HW9

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1. True standard deviation (σ) = 0.75

Samples taken follow a normal distribution.

(a) As we need to compute a 95%-CI,

$\therefore \boxed{\gamma = 0.95}$

$\boxed{n = 20}$

Sample mean ($\bar{X}_n$) = 4.85

To compute 95% CI for the true average porosity of a certain seam, we need the following formulations,

$$\boxed{\bar{X}_n - \frac{\sigma}{\sqrt{n}} q(1-\alpha_2) \leq \mu \leq \bar{X}_n - \frac{\sigma}{\sqrt{n}} q(\alpha_1)} \quad (1)$$

where,

$$\boxed{\alpha_1 + \alpha_2 = 1 - \gamma} \quad (2)$$

As we will consider this to be a balanced CI,

$\therefore \boxed{\alpha_1 = \alpha_2}$

So,

$$\alpha_1 + \alpha_2 = 1 - 0.95 = 0.05$$

$$2\alpha_1 = 0.05 \Rightarrow \boxed{\alpha_1 = 0.025}$$

$\because \alpha_1 = \alpha_2$, $\therefore \boxed{\alpha_1 = \alpha_2 = 0.025}$

Considering the 1st equation above and putting in the values,

$$4.85 - \frac{0.75}{\sqrt{20}} q(1 - 0.025) \leq \mu \leq 4.85 - \frac{0.75}{\sqrt{20}} q(0.025)$$

Solving this,
and considering the normal distribution table,

$$q(1-0.025) = q(0.975) = \underline{1.96}$$

$$\& \ q(0.025) = \underline{-1.96}$$

$\Rightarrow$

$$4.85 - \frac{0.75}{\sqrt{20}} \times (1.96) \leq \mu \leq 4.85 - \frac{0.75}{\sqrt{20}} \times (-1.96)$$

$$\boxed{4.52 \leq \mu \leq 5.18}$$

This is the 95% CI for $\mu \Rightarrow \underline{[4.52, 5.18]}$

Ans.

(b) Now, to compute the 98% CI for true average porosity of another seam,

$$\boxed{\gamma = 0.98}$$

$\underline{n = 16}$, sample average porosity $(\bar{X}_n) = 4.56$.

$\underline{\sigma = 0.75}$

Considering the (1)st and (2)nd equation in part (a), when we are considering a balanced CI here,

$$\alpha_1 + \alpha_2 = 1 - 0.98 = 0.02$$

$[\because \alpha_1 = \alpha_2]$

$$\therefore 2\alpha_1 = 0.02 \Rightarrow \boxed{\alpha_1 = 0.01}$$

$$\boxed{\therefore \alpha_1 = \alpha_2 = 0.01}$$

$$4.56 - \frac{0.75}{\sqrt{16}} q(1-0.01) \leq \mu \leq 4.56 - \frac{0.75}{\sqrt{16}} q(0.01)$$

from the normal distribution table,

$$q(1-0.01) = q(0.99) = \underline{2.33}$$

&

$$q(0.01) = -\underline{2.33}$$

$\Rightarrow$

$$4.56 - \frac{0.75}{\sqrt{16}} \times (2.33) \leq \mu \leq 4.56 - \frac{0.75}{\sqrt{16}} \times (-2.33)$$

$$4.124 \leq \mu \leq 4.996$$

$$\boxed{4.12 \leq \mu \leq 5.0}$$

This is the 98% CI for $\mu \Rightarrow \underline{[4.12, 5.00]}$ Ans.

(c) for a 95% interval, $\boxed{\gamma = 0.95}$

$\underline{\sigma = 0.75}$.

Width of the interval $(L) = \underline{0.40}$

To find how large the sample size (n) is $\Rightarrow$

from the previous parts, we know that the confidence intervals are given by:-

$$\left(\bar{X}_n - \frac{\sigma}{\sqrt{n}} q(N(0,1), 1-\alpha_2), \; \bar{X}_n - \frac{\sigma}{\sqrt{n}} q(N(0,1), \alpha_1) \right)$$

$\therefore$ the width of the interval can be computed by $\Rightarrow$

$$\boxed{L = \left[\bar{X}_n - \frac{\sigma}{\sqrt{n}} q(N(0,1), \alpha_1)\right] - \left[\bar{X}_n - \frac{\sigma}{\sqrt{n}} q(N(0,1), 1-\alpha_2)\right]} \quad (3)$$

$\because$ we will consider this as a balanced CI, $(\alpha_1 = \alpha_2)$

$$\alpha_1 + \alpha_2 = 1 - 0.95 = 0.05$$

$$\Rightarrow 2\alpha_1 = 0.05 \Rightarrow \alpha_1 = 0.025$$

$$\Rightarrow \boxed{\alpha_1 = \alpha_2 = 0.025}$$

Putting in all the values in the 3rd equation above, & solve.

$$0.40 = \left[\cancel{\bar{X}_n} - \cancel{\bar{X}_n} - \frac{\sigma}{\sqrt{n}} q(N(0,1), \alpha_1) + \frac{\sigma}{\sqrt{n}} q(N(0,1), 1-\alpha_2)\right]$$

$$0.40 = \frac{\sigma}{\sqrt{n}}\left[q(N(0,1), 1-\alpha_2) - q(N(0,1), \alpha_1)\right]$$

$$\Rightarrow \sqrt{n} = \frac{0.75}{0.40}\left[q(N(0,1), 1-\alpha_2) - q(N(0,1), \alpha_1)\right]$$

$$q(N(0,1), 1-\alpha_2) = q(N(0,1), 1-0.025)$$

$$\Rightarrow q(N(0,1), 0.975) = \underline{\underline{1.96}}$$

$$\because q(N(0,1), 0.975) = -q(N(0,1), 0.025)$$

$$\therefore q(N(0,1), 0.025) = \underline{\underline{-1.96}}$$

$$\Rightarrow \sqrt{n} = \frac{0.75}{0.40}\left[1.96 - (-1.96)\right] = \frac{0.75}{0.40} \times 3.92$$

$$n = \left(\frac{0.75}{0.40} \times 3.92 \right)^2 = 54.0225$$

Rounding up the value of n

$$\boxed{n \approx 55}$$

$\therefore$ The sample size (n) should be 55.

Ans.

(d) for a 99% confidence interval,

$$\gamma = 0.99 \quad , \quad \sigma = 0.75$$

As we need to find the sample size necessary to estimate true average porosity (μ) to be within 0.2.

As this is a balanced CI for a normal distribution, it will be the same on both the sides from μ,

$$\therefore L = 2 \times 0.2 = 0.4$$

Again, this is a balanced CI ($\alpha_1 = \alpha_2$),

$$\alpha_1 + \alpha_2 = 1 - \gamma = 1 - 0.99 = 0.01$$

$$\Rightarrow 2\alpha_1 = 0.01 \Rightarrow \boxed{\alpha_1 = 0.005}$$

$$\Rightarrow \boxed{\alpha_1 = \alpha_2 = 0.05}$$

Using the equation (3) in part (c),

$$0.4 = \left[\cancel{\bar{X}_n} - \cancel{\bar{X}_n} - \frac{\sigma}{\sqrt{n}} q(N(0,1), \alpha_1) + \frac{\sigma}{\sqrt{n}} q(N(0,1), 1-\alpha_2) \right]$$

$$0.4 = \frac{0.75}{\sqrt{n}}\left[q(N(0,1), 1-0.005) - q(N(0,1), 0.005)\right]$$

$$\Rightarrow \quad 0.4 = \frac{0.75}{\sqrt{n}}\left[q(N(0,1), 0.995) - q(N(0,1), 0.005)\right]$$

$$\Rightarrow \quad 0.4 = \frac{0.75}{\sqrt{n}}\left[2.58 - (-2.58)\right]$$

$\because$ $\underline{q(N(0,1), 0.995) = 2.58}$

& $q(N(0,1), 0.995) = -q(N(0,1), 0.005)$

$\therefore$ $\underline{q(N(0,1), 0.005) = -2.58}$

$$\Rightarrow \quad \sqrt{n} = \frac{0.75}{0.4} \times 2 \times 2.58$$

$$\Rightarrow \quad n = \left(\frac{0.75}{0.4} \times 2 \times 2.58\right)^2 = \underline{\underline{93.61}}$$

We will round up this value,

$$\boxed{\underline{n \approx 94}}$$

<u>Ans.</u>

(2.) Sample mean compressive strengths ($\bar{X}_n$) = 64.41 MPa

Sample standard deviation (S_n) = 10.32 MPa

$n = 18$ tested specimens.

— The distribution for compressive strength was found to be normal.

(a) Confidence level (γ) = 0.98 (98%)

— To calculate the confidence interval for the true average compressive strength, we will go through the steps 1-4 as taught in the class.
(98% CI)

Step (1):- find a pivot for the unknown parameters,

— here we have 2 unknown parameters, both, μ and σ^2 (mean and variance).

$$\boxed{I(X_1, \ldots, X_n; \mu, \sigma^2) = \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}}}$$ where,

$\bar{X}_n$ is sample mean, S_n is sample variance and μ is the true mean/ average.

— This is considered here (S_n instead of σ) because here we have both μ and σ unknown and $\bar{X}_n$ and S_n have been provided.

Step ②:- To see the distribution of the pivot,

$\because$ we are conserned with true average,

the distribution of the pivot ($\mathcal{I}$) is t_{n-1} distribution.

(t- distribution with $n-1$ degrees of freedom).

(As shown in class),

$$\boxed{\mathcal{I}(X_1, \dots, X_n; \mu) = \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \sim t_{n-1}}$$

As we can see above ~~that~~ the pivot is dependent on μ but the distribution of the pivot is not!

Also,

Let q be its quantile function,

$q(\alpha)$ = The value of x such that

$$\boxed{P(t_{n-1} \leq x) = \alpha}$$

Step ③:- Since we need to calculate the confidence interval with an upper and a lower bound, we will choose some plausible values for α_1 and α_2.

We know:-

$$\boxed{\alpha_1 + \alpha_2 = 1 - \gamma}$$

$$\Rightarrow P(q(t_{n-1}, \alpha_1) \leq \mathcal{I}(X_1, \dots, X_n; \mu, \sigma^2) \leq q(t_{n-1}, 1-\alpha_2)) = \gamma$$

Substituting the value of the pivot in the equation above:-

$$\boxed{P\left(q(t_{n-1}, \alpha_1) \leq \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \leq q(t_{n-1}, 1-\alpha_2)\right) = \gamma}$$

Step 4:-

Solving the inequality for μ, we get:-

Multiplying both sides by $S_n/\sqrt{n}$:-

$\Rightarrow$

$$P\left(\frac{S_n}{\sqrt{n}} q(t_{n-1}, \alpha_1) \leq \bar{X}_n - \mu \leq \frac{S_n}{\sqrt{n}} q(t_{n-1}, 1-\alpha_2)\right) = \gamma$$

Subtracting $\bar{X}_n$ from ~~p~~ both the sides of the inequality,

$$P\left(\frac{S_n}{\sqrt{n}} q(t_{n-1}, \alpha_1) - \bar{X}_n \leq -\mu \leq \frac{S_n}{\sqrt{n}} q(t_{n-1}, 1-\alpha_2) - \bar{X}_n\right) = \gamma$$

Multiplying the inequality by -1, we get:-

$$P\left(\bar{X}_n - \frac{S_n}{\sqrt{n}} q(t_{n-1}, 1-\alpha_2) \leq \mu \leq \bar{X}_n - \frac{S_n}{\sqrt{n}} q(t_{n-1}, \alpha_1)\right) = \gamma$$

Thus, the confidence interval for μ can be given as:-

$$\boxed{\bar{X}_n - \frac{S_n}{\sqrt{n}} q(t_{n-1}, 1-\alpha_2) \leq \mu \leq \bar{X}_n - \frac{S_n}{\sqrt{n}} q(t_{n-1}, \alpha_1)} \quad (i)$$

$\because$ confidence level is 98%, $\boxed{\gamma = 0.98}$

Now, we will put in all the values in equation (1) above,

Also, $\because$ we know:-

$$\alpha_1 + \alpha_2 = 1 - \gamma = 1 - 0.98 = 0.02$$

$\because$ this is a balanced CI $(\alpha_1 = \alpha_2)$,

$$2\alpha_1 = 0.02 \Rightarrow \alpha_1 = 0.01$$

$$\Rightarrow \boxed{\alpha_1 = \alpha_2 = 0.01}$$

Now, from equation (1),

$$64.41 - \frac{10.32}{\sqrt{18}} q(t_{n-1}, 1 - 0.01) \leq \mu \leq 64.41 - \frac{10.32}{\sqrt{18}} \cdot q(t_{n-1}, 0.01)$$

$\because$ we know,

$$q(t_{n-1}, 0.99) = \underline{\underline{2.567}}$$

& $q(t_{n-1}, 0.99) = -q(t_{n-1}, 0.01)$

$$\therefore q(t_{n-1}, 0.01) = \underline{\underline{-2.567}}$$

$\Rightarrow$

$$64.41 - \frac{10.32}{\sqrt{18}} \times 2.567 \leq \mu \leq 64.41 - \frac{10.32}{\sqrt{18}} \times (-2.567)$$

$$58.16 \leq \mu \leq 70.654$$

$$\boxed{58.16 \leq \mu \leq 70.65}$$ Ans

$\therefore$ The confidence interval is $\underline{[58.16, 70.65]}$ Ans.

(b) Since, now we consider $\gamma = 0.99$ as 99% CI of the form $\underline{(-\infty, R_n)}$.
$\underline{\text{(upper-bounded)}}$

Also, we know that:-

$$\alpha_1 + \alpha_2 = 1 - \gamma$$

$$\Rightarrow \alpha_1 + \alpha_2 = 1 - 0.99 = 0.01 \Rightarrow \boxed{\alpha_1 + \alpha_2 = 0.01} \quad (2)$$

In order to get the interval as $(-\infty, R_n)$, the lower bound of the CI should be $-\infty$.

Considering the equation (1) from the part (a),

$$\bar{X}_n - \frac{S_n}{\sqrt{n}} q(t_{n-1}, 1-\alpha_2) = -\infty$$

($\because \bar{X}_n$ is finite)

$$\Rightarrow \frac{S_n}{\sqrt{n}} q(t_{n-1}, 1-\alpha_2) = \infty$$

$\because S_n$ and n is finite $(n \neq 0)$, $\therefore q(t_{n-1}, 1-\alpha_2) = \infty$

$\because$ we know that $\boxed{q(1) = \infty}$.

$$\therefore 1 - \alpha_2 = 1 \Rightarrow \boxed{\alpha_2 = 0}$$

Putting this in equation (2) above,

$$\alpha_1 + 0 = 0.99 \Rightarrow \boxed{\alpha_1 = 0.01}$$

Putting back the lower bound of ∞ and other values in RHS of the inequality (I),

$$-\infty \leq \mu \leq 64.41 - \frac{10.32}{\sqrt{18}} q(t_{n-1}, 0.01)$$

$$-\infty \leq \mu \leq 64.41 - \frac{10.32}{\sqrt{18}} \times (-2.567)$$

$$\left[\because q(t_{n-1}, 0.01) = -2.567\right]$$

$$\Rightarrow \quad -\infty \leq \mu \leq 70.654$$

$$\Rightarrow \quad \boxed{-\infty < \mu \leq 70.65}$$

Ans.

$\therefore$ The required CI is $\Rightarrow (-\infty, 70.65)$

where R_n comes out to be 70.65

Ans.

3.

Let $X_1, X_2, \dots, X_n$ be a random sample from a uniform distribution on the interval $[0, \theta]$, so, that:-

$$f(x) = \begin{cases} 1/\theta, & 0 \le x \le \theta \\ 0, & \text{otherwise} \end{cases}$$

If $Y = \max(X_i)$,

Let U be a random variable (rv) such that $U = Y/\theta$,

$$f_U(u) = \begin{cases} nu^{n-1}, & 0 \le u \le 1 \\ 0, & \text{otherwise} \end{cases}$$

(a) first, let's consider the LHS of the inequality:-

$$P\left(\left(\frac{\alpha}{2}\right)^{1/n} < Y/\theta \le \left(1 - \frac{\alpha}{2}\right)^{1/n}\right)$$

$$[\because U = Y/\theta]$$

$$= P\left(\left(\frac{\alpha}{2}\right)^{1/n} < U \le \left(1 - \frac{\alpha}{2}\right)^{1/n}\right)$$

$$= F\left(\left(1 - \frac{\alpha}{2}\right)^{1/n}\right) - F\left(\left(\frac{\alpha}{2}\right)^{1/n}\right) \quad \{F \text{ is the cdf}\}$$

Thus, this can be done by considering the CDF between $(1-\frac{\alpha}{2})^{1/n}$ and $(\alpha/2)^{1/n}$ interval. But we only want this!

Thus,

$$F_U(u) = \int_{-\infty}^{\infty} f(u)\,du = \int_{(\alpha/2)^{1/n}}^{(1-\frac{\alpha}{2})^{1/n}} nu^{n-1}\,du$$

$$\Rightarrow n \int_{(\alpha/2)^{1/n}}^{(1-\frac{\alpha}{2})^{1/n}} u^{n-1}\, du = n \left[\frac{u^{n-1+1}}{n-1+1} \right] \Bigg|_{(\alpha/2)^{1/n}}^{(1-\frac{\alpha}{2})^{1/n}}$$

$$\Rightarrow \cancel{n} \left[\frac{u^n}{\cancel{n}} \right] \Bigg|_{(\alpha/2)^{1/n}}^{(1-\frac{\alpha}{2})^{1/n}} = \left[\left(1-\frac{\alpha}{2}\right)^{1/n} \right]^n - \left[\left(\frac{\alpha}{2}\right)^{1/n} \right]^n$$

$$\Rightarrow 1 - \frac{\alpha}{2} - \frac{\alpha}{2} = \underline{\underline{1-\alpha}}$$

$$\therefore P\left(\left(\frac{\alpha}{2}\right)^{1/n} < \frac{Y}{\theta} \leq \left(1-\frac{\alpha}{2}\right)^{1/n} \right) = 1-\alpha$$

Hence, verified!

Now,

to derive the $100(1-\alpha)\%$ CI for θ,

Let's begin by considering the equality we verified above.

$$\Rightarrow P\left(\left(\frac{\alpha}{2}\right)^{1/n} < \frac{Y}{\theta} \leq \left(1-\frac{\alpha}{2}\right)^{1/n} \right) = 1-\alpha$$

Dividing the inequality by Y we get:-

$$P\left(\frac{(\alpha/2)^{1/n}}{Y} < \frac{1}{\theta} \leq \frac{(1-\alpha/2)^{1/n}}{Y} \right) = 1-\alpha$$

Taking reciprocals and redistributing in the inequality above:-

$$P\left(\frac{Y}{(1-\frac{\alpha}{2})^{1/n}} < \theta \leq \frac{Y}{(\alpha/2)^{1/n}} \right) = 1-\alpha$$

$\therefore$ The required confidence interval that we get is:-

$[\because Y = \max(X_i)]$

$$\Rightarrow \left(\frac{\max(X_i)}{\left(1-\frac{\alpha}{2}\right)^{1/n}}, \frac{\max(X_i)}{\left(\alpha/2\right)^{1/n}} \right)$$

Ans.

(b.)

To verify:-

$$P\left(\alpha^{1/n} \leq \frac{Y}{\theta} \leq 1\right) = 1-\alpha$$

we consider the LHS first,

$$\Rightarrow P\left(\alpha^{1/n} \leq \frac{Y}{\theta} \leq 1\right)$$

$\boxed{\because \frac{Y}{\theta} = U}$

$$\Rightarrow P\left(\alpha^{1/n} \leq U \leq 1\right) = F(1) - F(\alpha^{1/n})$$

We can compute by calculating the CDF between this interval which boils down to $\Rightarrow$

$$\Rightarrow \int_{\alpha^{1/n}}^{1} f_U(u).\, du$$

$$\Rightarrow \int_{\alpha^{1/n}}^{1} nu^{n-1}.\, du = n \left[\frac{u^{n-1+1}}{n-1+1}\right]_{\alpha^{1/n}}^{1} = \cancel{n} \left[\frac{u^n}{\cancel{n}}\right]_{\alpha^{1/n}}^{1}$$

$$\Rightarrow 1^n - (\alpha^{1/n})^n \Rightarrow \underline{\underline{1-\alpha}}$$

$\therefore$ We have verified that:-

$$P\left(\alpha^{1/n} \leq \frac{Y}{\theta} \leq 1\right) = 1-\alpha$$

Ans.

Now,

we have to derive $100(1-\alpha)\%$ CI for θ,

$\therefore$

$$P\left(\alpha^{1/n} \leq \frac{Y}{\theta} \leq 1\right) = 1-\alpha$$

Dividing the inequality by Y,

$$P\left(\frac{\alpha^{1/n}}{Y} \leq \frac{1}{\theta} \leq \frac{1}{Y}\right) = 1-\alpha$$

Taking reciprocals of the inequality above, we get:-

$$P\left(Y \leq \theta \leq \frac{Y}{\alpha^{1/n}}\right) = 1-\alpha$$

$\because$ $Y = \max(X_i)$,

$\therefore$

$$P\left(\max(X_i) \leq \theta \leq \frac{\max(X_i)}{\alpha^{1/n}}\right) = 1-\alpha$$

$\therefore$ The required confidence interval is:

$$\left(\max(X_i), \frac{\max(X_i)}{\alpha^{1/n}}\right)$$

Ans

(c) Confidence Interval for part (a) is :-

$$\left(\frac{\max(X_i)}{(1-\alpha/2)^{1/n}}, \frac{\max(X_i)}{(\alpha/2)^{1/n}} \right) \quad \text{---} (1)$$

~~Confidence~~ Confidence Interval for part (b) is :-

$$\left(\max(X_i), \frac{\max(X_i)}{\alpha^{1/n}} \right) \quad \text{---} (2)$$

for finding which of these intervals is shorter,
for which we need to find the length of the intervals first which is,
(Upper bound - lower bound).

Let L_1 be the length of CI from (1),

$$L_1 = \frac{\max(X_i)}{(\alpha/2)^{1/n}} - \frac{\max(X_i)}{(1-\alpha/2)^{1/n}} = \max(X_i)\left[\frac{1}{(\alpha/2)^{1/n}} - \frac{1}{(1-\alpha/2)^{1/n}} \right]$$

$$= \max(X_i) \cdot 2^{1/n} \cdot \left[\frac{1}{\alpha^{1/n}} - \frac{1}{(2-\alpha)^{1/n}} \right]$$

$$\therefore \quad L_1 = 2^{1/n} \cdot \max(X_i) \cdot \left[\frac{(2-\alpha)^{1/n} - \alpha^{1/n}}{\alpha^{1/n}(2-\alpha)^{1/n}} \right] \quad \text{---} (3)$$

Let L_2 be the length of CI from (2),

$$L_2 = \frac{\max(X_i)}{\alpha^{1/n}} - \max(X_i) = \max(X_i)\left[\frac{1}{\alpha^{1/n}} - 1 \right]$$

$$\therefore \quad L_2 = \max(X_i)\left[\frac{1-\alpha^{1/n}}{\alpha^{1/n}} \right] \quad \text{---} (4)$$

Multiplying (3) and (4) by $\alpha^{1/n} \cdot (2-\alpha)^{1/n}$, we get:-

$$L_1 (\alpha^{1/n})(2-\alpha)^{1/n} = 2^{1/n} \cdot \max(X_i)\, \cancel{\alpha^{1/n}} \cdot \cancel{(2-\alpha)^{1/n}} \left[\frac{(2-\alpha)^{1/n} - \alpha^{1/n}}{\cancel{\alpha^{1/n}} \cdot \cancel{(2-\alpha)^{1/n}}}\right]$$

$\Rightarrow$

$$L_1 \alpha^{1/n} \cdot (2-\alpha)^{1/n} = 2^{1/n} \cdot \max(X_i) \left[(2-\alpha)^{1/n} - \alpha^{1/n}\right] \text{ --- } (5)$$

And for eq (4),

$$L_2 \cdot (\alpha^{1/n}) \cdot (2-\alpha)^{1/n} = \max(X_i)\, \cancel{\alpha^{1/n}} \cdot (2-\alpha)^{1/n} \cdot \left[\frac{1-\alpha^{1/n}}{\cancel{\alpha^{1/n}}}\right]$$

$\Rightarrow$

$$L_2 \cdot \alpha^{1/n} \cdot (2-\alpha)^{1/n} = (2-\alpha)^{1/n} \cdot \max(X_i) \cdot \left[1-\alpha^{1/n}\right] \text{ --- } (6)$$

Now, to determine a relationship between (5)th and (6)th, we need to find a definite relationship among their individual terms, we might need to use the fact that,

$$0 < \alpha < 1 \text{ --- } (7)$$

Multiplying inequality (7) by -1,

$$0 > -\alpha > -1$$

Add 2 to both sides of the inequality (full inequality),

$$2 > 2-\alpha > 2-1$$

$$\Rightarrow \underline{2 > 2-\alpha > 1}$$

Now, we will take the power $1/n$ on the full inequality,

$$2^{1/n} > (2-\alpha)^{1/n} > 1$$

$$\therefore \quad 1 < (2-\alpha)^{1/n} < 2^{1/n} \text{ --- } (8)$$

From eq^n (5),

$$L_1 \alpha^{1/n} (2-\alpha)^{1/n} = 2^{1/n} \cdot \max(X_i) \cdot \left[(2-\alpha)^{1/n} - \alpha^{1/n}\right]$$

If we use the latter half of the inequality (8) in the above equation, it becomes:-

$$L_1 \alpha^{1/n} \cdot (2-\alpha)^{1/n} > \max(X_i) \cdot (2-\alpha)^{1/n} \left[(2-\alpha)^{1/n} - \alpha^{1/n} \right]$$

$$\left[\because (2-\alpha)^{1/n} < 2^{1/n} \right]$$

Now, from inequality (8), $\because 1 < (2-\alpha)^{1/n}$,

$$L_1 \alpha^{1/n} \cdot (2-\alpha)^{1/n} > (2-\alpha)^{1/n} \cdot \max(X_i) \left[1 - \alpha^{1/n} \right]$$

From eqn (6), we know that RHS of the above inequality becomes,

$$L_1 \alpha^{1/n} \cdot (2-\alpha)^{1/n} > L_2 \alpha^{1/n} (2-\alpha)^{1/n}$$

Dividing both sides of the above inequality by $\alpha^{1/n}(2-\alpha)^{1/n}$ and since both are positive terms, the sign doesn't change:-

$$\boxed{L_1 > L_2}$$

$\therefore$ we have found that L_2 has the narrower length between L_1 and L_2.

The narrower length is $\boxed{L_2 = \max(X_i) \left[\frac{1 - \alpha^{1/n}}{\alpha^{1/n}} \right]}$

Ans.

Using the data provided in the problem as follows:-

$x_1 = 4.2$, $x_2 = 3.5$, $x_3 = 1.7$, $x_4 = 1.2$, $x_5 = 2.5$

and $\gamma = 0.95$ and $n = 5$.

Now, CI with narrower length is $\Rightarrow$

$$CI_2 = \left(\max(X_i), \frac{\max(X_i)}{\alpha^{1/n}} \right) \quad \text{—— (2)}$$

$\because 1 - \alpha = \gamma \Rightarrow 1 - \alpha = 0.95 \Rightarrow \boxed{\alpha = 0.05}$

And,

$\max(X_i) = \max(4.2, 3.5, 1.7, 1.2, 2.4)$

$= 4.2$

Substituting these values in CI-(2),

$$CI = \left(4.2, \frac{4.2}{(0.05)^{1/5}}\right) \quad (\because n = 5)$$

$$\Rightarrow \boxed{CI = (4.2, 7.65)}$$

Ans.

$\therefore$ 95% CI for θ with above values is $\Rightarrow$ $(4.2, 7.65)$

Ans.

4.

Let $X_1, \dots, X_n$ be iid $N(\mu, \sigma^2)$ – distributed.

Also, μ and σ^2 are unknown parameters here.

Now, to construct the CI for this:–

<u>Step (1)</u>:– find a pivot for unknown parameter and to calculate the CI for σ^2, we get the pivot:–

$$\boxed{\Phi = \frac{(n-1) S_n^2}{\sigma^2}}$$ (As shown in the class).

<u>Step (2)</u>:– Determining the distribution of the pivot,

As shown in the class, the distribution of Φ is basically χ^2_{n-1} distribution. (with $n-1$ degrees of freedom).

Now, let q be the quantile function.

$q(\alpha)$ = The value of x such that,

$$\boxed{P\left(\chi^2_{n-1} \leq x\right) = \alpha}$$

<u>Step (3)</u>:–

As we need to construct a confidence interval, we need to choose α_1 and α_2.

And, $\boxed{\alpha_1 + \alpha_2 = 1 - \gamma}$ holds where γ is the confidence level.

Now, χ^2_{n-1}

$$P\left(q_{\chi}(\alpha_1) \leq \Phi(X_1, \dots, X_n; \mu, \sigma^2) \leq q(\chi^2_{n-1}, 1-\alpha_2)\right) = \gamma$$

Substituting the pivot in the equation above,

$$P\left(q(\chi^2_{n-1}, \alpha_1) \le \frac{(n-1)S_n^2}{\sigma^2} \le q(\chi^2_{n-1}, 1-\alpha_2)\right) = \gamma$$

Step ④:- Now, solving for σ^2 as we need to create CI for σ^2.

$$P\left(q(\chi^2_{n-1}, \alpha_1) \le \frac{(n-1)S_n^2}{\sigma^2} \le q(\chi^2_{n-1}, 1-\alpha_2)\right) = \gamma$$

Dividing the inequality by $(n-1)S_n^2$, we get:-

$$P\left(\frac{q(\chi^2_{n-1}, \alpha_1)}{(n-1)S_n^2} \le \frac{1}{\sigma^2} \le \frac{q(\chi^2_{n-1}, 1-\alpha_2)}{(n-1)S_n^2}\right) = \gamma$$

Now, we take reciprocals of the terms in the inequality through cross-products =>

$$P\left(\frac{(n-1)S_n^2}{q(\chi^2_{n-1}, 1-\alpha_2)} \le \sigma^2 \le \frac{(n-1)S_n^2}{q(\chi^2_{n-1}, \alpha_1)}\right) = \gamma$$

Therefore,
the confidence interval is =>

$$\left[\frac{(n-1)S_n^2}{q(\chi^2_{n-1}, 1-\alpha_2)}, \frac{(n-1)S_n^2}{q(\chi^2_{n-1}, \alpha_1)}\right]$$

Ans.

Now, for the calculation part,

$\gamma = 0.99$, $n = 15$ and $S_n^2 = 20$

We need a CI of the form $(0, R_n)$.

$\because \alpha_1 + \alpha_2 = 1 - \gamma \Rightarrow \alpha_1 + \alpha_2 = 1 - 0.99 = 0.01$

for the $(0, R_n)$ interval, the lower bound of the CI found should be zero,

$$\therefore \frac{(n-1) S_n^2}{q(\chi^2_{n-1}, 1-\alpha_2)} = 0$$

$\because S_n$ and n, both are finite and $n \neq 1$,

$$\Rightarrow \left(\therefore q(\chi^2_{n-1}, 1-\alpha_2) = \infty \right)$$

$\because$ we know that:- $q(1) = \infty$,

$\therefore 1 - \alpha_2 = 1 \Rightarrow \boxed{\alpha_2 = 0}$

$\therefore \alpha_1 + 0 = 0.01 \Rightarrow \boxed{\alpha_1 = 0.01}$

$$\therefore R_n \Rightarrow \frac{(n-1) S_n^2}{q(\chi^2_{n-1}, 0.01)}$$

$$CI \Rightarrow \left(0, \frac{(n-1) S_n^2}{q(\chi^2_{n-1}, 0.01)} \right)$$

Putting in the values,

$(n-1)$ degrees of freedom $\Rightarrow (15-1) = 14$ degrees of freedom.

$$\therefore CI \Rightarrow \left(0, \frac{14 \times (20)^2}{q(\chi^2_{15-1}, 0.01)}\right)$$

$$\Rightarrow \left(0, \frac{14 \times 20^2}{4.66}\right) \quad \left[\because q(\chi^2_{14}, 0.01) = 4.66\right]$$

$$\Rightarrow \left(0, \frac{5600}{4.66}\right)$$

$$\therefore CI \Rightarrow (0, 1201.7167)$$

$$\Rightarrow \boxed{(0, 1201.72)}$$

Ans

(5.) $\because n = 50$, $\bar{X}_n = 634.16$ and $S_n = 164.43$

Here σ^2 and μ are unknown variables.

Now, we will try to construct the confidence interval for $\mu \Rightarrow$.

Step (1):- find a pivot for the unknown parameters,

$$\Phi(X_1, \ldots, X_n; \mu, \sigma^2) = \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}}$$ $\longrightarrow$ This will be the approximate pivot when considering n as large.

Step (2):- finding the distribution of the pivot,

The distribution of Φ is approximately $N(0,1)$ [as shown in class]

$$\boxed{\Phi = \frac{\bar{X}_n - \mu}{S_n/\sqrt{n}} \overset{approx}{\sim} N(0,1)}$$

Let q be the quantile function,

$q(\alpha) =$ The value of x such that,

$$\boxed{P(N(0,1) \leq x) = \alpha}$$

Step (3):- As we need to construct a confidence interval, we need to choose α_1 & α_2, such that,

$$\boxed{\alpha_1 + \alpha_2 = 1 - \gamma}$$

Now,

$$P\left(q(N(0,1),\alpha_1) \le \Phi(x_1,\dots,x_n;\sigma^2,\mu) \le q(N(0,1),1-\alpha_2)\right) = \gamma$$

$\Rightarrow$

$$\boxed{P\left(q(N(0,1),\alpha_1) \le \frac{\bar{X}_n-\mu}{S_n/\sqrt{n}} \le q(N(0,1),1-\alpha_2)\right)=\gamma} \quad \text{--- (1)}$$

Step (4) :- Solving for μ,

Multiplying the inequality in eqn (1) above by $S_n/\sqrt{n}$,

$$P\left(\frac{S_n}{\sqrt{n}}q(N(0,1),\alpha_1) \le \bar{X}_n-\mu \le \frac{S_n}{\sqrt{n}}q(N(0,1),1-\alpha_2)\right)=\gamma$$

Subtracting $\bar{X}_n$ from all sides of the inequality above,

$$P\left(\frac{S_n}{\sqrt{n}}q(N(0,1),\alpha_1)-\bar{X}_n \le -\mu \le \frac{S_n}{\sqrt{n}}q(N(0,1),1-\alpha_2)-\bar{X}_n\right)=\gamma$$

Multiplying all the sides by -1, we get:-

$$\underbrace{P\left(\bar{X}_n-\frac{S_n}{\sqrt{n}}q(N(0,1),1-\alpha_2) \le \mu \le \bar{X}_n-\frac{S_n}{\sqrt{n}}q(N(0,1),\alpha_1)\right)=\gamma}_{(2)}$$

$\because$ $\gamma = 0.95$ (95%) and $\alpha_1+\alpha_2 = 1-\gamma$,

we get $\Rightarrow$ $\alpha_1+\alpha_2 = 1-0.95 \Rightarrow \boxed{\alpha_1+\alpha_2 = 0.05}$

We consider this to be a balanced CI,

$$(\alpha_1=\alpha_2) \Rightarrow \alpha_1 + \alpha_1 = 0.05$$

$$\Rightarrow 2\alpha_1 = 0.05 \Rightarrow \boxed{\alpha_1 = 0.025}$$

$$\boxed{\alpha_1=\alpha_2=0.025}$$

Putting all the values in the inequality ② above,

$$654.16 - \frac{164.43}{\sqrt{50}}\, q(N(0,1), 0.975) \le \mu \le 654.16 - \frac{164.43}{\sqrt{50}}\, q(N(0,1), 0.025)$$

$$\because q(N(0,1), 0.975) = \underline{1.96}$$

and $q(N(0,1), 0.975) = -q(N(0,1), 0.025)$

$$\therefore q(N(0,1), 0.025) = \underline{\underline{-1.96}}$$

Therefore,

$$654.16 - \frac{164.43}{\sqrt{50}} \times (1.96) \le \mu \le 654.16 - \frac{164.43}{\sqrt{50}} \times (-1.96)$$

$$\Rightarrow \boxed{608.58 \le \mu \le 699.74}$$

Ans.

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(b.) $\sigma = 175$, $L = 50$ ppm , $\gamma = 0.95$

for constructing the CI for this:-

Step (1):- find a pivot for the unknown parameters,

$$\Phi(X_1, \dots, X_n; \mu, \sigma^2) = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \quad \text{(Approx. pivot)}.$$

Step (2):- find the distribution of the pivot,

The distribution for Φ is $N(0,1)$.

$$\boxed{\Phi = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)}$$

Let q be its quantile function,

$q(\alpha) =$ The value of x such that:-

$$P(N(0,1) \le x) = \alpha$$

Step (3):- Choosing α_1 and α_2 as we need to construct bounds for CI such that:-

$$\boxed{\alpha_1 + \alpha_2 = 1 - \gamma}$$

$$P(q(\alpha_1) \le \Phi(X_1, \dots, X_n; \mu) \le q(1-\alpha_2)) = \gamma$$

Substituting the value of the pivot,

$$P\left(q(N(0,1), \alpha_1) \le \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} \le q(N(0,1), 1-\alpha_2)\right) = \gamma$$

Step (4):- Solving the inequality for μ,

$$P\left(\frac{\sigma}{\sqrt{n}}\, q(N(0,1),\alpha_1) \le \bar{X}_n - \mu \le \frac{\sigma}{\sqrt{n}}\, q(N(0,1),1-\alpha_2)\right) = \gamma$$

$$P\left(\frac{\sigma}{\sqrt{n}}\, q(N(0,1),\alpha_1) - \bar{X}_n \le -\mu \le \frac{\sigma}{\sqrt{n}}\, q(N(0,1),1-\alpha_2) - \bar{X}_n\right) = \gamma$$

$\Rightarrow$

$$P\left(\bar{X}_n - \frac{\sigma}{\sqrt{n}}\, q(N(0,1),1-\alpha_2) \le \mu \le \bar{X}_n - \frac{\sigma}{\sqrt{n}}\, q(N(0,1),\alpha_1)\right) = \gamma$$

$$\therefore \text{CI} \Rightarrow \left(\bar{X}_n - \frac{\sigma}{\sqrt{n}}\, q(N(0,1),1-\alpha_2),\ \bar{X}_n - \frac{\sigma}{\sqrt{n}}\, q(N(0,1),\alpha_1)\right)$$

$\therefore$ Length of CI $(L) \Rightarrow$

$$= \left[\cancel{\bar{X}_n} - \frac{\sigma}{\sqrt{n}}\, q(N(0,1),\alpha_1) - \left(\cancel{\bar{X}_n} - \frac{\sigma}{\sqrt{n}}\, q(N(0,1),1-\alpha_2)\right)\right]$$

$$= \frac{\sigma}{\sqrt{n}}\, q(N(0,1),1-\alpha_2) - \frac{\sigma}{\sqrt{n}}\, q(N(0,1),\alpha_1)$$

$$L = \frac{\sigma}{\sqrt{n}}\left[q(N(0,1),1-\alpha_2) - q(N(0,1),\alpha_1)\right]$$

$$\because q(N(0,1),1-\alpha_2) = -q(N(0,1),\alpha_1)$$

$$\therefore L = \frac{2\sigma}{\sqrt{n}}\, q(N(0,1),1-\alpha_2)$$

$$\because \alpha_1 + \alpha_2 = 1-\gamma \Rightarrow \alpha_1 + \alpha_2 = 1 - 0.95 = 0.05$$

$\therefore$ Balanced CI $(\alpha_1 = \alpha_2) \Rightarrow$ $\alpha_1 = \alpha_2 = \frac{0.05}{2} = 0.025$

for n,

$$n = \left(\frac{2\sigma}{L} q\left(N(0,1), 1-\alpha_2\right)\right)^2$$

$$n = \left(\frac{2 \times 175}{50} q\left(N(0,1), 1-0.025\right)\right)^2$$

$\Rightarrow$

$$n = \left(\frac{350}{50} \times 1.96\right)^2 \quad [\because q(N(0,1), 0.975) = 1.96]$$

$$n = 188.2384$$

(Rounded up)

$$\Rightarrow \boxed{n \simeq 189}$$

$\therefore$ Repuired sample size $\Rightarrow$ 189

Ans.

6. Let $X_1, \ldots, X_n$ form a random sample from the exponential distribution with unknown parameter λ.

We know that the pdf $f(x)$ for the exponential distribution is given by:-

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , x > 0 \\ 0 & , \text{otherwise} \end{cases}$$

Here, λ is the unknown parameter.

Now, let $y = 2\lambda X$,

the pdf $g(y)$ of the exponential distribution can now be represented as:-

$$g(y) = \begin{cases} 1/2 \, e^{-y/2} & , y > 0 \\ 0 & , \text{otherwise} \end{cases} \quad \text{—— (1)}$$

Also, we know that,

$$\text{Gamma}(1, \lambda) = \text{Exp}(\lambda) \quad \text{—— (2)}$$

and, $$\chi^2_m = \text{Gamma}\left(\frac{m}{2}, \frac{1}{2}\right) \quad \text{—— (3)}$$

from the cheatsheet provided.

Putting $m = 2$ in equation (3), we get:-

$$\chi^2_2 = \text{Gamma}(1, 1/2)$$

Using equation (2),

$$\chi^2_2 = \text{Gamma}(1, 1/2) = \text{Exp}(1/2) \quad \text{—— (4)}$$

There are huge similarities between the pdf $f(x)$ and pdf $g(y)$. It is clear that $g(y)$ follows an exponential distribution but with parameter $1/2$, since

$f(x) = \lambda e^{-\lambda x}$ $\quad x > 0$ $\quad$ [0, otherwise]

$g(y) = \frac{1}{2} e^{-y/2}$ $\quad y > 0$ $\quad$ [0, otherwise]

from equation (4), we can come to the conclusion that $f(y)$ has a chi-squared distribution with degree of freedom 2.

Now, let's try to construct the required confidence intervals.

Step (1):– find an appropriate pivot for the unknown parameter,

$$\boxed{I(X_1, \ldots, X_n; \lambda) = 2\lambda \sum_{i=1}^{n} X_i}$$
$$= \sum_{i=1}^{n} y_i \qquad [\because y = 2\lambda x \text{ as assumed before}]$$

Step (2):– find the distribution of the pivot,

As seen above,

$$I(X_1, \ldots, X_n; \lambda) = 2\lambda \sum_{i=1}^{n} X_i = \sum_{i=1}^{n} y_i$$

As discussed before, $y_i = 2\lambda X_i$ and it follows an exponential distribution with parameter 1/2. This is equivalent to chi-squared distribution with 2 degrees of freedom.

Further all y_i's are independent, $\therefore$, the pivot I follows a chi-squared distribution with $2n$ degrees of freedom.

$$\Phi(X_1, \ldots, X_n; \lambda) \sim \chi^2_{2n}$$

Let q be its quantile function such that:-

$q(\alpha) =$ the value of x such that:-

$$P(\chi^2_{2n} \leq x) = \alpha$$

Step ③:- Choose values of α_1 and α_2 such that:-

$$\boxed{\alpha_1 + \alpha_2 = 1 - \gamma}$$

$$P(q(\chi^2_{2n}, \alpha_1) \leq \Phi(X_1, \ldots X_n; \lambda) \leq q(\chi^2_{2n}, 1-\alpha_2)) = \gamma$$

Substituting the value of the pivot in the equation above:-

$$P(q(\chi^2_{2n}, \alpha_1) \leq 2\lambda \sum_{i=1}^{n} X_i \leq q(\chi^2_{2n}, 1-\alpha_2)) = \gamma$$

Step ④:- Solve the inequality for λ, we get:-

Dividing both sides by $2\sum_{i=1}^{n} X_i$, we get:-

$$P\left(\frac{1}{2\sum_{i=1}^{n} X_i} q(\chi^2_{2n}, \alpha_1) \leq \lambda \leq \frac{1}{2\sum_{i=1}^{n} X_i} q(\chi^2_{2n}, 1-\alpha_2)\right) = \gamma$$

$\therefore$ The CI is $\Rightarrow \left(\dfrac{q(\chi^2_{2n}, \alpha_1)}{2\sum_{i=1}^{n} X_i}, \dfrac{q(\chi^2_{2n}, 1-\alpha_2)}{2\sum_{i=1}^{n} X_i} \right)$

Ans.

⑦ Sample mean $(\bar{X}_n) = 3.2$ Here the unknown parameter is $\underline{\underline{\theta}}$.

$n = 100$ (100 iid poisson variables).

Step(1):
Since, we want to find an approximate confidence interval, we can consider the pivot to be $\Rightarrow$

$$\boxed{\Phi(X_1, \ldots X_n; \mu, \sigma) \Rightarrow \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}}}$$

Step(2): finding the distribution of the pivot,
$\because$ we have taken the above pivot as an approximate considering that n is large,

$$\boxed{\Phi = \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \sim N(0,1)}$$

Let $q(\alpha)$ be the quantile function such that:-

$$\underline{P(N(0,1) \leq x) = \alpha}$$

Step (3): But first we need to find μ and σ for the poisson distribution, ($\because$ this is a discrete function)

$$\therefore E[X] \stackrel{!}{=} \mu \Rightarrow \boxed{E[X] = \sum_{x \geq 0} x f(x)}$$

for Poisson distribution, we know that:

$$\boxed{f(x) = \frac{e^{-\theta} \theta^{x}}{x!}}$$

$$\therefore E[X] = \sum_{x \geq 0} \frac{x \cdot e^{-\theta} \cdot \theta^x}{x!} = \sum_{x \geq 0} \frac{\cancel{x} \cdot e^{-\theta} \cdot \theta^x}{\cancel{x} \cdot ((x-1)!)}$$

$$\Rightarrow E[X] = \sum_{x \geq 1} \frac{e^{-\theta} \cdot \theta^x}{(x-1)!} = e^{-\theta} \cdot \theta \sum_{x \geq 1} \frac{\theta^{x-1}}{(x-1)!}$$

By Taylor series expansion,

$$\boxed{\sum_{x \geq 1} \frac{\theta^{x-1}}{(x-1)!} = e^{\theta}}$$

$$\therefore \Rightarrow E[X] = \cancel{e^{-\theta}} \cdot \theta \cdot \cancel{e^{\theta}} = \theta$$

$$\boxed{\therefore E[X] = \theta}$$

Now, we need to compute $V[X]$ (Variance),

$$\boxed{V[X] = E[X^2] - (E[X])^2} \longleftarrow (1)$$

$\because$ we know $E[X]$, let's compute $E[X^2]$.

$$E[X^2] = \sum_{x \geq 0} x^2 f(x) = \sum_{x \geq 0} \frac{x^2 \cdot e^{-\theta} \cdot \theta^x}{x!}$$

$$= \sum_{x \geq 0} \frac{\cancel{x} \cdot x \cdot e^{-\theta} \cdot \theta^{x-1} \cdot \theta}{\cancel{x}((x-1)!)} = e^{-\theta} \cdot \theta \sum_{x \geq 1} \frac{x \theta^{x-1}}{(x-1)!}$$

$\Rightarrow$

$$E[X^2] = e^{-\theta} \cdot \theta \sum_{x \geq 1} \frac{(x-1+1)\,\theta^{x-1}}{(x-1)!} = e^{-\theta} \cdot \theta \left[\sum_{x \geq 1} \frac{(x-1) \cdot \theta^{x-1}}{(x-1)!} + \sum_{x \geq 1} \frac{1 \cdot \theta^{x-1}}{(x-1)!} \right]$$

By Taylor series expansion, we know that:

$$\sum_{x \geq 1} \frac{\theta^{x-1}}{(x-1)!} = e^{\theta}$$

$$\therefore \Rightarrow E[X^2] = e^{-\theta} \cdot \theta \left[\sum_{x \geq 1} \frac{\cancel{(x-1)} \cdot \theta^{x-2} \cdot \theta}{\cancel{(x-1)}(x-2)!} + e^{\theta} \right]$$

$$\Rightarrow E[X^2] = e^{-\theta} \cdot \theta \left[\theta \underbrace{\sum_{x \geq 2} \frac{\theta^{x-2}}{(x-2)!}}_{\boxed{e^{\theta}}} + e^{\theta} \right]$$

$$\Rightarrow E[X^2] = e^{-\theta} \cdot \theta \left[\theta e^{\theta} + e^{\theta} \right] = \cancel{e^{-\theta}} \cdot \theta \cdot \cancel{e^{\theta}} \left[\theta + 1 \right]$$
$$= \theta(\theta + 1)$$

Putting values in (1),

$$V[X] = \theta(\theta+1) - (\theta)^2 = \cancel{\theta^2} + \theta - \cancel{\theta^2} = \theta$$

$$\boxed{\therefore V[X] = \theta}$$

Also, we will choose α_1 and α_2 as we need to construct bounds for CI such that:-

$$\boxed{\alpha_1 + \alpha_2 = 1 - \gamma}$$

$$P(q(N(0,1), \alpha_1) \leq I(X_1, \ldots, X_n; \mu, \sigma) \leq q(N(0,1), 1-\alpha_2)) = \gamma$$

Substituting the value of the pivot,

$$P\left(q(N(0,1),\alpha_1) \le \frac{\bar{X}_n - \mu}{\sigma/\sqrt{n}} \le q(N(0,1),1-\alpha_2)\right) = \gamma$$

Step ④:- Now, we can put in the values for μ and σ there. (And start solving for θ).

~~Also, by the law of large numbers (CLT), we know that sample mean becomes the actual mean population mean when n is large, which is the case here.~~

As computed before, $E[X] = \mu = \theta$ and $V[X] = \sigma^2 = \theta$,

Thus,

$$P\left(q(N(0,1),\alpha_1) \le \frac{\bar{X}_n - \theta}{\sqrt{\theta}/\sqrt{n}} \le q(N(0,1),1-\alpha_2)\right) = \gamma$$

Multiplying inequality by $\sqrt{\frac{\theta}{n}}$,

$$P\left(\sqrt{\frac{\theta}{n}}\, q(N(0,1),\alpha_1) \le \bar{X}_n - \theta \le \sqrt{\frac{\theta}{n}}\, q(N(0,1),1-\alpha_2)\right) = \gamma$$

Subtracting $\bar{X}_n$ from the inequality,

$$P\left(\sqrt{\frac{\theta}{n}}\, q(N(0,1),\alpha_1) - \bar{X}_n \le -\theta \le \sqrt{\frac{\theta}{n}}\, q(N(0,1),1-\alpha_2) - \bar{X}_n\right) = \gamma$$

Multiplying inequality by -1,

$$P\left(\bar{X}_n - \sqrt{\frac{\theta}{n}}\, q(N(0,1),1-\alpha_2) \le \theta \le \bar{X}_n - \sqrt{\frac{\theta}{n}}\, q(N(0,1),\alpha_1)\right) = \gamma$$

Now, by the law of large numbers (CLT), we know that sample mean becomes the population mean and $\therefore$,

$E[X] = V[X] = \theta$, $\therefore$ $\underline{\theta = \bar{X}_n}$,

So,
it becomes,

$$P\left(\bar{X}_n - \sqrt{\frac{\bar{X}_n}{n}}\, q(N(0,1), 1-\alpha_2) \le \theta \le \bar{X}_n - \sqrt{\frac{\bar{X}_n}{n}}\, q(N(0,1), \alpha_1)\right) = \gamma$$

Also,

$\because \alpha_1 + \alpha_2 = 1 - \gamma$ and $\gamma = 0.95$

$\Rightarrow \alpha_1 + \alpha_2 = 1 - 0.95 = 0.05$

for a balanced CI,

$$\boxed{\alpha_1 = \alpha_2}$$

$\therefore\ 2\alpha_1 = 0.05 \Rightarrow \boxed{\alpha_1 = 0.025}$

$\Rightarrow \boxed{\alpha_1 = \alpha_2 = 0.025}$

To calculate 95% CI, we substitute all the value for the confidence interval,

$$CI \Rightarrow \left(\bar{X}_n - \sqrt{\frac{\bar{X}_n}{n}}\, q(N(0,1), 1-\alpha_2) \le \theta \le \bar{X}_n - \sqrt{\frac{\bar{X}_n}{n}} \cdot q(N(0,1), \alpha_1)\right)$$

$$CI \Rightarrow \left(3.2 - \sqrt{\frac{3.2}{100}}\, q(N(0,1), 1-0.025) \le \theta \le 3.2 - \sqrt{\frac{3.2}{100}}\, q(N(0,1), 0.025)\right)$$

$\because q(N(0,1), 1-0.025) = q(N(0,1), 0.975) = \underline{1.96}$

and

$q(N(0,1), 0.025) = \underline{-1.96}$

$\therefore$

$\Rightarrow$

$$CI = \left(3.2 - 1.96 \times \sqrt{\frac{3.2}{100}} \leq \theta \leq 3.2 + 1.96\sqrt{\frac{3.2}{100}}\right)$$

$$\Rightarrow \boxed{2.8494 \leq \theta \leq 3.5506}$$

$\therefore$ 95%-CI for θ of the poisson distribution

is $\Rightarrow (2.8494, 3.5506)$

$= \underline{(2.85, 3.55)}$

Ans.