Sheet 3: Combinatorics, Conditional Probability and Independence

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NOTE: All results should be rounded to two decimal places unless otherwise stated. If a number or result has fewer decimal places, it is okay to keep fewer. For probabilities, give two decimal places when expressed in percentage (e.g., 12.34%) and four decimal places when expressed as numbers (e.g., 0.1234).

Exercise 1
[D, Section 2.4, Exercise 45]

Exercise 2
[D, Section 2.4, Exercise 50]

Exercise 3
[D, Section 2.5, Exercise 88]

Exercise 4
Suppose there are n students in this class. We are interested in the probability of the event A that at least two students have the same birthday. For simplicity, we assume that no student was born on February 29 and that all students were born independently of each other (no twins!). We also assume that each day in the year is equally probable for a birthday.

(a) The instructor asks all students for their birthdays. How many possible outcomes are there?

(b) Describe in words what A' means.

(c) Compute $P(A')$ in dependence of n.

(d) Compute $P(A)$ in dependence of n.

If you have a good calculator, you can verify that $P(A) \geq 50\%$ already at $n = 23$ and that $P(A) \geq 99\%$ already at $n = 57$.

(e) Birth statistics show that not each day in the year is equally likely as a birthday, see the following chart in Figure e. Give a plausible explanation for the Top 10 and the Bottom 10.

Exercise 5
[D, Section 2.3, Exercise 30]
It suffices to state the results as a sum/product of numbers, factorials, binomial coefficients ... For example, $6 \times 9!$ is sufficient, no need to calculate this further. Read Example 2.23 before solving this exercise.

Exercise 6
Consider the Monty Hall problem, except that when Monty has a choice between opening doors 2 and 3, he opens door 2 with probability p, where $\frac{1}{2} \leq p \leq 1$. To recap: there are three doors, behind one of which there is a car (which you want), and behind the other two of which there are goats (which you don't want). Initially, all possibilities are equally likely for where the car is. You

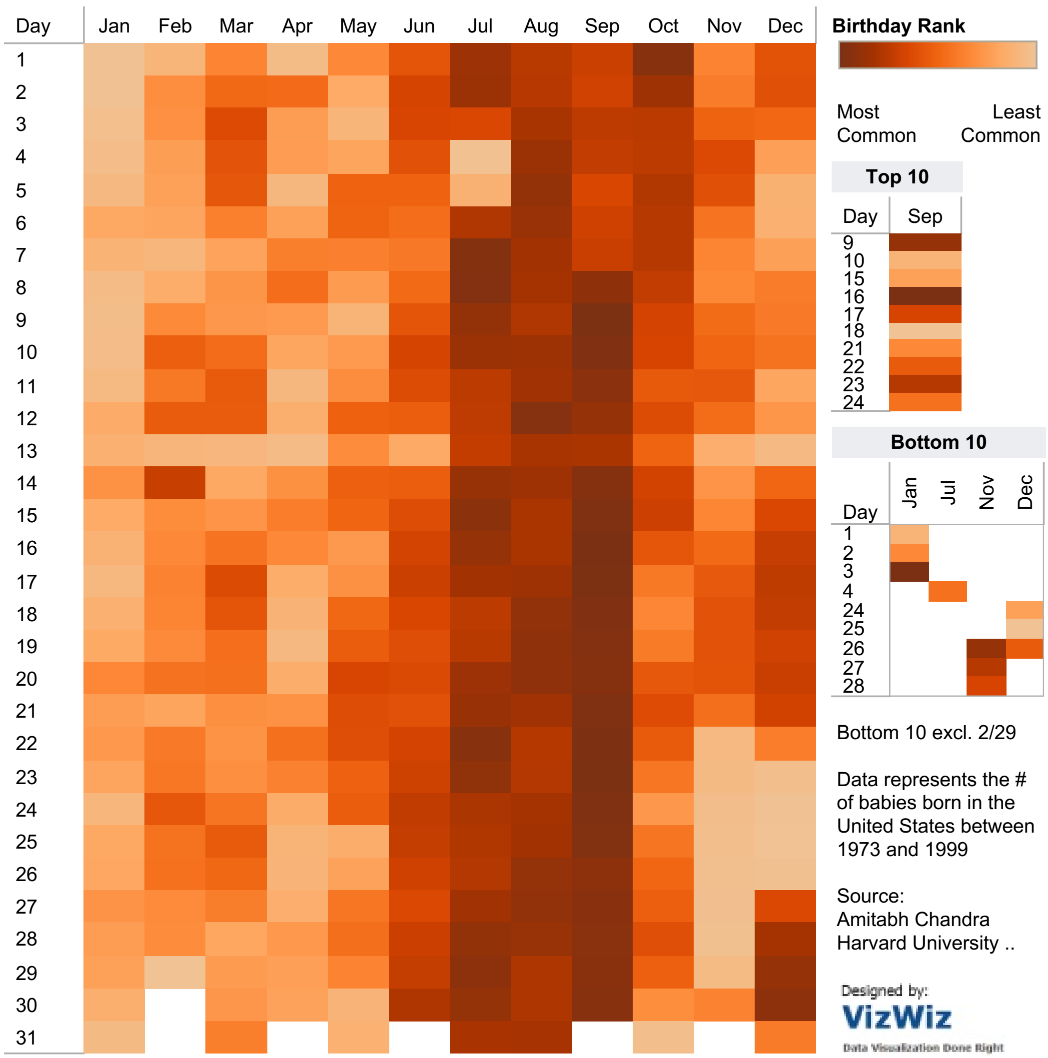

Figure 1: Birth statistics chart

choose a door, which for concreteness we assume is door 1. Monty Hall then opens a door to reveal a goat, and offers you the option of switching. Assume that Monty Hall knows which door has the car, will always open a goat door and offer the option of switching, and as above assume that if Monty Hall has a choice between opening door 2 and door 3, he chooses door 2 with probability p.

(a) Find the unconditional probability that the strategy of always switching succeeds (unconditional in the sense that we do not condition on which of doors 2 or 3 Monty opens).

(b) Find the probability that the strategy of always switching succeeds, given that Monty opens door 2.

(c) Find the probability that the strategy of always switching succeeds, given that Monty opens door 3.

(d) Given that Monty opens door 2, is there a value of p such that switching has a lower success probability than staying with the initial choice?

(e) Given that Monty opens door 3, is there a value of p such that switching has a lower success probability than staying with the initial choice?

Exercise 7

You are the contestant on the Monty Hall show. Monty is trying out a new version of his game, with rules as follows. You get to choose one of three doors. One door has a car behind it, another has a computer, and the other door has a goat (with all permutations equally likely). Monty, who knows which prize is behind each door, will open a door (but not the one you chose) and then let you choose whether to switch from your current choice to the other unopened door. Assume that you prefer the car to the computer, the computer to the goat, and the car to the goat.

(a) Suppose for this part only that Monty always opens the door that reveals your less preferred prize out of the two alternatives, e.g., if he is faced with the choice between revealing the goat or the computer, he will reveal the goat. Monty opens a door, revealing a goat (this is again for this part only). Given this information, should you switch? If you do switch, what is your probability of success in getting the car?

(b) Now suppose that Monty reveals your less preferred prize with probability $p \in [0, 1]$, and your more preferred prize with probability $q = 1 - p$. Monty opens a door, revealing a computer. Given this information, should you switch (your answer can depend on p)? If you do switch, what is your probability of success in getting the car (in terms of p)?

Exercise 8

(a) In the vaccine example of the class, find A, B and C such that

$$\frac{P(A \mid B)}{P(A \mid B')} > \frac{P(A \mid B, C)}{P(A \mid B', C)} \qquad \text{and} \qquad \frac{P(A \mid B)}{P(A \mid B')} > \frac{P(A \mid B, C')}{P(A \mid B', C')}.$$

This is a weaker version of Simpson's paradox. You need to verify that these properties are satisfied. For this exercise, consider relative frequencies as the same as probabilities.

(b) Show that the events from (b) do *not* satisfy the stronger definition from the class, which is

$$P(A \mid B, C) < P(A \mid B', C) \quad \text{and} \quad P(A \mid B, C') < P(A \mid B', C') \quad \text{but} \quad P(A \mid B) > P(A \mid B').$$

Note: Under the weaker version of Simpson's paradox, the ratio of $P(A \mid B)/P(A \mid B')$ decreases *both* when we condition on C *and* when we condition on C'. Under the stronger version, this ratio even decreases from a value larger than 1 to a value smaller than 1 under conditioning.

Exercise 9

(a) How many possibilities are there to give 30 identical cookies to 4 kids?

(b) How many possibilities are there to give 30 identical cookies to 4 kids if you make sure that each kids gets at least one cookie?

(c) Now you give 30 cookies to 40 kids. Because demand exceeds supply, you give at most one cookie to each kid. How many possibilities do you have?

Write your solution in terms of binomial coefficients. No justification needed.