac{z}{\alpha}\frac{\partial}{\partial z}\right)\\
&= \frac{1}{\varepsilon\alpha B\left(\varepsilon\alpha z\right)}\left(\frac{\partial}{\partial z}\frac{\partial}{\partial\alpha} - \frac{\partial}{\partial z}\left(\frac{z}{\alpha}\frac{\partial}{\partial z}\right)\right)\\
&= \frac{1}{\varepsilon\alpha B\left(\varepsilon\alpha z\right)}\left(\frac{\partial^2}{\partial z\,\partial\alpha} - \frac{1}{\alpha}\frac{\partial}{\partial z} - \frac{z}{\alpha}\frac{\partial^2}{\partial z^2}\right),\\
\frac{\partial^2}{\partial\alpha^2} &= \frac{\partial}{\partial\alpha}\left(\frac{\partial}{\partial\alpha} - \frac{z}{\alpha}\frac{\partial}{\partial z}\right)\\
&= \left(\frac{\partial}{\partial\alpha} - \frac{z}{\alpha}\frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial\alpha} - \frac{z}{\alpha}\frac{\partial}{\partial z}\right)\\
&= \frac{\partial^2}{\partial\alpha^2} - \frac{2z}{\alpha}\frac{\partial^2}{\partial z\,\partial\alpha} - 2\frac{\partial}{\partial\alpha}\left(\frac{z}{\alpha}\frac{\partial}{\partial z}\right) + \frac{z^2}{\alpha^2}\frac{\partial^2}{\partial z^2}\\
&= \frac{\partial^2}{\partial\alpha^2} - \frac{2z}{\alpha}\frac{\partial^2}{\partial z\,\partial\alpha} + \frac{2z}{\alpha^2}\frac{\partial}{\partial z} + \frac{z^2}{\alpha^2}\frac{\partial^2}{\partial z^2}.
\end{aligned}
$$

The initial condition at $\tau = 0$, $\delta(f-K) = 0$ for all $f \neq K$. Looking at (9.16) and (9.17), $z = 0$ when $f = K$, i.e $B(0) = C(K)$. Therefore, as a function of z

$$\delta(f-K) = \delta\left(\varepsilon\alpha z C(K)\right) = \frac{1}{\varepsilon\alpha C(K)}\delta(z). \tag{9.20}$$

The first equality arises from the definition of the Dirac delta function as an impulse function and the

[7] To avoid confusion in doing the transformation of the first order partial derivatives, we will refer to variables f, α and z with temporary subscripts indicating whether they are the new or old variables, then revert back to the original variables. We have that

$$z_{\text{new}} = \frac{1}{\varepsilon\alpha_{\text{old}}}\int_K^{f_{\text{old}}}\frac{dp}{C(p)}.$$

Then

$$
\begin{aligned}
\frac{\partial}{\partial f_{\text{old}}} &= \frac{\partial z_{\text{new}}}{\partial f_{\text{old}}}\frac{\partial}{\partial z_{\text{new}}} + \frac{\partial\alpha_{\text{new}}}{\partial f_{\text{old}}}\frac{\partial}{\partial\alpha_{\text{new}}} = \frac{1}{\varepsilon\alpha_{\text{old}}C(f_{\text{old}})}\frac{\partial}{\partial z_{\text{new}}} = \frac{1}{\varepsilon\alpha_{\text{old}}B\left(\varepsilon\alpha z_{\text{new}}\right)}\frac{\partial}{\partial z_{\text{new}}},\\
\frac{\partial}{\partial\alpha_{\text{old}}} &= \frac{\partial\alpha_{\text{new}}}{\partial\alpha_{\text{old}}}\frac{\partial}{\partial\alpha_{\text{new}}} + \frac{\partial z_{\text{new}}}{\partial\alpha_{\text{old}}}\frac{\partial}{\partial z_{\text{new}}} = \frac{\partial}{\partial\alpha_{\text{new}}} - \frac{z_{\text{new}}}{\alpha_{\text{old}}}\frac{\partial}{\partial z_{\text{new}}}.
\end{aligned}
$$

Now, $f_{\text{old}} = f$, $\alpha_{\text{old}} = \alpha_{\text{new}} = \alpha$ and $z_{\text{new}} = z$. So substituting these into the above partial derivatives result in (9.18) and (9.19).

second equality is from the property:

$$\int_{-\infty}^{\infty} \delta(x)dx = 1.$$

Thus, the pricing formula becomes

$$V(t,f,\alpha) = (f-K)^{+} + \frac{1}{2}\varepsilon^2 C^2(K)\int_0^{\tau_{\mathrm{ex}}} P(\tau,z,\alpha)d\tau, \tag{9.21}$$

where P is the solution of

$$\begin{aligned}
P_\tau &= \frac{1}{2}\varepsilon^2\alpha^2 C^2(f)P_{ff} + \varepsilon^2\rho\nu\alpha^2 C(f)P_{f\alpha} + \frac{1}{2}\varepsilon^2\nu^2\alpha^2 P_{\alpha\alpha} \\
&= \frac{1}{2}\varepsilon^2\alpha^2 B^2(\varepsilon\alpha z)\frac{1}{\varepsilon^2\alpha^2 B^2(\varepsilon\alpha z)}\left(\frac{\partial^2}{\partial z^2} - \varepsilon\alpha\frac{B'(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\frac{\partial}{\partial z}\right)P \\
&+ \varepsilon^2\rho\nu\alpha^2 B(\varepsilon\alpha z)\frac{1}{\varepsilon\alpha B(\varepsilon\alpha z)}\left(\frac{\partial^2}{\partial z\,\partial\alpha} - \frac{1}{\alpha}\frac{\partial}{\partial z} - \frac{z}{\alpha}\frac{\partial^2}{\partial z^2}\right)P \\
&+ \frac{1}{2}\varepsilon^2\nu^2\alpha^2\left(\frac{\partial^2}{\partial\alpha^2} - \frac{2z}{\alpha}\frac{\partial^2}{\partial z\,\partial\alpha} + \frac{2z}{\alpha^2}\frac{\partial}{\partial z} + \frac{z^2}{\alpha^2}\frac{\partial^2}{\partial z^2}\right)P \\
&= \frac{1}{2}P_{zz} - \frac{1}{2}\varepsilon\alpha\frac{B'(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}P_z + \varepsilon\rho\nu\alpha P_{z\alpha} - \varepsilon\rho\nu P_z - z\varepsilon\rho\nu P_{zz} \\
&+ \frac{1}{2}\varepsilon^2\nu^2\alpha^2 P_{\alpha\alpha} - \varepsilon^2\nu^2\alpha z P_{z\alpha} + \varepsilon^2\nu^2 z P_z + \frac{1}{2}\varepsilon^2\nu^2 z^2 P_{zz} \\
&= \frac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)P_{zz} - \frac{1}{2}\varepsilon\alpha\frac{B'}{B}P_z + \left(\varepsilon\rho\nu - \varepsilon^2\nu^2 z\right)\left(\alpha P_{z\alpha} - P_z\right) + \frac{1}{2}\varepsilon^2\nu^2\alpha^2 P_{\alpha\alpha},
\end{aligned}$$

for $\tau > 0$, subject to the initial condition ($\tau = 0$):

$$P = \frac{\alpha^2}{\varepsilon\alpha C(K)}\delta(z) = \frac{\alpha}{\varepsilon C(K)}\delta(z).$$

Now, we transform the dependent variable by defining

$$\hat{P} := \frac{\varepsilon}{\alpha}C(K)P. \tag{9.22}$$

Then,

$$P = \frac{\alpha}{\varepsilon C(K)}\hat{P},$$

and substituting this into (9.10) and once again using the above partial derivatives, we get that for $\tau > 0$,

$\hat{P}$ solves

$$
\begin{aligned}
\frac{\alpha}{\varepsilon C(K)}\hat{P}_\tau &= \frac{1}{2}\varepsilon^2\alpha^2B^2(\varepsilon\alpha z)\frac{1}{\varepsilon^2\alpha^2B^2(\varepsilon\alpha z)}\left(\frac{\partial^2}{\partial z^2}-\varepsilon\alpha\frac{B'(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\frac{\partial}{\partial z}\right)\left(\frac{\alpha}{\varepsilon C(K)}\hat{P}\right) \\
&\quad+\varepsilon^2\rho\nu\alpha^2B(\varepsilon\alpha z)\frac{1}{\varepsilon\alpha B(\varepsilon\alpha z)}\left(\frac{\partial^2}{\partial z\,\partial\alpha}-\frac{1}{\alpha}\frac{\partial}{\partial z}-\frac{z}{\alpha}\frac{\partial^2}{\partial z^2}\right)\left(\frac{\alpha}{\varepsilon C(K)}\hat{P}\right) \\
&\quad+\varepsilon^2\rho\nu\alpha^2B(\varepsilon\alpha z)\frac{1}{\varepsilon\alpha B(\varepsilon\alpha z)}\left(\frac{\partial^2}{\partial z\,\partial\alpha}-\frac{1}{\alpha}\frac{\partial}{\partial z}-\frac{z}{\alpha}\frac{\partial^2}{\partial z^2}\right)\left(\frac{\alpha}{\varepsilon C(K)}\hat{P}\right) \\
&\quad+\frac{1}{2}\varepsilon^2\nu^2\alpha^2\left(\frac{\partial^2}{\partial\alpha^2}-\frac{2z}{\alpha}\frac{\partial^2}{\partial z\,\partial\alpha}+\frac{2z}{\alpha^2}\frac{\partial}{\partial z}+\frac{z^2}{\alpha^2}\frac{\partial^2}{\partial z^2}\right)\left(\frac{\alpha}{\varepsilon C(K)}\hat{P}\right) \\
&= \frac{1}{2}\frac{\alpha}{\varepsilon C(K)}\hat{P}_{zz}-\frac{1}{2}\varepsilon\alpha\frac{B'(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\frac{\alpha}{\varepsilon C(K)}\hat{P}_z+\varepsilon\rho\nu\alpha\left(\frac{\partial}{\partial\alpha}\left(\frac{\alpha}{\varepsilon C(K)}\hat{P}_z\right)\right. \\
&\quad\left.-\frac{1}{\alpha}\frac{\alpha}{\varepsilon C(K)}\hat{P}_z-\frac{z}{\alpha}\frac{\alpha}{\varepsilon C(K)}\hat{P}_{zz}\right)+\frac{1}{2}\varepsilon^2\nu^2\alpha^2\left(\frac{\partial^2}{\partial\alpha^2}\left(\frac{\alpha C(K)}{\varepsilon}\hat{P}\right)-\frac{2z}{\alpha}\frac{\partial}{\partial\alpha}\left(\frac{\alpha}{\varepsilon C(K)}\hat{P}\right)\right. \\
&\quad\left.+\frac{z^2}{\alpha^2}\frac{\alpha}{\varepsilon C(K)}\hat{P}_{zz}+\frac{2z}{\alpha^2}\frac{\alpha}{\varepsilon C(K)}\hat{P}_z\right) \\
&= \frac{1}{2}\frac{\alpha}{\varepsilon C(K)}\hat{P}_{zz}-\frac{1}{2}\varepsilon\alpha\frac{B'}{B}\frac{\alpha}{\varepsilon C(K)}\hat{P}_z+\varepsilon\rho\nu\alpha\left(\frac{C(K)}{\varepsilon}\hat{P}_z+\frac{\alpha}{\varepsilon C(K)}\hat{P}_{z\alpha}\right. \\
&\quad\left.-\frac{1}{\alpha}\frac{\alpha}{\varepsilon C(K)}\hat{P}_z-\frac{z}{\alpha}\frac{\alpha}{\varepsilon C(K)}\hat{P}_{zz}\right)+\frac{1}{2}\varepsilon^2\nu^2\alpha^2\left(\frac{2}{\varepsilon C(K)}\hat{P}_\alpha+\frac{\alpha}{\varepsilon C(K)}\hat{P}_{\alpha\alpha}\right.{}^{8} \\
&\quad\left.-\frac{2z}{\alpha}\left(\frac{1}{\varepsilon C(K)}\hat{P}_z+\frac{\alpha}{\varepsilon C(K)}\hat{P}_{z\alpha}\right)+\frac{z^2}{\alpha^2}\frac{\alpha}{\varepsilon C(K)}\hat{P}_{zz}+\frac{2z}{\alpha^2}\frac{\alpha}{\varepsilon C(K)}\hat{P}_z\right) \\
&= \frac{\alpha}{\varepsilon C(K)}\left(\frac{1}{2}\hat{P}_{zz}-\frac{1}{2}\varepsilon\alpha\frac{B'}{B}\hat{P}_z+\varepsilon\rho\nu\hat{P}_z+\varepsilon\rho\nu\alpha\hat{P}_{z\alpha}-\varepsilon\rho\nu\hat{P}_z-\varepsilon\rho\nu z\hat{P}_{zz}+\varepsilon^2\nu^2\alpha\hat{P}_\alpha\right. \\
&\quad\left.+\frac{1}{2}\varepsilon^2\nu^2\alpha^2\hat{P}_{\alpha\alpha}-\frac{2z}{\alpha^2}\hat{P}_z-\frac{2z}{\alpha}\hat{P}_{z\alpha}+\frac{z^2}{\alpha^2}\hat{P}_{zz}+\frac{2z}{\alpha^2}\hat{P}_z\right) \\
\Rightarrow \hat{P}_\tau &= \frac{1}{2}\left(1-2\varepsilon\rho\nu z+\varepsilon^2\nu^2z^2\right)\hat{P}_{zz}-\frac{1}{2}\varepsilon\alpha\frac{B'}{B}\hat{P}_z+\left(\varepsilon\rho\nu-\varepsilon^2\nu^2z\right)\alpha\hat{P}_{z\alpha}+\frac{1}{2}\varepsilon^2\nu^2\left(\alpha^2\hat{P}_{\alpha\alpha}+2\alpha\hat{P}_\alpha\right)
\end{aligned}
$$

(9.11) at $\tau = 0$ then becomes

$$\hat{P} = \delta(z),$$

and the option price is then given as

$$V(t,f,\alpha) = (f-K)^+ + \frac{1}{2}\varepsilon\alpha C(K)\int_0^{\tau_{\text{ex}}}\hat{P}(\tau,z,\alpha)d\tau. \tag{9.23}$$

Looking at the above PDE for $\hat{P}$, we see that up to leading order, $\hat{P}$ solves the standard diffusion $\hat{P}_\tau = \frac{1}{2}\hat{P}_{zz}$ with $\hat{P} = \delta(z)$ at $\tau = 0$.

---

[8] Here, we calculate $\frac{\partial^2}{\partial\alpha^2}\left(\frac{\alpha}{\varepsilon C(K)}\hat{P}\right)$:

$$
\begin{aligned}
\frac{\partial}{\partial\alpha}\left(\frac{\alpha}{\varepsilon C(K)}\hat{P}\right) &= \frac{1}{\varepsilon C(K)}\hat{P}+\frac{\alpha}{\varepsilon C(K)}\hat{P}_\alpha, \\
\frac{\partial^2}{\partial\alpha^2}\left(\frac{1}{\varepsilon C(K)}\hat{P}+\frac{\alpha}{\varepsilon C(K)}\hat{P}_\alpha\right) &= \frac{2}{\varepsilon C(K)}\hat{P}_\alpha+\frac{\alpha}{\varepsilon C(K)}\hat{P}_{\alpha\alpha}.
\end{aligned}
$$

Next, we are required to transform the problem to a standard diffusion at $O(\varepsilon)$, and then $O(\varepsilon^2)$. The perturbed solution for $\hat{P}(\tau, z, \alpha, \varepsilon)$ can be expanded as

$$\hat{P}(\tau, z, \alpha, \varepsilon) = {}^9\hat{P}^0(\tau, z) + \varepsilon \hat{P}^1(\tau, z, \alpha) + \ldots$$

Notice that α enters the problem at $O(\varepsilon)$ and consequently, all partial derivatives $\hat{P}_\alpha$, $\hat{P}_{z\alpha}$ and $\hat{P}_{\alpha\alpha}$ are $O(\varepsilon)$. Therefore, for $\tau > 0$ up to $O(\varepsilon^2)$, the PDE becomes

$$\hat{P}_\tau = \frac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)\hat{P}_{zz} - \frac{1}{2}\varepsilon\alpha\frac{B'}{B}\hat{P}_z + \varepsilon\rho\nu\alpha\hat{P}_{z\alpha}$$

In order to remove the term $\frac{1}{2}\varepsilon\alpha\frac{B'}{B}\hat{P}_z$, we define $H(\tau, z, \alpha)$ by

$$\hat{P} = \sqrt{C(f)/C(K)}H := \sqrt{B(\varepsilon\alpha z)/B(0)}H. \tag{9.24}$$

Finding all the partial derivatives:

$$\begin{aligned}
\hat{P}_z &= \frac{\partial}{\partial z}\left(\sqrt{B(\varepsilon\alpha z)/B(0)}H\right) \\
&= \sqrt{B(\varepsilon\alpha z)/B(0)}H_z + \frac{1}{2}\frac{1}{\sqrt{B(\varepsilon\alpha z)/B(0)}}\frac{B_z(\varepsilon\alpha z)}{B(0)}\varepsilon\alpha H \\
&= \sqrt{B(\varepsilon\alpha z)/B(0)}H_z + \frac{1}{2}\sqrt{B(0)/B(\varepsilon\alpha z)}\frac{B_z(\varepsilon\alpha z)}{B(0)}\varepsilon\alpha\sqrt{B(\varepsilon\alpha z)/B(\varepsilon\alpha z)}H \\
&= \sqrt{B(\varepsilon\alpha z)/B(0)}\left(H_z + \frac{1}{2}\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon\alpha H\right),
\end{aligned}$$

$$\begin{aligned}
\hat{P}_{zz} &= \frac{\partial}{\partial z}\left(\sqrt{B(\varepsilon\alpha z)/B(0)}\left(H_z + \frac{1}{2}\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon\alpha H\right)\right) \\
&= \frac{\partial}{\partial z}\left(\sqrt{B(\varepsilon\alpha z)/B(0)}\right)\left(H_z + \frac{1}{2}\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon\alpha H\right) + \sqrt{B(\varepsilon\alpha z)/B(0)}\frac{\partial}{\partial z}\left(H_z + \frac{1}{2}\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon\alpha H\right) \\
&= \sqrt{B(\varepsilon\alpha z)/B(0)}\Bigg(\frac{1}{2}\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon\alpha\left(H_z + \frac{1}{2}\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon\alpha H\right) + H_{zz} \\
&\quad + \frac{1}{2}\varepsilon\alpha\left(\varepsilon\alpha\frac{B_{zz}(\varepsilon\alpha z)}{B(\varepsilon\alpha z)} - \varepsilon\alpha\left(\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\right)^2\right)H + \frac{1}{2}\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon\alpha H_z\Bigg) \\
&= \sqrt{B(\varepsilon\alpha z)/B(0)}\left(H_{zz} + \frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon\alpha H_z + \varepsilon^2\alpha^2\left(\frac{B_{zz}(\varepsilon\alpha z)}{2B(\varepsilon\alpha z)} - \frac{B_z^2(\varepsilon\alpha z)}{4B^2(\varepsilon\alpha z)}\right)H\right),
\end{aligned}$$

[9] The superscripts of $\hat{P}$ in the perturbation expansion are not powers of that function, but rather orders. This notation is adopted here, and further on in the text, to avoid confusion of subscripts that will be used for partial differentiation.

$$
\begin{aligned}
\hat{P}_{z\alpha} &= \frac{\partial}{\partial\alpha}\left(\sqrt{B(\varepsilon\alpha z)/B(0)}\left(H_z + \frac{1}{2}\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon\alpha H\right)\right) \\
&= \frac{\partial}{\partial\alpha}\left(\sqrt{B(\varepsilon\alpha z)/B(0)}\right)\left(H_z + \frac{1}{2}\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon\alpha H\right) + \sqrt{B(\varepsilon\alpha z)/B(0)}\frac{\partial}{\partial\alpha}\left(H_z + \frac{1}{2}\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon\alpha H\right) \\
&= \frac{1}{2}\frac{1}{\sqrt{B(\varepsilon\alpha z)/B(0)}}\frac{B_\alpha(\varepsilon\alpha z)}{B(0)}\varepsilon\alpha\sqrt{B_\alpha(\varepsilon\alpha z)/B(\varepsilon\alpha z)}\left(H_z + \frac{1}{2}\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon\alpha H\right) \\
&+ \sqrt{B(\varepsilon\alpha z)/B(0)}\Bigg(H_{z\alpha} + \frac{1}{2}\varepsilon\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}H + \frac{1}{2}\varepsilon\alpha\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}H_\alpha \\
&+ \frac{1}{2}\varepsilon\alpha\left(\frac{B_{z\alpha}(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon z - \frac{B_z(\varepsilon\alpha z)B_\alpha(\varepsilon\alpha z)}{B^2(\varepsilon\alpha z)}\varepsilon z\right)H\Bigg) \\
&= \sqrt{B(\varepsilon\alpha z)/B(0)}\left(\frac{1}{2}\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon\alpha\left(H_z + \frac{1}{2}\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}\varepsilon\alpha H\right)\right) + \sqrt{B(\varepsilon\alpha z)/B(0)}\Bigg(H_{z\alpha} + \frac{1}{2}\varepsilon\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}H \\
&+ \frac{1}{2}\varepsilon\alpha\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}H_\alpha + \frac{1}{2}\varepsilon^2\alpha z\frac{B_{z\alpha}(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}H - \frac{1}{2}\varepsilon^2\alpha z\frac{B_z(\varepsilon\alpha z)B_\alpha(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}H\Bigg) \\
&= \sqrt{B(\varepsilon\alpha z)/B(0)}\left(H_{z\alpha} + \frac{1}{2}\varepsilon z\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}H_z + \frac{1}{2}\varepsilon\alpha\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}H_\alpha + \frac{1}{2}\varepsilon\frac{B_z(\varepsilon\alpha z)}{B(\varepsilon\alpha z)}H + O(\varepsilon^2)\right)
\end{aligned}
$$

So, expressing the PDE for $\hat{P}$ in terms of H, we get that[10]

$$
\begin{aligned}
&\sqrt{B(\varepsilon\alpha z)/B(0)}H_\tau \\
&= \sqrt{B(\varepsilon\alpha z)/B(0)}\frac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)\left(H_{zz} + \frac{B'}{B}\varepsilon\alpha H_z + \varepsilon^2\alpha^2\left(\frac{B''}{2B} - \frac{B'^2}{4B^2}\right)H\right) \\
&- \frac{1}{2}\varepsilon\alpha\frac{B'}{B}\frac{\alpha C(K)}{\varepsilon}\sqrt{B(\varepsilon\alpha z)/B(0)}\left(H_z + \frac{1}{2}\frac{B'}{B}\varepsilon\alpha H\right) \\
&+ \sqrt{B(\varepsilon\alpha z)/B(0)}\alpha\varepsilon\rho\nu\left(H_{z\alpha} + \frac{1}{2}\varepsilon z\frac{B'}{B}H_z + \frac{1}{2}\varepsilon\alpha\frac{B'}{B}H_\alpha + \frac{1}{2}\varepsilon\frac{B'}{B}H + O(\varepsilon^2)\right)
\end{aligned}
$$

For $\tau > 0$ up to $O(\varepsilon^2)$,

$$
\begin{aligned}
H_\tau &= \frac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)H_{zz} - \frac{1}{2}\varepsilon^2\rho\nu\alpha\frac{B'}{B}\left(zH_z - H\right) \\
&+ \varepsilon^2\alpha^2\left(\frac{1}{4}\frac{B''}{B} - \frac{3}{8}\frac{B'^2}{B^2}\right)H + \varepsilon\rho\nu\alpha\left(H_{z\alpha} + \frac{1}{2}\varepsilon\alpha\frac{B'}{B}H_\alpha\right),
\end{aligned} \tag{9.25}
$$

with initial condition:

$$H = \delta(z).$$

Using the transformations given by (9.22) and (9.24) for P, the undiscounted option price (9.21) then

[10] Since we only have derivatives with respect to z and for notational simplicity, we express $B_z(\varepsilon\alpha z)$ as B' and $B_{zz}(\varepsilon\alpha z)$ as B''.

becomes

$$
\begin{aligned}
V(t,f,\alpha) &= (f-K)^+ + \frac{1}{2}\varepsilon^2 C^2(K)\int_0^{\tau_{\text{ex}}} P(\tau,z,\alpha)d\tau \\
&= (f-K)^+ + \frac{1}{2}\varepsilon^2 C^2(K)\int_0^{\tau_{\text{ex}}} \frac{\alpha}{\varepsilon C(K)}\hat{P}(\tau,z,\alpha)d\tau \\
&= (f-K)^+ + \frac{1}{2}\varepsilon^2 C^2(K)\int_0^{\tau_{\text{ex}}} \frac{\alpha}{\varepsilon C(K)}\sqrt{B(\varepsilon\alpha z)/B(0)}H(\tau,z,\alpha)d\tau \\
&= (f-K)^+ + \frac{1}{2}\varepsilon\alpha\sqrt{B(\varepsilon\alpha z)B(0)}\int_0^{\tau_{\text{ex}}} H(\tau,z,\alpha)d\tau, \qquad (9.26)
\end{aligned}
$$

using the fact that $\sqrt{B(\varepsilon\alpha z)/B(0)} = \sqrt{C(f)/C(K)}$.

Once again, the above PDE in H is independent of α to leading order, and at $O(\varepsilon)$, it depends on α through the term $\varepsilon\rho\nu\alpha\left(H_{z\alpha} + \frac{1}{2}\varepsilon\alpha\frac{B'}{B}H_\alpha\right)$. Therefore, the derivatives $H_{z\alpha}$ and $H_{\alpha\alpha}$ are no larger than $O(\varepsilon)$ and the last term is no larger than $O(\varepsilon^2)$. Thus, H is independent of α until $O(\varepsilon^2)$ which implies that the α derivatives are no larger than $O(\varepsilon^2)$. This then means that the last term is $O(\varepsilon^3)$ which can then be neglected. To see this, we expand $H(\tau,z,\alpha,\varepsilon)$ and substitute it into (9.25) and equate like powers of α:

$$
H(\tau,z,\alpha,\varepsilon) = H^0 + \varepsilon H^1 + \varepsilon^2 H^2 + \dots
$$

Then PDE (9.25) becomes:

$$
\begin{aligned}
&H^0_\tau + \varepsilon H^1_\tau + \varepsilon^2 H^2_\tau \\
&= \frac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)\left(H^0_{zz} + \varepsilon H^1_{zz} + \varepsilon^2 H^2_{zz}\right) - \frac{1}{2}\varepsilon^2\rho\nu\alpha\frac{B'}{B}\left(zH^0_z + z\varepsilon H^1_z + z\varepsilon^2 H^2_z\right) \\
&+ \frac{1}{2}\varepsilon^2\rho\nu\alpha\frac{B'}{B}\left(H^0 + \varepsilon H^1 + \varepsilon^2 H^2\right) + \varepsilon^2\alpha^2\left(\frac{1}{4}\frac{B''}{B} - \frac{3}{8}\frac{B'^2}{B^2}\right)\left(H^0 + \varepsilon H^1 + \varepsilon^2 H^2\right) \\
&+ \varepsilon\rho\nu\alpha\left(H^0_{z\alpha} + \varepsilon H^1_{z\alpha} + \varepsilon^2 H^2_{z\alpha} + \frac{1}{2}\varepsilon\alpha\frac{B'}{B}\left(H^0_\alpha + \varepsilon H^1_\alpha + \varepsilon^2 H^2_\alpha\right)\right) \qquad (9.27)
\end{aligned}
$$

At $O(1)$:

$$
\begin{aligned}
H^0_\tau &= \frac{1}{2}H^0_{zz}, \\
\Rightarrow H^0 &= H^0(\tau,z).
\end{aligned}
$$

At $O(\varepsilon)$:

$$
\begin{aligned}
H^1_\tau &= -\rho\nu z H^0_{zz} + \frac{1}{2}H^1_{zz} + \rho\nu\alpha H^0_{z\alpha} \\
&= -\rho\nu z H^0_{zz} + \frac{1}{2}H^1_{zz} \\
\Rightarrow H^1 &= H^1(\tau,z).
\end{aligned}
$$

At $O(\varepsilon^2)$:

$$
\begin{aligned}
H^2_\tau &= \nu^2 z^2 H^0_{zz} - \rho\nu z H^1_{zz} - \frac{1}{2}\rho\nu\alpha\frac{B'}{B}zH^0_z + \frac{1}{2}\rho\nu\alpha\frac{B'}{B}H^0 + \alpha^2\left(\frac{1}{4}\frac{B''}{B} - \frac{3}{8}\frac{B'^2}{B^2}\right)H^0 + \rho\nu\alpha H^1_{z\alpha} + \frac{1}{2}\rho\nu\alpha^2\frac{B'}{B}H^0_\alpha \\
&= \nu^2 z^2 H^0_{zz} - \rho\nu z H^1_{zz} - \frac{1}{2}\rho\nu\alpha\frac{B'}{B}zH^0_z + \frac{1}{2}\rho\nu\alpha\frac{B'}{B}H^0 + \alpha^2\left(\frac{1}{4}\frac{B''}{B} - \frac{3}{8}\frac{B'^2}{B^2}\right)H^0
\end{aligned}
$$

$\Rightarrow H^2 = H^2(\tau,z,\alpha)$.

From (9.27), we see that the derivatives of H^2 with respect to α only occur at $O(\varepsilon^3)$. Therefore, they can be dropped.

So, for $\tau > 0$, the PDE becomes

$$H_\tau = \frac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right) H_{zz} - \frac{1}{2}\varepsilon^2\rho\nu\alpha\frac{B'}{B}\left(zH_z - H\right) + \varepsilon^2\alpha^2\left(\frac{1}{4}\frac{B''}{B} - \frac{3}{8}\frac{B'^2}{B^2}\right) H. \tag{9.28}$$

Clearly, there are no longer any α derivatives in the above equation and consequently, α can be treated as a constant, as opposed to an independent variable.

We next remove the H_z term to $O(\varepsilon^2)$. By noting that $\frac{B'}{B}$ and $\frac{B''}{B}$ are constants to leading order, we replace them with

$$b_1 = \frac{B'(\varepsilon\alpha z_0)}{B(\varepsilon\alpha z_0)}, \quad b_2 = \frac{B''(\varepsilon\alpha z_0)}{B(\varepsilon\alpha z_0)}, \tag{9.29}$$

with the error being $O(\varepsilon)$[11]. The constant z_0 will be determined later. Next, we perform another transformation by defining $\hat{H}$ by

$$H := e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}\hat{H}. \tag{9.30}$$

Calculating the z partial derivatives of H in terms of $\hat{H}$, we get that

$$\begin{aligned}
H_z &= \frac{\partial}{\partial z}\left(e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}\hat{H}\right)\\
&= e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}\left(\hat{H}_z + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1 z\hat{H}\right),\\
H_{zz} &= \frac{\partial}{\partial z}\left(e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}\left(\hat{H}_z + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1 z\hat{H}\right)\right)\\
&= e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}\left(\frac{1}{2}\varepsilon^2\rho\nu\alpha b_1 z\left(\hat{H}_z + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1 z\hat{H}\right) + \hat{H}_{zz} + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1\hat{H} + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1 z\hat{H}_z\right)\\
&= e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}\left(\varepsilon^2\rho\nu\alpha b_1 z\hat{H}_z + \hat{H}_{zz} + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1\hat{H} + O(\varepsilon^4)\right)
\end{aligned}$$

Substituting this into (9.28) and dividing through by $e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}$, we get the following PDE in $\hat{H}$ for $\tau > 0$:

$$\begin{aligned}
\hat{H}_\tau &= \frac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)\left(\varepsilon^2\rho\nu\alpha b_1 z\hat{H}_z + \hat{H}_{zz} + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1\hat{H}\right)\\
&\quad - \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1\left(z\hat{H}_z + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1 z^2\hat{H} - \hat{H}\right) + \varepsilon^2\alpha^2\left(\frac{1}{4}b_2 - \frac{3}{8}b_1^2\right)\hat{H}\\
&= \frac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)\hat{H}_{zz} + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1 z\hat{H}_z + \frac{1}{4}\varepsilon^2\rho\nu\alpha b_1\hat{H} - \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1 z\hat{H}_z\\
&\quad + \frac{1}{2}\varepsilon^2\rho\nu\alpha b_1\hat{H} + \varepsilon^2\alpha^2\left(\frac{1}{4}b_2 - \frac{3}{8}b_1^2\right)\hat{H}\\
\Rightarrow \hat{H}_\tau &= \frac{1}{2}\left(1 - 2\varepsilon\rho\nu z + \varepsilon^2\nu^2 z^2\right)\hat{H}_{zz} + \frac{3}{4}\varepsilon^2\rho\nu\alpha b_1\hat{H} + \varepsilon^2\alpha^2\left(\frac{1}{4}b_2 - \frac{3}{8}b_1^2\right)\hat{H}
\end{aligned}$$

[11] Since

$$\varepsilon\alpha z = \int_K^f \frac{dp}{C(p)}$$

which is $O(\varepsilon)$, the error in replacing $\frac{B'}{B}$ and $\frac{B''}{B}$ with constants is therefore also $O(\varepsilon)$.

So, using the definition of $\hat{H}$, the option price then becomes:

$$V(t,f,\alpha) = (f-K)^{+} + \frac{1}{2}\varepsilon\alpha\sqrt{B(\varepsilon\alpha z)B(0)}e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}\int_0^{\tau_{\text{ex}}}\hat{H}(\tau,z)d\tau. \tag{9.31}$$

Next, define a new independent variable x by

$$x := \frac{1}{\varepsilon\nu}\int_0^{\varepsilon\nu z}\frac{d\xi}{\sqrt{1-2\rho\xi+\xi^2}}. \tag{9.32}$$

By using the identity:

$$\int\frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right| + C,$$

where C is a constant of integration, the above integral can be solved. Begin by completing the square:

$$\xi^2 - 2\rho\xi + 1 = (\xi-\rho)^2 - (\rho^2-1).$$

In this case, let $u = (\xi-\rho)$ and $a = \sqrt{\rho^2-1}$. The limits of integration shift by $-\rho$. Thus,

$$\begin{aligned}
x &= \frac{1}{\varepsilon\nu}\int_0^{\varepsilon\nu z}\frac{d\xi}{\sqrt{1-2\rho\xi+\xi^2}}\\
&= \frac{1}{\varepsilon\nu}\int_{-\rho}^{\varepsilon\nu z-\rho}\frac{du}{\sqrt{u^2-a^2}}\\
&= \frac{1}{\varepsilon\nu}\ln\left|u+\sqrt{u^2-a^2}\right|\Big|_{-\rho}^{\varepsilon\nu z-\rho}\\
&= \frac{1}{\varepsilon\nu}\left(\ln\left(\varepsilon\nu z-\rho+\sqrt{(\varepsilon\nu z-\rho)^2-(\rho^2-1)}\right) - \ln\left(-\rho+\sqrt{(-\rho)^2-(\rho^2-1)}\right)\right)\\
&= \frac{1}{\varepsilon\nu}\left(\ln\left(\varepsilon\nu z-\rho+\sqrt{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}\right) - \ln(1-\rho)\right)\\
&= \frac{1}{\varepsilon\nu}\ln\left(\frac{\varepsilon\nu z-\rho+\sqrt{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}}{1-\rho}\right)
\end{aligned}$$

Using the definitions of the hyperbolic sine and cosine functions, we can solve for $\varepsilon\nu z$ implicitly. We have that,

$$\sinh x = \frac{e^x-e^{-x}}{2}, \tag{9.33}$$

$$\cosh x = \frac{e^x+e^{-x}}{2}. \tag{9.34}$$

Then, simplifying the expression above for x:

$$\begin{aligned}
\varepsilon\nu x &= \ln\left(\frac{\varepsilon\nu z-\rho+\sqrt{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}}{1-\rho}\right)\\
(1-\rho)e^{\varepsilon\nu x} &= \varepsilon\nu z-\rho+\sqrt{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}\\
((1-\rho)e^{\varepsilon\nu x}-\varepsilon\nu z+\rho)^2 &= \varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1\\
(1-\rho)^2e^{2\varepsilon\nu x}-2(1-\rho)e^{\varepsilon\nu x}(\varepsilon\nu z-\rho)+(\varepsilon\nu z-\rho)^2 &= \varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1
\end{aligned}$$

Squaring the left hand side and simplifying, we get

$$\begin{aligned}
(1-\rho)^2e^{2\varepsilon\nu x}-2e^{\varepsilon\nu x}(\varepsilon\nu z-\rho)+2\rho e^{\varepsilon\nu x}(\varepsilon\nu z-\rho)+\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+\rho^2 &= \varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1\\
(1-\rho)^2e^{2\varepsilon\nu x}-2e^{\varepsilon\nu x}\varepsilon\nu z+2\rho e^{\varepsilon\nu x}+2\rho e^{\varepsilon\nu x}\varepsilon\nu z-2\rho^2e^{\varepsilon\nu x}+\rho^2 &= 1
\end{aligned}$$

Gathering terms in $\varepsilon\nu z$ on the left hand side, then dividing through first by $e^{\varepsilon\nu x}$, then by $(\rho-1)/2$, equation becomes

$$
\begin{aligned}
2\varepsilon\nu z e^{\varepsilon\nu x}(\rho-1) &= 1-\rho^2+2\rho^2 e^{\varepsilon\nu x}-(1-\rho)^2 e^{2\varepsilon\nu x}-2\rho e^{\varepsilon\nu x}\\
2\varepsilon\nu z(\rho-1) &= \left(1-\rho^2\right)e^{-\varepsilon\nu x}+2\rho(\rho-1)-(1-\rho)^2 e^{\varepsilon\nu x}\\
\varepsilon\nu z &= -(\rho+1)\frac{e^{-\varepsilon\nu x}}{2}+\rho-(\rho-1)\frac{e^{\varepsilon\nu x}}{2}\\
&= \frac{e^{\varepsilon\nu x}-e^{-\varepsilon\nu x}}{2}-\rho\left(\frac{e^{\varepsilon\nu x}+e^{-\varepsilon\nu x}}{2}-1\right)\\
&= \sinh\varepsilon\nu x-\rho\left(\cosh\varepsilon\nu x-1\right),
\end{aligned}
$$

where in the last line, the definitions of the hyperbolic sine and cosine were used.

Once again, we find the partial derivatives of $\hat{H}$ with respect to the new variable, $x=x(z)$. So, using the Leibnitz rule (Abramowitz & Stegun 1974), we get

$$
\begin{aligned}
\frac{\partial}{\partial z} &\overset{12}{=} \frac{dx}{dz}\frac{\partial}{\partial x}\\
&= \frac{d}{dz}\left(\frac{1}{\varepsilon\nu}\int_0^{\varepsilon\nu z}\frac{d\xi}{\sqrt{1-2\rho\xi+\xi^2}}\right)\frac{\partial}{\partial x}\\
&= \frac{1}{\varepsilon\nu}\left(\frac{\varepsilon\nu}{\sqrt{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}}\right)\frac{\partial}{\partial x}\\
&= \frac{1}{\sqrt{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}}\frac{\partial}{\partial x},
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{\partial^2}{\partial z^2} &= \frac{\partial}{\partial z}\left(\frac{dx}{dz}\frac{\partial}{\partial x}\right)\\
&= \frac{d^2x}{dz^2}\frac{\partial}{\partial x}+\left(\frac{dx}{dz}\right)^2\frac{\partial^2}{\partial x^2}\\
&= -\frac{1}{2}\frac{1}{\sqrt{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}}\frac{2\varepsilon^2\nu^2z-2\varepsilon\rho\nu}{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}\frac{\partial}{\partial x}+\frac{1}{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}\frac{\partial^2}{\partial x^2}\\
&= \frac{1}{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}\left(\frac{\partial^2}{\partial x^2}-\frac{\varepsilon^2\nu^2z-\varepsilon\rho\nu}{\sqrt{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}}\frac{\partial}{\partial x}\right)
\end{aligned}
$$

Then, the PDE for $\hat{H}(\tau,x)$ becomes

$$
\begin{aligned}
\hat{H}_\tau &= \frac{1}{2}\left(1-2\varepsilon\rho\nu z+\varepsilon^2\nu^2z^2\right)\frac{1}{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}\left(\frac{\partial^2}{\partial x^2}-\frac{\varepsilon^2\nu^2z-\varepsilon\rho\nu}{\sqrt{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}}\frac{\partial}{\partial x}\right)\hat{H}\\
&+\frac{3}{4}\varepsilon^2\rho\nu\alpha b_1\hat{H}+\varepsilon^2\alpha^2\left(\frac{1}{4}b_2-\frac{3}{8}b_1^2\right)\hat{H}\\
&= \frac{1}{2}\hat{H}_{xx}-\frac{1}{2}\frac{\varepsilon^2\nu^2z-\varepsilon\rho\nu}{\sqrt{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}}\hat{H}_x+\frac{3}{4}\varepsilon^2\rho\nu\alpha b_1\hat{H}+\varepsilon^2\alpha^2\left(\frac{1}{4}b_2-\frac{3}{8}b_1^2\right)\hat{H},
\end{aligned}
$$

and the undiscounted option price can be rewritten in terms of x by substituting By defining

$$
I(\zeta) := \sqrt{\zeta^2-2\rho\zeta+1} \tag{9.35}
$$

[12] We use total differentiation, $\frac{dx}{dz}$, as opposed to partial, $\frac{\partial x}{\partial z}$, since $x=x(z)$.

The above PDE can be simplified for ($\tau > 0$) as:

$$\hat{H}_\tau = {}^{13}\frac{1}{2}\hat{H}_{xx} - \frac{1}{2}\varepsilon\nu I'(\varepsilon\nu z)\hat{H}_x + \frac{3}{4}\varepsilon^2\rho\nu\alpha b_1\hat{H} + \varepsilon^2\alpha^2\left(\frac{1}{4}b_2 - \frac{3}{8}b_1^2\right)\hat{H} \tag{9.36}$$

with

$$\hat{H}(0,x) = \delta(x). \tag{9.37}$$

In terms of x, the option price (9.31) can be expressed as:

$$V(t,f,\alpha) = (f-K)^+ + \frac{1}{2}\varepsilon\alpha\sqrt{B(\varepsilon\alpha z)B(0)}e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}\int_0^{\tau_{\text{ex}}}\hat{H}(\tau,x)d\tau. \tag{9.38}$$

The next step is to transform the dependent variable by defining Q by

$$\hat{H} := I^{1/2}\left(\varepsilon\nu z(x)\right)Q = \left(\varepsilon^2\nu^2z^2 - 2\varepsilon\rho\nu z + 1\right)^{1/4}Q. \tag{9.39}$$

To rewrite (9.36) in terms of Q, we first note that

$$\frac{dz}{dx} = \frac{1}{\frac{dx}{dz}} = \sqrt{\varepsilon^2\nu^2z^2 - 2\rho\varepsilon\nu z + 1}.$$

Therefore, we find that

$$\begin{aligned}
\hat{H}_x &= \frac{\partial}{\partial x}\left(I^{1/2}\left(\varepsilon\nu z\right)Q\right)\\
&= I^{1/2}\left(\varepsilon\nu z\right)Q_x + \frac{1}{2}I^{-1/2}\left(\varepsilon\nu z\right)\varepsilon\nu I'(\varepsilon\nu z)\frac{dz}{dx}Q\\
&= I^{1/2}\left(\varepsilon\nu z\right)Q_x + \frac{1}{2}I^{-1/2}\left(\varepsilon\nu z\right)\varepsilon\nu I'(\varepsilon\nu z)I(\varepsilon\nu z)Q\\
&= I^{1/2}\left(\varepsilon\nu z\right)\left(Q_x + \frac{1}{2}\varepsilon\nu I'(\varepsilon\nu z)Q\right),
\end{aligned}$$

and

$$\begin{aligned}
\hat{H}_{xx} &= \frac{\partial}{\partial x}\left(I^{1/2}\left(\varepsilon\nu z\right)Q_x + \frac{1}{2}\varepsilon\nu I'(\varepsilon\nu z)I^{1/2}\left(\varepsilon\nu z\right)Q\right)\\
&= \frac{1}{2}\varepsilon\nu I'(\varepsilon\nu z)I^{1/2}\left(\varepsilon\nu z\right)Q_x + I^{1/2}\left(\varepsilon\nu z\right)Q_{xx} + \frac{1}{2}\varepsilon\nu\Bigg(\left(I''(\varepsilon\nu z)\varepsilon\nu I(\varepsilon\nu z)\right)I^{1/2}\left(\varepsilon\nu z\right)Q\\
&+ I'(\varepsilon\nu z)\left(\frac{1}{2}\varepsilon\nu I'(\varepsilon\nu z)I^{1/2}\left(\varepsilon\nu z\right)\right)Q + I'(\varepsilon\nu z)I^{1/2}\left(\varepsilon\nu z\right)Q_x\Bigg)\\
&= I^{1/2}\left(\varepsilon\nu z\right)\left(Q_{xx} + \varepsilon\nu I'(\varepsilon\nu z)Q_x + \varepsilon^2\nu^2\left(\frac{1}{2}I''(\varepsilon\nu z)I(\varepsilon\nu z) + \frac{1}{4}I'(\varepsilon\nu z)I'(\varepsilon\nu z)\right)Q\right).
\end{aligned}$$

[13] Using the definition of $I(\zeta)$, we see that

$$I'(\zeta) = \frac{1}{2}\frac{2\zeta - 2\rho}{\sqrt{\zeta^2 - 2\rho\zeta + 1}} = \frac{\zeta - \rho}{\sqrt{\zeta^2 - 2\rho\zeta + 1}}.$$

Thus,

$$\varepsilon\nu I'(\varepsilon\nu z) = \frac{\varepsilon^2\nu^2 z - \rho\varepsilon\nu}{\sqrt{\varepsilon^2\nu^2z^2 - 2\rho\varepsilon\nu z + 1}}.$$

Substituting this into (9.36)[14], we get

$$
\begin{aligned}
I^{1/2}Q_\tau &= \frac{1}{2}I^{1/2}\left(Q_{xx}+\varepsilon\nu I'Q_x+\varepsilon^2\nu^2\left(\frac{1}{2}I''I+\frac{1}{4}I'I'\right)Q\right)-\frac{1}{2}\varepsilon\nu I'\left(I^{1/2}\left(Q_x+\frac{1}{2}\varepsilon\nu I'Q\right)\right)\\
&\quad+\frac{3}{4}\varepsilon^2\rho\nu\alpha b_1 I^{1/2}Q+\varepsilon^2\alpha^2\left(\frac{1}{4}b_2-\frac{3}{8}b_1^2\right)I^{1/2}Q\\
\Rightarrow Q_\tau &= \frac{1}{2}Q_{xx}+\varepsilon^2\nu^2\left(\frac{1}{4}I''I-\frac{1}{8}I'I'\right)Q+\frac{3}{4}\varepsilon^2\rho\nu\alpha b_1 Q+\varepsilon^2\alpha^2\left(\frac{1}{4}b_2-\frac{3}{8}b_1^2\right)Q
\end{aligned}
\tag{9.40}
$$

with

$$
Q(0,x)=\delta(x), \tag{9.41}
$$

and (9.38) becomes

$$
V(t,f,\alpha)=(f-K)^+ +\frac{1}{2}\varepsilon\alpha\sqrt{B(\varepsilon\alpha z)B(0)}I^{1/2}(\varepsilon\nu z)e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}\int_0^{\tau_{\text{ex}}}Q(\tau,x)d\tau. \tag{9.42}
$$

As was done previously, replace z in $I(\varepsilon\nu z)$, $I'(\varepsilon\nu z)$ and $I''(\varepsilon\nu z)$ with a constant z_0 which will be chosen later. In doing so, the error is once again $O(\varepsilon)$. Now, define a constant κ by

$$
\kappa:=\nu^2\left(\frac{1}{4}I''(\varepsilon\nu z_0)I(\varepsilon\nu z_0)-\frac{1}{8}\left(I'(\varepsilon\nu z_0)\right)^2\right)+\frac{3}{4}\rho\nu\alpha b_1+\alpha^2\left(\frac{1}{4}b_2-\frac{3}{8}b_1^2\right) \tag{9.43}
$$

Then, for $\tau>0$, (9.40) can be simplified to

$$
Q_\tau=\frac{1}{2}Q_{xx}+\varepsilon^2\kappa Q, \tag{9.44}
$$

subject to

$$
Q(0,x)=\delta(x). \tag{9.45}
$$

To solve this, begin by making the substitution:

$$
Q=e^{\varepsilon^2\kappa\tau}W.
$$

Then,

$$
\begin{aligned}
Q_\tau &= e^{\varepsilon^2\kappa\tau}\left(W_\tau+\varepsilon^2\kappa W\right),\\
Q_{xx} &= e^{\varepsilon^2\kappa\tau}W_{xx}.
\end{aligned}
$$

So, (9.44) becomes

$$
\begin{aligned}
e^{\varepsilon^2\kappa\tau}\left(W_\tau+\varepsilon^2\kappa W\right) &= \frac{1}{2}e^{\varepsilon^2\kappa\tau}W_{xx}+\varepsilon^2\kappa e^{\varepsilon^2\kappa\tau}W\\
\Rightarrow W_\tau &= \frac{1}{2}W_{xx},
\end{aligned}
$$

which is basically the heat equation with initial condition (9.45). This is commonly referred to as a Cauchy problem (initial value problem) and can be solved via a convolution of the heat kernel and the initial condition (Hulley & Lotter 2004, §7.8.2). We first define the heat kernel:

[14] For notational simplicity, $I(\varepsilon\nu z)=I$

Definition 13 ***Fundamental Solution***

(Hulley & Lotter 2004, §7.8.1) The expression

$$W_\delta(\tau, x) = \frac{1}{\sqrt{4\pi\tau}} \exp\left(-\frac{x^2}{4\tau}\right) \tag{9.46}$$

is called the fundamental solution of the heat equation, or the heat kernel.

To get the PDE into the standard format, let $\bar{\tau} = \frac{1}{2}\tau$. Then

$$\frac{\partial W}{\partial \tau} = \frac{\partial W}{\partial \bar{\tau}}\frac{\partial \bar{\tau}}{\partial \tau} = \frac{1}{2}\frac{\partial W}{\partial \bar{\tau}}.$$

So, the PDE is now:

$$\begin{aligned} W_{\bar{\tau}} &= W_{xx}, \\ W_0 = W(0,x) &= \delta(x). \end{aligned}$$

The solution of this is a convolution of the initial condition, W_0, and the heat kernel:

$$\begin{aligned} W(\bar{\tau}, x) = \left(W_0 * W_\delta(\bar{\tau}, \cdot)\right)(x) &:= \int_{-\infty}^{\infty} W(0,x) * W_\delta(\bar{\tau}, x-y)dy \\ &= \int_{-\infty}^{\infty} W(0,x-y) * W_\delta(\bar{\tau}, y)dy \\ &= \frac{1}{\sqrt{4\pi\bar{\tau}}}\int_{-\infty}^{\infty} \delta(x-y)\exp\left(-\frac{x^2}{4\bar{\tau}}\right)dy \\ &= \frac{1}{\sqrt{4\pi\bar{\tau}}}\exp\left(-\frac{x^2}{4\bar{\tau}}\right) \end{aligned}$$

Now substituting back for τ:

$$W(\tau, x) = \frac{1}{\sqrt{2\pi\tau}} e^{-x^2/2\tau}. \tag{9.47}$$

So,

$$Q = \frac{1}{\sqrt{2\pi\tau}} e^{-x^2/2\tau} e^{\varepsilon^2\kappa\tau} =^{15} \frac{1}{\sqrt{2\pi\tau}} e^{-x^2/2\tau} \frac{1}{\left(1 - \frac{2}{3}\varepsilon^2\kappa\tau + \ldots\right)^{3/2}} \tag{9.48}$$

The option price then becomes

$$V(t,f,\alpha) = (f-K)^+ + \frac{1}{2}\varepsilon\alpha\sqrt{B(\varepsilon\alpha z)B(0)} I^{1/2}(\varepsilon\nu z) e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}\int_0^{\tau_{\text{ex}}} \frac{1}{\sqrt{2\pi\tau}} e^{-x^2/2\tau} e^{\varepsilon^2\kappa\tau} d\tau. \tag{9.49}$$

---

[15] Consider the expansion of $e^{\varepsilon^2\kappa\tau}$ up to $O(\varepsilon^2)$:

$$e^{\varepsilon^2\kappa\tau} = 1 + \varepsilon^2\kappa\tau + \ldots$$

So,

$$\begin{aligned} \frac{1}{\left(1 - \frac{2}{3}\varepsilon^2\kappa\tau + \ldots\right)^{3/2}} = \left(1 - \frac{2}{3}\varepsilon^2\kappa\tau + \ldots\right)^{-3/2} &= 1 - \frac{2}{3}\left(-\frac{3}{2}\right)\varepsilon^2\kappa\tau + \ldots \\ &= 1 + \varepsilon^2\kappa\tau + \ldots \end{aligned}$$

as required.

To simplify this expression, define θ by

$$\begin{aligned}
\varepsilon^2\theta &:= \ln\left(\frac{\varepsilon\alpha z}{f-K}\sqrt{B(\varepsilon\alpha z)B(0)}\right) + \ln\left(\frac{xI^{1/2}(\varepsilon\nu z)}{z}\right) + \frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2 \\
\Rightarrow e^{\varepsilon^2\theta} &= \frac{\varepsilon\alpha z}{f-K}\sqrt{B(\varepsilon\alpha z)B(0)}\frac{xI^{1/2}(\varepsilon\nu z)}{z}e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}. \qquad (9.50)
\end{aligned}$$

Then, (9.49) can be simplified:

$$\begin{aligned}
&V(t,f,\alpha) \\
&= (f-K)^+ + \frac{1}{2}\varepsilon\alpha\sqrt{B(\varepsilon\alpha z)B(0)}I^{1/2}(\varepsilon\nu z)e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}\int_0^{\tau_{\mathrm{ex}}}\frac{1}{\sqrt{2\pi\tau}}e^{-x^2/2\tau}e^{\varepsilon^2\kappa\tau}\,d\tau \\
&= (f-K)^+ + \frac{1}{2}\frac{f-K}{x}\left(\frac{\varepsilon\alpha z}{f-K}\sqrt{B(\varepsilon\alpha z)B(0)}\frac{xI^{1/2}(\varepsilon\nu z)}{z}e^{\varepsilon^2\rho\nu\alpha b_1 z^2/4}\right)\int_0^{\tau_{\mathrm{ex}}}\frac{1}{\sqrt{2\pi\tau}}e^{-x^2/2\tau}e^{\varepsilon^2\kappa\tau}\,d\tau \\
&= (f-K)^+ + \frac{1}{2}\frac{f-K}{x}\int_0^{\tau_{\mathrm{ex}}}\frac{1}{\sqrt{2\pi\tau}}e^{-x^2/2\tau}e^{\varepsilon^2\theta}e^{\varepsilon^2\kappa\tau}\,d\tau \qquad (9.51)
\end{aligned}$$

In order to simplify this expression, the following simplification which is valid up to $O(\varepsilon^2)$, will be used:

$$e^{\varepsilon^2\kappa\tau} = \frac{1}{\left(1-\frac{2}{3}\kappa\varepsilon^2\tau\right)^{3/2}} = \frac{1}{\left(1-2\varepsilon^2\tau\frac{\theta}{x^2}\right)^{3/2}}. \qquad (9.52)$$

This implies that $\varepsilon^2\kappa = 3\varepsilon^2\frac{\theta}{x^2}$. To verify this, we begin by expanding $\varepsilon^2\frac{\theta}{x^2}$. Using the Definition (9.32), we obtain an expansion for $\frac{1}{x^2}$ up to $O(\varepsilon^2)$ by expanding the integrand using the binomial theorem (Goldberg 1976, §8.6):

$$\begin{aligned}
x^{-2} &= \left(\frac{1}{\varepsilon\nu}\int_0^{\varepsilon\nu z}\left(1-2\rho\xi+\xi^2\right)^{-1/2}d\xi\right)^{-2} \\
&= \left(\frac{1}{\varepsilon\nu}\int_0^{\varepsilon\nu z}\left(1-\frac{1}{2}\left(-2\rho\xi+\xi^2\right)-\frac{1}{2}\left(-\frac{3}{2}\right)\frac{1}{2}\left(-2\rho\xi+\xi^2\right)^2+\dots\right)d\xi\right)^{-2} \\
&= \left(\frac{1}{\varepsilon\nu}\int_0^{\varepsilon\nu z}\left(1+\rho\xi-\frac{1}{2}\xi^2+\frac{3}{2}\rho^2\xi^2+\dots\right)d\xi\right)^{-2} \\
&= \left(\frac{1}{\varepsilon\nu}\left(\xi+\frac{\rho}{2}\xi^2-\frac{1}{6}\xi^3+\frac{1}{2}\rho^2\xi^3\Big|_0^{\varepsilon\nu z}\right)\right)^{-2} \\
&= \left(z+\frac{\rho}{2}\varepsilon\nu z^2-\frac{1}{6}(\varepsilon\nu)^2z^3+\frac{1}{2}\rho^2(\varepsilon\nu)^2z^3\right)^{-2} \\
&= \frac{1}{z^2}\left(1+\frac{\rho}{2}\varepsilon\nu z-\frac{1}{6}(\varepsilon\nu z)^2+\frac{1}{2}\rho^2(\varepsilon\nu z)^2\right)^{-2} \\
&= \frac{1}{z^2}\left(1+(-2)\left(\frac{\rho}{2}\varepsilon\nu z-\frac{1}{6}(\varepsilon\nu z)^2+\frac{1}{2}\rho^2(\varepsilon\nu z)^2\right)+\frac{(-2)(-3)}{2}\left(\frac{\rho^2}{4}(\varepsilon\nu z)^2+\dots\right)\right) \\
&= \frac{1}{z^2}\left(1-\rho\varepsilon\nu z-\frac{1}{4}\rho^2(\varepsilon\nu z)^2\right). \qquad (9.53)
\end{aligned}$$

Since $\varepsilon^2\theta$ is already $O(\varepsilon^2)$, we can conclude that the only term that will make a contribution to the expansion is $\frac{1}{z^2}$. Each of the three terms of $\varepsilon^2\theta$ will be expanded separately.

1. Consider the first term:

$$\ln\left(\frac{\varepsilon\alpha z}{f-K}\sqrt{B(0)B(\varepsilon\alpha z)}\right).$$

Since

$$f - K = {}^{16}\varepsilon\alpha \int_0^z B(\varepsilon\alpha p)dp \tag{9.54}$$

up to $O(\varepsilon)$, we expand $(f-K)^{-1}$ as well as $B(\varepsilon\alpha z)$ to simplify this term.

$$\begin{aligned}
&\left(\varepsilon\alpha \int_0^z B(\varepsilon\alpha p)dp\right)^{-1} \\
&= \left(\varepsilon\alpha \int_0^z \left(B(0) + \varepsilon\alpha p B'(0) + \frac{(\varepsilon\alpha p)^2}{2} B''(0) + \dots\right) dp\right)^{-1} \\
&= \left(\varepsilon\alpha \left(pB(0) + \frac{\varepsilon\alpha}{2} p^2 B'(0) + \frac{(\varepsilon\alpha)^2}{6} p^3 B''(0)\Big|_0^z\right)\right)^{-1} \\
&= \frac{1}{\varepsilon\alpha z B(0)} \left(1 + \frac{\varepsilon\alpha z}{2}\frac{B'(0)}{B(0)} + \frac{(\varepsilon\alpha z)^2}{6}\frac{B''(0)}{B(0)}\right)^{-1}
\end{aligned}$$

Clearly, the choice of $z_0 = 0$ in (9.29) and (9.43) is the sufficient to guarantee accuracy up to $O(\varepsilon)$. Therefore, using (9.29), we have that

$$\begin{aligned}
(f-K)^{-1} &= \frac{1}{\varepsilon\alpha z B(0)} \left(1 + \frac{\varepsilon\alpha z}{2} b_1 + \frac{(\varepsilon\alpha z)^2}{6} b_2\right)^{-1} \\
&= \frac{1}{\varepsilon\alpha z B(0)} \left(1 - \frac{\varepsilon\alpha z}{2} b_1 - \frac{(\varepsilon\alpha z)^2}{6} b_2 + \frac{(-1)(-2)}{2}\left(\frac{(\varepsilon\alpha z)^2}{4} b_1^2 + \dots\right)\right) \\
&= \frac{1}{\varepsilon\alpha z B(0)} \left(1 - \frac{\varepsilon\alpha z}{2} b_1 - \frac{(\varepsilon\alpha z)^2}{6} b_2 + \frac{(\varepsilon\alpha z)^2}{4} b_1^2\right)
\end{aligned} \tag{9.55}$$

---

[16] Since

$$\begin{aligned}
\frac{dz}{df} &= \frac{1}{\varepsilon\alpha C(f)} = \frac{1}{\varepsilon\alpha B(\varepsilon\alpha z)}, \\
\Rightarrow \frac{df}{dz} &= \varepsilon\alpha B(\varepsilon\alpha z) \\
\Rightarrow f - K &= \varepsilon\alpha \int_0^z B(\varepsilon\alpha p)dp.
\end{aligned}$$

Using a Taylor series expansion around $z = 0$, we have that

$$\begin{aligned}
f - K = \varepsilon\alpha \int_0^z B(\varepsilon\alpha p)dp &= \varepsilon\alpha z B(0) \\
&= \varepsilon\alpha z C(K).
\end{aligned}$$

This is consistent with Definition (9.16) of z up to $O(\varepsilon)$ since up to order ε, $z = \frac{1}{\varepsilon\alpha}\frac{f-K}{C(K)}$.

Next, consider $\sqrt{B(\varepsilon\nu z)}$. Expanding around $z=0$:

$$\begin{aligned}
B^{1/2}(\varepsilon\nu z) &= \left(B(0)+\varepsilon\nu z B'(0)+\frac{(\varepsilon\nu z)^2}{2}B''(0)+\dots\right)^{1/2}\\
&= B^{1/2}(0)\left(1+\varepsilon\nu z b_1+\frac{(\varepsilon\nu z)^2}{2}b_2+\dots\right)^{1/2}\\
&= B^{1/2}(0)\left(1+\frac{1}{2}\left(\varepsilon\nu z b_1+\frac{(\varepsilon\nu z)^2}{2}b_2\right)+\frac{1}{2}\left(-\frac{1}{2}\right)\frac{1}{2}\left((\varepsilon\nu z)^2 b_1^2+\dots\right)\right)\\
&= B^{1/2}(0)\left(1+\frac{\varepsilon\nu z}{2}b_1+\frac{(\varepsilon\nu z)^2}{4}b_2-\frac{(\varepsilon\nu z)^2}{8}b_1^2\right) \qquad (9.56)
\end{aligned}$$

Using the property of the natural logarithm that $\ln(ab)=\ln a+\ln(b)$ and the Taylor series expansion for small x:

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\dots$$

we get that

$$\begin{aligned}
&\ln\left(\frac{\varepsilon\alpha z}{f-K}\sqrt{B(0)B(\varepsilon\alpha z)}\right)\\
&\approx \ln\left(\varepsilon\alpha z B(0)\left(1+\frac{\varepsilon\nu z}{2}b_1+\frac{(\varepsilon\nu z)^2}{4}b_2-\frac{(\varepsilon\nu z)^2}{8}b_1^2\right)\frac{1}{\varepsilon\alpha z B(0)}\left(1-\frac{\varepsilon\alpha z}{2}b_1-\frac{(\varepsilon\alpha z)^2}{6}b_2+\frac{(\varepsilon\alpha z)^2}{4}b_1^2\right)\right)\\
&= \ln\left(\left(1+\frac{\varepsilon\nu z}{2}b_1+\frac{(\varepsilon\nu z)^2}{4}b_2-\frac{(\varepsilon\nu z)^2}{8}b_1^2\right)\left(1-\frac{\varepsilon\alpha z}{2}b_1-\frac{(\varepsilon\alpha z)^2}{6}b_2+\frac{(\varepsilon\alpha z)^2}{4}b_1^2\right)\right)\\
&= \ln\left(1+\frac{\varepsilon\nu z}{2}b_1+\frac{(\varepsilon\nu z)^2}{4}b_2-\frac{(\varepsilon\nu z)^2}{8}b_1^2\right)+\ln\left(1-\frac{\varepsilon\alpha z}{2}b_1-\frac{(\varepsilon\alpha z)^2}{6}b_2+\frac{(\varepsilon\alpha z)^2}{4}b_1^2\right)\\
&= \left(\frac{\varepsilon\nu z}{2}b_1+\frac{(\varepsilon\nu z)^2}{4}b_2-\frac{(\varepsilon\nu z)^2}{8}b_1^2-\frac{1}{2}\frac{(\varepsilon\nu z)^2}{4}b_1^2+\dots\right)\\
&+\left(-\frac{\varepsilon\alpha z}{2}b_1-\frac{(\varepsilon\alpha z)^2}{6}b_2+\frac{(\varepsilon\alpha z)^2}{4}b_1^2-\frac{1}{2}\frac{(\varepsilon\alpha z)^2}{4}b_1^2+\dots\right)\\
&\approx \frac{(\varepsilon\alpha z)^2}{12}b_2-\frac{(\varepsilon\alpha z)^2}{8}b_1^2 \qquad (9.57)
\end{aligned}$$

Therefore, up to order ε^2 using (9.53), we have that

$$\frac{3}{x^2}\ln\left(\frac{\varepsilon\alpha z}{f-K}\sqrt{B(0)B(\varepsilon\alpha z)}\right)=\frac{1}{z^2}\left(\frac{(\varepsilon\alpha z)^2}{4}b_2-\frac{3\,(\varepsilon\alpha z)^2}{8}b_1^2\right)=\varepsilon^2\alpha^2\left(\frac{1}{4}b_2-\frac{3}{8}b_1^2\right)$$

which is clearly ε^2 multiplied by the second term of (9.43).

2. Consider the second term of (9.50):

$$\ln\left(\frac{xI^{1/2}(\varepsilon\nu z)}{z}\right).$$

The requirement is to expand $I^{1/2}(\varepsilon\nu z)$ and x around $z=0$ to obtain the expansion up to order ε^2. The derivatives $I'(0)$ and $I''(0)$ are required for this expansion. Given that

$$I(\xi)=\left(1-2\rho\xi+\xi^2\right)^{1/2},$$

we have that

$$
\begin{aligned}
I'(\xi) &= \frac{1}{2}\left(1-2\rho\xi+\xi^2\right)^{-1/2}(2\xi-2\rho) \\
&= \left(1-2\rho\xi+\xi^2\right)^{-1/2}(\xi-\rho) \\
I''(\xi) &= \left(1-2\rho\xi+\xi^2\right)^{-1/2} - (\xi-\rho)^2\left(1-2\rho\xi+\xi^2\right)^{-3/2}.
\end{aligned}
$$

So,

$$
\begin{aligned}
I'(0) &= -\rho, \\
I''(0) &= 1-\rho^2.
\end{aligned}
$$

Using the binomial theorem (Goldberg 1976, §8.6) and expanding around $z=0$, we have that

$$
\begin{aligned}
I^{1/2}(\varepsilon\nu z) &= \left(I(0)+\varepsilon\nu z I'(0)+\frac{(\varepsilon\nu z)^2}{2}I''(0)+\dots\right)^{1/2} \\
&= I^{1/2}(0)\left(1+\varepsilon\nu z\frac{I'(0)}{I(0)}+\frac{(\varepsilon\nu z)^2}{2}\frac{I''(0)}{I(0)}+\dots\right)^{1/2} \\
&= I^{1/2}(0)\left(1+\frac{\varepsilon\nu z}{2}\frac{I'(0)}{I(0)}+\frac{(\varepsilon\nu z)^2}{4}\frac{I''(0)}{I(0)}-\frac{(\varepsilon\nu z)^2}{8}\left(\frac{I'(0)}{I(0)}\right)^2+\dots\right) \\
&= 1-\frac{\rho}{2}\varepsilon\nu z+\frac{(\varepsilon\nu z)^2}{8}\left(2-3\rho^2\right).
\end{aligned}
\tag{9.58}
$$

Using Definition (9.32) and the binomial theorem, we expand $I^{-1}(\xi)$ around $z=0$:

$$
\begin{aligned}
x &= \frac{1}{\varepsilon\nu}\int_0^{\varepsilon\nu z} I^{-1}(\xi)d\xi \\
&= \frac{1}{\varepsilon\nu}\int_0^{\varepsilon\nu z}\left(I(0)+\xi I'(0)+\frac{\xi^2}{2}I''(0)+\dots\right)^{-1}d\xi \\
&= \frac{1}{\varepsilon\nu I(0)}\int_0^{\varepsilon\nu z}\left(1+\xi\frac{I'(0)}{I(0)}+\frac{\xi^2}{2}\frac{I''(0)}{I(0)}+\dots\right)^{-1}d\xi \\
&= \frac{1}{\varepsilon\nu}\int_0^{\varepsilon\nu z}\left(1-\xi\frac{I'(0)}{I(0)}-\frac{\xi^2}{2}\frac{I''(0)}{I(0)}+\xi^2\left(\frac{I'(0)}{I(0)}\right)^2\dots\right)d\xi \\
&\approx \frac{1}{\varepsilon\nu}\int_0^{\varepsilon\nu z}\left(1+\xi\rho+\frac{\xi^2}{2}\left(3\rho^2-1\right)\right)d\xi \\
&= \frac{1}{\varepsilon\nu}\left(\xi+\frac{\rho}{2}\xi^2+\frac{\xi^3}{6}\left(3\rho^2-1\right)\Big|_0^{\varepsilon\nu z}\right) \\
&= z\left(1+\frac{\rho}{2}\varepsilon\nu z+\frac{(\varepsilon\nu z)^2}{6}\left(3\rho^2-1\right)\right)
\end{aligned}
$$

Therefore,

$$
\frac{x}{z} = 1+\frac{\rho}{2}\varepsilon\nu z+\frac{(\varepsilon\nu z)^2}{6}\left(3\rho^2-1\right). \tag{9.59}
$$

So, using (9.58) and (9.59), we have that

$$\begin{aligned}
\ln\left(\frac{xI^{1/2}(\varepsilon\nu z)}{z}\right) &= \ln\left(\left(1+\frac{\rho}{2}\varepsilon\nu z-\frac{(\varepsilon\nu z)^2}{6}\left(1-3\rho^2\right)\right)\left(1-\frac{\rho}{2}\varepsilon\nu z-\frac{(\varepsilon\nu z)^2}{8}\left(3\rho^2-2\right)\right)\right)\\
&= \ln\left(1+\frac{\rho}{2}\varepsilon\nu z-\frac{(\varepsilon\nu z)^2}{6}\left(1-3\rho^2\right)\right)+\ln\left(1-\frac{\rho}{2}\varepsilon\nu z-\frac{(\varepsilon\nu z)^2}{8}\left(3\rho^2-2\right)\right)\\
&= \left(\frac{\rho}{2}\varepsilon\nu z-\frac{(\varepsilon\nu z)^2}{6}\left(1-3\rho^2\right)-\frac{1}{2}\frac{\rho^2}{4}(\varepsilon\nu z)^2+\dots\right)\\
&+\left(-\frac{\rho}{2}\varepsilon\nu z-\frac{(\varepsilon\nu z)^2}{8}\left(3\rho^2-2\right)-\frac{1}{2}\frac{\rho^2}{4}(\varepsilon\nu z)^2+\dots\right)\\
&= (\varepsilon\nu z)^2\left(\frac{1}{12}-\frac{\rho^2}{8}\right).
\end{aligned}$$

So, using (9.53):

$$\frac{3}{x^2}\ln\left(\frac{xI^{1/2}(\varepsilon\nu z)}{z}\right)=\frac{1}{z^2}(\varepsilon\nu z)^2\left(\frac{1}{4}-\frac{3}{8}\rho^2\right)=\varepsilon^2\nu^2\left(\frac{1}{4}-\frac{3}{8}\rho^2\right),$$

which is ε^2 multiplied by the first term of κ up to order ε^2, where $z_0=0$:

$$\begin{aligned}
\varepsilon^2\nu^2\left(\frac{1}{4}I''(0)I(0)-\frac{1}{8}\left(I'(0)\right)^2\right) &= \varepsilon^2\nu^2\left(\frac{1}{4}(1-\rho^2)-\frac{1}{8}\rho^2\right)\\
&= \varepsilon^2\nu^2\left(\frac{1}{4}-\frac{3}{8}\rho^2\right).
\end{aligned}$$

3. For last term of (9.50), using (9.53), we have that

$$\frac{3}{x^2}\left(\frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2\right)\approx\frac{3}{4}\varepsilon^2\rho\nu\alpha b_1,$$

which is ε^2 multiplied by the last term of κ up to order ε^2.

Therefore, (9.52) holds up to $O(\varepsilon^2)$. Using this to simplify (9.51), we get that

$$V(t,f,\alpha)=(f-K)^++\frac{1}{2}\frac{f-K}{x}\int_0^{\tau_{\text{ex}}}\frac{1}{\sqrt{2\pi\tau}}e^{-x^2/2\tau}e^{\varepsilon^2\theta}\frac{d\tau}{\left(1-\frac{2\tau}{x^2}\varepsilon^2\theta\right)^{3/2}}.$$

Changing integration variables, let

$$\begin{aligned}
q &= \frac{x^2}{2\tau}>0,\\
\Rightarrow \sqrt{q} &= \frac{|x|}{\sqrt{2\tau}},\\
dq = -\frac{x^2}{2\tau^2}d\tau &= -\frac{q}{\tau}d\tau,\\
\Rightarrow d\tau &= -\frac{\tau}{q}dq.
\end{aligned}$$

Changing limits of integration:

$$\begin{aligned}
\tau=0 &\Rightarrow q=\infty,\\
\tau=\tau_{\text{ex}} &\Rightarrow q=\frac{x^2}{2\tau_{\text{ex}}}.
\end{aligned}$$

In doing so, it is important to take the signs of x and $f-K$ into account as $q \geq 0$. Since $\frac{f-K}{x} > 0$, then

$$\frac{f-K}{x} = \left|\frac{f-K}{x}\right| = \frac{|f-K|}{|x|} = \frac{|f-K|}{\sqrt{2\tau q}}.$$

So, upon substitution of q and interchanging the limits of integration, we get that

$$\begin{aligned} V(t,f,\alpha) &= (f-K)^+ + \frac{1}{2}\left|\frac{f-K}{x}\right| \int_0^{\tau_{\text{ex}}} \frac{1}{\sqrt{2\pi\tau}} e^{-x^2/2\tau} e^{\varepsilon^2\theta} \frac{d\tau}{\left(1-\frac{2\tau}{x^2}\varepsilon^2\theta\right)^{3/2}} \\ &= (f-K)^+ - \frac{1}{2}\frac{|f-K|}{\sqrt{\pi}} \int_{\frac{x^2}{2\tau_{\text{ex}}}}^{\infty} \frac{1}{|x|\sqrt{2\tau}} e^{-q+\varepsilon^2\theta} \frac{\tau}{q} \frac{-dq}{\left(1-\frac{\varepsilon^2\theta}{q}\right)^{3/2}} \\ &= (f-K)^+ + \frac{1}{2}\frac{|f-K|}{\sqrt{\pi}} \int_{\frac{x^2}{2\tau_{\text{ex}}}}^{\infty} e^{-q+\varepsilon^2\theta} \frac{\tau\sqrt{q}}{|x|\sqrt{2\tau}} \frac{dq}{(q-\varepsilon^2\theta)^{3/2}} \\ &= (f-K)^+ + \frac{1}{4}\frac{|f-K|}{\sqrt{\pi}} \int_{\frac{x^2}{2\tau_{\text{ex}}}}^{\infty} e^{-q+\varepsilon^2\theta} \frac{dq}{(q-\varepsilon^2\theta)^{3/2}}. \end{aligned}$$

Note that in the second last line,

$$\frac{\tau\sqrt{q}}{|x|\sqrt{2\tau}} = \frac{\sqrt{\tau}|x|}{\sqrt{2}|x|\sqrt{2\tau}} = \frac{1}{2}.$$

Let $\hat{q} = q - \varepsilon^2\theta$, we get that up to order ε^2,

$$V(t,f,\alpha) = (f-K)^+ + \frac{1}{4}\frac{|f-K|}{\sqrt{\pi}} \int_{\frac{x^2}{2\tau_{\text{ex}}}-\varepsilon^2\theta}^{\infty} \frac{e^{-\hat{q}}}{\hat{q}^{3/2}} d\hat{q}. \tag{9.60}$$

9.2.3 Equivalent Normal Volatility

In order for (9.60) to be useful, the price will be converted to the equivalent implied Black volatility. This is achieved in two steps: we first convert to the implied normal volatility, and then convert to the implied Black volatility.

Consider the ordinary normal model:

$$d\hat{F} = \sigma_{\text{N}} dW, \quad \hat{F}(0) = f,$$

where σ_{N} is constant. So, there is only one source of randomness and the Gaussian probability density function describing the evolution of $\hat{F}$ has a constant standard deviation. Therefore, the diffusion coefficient, $C(f) = 1$, $\varepsilon\alpha = \sigma_{\text{N}}$ and $\nu = 0$. Much of the last section can be simplified since we are working with one Brownian motion (one dimension). This means that the expansion that is required is of the form:

$$\frac{1}{\sqrt{2\pi\sigma_{\text{N}}^2\tau}} \exp\left(-\frac{(f-K)^2}{2\sigma_{\text{N}}^2\tau}\right)(1+\ldots).$$

Substituting the values of $C(f)$, $\varepsilon\alpha$ and ν into (9.60) and (9.50), the option price for the normal model can be found. A more detailed approach will be given below.

The probability density function, $p(t,f;T,F)$ satisfies the forward Kolmogorov equation for all $T > t$:

$$p_{\text{T}} = \frac{1}{2}\sigma_{\text{N}}^2 p_{\text{FF}} \tag{9.61}$$

subject to

$$p = \delta(F - f)$$

at $T = t$. The undiscounted call option price can then be given by:

$$V(t,f) = \mathbb{E}\left[\left((\hat{F}(t_{\text{ex}}) - K\right)^{+} \Big| \hat{F}(t) = f\right] = \int_{K}^{\infty} (F - K) p(t, f; t_{\text{ex}}, F) dF.$$

Since

$$p(t, f; t_{\text{ex}}, F) = \delta(F - f) + \int_{t}^{t_{\text{ex}}} p_{\text{T}}(t, f; T, F) dT,$$

$$\begin{aligned}
V(t,f) &= \int_{K}^{\infty} (F - K) \left(\delta(F - f) + \int_{t}^{t_{\text{ex}}} p_{\text{T}}(t, f; T, F) dT\right) dF \\
&= \int_{K}^{\infty} (F - K)\delta(F - f) dF + \int_{K}^{\infty} \int_{t}^{t_{\text{ex}}} (F - K) p_{\text{T}}(t, f; T, F) dT dF \\
&= (f - K)^{+} + \frac{1}{2}\sigma_{\text{N}}^{2} \int_{t}^{t_{\text{ex}}} \int_{K}^{\infty} (F - K) p_{\text{FF}}(t, f; T, F) dF dT.
\end{aligned}$$

The last line results from substitution of (9.61). Now, performing integration by parts with respect to F and using the fact that $\lim_{x\to\infty} p(t, f; T, x) = \infty$, we get that

$$\begin{aligned}
V(t,f) &= (f - K)^{+} + \frac{1}{2}\sigma_{\text{N}}^{2} \int_{t}^{t_{\text{ex}}} \lim_{x\to\infty} \left((F - K)p_{\text{F}}\Big|_{K}^{x} - \int_{K}^{x} p_{\text{F}} dF\right) dT \\
&= (f - K)^{+} + \frac{1}{2}\sigma_{\text{N}}^{2} \int_{t}^{t_{\text{ex}}} (0 - (0 - p(t, f; T, K)))\, dT \\
&= (f - K)^{+} + \frac{1}{2}\sigma_{\text{N}}^{2} \int_{t}^{t_{\text{ex}}} p(t, f; T, K) dT. \qquad (9.62)
\end{aligned}$$

Since p depends on $T - t$, let $T - t = \tau$ and $\tau_{\text{ex}} = t_{\text{ex}} - t$. Therefore,

$$V(t,f) = (f - K)^{+} + \frac{1}{2}\sigma_{\text{N}}^{2} \int_{0}^{\tau_{\text{ex}}} p(\tau, f; K) d\tau, \qquad (9.63)$$

and

$$p_{\tau} = \frac{1}{2}\sigma_{\text{N}}^{2} p_{\text{FF}}$$

subject to

$$p = \delta(F - f)$$

at $\tau = 0$. It is also the case, from Proposition (2), that p follows the backward Kolmogorov equation. For $t < T$, we have that

$$\begin{aligned}
p_{\text{t}} + \frac{1}{2}\sigma_{\text{N}}^{2} p_{\text{ff}} &= 0, \\
\Rightarrow p_{\tau} &= \frac{1}{2}\sigma_{\text{N}}^{2} p_{\text{ff}}
\end{aligned}$$

subject to

$$p = \delta(f - K)$$

at $\tau = 0$. By defining

$$z := {}^{17}\frac{f - K}{\sigma_{\mathrm{N}}},$$

$$\Rightarrow \frac{\partial}{\partial f} = \frac{\partial z}{\partial f}\frac{\partial}{\partial z} = \frac{1}{\sigma_{\mathrm{N}}}\frac{\partial}{\partial z},$$

$$\frac{\partial^2}{\partial f^2} = \frac{\partial}{\partial f}\left(\frac{1}{\sigma_{\mathrm{N}}}\frac{\partial}{\partial z}\right) = \frac{1}{\sigma_{\mathrm{N}}^2}\frac{\partial^2}{\partial z^2}.$$

and

$$\delta(f - K) = \delta(\sigma_{\mathrm{N}} z) = \frac{1}{\sigma_{\mathrm{N}}}\delta(z).$$

So, in terms of z,

$$p_\tau = \frac{1}{2}\sigma_{\mathrm{N}}^2\left(\frac{1}{\sigma_{\mathrm{N}}^2}p_{zz}\right) = \frac{1}{2}p_{zz},$$

with

$$p = \frac{1}{\sigma_{\mathrm{N}}}\delta(z)$$

at $\tau = 0$. Using Definition (13), $p(\tau, z)$ can be solved for. Let $\bar{\tau} = \frac{1}{2}\tau$. Then

$$\frac{\partial p}{\partial \tau} = \frac{\partial p}{\partial \bar{\tau}}\frac{\partial \bar{\tau}}{\partial \tau} = \frac{1}{2}\frac{\partial p}{\partial \bar{\tau}}.$$

So, the PDE is now:

$$p_{\bar{\tau}} = p_{zz},$$

$$p_0 = p(0, z) = \frac{1}{\sigma_{\mathrm{N}}}\delta(z).$$

The solution of this is:

$$\begin{aligned} p(\bar{\tau}, z) = \left(p_0 * p_\delta(\bar{\tau}, \cdot)\right)(z) &:= \int_{-\infty}^{\infty} p(0, z) * p_\delta(\bar{\tau}, z - y)dy \\ &= \int_{-\infty}^{\infty} p(0, z - y) * p_\delta(\bar{\tau}, z)dy \\ &= \frac{1}{\sqrt{4\pi\bar{\tau}}\sigma_{\mathrm{N}}}\int_{-\infty}^{\infty}\delta(z - y)\exp\left(-\frac{z^2}{4\bar{\tau}}\right)dy \\ &= \frac{1}{\sqrt{4\pi\bar{\tau}}\sigma_{\mathrm{N}}}\exp\left(-\frac{z^2}{4\bar{\tau}}\right) \end{aligned}$$

Now substituting back for τ:

$$p(\tau, z) = \frac{1}{\sqrt{2\pi\tau}\sigma_{\mathrm{N}}}e^{-z^2/2\tau}. \tag{9.64}$$

[17] Since $f - K$ is of order ε, z can be expressed as an integral which is correct up to order ε^2. For simplicity, we use the above definition. Note that z has the same sign as $f - K$.

Substituting this into (9.62), the call option price is then

$$V(t,f) = (f-K)^+ + \frac{1}{2}\sigma_{\mathrm{N}} \int_0^{\tau_{\mathrm{ex}}} \frac{1}{\sqrt{2\pi\tau}} e^{-z^2/2\tau} d\tau.$$

Let $q = \frac{z^2}{2\tau} > 0$. Then $dq = -\frac{q}{\tau} d\tau$, and the limits of integration change to

$$\begin{aligned} \tau = 0 \quad &\Rightarrow q = \infty, \\ \tau = \tau_{\mathrm{ex}} \quad &\Rightarrow q = \tfrac{z^2}{2\tau_{\mathrm{ex}}} \quad = \frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}. \end{aligned}$$

Substituting this into the option price and changing the limits of integration, we get that

$$V(t,f) = (f-K)^+ - \frac{1}{2\sqrt{2\pi}}\sigma_{\mathrm{N}} \int_{\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}}^{\infty} -e^{-q}\frac{\sqrt{\tau}}{q} dq.$$

Using the fact that

$$\sigma_{\mathrm{N}} = \left|\frac{f-K}{z}\right| = \frac{|f-K|}{|z|} = \frac{|f-K|}{\sqrt{2\tau q}},$$

we get that

$$\begin{aligned} V(t,f) &= (f-K)^+ + \frac{1}{2\sqrt{2\pi}}\left|\frac{f-K}{z}\right| \int_{\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}}^{\infty} e^{-q}\frac{\sqrt{\tau}}{q} dq \\ &= (f-K)^+ + \frac{|f-K|}{2\sqrt{2\pi}} \int_{\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}}^{\infty} \frac{1}{\sqrt{2\tau q}} e^{-q}\frac{\sqrt{\tau}}{q} dq \\ &= (f-K)^+ + \frac{|f-K|}{4\sqrt{\pi}} \int_{\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}}^{\infty} \frac{e^{-q}}{q^{3/2}} dq. \end{aligned} \tag{9.65}$$

To evaluate the above integral, first perform integration by parts:

$$\begin{aligned} &\frac{|f-K|}{4\sqrt{\pi}} \int_{\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}}^{\infty} \frac{e^{-q}}{q^{3/2}} dq \\ &= \frac{|f-K|}{4\sqrt{\pi}} \lim_{x\to\infty} \left(-2q^{-1/2}e^{-q}\Big|_{\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}}^{x} - 2\int_{\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}}^{x} \frac{e^{-q}}{q^{1/2}} dq \right) \\ &= \frac{|f-K|}{4\sqrt{\pi}} \left(2\left(\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}\right)^{-1/2} \exp\left(-\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}\right)\right) - \frac{|f-K|}{2\sqrt{\pi}} \int_{\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}}^{\infty} \frac{e^{-q}}{q^{1/2}} dq \\ &= \frac{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}{\sqrt{2\pi}} \exp\left(-\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}\right) - \frac{|f-K|}{2\sqrt{\pi}} \int_{\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}}^{\infty} \frac{e^{-q}}{q^{1/2}} dq \end{aligned}$$

To evaluate the second integral, let $h = \sqrt{2q}$. Then $hdh = dq$ and

$$\begin{aligned} q = \frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}} \quad &\Rightarrow h = \tfrac{|f-K|}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}, \\ q = \infty \quad &\Rightarrow h = \infty. \end{aligned}$$

Then,

$$
\begin{aligned}
&\frac{|f-K|}{4\sqrt{\pi}}\int_{\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}}^{\infty}\frac{e^{-q}}{q^{3/2}}dq\\
&=\frac{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}{\sqrt{2\pi}}\exp\left(-\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}\right)-\frac{|f-K|}{2\sqrt{\pi}}\int_{\frac{|f-K|}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}}^{\infty}e^{-h^2/2}\frac{\sqrt{2}hdh}{h}\\
&=\frac{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}{\sqrt{2\pi}}\exp\left(-\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}\right)-\frac{|f-K|}{\sqrt{2\pi}}\int_{\frac{|f-K|}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}}^{\infty}e^{-h^2/2}dh\\
&=\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}\phi\left(\frac{f-K}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}\right)-|f-K|\left(1-\Phi\left(\frac{|f-K|}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}\right)\right),
\end{aligned}
$$

where Φ denotes the normal distribution, and ϕ denotes the Gaussian density function:

$$\phi(q) := \frac{1}{\sqrt{2\pi}}e^{-q^2/2}.$$

The option price is then

$$V(t,f)=(f-K)^{+}+\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}\phi\left(\frac{f-K}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}\right)-|f-K|\left(1-\Phi\left(\frac{|f-K|}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}\right)\right).$$

Considering the case when $f>K$, then $(f-K)^{+}=f-K$ and $|f-K|=f-K$. Then,

$$
\begin{aligned}
V(t,f)&=(f-K)+\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}\phi\left(\frac{f-K}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}\right)-(f-K)\left(1-\Phi\left(\frac{f-K}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}\right)\right)\\
&=\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}\phi\left(\frac{f-K}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}\right)+(f-K)\Phi\left(\frac{f-K}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}\right).
\end{aligned}
$$

Alternatively, if $f<K$, then $(f-K)^{+}=0$ and $|f-K|=-(f-K)$. therefore,

$$
\begin{aligned}
V(t,f)&=\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}\phi\left(\frac{f-K}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}\right)+(f-K)\left(1-\Phi\left(\frac{-(f-K)}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}\right)\right)\\
&=\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}\phi\left(\frac{f-K}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}\right)+(f-K)\Phi\left(\frac{f-K}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}\right).
\end{aligned}
$$

using the property of the cumulative normal distribution function:

$$\Phi(x)+\Phi(-x)=1.$$

Combining the above two scenarios,

$$V(t,f)=\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}\phi\left(\frac{f-K}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}\right)+(f-K)\Phi\left(\frac{f-K}{\sigma_{\mathrm{N}}\sqrt{\tau_{\mathrm{ex}}}}\right). \tag{9.66}$$

By equating the above price which is under the normal model, and that under the SABR model (9.60), the normal volatility can be found by equating the limits of integration:

$$
\begin{aligned}
\frac{(f-K)^2}{2\sigma_{\mathrm{N}}^2\tau_{\mathrm{ex}}}&=\frac{x^2}{2\tau_{\mathrm{ex}}}-\varepsilon^2\theta\\
\Rightarrow\frac{1}{\sigma_{\mathrm{N}}^2}&=\frac{x^2}{(f-K)^2}\left(1-2\varepsilon^2\frac{\theta}{x^2}\tau_{\mathrm{ex}}\right)
\end{aligned}
$$

Therefore, we get that

$$\begin{aligned}\sigma_{\mathrm{N}} &= \frac{f-K}{x}\left(1-2\varepsilon^2\frac{\theta}{x^2}\tau_{\mathrm{ex}}\right)^{-1/2}\\ &= \frac{f-K}{x}\left(1+\varepsilon^2\frac{\theta}{x^2}\tau_{\mathrm{ex}}+\dots\right)\end{aligned}\tag{9.67}$$

up to $O(\varepsilon^2)$. From Definition (9.32), we see that

$$\begin{aligned}x = \frac{1}{\varepsilon\nu}\int_0^{\varepsilon\nu z}\frac{d\xi}{\sqrt{1-2\rho\xi+\xi^2}} &= \frac{1}{\varepsilon\nu}\left(0+\varepsilon\nu z+\dots\right)\\ &= z+O(\varepsilon).\end{aligned}$$

So, the above expression for σ_{N} can be simplified further to obtain:

$$\sigma_{\mathrm{N}} = \left(\frac{f-K}{z}\right)\left(\frac{z}{x(z)}\right)\left(1+\varepsilon^2\left(\varphi_1+\varphi_2+\varphi_3\right)\tau_{\mathrm{ex}}+\dots\right),\tag{9.68}$$

where remaining coefficients, φ_1, φ_2 and φ_3 are corrections up to $O(\varepsilon^2)$ and will be derived below.

Using Definition (9.16), we see that

$$\frac{f-K}{z} = \frac{\varepsilon\alpha(f-K)}{\int_K^f\frac{dp}{C(p)}} = \left(\frac{1}{f-K}\int_K^f\frac{dp}{\varepsilon\alpha C(p)}\right)^{-1}.\tag{9.69}$$

The next factor, $\frac{z}{x}$, can be expressed using Definitions (9.32) and (9.16):

$$\frac{z}{x} = \frac{\varepsilon\nu z}{\ln\left(\frac{\varepsilon\nu z-\rho+\sqrt{\varepsilon^2\nu^2z^2-2\varepsilon\rho\nu z+1}}{1-\rho}\right)},$$

where, using a Taylor series expansion, we can define $\psi=\varepsilon\nu z$

$$\varepsilon\nu z = \varepsilon\nu\frac{1}{\varepsilon\alpha}\int_K^f\frac{dp}{C(p)} =^{18}\frac{\nu}{\alpha}\frac{f-K}{C(f_{\mathrm{av}})}\left(1+O(\varepsilon^2)\right),\tag{9.70}$$

and f_{av} can be either the geometric ($\sqrt{fK}$) or arithmetic average of f and K.

Then,

$$\frac{z}{x} = \frac{\psi}{\ln\left(\frac{\psi-\rho+\sqrt{\psi^2-2\rho\psi+1}}{1-\rho}\right)}.\tag{9.71}$$

[18] Consider that $K=\frac{1}{2}(f+K)-\frac{1}{2}(f-K)$, then

$$\begin{aligned}C^{-1}(K) = C^{-1}\left(f_{\mathrm{av}}-\frac{1}{2}(f-K)\right) &= \left(C\left(f_{\mathrm{av}}\right)-\frac{1}{2}(f-K)C'\left(f_{\mathrm{av}}\right)+O(\varepsilon^2)\right)^{-1}\\ &= \frac{1}{C\left(f_{\mathrm{av}}\right)}\left(1-\frac{1}{2}(f-K)\frac{C'}{C}+O(\varepsilon^2)\right)^{-1}\\ &= \frac{1}{C\left(f_{\mathrm{av}}\right)}\left(1+\frac{1}{2}(f-K)\frac{C'}{C}+O(\varepsilon^2)\right).\end{aligned}$$

Therefore, $\frac{f-K}{C(f_{\mathrm{av}})}$ is correct up to $O(\varepsilon^2)$.

The two factors, (9.69) and (9.71), are dominant and the remaining terms (to be derived below) provide small corrections.

We now derive φ_1, φ_2 and φ_3 up to $O(\varepsilon^2)$.

Consider the term $\varepsilon^2\varphi_1$. This can be equated, up to $O(\varepsilon^2)$, with the first term of $\varepsilon^2\frac{\theta}{x^2}$.

From §9.2.2, (9.53) showed that $\frac{1}{x^2} = \frac{1}{z^2} + O(\varepsilon^2)$. Therefore, using Definitions (9.50), (9.16) and (9.17),

$$\varepsilon^2\varphi_1 = \frac{1}{z^2}\ln\left(\frac{\varepsilon\alpha z}{f-K}\sqrt{C(f)C(K)}\right) = \frac{1}{z^2}\ln\left(\frac{\varepsilon\alpha z}{f-K}\sqrt{B(\varepsilon\alpha z)B(0)}\right). \tag{9.72}$$

Since $z = \frac{f-K}{\varepsilon\alpha C(f_{\text{av}})}\left(1+O(\varepsilon^2)\right)$, we have that

$$\begin{aligned}\frac{1}{z^2} &= \left(\frac{f-K}{\varepsilon\alpha C\left(f_{\text{av}}\right)}\left(1+O(\varepsilon^2)\right)\right)^{-2}\\ &= \frac{\varepsilon^2\alpha^2C^2\left(f_{\text{av}}\right)}{(f-K)^2}\left(1+O(\varepsilon^2)\right),\end{aligned} \tag{9.73}$$

and

$$\frac{\varepsilon\alpha z}{f-K} = \varepsilon\alpha\frac{1}{\varepsilon\alpha C\left(f_{\text{av}}\right)}\left(1+O(\varepsilon^2)\right) = \frac{1}{C\left(f_{\text{av}}\right)}\left(1+O(\varepsilon^2)\right). \tag{9.74}$$

As in (9.57), we had that

$$\ln\left(\frac{\varepsilon\alpha z}{f-K}\sqrt{B(\varepsilon\alpha z)B(0)}\right) = \frac{(\varepsilon\alpha z)^2}{12}b_2 - \frac{(\varepsilon\alpha z)^2}{8}b_1^2,$$

where

$$\begin{aligned}b_1 &= \frac{B'(\varepsilon\alpha z_0)}{B(\varepsilon\alpha z_0)},\\ b_2 &= \frac{B''(\varepsilon\alpha z_0)}{B(\varepsilon\alpha z_0)}.\end{aligned}$$

z_0 can be chosen to correspond to f_{av} (i.e. it can be chosen as the midpoint of 0 and z.)

Using (9.54)

$$f-K = \varepsilon\alpha\int_0^z B(\varepsilon\alpha p)dp,$$

we will now expand $B^{1/2}(0)$, $B^{1/2}(\varepsilon\alpha z)$ and (9.54) around the midpoint of 0 and $\varepsilon\alpha z$, $\varepsilon\alpha\frac{z}{2}$. This corresponds to $B\left(\varepsilon\alpha\frac{z}{2}\right) = C(f_{\text{av}})$.

1. Expanding $B(0)$ around $\varepsilon\alpha\frac{z}{2} = f_{\text{av}}$ up to $O(\varepsilon^2)$, we get

$$\begin{aligned}B^{1/2}(0) &= B^{1/2}\left(\varepsilon\alpha\frac{z}{2} + \varepsilon\alpha\left(0-\frac{z}{2}\right)\right)\\ &= \left(B\left(\varepsilon\alpha\frac{z}{2}\right) - \frac{1}{2}\varepsilon\alpha zB'\left(\varepsilon\alpha\frac{z}{2}\right) + \frac{1}{8}\left(\varepsilon\alpha z\right)^2B''\left(\varepsilon\alpha\frac{z}{2}\right) + \ldots\right)^{1/2}\\ &= B^{1/2}\left(\varepsilon\alpha\frac{z}{2}\right)\left(1 - \frac{1}{2}\varepsilon\alpha z\gamma_1 + \frac{1}{8}\left(\varepsilon\alpha z\right)^2\gamma_2 + \ldots\right)^{1/2}\\ &= B^{1/2}\left(\varepsilon\alpha\frac{z}{2}\right)\left(1 - \frac{1}{4}\varepsilon\alpha z\gamma_1 + \frac{1}{16}\left(\varepsilon\alpha z\right)^2\gamma_2 - \frac{1}{8}\left(\frac{1}{4}\left(\varepsilon\alpha z\right)^2\gamma_1^2 + \ldots\right)\right)\\ &= B^{1/2}\left(\varepsilon\alpha\frac{z}{2}\right)\left(1 - \frac{1}{4}\varepsilon\alpha z\gamma_1 + \frac{2\gamma_2-\gamma_1^2}{32}\left(\varepsilon\alpha z\right)^2 + \ldots\right)\end{aligned}$$

where

$$\gamma_1 = \frac{B'\left(\varepsilon\alpha\frac{z}{2}\right)}{B\left(\varepsilon\alpha\frac{z}{2}\right)},$$

$$\gamma_2 = \frac{B''\left(\varepsilon\alpha\frac{z}{2}\right)}{B\left(\varepsilon\alpha\frac{z}{2}\right)}.$$

2. Next, we expand $B^{1/2}(\varepsilon\alpha z)$:

$$\begin{aligned}
B^{1/2}(\varepsilon\alpha z) &= B^{1/2}\left(\varepsilon\alpha\frac{z}{2} + \varepsilon\alpha\frac{z}{2}\right) \\
&= \left(B\left(\varepsilon\alpha\frac{z}{2}\right) + \frac{1}{2}\varepsilon\alpha z B'\left(\varepsilon\alpha\frac{z}{2}\right) + \frac{1}{8}\left(\varepsilon\alpha z\right)^2 B''\left(\varepsilon\alpha\frac{z}{2}\right) + \dots\right)^{1/2} \\
&= B^{1/2}\left(\varepsilon\alpha\frac{z}{2}\right)\left(1 + \frac{1}{2}\varepsilon\alpha z\gamma_1 + \frac{1}{8}\left(\varepsilon\alpha z\right)^2\gamma_2 + \dots\right)^{1/2} \\
&= B^{1/2}\left(\varepsilon\alpha\frac{z}{2}\right)\left(1 + \frac{1}{4}\varepsilon\alpha z\gamma_1 + \frac{1}{16}\left(\varepsilon\alpha z\right)^2\gamma_2 - \frac{1}{8}\left(\frac{1}{4}\left(\varepsilon\alpha z\right)^2\gamma_1^2 + \dots\right)\right) \\
&= B^{1/2}\left(\varepsilon\alpha\frac{z}{2}\right)\left(1 + \frac{1}{4}\varepsilon\alpha z\gamma_1 + \frac{2\gamma_2 - \gamma_1^2}{32}\left(\varepsilon\alpha z\right)^2 + \dots\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sqrt{B(\varepsilon\alpha z)B(0)} &= B\left(\varepsilon\alpha\frac{z}{2}\right)\left(1 + \frac{1}{4}\varepsilon\alpha z\gamma_1 + \frac{2\gamma_2 - \gamma_1^2}{32}\left(\varepsilon\alpha z\right)^2\right)\left(1 - \frac{1}{4}\varepsilon\alpha z\gamma_1 + \frac{2\gamma_2 - \gamma_1^2}{32}\left(\varepsilon\alpha z\right)^2\right) \\
&:= B\left(\varepsilon\alpha\frac{z}{2}\right)XY,
\end{aligned}$$

where

$$X = 1 + \frac{1}{4}\varepsilon\alpha z\gamma_1 + \frac{2\gamma_2 - \gamma_1^2}{32}\left(\varepsilon\alpha z\right)^2,$$

and

$$Y = 1 - \frac{1}{4}\varepsilon\alpha z\gamma_1 + \frac{2\gamma_2 - \gamma_1^2}{32}\left(\varepsilon\alpha z\right)^2.$$

3. Lastly, consider the expression for $f - K$:

$$\begin{aligned}
f - K &= \varepsilon\alpha\int_0^z B(\varepsilon\alpha p)dp \\
&= \varepsilon\alpha\int_0^z B\left(\varepsilon\alpha\frac{z}{2} + \varepsilon\alpha\left(p - \frac{z}{2}\right)\right)dp \\
&= \varepsilon\alpha\int_0^z\left(B\left(\varepsilon\alpha\frac{z}{2}\right) + \varepsilon\alpha\left(p - \frac{z}{2}\right)\ B'\left(\varepsilon\alpha\frac{z}{2}\right) + \frac{1}{2}\left(\varepsilon\alpha\right)^2\left(p - \frac{z}{2}\right)^2 B''\left(\varepsilon\alpha\frac{z}{2}\right) + \dots\right)dp \\
&= \varepsilon\alpha B\left(\varepsilon\alpha\frac{z}{2}\right)\int_0^z\left(1 + \frac{\varepsilon\alpha}{2}\left(2p - z\right)\gamma_1 + \frac{\left(\varepsilon\alpha\right)^2}{2}\left(p^2 - pz + \frac{z^2}{4}\right)\gamma_2 + \dots\right)dp \\
&= \varepsilon\alpha B\left(\varepsilon\alpha\frac{z}{2}\right)\left(p + \frac{\varepsilon\alpha}{2}\left(p^2 - pz\right)\gamma_1 + \frac{\left(\varepsilon\alpha\right)^2}{2}\left(\frac{p^3}{3} - \frac{p^2}{2}z + p\frac{z^2}{4}\right)\gamma_2\Bigg|_0^z\right) \\
&= \varepsilon\alpha z B\left(\varepsilon\alpha\frac{z}{2}\right)\left(1 + \frac{1}{24}\left(\varepsilon\alpha z\right)^2\gamma_2\right) \\
\Rightarrow \frac{f - K}{\varepsilon\alpha z} &= B\left(\varepsilon\alpha\frac{z}{2}\right)\left(1 + \frac{1}{24}\left(\varepsilon\alpha z\right)^2\gamma_2\right)
\end{aligned}$$

Then,

$$
\begin{aligned}
&-\frac{1}{z^2}\ln\left(\frac{f-K}{\varepsilon\alpha z\sqrt{B(\varepsilon\alpha z)B(0)}}\right)\\
&=-\frac{1}{z^2}\ln\left(B\left(\varepsilon\alpha\frac{z}{2}\right)\left(1+\frac{1}{24}\left(\varepsilon\alpha z\right)^2\gamma_2\right)\frac{1}{B\left(\varepsilon\alpha\frac{z}{2}\right)XY}\right)\\
&=-\frac{1}{z^2}\ln\left(1+\frac{1}{24}\left(\varepsilon\alpha z\right)^2\gamma_2\right)+\frac{1}{z^2}\ln\left[\left(1+\frac{1}{4}\varepsilon\alpha z\gamma_1+\frac{2\gamma_2-\gamma_1^2}{32}\left(\varepsilon\alpha z\right)^2\right)\left(1-\frac{1}{4}\varepsilon\alpha z\gamma_1+\frac{2\gamma_2-\gamma_1^2}{32}\left(\varepsilon\alpha z\right)^2\right)\right]\\
&=-\frac{1}{z^2}\left(\frac{1}{24}\left(\varepsilon\alpha z\right)^2\gamma_2+\ldots\right)+\frac{1}{z^2}\left(\frac{1}{4}\varepsilon\alpha z\gamma_1+\frac{2\gamma_2-\gamma_1^2}{32}\left(\varepsilon\alpha z\right)^2-\frac{1}{2}\left(\frac{1}{16}\left(\varepsilon\alpha z\right)^2\gamma_1^2+\ldots\right)\right)\\
&+\frac{1}{z^2}\left(-\frac{1}{4}\varepsilon\alpha z\gamma_1+\frac{2\gamma_2-\gamma_1^2}{32}\left(\varepsilon\alpha z\right)^2-\frac{1}{2}\left(\frac{1}{16}\left(\varepsilon\alpha z\right)^2\gamma_1^2+\ldots\right)\right)\\
&=\frac{2\gamma_2-3\gamma_1^2}{24}\varepsilon^2\alpha^2 \qquad (9.75)
\end{aligned}
$$

Converting everything back from $B(\varepsilon\alpha\frac{z}{2})$, we require γ_1 and γ_2 in in terms $C\left(f_{\text{av}}\right)$. Thus, we find $B'(\varepsilon\alpha\frac{z}{2})$ and $B''(\varepsilon\alpha\frac{z}{2})$:

$$
\begin{aligned}
B'(\varepsilon\alpha\frac{z}{2}) &= \frac{d}{d\left(\varepsilon\alpha\frac{z}{2}\right)}B(\varepsilon\alpha\frac{z}{2}) = \frac{df}{d\left(\varepsilon\alpha\frac{z}{2}\right)}\frac{d}{df}C\left(f_{\text{av}}\right) = C\left(f_{\text{av}}\right)C'\left(f_{\text{av}}\right)\\
\Rightarrow \gamma_1 &= C'\left(f_{\text{av}}\right),\\
B''(\varepsilon\alpha\frac{z}{2}) &= \frac{d}{d\left(\varepsilon\alpha\frac{z}{2}\right)}\left(C\left(f_{\text{av}}\right)C'\left(f_{\text{av}}\right)\right) = \frac{df}{d\left(\varepsilon\alpha\frac{z}{2}\right)}\frac{d}{df}\left(C\left(f_{\text{av}}\right)C'\left(f_{\text{av}}\right)\right)\\
&= C\left(f_{\text{av}}\right)\left(\frac{d}{df}C\left(f_{\text{av}}\right)\right)C'\left(f_{\text{av}}\right)+C^2\left(f_{\text{av}}\right)\left(\frac{d}{df}C'\left(f_{\text{av}}\right)\right)\\
&= C\left(f_{\text{av}}\right)\left(C'\left(f_{\text{av}}\right)\right)^2+C^2\left(f_{\text{av}}\right)C''\left(f_{\text{av}}\right)\\
\Rightarrow \gamma_2 &= \left(C'\left(f_{\text{av}}\right)\right)^2+C\left(f_{\text{av}}\right)C''\left(f_{\text{av}}\right).
\end{aligned}
$$

Substituting this into (9.75), we get that

$$
\begin{aligned}
-\frac{1}{z^2}\ln\left(\frac{f-K}{\varepsilon\alpha z\sqrt{B(\varepsilon\alpha z)B(0)}}\right) &= \frac{2\gamma_2-3\gamma_1^2}{24}\varepsilon^2\alpha^2\\
&= \frac{1}{12}\left(\left(C'\left(f_{\text{av}}\right)\right)^2+C\left(f_{\text{av}}\right)C''\left(f_{\text{av}}\right)\right)-\frac{1}{8}\left(C'\left(f_{\text{av}}\right)\right)^2\\
&= \frac{2\varrho_2-\varrho_1^2}{24}\varepsilon^2\alpha^2C^2\left(f_{\text{av}}\right), \qquad (9.76)
\end{aligned}
$$

where

$$
\varrho_1 = \frac{C'\left(f_{\text{av}}\right)}{C\left(f_{\text{av}}\right)}, \qquad (9.77)
$$

$$
\varrho_2 = \frac{C''\left(f_{\text{av}}\right)}{C\left(f_{\text{av}}\right)}. \qquad (9.78)
$$

The next term, $\varepsilon^2\varphi_2$ is to be equated with the second term of $\varepsilon^2\frac{\theta}{x^2}$:

$$
\varepsilon^2\varphi_2 = \frac{1}{z^2}\ln\left(\frac{x}{z}I^{1/2}(\varepsilon\nu z)\right).
$$

Since the expressions for $\frac{x}{z}$ and $I^{1/2}(\varepsilon\nu z)$ up to $O(\varepsilon^2)$ have already been calculated in (9.59) and (9.58) respectively, we have that

$$
\begin{aligned}
\varepsilon^2\varphi_2 &= \frac{1}{z^2}\ln\left(\frac{x}{z}\left(1-2\varepsilon\rho\nu z+\varepsilon^2\nu^2z^2\right)^{1/4}\right)\\
&= \frac{1}{z^2}\ln\left(\left(1+\frac{1}{2}\rho\varepsilon\nu z+\frac{(\varepsilon\nu z)^2}{6}\left(3\rho^2-1\right)+\dots\right)\left(1-\frac{1}{2}\rho\varepsilon\nu z-\frac{(\varepsilon\nu z)^2}{8}\left(3\rho^2-2\right)+\dots\right)\right)\\
&= \frac{1}{z^2}\ln\left(1+\frac{1}{2}\rho\varepsilon\nu z+\frac{(\varepsilon\nu z)^2}{6}\left(3\rho^2-1\right)-\frac{1}{2}\rho\varepsilon\nu z-\frac{1}{4}\rho^2(\varepsilon\nu z)^2-\frac{(\varepsilon\nu z)^2}{8}\left(3\rho^2-2\right)+\dots\right)\\
&= \frac{1}{z^2}\ln\left(1+\frac{(\varepsilon\nu z)^2}{24}\left(2-3\rho^2\right)+\dots\right)\\
&= \varepsilon^2\nu^2\frac{\left(2-3\rho^2\right)}{24}
\end{aligned}
\tag{9.79}
$$

Lastly, $\varepsilon^2\varphi_3$ is equated with the third term of $\varepsilon^2\frac{\theta}{x^2}$. We have that

$$
\varepsilon^2\varphi_3 = \frac{1}{z^2}\left(\frac{1}{4}\varepsilon^2\rho\nu\alpha b_1 z^2\right) = \frac{1}{4}\varepsilon^2\rho\nu\alpha\frac{B'(\varepsilon\alpha z_0)}{B(\varepsilon\alpha z_0)}.
$$

We choose $B(\varepsilon\alpha z_0) = B\left(\varepsilon\alpha\frac{z}{2}\right) = C\left(f_{\text{av}}\right)$. Then as before,

$$
\frac{B'\left(\varepsilon\alpha\frac{z}{2}\right)}{B\left(\varepsilon\alpha\frac{z}{2}\right)} = \left(\frac{d}{d(\varepsilon\alpha\frac{z}{2})}B\left(\varepsilon\alpha\frac{z}{2}\right)\right)\frac{1}{B\left(\varepsilon\alpha\frac{z}{2}\right)} = \left(\frac{df}{d(\varepsilon\alpha\frac{z}{2})}\frac{d}{df}C\left(f_{\text{av}}\right)\right)\frac{1}{C\left(f_{\text{av}}\right)} = C'\left(f_{\text{av}}\right).
$$

$$
\varepsilon^2\varphi_3 = \frac{1}{4}\varepsilon^2\rho\nu\alpha C'\left(f_{\text{av}}\right) = \frac{1}{4}\varepsilon^2\rho\nu\alpha\varrho_1 C\left(f_{\text{av}}\right)+\dots
\tag{9.80}
$$

Therefore, up to $O(\varepsilon^2)$, the normal implied volatility as function of strike K is given as

$$
\sigma_{\text{N}}(K) = \frac{\varepsilon\alpha\,(f-K)}{\int_K^f\frac{dp}{C(p)}}\left(\frac{\psi}{x(\psi)}\right)\left(1+\left(\frac{2\varrho_2-\varrho_1^2}{24}\alpha^2C^2\left(f_{\text{av}}\right)+\nu^2\frac{\left(2-3\rho^2\right)}{24}+\frac{1}{4}\rho\nu\alpha\varrho_1C\left(f_{\text{av}}\right)\right)\varepsilon^2\tau_{\text{ex}}+\dots\right).
\tag{9.81}
$$

9.2.4 Equivalent Black Volatility

The next case, we consider the log-normal model:

$$
d\hat{F} = \varepsilon\sigma_{\text{B}}\hat{F}dW,\quad \hat{F}(0) = f,
$$

where $\varepsilon\sigma_{\text{B}}$ is the volatility. For Black's model, the undiscounted European option prices, with strike K and exercise date τ_{ex}, is given by

$$
\begin{aligned}
V_{\text{call}} &= f\Phi(d_1)-K\Phi(d_2),\\
V_{\text{put}} &= V_{\text{call}}+Z(0,\tau_{\text{ex}})(K-f),
\end{aligned}
$$

with

$$
d_{1,2} = \frac{\ln\frac{f}{K}\pm\frac{1}{2}\varepsilon^2\sigma_{\text{B}}^2\tau_{\text{ex}}}{\varepsilon\sigma_{\text{B}}\sqrt{\tau_{\text{ex}}}},
$$

and $Z(0,\tau_{\text{ex}})$ is the discount factor. To obtain the implied volatility, we require that $C(f)=f$ and $\nu=0$ in (9.67). Since $x=z\,(1+O(\varepsilon))$, then

$$x=z=\frac{1}{\varepsilon\sigma_{\text{B}}}\int_K^f\frac{dp}{p}=\frac{1}{\varepsilon\sigma_{\text{B}}}\ln\left(\frac{f}{K}\right).$$

Also, we have that

$$\varrho_1=\frac{C'\left(f_{\text{av}}\right)}{C\left(f_{\text{av}}\right)}=\frac{1}{f_{\text{av}}},$$

and $\varrho_2=0$. Therefore,

$$\begin{aligned}\varepsilon^2\varphi_1&=\frac{2\varrho_2-\varrho_1^2}{24}\varepsilon^2\sigma_{\text{B}}^2f_{\text{av}}^2+\ldots\\&=-\frac{1}{24}\varepsilon^2\sigma_{\text{B}}^2+\ldots\end{aligned}$$

Also, φ_2 and φ_3 are zero. Therefore, up to $O(\varepsilon^2)$, the implied normal volatility for Black's model is given by

$$\begin{aligned}\sigma_{\text{N}}(K)&=\frac{f-K}{x}\left(1+\varepsilon^2\left(\varphi_1+\varphi_2+\varphi_3\right)\tau_{\text{ex}}+\ldots\right)\\&=\frac{\varepsilon\sigma_{\text{B}}(f-K)}{\ln\left(\frac{f}{K}\right)}\left(1-\frac{1}{24}\varepsilon^2\sigma_{\text{B}}^2\tau_{\text{ex}}+\ldots\right)\end{aligned}\tag{9.82}$$

By setting $\varepsilon=1$, the implied volatility can be obtained in the original variables. To obtain the implied Black volatility for the SABR model, we equate σ_{N} in (9.82) and (9.81):

$$\begin{aligned}&\frac{\varepsilon\sigma_{\text{B}}(f-K)}{\ln\left(\frac{f}{K}\right)}\left(1-\frac{1}{24}\varepsilon^2\sigma_{\text{B}}^2\tau_{\text{ex}}+\ldots\right)\\&=\frac{\varepsilon\alpha\,(f-K)}{\int_K^f\frac{dp}{C(p)}}\left(\frac{\psi}{x(\psi)}\right)\left(1+\left(\frac{2\varrho_2-\varrho_1^2}{24}\alpha^2C^2\left(f_{\text{av}}\right)+\nu^2\frac{\left(2-3\rho^2\right)}{24}+\frac{1}{4}\rho\nu\alpha\varrho_1C\left(f_{\text{av}}\right)\right)\varepsilon^2\tau_{\text{ex}}+\ldots\right)\end{aligned}$$

Therefore,

$$\begin{aligned}\sigma_{\text{B}}=&\frac{\alpha\ln\left(\frac{f}{K}\right)}{\int_K^f\frac{dp}{C(p)}}\left(\frac{\psi}{x(\psi)}\right)\cdot\\&\left(1+\left(\frac{2\varrho_2-\varrho_1^2}{24}\alpha^2C^2\left(f_{\text{av}}\right)+\nu^2\frac{\left(2-3\rho^2\right)}{24}+\frac{1}{4}\rho\nu\alpha\varrho_1C\left(f_{\text{av}}\right)\right)\varepsilon^2\tau_{\text{ex}}+\ldots\right)\left(1-\frac{1}{24}\varepsilon^2\sigma_{\text{B}}^2\tau_{\text{ex}}+\ldots\right)^{-1}.\end{aligned}$$

Now,

$$\begin{aligned}&\left(1+\left(\frac{2\varrho_2-\varrho_1^2}{24}\alpha^2C^2\left(f_{\text{av}}\right)+\nu^2\frac{\left(2-3\rho^2\right)}{24}+\frac{1}{4}\rho\nu\alpha\varrho_1C\left(f_{\text{av}}\right)\right)\varepsilon^2\tau_{\text{ex}}+\ldots\right)\left(1-\frac{1}{24}\varepsilon^2\sigma_{\text{B}}^2\tau_{\text{ex}}+\ldots\right)^{-1}\\&=\left(1+\left(\frac{2\varrho_2-\varrho_1^2}{24}\alpha^2C^2\left(f_{\text{av}}\right)+\nu^2\frac{\left(2-3\rho^2\right)}{24}+\frac{1}{4}\rho\nu\alpha\varrho_1C\left(f_{\text{av}}\right)\right)\varepsilon^2\tau_{\text{ex}}+\ldots\right)\left(1+\frac{\alpha^2C^2\left(f_{\text{av}}\right)}{24f_{\text{av}}^2}\varepsilon^2\tau_{\text{ex}}+\ldots\right)\\&=\left(1+\left(\frac{2\varrho_2-\varrho_1^2+1/f_{\text{av}}^2}{24}\alpha^2C^2\left(f_{\text{av}}\right)+\nu^2\frac{\left(2-3\rho^2\right)}{24}+\frac{1}{4}\rho\nu\alpha\varrho_1C\left(f_{\text{av}}\right)\right)\varepsilon^2\tau_{\text{ex}}+\ldots\right).\end{aligned}$$

In the second line, the term $-\frac{1}{24}\sigma_{\text{B}}^2$ is being replaced with

$$\frac{2\varrho_2-\varrho_1^2}{24}\alpha^2 C^2\left(f_{\text{av}}\right)=-\frac{\alpha^2 C^2\left(f_{\text{av}}\right)}{24 f_{\text{av}}^2}. \tag{9.83}$$

Thus,

$$\sigma_{\text{B}}=\frac{\alpha \ln\left(\frac{f}{K}\right)}{\int_K^f \frac{dp}{C(p)}}\left(\frac{\psi}{x(\psi)}\right)\left(1+\left(\frac{2\varrho_2-\varrho_1^2+1/f_{\text{av}}^2}{24}\alpha^2 C^2\left(f_{\text{av}}\right)+\nu^2\frac{\left(2-3\rho^2\right)}{24}+\frac{1}{4}\rho\nu\alpha\varrho_1 C\left(f_{\text{av}}\right)\right)\varepsilon^2\tau_{\text{ex}}+\ldots\right) \tag{9.84}$$

9.2.5 Stochastic β Model

This is the special case of $C(f)=f^{\beta}$. So, the two-factor model is:

$$\begin{aligned}
d\hat{F}&=\varepsilon\hat{\alpha}\hat{F}^{\beta}dW_1,\\
d\hat{\alpha}&=\varepsilon\nu\hat{\alpha}dW_2,\\
dW_1dW_2&=\rho dt,
\end{aligned}$$

with $\hat{F}(0)=f$ and $\hat{\alpha}(0)=\alpha$. To obtain the implied volatility, we substitute the above parameters into (9.81). We have that

$$\begin{aligned}
\frac{\varepsilon\alpha\left(f-K\right)}{\int_K^f\frac{dp}{C(p)}}&=\varepsilon\alpha(f-K)\left(\int_K^f p^{-\beta}dp\right)^{-1}\\
&=\varepsilon\alpha(f-K)\left(\frac{1}{1-\beta}p^{1-\beta}\Big|_K^f\right)^{-1}\\
&=\varepsilon\alpha(f-K)(1-\beta)\left(f^{1-\beta}-K^{1-\beta}\right)^{-1}\\
&=\frac{\varepsilon\alpha(f-K)(1-\beta)}{f^{1-\beta}-K^{1-\beta}}.
\end{aligned} \tag{9.85}$$

Also, from (9.70),

$$\psi=\varepsilon\nu z=\frac{\nu}{\alpha}\frac{f-K}{C(f_{\text{av}})}=\frac{\nu}{\alpha}\frac{f-K}{f_{\text{av}}^{\beta}}. \tag{9.86}$$

Given that $C(\hat{F})=\hat{F}^{\beta}$, we have that

$$\begin{aligned}
C'(\hat{F})&=\beta\hat{F}^{\beta-1},\\
C''(\hat{F})&=\beta(\beta-1)\hat{F}^{\beta-2},\\
\Rightarrow\varrho_1&=\frac{C'(f_{\text{av}})}{C(f_{\text{av}})}=\frac{\beta f_{\text{av}}^{\beta-1}}{f_{\text{av}}^{\beta}}=\frac{\beta}{f_{\text{av}}},\\
\Rightarrow\varrho_2&=\frac{C''(f_{\text{av}})}{C(f_{\text{av}})}=\frac{\beta(\beta-1)f_{\text{av}}^{\beta-2}}{f_{\text{av}}^{\beta}}=\frac{\beta(\beta-1)}{f_{\text{av}}^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{2\varrho_2-\varrho_1^2}{24}\alpha^2C^2\left(f_{\mathrm{av}}\right) &= \frac{1}{24}\left(\frac{2\beta(\beta-1)}{f_{\mathrm{av}}^2}-\frac{\beta^2}{f_{\mathrm{av}}^2}\right)\alpha^2 f_{\mathrm{av}}^{2\beta} \\ &= \frac{1}{24}\left(\frac{\beta^2-2\beta}{f_{\mathrm{av}}^{2-2\beta}}\right)\alpha^2 \\ &= \frac{-\beta(2-\beta)\alpha^2}{24f_{\mathrm{av}}^{2-2\beta}},\end{aligned} \tag{9.87}$$

$$\frac{1}{4}\rho\nu\alpha\varrho_1C\left(f_{\mathrm{av}}\right) = \frac{1}{4}\rho\nu\alpha\frac{\beta}{f_{\mathrm{av}}}f_{\mathrm{av}}^{\beta} = \frac{\rho\nu\alpha\beta}{4f_{\mathrm{av}}^{1-\beta}}. \tag{9.88}$$

Thus, using (9.85), (9.86), (9.87) and (9.88), the implied normal volatility for this model is given by

$$\sigma_{\mathrm{N}}(K) = \frac{\varepsilon\alpha(f-K)(1-\beta)}{f^{1-\beta}-K^{1-\beta}}\left(\frac{\psi}{x(\psi)}\right)\left(1+\left(\frac{-\beta(2-\beta)\alpha^2}{24f_{\mathrm{av}}^{2-2\beta}}+\nu^2\frac{\left(2-3\rho^2\right)}{24}+\frac{\rho\nu\alpha\beta}{4f_{\mathrm{av}}^{1-\beta}}\right)\varepsilon^2\tau_{\mathrm{ex}}+\dots\right). \tag{9.89}$$

Using the Taylor series expansion for $\sinh x = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$, the above can be further simplified, we expand

$$\begin{aligned}f-K &= {}^{19}2\sqrt{fK}\sinh\left(\frac{1}{2}\ln\left(\frac{f}{K}\right)\right) \\ &= 2\sqrt{fK}\left(\frac{1}{2}\ln\left(\frac{f}{K}\right)+\frac{1}{3!}\left(\frac{1}{2}\ln\left(\frac{f}{K}\right)\right)^3+\frac{1}{5!}\left(\frac{1}{2}\ln\left(\frac{f}{K}\right)\right)^5+\dots\right) \\ &= \sqrt{fK}\ln\left(\frac{f}{K}\right)\left(1+\frac{1}{24}\ln^2\left(\frac{f}{K}\right)+\frac{1}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)\end{aligned}$$

and

$$\begin{aligned}f^{1-\beta}-K^{1-\beta} &= {}^{20}2\left(fK\right)^{\frac{1-\beta}{2}}\sinh\left(\frac{(1-\beta)}{2}\ln\left(\frac{f}{K}\right)\right) \\ &= 2\left(fK\right)^{\frac{1-\beta}{2}}\left(\frac{(1-\beta)}{2}\ln\left(\frac{f}{K}\right)+\frac{1}{3!}\left(\frac{(1-\beta)}{2}\ln\left(\frac{f}{K}\right)\right)^3+\frac{1}{5!}\left(\frac{(1-\beta)}{2}\ln\left(\frac{f}{K}\right)\right)^5+\dots\right) \\ &= \left(fK\right)^{\frac{1-\beta}{2}}(1-\beta)\ln\left(\frac{f}{K}\right)\left(1+\frac{(1-\beta)^2}{24}\ln^2\left(\frac{f}{K}\right)+\frac{(1-\beta)^4}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)\end{aligned}$$

19

$$\begin{aligned}2\sqrt{fK}\sinh\left(\frac{1}{2}\ln\left(\frac{f}{K}\right)\right) &= 2\sqrt{fK}\left(\frac{\exp\left(\frac{1}{2}\ln\left(\frac{f}{K}\right)\right)-\exp\left(-\frac{1}{2}\ln\left(\frac{f}{K}\right)\right)}{2}\right) \\ &= \sqrt{fK}\left(\sqrt{\frac{f}{K}}-\sqrt{\frac{K}{f}}\right) \\ &= f-K.\end{aligned}$$

This then leads to the simplification of $\frac{(1-\beta)f-K}{f^{1-\beta}-K^{1-\beta}}$:

$$\begin{aligned}\frac{(1-\beta)\,f-K}{f^{1-\beta}-K^{1-\beta}} &= \frac{(1-\beta)\sqrt{fK}\ln\left(\frac{f}{K}\right)}{(fK)^{\frac{1-\beta}{2}}\,(1-\beta)\ln\left(\frac{f}{K}\right)}\frac{\left(1+\frac{1}{24}\ln^2\left(\frac{f}{K}\right)+\frac{1}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)}{\left(1+\frac{(1-\beta)^2}{24}\ln^2\left(\frac{f}{K}\right)+\frac{(1-\beta)^4}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)}\\ &= (fK)^{\frac{\beta}{2}}\frac{\left(1+\frac{1}{24}\ln^2\left(\frac{f}{K}\right)+\frac{1}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)}{\left(1+\frac{(1-\beta)^2}{24}\ln^2\left(\frac{f}{K}\right)+\frac{(1-\beta)^4}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)}\end{aligned}$$

Thus, by taking the geometric average, $f_{\text{av}}=\sqrt{fK}$, (9.89) can be simplified to:

$$\begin{aligned}\sigma_{\text{N}}(K) = \varepsilon\alpha\,(fK)^{\frac{\beta}{2}}\frac{\left(1+\frac{1}{24}\ln^2\left(\frac{f}{K}\right)+\frac{1}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)}{\left(1+\frac{(1-\beta)^2}{24}\ln^2\left(\frac{f}{K}\right)+\frac{(1-\beta)^4}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)}\left(\frac{\psi}{x(\psi)}\right)\cdot\\ \left(1+\left(\frac{-\beta(2-\beta)\alpha^2}{24\,(fK)^{1-\beta}}+\nu^2\frac{\left(2-3\rho^2\right)}{24}+\frac{\rho\nu\alpha\beta}{4\,(fK)^{\frac{1-\beta}{2}}}\right)\varepsilon^2\tau_{\text{ex}}+\dots\right),\end{aligned} \tag{9.90}$$

where ψ is given by

$$\psi=\frac{\nu}{\alpha}\frac{f-K}{f_{\text{av}}^{\beta}}=\frac{\nu}{\alpha}\frac{\sqrt{fK}\ln\left(\frac{f}{K}\right)}{(fK)^{\beta}}=\frac{\nu}{\alpha}(fK)^{\frac{(1-\beta)}{2}}\ln\left(\frac{f}{K}\right), \tag{9.91}$$

using the expansion for $f-K$ up to second order, neglecting higher order terms, and

$$x(\psi)=\int_0^{\psi}\frac{d\xi}{\sqrt{\xi^2-2\rho\xi+1}}=\ln\left(\frac{\sqrt{\psi^2-2\rho\psi+1}+\psi-\rho}{1-\rho}\right). \tag{9.92}$$

To obtain the implied Black volatility, equate the implied normal volatility for the SABR model, $\sigma_{\text{N}}(K)$ in (9.90), with the implied normal volatility for Blacks model (9.82). Substituting in the expansion for $f-K$ and by noting that since $C(f_{\text{av}})=f_{\text{av}}^{\beta}=(fK)^{\frac{\beta}{2}}$, from (9.83), we have that

$$\frac{2\varrho_2-\varrho_1^2}{24}\alpha^2C^2(f_{\text{av}})=-\frac{\alpha^2C^2(f_{\text{av}})}{24f_{\text{av}}^2}=-\frac{\alpha^2f_{\text{av}}^{2\beta}}{24f_{\text{av}}^2}=-\frac{\alpha^2}{24\,(fK)^{1-\beta}}.$$

So, we get that

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[20]

$$\begin{aligned}2\,(fK)^{\frac{1-\beta}{2}}\sinh\left(\frac{(1-\beta)}{2}\ln\left(\frac{f}{K}\right)\right) &= 2\,(fK)^{\frac{1-\beta}{2}}\left(\frac{\exp\left(\frac{(1-\beta)}{2}\ln\left(\frac{f}{K}\right)\right)-\exp\left(-\frac{(1-\beta)}{2}\ln\left(\frac{f}{K}\right)\right)}{2}\right)\\ &= (fK)^{\frac{1-\beta}{2}}\left(\left(\frac{f}{K}\right)^{\frac{1-\beta}{2}}-\left(\frac{K}{f}\right)^{\frac{1-\beta}{2}}\right)\\ &= f^{1-\beta}-K^{1-\beta}.\end{aligned}$$

$$\frac{\varepsilon\sigma_{\mathrm{B}}(f-K)}{\ln\left(\frac{f}{K}\right)}\left(1-\frac{1}{24}\varepsilon^2\sigma_{\mathrm{B}}^2\tau_{\mathrm{ex}}+\dots\right)$$

$$=\frac{\varepsilon\sigma_{\mathrm{B}}}{\ln\left(\frac{f}{K}\right)}\sqrt{fK}\ln\left(\frac{f}{K}\right)\left(1+\frac{1}{24}\ln^2\left(\frac{f}{K}\right)+\frac{1}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)\left(1-\frac{\alpha^2}{24\,(fK)^{1-\beta}}\varepsilon^2\tau_{\mathrm{ex}}+\dots\right)$$

$$=\varepsilon\sigma_{\mathrm{B}}\sqrt{fK}\left(1+\frac{1}{24}\ln^2\left(\frac{f}{K}\right)+\frac{1}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)\left(1-\frac{\alpha^2}{24\,(fK)^{1-\beta}}\varepsilon^2\tau_{\mathrm{ex}}+\dots\right)$$

Equating this with (9.90), we get that up to $O(\varepsilon^2)$,

$$\varepsilon\sigma_{\mathrm{B}}\sqrt{fK}\left(1+\frac{1}{24}\ln^2\left(\frac{f}{K}\right)+\frac{1}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)\left(1-\frac{\alpha^2}{24\,(fK)^{1-\beta}}\varepsilon^2\tau_{\mathrm{ex}}+\dots\right)$$

$$=\varepsilon\alpha\,(fK)^{\frac{\beta}{2}}\,\frac{\left(1+\frac{1}{24}\ln^2\left(\frac{f}{K}\right)+\frac{1}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)}{\left(1+\frac{(1-\beta)^2}{24}\ln^2\left(\frac{f}{K}\right)+\frac{(1-\beta)^4}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)}.$$

$$\left(\frac{\psi}{x(\psi)}\right)\left(1+\left(\frac{-\beta(2-\beta)\alpha^2}{24\,(fK)^{1-\beta}}+\nu^2\frac{\left(2-3\rho^2\right)}{24}+\frac{\rho\nu\alpha\beta}{4\,(fK)^{(1-\beta)/2}}\right)\varepsilon^2\tau_{\mathrm{ex}}+\dots\right)$$

$$\Rightarrow\sigma_{\mathrm{B}}(f,K)=\frac{\alpha}{(fK)^{(1-\beta)/2}}\left(1+\frac{(1-\beta)^2}{24}\ln^2\left(\frac{f}{K}\right)+\frac{(1-\beta)^4}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)^{-1}\cdot\left(\frac{\psi}{x(\psi)}\right)$$

$$\left(1+\left(\frac{-\beta(2-\beta)\alpha^2}{24\,(fK)^{1-\beta}}+\nu^2\frac{\left(2-3\rho^2\right)}{24}+\frac{\rho\nu\alpha\beta}{4\,(fK)^{(1-\beta)/2}}\right)\varepsilon^2\tau_{\mathrm{ex}}+\dots\right)\left(1-\frac{\alpha^2}{24\,(fK)^{1-\beta}}\varepsilon^2\tau_{\mathrm{ex}}+\dots\right)^{-1}$$

$$=\frac{\alpha}{(fK)^{(1-\beta)/2}}\left(1+\frac{(1-\beta)^2}{24}\ln^2\left(\frac{f}{K}\right)+\frac{(1-\beta)^4}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)^{-1}\cdot\left(\frac{\psi}{x(\psi)}\right)$$

$$\left(1+\left(\frac{-\beta(2-\beta)\alpha^2}{24\,(fK)^{1-\beta}}+\nu^2\frac{\left(2-3\rho^2\right)}{24}+\frac{\rho\nu\alpha\beta}{4\,(fK)^{(1-\beta)/2}}\right)\varepsilon^2\tau_{\mathrm{ex}}+\dots\right)\left(1+\frac{\alpha^2}{24\,(fK)^{1-\beta}}\varepsilon^2\tau_{\mathrm{ex}}+\dots\right)$$

$$=\frac{\alpha}{(fK)^{(1-\beta)/2}}\left(1+\frac{(1-\beta)^2}{24}\ln^2\left(\frac{f}{K}\right)+\frac{(1-\beta)^4}{1920}\ln^4\left(\frac{f}{K}\right)+\dots\right)^{-1}\cdot\left(\frac{\psi}{x(\psi)}\right)$$

$$\left(1+\left(\frac{(1-\beta)^2\alpha^2}{24\,(fK)^{1-\beta}}+\nu^2\frac{\left(2-3\rho^2\right)}{24}+\frac{\rho\nu\alpha\beta}{4\,(fK)^{(1-\beta)/2}}\right)\varepsilon^2\tau_{\mathrm{ex}}+\dots\right)\tag{9.93}$$

which is the implied Black volatility as a function of strike when the current forward value is f. By letting $\varepsilon=1$, the original units are recovered. By setting $K=f$, the at-the-money (ATM) Black implied volatility, with current forward f, is given by

$$\sigma_{\mathrm{ATM}}={}^{21}\sigma_{\mathrm{B}}(f,f)=\frac{\alpha}{f^{1-\beta}}\left(1+\left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{f^{1-\beta}}+\frac{\left(2-3\rho^2\right)}{24}\nu^2\right)\tau_{\mathrm{ex}}+\dots\right).\tag{9.94}$$

The parameter α is a function of the current futures level and ATM volatility. Assuming we have already solved for ν and ρ, the above equation can be inverted and α is then found to be the root of a cubic

[21] Looking at $\lim_{K\to f}\frac{\psi}{x(\psi)}$: With ψ and $x(\psi)$ given by (9.91) and (9.92) respectively, we require L' Hopital's rule to show that $\lim_{K\to f}\frac{\psi}{x(\psi)}=1$. Firstly,

$$\frac{d\psi}{dK}=\frac{d}{dK}\left(\frac{\nu}{\alpha}\,(fK)^{\frac{(1-\beta)}{2}}\ln\left(\frac{f}{K}\right)\right)=\frac{\nu}{\alpha}\frac{(1-\beta)}{2}f^{\frac{(1-\beta)}{2}}K^{-\frac{(1+\beta)}{2}}\ln\left(\frac{f}{K}\right)-\frac{\nu}{\alpha}\,(fK)^{\frac{(1-\beta)}{2}}\,\frac{1}{K},$$

equation:

$$\frac{(1-\beta)^2\tau_{\text{ex}}}{24f^{2-2\beta}}\alpha^3+\frac{\rho\beta\nu\tau_{\text{ex}}}{4f^{1-\beta}}\alpha^2+\left(1+\frac{2-3\rho^2}{24}\nu^2\tau_{\text{ex}}\right)\alpha-\sigma_{\text{ATM}}f^{1-\beta}=0. \tag{9.95}$$

α is chosen as the either the only real root of the equation or the smallest positive root.

Reformulating the option skew with σ_{ATM} and not α as an input, we find that the skew volatility is invariant under $\frac{K}{f}$, i.e. $\sigma(f,K)=\sigma(1,K/f)$. Therefore, calibration is performed on relative strikes. To see this, we consider the more general case. For $\lambda>0$, we show that $\sigma(K,f)=\sigma(\lambda K,\lambda f)$. In the above case, $\lambda=\frac{1}{f}$. Firstly, consider (9.95), the cubic equation satisfied by α. α is clearly a function of f and σ_{ATM} and we require α as a function of λ i.e. $\alpha=\alpha(\lambda)$. Begin by multiplying this equation by $\lambda^{1-\beta}$:

$$\frac{(1-\beta)^2\tau_{\text{ex}}}{24f^{2-2\beta}}\lambda^{1-\beta}\alpha^3+\frac{\rho\beta\nu\tau_{\text{ex}}}{4f^{1-\beta}}\lambda^{1-\beta}\alpha^2+\left(1+\frac{2-3\rho^2}{24}\nu^2\tau_{\text{ex}}\right)\lambda^{1-\beta}\alpha-\sigma_{\text{ATM}}\lambda^{1-\beta}f^{1-\beta}=0.$$

Multiplying the top and bottom of the first term by $\lambda^{2-2\beta}$ and the top and bottom of the second term by $\lambda^{1-\beta}$, we get

$$\frac{(1-\beta)^2\tau_{\text{ex}}}{24(\lambda f)^{2-2\beta}}\left(\lambda^{1-\beta}\alpha\right)^3+\frac{\rho\beta\nu\tau_{\text{ex}}}{4(\lambda f)^{1-\beta}}\left(\lambda^{1-\beta}\alpha\right)^2+\left(1+\frac{2-3\rho^2}{24}\nu^2\tau_{\text{ex}}\right)\lambda^{1-\beta}\alpha-\sigma_{\text{ATM}}(\lambda f)^{1-\beta}=0,$$

$$\alpha(\lambda)=\lambda^{1-\beta}\alpha.$$

This gives the substitution for $\alpha\to\alpha(\lambda)$ that satisfies (9.95). The next step is to make the substitutions $f=\lambda f$, $K=\lambda K$ and $\alpha=\alpha(\lambda)$ into the closed-form formulae for the implied volatility:

So,

$$\lim_{K\to f}\frac{d\psi}{dK}=-\frac{\nu}{\alpha}\frac{1}{f^\beta}.$$

Then, using the Leibnitz rule (Abramowitz & Stegun 1974):

$$\begin{aligned}\frac{dx(\psi)}{dK}&=\frac{d}{dK}\left(\int_0^\psi\frac{d\xi}{\sqrt{\xi^2-2\rho\xi+1}}\right)=\frac{1}{\sqrt{\psi^2-2\rho\psi+1}}\frac{d\psi}{dK}\\&=\left(\left(\frac{\nu}{\alpha}\right)^2(fK)^{(1-\beta)}\ln^2\left(\frac{f}{K}\right)-2\rho\frac{\nu}{\alpha}(fK)^{\frac{(1-\beta)}{2}}\ln\left(\frac{f}{K}\right)+1\right)^{-1/2}\frac{d\psi}{dK}.\end{aligned}$$

Thus,

$$\begin{aligned}\lim_{K\to f}\frac{dx(\psi)}{dK}&=\lim_{K\to f}\left[\left(\left(\frac{\nu}{\alpha}\right)^2(fK)^{(1-\beta)}\ln^2\left(\frac{f}{K}\right)-2\rho\frac{\nu}{\alpha}(fK)^{\frac{(1-\beta)}{2}}\ln\left(\frac{f}{K}\right)+1\right)^{-1/2}\frac{d\psi}{dK}\right]\\&=\lim_{K\to f}\left(1\cdot\frac{d\psi}{dK}\right)\\&=-\frac{\nu}{\alpha}\frac{1}{f^\beta}.\end{aligned}$$

In conclusion,

$$\lim_{K\to f}\frac{\psi}{x(\psi)}=1.$$

$$\sigma_{\mathrm{B}}(f,K) = \frac{\alpha\left(1+\left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{(fK)^{1-\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{(fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right)\tau_{\mathrm{ex}}\right)}{(fK)^{(1-\beta)/2}\left[1+\frac{(1-\beta)^2}{24}\ln^2\frac{f}{K}+\frac{(1-\beta)^4}{1920}\ln^4\frac{f}{K}\right]}\frac{z}{x(z)}, \tag{9.96}$$

$$z = \frac{\nu}{\alpha}(fK)^{(1-\beta)/2}\ln\frac{f}{K}, \tag{9.97}$$

$$x(z) = \ln\left(\frac{\sqrt{1-2\rho z+z^2}+z-\rho}{1-\rho}\right). \tag{9.98}$$

Finding the transformed value of z then $x(z)$:

$$z(\lambda) = \frac{\nu}{\lambda^{1-\beta}\alpha}\left(\lambda^2 fK\right)^{(1-\beta)/2}\ln\frac{f}{K} = z,$$

indicating that $x(z)$ is also left unaltered. Therefore, in terms of λ, (9.96) becomes:

$$\begin{aligned}\sigma_{\mathrm{B}}(\lambda f,\lambda K) &= \frac{\lambda^{1-\beta}\alpha\left(1+\left(\frac{(1-\beta)^2}{24}\frac{\left(\lambda^{1-\beta}\alpha\right)^2}{(\lambda^2 fK)^{1-\beta}}+\frac{1}{4}\frac{\lambda^{1-\beta}\rho\beta\nu\alpha}{(\lambda^2 fK)^{(1-\beta)/2}}+\frac{2-3\rho^2}{24}\nu^2\right)\tau_{\mathrm{ex}}\right)}{(\lambda^2 fK)^{(1-\beta)/2}\left[1+\frac{(1-\beta)^2}{24}\ln^2\frac{f}{K}+\frac{(1-\beta)^4}{1920}\ln^4\frac{f}{K}\right]}\frac{z}{x(z)}\\ &= \sigma_{\mathrm{B}}(f,K).\end{aligned}$$

Since this hold for any $\lambda > 0$, it holds for $\lambda = \frac{1}{f}$, proving that the skew volatility is invariant under $\frac{K}{f}$.

The following analysis deal with two cases: the stochastic normal model ($\beta = 0$) and the stochastic log-normal model ($\beta = 1$).

Stochastic Normal Model

By substituting $\beta = 0$ into (9.90) and (9.93), the implied normal and implied Black volatilities are respectively:

$$\sigma_{\mathrm{N}}(f,K) = \varepsilon\alpha\left(1+\frac{(2-3\rho^2)}{24}\varepsilon^2\nu^2\tau_{\mathrm{ex}}+\dots\right), \tag{9.99}$$

and

$$\sigma_{\mathrm{B}}(f,K) = \varepsilon\alpha\frac{f-K}{\ln\left(\frac{f}{K}\right)}\left(\frac{\psi}{x(\psi)}\right)\left(1+\left(\frac{\alpha^2}{24fK}+\frac{(2-3\rho^2)}{24}\nu^2\right)\varepsilon^2\tau_{\mathrm{ex}}+\dots\right), \tag{9.100}$$

where

$$\psi = \frac{\nu}{\alpha}\sqrt{fK}\ln\left(\frac{f}{K}\right).$$

Stochastic Log-normal Model

Once again, substitute $\beta = 1$ into (9.90) and (9.93) to obtain the implied normal and implied Black volatilities are respectively:

$$\sigma_{\mathrm{N}}(f,K) = \varepsilon\alpha\frac{f-K}{\ln\left(\frac{f}{K}\right)}\left(\frac{\psi}{x(\psi)}\right)\left(1+\left(-\frac{1}{24}\alpha^2+\frac{1}{4}\rho\alpha\nu+\frac{(2-3\rho^2)}{24}\nu^2\right)\varepsilon^2\tau_{\mathrm{ex}}+\dots\right), \tag{9.101}$$

and

$$\sigma_{\mathrm{B}}(f,K)=\varepsilon\alpha\left(1+\left(\frac{1}{4}\rho\alpha\nu+\frac{(2-3\rho^2)}{24}\nu^2\right)\varepsilon^2\tau_{\mathrm{ex}}+\dots\right), \tag{9.102}$$

where

$$\psi=\frac{\nu}{\alpha}\ln\left(\frac{f}{K}\right).$$

9.3 Monte Carlo under SABR

In this section, we develop a Monte Carlo scheme to price options under the SABR model. Since path-dependent options have payoffs which depend on the underlying, we will require the SDE of S, and not F, which is what is given. For European options, the terminal value of S is the requirement and since $S(T)=F(T,T)$, where T is the maturity of the forward contract, Monte Carlo simulation under the forward measure can be easily implemented. Under this measure, $F(t)$ and $S(t)/Z(t,T)$ are martingales[22].

9.3.1 SDE of the Underlying

Given the SABR model for the forward $\hat{F}$, and volatility $\hat{\alpha}$, processes:

$$\begin{aligned} d\hat{F}&=\alpha C(\hat{F})dW_1,\\ d\hat{\alpha}&=\nu\hat{\alpha}dW_2,\\ dW_1dW_2&=\rho dt, \end{aligned}$$

where, as before, $C(\hat{F})$ is the diffusion coefficient, α is a 'volatility-like' parameter, ν is the volvol and ρ is the correlation between $\hat{F}$ and $\hat{\alpha}$. $\hat{F}(0)=f$ and $\hat{\alpha}(0)=\alpha$. In order to price any path-dependent option that has a payoff linked to the underlying process, we require the SDE for the underlying $\hat{S}$, not $\hat{F}$. Using the arbitrage free price of a forward contract at time t with maturity t_{ex}, we use the multi-dimensional version of Itô's formula (Björk 2004, §4.17) to derive the equation for $\hat{S}$. Let the constant risk free rate and dividend yield be r and q respectively.

Since

$$f(x,y,t)=\hat{S}=\hat{F}e^{-(r-q)(t_{\mathrm{ex}}-t)},$$

where $x=\hat{F}$ and $y=\hat{\alpha}$, we get that

$$\begin{aligned} &f_x=e^{-(r-q)(t_{\mathrm{ex}}-t)},\quad f_{xx}=0,\quad f_t=(r-q)\,\hat{F}e^{-(r-q)(t_{\mathrm{ex}}-t)},\\ &f_y=0,\quad f_{xy}=0,\quad f_{yy}=0. \end{aligned}$$

Therefore,

$$\begin{aligned} df&=(r-q)\,\hat{F}e^{-(r-q)(t_{\mathrm{ex}}-t)}dt+\hat{\alpha}\hat{F}^{\beta}e^{-(r-q)(t_{\mathrm{ex}}-t)}dW_1\\ \Rightarrow d\hat{S}&=(r-q)\hat{S}dt+\hat{\alpha}e^{(r-q)(\beta-1)(t_{\mathrm{ex}}-t)}\hat{S}^{\beta}dW_1. \end{aligned}$$

[22] The choice of numèraire asset is the zero-coupon bond maturing at time T.

The SABR model can be re-written as:

$$d\hat{S} = (r-q)\hat{S}dt + \hat{\alpha}e^{(r-q)(\beta-1)(t_{\mathrm{ex}}-t)}\hat{S}^{\beta}dW_1 \tag{9.103}$$

$$d\hat{\alpha} = \nu\hat{\alpha}dW_2 \tag{9.104}$$

$$dW_1 dW_2 = \rho dt,$$

9.3.2 Quasi-Monte Carlo

As in Chapter 7, we proceed to simulate the process of the underlying given by (9.103) and its volatility (9.104). The hybrid quasi-Monte Carlo technique will once again be used. Details are given in §7.4.2.

Since a scalar autonomous SDE written can be written in integral form as:

$$X(t) = X_0 + \int_0^t f(X(s))ds + \int_0^t g(X(s))dW(s), \quad 0 \le t \le T,$$

where the second integral is with respect to Brownian motion and X_0 is the initial condition, we can rewrite this as:

$$dX(t) = f(X(t))dt + g(X(t))dW(t), \quad X(0) = X_0 \quad 0 \le t \le T$$

Given the interval $[0, T]$ is to be discretized into N equally spaced intervals of size $\Delta t = \frac{T}{N}$. For $j = 1, \ldots, N$, (9.103) and (9.104) are discretised as using the following:

$$X_j = X_{j-1} + f(X_{j-1})\Delta t + g(X_{j-1})\left(W(\tau_j) - W(\tau_{j-1})\right)$$

where $\tau_j = j\Delta t$. Therefore,

$$\hat{S}_j = \hat{S}_{j-1} + (r-q)\hat{S}_{j-1}\Delta t + \hat{\alpha}_{j-1}e^{(r-q)(\beta-1)(t_{\mathrm{ex}}-j\Delta t)}\hat{S}_{j-1}^{\beta}u_j\sqrt{\Delta t} \tag{9.105}$$

$$\hat{\alpha}_j = \hat{\alpha}_{j-1} + \nu\hat{\alpha}_{j-1}\left(\rho u_j + \sqrt{1-\rho^2}v_j\right)\sqrt{\Delta t} \tag{9.106}$$

Here, u_j and v_j are the quasi-random $\Phi(0, 1)$ numbers which are generated as in chapter (7.1). The Choleski decomposition relates the two independent Brownian motions (u_j and v_j).

so, once β has been fixed, we use the at-the-money implied volatility and the current futures level to find ν, ρ and α as described in Chapter 10. The above scheme can be used to price European as well as path-dependent options. This can also be used to test the robustness of the SABR model which, after calibration, yields closed-form solutions to vanilla options.

Chapter 10

Calibration to Market Data

10.1 Source Data

We consider the equity futures market traded at the South African Futures Exchange (SAFEX). The data set that will be used for calibration and pricing is the implied volatility skews provided by one of the major equity option traders as given on 24 March 2005 for the futures contracts. Theses futures are on the TOP40 index (this index contains the biggest shares which are determined by free float market capitalization and liquidity).

We will be interested in the pricing of vanilla and exotic (barrier and Asian) options on the underlying index after calibration has been performed. The March expiry has been selected as it is the most liquid.

Futures options are American and fully margined (there is no initial premium but the options are subject to margin flow) and hence, the risk free rate does not appear in the pricing formulae given in (West 2005*b*, Chapter 10):

$$
\begin{aligned}
V_{\text{call}} &= f\Phi\left(d_1\right) - K\Phi\left(d_2\right), \\
V_{\text{put}} &= K\Phi\left(-d_2\right) - f\Phi\left(-d_1\right), \\
d_{1,2} &= \frac{\ln\frac{f}{K} \pm \sigma^2\tau_{\text{ex}}}{\sigma\sqrt{\tau_{\text{ex}}}},
\end{aligned}
$$

where the current futures level is f, the strike K, the volatility is σ and time to maturity, $\tau_{\text{ex}} = t_{\text{ex}} - t$. The maturity of the options and the futures contracts coincide.

10.2 Disk Contents

This section gives a brief description of the Excel spreadsheets and modules that are provided in both the local and stochastic volatility setting.

10.3 Local Volatility

Trees.xls is a spreadsheet that contains modules (which are a combination of subroutines and functions) to implement the implied tree models from Chapters 3 and 4. The 'Input' worksheet requires the underlying spot price at the current date, the expiry (of the final set of option inputs), the constant risk free rate, a constant dividend yield (applicable in the modules DermanKaniChriss and BarleCakici), skew data (implied volatility for strikes of liquid options) and a few other parameters that will be dealt with below. Each module contains a sub that routinely reads all the data in the input sheet and outputs, on a worksheet that corresponds to that module, a tree of spot prices, a tree of forward values, a tree of Arrow-Debreau prices, a tree of the associated risk-neutral associated probability of up and/or down movements within the tree and a tree of local volatilities. The input and output data may vary slightly, depending on the module.

The following modules are provided:

1. **DermanKani**: The module provided is discussed in detail in §3.6. The resulting binomial trees have equal spacing and consequently require that the input expiries are equally spaced.

2. **BarleCakici**: This module is similar to DermanKani, but has incorporated the modifications discussed in §3.7. The option expiries can be at unequal time intervals and the underlying can have a dividend yield.

3. **DermanKaniChriss**: This module is discussed in detail in §4.6. This local volatility tree will be used to check whether the vanilla option prices produced by the tree matches the observed data, as this tree has the added advantages of incorporating a time and/or state dependent implied volatility. The 'Input' sheet has an additional section for this module that incorporates a dividend yield (if required), a term or strike structure and an input as to the number of nodes required for the tree. The date of the futures expiry is also an additional input. This module allows for the option prices to be determined using either a trinomial tree or the Black-Scholes option pricing formula.

4. **modDKCExotic**: To price the exotic options of choice (up-and-out call option and an asian call option), the implied trinomial tree as well as the probabilities of the movements within the tree are required. This module contains the necessary code to do so, using the output of the DermanKaniChriss procedure. Details of the code that pertain specifically to each of the exotic options is given below in §10.5.3.

5. **modMakeSkew**: In order to determine whether the local volatility models do in fact demonstrate self-consistency, skews of various forms to be input into the tree models can be generated and compared with the output. Required input includes the current spot price, risk free rate, dividend yield, European option expiries, the ATM implied volatility for the first option expiry as well as the slope of the skew structure and term structures between each set of consecutive dates. Futures (or forward) levels are also of relevance in terms of interpolating the relative strikes to assign the ATM implied volatility to the correct strike (which must also be an input).

Certain cases, which include option prices or Arrow-Debreau that are equal to zero, lead to the code not running to completion. These are all numerical issues which arise in certain unpredictable cases. In these cases, very often minor modifications (for example, changing the number of steps required in the tree) will result in a valid calibration.

10.4 Stochastic Volatility

The second spreadsheet provided, Stochastic-Models.xls, contains modules that are a combination of Monte Carlo using the antithetic variates approach (for the Hull-White model), a hybrid quasi-Monte Carlo technique (for the Hull-White, Heston and SABR models) and Gauss-Legendre integration (for the Heston model). There are additional functions provided that use SABR hybrid quasi-Monte Carlo to price the exotic options which is further discussed in §10.5.3 below. The following modules are provided:

1. **modHullWhite**: This model runs a subroutine that calculates European call and put option prices using a Monte Carlo simulation with antithetic variates for variance reduction. Full details are provided in §7.4.1.

2. **modHullWhiteQuasiMC**: This module uses a hybrid quasi-Monte Carlo technique described in §7.4.2, to price European call and put options.

3. **modHGaussLegendre, modHestonIntegrands and modHestonIntegrals**: The first module contains a subroutine that calls the modules modHestonIntegrands and modHestonIntegrals, which outputs the prices of European call and put options. These first of two modules is required to determine the real part of the integrals being evaluated, while the second module integrates the result using Gauss-Legendre integration described in §8.5.2. The VBA code was obtained from (Vogt 2004).

4. **modHestonMC**: This module contains a subroutine that prices the European options using the above-mentioned hybrid quasi-Monte Carlo technique. Details are given in §8.5.1.

5. **modSABRVanilla**: This module contains a subroutine that evaluates European option prices using the hybrid quasi-Monte Carlo technique and the closed-form Black-Scholes option pricing formula. The input volatility for the second method is given by the SABR closed-form formula for the implied volatility for a given strike and futures level (after calibration has been performed). We compare this implied volatility with that obtained, using the Newton-Raphson Algorithm, from the simulated results.

6. **modUpandoutCall and modAsianOption**: The first of these modules contains a function that evaluates an up-and-out barrier call option. The second contains a function that prices an arithmetic average-rate call or put option. Both do so using the SABR model. This is further discussed in §10.5.3.

Some other functionality is used within the above modules for the determination of Black-Scholes option prices, interpolation methods and calibration of the SABR model. The module **mwakeupobjects** calls the appropriate dll, or uses cover functions, where the underlying object is in such a dll. They are as follows:

DLL	Object	Use
FMAModels	EquityOption	Used for pricing options and obtaining implied volatility using Newton-Raphson
FMAModels	Utilities	Simple mathematical procedures
YieldCurve	CurveInterpolate	Interpolation (raw)
SABR	BuildModel	To obtain the SABR parameters as in (West 2005*a*)
SABR	FitTraderSkew	SABR parameters
FMAModels	DateRules	To obtain the correct number of business days when pricing options
SABR	SABRfunctions	To evaluate skew volatility, obtain the α parameter

10.5 Model Calibration and Pricing Options

Using the local volatility model (trinomial tree) described in (Derman, Kani & Chriss 1996) and the SABR stochastic volatility model, we will begin by obtaining the parameters α, β, ν and ρ for the equity index data from the dealer for maturities of one and two years. This will be performed according to (West 2005*a*). Once these parameters are obtained, we will use the implied model skew in both SABR and the trinomial scheme to price both European and path-dependent options. The spot value for the index on 24 March 2005 is 11 963.

10.5.1 SABR Parameters

To obtain an implied volatility skew at the required maturities, we are first required to obtain the SABR parameters such that the model implied skew matches the dealer provided SAFEX implied skew for the required maturities[1] as in (West 2005*a*, §6). The methodology for each parameter is briefly described below. We note here that the current futures level f and σ_{ATM} are inputs into the model.

The β Parameter

The value of β is estimated from a log-log plot of $\sigma(f, f)$ and f:

$$\ln \sigma(f, f) = \ln \alpha - (1 - \beta) \ln f + \ldots$$

as given in (Hagan et al. 2002). Analysis of historical trade data (West 2005*a*) suggest a time weighted regression scheme. The results for the South African market suggests that $\beta = 0.7$. Once selected, this value does not change.

[1] Another approach is to obtain parameters such that the model implied skew matches historical market trade data which consists of single trades and trade sets.

The α parameter

This parameter is a function the current futures level f, the at-the-money (ATM) volatility σ_{ATM}, β, ρ, ν and τ_{ex}. Therefore, we have the ATM volatility as an input and use the model to calculate α. σ_{ATM} is given by

$$\sigma_{\text{ATM}} = \sigma_{\text{B}}(f,f) = \frac{\alpha}{f^{1-\beta}}\left(1+\left(\frac{(1-\beta)^2}{24}\frac{\alpha^2}{f^{2-2\beta}}+\frac{1}{4}\frac{\rho\beta\nu\alpha}{f^{1-\beta}}+\frac{\left(2-3\rho^2\right)}{24}\nu^2\right)\tau_{\text{ex}}\right).$$

Assuming we have already solved for ν and ρ, the above equation is inverted and α is found to be the root of the cubic equation (9.95):

$$\frac{(1-\beta)^2\,\tau_{\text{ex}}}{24f^{2-2\beta}}\alpha^3+\frac{\rho\beta\nu\tau_{\text{ex}}}{4f^{1-\beta}}\alpha^2+\left(1+\frac{2-3\rho^2}{24}\nu^2\tau_{\text{ex}}\right)\alpha-\sigma_{\text{ATM}}f^{1-\beta}=0. \tag{10.1}$$

α is chosen as the either the only real root of the equation or the smallest positive root.

The ν and ρ Parameters

Given that f and σ_{ATM} are inputs and β is chosen and fixed, given values for ν and ρ, we have that α is no longer an input. We seek values for the input pair (ν,ρ) such that the model best fits the SAFEX implied skew for one and two year maturities. The Nelder-Mead algorithm minimizes the error expression from the traded volatilities to the skew implied by the parameters.

10.5.2 Vanilla European Option Prices

Given that the valuation date (24 March 2005) is approximately one year prior to the March 2006 futures expiry (the third Thursday of the month) and correspondingly two years prior to the March 2007 expiry[2], we want to price a European call option on the underlying TOP40 index for maturities of approximately one and two years. The vanilla put option price can be calculated using put-call parity. We begin by using the dealer's skew data to calibrate the SABR model for the required maturities (obtain the parameters ν, ρ and α).

The following three methods are used to price both one and two year maturity options:

1. We first perform a hybrid quasi-Monte Carlo simulation described §9.3.2. This uses the parameters from the calibration to simulate the index movement throughout the life of the option. From the prices obtained, we back out the implied SABR volatility using the Newton-Raphson algorithm.

2. The next procedure will be to price the required options using the closed-form formula for the implied volatility given by (9.96). So, $\sigma_{\text{B}}(f,K)$ and other relevant inputs will be used in the standard Black-Scholes formulae for European call options.

[2] By convention OTC option expiries coincide with exchange expiries; and assuming this will avoid the need for interpolation.

3. The local volatility model described in (Derman, Kani & Chriss 1996) will also be calibrated using the SABR model implied skew, enabling the prices of the required vanilla options to be determined. For the one year option, the dealer's skew for 15 June 2005, 15 September 2005, 15 December 2005 and 16 March 2006 are used as inputs for the determination of the SABR parameters. Once this is done, the values for α, ν and ρ are used to obtain the SABR implied volatilities for a set of strikes at the above mentioned option expiries. For the two year option, the above four expiries and the 15 March 2007 skew data will be used.

The input data consists of the following:

1. Valuation date: 24 March 2005.
2. Spot: 11 963
3. Maturity dates:
   One year option: 16 March 2006,
   Two year option: 15 March 2007.
4. Continuously compounded risk free rate and dividend yields for the one and two years:

Input Value	16 March 2006	15 March 2007
r	7.44%	7.75%
q	3.50%	3.50%

5. Calibration using the data discussed in §10.1 yields the following SABR parameters at the input option expiries[3]:

Dates:	15-Jun-05	15-Sep-05	15-Dec-05	16-Mar-06	15-Jun-06	21-Sep-06	21-Dec-06	15-Mar-07
f_{ATM}	12 050	12 140	12 274	12 366	12 503	12 666	12 833	13 001
σ_{ATM}	14.00%	14.15%	13.50%	14.75%	15.00%	15.25%	15.75%	15.75%
α	2.0373	2.3904	2.2741	2.4727	2.5168	2.5619	2.6508	2.4567
ν	118.74%	90.42%	84.94%	79.45%	76.90%	74.14%	71.59%	69.23%
ρ	-54.71%	-78.09%	-70.87%	-63.65%	-62.32%	-60.88%	-59.55%	-58.32%

An example of the skew that is generated by the SABR model, contrasted to the dealer skew, is in Figure 10.1. The entire implied volatility surface from the SABR parameters is in Figure 10.2. The data that it represents is provided in Trees.xls, "mtm skew" spreadsheet.

Using this surface we obtain a local volatility surface from the Derman-Kani-Chriss as shown in Figure 10.3. This surface will vary depending on whether a term structure, skew structure, both or neither was used in the construction of the trees. There may also be slight differences if the option prices, used for determining the probabilities and subsequently the local volatility, are determined using a constant

[3]SAFEX equity expiries are for March, June, September and December of each year. The dealer provided skews for the June 2005, September 2005, December 2005, March 2006 and March 2007 expiries. The SABR model was fited to each of these expiries, and then the parameters found for Mach 2006 and March 2007 were interpolated for June 2006, September 2006 and December 2006. At the time of writing these option expiries had not yet traded in the formal market.

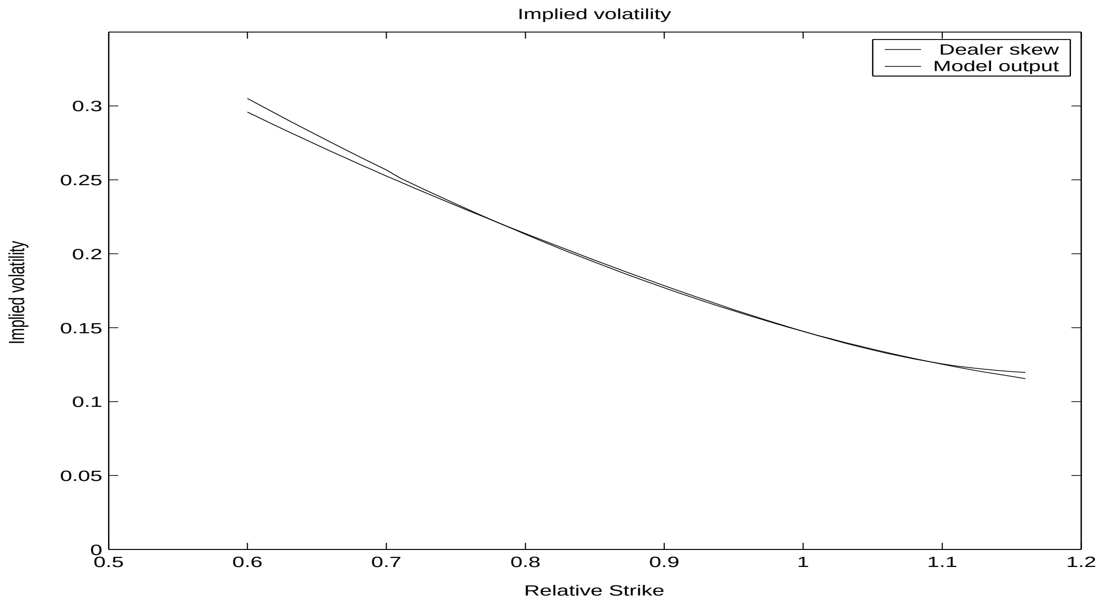

Figure 10.1: The SABR skew vs the dealer skew for option maturity 16 March 2006

volatility trinomial tree or the Black-Scholes option pricing formula. Increasing the number of nodes required has the effect of generating more detailed surfaces which can become unlikely or impractical as a result of interpolation or extrapolation errors.

1. For V_{call} with a maturity of approximately one year, we calculate the implied volatility from the SABR Monte Carlo simulation after calibration performed, the SABR implied volatility given by the closed-form solution and the implied volatility from the trinomial scheme. This local volatility models was calibrated using a skew derived from the SABR parameters.

2. For V_{call} with a maturity of approximately two years, paucity of data requires the ATM implied volatility for the futures expiries to be interpolated. We also linearly interpolate ρ and ν, which can then be used to find α, given the futures levels corresponding the dates. This data is provided above. Thus, we get the following:

It is important to stress that the implied volatilities may vary slightly in the implied tree scenario. This is as a result of the method one chooses to construct the tree, as well as the number of nodes required.

10.5.3 Exotic Equity Options

The following two exotic options will be priced:

(i) **Up-and-Out Call Option**:
This is a single barrier knock-out call option (up option) which entitles the holder of the option

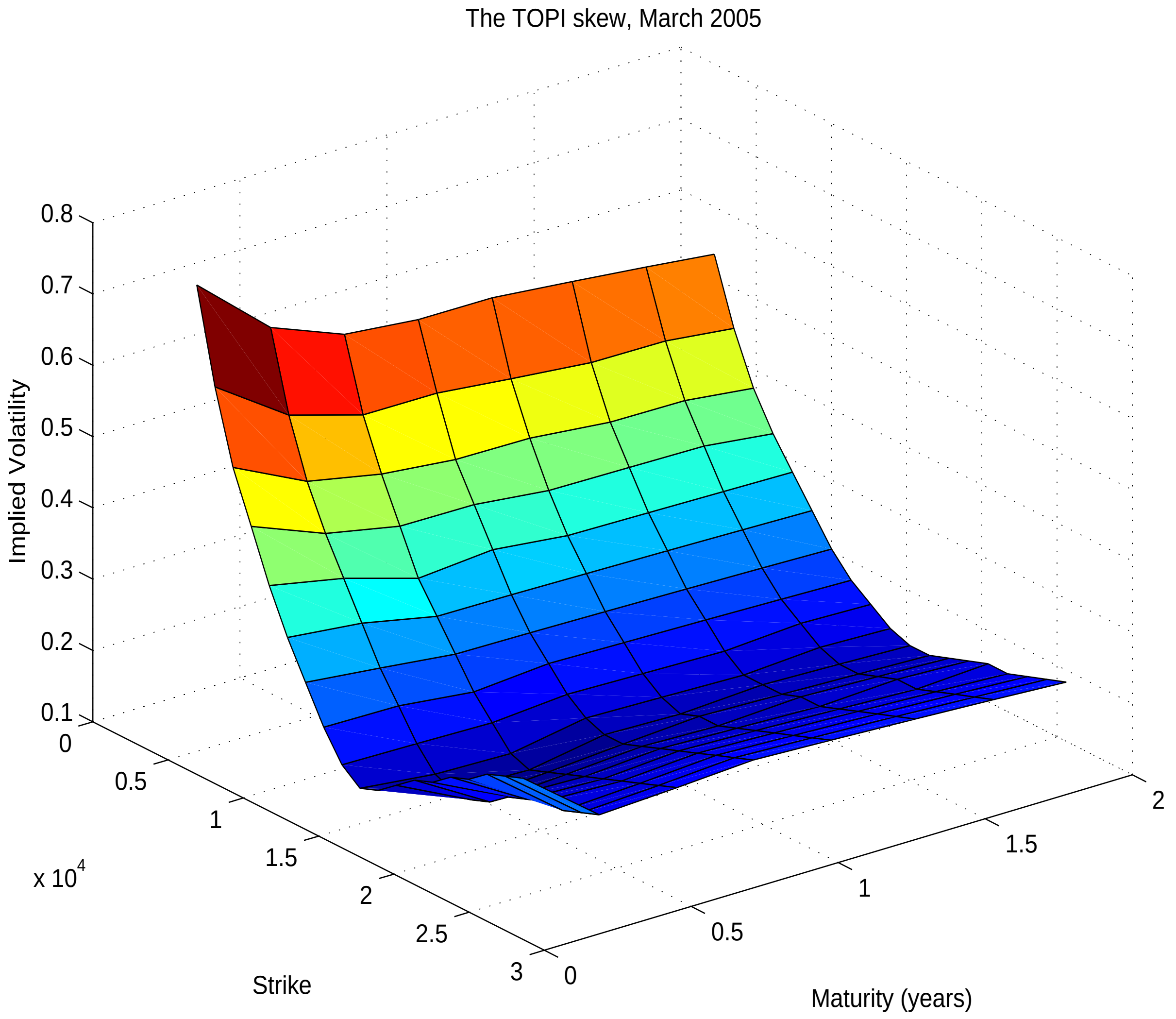

Figure 10.2: The TOP40 skew, by term and strike, March 2005

to the maximum of difference between the spot at maturity of the option and the strike, and zero. The option only pays off if the spot remains below the barrier during the life of the option, at the end of every business day. Note that the barrier must be above the strike for the option to have a value. The formula for such an option with constant volatility can be found in (Haug 1998, §2.10).

We will be considering a maturity of approximately two years with the SABR parameters given in §10.5.2, the barrier option is to be priced via the hybrid quasi-Monte Carlo simulation. This price will then be compared with the price obtained using the parameterized trinomial implied tree. The barrier option is checked at each of the constructed nodes in the tree and consequently, the price of this type of call option decreases as the number of nodes increases. This can be attributed to the limit of the discrete case being continuous. The continuous time simulation allows for the criterion to be validated at the end of every business day. As the number of sample paths in the Monte carlo

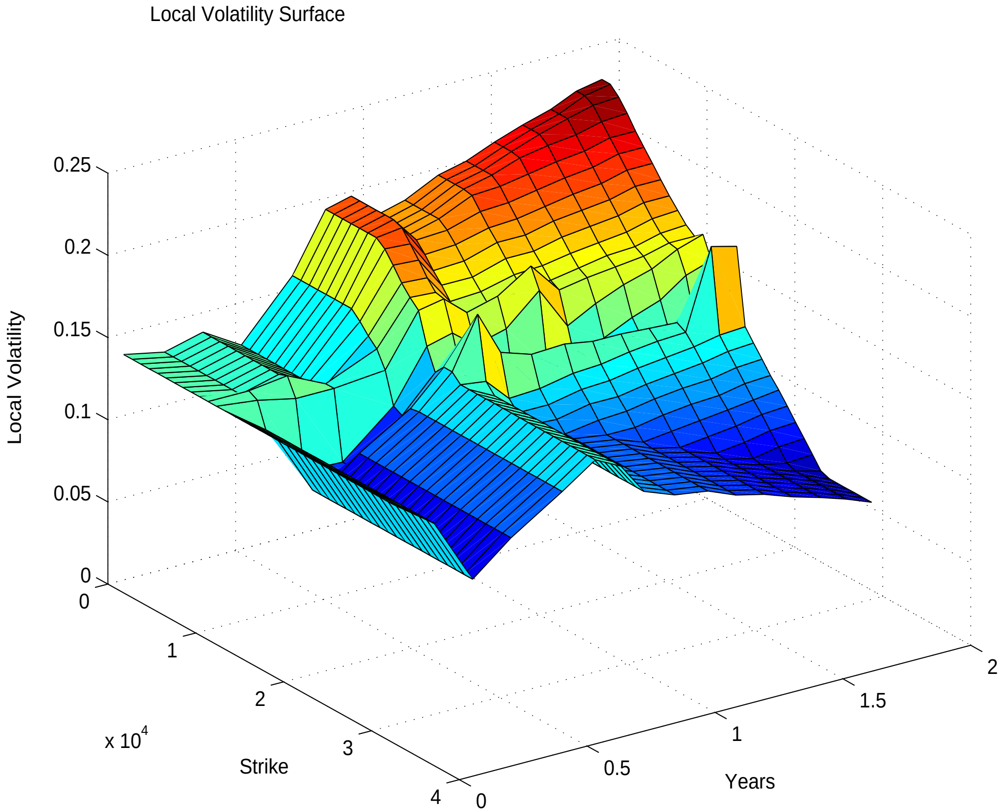

Figure 10.3: The local volatility surface obtained from the trinomial scheme. At each time node, for strikes which are above/below the maximum/minimum strke in the tree at that time node, we simply extrapolate from the volatility at such maximum/minimum.

simulation increases, the price increases. We will be using 10 000 sample paths for the Monte Carlo simulation and 20 nodes in the implied trinomial tree. We once again use the data provided by the dealer for 24 March 2005 to obtain the implied tree. The barrier is checked forward inductively, setting the probabilities of all paths that lead to a spot price greater than the barrier, to zero. The option price is then the discounted average of the sum of the probabilities multiplied by each of the payoffs.

Up-and-Out Call Option	**Value**
Futures	13001.00
Spot	11963
Valuation Date	24 March 2005
Expiry Date	15 March 2007
Barrier	14 000
Strike	10 000
SABR Price	706.83
DermanKaniChriss Price	570.10

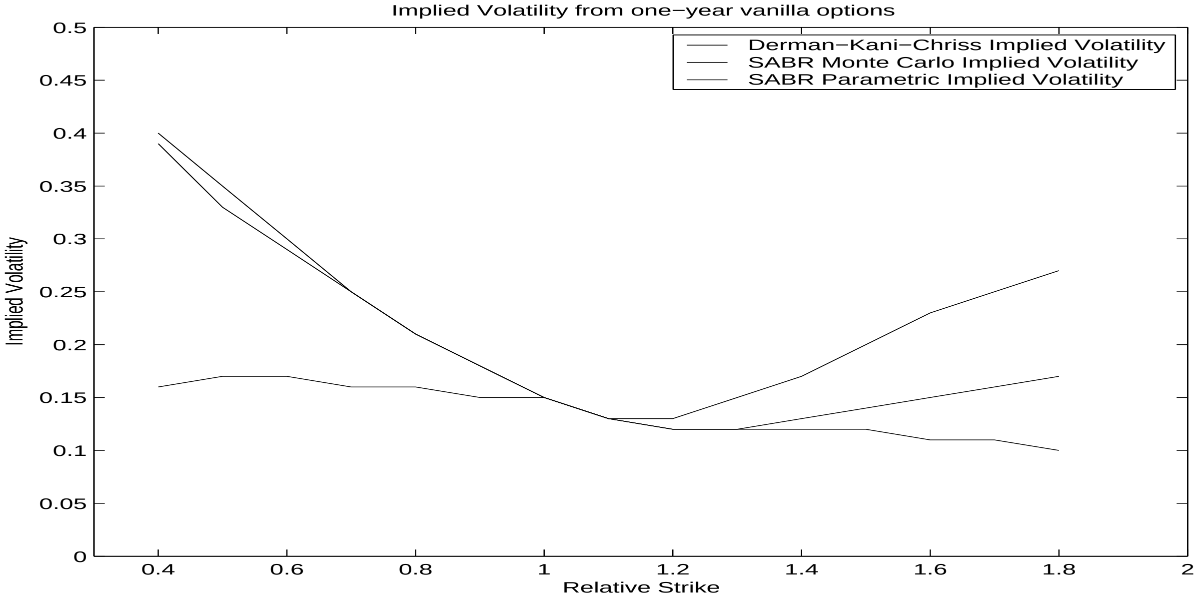

Figure 10.4: The implied volatilities of the one year vanilla option

As we can see, the price differences are quite large and increases as the number of tree nodes increase, as well as the number of sample paths increase.

(ii) **Arithmetic Average-Rate Option**:

Suppose the underlying is recorded at a few selected dates until maturity. The strike of such an option is set at the inception of the contract and is constant. At maturity, the holder of the call option will receive the maximum of the difference between the arithmetic average of all the spot prices recorded and the strike, and zero. The holder of a put option will receive the maximum of the difference between the strike and the arithmetic average of all the spot prices, and zero. The formula for such an option with constant volatility can be found in (Haug 1998, §2.12).

This exotic option is to be priced using the SABR Monte Carlo simulation, as well as the implied trinomial tree.

We briefly discuss the computational technique regarding the second method. Depending on the number of nodes N in the constructed tree, there will be 3^N paths, each with an associated probability. For simplicity, we assume the averaging is being performed on the dates that coincide with the nodes of the constructed tree. For computational efficiency, we limit the number of paths chosen to price the option. It was found that after about 20000 paths the computation time becomes onerous. If $N \leq 9$ then $3^N < 20000$, so all 3^N paths will be used. If $N > 9$, we randomly select 20 000 paths from the total of the 3^N paths.

At the terminal nodes of the tree for each of the paths, we require:

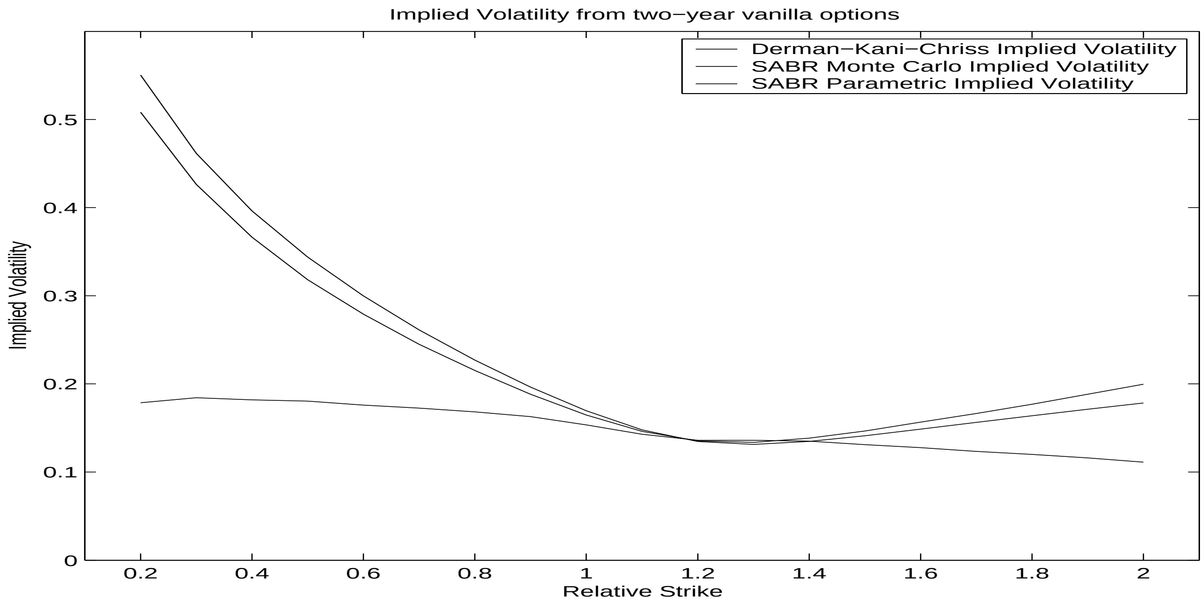

Figure 10.5: The implied volatilities of the two year vanilla option

1. The average of the spot prices that occur at each of the averaging nodes within the tree, which is simply the sum along the path divided by N[4],
2. The probability of the associated path, which is the product of the corresponding up, sideways or downwards probabilities.

Given a one/two year Asian option (valuation date is 24 March 2005 and expiry date is 16 March 2006/15 March 2007), we construct an eight/sixteen node implied trinomial tree. The terminal value of the option is taken to be the probability weighted average of the payoffs for each path, and this is then discounted to find the Asian option price.

The dates that correspond to the nodes of the tree are then used in the SABR Monte Carlo simulation. The averaging will occur over these days. Thus, using the relevant data for parameterization of the tree and for the simulation, the following prices for the one/two Asian options were obtained in Figures 10.6 and 10.7 respectively.

10.6 Conclusion

Bearing in mind that a mathematical model is an attempt to approximate reality, we can conclude that if a particular class of models accomplishes this to a better degree than its predecessor, it would be sensible to implement it. The model should be parsimonious, transparent and strike a balance between complexity and analytical tractability.

[4]By convention, the observed spot price at $t = 0$ does not contribute to the sum, it is only the subsequent nodes that do.

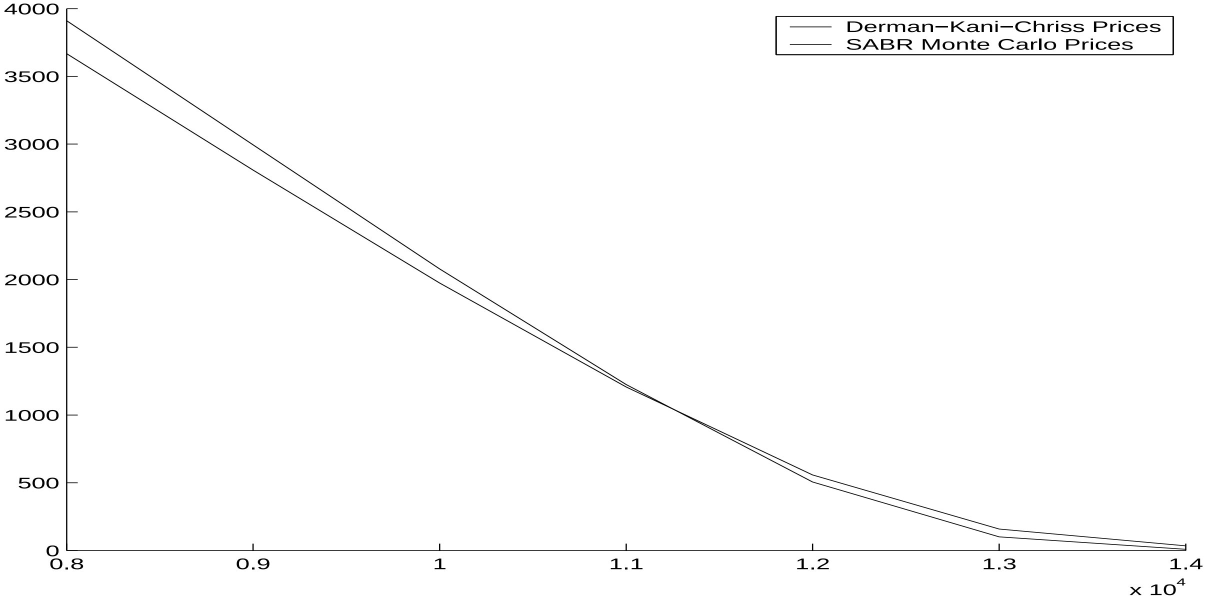

Figure 10.6: The prices obtained from the SABR Monte Carlo simulation and the eight node implied trinomial tree for the Asian option

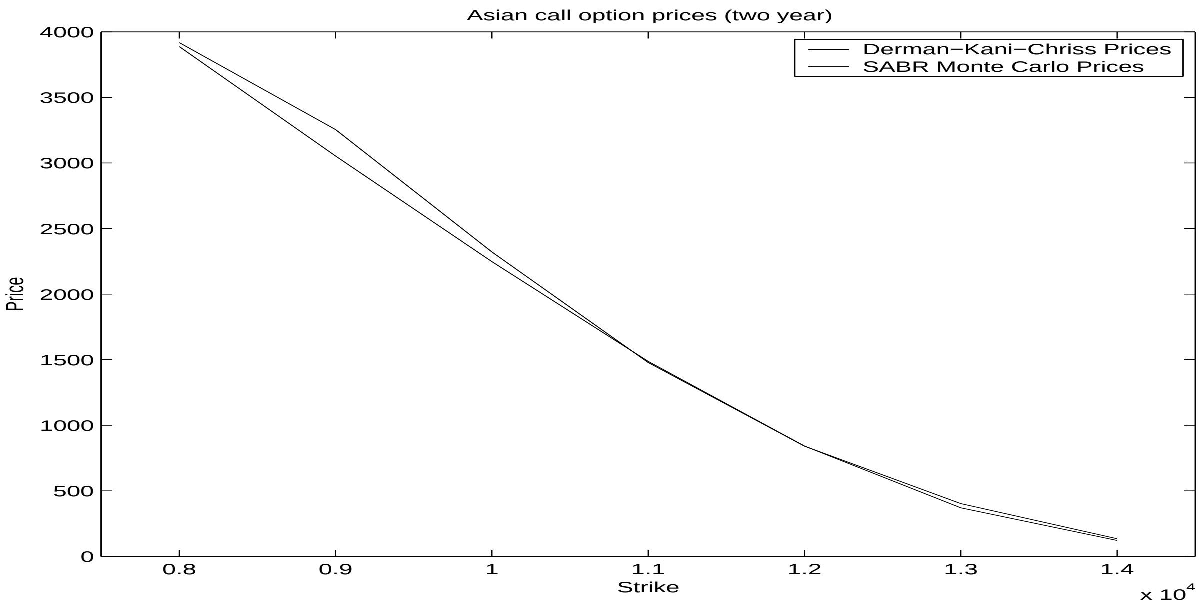

Figure 10.7: The prices obtained from the SABR Monte Carlo simulation and the sixteen node implied trinomial tree for the Asian option

The assumption of constant volatility in the original Black-Scholes model for asset prices is clearly inadequate for the above-mentioned purpose. This is evident from a number of observations, primarily the skew/smile that exists in the (vanilla) derivative markets. To better approximate the observed deviance from the base model, we have investigated a number of local and stochastic volatility option pricing models, the local models being a subset of the stochastic models. In principle, these models should output the observed vanilla option prices (implied volatilities) as they have been calibrated accordingly. This follows from the no-arbitrage requirement. Having calibrated the Derman-Kani-Chriss local volatility model and the SABR stochastic volatility model, it is evident that for strikes further away from the model, the local volatility model does not perform as well as the stochastic volatility model. This, of course, may vary slightly, depending on the input requirements for the construction of the tree. The prices for the exotic options, especially in illiquid markets that do not trade these instruments regularly, cannot be validated. It is interesting to note that the Asian option prices obtained from both methods were similar, regardless of the time to maturity. However, large disparities between the OTM vanilla options indicate that the results from the stochastic models are deemed more credible. The pricing procedure can be achieved in numerous ways. What is certainly a trickier task is the hedging of such instruments.

Bibliography

Abramowitz, M. & Stegun, I. A. (1974), *Handbook of Mathematical Functions, With Formulas, Graphs, and Mathematical Tables*, Dover.

Attari, M. (2004), Option Pricing Using Fourier Transforms: A Numerically Efficient Simplification. Charles River Associates, Boston.

Barle, S. & Cakici, N. (1995), 'Growing a smiling tree', *Risk* **8**(10).

Beasley, J. D. & Springer, S. G. (1977), 'Algorithm AS 111: The percentage points of the normal distribution', *Applied Statistics* **26**, 118–121.

Bingham, N. & Kiesel, R. (2004), *Risk-Neutral Valuation: Pricing and Hedging of Financial Derivatives*, second edn, Springer.

Björk, T. (2004), *Arbitrage Theory in Continuous Time*, second edn, Oxford University Press.

Black, F. (1976), 'The pricing of commodity contracts', *Journal of Financial Economics* **3**, 167–179.

Black, F. & Scholes, M. (1973), 'The pricing of options and corporate liabilities', *Journal of Political Economy* **73**(May-June), 637–659.

Brandt, M. W. & Wu, T. (2002), 'Cross-sectional tests of deterministic volatility functions', *Journal of Empirical Finance* **9**.

Breeden, D. & Litzenberger, R. (1978), 'Prices of state-contingent claims implicit in option prices', *Journal of Business* **51**, 621–651.

Burden, R. L. & Faires, J. D. (1997), *Numerical Analysis*, sixth edn, Brooks Cole.

Cheng, J. & Druzdzel, M. J. (1986), Computational investigation of low-discrepancy sequences in simulation algorithms for Bayesian networks. School of Information Sciences and Intelligent Systems Program, University of Pittsburgh.

Cole, J. D. & Kevorkian, J. (1981), *Perturbation Methods in Applied Mathematics*, Springer-Verlag.

Cox, J., Ross, S. & Rubinstein, M. (1979), 'Option pricing: a simplified approach', *Journal of Financial Economics* **7**, 229–263.

Derman, E. (1999), 'Regimes of volatility'. Goldman Sachs Quantitative Strategies Research Notes.

Derman, E. & Kani, I. (1994), 'Riding on a smile', *Risk* **7**(2).

Derman, E., Kani, I. & Chriss, N. (1996), 'Implied trinomial trees of the volatility smile', *Journal of Derivatives* **4**(summer).

Derman, E., Kani, I. & Zou, J. (1996), 'The local volatility surface: unlocking the information in index option prices', *Financial Analysts Journal* **July-Aug**, 25–36.

Dias, M. A. G. (2004), 'Quasi-Monte Carlo simulation'. Petrobras, Petróleo Brasileiro S.A.
*http://www.puc-rio.br/marco.ind/

Dupire, B. (1994), 'Pricing with a smile', *Risk* **7**(1).

Etheridge, A. (2002), *A Course in Financial Calculus*, Cambridge University Press.

Faure, H. (1982), 'Discrepance de suites associees a un systeme de numeration (en dimensions)', *Acta Arithmetica* **41**, 337–351.

Fouque, J.-P., Papanicolaou, G. & Sircar, K. R. (2000), *Derivatives in Financial Markets with Stochastic Volatility*, Cambridge University Press.

Gatheral, J. (2004). Lecture 1: Stochastic Volatility and Local Volatility.
*http://www.math.nyu.edu/fellows_fin_math/gatheral/lecture1_2004.pdf

Glasserman, P. (2004), *Monte Carlo Methods in Financial Engineering*, Springer.

Goldberg, R. R. (1976), *Methods of Real Analysis*, second edn, John Wiley and Sons.

Grimmet, G. & Stirzaker, D. (2001), *Probability and Random Processes*, third edn, Oxford University Press.

Hagan, P. S., Kumar, D., Lesniewski, A. S. & Woodward, D. E. (2002), 'Managing smile risk', *WILMOTT Magazine* **September**, 84–108.
*http://www.wilmott.com/pdfs/021118_smile.pdf

Hagan, P. S. & West, G. (2005), 'Interpolation methods for curve construction', *Applied Mathematical Finance* . To appear.
*http://www.finmod.co.za/interpolation.pdf

Hagan, P. S. & Woodward, D. E. (1998), 'Equivalent Black volatilities', *Applied Mathematical Finance* **6**, 147–157.

Hakala, J. & Wystup, U. (2002), *Foreign Exchange Risk: models, instruments and strategies*, Risk Books.

Haug, E. G. (1998), *The complete guide to option pricing formulas*, McGraw - Hill.

Heston, S. L. (1993), 'A closed form solution for options with stochastic volatility with applications to bond and currency options', *Review of Financial Studies* **6**(2).

Higham, D. J. (2001), 'An algorithmic introduction to numerical simulation of stochastic differential equations', *SIAM* **43**(3), 525–546.

Hull, J. & White, A. (1987), 'The pricing of options on assets with stochastic volatilities', *Journal of Finance* **42**, 281–300.

Hulley, H. & Lotter, G. (2004), 'Numerical Methods in Finance'. University of the Witwatersrand.

Jackson, M. & Staunton, M. (2001), *Advanced Modelling using Excel and VBA*, John Wiley and Sons.

James, G. (1999), *Advanced Modern Engineering Mathematics*, second edn, Addison-Wesley.

Kendall, M. G. (1945), *The Advanced Theory of Statistics*, Vol. 1, second edn, Charles Griffin & Company Limited.

Kwok, Y.-K. (1998), *Mathematical Models of Financial Derivatives*, Springer-Verlag.

Lewis, A. (2000), *Option Valuation under Stochastic Volatility : with Mathematica Code*, Finance Press.

Merton, R. C. (1973), 'The rational theory of option pricing', *Bell Journal of Economic and Management Science* **4**.

Moro, B. (February 1995), 'The full Monte', *Risk* **8**(2).

Nayfeh, A. H. (1981), *Introduction to Perturbation Techniques*, Wiley.

Oksendal, B. (2004), *Stochastic Differential Equations*, sixth edn, Springer-Verlag.

Owen, A. B. (1998), Monte Carlo Extension of Quasi-Monte Carlo. Department of Statistics, Stanford University.

Renault, E. & Touzi, N. (1996), 'Option Hedging and Implied Volatilities in a Stochastic Volatility Model', *Mathematical Finance* **6**, 279–302.

Rogers, L. C. G. & Williams, D. (2000), *Diffusions, Markov Processes and Martingales*, Vol. 1, Cambridge University Press.

Rubinstein, M. (1994), 'Implied binomial trees', *Journal of Finance* **49**(3), 771–818.

Rudin, W. (1987), *Real and Complex Analysis*, third edn, McGraw-Hill.

Schöbel, R and Zhu, J (1999), 'Stochastic Volatility with an Ornstein-Uhlenbeck Process: An extension', *European Finance Review* **3**, 23–46.

Sepp, A. (2003), Pricing European-Style Options under Jump Diffusion Processes with Stochastic Volatility: Applications of Fourier Transform. Faculty of Mathematics and Computer Science, University of Tartu, Estonia.
*www.hot.ee/seppar

Vogt, A. (2004), 'Optimized Excel solution for Heston's model using Gauss integration'.
*http://www.axelvogt.de/axalom/Heston93_opt.txt

Vose, D. (2000), *Risk Analysis: A Quantitative Guide*, second edn, Wiley.

West, G. (2005*a*), 'Calibration of the SABR model in illiquid markets', *Applied Mathematical Finance* . To appear.
*http://www.finmod.co.za/sabrilliquid.pdf

West, G. (2005*b*), 'The Mathematics of South African Financial Markets and Instruments'. Lecture notes, Honours in Mathematics of Finance, University of the Witwatersrand, Johannesburg.
*http://www.finmod.co.za/safm.pdf

Wilmott, P. (2000), *Paul Wilmott on Quantitative Finance*, John Wiley and Sons Ltd.