HW 8

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1. The pdf function is provided as:-

$$f(x;\theta) = \begin{cases} (\theta+1)x^{\theta} & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

To calculate the maximum likelihood estimator of θ, firstly,

$$\mathcal{L}_n(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{10} (\theta+1)x_i^{\theta}$$

As $(\theta+1)$ does not depend on i but it will multiplied 10 times, we can take that out of the product formulation,

$$\mathcal{L}_n(\theta) = (\theta+1)^{10} \prod_{i=1}^{10} x_i^{\theta}$$

Now, we take log on both the sides,

$$\log(\mathcal{L}_n(\theta)) = 10 \log(\theta+1) + \theta \sum_{i=1}^{10} \log x_i$$

Now we differentiate this w.r.t θ,

$$\frac{d(\log(\mathcal{L}_n(\theta)))}{d\theta} = \frac{10}{\theta+1} \cdot 1 + \sum_{i=1}^{10} \log x_i$$

We will equate this to zero to find the maxima of $\mathcal{L}_n(\theta)$ occurs at what θ which will be the maximum likelihood estimator of θ,

$$\Rightarrow \frac{10}{\theta+1} = -\sum_{i=1}^{10} \log x_i$$

$$\Rightarrow \frac{10}{-\sum_{i=1}^{10} \log x_i} = \theta + 1 \Rightarrow \theta = \frac{-10}{\sum_{i=1}^{10} \log x_i} - 1$$

$$\Rightarrow \boxed{\hat{\theta}_{MLE} = \frac{-10}{\sum_{i=1}^{10} \log x_i} - 1} \quad \text{or} \quad \boxed{\hat{\theta}_{MLE} = \frac{-n}{\sum_{i=1}^{n} \log x_i} - 1}$$

Ans.

Taking the value of x_i's, to find the estimate of θ,

$$\theta = \frac{-10}{\sum_{i=1}^{10} \log x_i} - 1 = \frac{-10}{\log x_1 + \log x_2 + \ldots + \log x_{10}} - 1$$

$$= \frac{-10}{(\log(0.92) + \log(0.79) + \log(0.90) + \log(0.65) + \log(0.86) + \log(0.47) + \log(0.73) + \log(0.97) + \log(0.94) + \log(0.77))} - 1$$

$$= \frac{-10}{\log(0.92 \times 0.79 \times 0.9 \times 0.65 \times 0.86 \times 0.47 \times 0.73 \times 0.97 \times 0.94 \times 0.77)} - 1$$

[Using the property of logarithms]

$$\theta = \frac{-10}{\log(0.0881)} - 1 = \frac{-10}{-2.4293} - 1$$

$$= 4.1164 - 1 = 3.1164 \approx 3.12$$

$\therefore$ The estimate of the given data $(\theta) = \boxed{3.12}$

Ans.

(2.) $X \rightarrow$ error in making a measurement of a physical characteristic or property

$E[X] = 0$, $X \rightarrow$ normal distribution.

pdf:-

$$f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta} \quad -\infty < x < \infty$$

n independent measurements made,

$X_1 = x_1, X_2 = x_2, \ldots, X_n = x_n$

Now, to compute MLE of θ,

$$L_n(\theta) = \prod_{i=1}^{n} f(x_i, \theta)$$

$$= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\theta}} \cdot e^{-x_i^2/2\theta}$$

$\because \frac{1}{\sqrt{2\pi\theta}}$ does not depend on i, we can take it out and as it will be multiplied n times, we take it as to the power of n.

$$= \frac{1}{(2\pi\theta)^{n/2}} \prod_{i=1}^{n} e^{-x_i^2/2\theta}$$

If we multiply e raised to the power something, then the powers get added,

$$\therefore \quad \mathcal{L}_n(\theta) = \frac{1}{(2\pi\theta)^{n/2}} \prod_{i=1}^{n} e^{-x_i^2/2\theta} = \frac{1}{(2\pi\theta)^{n/2}} \cdot e^{-\sum_{i=1}^{n} x_i^2/2\theta}$$

Now, taking logarithm on both the sides ($\ln$), we get:-

$$\ln \mathcal{L}_n(\theta) = -\frac{n}{2}\ln(2\pi\theta) - \frac{1}{2\theta}\sum_{i=1}^{n} x_i^2$$

Now, we will differentiate both the sides w.r.t θ,

$$\Rightarrow \frac{d\left(\ln(\mathcal{L}_n(\theta))\right)}{d\theta} = \underbrace{-\frac{n}{2}\cdot\left(\frac{1}{2\pi\theta}\right)\cdot 2\pi}_{} + \frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2$$

↳ By chain rule,

$$\left(\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx}\right)$$

Now, we will set this equal to 0, we get:-

$$-\frac{n}{2}\cdot\frac{1}{\theta} + \frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2 = 0$$

$$\frac{n}{2\theta} = \frac{1}{2\theta^2}\sum_{i=1}^{n} x_i^2$$

$\because$ θ cannot be equal to zero,

$$\boxed{\hat{\theta}_{MLE} = \frac{1}{n}\sum_{i=1}^{n} x_i^2}$$

To get the estimate of θ, we substitute the measurement errors:-

$$\boxed{\theta = \frac{1}{n}\left(x_1^2 + x_2^2 + \dots + x_n^2\right)}$$

Ans.

3. (b) $X_1, \dots, X_n$ are iid with pdf:-

$$f(x) = \frac{1}{x\sqrt{2\pi\theta_2}} e^{-\frac{(\log x - \theta_1)^2}{2\theta_2}}, \quad -\infty < x < \infty$$

with unknown parameters θ_1 and θ_2.

We have to find the MLE estimators of θ_1 and θ_2.

Step 1:-

$$L_n(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i; \theta_1, \theta_2)$$

$$= \prod_{i=1}^{n} \frac{1}{x_i\sqrt{2\pi\theta_2}} \cdot e^{-\frac{(\log x_i - \theta_1)^2}{2\theta_2}}$$

$$= \left(\frac{1}{(2\pi\theta_2)^{n/2}}\right) \cdot \prod_{i=1}^{n} \frac{1}{x_i} \cdot e^{-\frac{(\log x_i - \theta_1)^2}{2\theta_2}}$$

$$= \left(\frac{1}{(2\pi\theta_2)^{n/2}}\right) \cdot \left(\prod_{i=1}^{n} \frac{1}{x_i}\right) \cdot \left(e^{-\sum_{i=1}^{n} \frac{(\log x_i - \theta_1)^2}{2\theta_2}}\right)$$

Taking log on both the sides, we get:-

Step 2:-

$$\log(\mathcal{L}_n(\theta_1,\theta_2)) = -\frac{n}{2}\log(2\pi\theta_2) - \sum_{i=1}^{n}\log x_i - \frac{1}{2\theta_2}\sum_{i=1}^{n}(\log x_i - \theta_1)^2 \quad (1)$$

To find the derivative w.r.t θ_1, we get:-

$$\frac{d(\log(\mathcal{L}_n(\theta_1,\theta_2)))}{d\theta_1} = \frac{d}{d\theta_1}\left[-\frac{n}{2}\log(2\pi\theta_2) - \sum_{i=1}^{n}\log x_i - \frac{1}{2\theta_2}\sum_{i=1}^{n}(\log x_i - \theta_1)^2\right]$$

($\because$ ~~2nd and~~ ~~first 2~~ first 2 terms in RHS does not depend on θ_1, it's derivatives becomes zero.)

$$= \left[0 - 0 + \frac{d}{d\theta_1}\left(-\frac{1}{2\theta_2}\sum_{i=1}^{n}(\log x_i - \theta_1)^2\right)\right]$$

$$= \frac{d}{d\theta_1}\left[-\frac{1}{2\theta_2}\sum_{i=1}^{n}(\log^2 x_i + \theta_1^2 - 2\theta_1\log x_i)\right]$$

(Using $(a+b)^2 = a^2 + b^2 + 2ab$)

Now,

$$\Rightarrow -\frac{1}{2\theta_2}\frac{d}{d\theta_1}\left[\sum_{i=1}^{n}(\log^2 x_i - 2\theta_1\log x_i) + n\theta_1^2\right]$$

$$\Rightarrow -\frac{1}{2\theta_2}\left[0 - 2\sum_{i=1}^{n}\log x_i + 2n\theta_1\right]$$

$$\Rightarrow \frac{1}{2\theta_2}\left[2\sum_{i=1}^{n}\log x_i - 2n\theta_1\right]$$

Now, equating the above expression to 0, we get:-

$$\frac{1}{\theta_2}\left[\sum_{i=1}^{n}\log x_i - n\theta_1\right] = 0$$

Thus,

$$\boxed{\hat{\theta}_{1MLE} = \frac{1}{n}\sum_{i=1}^{n}\log x_i}$$

Ans

This is the MLE estimator for θ_1.

Now, to find the MLE w.r.t θ_2, we take derivative of (1) w.r.t θ_2:-

$$\frac{d}{d\theta_2}\left(\log\left(L_n(\theta_1,\theta_2)\right)\right) = \frac{-n}{2(2\pi\theta_2)}\cdot 2\pi - \frac{1}{2}\sum_{i=1}^{n}(\log x_i - \theta_1)^2\left(\frac{-1}{\theta_2^2}\right)$$

[we used the chain rule above to find the derivatives]

Equating the above equation to 0, we get:-

$$\frac{d}{d\theta_2}\left(\log\left(L_n(\theta_1,\theta_2)\right)\right) = \frac{-n}{2\theta_2} + \frac{1}{2\theta_2^2}\sum_{i=1}^{n}(\log x_i - \theta_1)^2 = 0$$

$$\frac{n}{2\theta_2} = \frac{1}{2\theta_2^2} \sum_{i=1}^{n} (\log x_i - \theta_1)^2$$

$(\because \theta_2 \neq 0)$

$$n = \frac{1}{\theta_2} \sum_{i=1}^{n} (\log x_i - \theta_1)^2$$

$$\theta_2 = \frac{1}{n} \sum_{i=1}^{n} (\log x_i - \theta_1)^2$$

Putting the estimator of θ_1 in RHS, we get:-

$$\boxed{\hat{\theta}_{2MLE} = \frac{1}{n} \sum_{i=1}^{n} \left(\log x_i - \frac{1}{n} \sum_{j=1}^{n} \log x_j\right)^2}$$

Ans.

4. $X_1, \ldots, X_n$ are iid
and, $U([a,b])$ – distributed $\Rightarrow$ Uniform distribution
with unknown parameters $\theta_1 = a$ and $\theta_2 = b$.

for uniform distribution, pdf is:-

$$f(x; \theta_1, \theta_2) = \begin{cases} 1/\theta_2 - \theta_1 & \theta_1 \leq x \leq \theta_2 \\ 0 & \text{otherwise} \end{cases}$$

The indicator function for the same will be $\Rightarrow$

$$I_{[\theta_1, \theta_2]}(x) = \begin{cases} 1 & x \in [\theta_1, \theta_2] \\ 0 & x \notin [\theta_1, \theta_2] \end{cases}$$

$\therefore$ the pdf becomes:-

$$f(x; \theta_1, \theta_2) = \frac{1}{\theta_2 - \theta_1} \cdot I_{[\theta_1, \theta_2]}(x)$$

To calculate the MLE's of θ_1 and $\theta_2 \Rightarrow$

$$L_n(\theta_1, \theta_2) = \prod_{i=1}^{n} f(x_i; \theta_1, \theta_2)$$

$$= \prod_{i=1}^{n} \frac{1}{\theta_2 - \theta_1} \cdot I_{[\theta_1, \theta_2]}(x_i)$$

As this part does not depend on i, we can factor that out and while doing that as it's multiplied n times, $\therefore$ we will raise it to the power n.

$$= \frac{1}{(\theta_2 - \theta_1)^n} \prod_{i=1}^{n} I_{[\theta_1, \theta_2]}(x_i)$$

The domain of the parameters and x_i's can be denoted as $\Rightarrow$

$$\left(-\infty \leq \theta_1 \leq x_i \leq \theta_2 < \infty \right)$$

and as the unknown parameters are independent, we can rewrite the indicator function $I_{[\theta_1, \theta_2]}(x_i)$ as multiplication of 2 indicator functions, one for θ_1 and another for θ_2. Also, we will change the indicator function to be written w.r.t the parameters.

$\therefore$ the equation becomes $\Rightarrow$

$$= \frac{1}{(\theta_2 - \theta_1)^n} \prod_{i=1}^{n} I_{[-\infty, x_i]}(\theta_1) \cdot I_{[x_i, \infty)}(\theta_2)$$

Now, we will first try and determine the MLE of θ_1, for that we consider θ_2 to be a constant, due to which,

$$I_{[x_i, \infty)}(\theta_2) = 1,$$

$\therefore$

$$= \frac{1}{(\theta_2 - \theta_1)^n} \prod_{i=1}^{n} I_{[-\infty, x_i]}(\theta_1) \cdot 1$$

Now,

$$\mathcal{L}_n(\theta_1, \theta_2) = 1 / (\theta_2 - \theta_1)^n \prod_{i=1}^{n} I_{(-\infty, x_i]}(\theta_1)$$

To remove the product function and maximize the L_n, we will have to consider the following:-

$$= \frac{1}{(\theta_2-\theta_1)^n} \prod_{i=1}^{n} I_{(-\infty, x_i]}(\theta_1)$$

$x_i \geq \theta_1$ for all i

and this is an exponential function.

(To get all the values as 1 from the indicator function, we will have to go as minimum as possible for x_i's $\because$ $x_i \geq \theta_1$ for all i)

Can be re-written as,

$$= \frac{1}{(\theta_2-\theta_1)^n} \cdot I_{(-\infty, \min(X_i)]}(\theta_1)$$

$\therefore$ the MLE for θ_1 becomes $\Rightarrow \boxed{\min(X_i)}$

Similarly, to calculate the MLE of $\theta_2 \Rightarrow$

we consider θ_1 to be a constant,

Thus, $\cancel{I_{[x_i, \infty)}(\theta_2)}$ $I_{(-\infty, x_i]}(\theta_1) = 1$

$\therefore$ the equation becomes,

$$\Rightarrow \frac{1}{(\theta_2-\theta_1)^n} \cdot \prod_{i=1}^{n} 1 . I_{[x_i, \infty)}(\theta_2)$$

Now, $\Rightarrow 1/(\theta_2-\theta_1)^n \cdot \prod_{i=1}^{n} I_{[x_i, \infty)}(\theta_2)$

To remove the product function, we will try and maximize L_n, we will have to consider the following:-

$$= \frac{1}{(\theta_2 - \theta_1)^n} \prod_{i=1}^{n} I_{[x_i, \infty)}(\theta_2)$$

$X_i \leq \theta_2$ for all i

and this is exponential function.

(To get all the values as 1 for the indicator function, we will have to go as maximum as possible for X_i's
$\therefore X_i \leq \theta_2$ for all i)

can be re-written as,

$$= \frac{1}{(\theta_2 - \theta_1)^n} \cdot I_{[\max(X_i), \infty)}(\theta_2)$$

$\therefore$ MLE of θ_2 becomes $\Rightarrow$ $\boxed{\max(X_i)}$

$\therefore$

$\boxed{\text{MLE of } \theta_1 = \min(X_i)}$

and,

$\boxed{\text{MLE of } \theta_2 = \max(X_i)}$

Ans.

(5) Let $X_1, \ldots, X_n$ be iid,

pdf can be written as,

$$f(x) = \begin{cases} K_{a,c}\, x^{-a-1} & , x \geq c \\ 0 & , \text{otherwise } (x < c) \end{cases}$$

with unknown parameters a and c.

(a) To determine $K_{a,c} \Rightarrow$

$\because f(x)$ is pdf,

the summation of all the probabilities over the full domain should be equal to 1.

$$\Rightarrow \int_{-\infty}^{\infty} f(x).dx = 1$$

$$\Rightarrow \int_{-\infty}^{c} \cancel{f(x)}^{\,0}.dx + \int_{c}^{\infty} f(x).dx = 1$$

↑ from the f(x) definition.

$$\Rightarrow \int_{c}^{\infty} K_{a,c}\, x^{-a-1}.\, dx = 1$$

$\because K_{a,c}$ will be in terms of a and c and does not involve x, we can take that out of the integral sign.

$$\Rightarrow K_{a,c} \int_c^{\infty} x^{-a-1}\, dx = 1$$

$$\Rightarrow K_{a,c} \left[\frac{x^{-a-1+1}}{-a-1+1} \right] \Bigg|_c^{\infty} = 1$$

$$\Rightarrow K_{a,c} \left[\frac{x^{-a}}{-a} \right] \Bigg|_c^{\infty} = 1$$

$$\Rightarrow K_{a,c} \left[\cancelto{0}{\frac{1}{-a(\infty)^a}} - \frac{1}{c^a(-a)} \right] = 1$$

$$K_{a,c} \cdot \frac{1}{c^a(a)} = 1 \Rightarrow \boxed{K_{a,c} = a c^a}$$

Ans.

(b.) To find the MLE of c,

$$L_n(a,c) = \prod_{i=1}^{n} f(x_i; a, c) \quad \text{———} \quad (1)$$

Also, we will have to consider an indicator function here,

$$I_{[c,\infty)}(x) = \begin{cases} 1 & x \in [c,\infty) \\ 0 & x \notin [c,\infty) \end{cases}$$

$$\therefore f(x; a, c) = K_{a,c}\, x^{-a-1} \cdot I_{[c,\infty)}(x)$$

Putting this in (1) above,

$$Z_n(a, c) = \prod_{i=1}^{n} K_{a,c}\, x_i^{-a-1} \cdot I_{[c,\infty)}(x_i)$$

from part (a),

$$\underline{K_{a,c} = a c^{a}},$$

$$\therefore\ Z_n(a, c) = \prod_{i=1}^{n} \left((a c^{a}) (x_i^{-a-1}) \cdot I_{[c,\infty)}(x_i) \right)$$

can be re-written as,

$$= (a c^{a})^{n} \prod_{i=1}^{n} \left((x_i^{-a-1}) \cdot I_{[c,\infty)}(x_i) \right)$$

$$\because\ \prod_{i=1}^{n} (x_i^{-a-1}) = \prod_{i=1}^{n} 1/x_i^{a+1} = \frac{1}{\prod_{i=1}^{n} x_i^{a+1}}$$

$\therefore$ the above equation becomes,

$$= \frac{(a c^{a})^{n}}{\prod_{i=1}^{n} x_i^{a+1}} \cdot \prod_{i=1}^{n} I_{[c,\infty)}(x_i)$$

The domain for the same is,

$$c \le x_i < \infty \iff -\infty < c \le x_i$$

$\therefore$ ~~to~~ indicator function can be written in terms of c,

$$\Rightarrow \frac{(ac^a)^n}{\prod_{i=1}^{n} X_i^{a+1}} \cdot \prod_{i=1}^{n} I_{[-\infty, X_i]}(c)$$

To remove the product function, we will try and maximize Z_n, we will have to consider the following:-

$$= \frac{(ac^a)^n}{\prod_{i=1}^{n} X_i^{a+1}} \cdot \prod_{i=1}^{n} I_{[-\infty, X_i]}(c) \quad \text{②}$$

$X_i \geq c$ for all i.

$\because$ this is an increasing function in c. So, $Z_n(a,c)$ will be maximum when c is maximum which will be as $\Rightarrow$

$$\underline{\underline{c \Rightarrow \min\{X_1, X_2, \dots X_n\}}}$$

Also,

$\prod_{i=1}^{n} I_{(-\infty, X_i]}(c)$ becomes $I_{(-\infty, \min(X_i)]}(c)$

as this would be the same (multiplication of 1's when in the domain).

$\therefore$ ②nd equation becomes,

$$= \left((ac^a)^n \Big/ \prod_{i=1}^{n} X_i^{a+1} \right) \cdot I_{(-\infty, \min(X_i)]}(c)$$

$\therefore$ MLE of c will be (when $\mathcal{L}_n(a,c)$ is being maximized) $\Rightarrow$

$$\boxed{\hat{c}_{MLE} = \min(X_i)}$$

Ans.

(c.) To calculate ~~the~~ MLE of a,

$$\mathcal{L}_n(a,c) = \prod_{i=1}^{n} f(x_i; a,c)$$

$$= \prod_{i=1}^{n} K_{a,c} X_i^{-a-1}$$

$\because K_{a,c} = ac^a$ and does not depend on i, we can take that out of the ~~summation~~ product and re-write the expression as,

$$\Rightarrow (ac^a)^n \prod_{i=1}^{n} X_i^{-a-1}$$

$$\because \prod_{i=1}^{n} X_i^{-a-1} = \prod_{i=1}^{n} 1/X_i^{a+1} = 1/\prod_{i=1}^{n} X_i^{a+1}$$

$$\therefore \Rightarrow \frac{(ac^a)^n}{\prod_{i=1}^{n} X_i^{a+1}}$$

($\because \mathcal{L}_n$ will be maximized when $c = \min(X_i)$ and then we have to see where it will be maximized w.r.t a),

$$\Rightarrow (a.(\min(X_i))^a)^n / \prod_{i=1}^{n} X_i^{a+1}$$

Taking logarithm on both sides of the equation,

$$\log(Z_n(a,c)) = n\log a + na\log(\min(X_i)) - (a+1)\sum_{i=1}^{n}\log(X_i)$$

To find ~~the~~ at what value of a, Z_n becomes maximum, we will differentiate ~~this~~ w.r.t a and equate the same to zero.

$$\frac{d}{da}(\log(Z_n(a,c))) = \frac{d}{da}\left(n\log a + na\log(\min(X_i)) - (a+1)\sum_{i=1}^{n}\log(X_i)\right) = 0$$

$$\Rightarrow \frac{n}{a} + n\log(\min(X_i)) - \sum_{i=1}^{n}\log(X_i) = 0$$

$$\Rightarrow \frac{n}{a} = \sum_{i=1}^{n}\log(X_i) - n\log(\min(X_i))$$

this is a constant.

Can be re-written as,

$$a = \frac{n}{\sum_{i=1}^{n}\log(X_i) - n\log(\min(X_i))}$$

$\because \sum_{i=1}^{n}\log(X_i) = \log\left(\prod_{i=1}^{n} X_i\right)$ by property of logarithms,

$$\therefore \boxed{\hat{a}_{MLE} = n \Big/ \left[\log\left(\prod_{i=1}^{n} X_i\right) - \log((\min(X_i))^n)\right]}$$

Ans

6. Let $X_1, \cdots, X_n$ be iid and, $U([0,\theta])$ - distributed with unknown parameter θ. (Uniform distribution).

$$\boxed{\hat{\theta}_{MLE} = \max X_i} \longrightarrow \text{from class.}$$

a) To find the MOM estimator $\hat{\theta}_{MOM}$,

pdf is:-

$$f(x) = \begin{cases} 1/\theta , & 0 < x < \theta \\ 0 & \text{otherwise.} \end{cases}$$

Step ①:- Calculating the moment,

$$E[X] = \frac{\theta}{2}$$ (this we know for a uniform distribution).

Step ②:- Equate the moment with sample moment,

$$E[X] = \hat{\mu}_1 = \frac{\theta}{2} \stackrel{!}{=} \bar{X}_n$$

Step ③:- Solving the equation,

$$E[X] = \bar{X}_n = \frac{\theta}{2}$$

$$\Rightarrow \frac{\theta}{2} = \bar{X}_n \Rightarrow \boxed{\hat{\theta}_{MOM} = 2\bar{X}_n}$$

Ans.

(b.) To see whether these estimators are biased or not, we know that an estimator (T) is unbiased if:-

$$\boxed{E[T] = \theta}$$

Let's consider $\hat{\theta}_{MOM}$ first,

$$\hat{\theta}_{MOM} = 2\bar{X}_n \quad \text{--- (1)}$$

We also know that mean of a uniform distribution with parameter θ is $\Rightarrow$ $\frac{0+\theta}{2} = \frac{\theta}{2}$

To find $E[\hat{\theta}_{MOM}]$,

$$E[\hat{\theta}_{MOM}] = E[2\bar{X}_n] = 2E[\bar{X}_n]$$ (By property of linearity)

Now,

$$E[\bar{X}_n] = \int_0^\theta x\, f(x).\,dx = \int_0^\theta x \cdot \frac{1}{\theta}.\,dx = \frac{1}{\theta}\int_0^\theta x\,dx$$

$$= \frac{1}{\theta}\left[\frac{x^2}{2}\right]\Big|_0^\theta = \frac{1}{\theta}\left[\frac{\theta^2}{2} - 0\right] = \frac{\theta}{2}$$

Now, $$E[\hat{\theta}_{MOM}] = 2E[\bar{X}_n] = 2 \times \frac{\theta}{2} = \theta$$

$\because$ $E[\hat{\theta}_{MOM}] = \theta$

$\therefore$ we can say that $\hat{\theta}_{MOM}$ is unbiased.

So, $$Bias[\hat{\theta}_{MOM}] = E[\hat{\theta}_{MOM}] - \theta = \theta - \theta = 0$$

$\boxed{\therefore Bias[\hat{\theta}_{MOM}] = 0}$ Ans.

Now, let's try to find whether $\hat{\theta}_{MLE}$ is biased or not.

We know that:-

$$\hat{\theta}_{MLE} = \max(X_i) = y \quad \text{(given)}.$$

Let's first try and find out the pdf of the estimated :-

cdf $\rightarrow$ $F_Y(y) = P(Y \leq y)$ {given that $Y \leq y$ iff. $x \leq y$}

$$= P(x_1 \leq y, x_2 \leq y, \ldots, x_n \leq y)$$

As these are independent,

$$\therefore F_Y(y) = P(x_1 \leq y).\ P(x_2 \leq y). \ldots P(x_n \leq y) \quad \text{—②}$$

Since X follows a uniform distribution,

$\therefore$ $P(x \leq y)$ over $[0, y]$

$$P(x \leq y) = \frac{0+y}{\theta} = y/\theta$$

Putting the above expression in ②nd eqution,

$$P(x_1 \leq y) \ldots P(x_n \leq y) = \left(\frac{y}{\theta}\right) \cdots \left(\frac{y}{\theta}\right) \quad \text{[n times]}$$

$$= \left(\frac{y}{\theta}\right)^n \quad (0 \leq y \leq \theta)$$

To find out the pdf, we will differentiate the cdf w.r.t y,

$$\frac{d}{dy}\left(\frac{y^n}{\theta^n}\right) = \frac{ny^{n-1}}{\theta^n}$$

$\therefore$

$$f(y) = \begin{cases} \frac{ny^{n-1}}{\theta^n} & 0 \le y \le \theta \\ 0 & \text{otherwise} \end{cases} \quad \text{— (3)}$$

Now, to find whether $\hat{\theta}_{MLE}$ is unbiased, we need to show that :-

$E[T] = \theta$ where estimator T will be unbiased if this holds.

$E[\max(X_i)] = E[y]$ $\{\because y = \max(X_i)\}$

$$E[y] = \int_{-\infty}^{\infty} y f(y) . dy$$

$$= \int_{-\infty}^{0} y \cancel{f(y)}^{0} . dy + \int_{0}^{\theta} y f(y) dy + \int_{\theta}^{\infty} y \cancel{f(y)}^{0} . dy$$

$$= \int_{0}^{\theta} y f(y) . dy = \int_{0}^{\theta} y\left(\frac{ny^{n-1}}{\theta^n}\right) . dy \quad \text{using (3)}$$

$$= \int_{0}^{\theta} \frac{ny^n}{\theta^n} . dy = \frac{n}{\theta^n}\int_{0}^{\theta} y^n . dy$$

$$= \frac{n}{\theta^n}\left[\frac{y^{n+1}}{n+1}\right]\Bigg|_{0}^{\theta} = \frac{n}{\theta^n}\left(\frac{\theta^{n+1}}{n+1} - \frac{0^{n+1}}{n+1}\right)$$

$$= \frac{n}{\theta^n}\left(\frac{\theta^{n+1}}{n+1}\right) = \frac{n\theta}{n+1} \neq \theta$$

$\therefore E[y] = \frac{n\theta}{n+1}$ which is not equal to θ.

we can say that $\hat{\theta}_{MLE} = y = \max(X_i)$ is a biased estimator.

$\therefore$ Bias for MLE $\Rightarrow E[\hat{\theta}_{MLE}] - \theta$

$$= \frac{n\theta}{n+1} - \theta$$

$$= \frac{n\theta - (n+1)\theta}{n+1}$$

$$= \frac{n\theta - n\theta - \theta}{n+1} = \frac{-\theta}{n+1}$$

$$\boxed{\text{Bias}_{MLE} = \frac{-\theta}{n+1}}$$ Ans.

(c.) We know that:-

$$MSE(T) = \sigma_T^2 + (Bias(T))^2$$

Let's calculate MSE for $\hat{\theta}_{MOM}$,

We know that,

$$V[x] = E[x^2] - (E[x])^2$$

$E[x] = \frac{\theta}{2}$ (for uniform distribution)

Also,

$$E[x^2] = \int_{-\infty}^{\infty} x^2 f(x).dx$$

$$= \int_{-\infty}^{0} x^2 \cancel{f(x)}^{\,0}.dx + \int_{0}^{\theta} x^2 f(x).dx + \int_{\theta}^{\infty} x^2 \cancel{f(x)}^{\,0}.dx$$

$$= \int_{0}^{\theta} x^2 f(x).dx$$

$$= \int_{0}^{\theta} x^2 . \frac{1}{\theta} .dx = \frac{1}{\theta} \left[\frac{x^3}{3}\right]\Big|_0^\theta$$

$$= \frac{1}{\theta}\left(\frac{\theta^3}{3} - \frac{0^3}{3}\right) = \frac{\theta^2}{3}$$

Now,

$$V[X] = \frac{\theta^2}{3} - \left(\frac{\theta}{2}\right)^2 = \frac{\theta^2}{3} - \frac{\theta^2}{4} = \frac{\theta^2}{12}$$

From part (b) we know that $\hat{\theta}_{MOM}$ is an unbiased estimator,

$$\therefore E[\hat{\theta}_{MOM}] = \theta$$

$$\therefore Bias(\hat{\theta}_{MOM}) = E[\hat{\theta}_{MOM}] - \theta = \theta - \theta = 0.$$

furthermore,

$$V[X] = \frac{\theta^2}{12}$$

$$V[\hat{\theta}_{MOM}] = V[2\bar{X}_n] = 4V[\bar{X}_n]$$

$$= 4 \cdot \frac{V[X]}{n} = \frac{4}{n} \cdot \frac{\theta^2}{12}$$

$$= \frac{\theta^2}{3n}$$

$$\therefore V[\hat{\theta}_{MOM}] = \frac{\theta^2}{3n}$$

Now, $MSE[\hat{\theta}_{MOM}] = V[\hat{\theta}_{MOM}] + (Bias(\hat{\theta}_{MOM}))^2$

$$= \frac{\theta^2}{3n} + (0)^2 = \frac{\theta^2}{3n}$$

$$\therefore \boxed{MSE[\hat{\theta}_{MOM}] = \frac{\theta^2}{3n}}$$

Ans.

Let's calculate the MSE of the MLE estimator,

$$\Rightarrow V[\hat{\theta}_{MLE}] = E[\hat{\theta}_{MLE}^2] - (E[\hat{\theta}_{MLE}])^2$$

We already know that :-

$$E[\hat{\theta}_{MLE}] = E[\max(X_i)] = E[Y] = \frac{n\theta}{n+1}$$

Let's calculate $E[\hat{\theta}_{MLE}^2] \Rightarrow$

$$E[\hat{\theta}_{MLE}^2] = E[\max^2(x_i)] = E[y^2]$$

$$= \int_{-\infty}^{\infty} y^2 f(y).dy$$

$$= \int_{-\infty}^{0} y^2 \cancel{f(y)}^{0}.dy + \int_{0}^{\theta} y^2 f(y).dy + \int_{\theta}^{\infty} y^2 \cancel{f(y)}^{0}.dy$$

$$= \int_{0}^{\theta} y^2 f(y).dy = \int_{0}^{\theta} y^2 \left(\frac{n y^{n-1}}{\theta^n}\right).dy$$

$$= \frac{n}{\theta^n} \int_{0}^{\theta} y^{n+1}.dy = \frac{n}{\theta^n}\left[\frac{y^{n+2}}{n+2}\right]\Bigg|_0^{\theta}$$

$$= \frac{n}{\theta^n}\left[\frac{\theta^{n+2}}{n+2} - \frac{0^{n+2}}{n+2}\right] = \frac{n}{\theta^n}\cdot\frac{\theta^{n+2}}{(n+2)}$$

$$= \frac{n\theta^2}{n+2}$$

Now,

$$V[\hat{\theta}_{MLE}] = \frac{n\theta^2}{n+2} - \left(\frac{n\theta}{n+1}\right)^2$$

$$= \frac{n\theta^2}{n+2} - \frac{n^2\theta^2}{(n+1)^2}$$

$$= \frac{n\theta^2 (n+1)^2 - n^2\theta^2 (n+2)}{(n+2)(n+1)^2}$$

$$= \frac{n\theta^2 (n^2+1+2n) - n^2\theta^2(n+2)}{(n+1)^2(n+2)}$$

$$= \frac{\cancel{n^3\theta^2} + n\theta^2 + \cancel{2n^2\theta^2} - \cancel{n^3\theta^2} - \cancel{2n^2\theta^2}}{(n+1)^2(n+2)}$$

$$= \frac{n\theta^2}{(n+1)^2(n+2)}$$

Also, we have to calculate Bias $(\hat{\theta}_{MLE})$ in part (b),

which is $\Rightarrow \frac{-\theta}{n+1}$

$$\therefore \quad MSE[\hat{\theta}_{MLE}] = \frac{n\theta^2}{(n+1)^2(n+2)} + \left(\frac{-\theta}{n+1}\right)^2$$

$$\left[\because \; MSE[\hat{\theta}_{MLE}] = Var[\hat{\theta}_{MLE}] + (Bias[\hat{\theta}_{MLE}])^2\right]$$

$$\Rightarrow MSE[\hat{\theta}_{MLE}] = \frac{n\theta^2}{(n+1)^2(n+2)} + \frac{\theta^2(n+2)}{(n+1)^2(n+2)}$$

$$= (2n\theta^2 + 2\theta^2) / ((n+1)^2(n+2))$$

$$= \frac{2\theta^2 (n+1)}{(n+1)^2 (n+2)} = \frac{2\theta^2}{(n+1)(n+2)}$$

$$\therefore \boxed{MSE(\hat{\theta}_{MLE}) = \frac{2\theta^2}{(n+1)(n+2)}}$$

Ans

(d) We know from part (c) that:-

$$\boxed{MSE[\hat{\theta}_{MOM}] = \frac{\theta^2}{3n}}$$

$$\boxed{MSE[\hat{\theta}_{MLE}] = \frac{2\theta^2}{(n+1)(n+2)}}$$

Now, let's try to find which one is smaller by assuming one of the cases,

Let's assume that:-

$$MSE[\hat{\theta}_{MLE}] < MSE[\hat{\theta}_{MOM}]$$

Then,

$$\frac{2\cancel{\theta^2}}{(n+1)(n+2)} < \frac{\cancel{\theta^2}}{3n} \quad (\because \theta \neq 0)$$

Solving,

$$\Rightarrow \quad 6n < (n+1)(n+2)$$

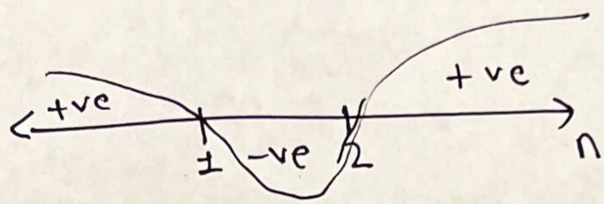
(multiplying both sides with expression of n won't change sign $\because n > 0$)

$$6n < n^2 + 2 + 3n$$

$$n^2 - 3n + 2 > 0$$

$$\Rightarrow \boxed{(n-2)(n-1) > 0}$$

Solving for this inequality,



(in these regions it's +ve/-ve)

Thus, for $n=1$ and $n=2$, $MSE[\hat{\theta}_{MLE}] = MSE[\hat{\theta}_{MOM}]$

Since n is the number of observations, they cannot be -ve.

$$\boxed{\text{for } n>2, \quad MSE[\hat{\theta}_{MLE}] < MSE[\hat{\theta}_{MOM}]}$$

This seems to be a valid assumption & holds true for $(n>2)$.

$$\boxed{\therefore \hat{\theta}_{MLE} \text{ has the smaller MSE.}}$$

Ans.