HW4 | Chandan Suri, CS4090

1.

Number of seats = 50
Number of passengers that have the tickets = 55
random variable $Y \Rightarrow$ Number of ticketed passengers who actually show up for the flight

$\Rightarrow$ We have been provided the probability mass function of Y.

(a.) P(flight will accomodate all the ticketed passengers who show up) $\Rightarrow P_A$

This can be achieved if at most 50 passengers show up.

$$P_A = P(Y \leq 50)$$
$$= P(Y=45) + P(Y=46) + P(Y=47) + P(Y=48) + P(Y=49) + P(Y=50)$$
$$= 0.05 + 0.10 + 0.12 + 0.14 + 0.25 + 0.17$$
$$= \underline{\underline{0.83}}$$ Ans.

(b) P(not all ticketed passengers who show up can be accommodated) $\Rightarrow P_B$

This is basically the complement of case (a) above,

So,

$$P_B = 1 - P(Y \leq 50)$$
$$= 1 - P_A$$
$$= 1 - 0.83 = \underline{\underline{0.17}}$$ Ans.

(c.) If I am the first person on the standby list who will get on the plane if there are any seats available,

Then, that would mean that I will be given the seat when at most 49 passengers have shown up.

So, the probability of that happenning would be as follows:-

$$P_C = P(Y \leq 49)$$

$$= P(Y=45) + P(Y=46) + P(Y=47) + P(Y=48) + P(Y=49)$$

$$= 0.05 + 0.10 + 0.12 + 0.14 + 0.25$$

$$= \underline{\underline{0.66}}$$ Ans.

Similarly, if I am the third person on the standby list, then the probability for me to get the seat can be computed as when at most 47 passengers have shown up for the flight that had tickets. (3 seats should be left)

Thus,

$$P_D = P(Y \leq 47)$$

$$= P(Y=45) + P(Y=46) + P(Y=47)$$

$$= 0.05 + 0.10 + 0.12$$

$$= \underline{\underline{0.27}}$$ Ans.

(2.) One to five forms are required for a building permit.
Y = Number of forms required of the next applicant.

Probability that y forms are required $\Rightarrow$

$$\boxed{p(y) = Ky} \qquad [\text{for } y = 1, \ldots, 5]$$

(a) To calculate the value of K,

Sum of all the probabilities should be equal to 1.
(as one-five forms are required, the only cases possible).

$$\sum_{y=1}^{5} p(y) = 1$$

$$P(Y=1) + P(Y=2) + P(Y=3) + P(Y=4) + P(Y=5) = 1$$

Putting the values of y in the function for the probability above,

$$1K + 2K + 3K + 4K + 5K = 1$$

$$15K = 1 \Rightarrow \boxed{K = 1/15}$$ Ans.

(b) for the probability that at most 3 forms are required,

$$\Rightarrow P(Y \leq 3)$$

$$\Rightarrow P(1) + P(2) + P(3)$$

$$\Rightarrow K + 2K + 3K = 6K$$

$$\Rightarrow 6 \times \frac{1}{15} = \frac{2}{5} = \underline{\underline{0.4}}$$ Ans.

(c) for the probability that between two and four (inclusive) forms are required,

$$\Rightarrow P(2 \leq y \leq 4) = P(2) + P(3) + P(4)$$

$$= 2K + 3K + 4K = 9K$$

$\Rightarrow 9 \times \frac{1}{15} = \frac{9}{15} = \frac{3}{5}$

$= \underline{\underline{0.6}}$ Ans

(d.)

for $p(y) = y^2/50$ with y values as 1, 2, 3, 4 or 5

to be the pmf of Y,

$\Rightarrow$ Sum of all the probabilities in the sample space should be equal to 1.

$$\sum_{y=1}^{5} p(y) = 1 \qquad \text{[should be equal to 1]}$$

$$\Rightarrow \sum_{y=1}^{5} \frac{y^2}{50}$$

$$\Rightarrow \frac{1^2}{50} + \frac{2^2}{50} + \frac{3^2}{50} + \frac{4^2}{50} + \frac{5^2}{50}$$

$$\Rightarrow \frac{1}{50}\left[1 + 4 + 9 + 16 + 25\right]$$

$$\Rightarrow \frac{55}{50} = \frac{11}{10} = \underline{\underline{1.1}}$$

This value comes to be greater than 1.

Thus, this distribution is not possible and $p(y) = y^2/50$ <u>cannot</u> be the pmf of Y.

Ans

③ Actual tracking weight of a stereo cartridge is set to track at 3g.

for a continuous rv X with pdf as shown below,

$$f(x) = \begin{cases} k[1-(x-3)^2] & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

ⓐ for determining the value of K,

we will assume that its a correct pdf.

(sum of all values in the domain should be equal to 1).

So,

$$\boxed{\int_{-\infty}^{\infty} f(x)\,.\,dx = 1}$$

we can break the integration above as,

$$\int_{-\infty}^{2} 0\,.\,dx + \int_{2}^{4} k[1-(x-3)^2]\,.\,dx + \int_{4}^{\infty} 0\,.\,dx = 1$$

The first and third term above will become zero.

$$\therefore \int_{2}^{4} k[1-(x-3)^2]\,.\,dx = 1$$

$$\Rightarrow k\int_{2}^{4}(1-(x-3)^2)\,.\,dx = 1$$

$$\Rightarrow k\left[\int_{2}^{4} 1\,.\,dx - \int_{2}^{4}(x-3)^2\,.\,dx\right] = 1$$

$$\Rightarrow k\left[x\Big|_{2}^{4} - \frac{(x-3)^3}{3}\Big|_{2}^{4}\right] = 1$$

$$\Rightarrow k\left[(4-2) - \left(\frac{(4-3)^3}{3} - \frac{(2-3)^3}{3}\right)\right] = 1$$

$$\Rightarrow k\left[2 - \left(\frac{1}{3} - \frac{(-1)}{3}\right)\right] = 1$$

$$\Rightarrow k\left[2 - \left(\frac{1}{3} + \frac{1}{3}\right)\right] = 1 \Rightarrow k\left[2 - \frac{2}{3}\right] = 1$$

$$\Rightarrow k\left[\frac{6-2}{3}\right] = 1 \Rightarrow k \cdot \frac{4}{3} = 1$$

$$\frac{4k}{3} = 1 \Rightarrow \boxed{k = 3/4} \Rightarrow \boxed{k = 0.75}$$

Ans.

(b) Graph of $f(x)$ can be shown as follows:-

firstly, after putting the value of k, $f(x)$ becomes:-

$$f(x) = \begin{cases} 0.75[1-(x-3)^2] & 2 \le x \le 4 \\ 0 & \text{otherwise} \end{cases}$$

Let's try to analyze the quadratic function above which looks like a downward facing parabola.

Let's see if it intersects the x-axis,

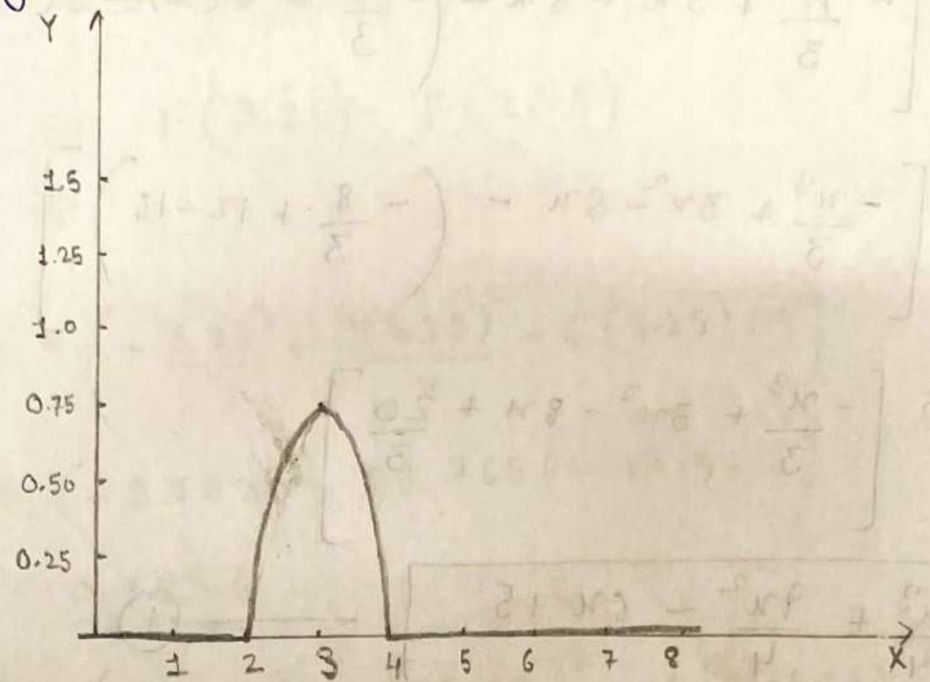
$$0.75[1-(x-3)^2] = 0 \Rightarrow (x-3)^2 = 1$$

$$x^2 - 6x + 9 - 1 = 0 \Rightarrow x^2 - 6x + 8 = 0 \Rightarrow x^2 - 4x - 2x + 8 = 0$$

$$(x-4)(x-2) = 0 \Rightarrow \underline{x = 2 \text{ or } 4}$$

$\therefore$ the graph should be a quadratic function, downward facing parabola with root at (maximum value) $x = 3$ and intersects the x-axis at $\underline{x = 2 \text{ and } 4.}$

graph of $f(x) \Rightarrow$



Ans.

(c) P(Actual tracking weight is greater than the prescribed weight) $\Rightarrow$

$$\boxed{P(X > 3)}$$

But, for computing this, we need to find the cdf (Cumulative distribution) $\Rightarrow$

$$F(x) = P(x \le X) = \int_{-\infty}^{x} f(y).dy$$

[changed the variable for clarity]

Thus,

$$\int_{-\infty}^{x} f(y).dy \Rightarrow \cancel{\int_{-\infty}^{2} 0.dy}^{\,0} + \int_{2}^{x} 0.75\left[1-(y-3)^2\right].dy$$

$$\Rightarrow 0.75 \int_{2}^{x} \left(1-(y-3)^2\right).dy$$

$$\Rightarrow 0.75 \left[\int_{2}^{x} \left(1-(y^2-6y+9)\right).dy\right]$$

$$\Rightarrow 0.75 \int_{2}^{x} (-y^2+6y-8).dy \Rightarrow 0.75\left[-\frac{y^3}{3}+\frac{6y^2}{2}-8y\right]\Bigg|_{2}^{x}$$

$$\Rightarrow 0.75\left[-\frac{x^3}{3}+3x^2-8x-\left(-\frac{2^3}{3}+3(2)^2-8(2)\right)\right]$$

$$\Rightarrow 0.75\left[-\frac{x^3}{3}+3x^2-8x-\left(-\frac{8}{3}+12-16\right)\right]$$

$$\Rightarrow 0.75\left[-\frac{x^3}{3}+3x^2-8x+\frac{20}{3}\right]$$

$$\Rightarrow \boxed{-\frac{x^3}{4}+\frac{9x^2}{4}-6x+5} \longrightarrow (1)$$

Ans.

Now,

for finding $P(X>3) \Rightarrow$

$$P(X>3) = 1-P(X\le 3)$$

$$= 1-F(3)$$

[Using equation (1) above for cdf]

$$= 1-\left[-\frac{3^3}{4}+\frac{9(3)^2}{4}-6(3)+5\right]$$

$$= 1-\left[-\frac{27}{4}+\frac{81}{4}-18+5\right]$$

$$= 1-\left[\frac{-27+81-52}{4}\right]$$

$$= 1-\left[\frac{2}{4}\right] = 1-\frac{1}{2} = \frac{1}{2} = \boxed{0.5}$$

Ans

(d) P(Actual weight is between 0.25g of the prescribed weight of 3g) $\Rightarrow$

$$P(2.75 \le X \le 3.25) \Rightarrow$$

$$\Rightarrow F(3.25) - F(2.75)$$

$$\Rightarrow$$ [Using equation (1) above]

$$F(3.25) = \left(-\frac{3.25^3}{4} + \frac{9(3.25)^2}{4} - 6(3.25) + 5\right)$$

$$= (-8.5820 + 23.7656 - 19.5 + 5)$$

$$= \underline{\underline{0.6836}}$$

$$F(2.75) = \left(-\frac{2.75^3}{4} + \frac{9(2.75)^2}{4} - 6(2.75) + 5\right)$$

$$= (-5.1992 + 17.0156 - 16.5 + 5)$$

$$= \underline{0.3164}$$

$\therefore$

$$P(2.75 \le X \le 3.25) \Rightarrow 0.6836 - 0.3164$$

$$= \underline{\underline{0.3672}}$$

Ans.

(e) P(Actual weight is ~~between~~ over 0.5g of the prescribed weight of 3g) $\Rightarrow$ (thus, for $X < 2.5$ and $X > 3.5$)

Thus, we need to find $P(2.5 \le X \le 3.5)$, such that,

$$P(\{X < 2.5\} \cup \{X > 3.5\}) = P(X \le 2.5) + P(X >= 3.5)$$

$$= \underline{\underline{1 - P(2.5 \le X \le 3.5)}}$$

So, for $P(2.5 \le X \le 3.5) \Rightarrow F(3.5) - F(2.5)$

$$\Rightarrow$$

$$F(3.5) \Rightarrow \left[-\frac{3.5^3}{4} + \frac{9.(3.5)^2}{4} - 6(3.5) + 5\right]$$

$$= \left[-10.7188 + 27.5625 - 21 + 5\right]$$

$$= \underline{\underline{0.8437}}$$

$$F(2.5) \Rightarrow \left[-\frac{2.5^3}{4} + \frac{9(2.5)^2}{4} - 6(2.5) + 5\right]$$

$$= \left[-3.9063 + 14.0625 - 15 + 5\right]$$

$$= \underline{\underline{0.1562}}$$

$$\therefore P(2.5 \le X \le 3.5) = F(3.5) - F(2.5)$$

$$= 0.8437 - 0.1562$$

$$= \underline{\underline{0.6875}}$$

$\therefore$ The required probability $\Rightarrow$

$$P(\{X < 2.5\} \cup \{X > 3.5\})$$

$$= 1 - P(2.5 \le X \le 3.5)$$

$$= 1 - 0.6875$$

$$= \underline{\underline{0.3125}}$$

<u>Ans.</u>

④ $X \rightarrow$ amount of time a book on two-hour reserve is actually checked out.

cdf:-

$$F(x) = \begin{cases} 0 & x < 0 \\ x^2/4 & 0 \le x < 2 \\ 1 & 2 \le x \end{cases}$$

ⓐ $P(X \le 1) \Rightarrow$

As we are already provided with the cumulative distribution:-

so, $P(X \le 1) = F(1)$

$$= \frac{(1)^2}{4}$$

$$= \underline{\underline{0.25}}$$ Ans.

[As x lies in the range $0 \le x < 2$] we take $F(x) = x^2/4$

ⓑ $P(0.5 \le X \le 1) \Rightarrow$

As we are already provided with the cumulative distribution:-

so,

$$P(0.5 \le X \le 1) \Rightarrow F(1) - F(0.5)$$

According to the cdf provided above, we will take

$$F(x) = x^2/4$$

$$\Rightarrow \frac{(1)^2}{4} - \frac{(0.5)^2}{4} = \frac{1}{4} - \frac{0.25}{4} = \frac{0.75}{4}$$

$$= \underline{\underline{0.1875}}$$ Ans

ⓒ for calculating $P(X > 1.5) \Rightarrow$

we can do $\Rightarrow$

$$= 1 - P(X \le 1.5)$$

$$= 1 - F(1.5)$$

$$= 1 - \frac{(1.5)^2}{4} = 1 - 0.5625$$

(As sum of all probabilities should be 1 $F(x) = 1$ for $2 \not\le x$.)

$\Rightarrow 1 - 0.5625 = \underline{\underline{0.4375}}$ Ans.

(d.) for calculating the median checkout duration,

$$F(\tilde{\mu}) = 0.5$$

As for $x<0$, it's 0 and $x \geq 2$, it's 1, median should lie somewhere in $0 \leq \tilde{\mu} < 2$, so,

$$\Rightarrow \frac{\tilde{\mu}^2}{4} = 0.5$$

$$\Rightarrow \tilde{\mu}^2 = 0.5 \times 4 = 2.0$$

$$\Rightarrow \tilde{\mu} = \sqrt{2} = \underline{\underline{1.4142}}$$ Ans.

(e.) for obtaining the density function $f(x)$,
As we have already been provided with the cdf, we would need to differentiate it,

for $x<0$,
$$f(x) = F'(x) = \frac{d}{dx}(0) = \underline{0}$$ Ans

for $0 \leq x < 2$,
$$f(x) = F'(x) = \frac{d}{dx}\left(\frac{x^2}{4}\right) = \frac{1}{4}\frac{d}{dx}(x^2)$$

$$= \frac{1}{4} \cdot 2x = \frac{x}{2} = \underline{\underline{0.5x}}$$ Ans.

for $x \geq 2$,
$$f(x) = F'(x) = \frac{d}{dx}(1) = \underline{0}$$ Ans

So,
$$\boxed{f(x) = \begin{cases} 0.5x & 0 \leq x < 2 \\ 0 & \text{otherwise} \end{cases}}$$ Ans

(5.) Probability that a goblet selected has cosmetic flaws and must be classified as "seconds" (p) =>

$$\underline{p = 0.1} \qquad [10\%]$$

(a) If six goblets are randomly selected, i.e.,

$$\underline{n = 6}.$$

Probability that only one of the 6 goblets selected is a "second" =>

P (only one goblet is a second) =>

[As this seems like a ~~bernoulli~~ binomial distribution where a goblet either is a "second" or is not,
so, probability that a goblet is not a second will be => $1-p = 1-0.1 = \underline{0.9}$]

For only one of six goblets to be "second", others will not be a second =>

$$P(X=1) \Rightarrow \binom{6}{1}(0.1)^1(0.9)^{6-1}$$

$$= \binom{6}{1}(0.1)(0.9)^5$$

$$= \frac{6!}{5!\,1!}(0.1)(0.9)^5$$

$$= 6(0.1)(0.9)^5$$

$$= 6(0.1)(0.59049)$$

$$= \underline{0.3543} \quad \text{Ans.}$$

(b) If six goblets are randomly selected (n=6), probability that atleast two are seconds =>

P (at least two of the goblets are seconds) =>

$$= P(X \geq 2)$$

$$= 1 - P(X < 2)$$

[As it will be easier to compute $P(X<2)$]

$$= 1 - [\underbrace{P(X=0)}_{\text{no goblet is a "second"}} + \underbrace{P(X=1)}_{\text{only one goblet is a "second"}}]$$

=> no goblet is a "second" only one goblet is a "second".

$$= 1 - \left[\binom{6}{0}(0.1)^0(0.9)^{6-0} + \binom{6}{1}(0.1)^1(0.9)^{6-1}\right]$$

$$= 1 - \left[\frac{6!}{6!\,0!} \cdot 1 \cdot (0.9)^6 + \frac{6!}{5!\,6!} \cdot (0.1)^1(0.9)^5\right]$$

$$= 1 - \left[(0.9)^6 + 0.3543\right]$$

$$= 1 - [0.5314 + 0.3543] = 1 - 0.8857$$

$$= 0.1143$$

Ans.

(c) If goblets are selected one by one,

The probability that at most five must be selected to find four that are not "seconds" =>

This would have 2 cases as follows:-

Case (1):- four goblets are selected such that none of them are not "seconds" =>

$$P(X=0) \Rightarrow \binom{4}{0}(0.1)^0(0.9)^{4-0} = \frac{6!}{6!\,0!} \cdot 1 \cdot (0.9)^4$$

$\Rightarrow 1 \times 1 \times (0.9)^4 = (0.9)^4 = \underline{\underline{0.6561}}$

$\cancel{\Rightarrow (0.9)^6 = 0.5314}$

<u>Case ②</u> :- five goblets are selected such that only one of them is a "second".

$P(X=1) \Rightarrow \binom{5}{1} (0.1)^1 (0.9)^{5-1}$

$= \frac{5!}{4!\,1!} (0.1)(0.9)^4$

$= 5(0.1)(0.9)^4 = 0.32805$

$\approx \underline{\underline{0.3281}}$

Adding the probabilities for the 2 cases above would give us the final probability $\Rightarrow$

$\Rightarrow 0.6561 + 0.3281$

$\Rightarrow$

<u>Case ②</u>:- As we have to stop at the 5th one when we get a goblet that is not a "second".

This would mean that there will be only (at most) one second in the first four goblets selected.

If there needs to be one second among the first four, that would mean that it can occur at the 1st - 4th place, thus, we will have 4 different combination for it. (and <u>5th one will be not a "second"</u>).

$\therefore$ Probability $\Rightarrow \left[\binom{4}{1}(0.1)^1 (0.9)^{4-1}\right] \times \boxed{0.9}$

$$\Rightarrow \left[\frac{4!}{3!\,1!} \times (0.1)^1 (0.9)^3\right] \times 0.9$$

$$= 4 \times 0.1 \times (0.9)^4 = \underline{\underline{0.2624}}$$

Adding the probabilities for the 2 cases above will give us the final probability ⇒

$$= [0.6561 + 0.2624]$$

$$= \underline{\underline{0.9185}}$$

<u>Ans.</u>

(6.) Probability that a randomly selected box of cereal has a particular prize $(p) = 0.2$

The event of getting a box with the prize is deemed as success here. (x)

As we are purchasing one box after the other until we obtain two of these prizes,

it looks like a negative binomial distribution, where x is unknown (number of trials),

probability of getting the prized box $= p = \underline{0.2}$.

and number of successes (prized boxes) needed $= r = \underline{\underline{2}}$.

(a) $\therefore$ probability that we purchase x boxes that do not have the desired prize $\Rightarrow$

$\Rightarrow$ following the formula for negative binomial distribution,

$$nb(x; r, p) = nb(x; 2, 0.2)$$

$$= \binom{x+2-1}{2-1} (0.2)^2 (1-0.2)^x$$

$$= \binom{x+1}{1} (0.2)^2 (0.8)^x$$

$$= \frac{(x+1)!}{x!\, 1!} (0.2)^2 (0.8)^x$$

$$= \underline{(x+1)(0.2)^2 (0.8)^x}, \text{ where } x = 0, 1, 2, \ldots$$

Ans

(b) The probability that we purchase 4 boxes $\Rightarrow$

As we stop when we have the 2nd prized box, that would mean that we came across $(4-2)=2$ non-prized boxes $\Rightarrow$

Using the formula for negative binomial distribution $\Rightarrow$

$$nb(x; r, p) = nb(2; 2, p) = nb(2; 2, 0.2)$$

$$= \binom{2+2-1}{2-1} (0.2)^2 (1-0.2)^2$$

$$= \binom{3}{1} (0.2)^2 (0.8)^2$$

$$= \frac{3!}{2!\,1!} (0.2)^2 (0.8)^2$$

$$= 3 (0.2)^2 (0.8)^2$$

$$= \underline{\underline{0.0768}}$$ Ans.

(c) The probability that we purchase at most 4 boxes $\Rightarrow$

$\therefore$ we can get at most 2 non-prized boxes $\Rightarrow$

$$P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$$

Using the formula for negative binomial distribution,

$$P(X \le 2) = nb(0; 2, 0.2) + nb(1; 2, 0.2) + nb(2; 2, 0.2)$$

$\therefore$

$$P(X \le 2) = \binom{0+2-1}{2-1}(0.2)^2(1-0.2)^0$$

$$+ \binom{1+2-1}{2-1}(0.2)^2(1-0.2)^1$$

$$+ \binom{2+2-1}{2-1}(0.2)^2(1-0.2)^2$$

$$= 1\times(0.2)^2\times 1 + 2\times(0.2)^2\times(0.8) + 3\times(0.2)^2\times(0.8)^2$$

$$= (0.2)^2 + 2(0.2)^2(0.8) + 3(0.2)^2(0.8)^2$$

$$= 0.04 + 0.064 + 0.0768$$

$$= \underline{\underline{0.1808}}$$

<u>Ans.</u>

(7)

X= number of children in the family

family decides to have children until it has three children of the same gender.

Given, $\boxed{P(B) = P(G) = 0.5}$

probability of a boy

probability of a girl

We need to find the pmf. of X.

As there are 2 genders (B and G) here, that would mean that it can have at most 5 children and 5th will have to be the third child of one of the genders.

(2 of each B and G + 1 of either B/G)

And, as it needs atleast 3 children of the same gender, X cannot be less than 3.

Thus, $3 \leq X \leq 5$ and $X \in I$,

so, either of the 3 cases hold,

$X = 3, 4$ or 5

To calculate the pmf of X, I will have to cover those 3 cases then, (furthermore, we will divide each case in 2 more cases as having a boy is success or having a girl is success).

Case (1):- $X = 3$,

when we get all boys $\Rightarrow$ $P_B = (0.5)^3$
(BBB)

$= 0.125$

When we get all girls (GGG) $\Rightarrow P_G = (0.5)^3$

$= 0.125$

Adding the 2 sub-cases above,

$\therefore P(X=3) = 0.125 + 0.125$

$= \underline{\underline{0.25}}$ Ans

Case ②:- $X=4$,

ⓐ When having a boy is a success and we know that third child is a boy when we have stopped. ~~$(n=3, k=2)$~~, ~~$k \rightarrow$ number of successes. (last one is a boy).~~

Using negative binomial distribution

$r=3,\ x=1$, third child is a boy.

$$P_B = \binom{1+3-1}{3-1}(0.5)^3(0.5)^1$$

$$= \frac{3!}{2!\,1!}(0.5)^3(0.5)^1$$

$$= 3(0.5)^3(0.5)^1 = 3(0.5)^4$$

$$= \underline{\underline{0.1875}}$$

ⓑ When having a girl is a success and we know that third child is a girl when we have stopped.

Using negative binomial distribution,

$r=3,\ x=1$, third child is a girl.

$$P_G = \binom{1+3-1}{3-1}(0.5)^3(0.5)^1 = 3(0.5)^4$$

$$= \underline{0.1875}$$

for case (2),

we will add probabilities for the 2 subcases above $(P_B + P_G)$

$\Rightarrow 0.1875 + 0.1875$

$= \underline{\underline{0.375}}$ Ans $(\underline{P(X=4)})$

Case (3):-

$X=5,$

(a) When having a boy is a success and we know that third child is a boy,

Using negative binomial distribution,

$r=3,\ x=2,\ p=0.5.$

$$P_B = nb(2;3,0.5) = \binom{2+3-1}{3-1}(0.5)^3(0.5)^2$$

$$= \binom{4}{2}(0.5)^5$$

$$= \frac{4!}{2!\,2!}(0.5)^5 = 3\times 2\times(0.5)^5$$

$$= \underline{6(0.5)^5}$$

(b) When having a girl is a success and we know that third child is a girl,

Using negative binomial distribution,

$r=3,\ x=2,\ p=0.5$

$$P_G = nb(2;3,0.5) = \binom{2+3-1}{3-1}(0.5)^3(0.5)^2$$

$$= \frac{4!}{2!\,2!}(0.5)^5 = \underline{\underline{6(0.5)^5}}$$

for case (3),

we add the probabilities,

$$P(X=5) = P_B + P_9$$
$$= 6(0.5)^5 + 6(0.5)^5$$
$$= 12(0.5)^5$$
$$= \underline{0.375}$$

Ans.

$\therefore$ The pmf can be shown as:-

X	3	4	5
$P(x)$	0.25	0.375	0.375

Ans

(8.) Information operators available = 5
Receiving requests according to a Poisson process with rate $\alpha = 2/min$.

Let $X \Rightarrow$ a Poisson random variable signifying the number of requests in a 1-minute interval.

As the period is taken to be 1-min and $\alpha = 2/min$.

$$\therefore \lambda = \alpha t$$
$$= (2)(1) = 2$$

(a) Probability that during a given 1-min period, the 1st operator receives no requests $\Rightarrow$

$$P(X=0) \Rightarrow \frac{e^{-\lambda} \lambda^{x}}{x!} \quad \text{[as it's a poisson process!]}$$

$$= \frac{e^{-2} 2^{0}}{0!} = e^{-2} = 0.1353$$

Ans

(b) Probability that during a given 1-min period, exactly 4 out of 5 operators receives no requests $\Rightarrow$

If we consider receiving no requests a success event,

~~then~~,
Using binomial distribution formulation,
and p (receiving no requests) $= 0.1353$ and $n=5$, $K=4$.

$$\Rightarrow \binom{5}{4} (0.1353)^4 (1-0.1353)^1$$

$$\Rightarrow 5\,(0.1353)^4 (0.8647) = 0.001448 \approx 0.0015$$

Ans

(c.) If all the operators receive exactly the same number of requests (x),

Probability of this happening during the 1-min period $\Rightarrow$

As this happens for all the operators simultaneously, we can calculate the probability of this happening to one of the operators & then multiply all.

$\therefore$ P(Operator i receivs x requests) $\Rightarrow$

$$= \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= \left(\frac{e^{-2}.2^{x}}{x!}\right)$$

$\therefore$ P(All operators receivs x requests) $\Rightarrow$

$$= \prod_{i=1}^{5} P(\text{Operator i receives } x \text{ requests})$$

[As all are independent]

$$= \left[P(\text{operator i receivs } x \text{ requests})\right]^{5}$$

And as x can vary, we need to sum those. (all the possibilities)

$$= \sum_{x=0}^{\infty}\left[\frac{e^{-2}.2^{x}}{x!}\right]^{5}$$

Ans.

9.

a) $X \Rightarrow$ geometric distribution,

$p \in (0,1)$

Negative binomial when $r = 1$,

pmf $\Rightarrow$

$$\boxed{nb(x; 1, p) = (1-p)^x p}, \quad x = 0, 1, 2, \ldots$$

$x \Rightarrow$ number of failures.

To prove:- $P(X \leq n) = 1 - (1-p)^{n+1}$

Now, we will handle the LHS:-

$$P(X \leq n) = 1 - P(X > n)$$

$$= 1 - \sum_{x=n+1}^{\infty} \underbrace{p(1-p)^x}_{\text{Using negative binomial distribution.}}$$

[$\because P(X > n)$ is the sum over all $P(X)$ from $n+1$ to ∞.]

Opening the sum (sigma) as follows:-

$$= 1 - \left[p(1-p)^{n+1} + p(1-p)^{n+2} + p(1-p)^{n+3} + \ldots \right]$$

Taking $p.(1-p)^{n+1}$ common from the bracket,

$$\Rightarrow 1 - p(1-p)^{n+1} \underbrace{\left[1 + (1-p) + (1-p)^2 + (1-p)^3 + \ldots \right]}$$

Infinite geometric progression (GP) with common ratio (r), $\boxed{(1-p)}$

As Sum of an infinite GP is given as:- $\boxed{a/1-r}$

$$P(X \leq n) = 1 - p(1-p)^{n+1} \cdot \frac{1}{1-(1-p)} = 1 - (1-p)^{n+1}$$

$$\therefore \boxed{P(X \leq n) = 1 - (1-p)^{n+1}}$$

Hence, Proved!

Ans.

(b).

$X_n \rightarrow$ has a geometric distribution with parameter λ/n, $\lambda \in (0, 1)$.

To prove:-

$$P\left(\frac{X_n}{n} \leq x\right) \longrightarrow P(X \leq x)$$

for the distribution function of an exponential random variable X with parameter λ,

for $x > 0$,

$P(X \leq x) \Rightarrow$ Cumulative distribution of exponential random variable X is. $\Rightarrow$

$$\boxed{(1 - e^{-\lambda x})}$$

$$F(x) = P(X \leq x) = (1 - e^{-\lambda x})$$

we have the RHS above!

Let's try and solve the LHS part and see if it converges to the RHS.

for distribution function of $\boxed{X_n/n}$ $\rightarrow$ random variable

where parameter is λ/n,

and for $x > 0$,

$$P\left(\frac{X_n}{n} \leq x\right) = P(X_n \leq nx)$$

$$= 1 - P(X_n > nx) \quad \text{— (1)}$$

$\Rightarrow$ As the pmf of geometric distribution is $\Rightarrow (1-p)^x p$.

for finding the cdf of the geometric distribution, we can do that as follows:-

$$\Rightarrow \sum_{-\infty}^{\infty} (1-p)^x p$$

As $x>0$ and we are trying to find, $P(X_n < x)$

$$\Rightarrow \sum_{x=0}^{x} (1-p)^x p$$

$$\Rightarrow p\left[(1-p)^0 + (1-p)^1 + (1-p)^2 + \cdots + (1-p)^x\right]$$

$$\Rightarrow p\left[\underbrace{1 + (1-p)^1 + (1-p)^2 + \cdots + (1-p)^x}_{\text{geometric progression,}}\right]$$

the sum of a geometric progression when $r<1$, (here $1-p<1$ as $p>0$.)

can be computed as $\Rightarrow \dfrac{a(1-r^n)}{(1-r)}$

$$\Rightarrow p\left[\frac{1(1-(1-p)^x)}{1-(1-p)}\right] = \underline{\underline{1-(1-p)^x}}$$

Then,

$$P(X_n \leq x) = 1-(1-p)^x$$

So,

$$P(X_n > x) = 1 - P(X_n \leq x) = 1-(1-(1-p)^x)$$
$$= \underline{(1-p)^x}.$$

Using the formula here,
and putting it in equation (1),
where parameter = λ/n.

So, $P(X_n > nx) = \left(1 - \frac{\lambda}{n}\right)^{nx}$

[$\because$ nx is a natural number]

So,

$\Rightarrow 1 - \left(1 - \frac{\lambda}{n}\right)^{nx}$

Now, for computing $P\left(\frac{X_n}{n} \leq x\right) \Rightarrow$

$$= \lim_{n\to\infty} \left(1 - \left(1 - \frac{\lambda}{n}\right)^{nx}\right)$$

$$= 1 - \lim_{n\to\infty} \left(1 - \frac{\lambda}{n}\right)^{nx}$$

$$\because \lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^{n} = e^{x}$$

So,

$$= \underline{\underline{1 - e^{-\lambda x}}}.$$

$\therefore$ LHS converges to RHS when $n \to \infty$.

$P\left(\frac{X_n}{n} \leq x\right)$ when $n \to \infty$ becomes $\Rightarrow (1 - e^{-\lambda x})$

and $P(X \leq x) = (1 - e^{-\lambda x})$ when X is an exponential random variable.

Both of these are equal!

Hence, Proved.

Ans.

(c.)

(*) Geometric distribution models in a binomial process:-

As we know, binomial distribution describes the number of successes k achieved in n trials, where probability of success is p.

There is a special case of binomial distribution, negative binomial distribution which describes the number of ~~successes~~ failures k until observing r ~~failures~~ successes, where probability of success is p.

Furthermore, geometric distribution is also a special case of negative binomial distribution, where the experiment is stopped at first ~~failure~~ success $(r=1)$.

Thus, geometric distribution models the case when 1st success is achieved after $(n-1)$ trials resulted in failures and nth trial resulted in a successful one.

(*) Exponential distribution models in a Poisson process:-

As we know, poisson distribution deals with the number of ~~successes~~ occurrences in a fixed period of time, while the exponential distribution deals with the time between occurrences of successive events as the time flows by continuously.

Thus, the exponential distribution models the parameter of the poisson distribution which is based on the time intervals between the occurrences of the successive events. (t/n). This parameter, inturn, helps us to calculate the number of occurrences.

Also, as $n \to \infty$, the poisson distribution fits the exponential function really well and vice-versa is also true.

—— Ans.