Divide & Conquer

- Divide matrix into smaller square matrix.
- solve that smaller square matrix.
- Merge into larger result.

Matrix Multiplication using Strassen's method.

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$C_{11} = S_1 + S_4 - S_5 + S_7$$
$$C_{12} = S_3 + S_5$$
$$C_{21} = S_2 + S_4$$
$$C_{22} = S_1 + S_3 - S_2 + S_6$$

where

$$S_1 = (A_{11} + A_{22}) * (B_{11} + B_{22})$$
$$S_2 = (A_{21} + A_{22}) * B_{11}$$
$$S_3 = A_{11} * (B_{12} - B_{22})$$
$$S_4 = A_{22} * (B_{21} - B_{11})$$
$$S_5 = (A_{11} + A_{12}) * B_{22}$$
$$S_6 = (A_{21} - A_{11}) * (B_{11} + B_{12})$$
$$S_7 = (A_{12} - A_{22}) * (B_{21} + B_{22})$$

Analysis

$$T(n) = \begin{cases} 1 & n \leq 2 \\ 7T(n/2) + n^2 & n \geq 2 \end{cases}$$

$\log_2 7 = 2.81 \quad k = 2.$

$O(n^{\log_2 7})$

$$\boxed{T(n) = O(n^{2.81})}$$

Whereas normal matrix multiplication requires $\boxed{O(n^3)}$

Example: Strassen's matrix multiplication

$$A = \begin{bmatrix} 0 & 0 & 3 & 4 \\ 5 & 6 & 0 & 0 \\ 0 & 0 & 6 & 5 \\ 4 & 3 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 & 0 & 6 & 5 \\ 4 & 3 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 5 & 6 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \times \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}.$$

$$= \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix} & \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 4 & 3 \end{bmatrix} & \begin{bmatrix} 6 & 5 \\ 0 & 0 \end{bmatrix} \end{bmatrix} \times \begin{bmatrix} \begin{bmatrix} 0 & 0 \\ 4 & 3 \end{bmatrix} & \begin{bmatrix} 6 & 5 \\ 0 & 0 \end{bmatrix} \\ \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix} & \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} \end{bmatrix}$$

$$S_1 = A_{11} + A_{22} * B_{11} + B_{22}$$

$$= \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 6 & 5 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 18+20 & 24+15 \\ 15+24 & 20+18 \end{bmatrix} = \begin{bmatrix} 38 & 39 \\ 39 & 38 \end{bmatrix}$$

$$S_2 = A_{21} + A_{22} \times B_{11}$$

$$= \begin{bmatrix} 0 & 0 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 6 & 5 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 20 & 15 \\ 12 & 9 \end{bmatrix}$$

$$S_3 = A_{11} \times B_{12} - B_{22}$$

$$= \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 6 & 5 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 15 & 5 \end{bmatrix}$$

$$S_4 = A_{22} \times B_{21} - B_{11}$$

$$\begin{bmatrix} 6 & 5 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 4 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 15 \\ 0 & 0 \end{bmatrix}$$

$$S_5 = A_{11} + A_{12} \times B_{22}$$

$$\begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 9 & 12 \\ 15 & 20 \end{bmatrix}$$

$$S_6 = A_{11} + A_{12} * B + B_{12}$$

$$\begin{bmatrix} 0 & 0 \\ 4 & 3 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix} \times \begin{bmatrix} 0 & 0 & & 6 & 5 \\ 4 & 3 & + & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -18 & -14 \end{bmatrix}$$

$$S_7 = A_{12} - A_{22} * B_{21} + B_{22}$$

$$\begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 6 & 5 \\ 0 & 0 \end{bmatrix} \times \begin{bmatrix} 0 & 0 \\ 5 & 6 \end{bmatrix} - \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -14 & -18 \\ 0 & 0 \end{bmatrix}$$