HW 7 | Chandan Suri, CS 4090

1.

The data for beam strength is denoted by $X_1, \dots, X_m$ and data is as follows:-

5.9 7.2 7.3 6.3 8.1 6.8 7.0 7.6 6.8

6.5 7.0 6.3 7.9 9.0 8.2 8.7 7.8 9.7 7.4

7.7 9.7 7.8 7.7 11.6 11.3 11.8 10.7

$m = 27$

The data for cylinder strength is denoted by $Y_1, \dots, Y_n$ and data is as follows:-

6.1 5.8 7.8 7.1 7.2 9.2 6.6 8.3 7.0 8.3 7.8

8.1 7.4 8.5 8.9 9.8 9.7 14.1 12.6 11.2

$n = 20$

X_i's constitute a random sample from a distribution with mean μ_1 and standard deviation σ_1 and Y_i's form a random sample (independent) form another distribution with mean μ_2 and standard deviation σ_2.

(a) $\because$ $X_1, X_2, \dots, X_m$ is a random sample from a distribution with mean μ_1, then $\bar{X}$ is an unbiased estimator of μ_1. Similarly,

$\because$ $Y_1, Y_2, \dots, Y_n$ is a random sample from a distribution with mean μ_2, then $\bar{Y}$ is an unbiased estimator of μ_2.

$\therefore$ $\bar{X} - \bar{Y}$ will form a distribution (random sample) with mean $\mu_1 - \mu_2$. So, $\bar{X} - \bar{Y}$ is an unbiased estimator of $\mu_1 - \mu_2$.

To calculate the estimate of the given data, we need to calculate the sample means,

Sample mean for beam strength ($\bar{X}$) =>

$$\bar{X} = \frac{1}{m} \sum_{i=1}^{27} x_i \qquad (m = 27)$$

$$\bar{X} = \frac{1}{27}[5.9 + 7.2 + 7.3 + 6.3 + 8.1 + 6.8 + 7.0 + 7.6 + 6.8 + 6.5 + 7.0 + 6.3 + 7.9 + 9.0 + 8.2 + 8.7 + 7.8 + 9.7 + 7.4 + 7.7 + 9.7 + 7.8 + 7.7 + 11.6 + 11.3 + 11.8 + 10.7]$$

$$= \frac{219.8}{27} = 8.1407 \simeq \underline{8.14}$$ Ans

Sample mean for cylinder strength ($\bar{Y}$) =>

$$\bar{Y} = \frac{1}{n} \sum_{i=1}^{20} y_i \qquad (n = 20)$$

$$\bar{Y} = \frac{1}{20}[6.1 + 5.8 + 7.8 + 7.1 + 7.2 + 9.2 + 6.6 + 8.3 + 7.0 + 8.3 + 7.8 + 8.1 + 7.4 + 8.5 + 8.9 + 9.8 + 9.7 + 14.1 + 12.6 + 11.2]$$

$$= \frac{171.5}{20} = 8.575 \approx \underline{8.58}$$ Ans

$\because$ $\bar{X}$ and $\bar{Y}$ are unbiased estimators of μ_1 and μ_2 respectively. Thus,

$$E[\bar{X}] = \mu_1 \quad \& \quad E[\bar{Y}] = \mu_2$$

Using the rule of expected value and the property of linearity, unbiased estimator of expectation of $\bar{X} - \bar{Y}$ is calculated as:-

$$E[\bar{X}-\bar{Y}] = E[\bar{X}] - E[\bar{Y}] = \mu_1 - \mu_2$$

Thus, estimate will be $\Rightarrow$

$$\bar{X} - \bar{Y} = 8.14 - 8.58$$

$$= -0.44$$

Ans.

(b) To calculate the variance of the estimator in part (a) which was $\bar{X} - \bar{Y} \Rightarrow$

$$V[\bar{X}-\bar{Y}] = V[\bar{X}] + V[\bar{Y}]$$

$$= \sigma_{\bar{X}}^2 + \sigma_{\bar{Y}}^2$$

(Using the property of variance and linearity)

$\Rightarrow$

$\because$ it follows a random sample with normal distribution, we can write the variance of the distribution of sample means as $\Rightarrow$

$$\sigma_{\bar{X}}^2 = \frac{\sigma_1^2}{m} \quad ; \quad \sigma_{\bar{Y}}^2 = \frac{\sigma_2^2}{n}$$

$$\therefore V[\bar{X}-\bar{Y}] = \boxed{\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}}$$

Ans.

$\therefore$ The standard deviation of estimator in part (a) $\Rightarrow$

$$\sigma_{\bar{X}-\bar{Y}} = \sqrt{V[\bar{X}-\bar{Y}]}$$

$$= \sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}$$

Ans.

$\therefore$ The sample variance of X is =>

$$\sigma_1^2 = \frac{1}{m-1} \sum_{i=1}^{m} (x_i - \bar{x})^2$$

$[\because \underline{m = 27}]$

$$\sigma_1^2 = \frac{1}{27-1} \Big[(5.9-8.14)^2 + (7.2-8.14)^2 + (7.3-8.14)^2 + (6.3-8.14)^2 +$$
$$(8.1-8.14)^2 + (6.8-8.14)^2 + (7-8.14)^2 + (7.6-8.14)^2$$
$$+ (6.8-8.14)^2 + (6.5-8.14)^2 + (7-8.14)^2 + (6.3-8.14)^2$$
$$+ (7.9-8.14)^2 + (9-8.14)^2 + (8.2-8.14)^2 + (8.7-8.14)^2$$
$$+ (7.8-8.14)^2 + (9.7-8.14)^2 + (7.4-8.14)^2 + (7.7-8.14)^2$$
$$+ (9.7-8.14)^2 + (7.8-8.14)^2 + (7.7-8.14)^2 + (11.6-8.14)^2$$
$$+ (11.3-8.14)^2 + (11.8-8.14)^2 + (10.7-8.14)^2 \Big]$$

$$= \frac{1}{26} \Big[5.0176 + 0.8836 + 0.7056 + 3.3856 + 0.0016$$
$$+ 1.7956 + 1.2996 + 0.2916 + 1.7956 + 2.6896 +$$
$$1.2996 + 3.3856 + 0.0576 + 0.7396 + 0.0036$$
$$+ 0.3136 + 0.1156 + 2.4336 + 0.5476 + 0.1936$$
$$+ 2.4336 + 0.1156 + 0.1936 + 11.9716$$
$$+ 9.9856 + 13.3956 + 6.5536 \Big]$$

$$= \frac{71.6052}{26} = 2.754 \approx \underline{2.75}$$

Ans

And,

The sample variance of Y is =>

$$\sigma_2^2 = \frac{1}{n-1} \sum_{i=1}^{n} (Y_i - \bar{Y})^2$$

$[\because n = 20]$

$$\sigma_2^2 = \frac{1}{20-1} \Big[(6.1-8.58)^2 + (5.8-8.58)^2 + (7.8-8.58)^2 + (7.1-8.58)^2 + (7.2-8.58)^2 + (9.2-8.58)^2 + (6.6-8.58)^2 + (8.3-8.58)^2 + (7-8.58)^2 + (8.3-8.58)^2 + (7.8-8.58)^2 + (8.1-8.58)^2 + (7.4-8.58)^2 + (8.5-8.58)^2 + (8.9-8.58)^2 + (9.8-8.58)^2 + (9.7-8.58)^2 + (14.1-8.58)^2 + (12.6-8.58)^2 + (11.2-8.58)^2 \Big]$$

$$= \frac{1}{19} \Big[6.1504 + 7.7284 + 0.6084 + 2.1904 + 1.9044 + 0.3844 + 3.9204 + 0.0784 + 2.4964 + 0.0784 + 0.6084 + 0.2304 + 1.3924 + 0.0064 + 0.1024 + 1.4884 + 1.2544 + 30.4704 + 16.1604 + 6.8644 \Big]$$

$$= \frac{\overset{84.118}{\cancel{68.6612}}}{19} = 4.427 \approx \underline{\underline{4.43}}$$

Ans.

$\therefore$ Standard deviation is $\Rightarrow$

$$= \sqrt{\frac{2.75}{27} + \frac{4.43}{20}}$$

$= 0.5686$

$\approx \underline{\underline{0.57}}$

Ans

(c.)

$\therefore$ Point estimate of ratio $\frac{\sigma_1}{\sigma_2} \Rightarrow$

$$\Rightarrow \frac{\sqrt{2.75}}{\sqrt{4.43}} = \frac{1.6583}{2.1047}$$

$$= 0.7879 \approx \underline{\underline{0.79}}$$

$\underline{\underline{\text{Ans.}}}$

(d.)

Point estimate of the variance of $X-Y$ is $\Rightarrow$

$V[X-Y] = V[X] + V[Y]$ (Using the property of variance & linearity)

$$= \sigma_1^2 + \sigma_2^2$$

$$= 2.75 + 4.43$$

$$= \underline{\underline{7.18}}$$

$\underline{\underline{\text{Ans.}}}$

(2.) Let $X_1, X_2, \ldots X_n$ represent a random sample from a Rayleigh distribution with pdf:-

$$f(x;\theta) = \frac{x}{\theta} e^{-x^2/2\theta} \quad ; x > 0$$

(a.) Given that, $E[X^2] = 2\theta$

We can calculate the value of the second moment as $\Rightarrow$ (unbiased estimator $\hat{\theta}$)

$$\frac{1}{n} \Sigma X_i^2 = 2\theta$$

$$\boxed{\hat{\theta} = \frac{\Sigma X_i^2}{2n}}$$

Ans

$\therefore$ The unbiased estimator of θ is $\frac{\Sigma X_i^2}{2n}$.

Furthermore, to show that this estimator is unbiased, we have to prove that :-

$$E[\hat{\theta}] = \theta \text{ for all } n,$$

$\therefore$

$$E[\hat{\theta}] = E\left[\frac{\Sigma X_i^2}{2n}\right] = \frac{E[\Sigma X_i^2]}{2n}$$

$$= \frac{\Sigma E[X_i^2]}{2n}$$ (Using the property of linearity)

$$= \frac{\Sigma 2\theta}{2n} = \frac{2n\theta}{2n} = \theta$$ $[\because \Sigma k = nk]$

$\because E[\hat{\theta}] = \theta$, $\therefore$ $\hat{\theta}$ is an unbiased estimator of θ. A

(b) $n = 10$ (number of observations)

Observations on vibratory stress of a turbine blade

$\Rightarrow$

16.88 10.23 4.59 6.66 13.68 14.23

19.87 9.40 6.51 10.95

$$\hat{\theta} = \frac{\Sigma x_i^2}{2n}$$

$$= \frac{\left[(16.88)^2 + (10.23)^2 + (4.59)^2 + (6.66)^2 + (13.68)^2 + (14.23)^2 + (19.87)^2 + (9.40)^2 + (6.51)^2 + (10.95)^2\right]}{2 \times 10}$$

$$= [284.934 + 104.653 + 21.068 + 44.356 + 187.142 + 202.493 + 394.817 + 88.36 + 42.38 + 119.903] / 20$$

$$= \frac{1490.106}{20} = \underline{\underline{74.51}}$$

Ans.

(3.) Let X denote the proportion of alloted time that a randomly selected student spends working on a certain aptitude test.

pdf is given by:-

$$f(x;\theta) = \begin{cases} (\theta+1)\,x^{\theta} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\underline{\underline{\theta > -1}}$$

(a) As the distribution here had only one unknown parameter,

we will calculate the first moment here only.

__Step 1:-__ Calculating the moment :-

$$E[X_1] = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$= \int_{-\infty}^{0} \cancel{x f(x)}^{\,0}\,dx + \int_{0}^{1} x f(x)\,dx + \int_{1}^{\infty} \cancel{x f(x)}^{\,0}\,dx$$

$$= \int_{0}^{1} x f(x)\,dx = \int_{0}^{1} x(\theta+1)x^{\theta}\,dx$$

$$= (\theta+1)\int_{0}^{1} x^{\theta+1}\,dx = (\theta+1)\left[\frac{x^{\theta+2}}{\theta+2}\right]\Bigg|_{0}^{1}$$

$$= \frac{(\theta+1)}{(\theta+2)}\left(1^{\theta+2} - 0^{\theta+2}\right)$$

$$= \frac{\theta+1}{\theta+2}$$

Ans

Step2:- Now, we will equate the moment with sample moment $\Rightarrow$

$$E[X_1] = \frac{\theta+1}{\theta+2} \overset{!}{=} \hat{\mu}_1 = \bar{X}_n$$

Step 3:- Now, we will solve the equation to determine the value of the estimator:-

$$\frac{\theta+1}{\theta+2} = \bar{X}_n$$

$$\theta+1 = \bar{X}_n(\theta+2)$$

$$\theta+1 = \theta\bar{X}_n + 2\bar{X}_n \Rightarrow \theta(1-\bar{X}_n) = 2\bar{X}_n - 1$$

$$\hat{\theta} = \frac{2\bar{X}_n - 1}{1-\bar{X}_n}$$

$\therefore$ $\hat{\theta}$ estimator equals $\frac{2\bar{X}_n - 1}{1-\bar{X}_n}$ Ans

Now, we are given the observations $\Rightarrow$

$x_1 = 0.92$, $x_2 = 0.79$, $x_3 = 0.90$, $x_4 = 0.65$, $x_5 = 0.86$, $x_6 = 0.47$, $x_7 = 0.73$, $x_8 = 0.97$, $x_9 = 0.94$, $x_{10} = 0.77$,

Let's first calculate mean $(\bar{X}_n) \Rightarrow$

$$\bar{X}_n = \sum x_i / n = \frac{(0.92 + 0.79 + 0.90 + 0.65 + 0.86 + 0.47 + 0.73 + 0.97 + 0.94 + 0.77)}{10}$$

$$= 8/10 = 0.8$$

Lets compute the estimator using the estimator $\hat{\theta} \Rightarrow$

$$\hat{\theta} = \frac{2\bar{X}_n - 1}{1 - \bar{X}_n} = \frac{2 \times (0.8) - 1}{1 - 0.8} = \frac{1.6 - 1}{0.2} = \frac{0.6}{0.2} = 3$$

$$\boxed{\therefore \hat{\theta} = 3}$$

$\therefore$ estimate equals $\underline{\underline{3}}$.

<u>Ans</u>.

4. a)

$\theta_{MLE} = \hat{\theta} = Y = \max(X_i)$

Given:- $Y \le y$ iff $x_1 \le y$,

~~for marginal distribution~~,

cdf

$F_Y(y) = P(Y \le y) = P(x_1 \le y, x_2 \le y, \dots, x_n \le y)$

As these are independent,

$$= \underline{P(x_1 \le y) \cdot P(x_2 \le y) \cdots P(x_n \le y)} \quad (1)$$

Since X follows a ~~normal~~ uniform distribution,

$\therefore P(x \le y)$ over $[0, y] \Rightarrow$

$$P(x \le y) = \frac{0 + y}{\theta} = y/\theta$$

Plugging this in equation (1),

$$\Rightarrow P(x_1 \le y) \cdot P(x_2 \le y) \cdots P(x_n \le y)$$

$$= \left(\frac{y}{\theta}\right) \cdot \left(\frac{y}{\theta}\right) \cdots \left(\frac{y}{\theta}\right) \quad \text{(n times)}$$

$$= \left(\frac{y}{\theta}\right)^n \quad ; \quad \underline{0 \le y \le \theta}$$

for pdf, we will differentiate cdf w.r.t y,

$$\Rightarrow \frac{d\left(y^n/\theta^n\right)}{dy} = \boxed{\frac{n\, y^{n-1}}{\theta^n}}$$

$\therefore$ pdf of $Y = \max(X_i)$,

$$\boxed{f_Y(y) = \begin{cases} \dfrac{n y^{n-1}}{\theta^n} & 0 \le y \le \theta \\ 0 & \text{otherwise} \end{cases}}$$

Hence, proved

Ans.

4. Let $X_1, \dots, X_n$ be a random sample from a uniform distribution on $[0, \theta]$.

$$\hat{\theta} = Y = \max(X_i)$$

$Y \le y$ iff each $X_i \le y$.

pdf of $Y = \max(X_i)$:-

$$f_Y(y) = \begin{cases} \dfrac{n y^{n-1}}{\theta^n} & 0 \le y \le \theta \\ 0 & \text{otherwise} \end{cases}$$

(b)

first we will find out the expected value of this pdf:-

$$E[f_y(y)] = \int_{-\infty}^{\infty} y f_Y(y) \cdot dy$$

$$= \int_{-\infty}^{0} y \cancel{f_Y(y)}^{\to 0} \cdot dy + \int_{0}^{\theta} y f_Y(y) \cdot dy + \int_{\theta}^{\infty} y \cancel{f_Y(y)}^{\to 0} \cdot dy$$

$$= \int_0^\theta y \cdot \left(\frac{n y^{n-1}}{\theta^n}\right) \cdot dy = \int_0^\theta \frac{n y^n}{\theta^n} \cdot dy$$

$$= \frac{n}{\theta^n} \int_0^\theta y^n \cdot dy = \frac{n}{\theta^n} \left[\frac{y^{n+1}}{n+1}\right]_0^\theta$$

$$= \frac{n}{\theta^n} \left[\frac{\theta^{n+1}}{n+1} - 0\right] = \frac{n}{\theta^n} \cdot \frac{\theta^{n+1}}{n+1}$$

$$= \boxed{\frac{n \cdot \theta}{n+1}}$$

Ans

Now, we need to prove that $\frac{(n+1) \max(X_i)}{n}$ is unbiased.

For this, we need to prove that $E[T] = \theta$ where T is the estimator.

Taking $T = \frac{(n+1)}{n} \max(X_i)$,

$$= E\left[\frac{(n+1)}{n} \cdot \max(X_i)\right]$$

$\because$ it's given that $Y = \max(X_i)$,

$$= E\left[\frac{(n+1)}{n} \cdot Y\right] = \frac{n+1}{n} E[Y]$$

($\because$ n does not depend on Y, we can take it outside and expectation of a constant is that constant).

$\because E[Y] = \frac{n\theta}{n+1}$

$$= \frac{n+1}{n} \cdot \frac{n\theta}{n+1}$$

$$= \theta$$

$\therefore$ $E[T] = \theta$

This implies that estimator is unbiased for all n, where the estimator was $\frac{(n+1) \max(X_i)}{n}$.

A

(5.) $X_1, \ldots, X_n \rightarrow$ iid

$U([a, b])$ – distributed with unknown params

as $\Rightarrow \theta_1 = a$, $\theta_2 = b$ (Uniform distribution).

(a)

Step 1

$$E[X] = \hat{\mu}_1 = \frac{a+b}{2}$$

(for uniform distribution).

$$E[X^2] = \hat{\mu}_2 = \int_{-\infty}^{\infty} x^2 \cdot f(x) \, dx$$

(for uniform distribution, using transformation rule).

$$= \int_{-\infty}^{a} x^2 \cancel{f(x)}^{\,0} dx + \int_{a}^{b} x^2 f(x) \, dx + \int_{b}^{\infty} x^2 \cancel{f(x)}^{\,0} dx$$

$$= \int_{a}^{b} x^2 \cdot \frac{1}{b-a} \, dx$$

[$\because$ pdf for uniform distribution $\Rightarrow$ $\frac{1}{b-a}$ $a \leq x \leq b$; 0 otherwise]

$$= \int_{a}^{b} \frac{x^2}{b-a} \cdot dx$$

$$= \frac{1}{b-a} \left[x^3/3 \right] \Big|_a^b = \frac{b^3 - a^3}{3(b-a)}$$

[$\because b^3 - a^3 = (b-a)(a^2 + b^2 + ab)$]

$$= \frac{(b-a)(a^2 + b^2 + ab)}{3(b-a)}$$

$$= \frac{a^2 + b^2 + ab}{3}$$

$\therefore E[x^2] = \dfrac{a^2+b^2+ab}{3}$

Step 2 $\Rightarrow$ Equating the terms $\Rightarrow$

$\because \theta_1 = a$ and $\theta_2 = b$,

$E[x] = \dfrac{a+b}{2} = \dfrac{\theta_1+\theta_2}{2} = \hat{\mu}_1 \stackrel{!}{=} \bar{X}_n$ (mean)

and,

$E[x^2] = \dfrac{a^2+b^2+ab}{3} = \dfrac{\theta_1^2+\theta_2^2+\theta_1\theta_2}{3} = \hat{\mu}_2$

$\stackrel{!}{=} V[x] + (\bar{x}_n)^2$ $\quad$ ($\because V[x] = E[x^2] - (E[x])^2$ and $E[x] = \bar{x}_n$)

$\therefore$

$E[x] = \dfrac{\theta_1+\theta_2}{2} = \bar{x}_n$

$\Rightarrow \boxed{\dfrac{\theta_1+\theta_2}{2} = \bar{x}_n}$ —— ①

$E[x^2] = \dfrac{\theta_1^2+\theta_2^2+\theta_1\theta_2}{3} = V[x] + (\bar{x}_n)^2$

$\Rightarrow \boxed{\dfrac{\theta_1^2+\theta_2^2+\theta_1\theta_2}{3} = V[x] + (\bar{x}_n)^2}$ —— ②

Step 3 $\Rightarrow$ Solving for equations ① and ② above to find the estimators for θ_1 and θ_2 $\Rightarrow$

from equation 1,

$$\theta_1 + \theta_2 = 2\bar{x}_n \Rightarrow \underline{\theta_1 = 2\bar{x}_n - \theta_2} \quad \text{———} \; 3$$

Putting the value of θ_1 from equation 3 in equation 2,

$$\frac{(2\bar{x}_n - \theta_2)^2 + \theta_2^2 + (2\bar{x}_n - \theta_2)\theta_2}{3} = V[x] + (\bar{x}_n)^2$$

$$4\bar{x}_n^2 + \cancel{\theta_2^2} - 4\bar{x}_n\theta_2 + \theta_2^2 + 2\bar{x}_n\theta_2 - \cancel{\theta_2^2} = 3V[x] + 3(\bar{x}_n)^2$$

$$4\bar{x}_n^2 - 3\bar{x}_n^2 + \theta_2^2 - 2\bar{x}_n\theta_2 = 3V[x]$$

$$\bar{x}_n^2 + \theta_2^2 - 2\bar{x}_n\theta_2 = 3V[x]$$

$(\because a^2 + b^2 - 2ab = (a-b)^2) \rightarrow$ generic terms a and b.

So,

$$(\theta_2 - \bar{x}_n)^2 = 3V[x]$$

$$\theta_2 - \bar{x}_n = \sqrt{3V[x]}$$

$$\theta_2 = \bar{x}_n + \sqrt{3}\sqrt{V[x]} \qquad [\because \sigma = \sqrt{V[x]}]$$

$\sigma \downarrow$ Standard deviation

$$\Rightarrow \boxed{\hat{\theta}_2 = \bar{x}_n + \sqrt{3}\,\sigma}$$

$$\boxed{T^{(2)} = \bar{x}_n + \sqrt{3}\,\sigma}$$

where $\sigma \rightarrow$ standard deviation

Putting value of θ_2 in equation 3,

$$\theta_1 = 2\bar{x}_n - \bar{x}_n - \sqrt{3}\,\sigma = \boxed{\bar{x}_n - \sqrt{3}\,\sigma}$$

$$\boxed{\therefore T^{(1)} = \bar{X}_n - \sqrt{3}\,\sigma}$$

$\therefore$ The estimators of θ_1 $(T^{(1)})$ and θ_2 $(T^{(2)})$ are as follows $\Rightarrow$

$$\boxed{T^{(1)} = \bar{X}_n - \sqrt{3}\,\sigma}$$

$$\boxed{T^{(2)} = \bar{X}_n + \sqrt{3}\,\sigma}$$

where $\bar{X}_n$ is mean and σ is the standard deviation.

Ans.

(b.) The sufficient condition for consistency of an estimator (T_n) for a parameter θ is $\Rightarrow$

$$\lim_{n\to\infty} E[T_n] = \theta$$

Here, for estimator $T^{(1)}$ for param $\theta_1 \Rightarrow$

$$\Rightarrow \lim_{n\to\infty} E[T^{(1)}] = E[\bar{X}_n - \sqrt{3}\sigma]$$

$$= E[\bar{X}_n] - \sqrt{3}\,E[\sigma]$$ (Using property of linearity)

$$\Rightarrow \lim_{n\to\infty} \left(E[\bar{X}_n] - \sqrt{3}\,E[\sigma] \right)$$

$$= \frac{\theta_1+\theta_2}{2} - \sqrt{3}\,E[\sigma]$$ (both parts does not depend on n)

$$\neq \theta_1$$

(As this cannot be equal to θ_1), $\therefore$ $T^{(1)}$ is inconsistent for θ_1.

Similarly,

for estimator $T^{(2)}$ for $\theta_2 \Rightarrow$

$$\lim_{n\to\infty} E[T^{(2)}] = \lim_{n\to\infty} E[\bar{X}_n + \sqrt{3}\,\sigma]$$

(Using the property of linearity)

$$= \lim_{n\to\infty} \left(E[\bar{X}_n] + \sqrt{3}\, E[\sigma] \right)$$

$$= \frac{\theta_1 + \theta_2}{2} + \sqrt{3}\, E[\sigma] \neq \theta_2$$

$\because$ it's not equal to θ_2,

thus, $T^{(2)}$ is not consistent for θ_2.

$\therefore$ $T^{(1)}$ and $T^{(2)}$ are inconsistent for θ_1 and θ_2 respectively.

Hence, Proved

Ans.

6

for an estimator T, (θ is the mean)

$$\underset{(T)}{MSE} = E[(T-\theta)^2] \quad \text{--- (1)}$$

$$Bias(T) = E[T] - \theta \quad \text{--- (2)}$$

From the proposition proof we know that the estimate of variance is the σ^2 (variance), thus,

$$V[T] = E[T^2] - (E[T])^2 \quad \text{--- (3)}$$

Let's try and solve the 1st equation to prove that :-

$$MSE(T) = V[T] + (Bias(T))^2$$

$\Rightarrow$

$$MSE(T) = E[(T-\theta)^2]$$

$$= E[T^2 + \theta^2 - 2T\theta]$$

$\left(\because (a-b)^2 = a^2+b^2 - 2ab\right)$

By the property of linearity,

$$= E[T^2] + E[\theta^2] - E[2T\theta]$$

$\because$ expectation of a constant is a constant and θ does not depend on T and is a constant, we can take some terms out, also, $E[\theta^2] = \theta^2$,

$$= E[T^2] + \theta^2 - 2\theta E[T]$$

Adding and subtracting $(E[T])^2$,

$$= E[T^2] + \theta^2 - 2\theta E[T] + (E[T])^2 - (E[T])^2$$

$$= E[T^2] - (E[T])^2 + (E[T])^2 + \theta^2 - 2\theta E[T]$$

From (3) equation, $V[T] = E[T^2] - (E[T])^2$,

$$= V[T] + (E[T] - \theta)^2$$

$\left(\because (a-b)^2 = a^2+b^2 - 2ab\right)$

From ② equation, we can rewrite the 2nd term as bias,

$$= V[T] + (Bias(T))^2$$

Thus,

$$\boxed{MSE(T) = V[T] + (Bias(T))^2}$$

As we could successfully derive this expression in RHS from LHS,

Hence, Proved.

Ans.

7. Let $X_1, \ldots, X_n$ be iid Bernoulli (p).

To prove:- There is no unbiased estimator of $\Theta = \log(p)$.

Let's prove this by assuming that if T were an unbiased estimator possible, we would have $E[T] = \Theta$ for all n.

$$E[T] = E[h(X_1, X_2, \ldots, X_n)]$$ ($\because$ estimator T is basically a func^n h over all r.v's).

$\because$ $E[x] = \sum x\, p(x)$

Using the transformation rule,

$$\Rightarrow E[T] = \underbrace{\sum_{\substack{x_1, x_2, \ldots x_n \\ \in \{0,1\}}}}_{2^n \text{ terms}} h(x_1, x_2, \ldots, x_n)\, \underbrace{p(X = x_1, \ldots, X_n = x_n)}_{\text{Joint pmf.}}$$

We will try and find the lower bound of this equation:-

We know that the 1st term consists of 2^n terms, furthermore, probabilities always lie between 0 and 1.

$\therefore$ The lower bound of this will be zero.

$$E[T] \geq \sum_{\substack{x_1, x_2 \ldots x_n \\ \in \{0,1\}}} \min |h(x_1, x_2, \ldots x_n)| . 0$$

$\therefore$ $E[T] \geq 0$

Now,

RHS:- $\theta = \log(p)$

We know that when $p = 0$,

$$\theta = \log 0 = -\infty \quad \text{(unbounded)}$$

We can see that $E[T] \geq 0$, however θ goes till $-\infty$.

$\therefore$ LHS $\neq$ RHS.

$\therefore$ T is an unbiased estimator.

Ans.