HW-5 | CS4090 – Chandan Suri

1. The probability density function is given as:-

$$p(x) = \begin{cases} 0.05 & x=1 \\ 0.10 & x=2 \\ 0.35 & x=4 \\ 0.40 & x=8 \\ 0.10 & x=16 \end{cases}$$

(a) As this is a discrete function, we can compute the expectation as:-

$$E(x) = \sum_{x \in E} x\, p(x)$$

, where E is the domain of x, encompassing all the possible values of x as mentioned in the pmf.

$$E(x) = (1 \times 0.05) + (2 \times 0.1) + (4 \times 0.35) + (8 \times 0.4) + (16 \times 0.10)$$

$$= 0.05 + 0.2 + 1.4 + 3.2 + 1.6 = \underline{\underline{6.45}}.$$

$$\therefore \underline{\underline{E(x) = 6.45 \text{ GB}}}$$ Ans.

(b) The variance when computed using the definition can be done as follows:-

$$V[x] = E\left[(x - E[x])^2\right]$$

$$\because E[x] = 6.45$$

$$\therefore V[x] = E\left[(x - 6.45)^2\right]$$

$$= \sum_{x \in E} (x - 6.45)^2\, p(x)$$

, where E is the domain of x, encompassing all the possible values of x as mentioned in the pmf.

$$\Rightarrow (1-6.45)^2 p(1) + (2-6.45)^2 p(2) + (4-6.45)^2 p(4)$$
$$+ (8-6.45)^2 p(8) + (16-6.45)^2 p(16)$$

$$\Rightarrow (1-6.45)^2 (0.05) + (2-6.45)^2 (0.1) + (4-6.45)^2 (0.35)$$
$$+ (8-6.45)^2 (0.4) + (16-6.45)^2 (0.1)$$

$$\Rightarrow 1.4851 + 1.9803 + 2.1009 + 0.961 + 9.1203$$

$$\Rightarrow 15.6476$$

$$\therefore V[x] = 15.6476 \approx \underline{\underline{15.65}} \ GB^2$$

Ans.

(c) The standard deviation of X :-

$$SD(x) = \sigma_X = \sqrt{V(x)}$$

$$\therefore, \ \sigma_X = \sqrt{15.6476} = 3.9557 \approx \underline{\underline{3.96}} \ GB$$

Ans.

(d) $V[x]$ calculation using the shortcut formula,

According to the shortcut formula for theoretical variance :-

$$V[X] = E[X^2] - (E[X])^2$$

we know that,

$$E[X] = 6.45$$

So, we need to compute $E[X^2]$,

By the rule of transformation,

$$E[h(x)] = \sum_{x \in E} h(x)\, p(x)$$

$$\therefore E[X^2] = \sum_{x \in E} x^2 p(x) = (1)^2 \times 0.05 + 2^2 \times 0.1 + 4^2 \times 0.35$$
$$+ 8^2 \times 0.4 + 16^2 \times 0.1$$

$$= 0.05 + 0.4 + 5.6 + 25.6 + 25.6 = 57.25$$

Using the values above for theoretical variance,

$$V[x] = 57.25 - (6.45)^2$$
$$= 57.25 - 41.6025$$
$$= 15.6475 \approx \underline{\underline{15.65\ GB^2}}$$

Ans.

(2.) X = rated capacity of a freezer of this brand sold at a certain store.

The pmf is specified as:-

$$p(x) = \begin{cases} 0.2 & x = 16 \\ 0.5 & x = 18 \\ 0.3 & x = 20 \end{cases}$$

(a) As this is a discrete function above, the expected value of X can be computed as:-

$$E[X] = \sum_{x \in E} x\,p(x),$$ where E is the domain of X, i.e., all the possible values of x as mentioned in the pmf.

$$= 16(0.2) + 18(0.5) + 20(0.3)$$
$$= 3.2 + 9 + 6$$
$$= \underline{\underline{18.2\ ft^3}}$$

Ans.

Using the transformation rule $E[h(x)] = \sum_{x \in E} h(x)\,p(x)$,

$$E[X^2] = \sum_{x \in E} x^2\,p(x)$$
$$= 16^2(0.2) + 18^2(0.5) + 20^2(0.3)$$
$$= 51.2 + 162 + 120 = \underline{333.2}\ ft^6$$

Ans.

And theoretical variance can be computed using the shortcut formula below:-

$$V[X] = E[X^2] - (E[X])^2$$

$$= 333.2 - (18.2)^2$$

$$= 333.2 - 331.24 = \underline{\underline{1.96}}\ ft^6$$

Therefore,

$$E[X] = 18.2\ ft^3$$

$$E[X^2] = 333.2\ ft^6$$

$$V[X] = 1.96\ ft^6$$

<u>Ans</u>.

(b) Given, the prize of the freezer having capacity $X \Rightarrow$ $70X - 650$

As we need to find the expected prize paid by the next customer to buy a freezer,

$$h(x) = 70X - 650$$

in,

$E[h(x)] = ?$ (this is what we need to find)

As the expectation is linear,

thus,

$$E[70X - 650] = 70\,E[X] - 650$$

[Using $E[aX + b] = a\,E[X] + b$]

$\Rightarrow 70 \times 18.2 - 650$

$\Rightarrow 1274 - 650$

$\Rightarrow \underline{\underline{624\ ft^3}}$ <u>Ans</u>.

(c.) For calculating the variance of the price paid by the next customer ⇒

Using, $V[aX+b] = a^2 V[X]$

$$\Rightarrow \quad V[70X - 650] = (70)^2 V[X]$$

$$= (70)^2 \times 1.96$$

$$= 4900 \times 1.96$$

$$= 9604 \text{ ft}^6$$

Ans.

(d.) $h(X) = X - 0.008X^2$

Actual capacity of the freezer is given by the above formula.

For calculating the expected actual capacity of the freezer purchased by the next customer ⇒

Using the transformation rule and the fact that expectation is linear ⇒

$$E[h(X)] = E[X - 0.008X^2]$$

$$= E[X] - E[0.008X^2]$$

$$= E[X] - 0.008\, E[X^2]$$

Using the values computed before,

$$= \cancel{333.2}\ 18.2 - 0.008 \times 333.2$$

$$= 18.2 - 2.6656$$

$$= 15.5344 \approx 15.53$$

Ans.

(3.) X = the headway between two randomly selected consecutive cars (sec).

The distribution (pdf) is given by:- (for time headway)

$$f(x) = \begin{cases} K/x^4 & x > 1 \\ 0 & x \leq 1 \end{cases}$$

(a) For a distribution $f(x)$ to be a legitimate pdf :-

$$\int_{-\infty}^{\infty} f(x).dx = 1$$

Distributing the integral for different parts of the domain of x as follows:-

$$\int_{-\infty}^{1} f(x).dx + \int_{1}^{\infty} f(x).dx = 1$$

$$\cancel{\int_{-\infty}^{1} 0.dx}^{\,0} + \int_{1}^{\infty} \frac{K}{x^4}.dx = 1$$

$$0 + \int_{1}^{\infty} K/x^4 .dx = 1$$

$$K \int_{1}^{\infty} \frac{1}{x^4}.dx = 1 \Rightarrow K \int_{1}^{\infty} x^{-4}.dx = 1$$

$$K \left[\frac{x^{-3}}{-3}\right]_{1}^{\infty} = 1 \Rightarrow K \left[\frac{-1}{3x^3}\right]_{1}^{\infty} = 1$$

$$\Rightarrow K \left[\frac{-1}{3(\infty)^3} + \frac{1}{3(1)^3}\right] = 1 \Rightarrow K \left[0 + \frac{1}{3}\right] = 1$$

$$\frac{K}{3} = 1 \Rightarrow \boxed{K = 3}$$

Ans.

(b) The cumulative distribution function is the summation of all the probabilities of values in the domain of X s.t.,

$$F(X) = P(X \leq x)$$

which can be written as,

$$F(X) = \int_{-\infty}^{x} f(x).dx = \int_{-\infty}^{1} f(x).dx + \int_{1}^{x} f(x).dx$$

Using the definition of $f(x)$ for values of x (domain),

$$F(X) = \int_{-\infty}^{1} 0.dx + \int_{1}^{x} \frac{K}{x^4}.dx \qquad \because K = 3$$

$$= 0 + \int_{1}^{x} \frac{3}{x^4}.dx = \int_{1}^{x} 3x^{-4}.dx$$

$$= 3\int_{1}^{x} x^{-4}.dx = 3\left[\frac{x^{-3}}{(-3)}\right]\Bigg|_{1}^{x}$$

$$= 3\left[\frac{-1}{3x^3}\right]\Bigg|_{1}^{x} = \cancel{3}\left[\frac{-1}{\cancel{3}x^3} + \frac{1}{\cancel{3}(1)^3}\right]$$

$$= -\frac{1}{x^3} + 1 = 1 - 1/x^3 = 1 - x^{-3}$$

when $x > 1$.

for all values $x \leq 1$, $F(x) = 0$.

$\therefore$ The cdf can be given as:-

$$F(X) = \begin{cases} 0 & x \leq 1 \\ 1 - x^{-3} & x > 1 \end{cases}$$

Ans.

(c) The probability that headway exceeds 2 sec,

$$P(X>2) = ?$$

We can rewrite this as:- (to use the formulation of cdf)

$$\Rightarrow P(X>2) = 1 - P(X \leq 2)$$

$$\because F(x) = P(X \leq x)$$

$$\therefore P(X>2) = 1 - F(2)$$

$$= 1 - \left(1 - \frac{1}{2^3}\right)$$

$$= \cancel{1} - \cancel{1} + \frac{1}{2^3} = \frac{1}{8} = \underline{\underline{0.125}}$$

Ans.

The probability that headway is between 2 and 3 secs can be written as $\Rightarrow$

$$= P(2 < X < 3)$$

Using the cdf formulation,

$$P(2<X<3) = F(3) - F(2)$$

$$= \left(1 - \frac{1}{3^3}\right) - \left(1 - \frac{1}{2^3}\right)$$

$$= \cancel{1} - \frac{1}{27} - \cancel{1} + \frac{1}{8}$$

$$= \frac{1}{8} - \frac{1}{27} = \frac{27-8}{8 \times 27}$$

$$= \frac{19}{8 \times 27} = 0.08796 = 8.796\%$$

$$\approx \underline{\underline{8.8\%}}$$ Ans.

(d.)

Mean value of headway = Expected value of X

Thus, (∵ this is a continuous variable)

$$E[X] = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$= \int_{-\infty}^{1} x f(x).dx + \int_{1}^{\infty} x f(x).dx$$

For $x < 1$, $f(x) = 0$, thus, the 1st term above converges to zero,

$$= \int_{-\infty}^{1} x \times 0 .dx + \int_{1}^{\infty} x\left(\frac{3}{x^4}\right).dx$$

$$= 0 + \int_{1}^{\infty} \frac{3}{x^3}.dx = \int_{1}^{\infty} \frac{3}{x^3}.dx$$

$$= \int_{1}^{\infty} 3x^{-3}.dx = 3\left[\frac{x^{-2}}{-2}\right]\Bigg|_{1}^{\infty}$$

$$= 3\left[\frac{-1}{2(\infty)^2} + \frac{1}{2(1)^2}\right] = 3\left[0 + \frac{1}{2}\right]$$

$$= 3/2 = \underline{\underline{1.5 \text{ sec}}}$$ Ans.

For calculating the standard deviation of headway, we will need to calculate the variance first, for which we need to calculate $E[X^2] \Rightarrow$

$$\left[\because E[h(x)] = \int_{-\infty}^{\infty} h(x) f(x).dx\right]$$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x).dx$$

$$\Rightarrow \int_{-\infty}^{\infty} x^2 f(x).dx$$

$\because f(x) = 0$ for $x \leq 1$ and $f(x) = \frac{3}{x^4}$ for $x > 1$,

$$\Rightarrow \int_{-\infty}^{1} x^2 . 0 . dx + \int_{1}^{\infty} x^2 \left(\frac{3}{x^4}\right).dx$$

$$\Rightarrow 0 + \int_{1}^{\infty} \frac{3}{x^2}.dx = \int_{1}^{\infty} 3x^{-2}.dx = 3\int_{1}^{\infty} x^{-2}.dx$$

$$\Rightarrow 3\left[\frac{x^{-1}}{-1}\right]\Bigg|_{1}^{\infty} = 3\left[\frac{-1}{x}\right]\Bigg|_{1}^{\infty}$$

$$\Rightarrow 3\left[-\frac{1}{\infty} + \frac{1}{1}\right] = \underline{\underline{3}} \text{ sec}^2$$

Ans.

Using the formule for variance,

$$V[X] = E[X^2] - (E[X])^2$$

$$= 3 - (1.5)^2 = 3 - 2.25$$

$$= \underline{\underline{0.75 \text{ sec}^2}}$$

Ans

Now, standard deviation is $\Rightarrow$

$$\sigma_x = \sqrt{V[X]} = \sqrt{0.75}$$

$$= 0.866 \approx \underline{\underline{0.87 \text{ sec}}}$$

Ans.

(e) Probability that the headway is within 1 standard deviation of the mean value (which is the expected value of x) =>

Thus, X should be within $E[x] \pm \sigma_x$,

$$\left(E[x] - \sigma_x < X < E[x] + \sigma_x \right)$$

$$E[x] - \sigma_x = 1.5 - 0.87 = 0.63$$

and,

$$E[x] + \sigma_x = 1.5 + 0.87 = 2.37$$

$\therefore$ probability can be written as =>

$$= P\left(E[x] - \sigma_x < X < E[x] + \sigma_x \right)$$

$$= P\left(0.63 < X < 2.37 \right)$$

We can compute this using the formulation for cdf,

$$\Rightarrow F(2.37) - F(0.63)$$

$\because F(x) = 0$ when $x \leq 1$,

$$\Rightarrow F(2.37) - 0 = F(2.37)$$

$$= 1 - \frac{1}{(2.37)^3} = 0.92488 \approx \underline{0.9249}$$

$$\therefore P(0.63 < X < 2.37) = \underline{\underline{0.9249}}$$

Ans.

(4.) (a) A continuous rv X is said to have a uniform distribution on the interval $[A, B]$ if the pdf of X is:-

$$f(x) = \begin{cases} \frac{1}{B-A} & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

for calculating the (100p)th percentile, we will ~~first~~ need to calculate cdf and then through quantile function we can do so,

$\therefore$ cdf can be computed as $\Rightarrow$

$$F(x) = \int_{-\infty}^{x} f(x)\, dx = \int_{-\infty}^{A} 0 . dx + \int_{A}^{x} \frac{1}{B-A} . dx$$

$$= 0 + \frac{1}{B-A} \int_{A}^{x} 1 . dx = \frac{1}{B-A} [x]_{A}^{x}$$

$$= \frac{x-A}{B-A}$$

$\therefore$ $F(x)$ is $\Rightarrow$

$$F(x) = \begin{cases} \frac{x-A}{B-A} & A \le x \le B \\ 0 & \text{otherwise} \end{cases}$$

To find out the (100p)th percentile, we will have to set $\Rightarrow$ $F(x) = p \Rightarrow (x-A)/(B-A) = p$

Solving for x,

$$x - A = p(B-A)$$

$$\underline{x = p(B-A) + A}$$

Thus, the $(100p)$th percentile is $\underline{p(B-A)+A}$.

Ans.

(b.) To calculate the expected value of X,

$$E[X] = \int_{-\infty}^{\infty} x f(x).dx$$

$$= \int_{-\infty}^{A} x.0.dx + \int_{A}^{B} \frac{x}{B-A}.dx + \int_{B}^{\infty} x.0.dx$$

[1st and 3rd term becomes zero]

$$= \int_{A}^{B} \frac{x}{B-A}.dx$$

$$= \frac{1}{B-A}\left[\frac{x^2}{2}\right]\Bigg|_{A}^{B} = \frac{B^2-A^2}{2(B-A)} = \frac{\cancel{(B-A)}(B+A)}{2\cancel{(B-A)}}$$

$$= \frac{A+B}{2}$$

Thus, $\underline{E[X] = \frac{A+B}{2}}$ [mean of the interval] $[A \le x \le B]$.

Now, for calculating $V[X]$, using the shortcut formula,

$$\underline{V[X] = E[X^2] - (E[X])^2}$$

Thus, we first need to calculate $E[X^2]$,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f(x).dx$$ [Using the transformation rule]

$$E[X^2] = \int_{-\infty}^{A} x^2.0.dx + \int_{A}^{B} \frac{x^2}{B-A}.dx + \int_{B}^{\infty} x^2.0.dx$$

$$= \frac{1}{B-A}\int_{A}^{B} x^2.dx = \frac{1}{B-A}\left[\frac{x^3}{3}\right]\Big|_{A}^{B}$$

$$= \frac{B^3-A^3}{3(B-A)}$$

$$\because B^3 - A^3 = (B-A)(A^2+AB+B^2)$$

$$\Rightarrow \frac{\cancel{(B-A)}(A^2+AB+B^2)}{3\cancel{(B-A)}}$$

$$\Rightarrow \frac{A^2+AB+B^2}{3}$$

Therefore,

$$V[X] = \frac{A^2+AB+B^2}{3} - \left(\frac{A+B}{2}\right)^2$$

$$= \frac{A^2+B^2+AB}{3} - \frac{A^2+B^2+2AB}{4}$$

$$= \frac{4A^2+4B^2+4AB - 3A^2 - 3B^2 - 6AB}{12}$$

$$= (A^2+B^2-2AB)/12 = \frac{(A-B)^2}{12} \text{ or } \frac{(B-A)^2}{12}$$

Ans

$\therefore$ Standard deviation $(\sigma_x) \Rightarrow$

$$= \sqrt{V[x]}$$

$$= \cancel{\sqrt{\frac{(B-A)}{12}}} \sqrt{\frac{(B-A)^2}{12}}$$

$$= \frac{B-A}{2\sqrt{3}}$$

Ans.

(c) for a positive integer n, we need to find $E[x^n]$,

$$E[x^n] = \int_{-\infty}^{\infty} \frac{x^n f(x)}{\cancel{B-A}} \cdot dx$$

[Using the transformation rule]

$$= \int_{-\infty}^{A} x^n \cdot 0 \cdot dx + \int_{A}^{B} \frac{x^n}{(B-A)} \cdot dx + \int_{B}^{\infty} x^n \cdot 0 \cdot dx$$

$$= \frac{1}{B-A} \int_{A}^{B} x^n \cdot dx = \frac{1}{B-A} \left[\frac{x^{n+1}}{n+1} \right]_A^B$$

$$= \frac{B^{n+1} - A^{n+1}}{(B-A)(n+1)}$$

$$\therefore E[x^n] = \frac{B^{n+1} - A^{n+1}}{(B-A)(n+1)}$$

Ans

(5.) X = number of hoses being used on the self-service island at a particular time.

Y = number of hoses on the full-service island being used at a particular time.

(a) For $P(X=1, Y=1) \Rightarrow$

We can refer directly to the table that holds joint pmf of X and Y. In the table, where $X=1$ and $Y=1$ at the same time, that will give us the joint probability needed here $\Rightarrow$

$$\therefore P(X=1, Y=1) = \underline{\underline{0.2}}$$

Ans.

(b) for $P(X \leq 1, Y \leq 1) \Rightarrow$

for $X \leq 1$, the possible values of X are $\{0, 1\}$

for $Y \leq 1$, the possible values of Y are $\{0, 1\}$.

So, for getting this probability, we will have to add up the table entries where $X \leq 1$ and $Y \leq 1$ simultaneously. (there are 4 such cases).

$$= P(X=1, Y=1) + P(X=1, Y=0) + P(X=0, Y=1) + P(X=0, Y=0)$$

$$= 0.2 + 0.08 + 0.04 + 0.1$$

$$= 0.42$$

$$\therefore P(X \leq 1, Y \leq 1) = \underline{\underline{0.42}}$$

Ans.

(c.)

Event=> $\{ X \neq 0 \text{ and } Y \neq 0 \}$

can be written as the event when the number of Hoses being used on the self-service island is ~~greater than equal to~~ atleast 1 (not equal to zero numbers of hoses) and the number of hoses being used on the full-service island is atleast 1 ($X \geq 1$ and $Y \geq 1$).

So, to compute the probability of this event =>

$$P(X \neq 0, Y \neq 0) = P(X \geq 1, Y \geq 1)$$

This is such that the domain of X and Y is:-

$$X \in \{0, 1, 2\} \quad ; \quad Y \in \{0, 1, 2\}$$

So, this can be computed as:-

$$P(X \neq 0, Y \neq 0) = P(X \geq 1, Y \geq 1)$$

$$= P(X=1, Y=1) + P(X=1, Y=2) + P(X=2, Y=1) + P(X=2, Y=2)$$

$$= 0.2 + 0.06 + 0.14 + 0.3$$

$$= 0.7$$

$$\therefore P(X \neq 0, Y \neq 0) = 0.7$$

Ans.

(d.) for computing the marginal pmf of X and Y =>

$$P_X(x) = P(X = x) = \sum_y p(x, y) \quad \&$$

$$P_Y(y) = P(Y = y) = \sum_x p(x, y)$$

Thus, marginal distribution for values in X will be as follows:-

$$P_X(0) = p(0,0) + p(0,1) + p(0,2)$$

$$= 0.1 + 0.04 + 0.02 = \underline{\underline{0.16}} \text{ Ans}$$

$$P_X(1) = p(1,0) + p(1,1) + p(1,2)$$

$$= 0.08 + 0.2 + 0.06 = \underline{\underline{0.34}} \text{ Ans.}$$

$$P_X(2) = p(2,0) + p(2,1) + p(2,2)$$

$$= 0.06 + 0.14 + 0.30 = \underline{\underline{0.5}} \text{ Ans.}$$

Thus, marginal distribution for values in Y will be as follows:-

$$P_y(0) = p(0,0) + p(1,0) + p(2,0)$$

$$= 0.1 + 0.08 + 0.06 = \underline{\underline{0.24}} \text{ Ans.}$$

$$P_y(1) = p(0,1) + p(1,1) + p(2,1)$$

$$= 0.04 + 0.2 + 0.14 = \underline{\underline{0.38}} \text{ Ans.}$$

$$P_y(2) = p(0,2) + p(1,2) + p(2,2)$$

$$= 0.02 + 0.06 + 0.3 = \underline{\underline{0.38}} \text{ Ans.}$$

Thus,

Marginal pmf of X is:-

$P_X(0) = 0.16 \quad P_X(1) = 0.34 \quad P_X(2) = 0.5$

Marginal pmf of Y is:-

$P_y(0) = 0.24 \quad P_y(1) = 0.38 \quad P_y(2) = 0.38$

Now, for $P(X \leq 1) \Rightarrow$ As $x \in \{0,1,2\}$,

$\therefore$

$$\underline{P(x \leq 1) = P_x(0) + P_x(1)}$$

$\Rightarrow$ $P(X \leq 1) = P_x(0) + P_x(1)$

$= 0.16 + 0.34 = \underline{\underline{0.5}}$. Ans.

(e.) Two random variables, such as X and Y can be said to be independent when any of them doesn't affect the probability of another one. So, probabilistically,

X and Y are independent if and only if :-

$$P(x,y) = P_x(x)\ P_y(y) \quad \text{for all values of } x \text{ and } y.$$

Technically, we will have to prove this over all the values of x and y. Let's start with:-

Let $x = 1$ and $y = 1$,

we get,

$P(1,1) = 0.2$ [from the table]

Marginal probabilities can be written as:-

$P_x(1) = 0.34$

$P_y(1) = 0.34$

So, $0.2 = P_x(1)\ P_y(1) \Rightarrow 0.2 = 0.34 \times 0.34$

$\Rightarrow 0.2 \neq 0.1156 \Rightarrow \underline{P_x(1)\ P_y(1) \neq P(1,1)}$

Thus, X and Y are not independent random variables. Ans

(6.)

X = the lifetime of the first bulb. (both in 1000s of hours).
Y = the lifetime of the second bulb.

Given:- X and Y are independent and each has an exponential distribution with parameter $\lambda = 1$.

(a.)

$\because$ X and Y are independent,

$\therefore$ the joint pdf of X & Y is given by:-

$$\underline{P(x,y) = P(X=x, y=y) = P(X=x)\,P(Y=y)}$$

for all x and y values possible.

$$\therefore\ P(x,y) = P(X=x)\,P(Y=y)$$

$\because f(x) = \lambda e^{-\lambda x}$ for an exponential distribution.

So,

$$P(x,y) = (\lambda e^{-\lambda x})(\lambda e^{-\lambda y})$$

$$[\because \lambda = 1.]$$

$$= e^{-x}\, e^{-y} = e^{-(x+y)}$$

$$\therefore\ \boxed{P(x,y) = e^{-x-y}}\quad (f(x,y))$$

Ans.

(b.) Probability that each bulb lasts at most 1000 hours,

$$P(X \le 1, y \le 1) = P(X \le 1)\,P(Y \le 1)\quad [\because X \text{ and } Y \text{ are independent}]$$

$$= F(X=1)\,F(Y=1)$$

$\therefore \underline{F(x) = 1 - e^{-\lambda x}}$ (for exponential distribution) ②

Now,

$$P(X \le 1) = F(x=1) = 1 - e^{-\lambda(1)} = \underline{1 - e^{-1}} \quad [\because \lambda = 1]$$

$$P(Y \le 1) = F(Y=1) = 1 - e^{-\lambda(1)} = 1 - e^{-1} \quad [\because \lambda = 1]$$

$$\therefore P(X \le 1, Y \le 1) = P(X \le 1)\, P(Y \le 1)$$

$$= (1 - e^{-1})(1 - e^{-1})$$

$$= (1 - 1/e)^2$$

$$= 0.399576$$

$$\approx \underline{\underline{0.3996}}$$

$$\therefore P(X \le 1, Y \le 1) = \underline{\underline{0.3996}}$$

Ans.

ⓒ Probability that the total lifetime of the two bulbs is atmost 2.

Thus, $\underline{X + Y \le 2} \Rightarrow \underline{\underline{Y \le 2 - X}}$

Thus, for calculating the total lifetime, we will have to integrate x from 0 to 2 and y from 0 to $2-x$. (since, both of them when put together should be atmost 2).

$$\therefore P(X + Y \le 2) \Rightarrow \int_0^2 \int_0^{2-x} f(x,y)\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} e^{-x-y} . dy . dx$$

$$= \int_0^2 e^{-x} \int_0^{2-x} e^{-y}\,dy\,dx$$

$$= \int_0^2 e^{-x} \left[\frac{e^{-y}}{-1}\right]\Bigg|_0^{2-x} dx$$

$$= \int_0^2 e^{-x} \left(\frac{e^{-2+x}}{-1} - \left(\frac{e^{0}}{-1}\right)\right) . dx$$

$$= \int_0^2 e^{-x} \left(-e^{x-2} + 1\right) . dx$$

$$= \int_0^2 e^{-x}\left(1 - e^{x-2}\right) . dx = \int_0^2 \left(e^{-x} - e^{\cancel{x}-2-\cancel{x}}\right) . dx$$

$$= \int_0^2 \left(e^{-x} - e^{-2}\right) . dx = \left[\frac{e^{-x}}{(-1)} - xe^{-2}\right]\Bigg|_0^2$$

$$= \left[\left(-e^{-2} - 2e^{-2}\right) - \left(-e^{0} - 0\right)\right]$$

$$= \left[-3e^{-2} + 1\right] = 1 - \frac{3}{e^2} = 0.593994$$

$$\approx 0.5940$$

$\therefore P(X+Y \leq 2) = \underline{\underline{0.5940}}$ **Ans.**

(d.) Probability that total lifetime is between 1 and 2 can be computed as:-

$$P(1 \leq X+Y \leq 2) = P(X+Y \leq 2) - P(X+Y \leq 1)$$

$\because P(X+Y \leq 2) = 0.5940$ [from part (c)]

$\therefore$ we need to compute:-

$$P(X+Y \leq 1) \Rightarrow$$

$\Rightarrow X+Y \leq 1$

$\Rightarrow \underline{Y \leq 1-X}$

[we will put the limits in the integral accordingly].

$$= \int_0^1 \int_0^{1-x} e^{-x-y}\, dy\, dx$$

$$= \int_0^1 e^{-x} \int_0^{1-x} e^{-y}\, dy . dx = \int_0^1 e^{-x} \left[\frac{e^{-y}}{(-1)} \right]_0^{1-x} dx$$

$$= \int_0^1 e^{-x} \left[\frac{e^{-1+x}}{-1} - \frac{e^0}{(-1)} \right] . dx = \int_0^1 e^{-x} \left[-e^{x-1} + 1 \right] . dx$$

$$= \int_0^1 e^{-x} (1 - e^{x-1}) . dx = \int_0^1 (e^{-x} - e^{\cancel{-x}+\cancel{x}-1}) . dx$$

$$= \int_0^1 (e^{-x} - e^{-1}) . dx = \left[\frac{e^{-x}}{(-1)} - x e^{-1} \right]_0^1$$

$$= \left[\left(\frac{e^{-1}}{(-1)} - e^{-1} \right) - \left(\frac{e^0}{(-1)} - 0 \right) \right]$$

$$= \left[(-e^{-1} - e^{-1}) - (-1) \right] = 1 - 2e^{-1}$$

$$= \underline{\underline{0.2642}}$$

Ans.

Now,

$$= P(X+Y \leq 2) - P(X+Y \leq 1)$$

$$= 0.5940 - 0.2642 = 0.3298$$

$$\Rightarrow \underline{\underline{P(1 \leq X+Y \leq 2) = 0.3298}}$$

Ans.

(7.)

X = Actual air pressure in the right tire
Y = Actual air pressure in the left tire.

Joint pdf is given by:-

$$f(x,y) = \begin{cases} K(x^2+y^2) & 20 \le x \le 30,\ 20 \le y \le 30 \\ 0 & \text{otherwise} \end{cases}$$

(a) for finding the value of K, we will assume that this is a legitimate joint pdf.

So, sum of all the probabilities should be equal to one.

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 1\right)$$

So, for values other than $20 \le x,y \le 30$, the integration will come out to be zero as $f(x,y) = 0$ for those ranges.

$$\therefore \int_{20}^{30}\int_{20}^{30} K(x^2+y^2)\,dx\,dy = K\int_{20}^{30}\int_{20}^{30}(x^2+y^2)\,dx\,dy$$

$$\Rightarrow K\int_{20}^{30}\int_{20}^{30}(x^2+y^2)\,dx\,dy = 1$$

$$\Rightarrow K\int_{20}^{30}\left[\frac{x^3}{3}+xy^2\right]\Bigg|_{20}^{30} dy = 1$$

$$\Rightarrow K\int_{20}^{30}\left[\frac{30^3}{3}+30y^2-\left(\frac{20^3}{3}+20y^2\right)\right]dy = 1$$

$$\Rightarrow \quad K\int_{20}^{30}\left[\frac{1}{3}(30^3-20^3)+(30-20)y^2\right]dy = 1$$

$$\Rightarrow \quad K\int_{20}^{30}\left(\frac{19000}{3}+10y^2\right)dy = 1$$

$$\Rightarrow \quad \cancel{K\int_{20}^{30}\frac{19000}{3}y+\frac{10y^3}{3}} \quad K\left[\frac{19000y}{3}+\frac{10y^3}{3}\right]\Bigg|_{20}^{30} = 1$$

$$\Rightarrow \quad K\left[\left(\frac{19000\times 30}{3}+\frac{10(30)^3}{3}\right)-\left(\frac{19000\times 20}{3}+\frac{10(20)^3}{3}\right)\right]=1$$

$$\Rightarrow \quad K\left[\frac{19000}{3}(30-20)+\frac{10}{3}\left((30)^3-(20)^3\right)\right] = 1$$

$$\Rightarrow \quad K\left[\frac{190000}{3}+\frac{10\times 19000}{3}\right] = 1$$

$$\Rightarrow \quad K\left(\frac{380000}{3}\right) = 1 \quad \Rightarrow \quad \boxed{K = \frac{3}{380000}}$$

Ans.

(b) Probability that both the tires are underfilled can be computed as:-

$$P(X \le 26, Y \le 26) = ?$$

$$\Rightarrow \int_{20}^{26}\int_{20}^{26} K(x^2+y^2)\,dx\,dy$$

$$= K\int_{20}^{26}\int_{20}^{26}(x^2+y^2)\,dx\,dy$$

$$= K\int_{20}^{26}\left[\frac{x^3}{3}+xy^2\right]\Bigg|_{20}^{26}.dy$$

$$= K\int_{20}^{26}\left[\frac{26^3}{3}+26y^2-\left(\frac{20^3}{3}+20y^2\right)\right]dy$$

$$= K\int_{20}^{26}\left(\frac{1}{3}(26^3-20^3)+y^2(26-20)\right).dy$$

$$= K\int_{20}^{26}\left(\frac{9576}{3}+6y^2\right).dy$$

$$= K\left[\frac{9576y}{3}+\frac{\overset{2}{\cancel{6}}y^3}{\cancel{3}}\right]\Bigg|_{20}^{26} = K\left[3192y+2y^3\right]\Bigg|_{20}^{26}$$

$$= K\left[\left(3192\times 26+2(26)^3\right)-\left(3192\times 20+2(20)^3\right)\right]$$

$$= K\left[3192\times 6+2\left(26^3-20^3\right)\right] = K\left[3192\times 6+2\times 9576\right]$$

$$= K\left[19152+19152\right] = \frac{38304\times 3}{380000} = \underline{\underline{0.3024}}\ \text{Ans}$$

(c) Probability that the difference in air pressure between the two tires is atmost 2psi :-

That would mean that:-

$$|X-Y| \le 2$$

[2 cases]

$X - Y \le 2$

$-Y \le 2 - X$

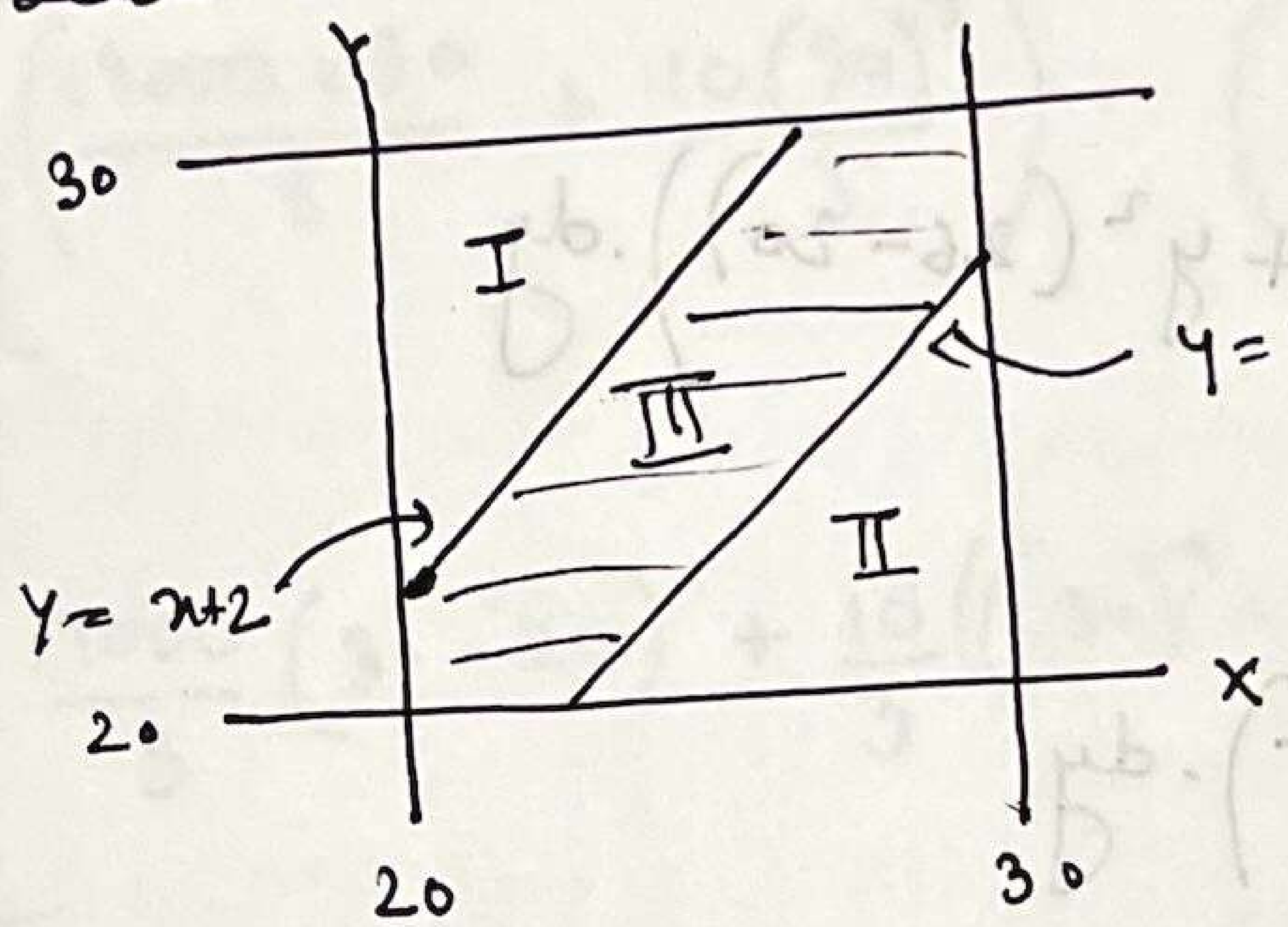
$Y \geq 2 - X$

$-(X-Y) \le 2$

$-X + Y \le 2$

$Y \le 2 + X$

These 2 cases will only be possible in the III rd region shown below :-



As sum of all the probabilities is equal to 1, we can determine the area of region III and thus, the probability that the difference in the air pressure between the two tires is at most 2 psi by the following formulation:-

$$P(|X-Y| \le 2) =$$

$$1 - \underbrace{\int_{20}^{28}\int_{x+2}^{30} f(x,y)\,dy\,dx}_{\text{Ist region}} - \underbrace{\int_{22}^{30}\int_{20}^{x-2} f(x,y)\,dy\,dx}_{\text{IInd region.}}$$

As we already know the pdf $f(x,y)$, we can determine the terms individually above one by one ⇒

for the area of I region =>

$$= \int_{20}^{28} \int_{x+2}^{30} k(x^2+y^2)\, dy\, dx$$

$$= k \int_{20}^{28} \int_{x+2}^{30} (x^2+y^2)\, dy\, dx \quad [\because k \text{ is a constant}]$$

$$= k \int_{20}^{28} \left[x^2 y + \frac{y^3}{3} \right]\Big|_{x+2}^{30} dx$$

$$= k \int_{20}^{28} \left[\left(30x^2 + \frac{(30)^3}{3} \right) - \left(x^2(x+2) + \frac{(x+2)^3}{3} \right) \right] dx$$

$$= k \int_{20}^{28} \left(30x^2 + 9000 - x^3 - 2x^2 - \frac{(x+2)^3}{3} \right) . dx$$

$$= k \int_{20}^{28} \left(28x^2 - x^3 - \frac{(x+2)^3}{3} + 9000 \right) . dx$$

$$= k \left[\frac{28x^3}{3} - \frac{x^4}{4} - \frac{(x+2)^4}{4\times 3} + 9000x \right]\Bigg|_{20}^{28}$$

$$= k \left[\frac{28x^3}{3} - \frac{x^4}{4} - \frac{(x+2)^4}{12} + 9000x \right]\Bigg|_{20}^{28}$$

$$= k \left[\left(\frac{(28)^4}{3} - \frac{28^4}{4} - \frac{30^4}{12} + 9000\times 28 \right) - \left(\frac{28(20)^3}{3} - \frac{20^4}{4} - \frac{22^4}{12} + 9000\times 20 \right) \right]$$

$$= k \left[\left(\frac{4(28)^4 - 3(28)^4}{12} - \frac{30^4}{12} + \frac{22^4}{12} - \frac{28(20)^3}{3} + \frac{20^4}{4} + 9000(28-20) \right) \right]$$

$$= K\left[\frac{28^4 - 30^4 + 22^4 + 3(20)^4 - 112(20)^3 + 72000\times 12}{12}\right]$$

$\because K = 3/380000$

$$= \frac{\cancel{3}}{380000}\left[\frac{38912 + 480000 - 896000 + 864000}{\cancel{12}\,4}\right]$$

$$= \boxed{\frac{486912}{1520000}}$$

For the area of the II region =>

$$= \int_{22}^{30}\int_{20}^{x-2} K(x^2+y^2)\,dy\,dx$$

$$= K\int_{22}^{30}\int_{20}^{x-2}(x^2+y^2)\,dy\,dx \qquad [\because K \text{ is a constant}]$$

$$= K\int_{22}^{30}\left[x^2y + \frac{y^3}{3}\right]\Bigg|_{20}^{x-2} dx$$

$$= K\int_{22}^{30}\left[\left(x^2(x-2) + \frac{(x-2)^3}{3}\right) - \left(20x^2 + \frac{20^3}{3}\right)\right].dx$$

$$= K\int_{22}^{30}\left(x^3 - 2x^2 + \frac{(x-2)^3}{3} - 20x^2 - \frac{20^3}{3}\right).dx$$

$$= K\int_{22}^{30}\left(x^3 - 22x^2 + \frac{(x-2)^3}{3} - \frac{20^3}{3}\right).dx$$

$$= K\left[\frac{x^4}{4} - \frac{22x^3}{3} + \frac{(x-2)^4}{12} - \frac{20^3 x}{3}\right]\Bigg|_{22}^{30}$$

$$= K\left[\left(\frac{30^4}{4} - \frac{22(30)^3}{3} + \frac{28^4}{12} - \frac{20^3(30)}{3}\right) - \left(\frac{22^4}{4} - \frac{22^4}{3} + \frac{20^4}{12} - \frac{20^3(22)}{3}\right)\right]$$

$$= K\left[\frac{3(30)^4 - 88(30)^3 + 28^4 - 120(20)^3 + 22^4 - 20^4 + 88(20)^3}{12}\right]$$

$\because K = 3/380000$

$$= \frac{\cancel{3}}{380000}\left[\frac{2430000 - 2376000 + 614656 - 960000 + 234256 - 160000 + 704000}{\cancel{12}\,4}\right]$$

$$= \boxed{\frac{486912}{1520000}}$$

Therefore,

to compute the area of the IIIrd region,

$$\Rightarrow 1 - \frac{486912}{1520000} - \frac{486912}{1520000}$$

$$\Rightarrow 1 - \frac{973824}{1520000} = \frac{546176}{1520000}$$

$= 0.3593$

$\therefore \boxed{P(|X-Y| \leq 2) = 0.3593}$

This is the probability that the difference in air pressure between the two tires (left & right or right & left) is atmost 2 psi.

Ans.

(d.) for finding the marginal distribution of air pressure in the right tire alone $\Rightarrow$

As x denotes the air pressure for the right tire.

$\therefore$ (marginal distribution of x can) be written as $\Rightarrow$

$$f_x(x) = \int_{20}^{30} f(x,y)\,dy$$

$$= k\int_{20}^{30} (x^2+y^2).dy$$

$$= \cancel{k\int_{20}^{30}\left(x^2y+\frac{y^3}{3}\right)} = k\left[x^2y+\frac{y^3}{3}\right]\Bigg|_{20}^{30}$$

$$= k\left[\left(30x^2+\frac{(30)^3}{3}\right)-\left(20x^2+\frac{(20)^3}{3}\right)\right]$$

$$= k\left[x^2(30-20)+\frac{(30)^3-(20)^3}{3}\right]$$

$$= k\left[10x^2+\frac{19000}{3}\right] = \frac{3}{380000}\left[10x^2+\frac{19000}{3}\right]$$

$$= \frac{3x^2}{38000}+\frac{1}{20}$$

$$\therefore \boxed{f_x(x) = \frac{1}{20}+\frac{3x^2}{38000}} \quad \left[\text{for } 20 \le x \le 30\right]$$

Ans: $f_x(x) = 0$ (otherwise)

(e.) If x and Y are independent r.v's, then,

$$f(x,y) = f_x(x)\, f_y(y)$$

So, we need to find the marginal distribution of Y, (left tire)

So, marginal distribution of Y ⟹

$$f_Y(y) = \int_{20}^{30} f(x,y)\,dx$$

$$= K\int_{20}^{30} (x^2+y^2)\,dx = K\left[\frac{x^3}{3} + xy^2\right]_{20}^{30}$$

$$= K\left[\left(\frac{30^3}{3} + 30y^2\right) - \left(\frac{(20)^3}{3} + 20y^2\right)\right]$$

$$= K\left[\frac{(30)^3 - (20)^3}{3} + y^2(30-20)\right]$$

$$= K\left[\frac{19000}{3} + 10y^2\right] \qquad \because K = \frac{3}{380000}$$

$$= \frac{3}{380000}\left[\frac{19000}{3} + 10y^2\right]$$

$$= \frac{1}{20} + \frac{3y^2}{38000}$$

$$\therefore \boxed{f_Y(y) = \frac{1}{20} + \frac{3y^2}{38000}} \quad \text{for } 20 \le y \le 30$$

Ans $f_Y(y) = 0$ (otherwise).

$$\Rightarrow \quad f_X(x)\, f_Y(y) = \left(\frac{1}{20} + \frac{3x^2}{38000}\right)\left(\frac{1}{20} + \frac{3y^2}{38000}\right)$$

$$= \frac{(1900 + 3x^2)(1900 + 3y^2)}{(38000)^2}$$

$$= \frac{(3x^2 + 1900)(3y^2 + 1900)}{1444000000}$$

$$\therefore \cancel{f(x,y)}\ f(x,y) = \frac{3}{380000}(x^2+y^2)$$

&

$$\frac{3}{380000}(x^2+y^2) \neq \frac{(3x^2+1900)(3y^2+1900)}{1444000000}$$

$$\therefore \boxed{f(x,y) \neq f_X(x)\, f_Y(y)}$$

This shows ~~that~~ X and Y are not independent random variables.

Ans.

(8.) joint pdf is given as:-

$$f(x,y) = \begin{cases} \frac{3}{380000}(x^2+y^2) & 20 \le x \le 30, 20 \le y \le 30 \\ 0 & \text{Otherwise} \end{cases}$$

(a) and,

marginal pdfs of X and Y were determined as follows:-

$$f_x(x) = \begin{cases} \frac{3x^2}{38000} + \frac{1}{20} & 20 \le x \le 30 \\ 0 & \text{otherwise.} \end{cases}$$

$$f_y(y) = \begin{cases} \frac{3y^2}{38000} + \frac{1}{20} & 20 \le y \le 30 \\ 0 & \text{otherwise} \end{cases}$$

⇒ Conditional pdf of Y given X=x can be computed as follows:-

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}$$

This, we will only compute for the range $20 \le y \le 30$ since, otherwise it will be zero.

$$\text{so, } f_{Y|X}(y|x) = \frac{\frac{3}{380000}(x^2+y^2)}{\frac{3x^2}{38000} + \frac{1}{20}}$$

$$f_{Y|X}(y|x) = \frac{\frac{3}{380000}(x^2+y^2)}{\left(\frac{3}{380000}\right)10x^2 + \frac{1}{20}}$$

$\because K = \frac{3}{380000}$, let's put this,

$$f_{Y|X}(y|x) = \frac{K(x^2+y^2)}{10Kx^2 + 1/20}$$

$$\therefore \boxed{f_{Y|X}(y|x) = \begin{cases} \frac{K(x^2+y^2)}{10Kx^2 + 1/20} & 20 \le y \le 30 \\ 0 & \text{otherwise} \end{cases}}$$

Ans.

$\Rightarrow$ for computing conditional pdf of X given that $Y=y$ can be done as follows:-

$$f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$$

This, we will only compute for the range $20 \le x \le 30$, since, otherwise this will be zero.

So,

$$f_{X|Y}(x|y) = \frac{\frac{3}{380000}(x^2+y^2)}{\frac{3y^2}{38000} + \frac{1}{20}}$$

$\Rightarrow$

$$= \frac{\frac{3}{380000}(x^2+y^2)}{\left(\frac{3}{380000}\right)10y^2 + \frac{1}{20}}$$

$\because k = \frac{3}{380000}$, let's put this,

$$= \frac{k(x^2+y^2)}{10ky^2 + 1/20}$$

$$\therefore \boxed{f_{X|Y}(x|y) = \begin{cases} \frac{k(x^2+y^2)}{10ky^2+1/20} & 20 \le x \le 30 \\ 0 & \text{otherwise} \end{cases}}$$

Ans.

(b.) Pressure in the right tire = 22psi

So, to find the probability that the left tire has a pressure of at least 25 psi, given that the pressure in the right tire is found to be 22psi.

Using the formula for conditional pdf of Y given X=22,

$$P(Y \geq 25 \mid X = 22) = \int_{25}^{30} f_{Y|X}(y|22)\, dy$$

$$= \int_{25}^{30} \frac{k(22^2+y^2)}{10k(22)^2 + 1/20} \cdot dy$$

$$= \int_{25}^{30} \frac{k(22^2+y^2)}{10k(22)^2+0.05} \cdot dy$$

$$= \frac{k}{10k(22)^2+0.05} \int_{25}^{30} (22^2+y^2) \cdot dy$$

$$= \frac{k}{10k(22)^2+0.05} \left[(22)^2 y + \frac{y^3}{3} \right] \Bigg|_{25}^{30}$$

$$= \frac{k}{(22)^2 10k+0.05} \left[484y + \frac{y^3}{3} \right] \Bigg|_{25}^{30}$$

$$= \frac{k}{(22)^2 10k+0.05} \left[\left(484 \times 30 + \frac{30^3}{3} \right) - \left(484 \times 25 + \frac{(25)^3}{3} \right) \right]$$

$$= \frac{k}{(22)^2 10k+0.05} \left[484(30-25) + \frac{30^3-25^3}{3} \right]$$

$$\because k = \frac{3}{380000}$$

$$\Rightarrow \frac{3/380000}{\frac{(22)^2 \times 1\cancel{0} \times 3}{3800\cancel{0}0} + \frac{1}{20}} \left[484 \times 5 + \frac{11375}{3} \right]$$

$$\Rightarrow \frac{3/\cancel{380000}}{\frac{(30(22)^2+19000)}{\cancel{380000}}} \left[2420 + \frac{11375}{3} \right]$$

$$\Rightarrow \frac{3}{30(22)^2+19000}\left[2420+11375/3\right]$$

$$\Rightarrow \frac{3}{33520}\left[2420+\frac{11375}{3}\right]$$

$$\Rightarrow \frac{1}{33520}\left[2420\times 3+11375\right] = \frac{18635}{33520} = \underline{\underline{0.5559}}$$

Ans.

$$\therefore \underline{\underline{P(Y \geq 25 \mid X=22) = 0.5559}}$$

Ans.

for $P(Y \geq 25) \Rightarrow$

$$P(Y \geq 25) = \int_{25}^{30} f_Y(y).dy$$

$$= \int_{25}^{30} \left(\frac{3y^2}{38000} + 0.05\right) dy$$

$$= \left[\frac{3}{38000}\left(\frac{y^3}{3}\right) + 0.05y\right]\Bigg|_{25}^{30}$$

$$= \left[\frac{y^3}{38000} + 0.05y\right]\Bigg|_{25}^{30}$$

$$= \left[\left(\frac{30^3}{38000} + 0.05\times 30\right) - \left(\frac{25^3}{38000} + 0.05\times 25\right)\right]$$

$$= \left[\frac{30^3 - 25^3}{38000} + 0.05\times 5\right] = \frac{11375}{38000} + 0.25$$

$= 0.2993 + 0.25$

$= \underline{\underline{0.5493}}$

$\therefore P(Y \geq 25) = 0.5493$

$\therefore \underline{P(Y \geq 25 \mid X = 22) \neq P(Y \geq 25)}$

and

$P(Y \geq 25 \mid X = 22) > P(Y \geq 25)$

(a little bit greater).

Ans.

(c) Pressure in the right tire = 22 psi.

$\therefore$ Expected pressure in the left tire, given that the pressure in the right tire is found to be 22 psi, can be calculated as follows:-

$$E(Y \mid X = 22) = \int_{20}^{30} y \, f_{Y|X}(y|22) \, dy$$

$\Rightarrow$ Using the conditional pdf of Y given X = 22,

$$= \int_{20}^{30} y \left(\frac{k(22^2 + y^2)}{10k(22)^2 + 0.05} \right) dy$$

$$= \frac{k}{4840k + 0.05} \int_{20}^{30} (22)^2 y + y^3 \, dy$$

$$= \frac{K}{4840K+0.05}\left[\frac{484\,y^2}{2} + \frac{y^4}{4}\right]\Big|_{20}^{30}$$

$$= \frac{K}{4840K+0.05}\left[242\,y^2 + \frac{y^4}{4}\right]\Big|_{20}^{30}$$

$$= \frac{K}{4840K+0.05}\left[\left(242(30)^2 + \frac{(30)^4}{4}\right) - \left(242(20)^2 + \frac{(20)^4}{4}\right)\right]$$

$$= \frac{K}{4840K+0.05}\left[242\left(30^2 - 20^2\right) + \frac{1}{4}\left(30^4 - 20^4\right)\right]$$

$$= \frac{K}{4840K+0.05}\left[\overset{121000}{\cancel{4598000}} + 162500\right] \qquad \left[\because K = \frac{3}{380000}\right]$$

$$= \frac{\overset{283500}{\cancel{4760500}}\,K}{4840K+0.05} = \frac{\dfrac{\overset{283500}{\cancel{4760500}} \times 3}{380000}}{\dfrac{4840\times 3}{380000} + \dfrac{1}{20}}$$

$$= \frac{\dfrac{\overset{283500}{\cancel{4760500}}\times 3}{\cancel{380000}}}{\dfrac{4840\times 3 + 19000}{\cancel{380000}}} = \frac{\overset{283500}{\cancel{4760500}}\times 3}{4840\times 3 + 19000}$$

$$= \frac{\overset{850500}{\cancel{14281500}}}{33520} = 25.3729 \simeq \underline{\underline{25.37}}\ \text{psi}$$

Ans.

⇒ Variance of the pressure in left tire can be calculated as follows:-

$$V(Y|X=22) = E(Y^2|X=22) - (E(Y|X=22))^2$$

$$\Rightarrow E(Y^2|X=22) = \int_{20}^{30} y^2 f_{Y|X}(y|22).dy$$ [Using the transformation rule]

$$= \int_{20}^{30} \frac{y^2 K(22^2+y^2)}{10K(22)^2+0.05}.dy$$

$$= \frac{K}{4840K+0.05} \int_{20}^{30} (484y^2+y^4).dy$$

$$= \frac{K}{4840K+0.05}\left[\frac{484y^3}{3}+\frac{y^5}{5}\right]\Bigg|_{20}^{30}$$

$$= \frac{K}{4840K+0.05}\left[\frac{484}{3}(30^3-20^3)+\frac{(30^5-20^5)}{5}\right]$$

$$= \frac{K}{4840K+0.05}\left[\frac{484\times 19000}{3} - \frac{21100000}{5}\right]$$

$$\because K = 3/380000$$

$$\Rightarrow \cancel{\frac{3/380000}{4840\times 3}}$$

$$\Rightarrow \frac{3/380000}{\frac{4840\times 3}{380000}+0.05}\left[\frac{9196000}{3}+\frac{21100000}{5}\right]$$

$$\Rightarrow \frac{3}{33520}\left[3065333.333+4220000\right]$$

$$\Rightarrow 652.0286 \approx \underline{652.03}$$ Ans.

Therefore, to find the variance,

$$V(Y|X=22) = 652.0286 - (25.3729)^2$$

$$= \underline{\underline{8.24}}$$ Ans.

Therefore, the standard deviation of the pressure in the left tire $\Rightarrow$

$$SD(Y|X=22) = \sqrt{V(Y|X=22)}$$

$$= \sqrt{8.24}$$

$$= \underline{\underline{2.87 \text{ psi}}}$$

Ans.