# CONTENTS


| Sl. No. | DETAILS | | Pg. No. |
|---|---|---|---|
| 1. | SET THEORY / RELATIONS / FUNCTIONS | ... | A.1-A.38 |
| 2. | POSETS/LATTICES/BOOLEAN ALGEBRA | ... | B.1-B.38 |
| 3. | PROPOSITIONAL / PREDICATE LOGIC | ... | C.1-C.24 |
| 4. | ALGEBRAIC STRUCTURES | ... | D.1-D.24 |
| 5. | RECURRENCE RELATIONS / COMBINATORICS | ... | E.1-E.28 |
| 6. | UNSOLVED EXAMINATION QUESTION PAPERS | ... | F.1-F.4 |





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# SYLLABUS

| | Topic |
|---|---|
| 1. | **Set Theory :** Introduction, Size of sets and Cardinals, Venn diagrams, Combination of sets, Multisets, Ordered pairs and Set Identities.<br>**Relation :** Definition, Operations on relations, Composite relations, Properties of relations, Equality of relations, Partial order relation.<br>**Functions :** Definition, Classification of functions, Operations on functions, Recursively defined functions. |
| 2. | **Posets, Hasse Diagram and Lattices :** Introduction, Partial ordered sets, Combination of Partial ordered sets, Hasse diagram, Introduction of lattices. Properties of lattices – Bounded, Complemented, Modular and Complete lattice.<br>**Boolean Algebra :** Introduction, Axioms and Theorems of Boolean algebra, Boolean functions. Simplification of Boolean functions, Karnaugh maps, Logic gates. |
| 3. | **Propositional :** Propositions, Truth tables, Tautology, Contradiction, Algebra of Propositions, Theory of Inference and Natural Detection.<br>**Predicate Logic :** Theory of Predicates, First order predicate, Predicate formulas, Quantifiers, Inference theory of predicate logic. |
| 4. | **Algebraic Structures :** Introduction to algebraic Structures and properties. Types of algebraic structures: Semi group, Monoid, Group, Abelian group and Properties of group. Subgroup, Cyclic group, Cosets, Permutation groups, Homomorphism and Isomorphism of groups.<br>**Rings and Fields :** Definition and elementary properties of Rings and Fields. |
| 5. | **Natural Numbers :** Introduction, Piano's axioms, Mathematical Induction, Strong Induction and Induction with Nonzero Base cases.<br>**Recurrence Relation & Generating Functions :** Introduction and properties of Generating Functions. Simple Recurrence relation with constant coefficients and Linear recurrence relation without constant coefficients. Methods of solving recurrences.<br>**Combinatorics :** Introduction, Counting techniques and Pigeonhole principle, Polya's Counting theorem. |

# SET THEORY / RELATION / FUNCTIONS

| | | |
|---|---|---|
| 1. | ◆ *Find the power set of set $X = \{1, 2, 3\}$.* | (2020-21) |
| | ◆ *Define the Power set* | (2018-19) |
| | *If $A = \{1, 2, 3\}$ find $P(A)$ and $n\{P(A)\}$.* | |
| | ◆ *What do you mean by power set? Illustrate with an example.* | (2017-18) |
| | ◆ *Define power set.* | (2015-16) |
| | ◆ *Define a power set. Illustrate with an example.* | (2011-12) |

**Power Set**

If $S$ is any set then the family of all subsets of $S$ is called the power set of $S$. The power set of $S$ is denoted by $P(S)$. Symbolically $P(S) = \{T{:}T\ S\}$. If the set $S$ is finite and contain $n$ elements, then the power set of $S$ will then contain $2^n$ elements.

**Example :** If $A = \{1, 2\}$ then $P(A) = \{\{1\}, \{2\}, \{1, 2\}, \{\}\}$

**Numerical Solution :** $A = \{1, 2, 3\}$

$P(A) = \{(1),(2),(3),(1,2),(1,3),(2,3),(1,2,3),\phi\}$

The number of elements in a power set of a set $A$ is $2^n$, where $n$ is number of elements in $A$.

$nP(A) = 2^3 = 8$ **(Ans.)**

| | | |
|---|---|---|
| 2. | *Discuss Inverse function with example.* | (2020-21) |

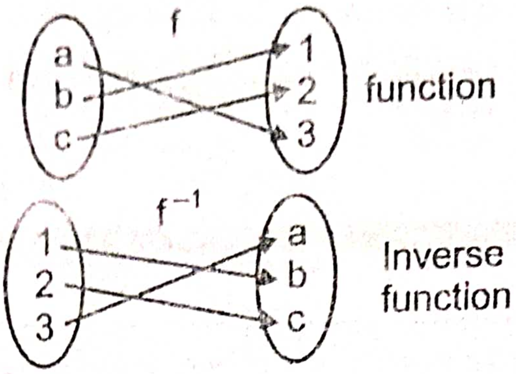
A function accepts values, perform particular operations on these values & generate on output.

The inverse function agrees with the resultant, operates & reaches back to the original function.

**Example :**

$$f(x) = 2x + 5 = y$$

$$g(y) = \frac{y-5}{2} = x \text{ is the inverse of } f(x)$$




(Figure)

**3. *Define the Binary Relation and Explain the properties of Binary Relation with example.***
**(2020-21)**

***Binary Relation :***

Binary Relation $R$ from $A$ to $B$ is a sub set of $A \times B$. If $R \subseteq A \times B$ and $(a,b) \in R$, we say that $a$ is related to $b$ by $R$.

***Properties of Binary Relation :***

**(1)** **Reflexive Relation :** A relation $R$ on set $A$ is reflexive if $(a, a) \in R$ for all $a \in A$.

**Example :** Let $A = (1, 2, 3)$ then $R$ is reflexive if $(1, 1)(2, 2)(3, 3) \in R$.

**(2)** **Symmetric Relation :** A relation $R$ on set $A$ is symmetric if whenever $a\,R\,b$ then $b\,R\,a$.

**Example :** Let $A = \{1, 2, 3, 4\}$

$R = \{(1, 2)(2, 1)(1, 1)(2, 2)\}$ is symmetric

**(3)** **Asymmetric Relation :** A relation $R$ on set $A$ is asymmetric if whenever $a\,R\,b$ then $b \not R\, a$

**Example :**

Let $A = \{1, 2, 3, 4\}$

$R = \{(1, 2)(3, 4)(4, 2)\}$ is asymmetric relation.

**(4)** **Antisymmetric Relation :** A relation $R$ on set $A$ is antisymmetric if whenever $aRb$ and $bRa$ then $a = b$.

**Example :** Let $A = \{1, 2, 3, 4\}$

$R = \{(1, 1)(2, 3)(4, 4)(3, 4)\}$ is antisymmetric relation.

**Transitive Relation :** A relation $R$ on set $A$ is transitive if whenever $aRb$ and $bRc$, then $aRc$

Let $A = \{1, 2, 3, 4\}$

$R = \{(1, 1)(1, 2)(2, 4)(1, 4)\}$ is a transitive relation because $1R2$ and $2R4$, then $1R4$ is available in set.

> **4.** ***A market research group conducted a survey of 1000 consumers and reported that, 720 consumers liked product X and 450 liked product Y. What is the least number that must have liked both products?*** **(2020-21)**

$$n(A \cup B) = n(A) + n(B) - n(A \cap B)$$

$$n(A \cup B) = 1000$$

$$n(A) = 720 = \text{Customer Liked Product } X$$

$$n(B) = 450 = \text{Customer Liked product } Y$$

$$1000 = 720 + 450 - n(A \cap B)$$

$$n(A \cap B) = 720 + 450 - 1000$$

$= 170$ customers like both product.

> **5.** ◆ ***Define the term function. Explain the difference between relation and function.*** **(2020-21)**
>
> ◆ ***What do you mean by function? Explain different types of functions with proper examples.*** **(2017-18)**

***Function***

A function $f$ from $X$ to $Y$ is set of ordered pairs $(x, y)$ such that $x \in X$ and $y \in Y$, which is subject to the following condition: every element of $X$ is the first component of one and only one ordered pair in the set. In other words, for every $x$ in $X$ there is exactly one element $y$ such that the ordered pair $(x, y)$ belongs to the set of pairs defining the function $f$.

In this definition, $X$ and $Y$ are respectively called the domain and the co-domain of the function $f$. If $(x, y)$ belongs to the set defining $f$, then $y$ is the value of $f$ for the argument $x$, or the image of $x$ under $f$. One says also that $y$ is the value of $f$ for the value $x$ of the variable.

This formal definition is a precise rendition of the idea that to each $x$ is associated an element $y$ of $Y$, namely the uniquely specified element $y$ with the property just mentioned.

A relation is a function if for every $x$ in the domain there is exactly one $y$ in the co-domain.

A vertical line through any element of the domain should intersect the graph of the function exactly once.

**Types of Functions**

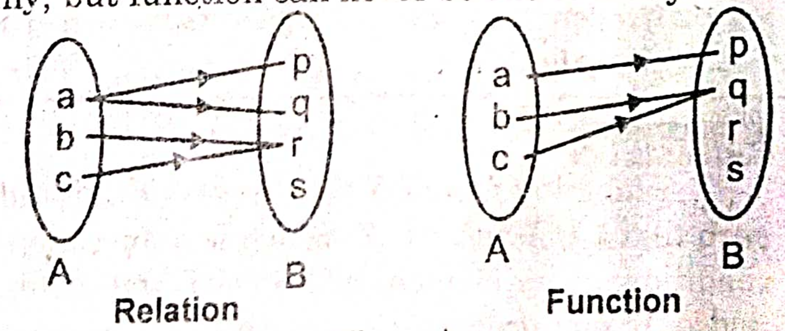
(1) A function is injective if for every $y$ in the co-domain $B$ there is at most one $x$ in the domain. A horizontal line should intersect the graph of the function at most once (i.e. not at all or once).

(2) A function is surjective if for every $y$ in the co-domain $B$ there is at least one $x$ in the domain. A horizontal line intersects the graph of the function at least once (i.e. once or more).The range and the co-domain are identical.

(3) A function is bijective if for every $y$ in the co-domain there is exactly one $x$ in the domain. A horizontal line through any element of the range should intersect the graph of the function exactly once.

**Difference Between Relation and Function**

Relation can always be one to one, one to many, many to many, but function can never be one to many.




(Figure )

## 6. *What is a Set and how it is classified?*

**Set**

A set is a collection of distinct and distinguishable objects around us. Or in another way it may defined as a group of items having some common characteristic(s) example.

(1) Set of all 26 alphabets i.e., $\{A, B, C, ...Z\}$

(2) Set of all planets near earth.

(3) Set of all the students having more than 70% marks in their mid-term examination.

A set can be represented by two ways :

(a) **Tabular Form :** In this form, each member of set is written one by within curly braces i.e., { } and the name of set may denoted by letter $A$ or $B$ or $S$.

**Example :** A set of even integers from 0 to 10 is shown as :

$S = \{0, 2, 4, 6, 8, 10\}$

(b) **Symbolic Form :** In this method, we will take a variable $n$ within { } as it stands for each of the member of set and then state the properties possesses by $n$.

**Example :** Set of odd integer up to 10.

$S = \{n : n \text{ is an odd integer, } n \text{ is less than } 11\}$

Or, in more symbolic fashion.

$S = \{n : n \% 2 \neq 0, n < 11\}$

**Types of Set**

**(1) Singleton Set :** A set having a single element is called singleton set. For example, set of odd integers between 4 and 6 will be given as :

$S = \{5\}$

Or, $S = \{n : n \% 2 \neq 0, 4 \leq n < 6\}$

**(2) Sub-Set :** A set $B$ is said to be a subset of another set $A$ if each element of $B$ also lies in $A$. example,

If $A = \{1, 2, 3, 4\}$

And, $B = \{1, 2, 3\}$

Then, $B \subseteq A$

Similarly,

If $A = \{1, 2, 3\}$

And, $B = \{2, 1, 3\}$

Then, $B \subseteq A$

**(3) Proper Sub-Set :** A set $B$ is said to be a proper sub-set of another set $A$ if :

All the elements of $B$ must lie at $A$.

There should be at least one element in $A$ which does not lie at $B$.

**Example, :**

If, $A = \{1, 2, 3, 4, 5\}$

And, $B = \{1, 2, 3, 4\}$

Then, $B \subset A$ (note the symbol $\subseteq$ and $\subset$)

(4) **Improper Sub-Set :** A set $B$ is said to be a improper sub-set of another set $A$ if $B$ have exactly same element as $A$.

(5) **Universal Set :** A set which is superset of all the sets under consideration is called universal set and denoted by $U$ (capital $U$). In other words, we may say a set consists of various sub-sets in called universal set.

**Example :**

Let $A = \{1, 2\}$
$B = \{4, 5, 6, 7\}$
$C = \{8\}$
$D = \{10, 11, 14, 15\}$

Then, $U = \{1, 2, 4, 5, 6, 7, 8, 9, 10, 11, 14, 15\}$ is superset of $A$, $B$, $C$ and $D$. Thus called universal set.

(6) **Null (or Empty or Void) Set :** A set having no member is called null set and represented by $\phi$ as :

$$\phi = \{\}$$

Or, $\phi = \{n : n \% 2 = 0, 4 < n < 6\}$

(7) **Pair Set :** A set having two members as $A = \{2, 3\}$

(8) **Finite Set :** A set having finite number of element is called finite set i.e., counting of members is feasible.

Example :

$A = \{1, 2, 3, 4\}$

Or, $A = \{n : n$ is an even number, $n$ is less than $1000\}$

(9) **Infinite Set :** A set having infinite members is called infinite set i.e., members can not be counted.

**Example :** $A = \{n : n$ is set of all natural numbers.$\}$

(10) **Power Set :** The power set of a set $A$ is defined as a set of all subsets of $A$ including null set. It is denoted by $P(A)$ as 'power of $A$'.

**Example :** if $A = \{1, 2, 3\}$

Then $P(A) = \{(1), (2), (3), (1, 2), (1, 3), (2, 3), (1, 2, 3), \phi\}$ Obviously, the number of elements in a power set of a set $A$ is $2^n$ where $n$ is number of elements in $A$.

(11) **Complement of a Set :** A set $S'$ is said to be a complement set of another set $S$ if it contains all those elements of universal set that doesn't lie in $S$.

**Example :**

If $U = \{1, 2, 3, 4, 5, 6, 7\}$

And $S = \{1, 4, 7\}$

Then $S' = \{2, 3, 5, 6\}$

i.e., $S' = \{n : n \in u, n \in s\}$

(12) **Disjoint Sets :** If two sets $A$ and $B$ is considered such that no element is common in both set then $A$ and $B$ is called Disjoint or mutually exclusive sets.

**Example :**

$A = \{1, 2, 3\}$

$B = \{4, 5, 6, 7, 8\}$

**7. *Define set and their Cardinality with examples.* (2013-14)**

**Set :** A well defined collection of objects is called a set. These objects are also called elements of the set.

**Cardinality :** Total number of elements in a set is called cardinality of that set.

**Example :** Let us consider a set of vowel alphabet if we denote this set with $V$ then

$V = \{a, e, i, o, u\}$

Since total vowel alphabet is 5, hence cardinality of the set $V$ is 5.

**8. *Give a brief description of various Set Theoretic Operations and law of Set Theory?***

The fundamental operations that we can perform on to sets are given as follows :

(1) **Union :** The union of two sets $A$ and $B$ contains the elements which are in either $A$ or $B$ or in both. It is denoted by $\cup$. Symbolically :

$$A \cup B = \{n : n \in A \text{ or } n \in B\}$$

(2) **Intersection :** The intersection of two sets $A$ and $B$ contains the elements which are common in both given set. i.e.

$$A \cap B = \{n : n \in A \text{ and } n \in B\}$$

Example : $A = \{1, 2, 3\}$

$B = \{2, 4, 6\}$

$(A \cup B) = \{1,2,3,4,6\}$

$(A \cap B) = \{2\}$

(3) **Difference :** Difference of two sets $A$ and $B$ contains the elements that exist in $A$ but not in $B$ as :

$$A - B = \{n : n \in A, n \in B\}$$

or, $B - A = \{n : n \in B, n \in A\}$

(4) **Cartesian Product :** The Cartesian product of two sets $A$ and $B$ yields a set containing all possible ordered pair from $A$ and $B$.

i.e. $A \times B = \{(a, b) : a \in A, b \in B\}$

Example : if $A = \{1, 2\}$, $B = \{3, 4\}$

$$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$$

## LAWS OF SET THEORY

(1) **Commutative Law :**

$$A \cap B = B \cap A$$
$$A \cup B = B \cup A$$

**Proof :** Let $n$ be an element of $A \cup B$ then, we may write :

$n : n \in (A \cup B)$ — n∈(A∪B)

$\Rightarrow \; n \in A \vee n \in B$ — n∈A or n∈B

$\Rightarrow \; n \in B \vee n \in A$ — n∈B or n∈A

$\Rightarrow \; n \in (B \cup A)$ — n∈(B∪A)

i.e. $n$ is an element of $(B \cup A)$ too. Thus $A \cup B = B \cup A$

Similarly if $n$ is an element of $A \cap B$ as :

$n : n \in (A \cap B)$ — n∈(A∩B)

$\Rightarrow \; n \in A \wedge n \in B$ — n∈B or n∈A

$\Rightarrow \; n \in B \wedge n \in A$

$\Rightarrow \; n \in (B \cap A)$ — n∈(B∩A)

i.e. $n$ is an element of $(B \cap A)$ too thus.

$$A \cap B = B \cap A$$

(2) **Idempotent Law :**

$$A \cup A = A$$
$$A \cap A = A$$

**Proof :** It can be proved easily by taking an element $n$ of $A$.

(3) **Associative Law :**

$$A \cup (B \cup C) = (A \cup B) \cup C$$
$$A \cap (B \cap C) = (A \cap B) \cap C$$

**Proof :**

(a) Let, $n$ be an element of $A \cup (B \cup C)$ then it implies :

$\Rightarrow \quad n \in A \cup (B \cup C)$ $\quad$ $n \in A \cup (B \cup C)$

$\Rightarrow \quad n \in A \vee n \in (B \cup C)$ $\quad$ $n \in A$ or $n \in (B$ or $C)$

$\Rightarrow \quad n \in A \vee n \in B \vee n \in C$ $\quad$ $n \in A$ or $n \in B$ or $n \in C$

$\Rightarrow \quad n \in (A \cup B) \vee n \in C$ $\quad$ $n \in A$ or $n \in B$ $n \in C$

$\Rightarrow \quad n \in (A \cup B) \cup C$ $\quad$ $n \in (A \cup B)$ or $n \in C$

Thus, $A \cup (B \cup C) = (A \cup B) \cup C$ $\quad$ $n \in (A \cup B) \cup C$

**(b)** Let, $n$ be an element of $A \cap (B \cap C)$ such that :

$\Rightarrow \quad n \in A \cap (B \cap C)$

$\Rightarrow \quad n \in A \wedge n \in (B \cap C)$

$\Rightarrow \quad n \in A \wedge n \in B \wedge n \in C$

$\Rightarrow \quad n \in (A \cap B) \wedge n \in C$

$\Rightarrow \quad n \in (A \cap B) \cap C$

Thus, $\quad A \cap (B \cap C) = (A \cap B) \cap C$.

**Note :** Student should not confused with the symbol '$\wedge$' and '$\vee$' since these refer to 'AND' and 'OR' operator respectively.

**(4) Distribution Law :**

$$A \cup (B \cap C) - (A \cup B) \cap (A \cup C)$$

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

**Proof :**

**(a)** Let $n$ be an element of $A \cup (B \cap C)$ so that

$\Rightarrow \quad n \in A \cup (B \cap C)$

$\Rightarrow \quad n \in A \cup (n \in (B \cap C))$

$\Rightarrow \quad n \in A \cup (n \in B \wedge n \in C)$

$\Rightarrow \quad n \in A \cup n \in B \wedge n \in A \cup n \in C$

$\quad \because A \cup (B \wedge C) = (A \cup B) \wedge (A \cup C)$

$\Rightarrow \quad n \in (A \cup B) \wedge n \in (A \cup C)$

$\Rightarrow \quad n \in (A \cup B) \cap (A \cup C)$

i.e. $\quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

**(b)** Let $n$ be an element of $A \cap (B \cup C)$ then

$= \quad n \in A \cap (B \cup C)$

$\Rightarrow \quad n \in A \wedge n \in B \cup n \in C$

$\Rightarrow \quad n \in A \wedge n \in B \cup n \in A \wedge n \in C$

$\Rightarrow \quad n \in (A \cap B) \cup n \in (A \cap C)$

$\Rightarrow \quad n \in (A \cap B) \cup (A \cap C)$

Thus, $\quad A \cap (B \cup C) = (A \cap B) \cap (A \cap C)$

(5) **D'morgan's Law**

$(A \cup B)' = A' \cap B'$

$(A \cap B)' = A' \cup B'$

**Proof :**

**(a)** Let $n$ be an element of $(A \cup B)'$

$\Rightarrow \quad n \in (A \cup B)'$    $n \in (A \cup B)'$

$\Rightarrow \quad n \in (A \cup B)$    $n \notin (A \cup B)$

$\Rightarrow \quad n \in A \wedge n \in B$    $n \notin A$ or $n \notin B$

$\Rightarrow \quad n \in A' \wedge n \in B'$    $n \in A'$ or $n \in'$

$\Rightarrow \quad n \in (A' \cap B')$    $n \in (A' \cap B)$

i.e. $\quad (A \cup B)' = A' \cap B'$

**(b)** Let, $n \in (A \cap B)'$

$\Rightarrow \quad n \in (A \cap B)$

$\Rightarrow \quad n \in A \cup n \in B$

$\Rightarrow \quad n \in A' \cup n \in B'$

$\Rightarrow \quad n \in (A' \cup B')$

i.e. $\quad (A \cap B)' = A \cap B' = B' - A'$

(6) $A - B = A \cap B' = B' - A'$

Let, $n$ be an element of $A - B$ as :

$\Rightarrow \quad n \in (A - B)$

$\Rightarrow \quad n \in A \wedge n \in B$

$\Rightarrow \quad n \in A \wedge n \in B'$

$\Rightarrow \quad n \in (A \cap B')$

$\Rightarrow \quad n \in A' \wedge n \in B'$

$\Rightarrow \quad n \in B' \wedge n \in A'$

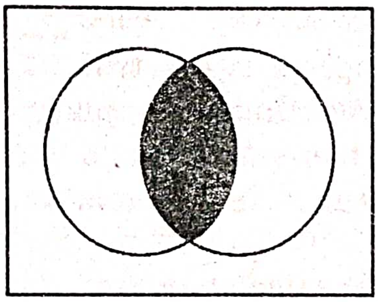
$\Rightarrow \quad n \in (B' - A')$ **(Proved)**

**9. *What do you understand by Venn diagram***

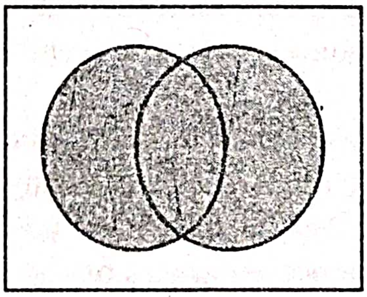
***Venn Diagram***

A Venn diagram, also called primary diagram, set diagram or logic diagram, is a diagram that shows all possible logical relations between a finite collection of different sets. These diagrams depict elements as points in the plane, and sets as regions inside closed curves. A Venn diagram consists of multiple overlapping closed curves, usually circles, each representing a set. The points inside a curve labeled $S$ represent elements of the set S, while points outside the boundary represent elements not in the set $S$. This lends itself to intuitive visualizations; for example, the set of all elements that are members of both sets $S$ and $T$, denoted $S \cap T$ and read "the intersection of $S$ and $T$", is represented visually by the area of overlap of the regions S and T. In Venn diagrams, the curves are overlapped in every possible way, showing all possible relations between the sets. They are thus a special case of Euler diagrams, which do not necessarily show all relations. Venn diagrams were conceived around 1880 by John Venn. They are used to teach elementary set theory, as well as illustrate simple set relationships in probability, logic, statistics, linguistics, and computer science.

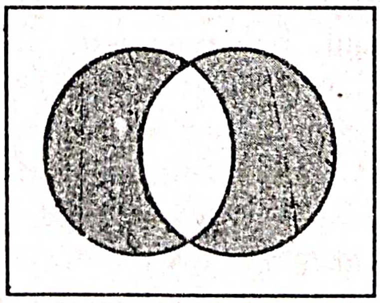
A Venn diagram in which the area of each shape is proportional to the number of elements it contains is called an area-proportional (or scaled Venn diagram).



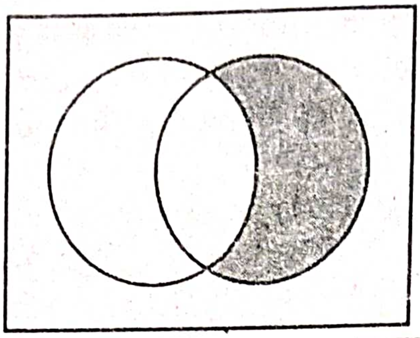
Intersection of two sets $\boxed{A \cap B}$



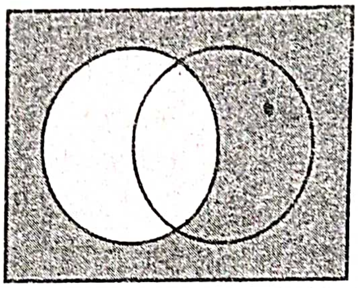
Union of two sets $\boxed{A \cup B}$



Symmetric difference of two sets $\boxed{A \Delta B}$



Relative complement of $A$ (left) in $B$ (right) $\boxed{A^c \cap B = B \setminus A}$



Absolute complement of A in U $\boxed{A^c = U \setminus A}$

A Venn diagram is constructed with a collection of simple closed curves drawn in a plane. According to Lewis, the "principle of these diagrams is that classes [or sets] be represented by regions in such relation to one another that all the possible logical relations of these classes can be indicated in the same diagram. That is, the diagram initially leaves room for any possible relation of the classes, and the actual or given relation, can then be specified by indicating that some particular region is null or is not-null".

Venn diagrams normally comprise overlapping circles. The interior of the circle symbolically represents the elements of the set, while the exterior represents elements that are not members of the set. For instance, in a two-set Venn diagram, one circle may represent the group of all wooden objects, while the other circle may represent the set of all tables. The overlapping region, or intersection, would then represent the set of all wooden tables. Shapes other than circles can be employed as shown below by Venn's own higher set diagrams. Venn diagrams do not generally contain information on the relative or absolute sizes (cardinality) of sets. That is, they are schematic diagrams generally not drawn to scale.

**10. *Prove that $A \cup A' = U$ and $A \cap A' = \phi$.* (2013-14)**

Any element of $A$ or $A'$ is in $U$. So $A \cup A' \subseteq U$

Let $x \notin U$ if $x$ is not $A$ i.e., $x \notin A$ then $x$ must be in $A'$

i.e., $x \in A \cup A'$ Hence $U \subseteq A \cup A'$

this means $A \cup A' \subseteq U$ and $U \subseteq A \cup A'$

i.e., $A \cup A' = U$

An element x in A if and only if x is not in $A'$

i.e., $A \cap A' = \phi$

**11. *Write the Set $A = \{2, -2, 3, -3, 4, -4.......\}$ in the Set Builder Form.* (2013-14)**

Given $A = \{2, -2, 3, -3, 4, -4...\}$

It is clear from the given set that $A$ is set of all integer except 0, -1, 1.

Hence,

$A = \{x : x = n \text{ or } x = -n \text{ for n } n \in Z \text{ and } x \neq 0, 1, -1...\}$

**12. *If set $A = \{x : x \in N, x$ is a factor of $10\}$ and $B = \{x : x \in N, x$ is a factor of $15\}$ where $N$ is set of Natural Numbers. Find $A - B$.* (2013-14)**

$A = \{x : x \in N, x \text{ is factor of } 10\}$

$A = \{1, 2, 5\}$

$B = \{x : x \in N, x \text{ is factor of } 15\}$

$B = \{1, 3, 5\}$

$A - B = \{1, 5\}$

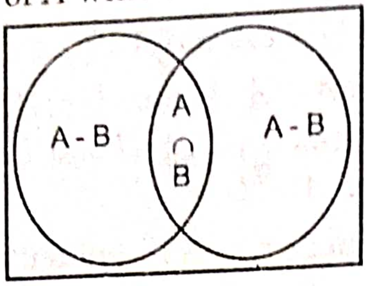
**13.**

- ***Show that for any two sets A and B in set theory : $A - (A \cap B) = A - B$.* (2018-19)**
- ***Show that for any two sets, A and B :* (2017-18)**

  $$A - (A \cap B) = A - B.$$

  ***Also draw Venn diagrams for both.***
- ***Show that for any two sets A and B. $A - (A \cap B) = A - B$. Also find the values $A - (A \cap B)$ and $A - B$ for set $A = \{1, 2, 3, 4, 5\}$ $B = \{2, 3, 4, 6\}$.* (2016-17)**

Now $A - B$ is the difference of the sets $A$ and $B$, If $A$ and $B$ are two sets then their difference $A - B$ is the set of all those elements of $A$ which do not belong to $A$.




(Figure)

$A \cap B$ is the intersection of two sets $A$ and $B$ which represent the set of all elements common to both $A$ and $B$.

Now from the figure, it is clear that $(A - B)$ & $A \cap B$ are disjoints sets and their union is $A$.

$\therefore (A - B) + A \cap B = A \rightarrow A - (A \cap B) = A - B$

$$A = \{1, 2, 3, 4, 5\} \text{ and } B = \{2, 3, 4, 6\}$$

So $A \cap B = \{2, 3, 4\} \Rightarrow A - (A \cap B) = \{1, 5\}$

$A - B = \{1, 5\} \Rightarrow \quad A - (A \cap B) = A - B$

14. ◆ ***Define the Cartesian Product of sets.*** **(2018-19)**

◆ ***If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{2, 4, 6, 8\}$, and $B = \{3, 5, 6, 7\}$ then find $A \times B$, $A - B$?***

**Cartesian Product of Sets**

If $A$ and $B$ are two non-empty sets, then their Cartesian product $A \times B$ is the set of all ordered pair of elements from $A$ and $B$.

$A \times B = \{(x, y) : x \in A, y \in B\}$

Suppose, if $A$ and $B$ are two non-empty sets, then the Cartesian product of two sets, $A$ and set $B$ is the set of all ordered pairs $(a, b)$ such that $a \in A$ and $b \in B$ which is denoted as $A \times B$.

**For Example :** If $A = \{7, 8\}$ and $B = \{2, 4, 6\}$, find $A \times B$.

**Solution :** $A \times B = \{(7, 2); (7, 4); (7, 6); (8, 2); (8, 4); (8, 6)\}$

The 6 ordered pairs thus formed can represent the position of points in a plane, if $A$ and $B$ are subsets of a set of real numbers.

**Numerical Solution :** If $U = \{1, 2, 3, 4, 5, 6, 7, 8\}$, $A = \{2, 4, 6, 8\}$, and $B = \{3, 5, 6, 7\}$

(1) $A \times B = \{2, 4, 6, 8\} \times \{3, 5, 6, 7\}$

$=\{(2, 3), (2, 5), (2, 6), (2, 7), (4, 3), (4, 5), (4, 6),$
$(4, 7), (6, 3), (6, 5), (6, 6), (6, 7), (8, 3), (8,5),$
$(8, 6), (8, 7)\}$ **(Ans.)**

(2) $A - B = \{2, 4, 6, 8\} - \{3, 5, 6, 7\}$
$=\{2, 4, 7\}$ **(Ans.)**

**15.** ***Define the Composite relation. And Let set $A = \{1, 2, 3\}$, $B = \{p, q, r\}$, $C = \{x, y, z\}$ and the relations are, $R = \{(1, p), (1, r), (2, q), (3, q)\}$ and $S = \{(p, y), (q, x), (r, z)\}$, then compute RoS.*** **(2018-19)**

**Composite Relation**

Let $R$ be a relation from a relation from a set $A$ to a set $B$ and $S$ a relation from $B$ to a set $C$. The composite of $R$ and $S$ is the relation consisting of the ordered pairs $(a, c)$ where $a \in A$ and $c \in C$, and for which there is $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of $R$ and $S$ by $S \circ R$.

**Example :**

**(1)** Let $A = \{1, 2, 3\}$, $B = \{0, 1, 2\}$ and $C = \{a, b\}$

**(2)** $R = \{(1, 0), (1, 2), (3, 1), (3, 2)\}$

**(3)** $S = \{(0, b), (1, a), (2, b)\}$

**(4)** $S \circ R = \{(1, b), (3, a), (3, b)\}$

**Numerical Solution :**

$A = \{1,2,3\}, B = \{p,q,r\}, C = \{x,y,z\}$

$R = \{(1,p),(1,r),(2,q),(3,q)\}$

$S = \{(p, y),(q,x),(r, z)\}$

$RoS = ?$

$RoS = \{(1, y),(1,z),(2, x),(3,x)\}$ **(Ans.)**

**16.** ***Let f and $g : R \to R$, be defined as follows :*** **(2017-18)**

$$f(x) = x + 2,\ g(x) = \frac{1}{(x^2+1)}.\ \textit{Compute } fog(x)$$

$fog = f(g(x))$

$$= f\left(\frac{1}{(x^2+1)}\right)$$

$$= \frac{1}{(x^2+1)} + 2$$

$$= \frac{2x^2+3}{(x^2+1)}$$

**17. *State and prove De Morgan's law for logic.*** (2017-18)

**De Morgan's Law**

The complement of the union of two sets is equal to the intersection of their complements and the complement of the intersection of two sets is equal to the union of their complements. These are called De Morgan's laws.

For any two finite sets $A$ and $B$;

(1) $(A \cup B)' = A' \cap B'$ (which is a De Morgan's law of union).

Let $P = (A \cup B)'$ and $Q = A' \cap B'$

Let $x$ be an arbitrary element of $P$ then $x \in P \Rightarrow x \in (A \cup B)'$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in A' \cap B'$

$\Rightarrow x \in Q$

Therefore, $P \subset Q$ ...(1)

Again, let $y$ be an arbitrary element of $Q$ then $y \in Q \Rightarrow y \in A' \cap B'$

$\Rightarrow y \in A'$ and $y \in B'$

$\Rightarrow y \notin A$ and $y \notin B$

$\Rightarrow y \notin (A \cup B)$

$\Rightarrow y \in (A \cup B)'$

$\Rightarrow y \in P$

Therefore, $Q \subset P$ ...(2)

Now combine (1) and (2) we get; $P = Q$ i.e. $(A \cup B)' = A' \cap B'$

(2) $(A \cap B)' = A' \cup B'$ (which is a De Morgan's law of intersection).

Let $M = (A \cap B)'$ and $N = A' \cup B'$

Let $x$ be an arbitrary element of $M$ then

$$x \in M \Rightarrow x \in (A \cap B)'$$
$$\Rightarrow x \notin (A \cap B)$$
$$\Rightarrow x \notin A \text{ or } x \notin B$$
$$\Rightarrow x \in A' \text{ or } x \in B'$$
$$\Rightarrow x \in A' \cup B'$$
$$\Rightarrow x \in N$$

Therefore, $M \subset N$ ...(1)

Again, let $y$ be an arbitrary element of $N$ then

$$y \in N \Rightarrow y \in A' \cup B'$$
$$\Rightarrow y \in A' \text{ or } y \in B'$$
$$\Rightarrow y \notin A \text{ or } y \notin B$$
$$\Rightarrow y \notin (A \cap B)$$
$$\Rightarrow y \in (A \cap B)'$$
$$\Rightarrow y \in M$$

Therefore, $N \subset M$ ...(2)

Now combine (1) and (2) we get; $M = N$ i.e. $(A \cap B)' = A' \cup B'$

| | | |
|---|---|---|
| 18. | ◆ | ***What is a Relation? Explain various types of Relations.*** |
| | ◆ | ***What do you mean by equivalence relations?*** **(2017-18)** |

## *Relation*

A relation between two sets $A$ and $B$ is a subset of Cartesian Product $(A \times B)$ of $A$ and $B$ i.e., $R \subseteq A \times B$

Here, we write $xRy$ if $(x, y) \in R$ and it is read as "$x$ is related to $y$ by the relation $R$." For example, consider two sets $A$ an $B$ as :

$A = \{a, b\}$
$B = \{c, d\}$

Then $A \times B = \{(a, c), (a, d), (b, c), (b, d)\} = C$

Now any sub-set of $C$ will be the relation of $A$ to $B$ i.e.:

$R_1 = \{(a, c)\}$

$R_2 = \{(a, c), (b, c)\}$

$R_3 = \{(a,d),(b, c), (b,d)\}$

**Some of the Properties of a Relation between Sets are**

**(1)** Number of relations from $A$ to $B$ is $2^{mn}$ where $m$ and $n$ refer to the number of elements in $A$ and $B$ respectively.

**Proof :** If $A$ and $B$ are two sets having $m$ and $n$ elements respectively then the Cartesian product of $AB$ i.e., $A \times B$ will have $mn$ elements and the set having $mn$ elements, will have $2^{mn}$ sub-sets. Since each subset of $A \times B$ is a relation from $A$ to $B$ thus obviously, there will be $2^{mn}$ relation from $A$ to $B$.

**(2)** Since $\phi$ (null set) is also an element of $A \times B$ and so a sub-set of $A \times B$ consequently $\phi$ is also a relation from $A$ to $B$ and generally known as void, null or empty relation.

**(3)** From $A \times B$ we also have :

$A \times B \in A \times B$

Thus, $A \times B$ also a Relation from $A$ to $B$ and known as Universal Relation.

**19.** ***Examples :***
***If $A = \{1, 2, 3\}$ and $B = \{2, 4\}$ then find the relation $R$ from $A$ to $B$ as $R$ is defined as "Is less than".***

**Solution**

$\because \quad A = \{1, 2, 3\}$

$B = \{2, 4\}$

Then, $\quad A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$

Since, $R$ is defined as "Is less than" so

$R = \{(1, 2), (1, 4), (2, 4), (3, 4)\}$

Or, it may written as :

$R = \{(x, y) : x < y \; x \in A, \; y \in B\}$

A relation $R$ between two sets can be represented by a matrix called Relation Matrix.

Let, $X = \{x_1, \; y_2 ... n_m\}$

And, $Y = \{y_1, \; y_2 ... y_n\}$

Then the relation matrix $M$ would be given as :

$$M = [a_{ij}]_{m \times n}$$

So that,

$$a_{ij} = 1$$

if, $(x_i, y_i) \in R$

And, $a_{ij} = 0$

if, $(x_i, y_i) \notin R$

For example, consider the sets $A$ and $B$ given in the previous problem. The relation matrix for the relation $R$ will be :

$$\begin{array}{c|cc} & 2 & 4 \\ \hline 1 & 1 & 0 \\ 2 & 0 & 1 \\ 3 & 0 & 1 \end{array}$$

## Domain and Range

If $R$ is a relation from $A$ to $B$ i.e., $R \subseteq A \times B$ then the set of all first elements of the ordered pairs belong to $R$ is called domain of $R$. Similarly, the set of all the last elements of the ordered pairs of $R$, is called Range of $R$.

**Example :** Let, we have :

$$A = \{1, 2, 3\}$$

And, $B = \{2, 3, 4\}$

Then, the relation "is equals" will be :

$$R = \{(2, 2), (3, 3)\}$$

Thus, the domain of $R$ :

$$D = \{2, 3\}$$

and Range $R = \{2, 3\}$

## Types of Relations

**(1) Inverse Relation :** If $R$ be a relation from set $A$ to $B$ then inverse of $R$ i.e., $R^{-1}$ will be a relation from $B$ to $A$ i.e., :

$$R^{-1} = \{(b,a) : (a,b) \in R\}$$

In other words, we may conclude that an inverse relation $(R^{-1})$ from set $B$ to $A$ contains ordered pairs which when reversed, produced an ordered pair lies at $R$.

**(2) Identity Relation :** An identify Relation $R$ in set $A$ to itself (i.e. $A \times A$) contains all those ordered pairs $(x, y)$ for which $x = y$

Thus, $R = \{(x, y) : x \in A,\ y \in A,\ x = y\} = I$

(An identity relation is generally denoted by $I$)

(3) **Reflexive Relation :** A relation $R$ in set $A$ is said to be reflexive if for all $a \in A$, $aRa$ holds i.e. '$a$' is related to itself by $R$.

For example, consider a set $A = \{a, b, c\}$ and its two relation to itself as :

$$R_1 = \{(a,b),(b,c),(b,b),(a,a)\}$$

$$R_2 = \{(a,a),(a,b),(b,c),(b,b),(c,c)\}$$

Here, $R_1$ is not Reflexive because for all $a \in A$ (i.e., $a$, $b$, $c$) it does not hold $aRa$. (There is no $(c, c)$ in $R_1$ as $c \in A$ but $R_2$ is Reflexive as it holds $aRa$ for all $c \in A$ let made is two Relations of the given set $A$ as :

$$R_1 = \{(a,a),(b,b),(c,c)\}$$

$$R_2 = \{(a,a),(b,b),(c,c);(a,b),(b,c)\}$$

The Relation $R_1$ is reflexive and also identity relation while $R_2$ is only reflexive because it doesn't identity relation. Thus, it is obvious now that it $I$ be the identity relation in $A$ and $R$ be the reflexive relation in $A$ and $R$ be the reflexive relation in $A$ then it must hold : $I \subset R$.

(4) **Symmetric Relation :** A relation $(R)$ in a set $A$ is said to be Anti-symmetric if for all $(a,b) \in R$, $aRb \Rightarrow bRa$.

(5) **Anti-Symmetric Relation :** A relation $R$ in $A$ is said to be Anti-symmetric if $(x,y) \in R$ ^ $(y,x) \in R \Rightarrow x = y$ i.e., a relation $R$ is said to be anti-symmetric if except for $s = y$, $R$ is not symmetric.

(6) **Transitive Relation :** A relation in set $A$ is said to be transitive if it holds :

$$(a,b) \in R, (b,c) \in R \Rightarrow (a,c) \in R$$

(7) **Equivalence Relation :** A relation $R$ is set $A$ is said to be Equivalence if :

(a) $R$ is Reflexive i.e., for all $a \in A$ $a R a$ holds

(b) $R$ is Symmetric i.e., for all $a, b \in A$ $aRb \Rightarrow bRa$

(c) $R$ is Transitive i.e., for all $a, b, c \in A$ $aRa$^$bRc \Rightarrow aRC$

An important implication of equivalence relation is : "If $R$ be an equivalence relation in set a then $R^{-1}$ is also an equivalence relation in $A$"

**Proof :** Given that $R$ is an equivalence Relation in $A$ i.e., $R$ is Reflexive, Symmetric and Transitive and to prove that $R^{-1}$ is also Equivalence, it must also hold Reflexive, Symmetric and Transitive Relation.

**(a)** Since $R$ is reflexive i.e., for all $a \in A, (a,a) \in R$

$$\Rightarrow \quad (a,a) \in R^{-1}$$

i.e., $R^{-1}$ is also reflexive.

**(b)** Since $R$ is symmetric i.e., for all $a,\ b \in A$

$$(a,b) \in R \Rightarrow (b,a) \in R$$

$$\Rightarrow \quad (b,\ a) \in R \Rightarrow (a,b) \in R \qquad (\because R \text{ is symmetric})$$

$$\Rightarrow \quad (a,b) \in R^{-1} \Rightarrow (b,a) \in R^{-1}$$

Thus, $R^{-1}$ is also symmetric.

**(c)** Since $R$ is transitive i.e., for all $(a,\ b,\ c) \leftarrow R$

$(a,\ b) \in R,\ (c,b) \in R \Rightarrow (a,\ b) \in R$

Since, $R$ is symmetric too. Thus, we may write :

$(b,a) \in R, (c,a) \in R \Rightarrow (a,b) \in R$

$(a,b) \in R^{-1},\ (b,\ c) \in R^{-1} \Rightarrow (a,c) \in R^{-1}$

i.e., $R^{-1}$ is transitive.

Since, $R^{-1}$ is Reflexive, Symmetric and Transitive. Thus, $R^{-1}$ is also Equivalence.

| 20. | ***Define linearly orders set.*** | **(2017-18)** |
|---|---|---|

***Linearly Orders Set***

A set in which a relation, as "less than or equal to," holds for all pairs of elements of the set. Also called chain, linearly ordered set, simply ordered set.

A total order (or "totally ordered set," or "linearly ordered set") is a set plus a relation on the set (called a total order) that satisfies the conditions for a partial order plus an additional condition known as the comparability condition. A relation $\leq$ is a total order on a set $S$ ("$\leq$ totally orders $S$") if the following properties hold.

(1) **Reflexivity :** $a \le a$ for all $a \in S$.
(2) **Antisymmetry :** $a \le b$ and $b \le a$ implies $a = b$.
(3) **Transitivity :** $a \le b$ and $b \le c$ implies $a \le c$.
(4) **Comparability (Trichotomy Law) :** For any, $a, b \in S$ either $a \le b$ or $b \le a$.

The first three are the axioms of a partial order, while addition of the trichotomy law defines a total order.

**21.** ◆ ***Prove that the relation "Congruence Modulo m", given by***
$'\Xi' = \{(x, y) \mid x - y \text{ is divisible by } m\}$ ***over the set of positive integers is an equivalence relation.***
◆ ***Also, show that if $x_1 \Xi y_1$ and $x_2 \Xi y_2$, then $(x_1 + x_2) \Xi (y_1 + y_2)$.*** **(2017-18)**

***Reflexive***

Let $x \in Z$, then $x \cong x(\text{mod}\, m)$ because $x - x = 0.m$ where $k = 0$

Thus $x \cong x(\text{mod}\, m) \forall x \in Z$

$\rightarrow xRx$. Hence congruence modulo $m$ is reflexive.

**Symmetric :** Let $xRy$, then $x \cong y(\text{mod}\, m)$

$\rightarrow x - y = km$ for $k \in Z$

$\rightarrow y - x = (-k)m$

Thus $y - x$ is divisible by $m$

$\rightarrow y \cong x(\text{mod}\, m)$

i.e. $xRy \rightarrow yRx$

Hence, congruence modulo $m$ is symmetric on $x$.

**Transitive :** Let $xRy$ and $yRz$ then $x \cong y(\text{mod}\, m)$ and $y \cong z(\text{mod}\, m)$

But $x \cong y(\text{mod}\, m) \rightarrow x - y = k_1 m$ for some $k_1 \in Z$

And $y \cong z(\text{mod}\, m) \rightarrow y - z = k_2 m$ for some $k_2 \in Z$

Therefore $x - z = (x - y) + (y - z)$

$= k_1 m + k_2 m = (k_1 + k_2)m$

i.e. $x \cong z(\text{mod}\, m)$

Hence, congruence modulo $m$ is transitive on $Z$. Thus, $R$ being reflexive, symmetric and transitive. Hence, it is an equivalence relation in $Z$.

**22. Define a relation R which is Reflexive, Symmetric, Anti-symmetric and Transitive for a set $A = \{1, 2, 3, 4, 5\}$.** (2016-17)

$R$ = [(1, 1) (1, 3), (3, 4) (1, 4) (2, 2) (3, 3) (4, 4) (5, 5)] is reflexive, Symmetric, anti – symmetric and transitive

**23. What is Composition of functions? Also prove that $f^1og^{-1} = (gof)^{-1}$ where $f: Q \to Q$ such that $f(x) = 4x$ and $f: Q \to Q$ such that $g(x) = x + 4$ are two functions.** (2016-17)

Let us consider function $f: A \to B$ and $g: B \to C$

Here the co-domain of $f$ is the domain of $g$. Then, we may define a new function from $A$ to $C$, called the composition of $f$ and $g$ and written $gof$ $(gof)(a) = g(f(a))$ i.e. we find the image of $a'$ under $f$ and then find the image of $f(a)$ under $g$. Since $f$ and $g$ are one-one and onto, then $f^{-1}$ and $g^{-1}$ exist and defined as :

$$f^{-1}: Q \to Q$$

such that $f^{-1}(x) = \frac{x}{4}$ and $g^{-1}: Q \to Q$

such that $g^{-1}(x) = x - 4$.

As we know that the product of two one-one onto mapping is one-one onto, therefore $gof: Q \to Q$ is one – one onto and $(gof)\, x = g(f(x)) = g(4x) = 4x + 4$

Since $gof$ is one – one onto therefore $(gof)^{-1}$ exists and defined as

$$(gof)^{-1}: Q \to Q \text{ such that } (gof)^{-1}(x) = \frac{x-4}{4} \quad ...(1)$$

Now $(f^{-1} o\, g^{-1}): Q \to Q$

$$\text{Where } (f^{-1} o\, g^{-1})(x) = f^{-1}(g^{-1}(x)) = f^{-1}(x-4) = \frac{x-4}{4} \quad ...(2)$$

From equation (1) and (2)

We get $(f^{-1} o\, g^{-1}) = (gof)^{-1}$

**24. *If A be non empty set with n elements then prove that the number of function from $A \to A$ is less than the number of relation from $A \to A$ i.e. $n^n < 2^{n^2}$.*** **(2016-17)**

Let the number of elements of $A$ and $B$ are '$m$' and '$n$' respectively. The number of elements of $A \times B$ is $mn$. Therefore, the number of elements of the powerset of $A \times B$ is $2^{mn}$.Thus $A \times B$ has $2^{mn}$ different subsets. We know every subset of $A \times B$ is a relation from $A$ to $B$ hence the number of different relations $A$ to $B$ is $2^{mn}$.

That is $2^{n^2}$ binary relation on a binary set $A$, because $A \times A$ has $n^2$ elements.

**25. *If $f : R \to R$ defined by $f(x) = 3x + 7$ is a one onto function, find $f^{-1}$.***

Since $f(x)$ is one – one onto function.
Hence $f^{-1}(x)$ exist and let us consider

$$f(x) = y = 3x + 7$$

$$x = \frac{y-7}{3}$$

$$f^{-1}(y) = \frac{y-7}{3}$$

$$f^{-1}(x) = \frac{x-7}{3}$$

Hence, the inverse is

$$f^{-1}(x) = \frac{x-7}{3}$$

**26. *If $A = \{a, b, c, d, e\}$, $B = \{a, c, e, g\}$ and $C = \{b, e, f, g\}$ then prove the following :***

**(1) $(A \cup B) \cap C \neq A \cup (B \cap C)$**

**(2) $(A - B) \cap (A - C) = A - (B \cup C)$** **(2015-16)**

$A = \{a, b, c, d, e\}$ $B = \{a, c, e, g\}$
$C = \{b, e, f, g\}$

**(1) $(A \cup B) \cap C \neq A \cup (B \cap C)$ :**

L.H.S. $= A \cup B = \{a, b, c, d, e, g\}$

We know $C = \{b, e, f, g\}$

So $(A \cup B) \cap C = \{a, b, c, d, e, g\} \cap \{b, e, f, g\}$
$= \{b, e, g\}$

Now R.H.S. $= B \cap C = \{a, c, e, g\} \cap \{b, e, f, g\}$
$= \{e, g\}$

And $A \cup (B \cap C) = \{a, b, c, d, e\} \cup \{e, g\}$
$= \{a, b, c, d, e, g\}$

Now it can be seen that
L.H.S. $\neq$ R.H.S. **(Hence Proved)**

**(2)** $(A - B) \cap (A - C) = A - (B \cup C)$

Now,
$A - B = \{a, b, c, d, e\} - \{a, c, e, g\} = \{b, d\}$
$A - C = \{a, b, c, d, e\} - \{b, e, f, g\} = \{a, c, d\}$

So,
L.H.S. $= (A - B) \cap (A - C) = \{b, d\} \cap \{a, c, d\} = \{d\}$
$B \cup C = \{a, c, e, g\} \cup \{b, e, f, g\} = \{a, b, c, e, f, g\}$
R.H.S $= A - (B \cup C) = \{a, b, c, d, e\} - \{a, b, c, e, f, g\} = \{d\}$

It can be seen that
L.H.S. = R.H.S. **(Hence Proved)**

**27. *Let R be an equivalence relation over set of integers I defined as R = {(a, b) | a – b is divisible by 5}. Find the equivalence classes of set I.***
**(2015-16)**

At first we prove that $R$ is equivalence relation over $I$ and defined as

$R = \{(a.b) \mid a - b \text{ is divisible by } 5\}$

**(1)** **Reflexivity** : Given any integer '$n$' we have $n - n = 0$ and 0 is divisible by 5. Thus $R$ is reflexive

**(2)** **Symmetry** : If $x - y$ is divisible by 5 then its negative $-(x - y)$ is also divisible by 5. i.e. $(y - x)$

**(3)** **Transitivity** : We know that $m - n = 5k$ and $n - y = 5j$. For some integer $k$ and $j$, thus $m - n + n - y = m - y = 5\,(y + j)$ i.e. divisible by 5.

Hence $R$ is equivalence relation.

There are exactly 5 equivalence classes which is as follows :

$$A_0 = \{...,-10,-5,0,5,10...\} = [0]$$
$$A_1 = \{...,-9,-4,1,6,11...\} = [1]$$
$$A_2 = \{...,-8,-3,2,7,12...\} = [2]$$
$$A_3 = \{...,-7,-2,3,8,13...\} = [3]$$
$$A_4 = \{...,-6,-1,4,9,14...\} = [4]$$
$$Z/R = \{[0],[1],[2],[3],[4]\}$$

**28. *Check whether the following function $f : R \to R$ are one – one onto :***
**(1) $f(x) = e^x$**
**(2) $f(x) = |x|$** **(2015-16)**

(1) $f(x) = e^x$

Let $x_1, x_2, \in R$ such that $f(x_1) = f(x_2)$

$\Rightarrow \quad e^{x1} = e^{x2}$

Taking log of both sides, we get

$x_1 \log_e^e = x_2 \log_e^e$

$\Rightarrow \quad x_1 = x_2$ Hence $f$ is one – one

Further let $y = e^x$

$x = \log y$

$f$ is not onto because logarithm of a negative real number is not a real number. Thus there are many elements in the co-domain which do not have their pre – images in the domain.

$\therefore \quad f$ is not onto

(2) $f(x) = |x| = \begin{cases} x & if\ x \geq 0 \\ -x & if\ x < 0 \end{cases}$

Let $x_1, x_2, \in R$ such that $f(x_1) = f(x_2)$

$|x_1| = |x_2|$

This does not imply that $x_1 = x_2$. Hence $f$ is not one – one $f$ is not onto because, there are many elements in the co – domain which do not have their pre – image in the domain. For example, there is no element in the domain whose image is –1, –2, –3... etc. Hence $f$ is not onto.

**29. *If $R^{-1}$ and $S^{-1}$ are the inverse of relation $R$ and $S$ respectively, then prove that***
**$(SoR)^{-1} = R^{-1}oS^{-1}$.** **(2015-16)**

Suppose $R$ is a relation from $A$ to $B$ and $R^{-1}$ is a relation from $B$ to $A$. Similarly, $S$ is a relation from $B$ to $C$. Thus $S^{-1}$ be the relation from $C$ to $B$.

Thus, the relation $S^{-1}$ or $R^{-1}$ is from $C$ to $A$. Now, $xRy$ and $yRz$ then $x\ (RoS)$ and $Z(RoS)^{-1}x$. But $z\ S^{-1}\ y$ and $y\ R^{-1}\ x$ so that $z\ (R^{-1} \text{ o } S^{-1})\ x$

Hence $(RoS)^{-1} = S^{-1}oR^{-1} \ \forall \ x \in A, Z \in C$

**30. *Show that relation "xRy iff ( x – y ) is divisible by 5" is an equivalence relation on the set of integers* (2014-15)**

$xRY$ *iff* $(x\text{-}y)$ is divisible by 5"

If $xRx \ \forall x$, then $\{x \in I\}$

$x - x = 0$, that is divisible by 5

So, $R$ is reflexive

Now, $xRy$ means $(x - y)$ is divisible by 5

$\Rightarrow$ $-(x - y)$ is divisible by 5.

$\Rightarrow$ $(y - x)$ is divisible by 5

$\Rightarrow$ $yRx$.

So, $R$ is symmetric.

Again, $xRy$ and $yRz$.

$\Rightarrow$ $(x - y)$ is divisible by 5 and $(y - z)$ is divisible by 5.

$\Rightarrow$ $[(x - y) + (y - z)]$ is divisible by 5.

$xRz$.

So $R$ is transitive

So, since $R$ is reflexive, symmetric and transitive hence we can say that $R$ is equivalence on set of integers.

**31. *In a class of 120 students, 80 students study mathematics, 45 study history and 30 students study both the subjects. Find the number of students who study neither mathematics nor history.* (2014-15)**

Let, Total no. of students $n(A) = 120$

Total mathematics students $n(B) = 80$

Total history student $n(C) = 45$

Total no. of students who study both mathematics and history $= n(B \cap C) = 30$

As, we know that

$$n(A \cup B) = n(A) + n(B) + n(B \cap C) + n(\overline{B} \cap \overline{C})$$

So, Here

$$n(A) = n(B) + n(C) - n(B \cap C) + n(\overline{B} \cap \overline{C})$$

$$120 = 80 + 45 - 30 + n(\overline{B} \cap \overline{C})$$

$$n(\overline{B} \cap \overline{C}) = 120 + 30 - 125 = 25$$

$$n(\overline{B} \cap \overline{C}) = 25$$

(Ans.)

**32.** ***Given $R = \{(1, 1), (2, 2), (2, 3), (3, 3), (3, 4), (4, 3), (3,1)\}$ for $A = \{1, 2, 3, 4\}$ make its relation matrix $M_R$ and check whether it is symmetric, reflexive or transitive.*** **(2014-15)**

$R$ {(1, 1), (2, 2), (2, 3), (3, 3), (3, 4), (4, 3), (3, 1)}

For $A = \{1, ,2, 3, 4\}$

So, Relation Matrix $M_R$ is :

$$M_r = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Now from $M_R$ Matrix we see that

**Symmetric :** There is relation (2, 3) but not (3, 2) hence relation is not symmetric.

**Reflexive :** Due to Absence of (4, 4) here this relation is not reflexive.

**Transitive :** Since there is (3, 4), (4, 3) and (3, 3) present but (2, 3), (3, 1) is there but there is not any relation like (2, 1) hence it is not transitive also.

**33.** ***If $A = \{1, 3, 4\}$, $B = \{2, 3, 4\}$, $C = \{1, 2, 3\}$ then find $(A \times B) - (A \times C)$ and $(B \times C) \cap (B \times A)$.*** **(2014-15)**

$A = \{1, 3, 4\}$, $B = \{2, 3, 4\}$, $C = \{1, 2, 3\}$

$A \times B = \{(1, 2), (1, 3), (1, 4), (3, 2), (3, 3), (3, 4),$
$(4, 2), (4, 3), (4, 4)\}$

$A \times C = \{(1, 1), (1, 2), (1, 3), (3, 1), (3, 2), (3, 3),$
$(4, 1), (4, 2), (4, 3)\}$

Now $(A \times B) - (A \times C) = \{(1, 4), (3, 4), (4, 4)\}$

Now $B \times C = \{(2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (3, 3),$
$(4, 1), (4, 2) (4, 3)\}$

Now $B \times A = \{(2, 1), (2, 3), (2, 4), (3, 1), (3, 3), (3, 4),$
$(4, 1), (4, 3) (4, 4)\}$

Hence, $(B \times C) \cap (B \times A) = \{(2, 1), (2, 3), (3, 1), (3, 3),$
$(4, 1), (4, 3)\}$

**34.** ***Let P and Q be two sets. If $P \to Q$ is one-one onto, then prove that $f^{-1} : Q \to P$ is also one-one onto.*** **(2014-15)**

Let $f$ be the function which is one-one onto

Hence, $f : P \to Q$ is one – one onto.

Hence $y = f(x) \Rightarrow f^{-1}(y) = x, x \in P, y \in Q$

Let $y_1 = f(x_1), y_2 = f(x_2), x_1, x_2 \in P; y_1, y_2 \in Q$

Then $x_1 = f^{-1}(y_1), x_2 = f^{-1}(y_2)$

$f^{-1}(y_1) = f^{-1}(y_2) \Rightarrow x_1 = x_2 \Rightarrow f(x_1) = f(x_2)(\because f$ is a function)

$\Rightarrow \quad y_1 = y_2$

$\therefore$ The mapping $f^{-1}$ is one-one

Again, let $x$ be an arbitrary element of $P$, then for mapping $F$ there exists an element $y$ in $Q$ such that

$y = F(x)$.

But

$f^{-1}(y) = x \forall x \in P.$

$\{f^{-1}(y) : y \in Q\} = P$

Thus $f^{-1} : Q \to P$ is onto

Hence, $f^{-1} : Q \to P$ is one – one onto mapping.

**35.** ***Let X = {1, 2, 3, 4} then Relation R = {(1, 3}, (1, 1), (3, 1), (1, 2), (3, 3), (4, 4)} is Reflexive or not, explain it.*** **(2013-14)**

Given $X = \{1, 2, 3, 4\}$ and Relation $R = \{(1, 3), (1, 1), (3, 1), (1, 2), (3, 3), (4, 4)\}$

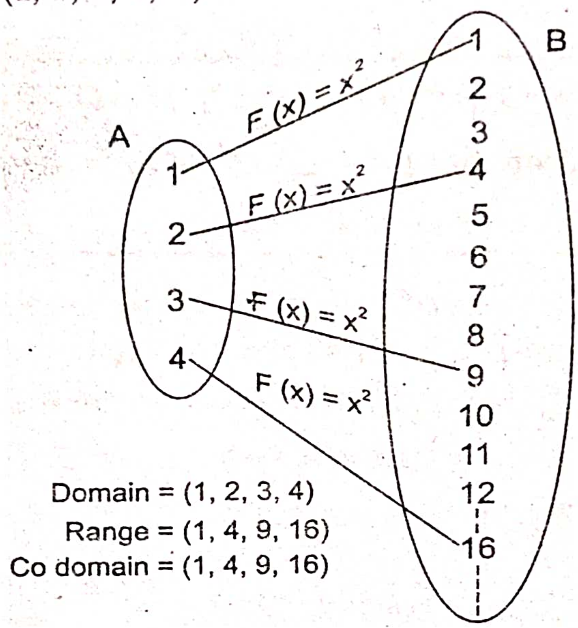
It is not reflexive because relation (2, 2) is not there.

**36.** ***Function $f : (A \to B)$ is defined $f(x) = x^2$ where set $A = \{1, 2, 3, 4\}$ and B = set of Natural numbers. Find Domain, Range and Co-domain of function f.*** **(2013-14)**

Function $f: (A \rightarrow B); f(x) = x^2$

$A = (1, 2, 3, 4,)$

$B = (1, 2, 3, 4, \ldots)$




(Figure)

| | |
|---|---|
| 37. | ◆ *Let a and b be integers and suppose Q (a, b) is defined Recursively by :* $$Q(a,b) = \begin{cases} 5 & \text{if } a < b \\ Q(a-b, b+2) + a & \text{if } a \geq b \end{cases}$$ ◆ *Find :* (1) $Q(2, 7)$ (2) $Q(5, 3)$ (3) $Q(15, 2)$ **(2013-14)** |

Here given $Q(a,b) = \begin{cases} 5 & \text{if} \quad a < b \\ Q \ (a-b, b+2) + a & \text{if } a \geq b \end{cases}$

**(1)** $\boldsymbol{Q}$ **(2, 7) :**

Here $a = 2, b = 7$ and $b > a$

Hence $Q\ (a, b) = 5$

**(2)** $\boldsymbol{Q}$ **(5, 3) :**

Here $a > b$, so

$Q\ (5, 3) = Q\ (5\text{-}3, 3 + 2) + 5$

$= Q\ (2, 5) + 5 \ \{Q\ (2, 5) = 5$

$\therefore a < b\} = 5 + 5 = 10$

**(3)** $\boldsymbol{Q}$ **(15, 2) :**

Here $a = 15, b = 3$ and $a > b$

$$Q(15, 2) = Q(15 - 2, 2 + 2) + 15$$
$$= Q(13, 4) + 15 = Q(13 - 4, 4 + 2) + 15 + 13$$
$$= Q(9, 6) + 28 = (9 - 6, 6 + 2) + 28 + 9$$
$$= Q(3, 8) + 37 = 5 + 37 = 42$$

38. ◆ *Consider the following four relations :*

*(1) Relation ≤ (less than or equal) on the set Z of integers.*

*(2) Set inclusion ⊆ on a collection C of sets.*

*(3) Relation ⊥ (perpendicular) on the set L of lines in the plane.*

◆ *Relation || (parallel) on the set L of lines in the plane. Determine which of the relations are Reflexive. Symmetric, Transitive and Anti-symmetric.* **(2013-14)**

Reflexive :

(1) Is reflexive,

(2) is reflexive

(3) Is reflexive and

(4) Is reflexive.

Symmetric :

(1) Is not symmetric

(2) Is also not symmetric

(3) Is symmetric.

(4) Is symmetric.

Transitive :

(1) Is transitive

(2) Is transitive

(3) Is not transitive

(4) Is transitive.

Antisymmetric : None except (i) are antisymmetric.

39. ◆ *Let function $f : A \to B$ and $g : B \to C$ be both one-one and Onto functions, then prove gof : $A \to C$ is also one- one and onto.* **(2013-14)**

◆ *If $f : A \to B$ and $g : B \to C$ be one-to-one onto function, then $g \circ f$ is also one-to-one onto and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.* **(2010-11)**

For this statement we have to prove

(1) *gof* is one – one

Let $x_1, x_2 \in A$ such that

$$(gof)\ (x_1) = gof\ (x_2)$$

$$\Rightarrow \quad g(f(x_1)) = g(f(x_2))$$

But $g$ is one-one, therefore $f(x_1) = f(x_2)$

Also $f$ is one – one, thus $x_1 = x_2$

$\therefore$ The mapping $gof$ is one – one.

**(2)** $g\ o\ f$ is onto

Let $z \in c$. Since $g$ is an onto mapping of $B$ to $C$ therefore exists $y \in B$ such that $g(y) = z$. Again, since $f$ is an onto mapping of $A$ to $B$, therefore there exists $x \in A$ such that $f(x) = y$. Hence for every $z \in c$ there exist $x \in A$ such that $(gof)(x) = g(f(x) = g(y) = z$

Thus $gof : A \rightarrow C$ is onto.

**(3)** Since, $g\ o\ f$ is one – one onto mapping hence $g\ o\ f$ is invertible. Let $z \in c$ such that $z = g(y)$

For all $y \in B$, then $y = g^{-1}(z)$

Also let $y = f(x) \Rightarrow x = f^{-1}(y)$

Now $(g\ o\ f)(x) = g\ (f(x)) = g(y) = z$

$$(gof)^{-1}\ (z) = x \qquad ...(1)$$

$$\text{Also } \left(f^{-1}og^{-1}\right)(z) = f^{-1}\left(g^{-1}(z)\right) = f^{-1}(y) = x$$

$$\left(f^{-1}og^{-1}\right)(z) = x \qquad ...(2)$$

From equation 1 and 2 we observe

$$(gof)^{-1} = f^{-1}og^{-1}$$

**40. *If* $A = \{1, 2, 4\}$, $B = \{2, 5, 7\}$ *and* $C = \{1, 3, 7\}$ *find* $(A \times B) \cap (A \times C)$. (2011-12)**

$A = \{1,2,4\}$ $B = \{2,5,7\}$ $C = \{1,3,7\}$

$A \times B = \{(1, 2), (1, 5), (1, 7), (2, 2), (2, 5), (2, 7), (4, 2), (4, 5), (4, 7)\}$

$A \times C = \{(1, 1), (1, 3), (1, 7), (2, 1), (2, 3), (2, 7), (4, 1), (4, 3), (4, 7)\}$

$(A \times B) \cap (A \times C) = \{(1, 7), (2, 7), (4, 7)\}$ **(Ans.)**

**41. *If* $f, g : R \rightarrow R$ *be defined as* : (2011-12)**

$$f(x) = ax + b,\ g(x) = 1 - x + x^2,\ \forall\ x \in R$$

***If* $(gof)\ (x) = 9x^2 - 9x + 3$, *determine* $(a, b)$.**

$f(x) = ax + b$, $g(x) = 1 - x + x^2$ and $(gof)\ (x) = 9x^2 - 9x + 3$

$$g(f(x)) = g(ax+b) = 1-(ax+b)+(ax+b)^2$$

$$9x^2 - 9x + 3 = 1 - ax - b + a^2x^2 + b^2 + 2axb$$

Comparing the coefficient of different power of $x$

We get $x^2 = 9$ and $2ab - a = -9$

So, value of $(a, b)$ are $(3, -1)$ and $(-3, 2)$. **(Ans.)**

**42. *Show that the relation "equality" defined in any set A is an equivalence relation.*** **(2011-12)**

Let $R$ is the relation on the set of strings of Hindi letters.

let $l(a) = l(b)$ where $l(x)$ is the length of string $x$.

Since $l(a) = l(b)$,it follows that $aRa$ whenever $a$ is the string so that $R$ is reflexive.

Suppose $aRb$, So that $l(a) = l(b)$. Then $bRa$, since $l(b) = l(a)$. Hence $R$ is symmetric.

Again suppose that $aRb$ and $bRc$ i.e. $l(a) = l(b)$ and $l(b) = l(c)$ hence

$l(a) = l(c)$ which implies $aRc$, Consequently $R$ is transitive

Since $R$ is reflexive, symmetric, transitive, it is an equivalence relation.

**43. *Give the power set of the set given below :***

***A={a, {b}}.***

$P(A) = \{\{a\}, \{b\} \{a, \{b\}\}, \{\ \}\}$.

**44. *Prove that the complement of union of two sets is the intersection of their complements.*** **(2011-12)**

Let $s$ be any arbitrary element of the set $(A \cup B)'$ then

$$X \in (A\cup B)' = X \notin (A \cup B) = X \notin A \text{ or } X \notin B$$
$$= X \in A' \text{ and } X \in B' = X \in (A' \cap B')$$

$(A \cup B)' \subseteq (A' \cap B')$ ... (1)

Conversely, let $X$ be any arbitrary element of the set $A' \cap B'$

The $X \in (A' \cap B') = X \in A'$ and $X \in B' = X \notin A$ and $X \notin B$

$= X$ does not belong to both $A$ and $B$

$= X \notin (A \cup B) = X \in (A \cup B)'$

$A' \cap B' \subseteq (A \cup B)'$ ... (2)

From equation (1) and (2) we prove that complement of the union of two sets is the intersection of their complements. **(Hence Proved)**

**45.** ***Let $X = \{1, 2, ..., 7\}$ and $R = \{(x, y) : x - y$ is divisible by 3\}, show that $R$ is an equivalence relation.***
**(2011-12)**

We can easily verify that $R$ is an equivalence relation on $Z$. We can determine the members of equivalent classes as follows.

For each integer $a$

$$
\begin{aligned}
[a] &= \{x \in Z : x \, Ra\} \\
&= (x \in Z : x - a \text{ is divisible by } 3) \\
&= (x \in Z : x - a = 3k \text{ ,for some integer } k) \\
&= (x \in Z : x = 3k + a, \text{for some integer } k)
\end{aligned}
$$

In particular

$$
\begin{aligned}
[0] &= (x \in Z : x = 3k + 0 \text{ ,for some integer } k) \\
&= (x \in Z : x = 3k \text{ ,for some integer } k) \\
&= \{\ldots\ldots\ldots -6, -3, 0, 3, 6, \ldots\ldots\ldots\} \\
[1] &= (x \in Z : x = 3k + 1 \text{ ,for some integer } k) \\
&= (\ldots\ldots\ldots -5, -2, 1, 4, 7, \ldots\ldots\ldots) \\
[2] &= (x \in Z : x = 3k + 2 \text{ ,for some integer } k) \\
&= (\ldots\ldots\ldots -4, -1, 2, 5, \ldots\ldots\ldots)
\end{aligned}
$$

There are no other equivalence classes because every integer is already accounted for, in one of [0], [1], [2].

**46.** ***What are the recursively defined functions? Give the recursive definition of factorial function.***
**(2011-12)**

Recursion refers to several related concepts in computer science and mathematics. One can use recursion to define sequences, functions and sets. The sequence 1, 3, 9, 27 ... for examples, can be defined explicitly by the formula $S(n) = 3^n$ for all integers $n > 0$, but the sequence can also be defined recursively as follows :

**(1)** $S(0) = 1$

**(2)** $S(n+1) = 3S(n)$ for all integers $n > 0$

A recursive definition has two parts :

**(a)** Definition of the smallest argument (usually $f(0)$ or $f(1)$).

**(b)** Definition of $f(n)$, given $f(n-1)$, $f(n-2)$, etc.

Here is an example of a recursively defined function :

$$\begin{cases} f(0) = 5 \\ f(n) = f(n-1) + 2 \end{cases}$$

We can calculate the values of this function :

$f(0) = 5$

$f(1) = f(0) + 2 = 5 + 2 = 7$

$f(2) = f(1) + 2 = 7 + 2 = 9$

$f(3) = f(2) + 2 = 9 + 2 = 11$

This recursively defined function is equivalent to the explicitly defined function $f(n) = 2n + 5$. However, the recursive function is defined only for non negative integers.

**47.** ***Let $X = \{1, 2, 3\}$, $Y = \{p, q\}$ and $Z = \{a, b\}$. Let $f : X \to Y$ be $f = \{(1, p), (2, p), (3, q)\}$ and $g : Y \to Z$ be given by $g = \{(p, b), (q, b)\}$. Find gof and show it pictorially.*** **(2011-12)**

Given $X = \{1, 2, 3\}$, $Y = \{p, q\}$, $Z = \{a, b\}$

$f : X \to Y,\ f = \{(1, p)(2, q), (3, q)\}$

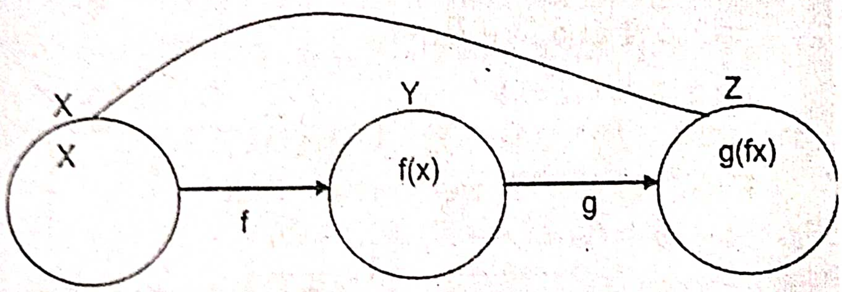
$g : Y \to Z,\ g = \{(p, q)(q, b)\}$

then gof : $X \to Z$ is defined by

$(gof)\ (1) = g(f(1)) = g(p) = b$

$(gof)\ (2) = g(f(2)) = g(q) = b$

$(gof)\ (3) = g(f(3)) = g(q) = b$




(Figure)

**48.** ***Prove that union of two countable infinite set is countably infinite.*** **(2011-12)**

An infinite set $A$ is said to be countably infinite (or denumerable) if it is equivalent to the set $N$ of natural numbers.

Consider the sets $A_i = \{a_{1i}, a_{2i}, a_{3i}, \ldots\}$, $i = 1,2,3, \ldots$ Each $A_i$, $i = 1,2,3$ is countable .The $k^{th}$ element of

$A_i$ is $a_{ki}$ .The elements of the countable union U $A_i$ of the sets $A_i$'s can be listed as $a_{11}$, $a_{12}$, $a_{13}$, $a_{22}$, $a_{31}$, $a_{14}$, $a_{23}$, $a_{32}$, $a_{41}$,...(the order has been taken according to the sum $i + j = k$, $k$ =2,3,...$i$, $j$ being the suffices of the element $a_{ij}$ ε $A_j$) .The one-one correspondence between the elements of U $A_i$ and we got the set of positive integers Hence the set U $A_i$ is countably infinite.

**49.** ***Prove that :*** **(2010-11)**

$$A - (B \cap C) = (A - B) \cup (A - C)$$

***For all sets A, B and C.***

Let $x \in A - (B \cap C)$ A − B

$\Rightarrow x \in A \,\text{and} \notin x (B \cap C)$

$\Rightarrow (x \in A) \,\text{and}\, (x \notin B \,\text{or}\, x \in C)$

$\Rightarrow (x \in A$ or $x \notin B)$and $(x \in A$ or $x \notin C)$

$\Rightarrow (x \in (A - B))$ and $(x \in (A - C))$

$\Rightarrow x \in [(A - B) \cup (A - C)]$

$\Rightarrow A - (B \cap C) \subseteq (A - B) \cup (A - C)$

Let $y$ be any element of $(A - B) \cup (A - C)$, then

$y \in (A - B) \cup (A - C) \Rightarrow [y \in (A - B)]\text{and}[y \in (A - C)]$

$\Rightarrow [y \in A \,\text{or}\, y \notin B]\text{and}[y \in A$ or $y \notin C]$

$\Rightarrow y \in A \,\text{and}[y \notin B$ or $y \notin c]$

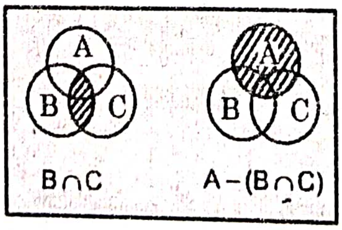
$\Rightarrow y \in A$ and $[y \notin B \cap C]$

$\Rightarrow y \in A - (B \cap C)$

$\Rightarrow (A - B) \cup (A - C) \subseteq A - (B \cap C)$

From (1) and (2) we conclude that

$A - (B \cap C) = (A - B) \cup (A - C)$




(Figure)

**50.** ***If I be the set of all integers and if the relation R be defined over the set I by x Ry if x − y is an even integer, where x, y ∈ I, show that R is an equivalence relation.*** **(2010-11)**

**Reflexive :** Let $x \in z$, then $x - x = 0 = 0.2 \Rightarrow (x - x)$ is even

$\Rightarrow xRx \, \forall x \in z$

Therefore $R$ is reflexive

**Symmetric :** It $xRy$, then $xRy \Rightarrow (x\text{-}y)$ is even

$\Rightarrow x - y = 2p$

$\Rightarrow y - x = 2(-p)$

$\Rightarrow y - x$ is even {for some $p \in z, -p \in z$}

$\Rightarrow yRx$

Therefore, $R$ is symmetric

**Transitive :** Let $xRy$ and $yRz$, then

$xRy \Rightarrow (x - y)$ is even

$\Rightarrow x - y = 2.p$

And $yRz \Rightarrow (y - z)$ is even

$\Rightarrow y - z = 2.q$

$\therefore$ $xRy$ and $yRz \Rightarrow x - y = 2.p$ and $y - z = 2q$

$\Rightarrow (x - y) + (y - z) = 2.p + 2.q = x - z = 2.(p + q)$

$= x - z$ is even

$\Rightarrow xRz$

Therefore, $R$ is reflexive, symmetric and transitive and hence is an equivalence relation in $Z$.

| | | |
|---|---|---|
| **51.** | ◆ | ***Consider the functions f, g : R → R, defined by f(x) = 2x + 3 and g(x) = x² + 1.*** **(2010-11)** |
| | ◆ | ***Find the composition function (g o f) (x) and (f o g) (x).*** |

Since $f : R \to R$ and $g : R \to R$, then obviously $g \circ f$ : $R \to R$ and fog : $R \to R$

Also $(gof)(x) = g(f(x) = g(2x + 3) = (2x + 3)^2 + 1$

$= 4x^2 + 12x + 10$

And $(fog)(x) = f(g(x)) = f(x^2 + 1) = 2(x^2 + 1) + 3$

$= 2x^2 + 5$

Clearly, $(gof)(x) \neq (fog)(x)$.

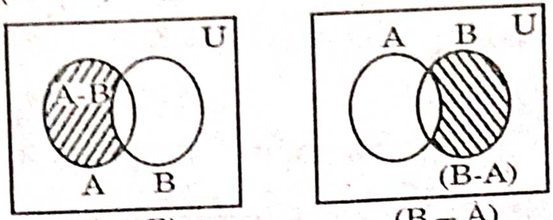
| | | |
|---|---|---|
| **52.** | **(1)** | ***When is A − B = B − A? Explain.*** |
| | **(2)** | ***Show that any positive integer n greater than or equal to 2 is either a prime or products of primes.*** |

(1) When is $A - B = B - A$

Let $A$ and $B$ be two sets. Thus,

$(A - B) = \{x : x \in A \text{ and } n \notin B)$

$(B - A) = \{x : x \in B \text{ and } n \notin A)$




(A – B) (B – A)

(Figure)

$(A - B) = (B - A)$ is possible if both set have the same element and result will be empty.

**Example :**

$A = \{1,2,3\}$, $B = \{1,2,3\}$

$A - B = \{\phi\}$, $B - A = \{\phi\}$

Now, both are equal. Otherwise not possible.

(2) Here $P(n)$ is the predicate '$n$' is a prime or $n$ is a product of primes.

**Step I : (Basis of Induction) :** Since 2 is a prime $P(2)$ is true.

**Step II : (Induction Hypothesis) :** Assume that $P(n)$ is true for any integer $n$ such that $2 \leq n \leq k$ i.e., $P(3)$, $P(4)$,....$P(k)$ are true.

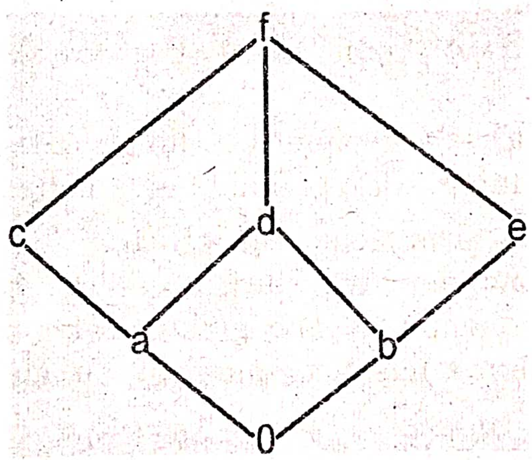
**Step III : (Induction Step) :** Now consider $P(k + 1)$. If $k + 1$ is a prime then $P(k + 1)$ is true. If $k + 1$ is not a prime, then $k + 1 = rs$, where, $2 \leq r \leq k$ and $2 \leq s \leq k$. But by air induction hypothesis, $P(r)$ is true and $P(s)$ is true.

Therefore, $r$ and $s$ are either primes or product of primes, and therefore $k + 1$ is a product of primes. So, $P(k + 1)$ is true. Therefore, $P(n)$ is true $\forall n \geq 2$.

□□

# Posets/Lattices/Boolean Algebra

1. ◆ *What do you mean by Distributive Lattices?* (2020-21)
   ◆ *Define Complemented lattice example.* (2018-19)
   ◆ *What do you mean by distributed lattice and complemented lattice? Consider the bounded lattice L, given below. Check whether it is distributive or not.* (2017-18)




(Figure)

**Distributive Lattice**

Distributive Lattice is a lattice in which the operations of join and meet distribute over each other. The prototypical examples of such structures are collections of sets for which the lattice operations can be given by set union and intersection. Indeed, these lattices of sets describe the scenery completely: every distributive lattice is up to isomorphism given as such a lattice of sets.

As in the case of arbitrary lattices, one can choose to consider a distributive lattice $L$ either as a structure of order theory or of universal algebra. Both views and their mutual correspondence are discussed in the article on lattices. In the present situation, the algebraic description appears to be more convenient :

A lattice $(L,\vee,\wedge)$ is distributive if the following additional identity holds for all $x$, $y$, and $z$ in $L$:

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

Viewing lattices as partially ordered sets, this says that the meet operation preserves non-empty finite joins. It is a basic fact of lattice theory that the above condition is equivalent to its dual :

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z) \text{ for all } x, y, \text{ and } z \text{ in } L$$

**Example :** Every Boolean algebra is a distributive lattice.

## *Complemented Lattice*

Complemented Lattice is a bounded lattice (with least element 0 and greatest element 1), in which every element a has a complement, i.e. an element $b$ satisfying $a \vee b = 1$ and $a \wedge b = 0$. Complements need not be unique.

A relatively complemented lattice is a lattice such that every interval $[c,d]$, viewed as a bounded lattice in its own right, is a complemented lattice.

A complemented lattice is a bounded lattice (with least element 0 and greatest element 1), in which every element a has a complement, i.e. an element $b$ such that

$$a \vee b = 1 \text{ and } a \wedge b = 0.$$

In general an element may have more than one complement. However, in a (bounded) distributive lattice every element will have at most one complement. A lattice in which every element has exactly one complement is called a uniquely complemented lattice.

A lattice with the property that every interval is complemented is called a relatively complemented lattice. In other words, a relatively complemented lattice is characterized by the property that for every element $a$ in an interval $[c,d]$ there is an element $b$ such that

$$a \vee b = d \text{ and } a \wedge b = c$$

Such an element $b$ is called a complement of $a$ relative to the interval.

A distributive lattice is complemented if and only if it is bounded and relatively complemented.

**Figure Solution :** In figure, it is not distributive lattice because $c$ has two complement $e$ and $b$.

**2. *Define Boolean algebra and illustrate it with example.*** **(2020-21)**

**Boolean Algebra**

A finite lattice in called a Boolean Algebra if it is isomorphic with $B_n$ for some non-negative integer $n$.

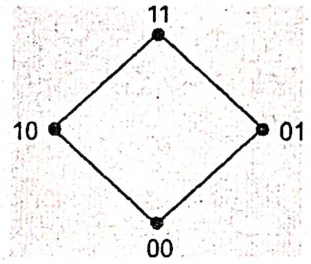
**Example :** $B_0$ 0

$n = 0$

$B_1$

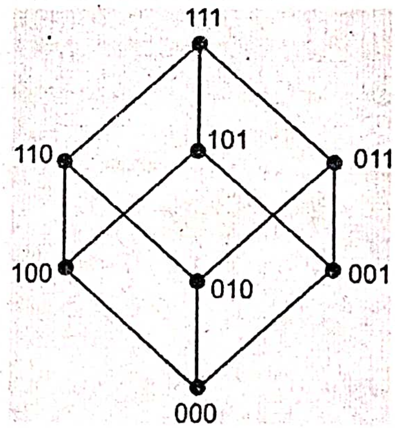

$n = 1$

$B_2$




$n = 2$

$B_3$

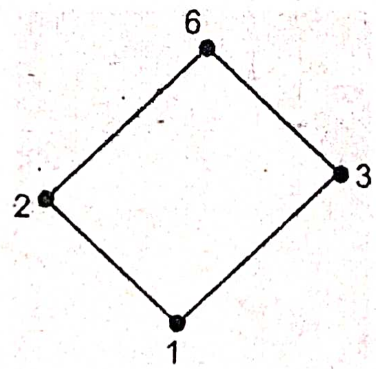



$n = 3$

**(Figure)**

If any lattice is isomorphic with $B_0, B_1, B_2, B_3$ then it is called as Boolean algebra.

Hasse Diagram of $D_6$ is as follows :



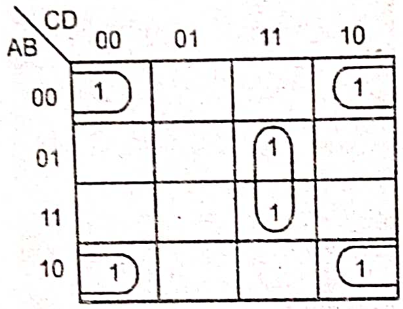

**(Figure)**

This diagram is isomorphic with $B_2$. So $D_6$ is a Boolean Algebra.

**3.** ***Use Karnaugh map representation to find minimal sum of products expression for the following Boolean function :***

$$F(A, B, C, D) = \Sigma(0, 2, 7, 8\ 10, 15)$$ **(2020-21)**

It uses 4 variable $k$ – Map




B' D' +BCD

**(Figure)**

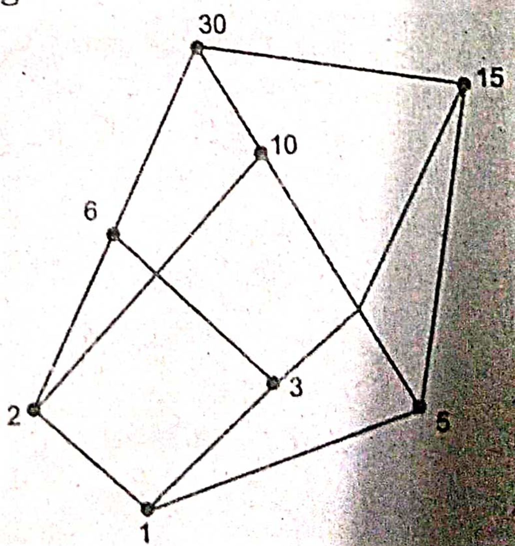
**4.** ***Let D be the set of all factors of 30 and let '/' be the divisibility relation on L. Then show that (D, '/') is a lattice.*** **(2020-21)**

**Lattice**

A lattice is poset in which every super subset $\{a, b\}$ consists of a $LUB$ and $GLB$

First we will draw Hasse Diagram for $D_{30}$

**In this Diagram :**




**(Figure)**

$I = 1 =$ Least Element

$O = 30 =$ Greatest Element

Here we can check of every pair, there exist a least upper bound & greatest lower Bound.

5. ◆ ***State and prove the De-Morgan's law of Boolean Algebra.*** **(2020-21)**
   ◆ ***State the De-Morgan's Laws of Boolean Algebra.*** **(2018-19)**
   ◆ ***Give a brief description of Laws of Equivalence.***

Two statement formula $A$ and $B$ which consists of variables $P_1, P_2, P_3 \ldots P_n (n \geq 1)$ are said to be equivalent, if they acquire same truth values for all possible values of $P_1, P_2 \ldots$

The equivalency between statement formula can be obtained either by drawing the truth table for each of given formula or by applying laws of equivalence. Here, the method of drawing truth table is quite insufficient and difficult in case of more than two variables thus we should prefer the later method.

***Equivalence Laws :***

Some of the important laws are given below :

(1) **Idempotent Law :**
   (a) $A \vee A \Leftrightarrow A$
   (b) $A \wedge A \Leftrightarrow A$

(2) **Commutative Law :**
   (a) $(A \vee B) \Leftrightarrow (B \vee A)$
   (b) $(A \wedge B) \Leftrightarrow (B \wedge A)$

(3) **Associative Laws :**
   (a) $A \vee (B \vee C) \Leftrightarrow (A \vee B) \vee C$
   (b) $A \wedge (B \wedge C) \Leftrightarrow (A \wedge B) \wedge C$

(4) **Distributive Laws :**
   (a) $A \vee (B \wedge C) \Leftrightarrow (A \vee B) \wedge (A \vee C)$
   (b) $A \wedge (B \vee C) \Leftrightarrow (A \wedge B) \vee (A \wedge C)$

(5) **D'morgan's Laws :**
   (a) $\rceil(P \vee Q) \Leftrightarrow \rceil P \wedge \rceil Q$
   (b) $\rceil(P \wedge Q) \Leftrightarrow \rceil P \vee \rceil Q$

(6) **Conditional Law :**
   $P \rightarrow Q \Leftrightarrow \rceil P \vee Q$

(7) **Bi-conditional Law :**
$P \leftrightarrow Q \Leftrightarrow (P \to Q) \wedge (Q \to P)$

6.. ◆ ***Define Partially Ordered Set. Explain the properties, a Poset must satisfy?*** **(2017-18)**
◆ ***Define partially ordered set with example.*** **(2011-12)**

A relation $R$ on a set $S$ is called a partial ordering if it is reflexive, anti-symmetric and transitive. That is

(1) $aRa$ for all $a \in S$ (reflexivity)
(2) $aRb$ and $bRa \to a=b$ (antisymmetry)
(3) $aRb$ and $bRc \to aRc$ (transitivity) for $a, b, c \in S$

A set $S$ together with a partial order relation $R$ is called partially ordered set or a Poset. It is denoted by $(S, R)$.

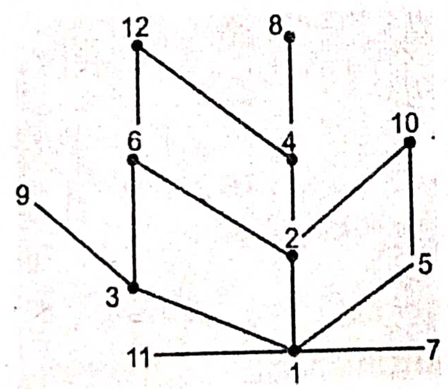
**Example :** The relation $\geq$ is a partial ordering on the set of integers $Z$ and the set $Z^+$ of all positive integers under divisibility relation forms a poset.

7. ***Let $S = \{1,2,3...,12\}$ be a poset under divisibility relation. Draw Hasse diagram & find first and last element. Also find upper bound, Lower bound, Least upper bound, Greatest Lower Bound for the subset $\{5, 7, 8\}$.*** **(2016-17)**

First element is 1 and last elements are 7, 8, 9, 10, 11, 12.

There is no upper bound because there is no integer in $S$ which is divisible by 5, 7 and 8. Lower bound is 1 since 5, 7, 8 are divisible by 1.

So $glb = \{1\}$ and lub $= \phi$




**(Figure)**

8. ◆ *Consider the Poset S = ({1, 2, 3, 4, 6, 9, 12, 18, 36},/). Find the Greatest Lower Bound and Least Upper Bound of the sets {6, 18} and {4, 6, 9}.* (2018-19)

◆ *If A = {1, 2, 3, 4, 6, 12, 18, 36} be ordered by the relation "a divides b". Then draw the Hasse diagram.*

◆ *Consider the partially order set,* (A, '|') where set A = {1, 2, 3, 4, 6, 9, 12, 18, 36} *ordered by divisibility and poset* (B, '|') *where* B = {2, 4, 6, 8} :

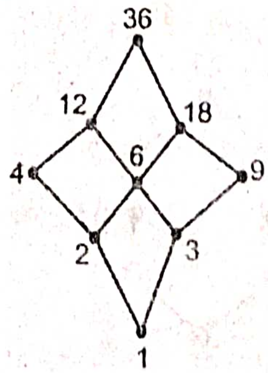
(1) *Draw Hasse diagram, of both poset. Find first and last element if exists.*

(2) *Find minimal and maximal element.*

(3) *Find least upper bound i.e. supremum and greatest lower bound i.e. infimum of every pair of elements of poset (B, '|').*

(4) *Is poset lattices?* (2013-14)

(1) Hasse diagram for set $A$ = {1, 2, 3, 4, 6, 9, 12, 18, 36}




First element = 1

Last element = 36

(Figure)

Hasse diagram for

$B$ = {2, 4, 6, 8}


First element = 2

Last element = 8

(Figure)

For $A$ minimal element = 1 and maximal element is = 36

(2) For $B$ minimal element is = 2 and maximal element is = 8.

(3) For (2, 4) : greatest lower bound = 4, least upper bound = 4;

For (4, 6) : greatest lower bound = 2, least upper bound = 8;

For (6, 8) : greatest lower bound = 2, least upper bound = 8;

For (2, 8) : greatest lower bound = 2, least upper bound = 8;

(4) Yes, these posets are lattices.

**Numerical Solution :** lub of {6, 18} = {36}

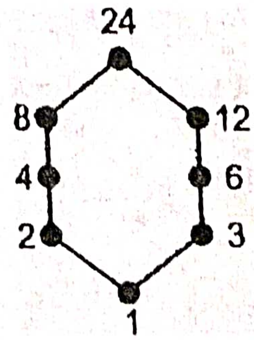
glb of {6, 18} ={1}

lub of {4, 6, 9} = {36}

glb of {4, 6, 9} = {1}.

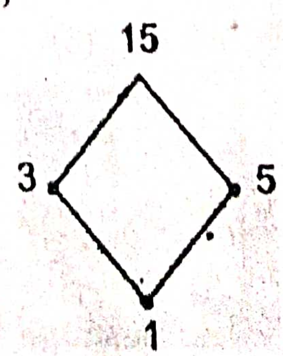
**9.** ***Let $D_m$ denote the positive divisors of integers $m$ ordered by divisibility. Draw the Hasse diagrams of* : (1) $D_{24}$, (2) $D_{15}$.** **(2018-19)**

(1) $D_{24}$ = {1, 2, 3, 4 6, 8, 12, 24}




**(Figure)**

(2) $D_{15}$ = {1, 3, 5, 15}



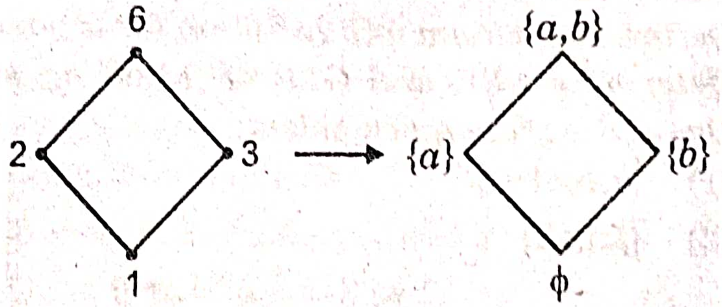

**(Figure)**

**10. *Let $A = \{1, 2, 3, 6\}$ and Let $\leq$ the divisibility relation on $A$ and let $B = \{\phi, \{a\}, \{b\}, \{a, b\}\}$ and the relation $\leq$ be the relation $\subseteq$. Then show that $(A, \leq)$ and $(B, \subseteq)$ are isomorphic posets.* (2018-19)**

$A = \{1, 2, 3, 6\}$ and $\leq$ be the relation /.

$B = \{\phi, \{a\}, \{b\}, \{a, b\}\}$ and $\leq$ be the relation $\subseteq$.

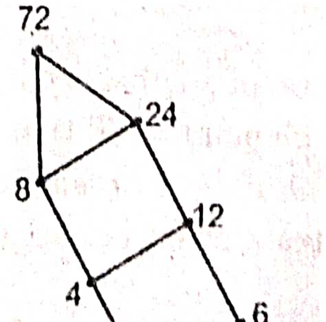
If $f(1) = \phi$, $f(2) = \{a\}$, $f(3) = \{b\}$, $f(6) = \{a, b\}$ then $f$ is an isomorphism. They have the same Hasse diagrams.




(Figure)

**Note** : A Hasse diagram uniquely determines the relations in a partially in a partially ordered set, so two isomorphic hasse diagrams represent two isomorphic posets, provided that they are isomorphic as directed graphs.

**11. *Draw the hasse diagram of poset $(D_{72}, '|')$. '|' represents the divisibility operation.* (2017-18)**



12. *Let* $A = \{1,2,3,4,5,6\}$. *Compute* $(4,1,3,5)o(5,6,3)$.

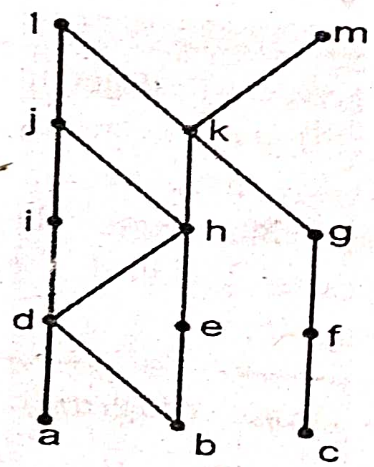
(2017-18)

| o | 5 | 6 | 3 |
|---|---|---|---|
| 4 | (4 5) | (4 6) | (4 3) |
| 1 | (1 5) | (1 6) | (1 3) |
| 3 | (3 5) | (3 6) | (3 3) |
| 5 | (5 5) | (5 6) | (5 3) |

13. ***Define Supremum and infimum for a partial order. Determine LUB and GLB of following subsets for Hasse diagram given below :*** (2017-18)

(1) $\{a,b,c\}$

(2) $\{f,g,h\}$




(Figure)

**Infimum**

The infimum of a subset $S$ of a partially ordered set $T$ is the greatest element in $T$ that is less than or equal to all elements of $S$, if such an element exists. Consequently, the term greatest lower bound (GLB) is also commonly used.

**Supremum :** The supremum of a subset $S$ of a partially ordered set $T$ is the least element in $T$ that is greater than or equal to all elements of $S$, if such an element exists. Consequently, the supremum is also referred to as the least upper bound (or LUB).

A lower bound of a subset $S$ of a partially ordered set $(P,\leq)$ is an element $a$ of $P$ such that

(1) $a \leq x$ for all $x$ in $S$.

A lower bound $a$ of $S$ is called an infimum (or greatest lower bound, or meet) of $S$ if

(2) For all lower bounds $y$ of $S$ in $P$, $y \leq a$ ($a$ is larger than any other lower bound).

Similarly, an upper bound of a subset $S$ of a partially ordered set $(P,\leq)$ is an element $b$ of $P$ such that

(3) $b \geq x$ for all $x$ in $S$.

An upper bound $b$. of $S$ is called a supremum (or least upper bound, or join) of $S$ if

(4) For all upper bounds $z$ of S in $P$, $z \geq b$ ($b$ is less than any other upper bound)

**Solution :** All the elements in the figure is LUB of $(a,b,c)$

No element in the figure is GLB of $(a,b,c)$.

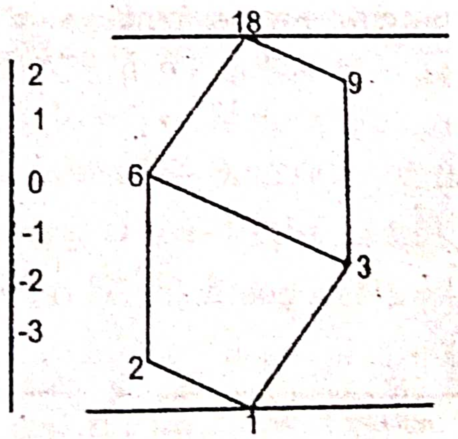
LUB of $(f,g,h)$ is $j$, $k$, $l$, $m$

No element in the figure is GLB of $(f,g,h)$

| 14. | *Define the bounded lattice.* | (2016-17) |
|---|---|---|

**Bounded Lattice**

Let '$L$' be a lattice w.r.t $R$ if there exists an element $I \in L$ such that $(aRI)$ $\forall$ $x \in L$, then $I$ is called upper bound of a lattice $L$.




(Figure)

Similarly if there exist an element $O \in L$ such that $(ORa)$ $\forall a \in L$, then $O$ is called lower bound of lattice $L$.

In a lattice if upper bound and lower exists then it is called bounded lattice.

**Example :**

(1) $[R;\leq]$ $R$ is the set of real number

(2) $D_{18}=\{1,2,3,6,9,18\}$

Here you can easily see For Ex. (1) there is no upper and lower bounds are present but in Ex. (2) both upper bound (18) and lower bound (1) are present.

**15. *If $(A,\leq)$ $(B,\leq)$ are Posets, then $(A\times B,\leq)$ is a poset with partial order defined by $(a,b)\leq(a',b')$ if $a\leq a'$ in $A$ and $b\leq b'$ in $B$.*** **(2016-17)**

To show that $(A \times B, \leq)$ is a poset we must show that $\leq$ is reflexive antisymmetric and transitive.

(1) Suppose $(a, b) \in A \times B$ be any element then $(a, b) \leq (a, b)$.
Since $a \leq a$ in $A$ and $b \leq b$ in $B$. Hence $\leq$ is reflexive in $A \times B$.

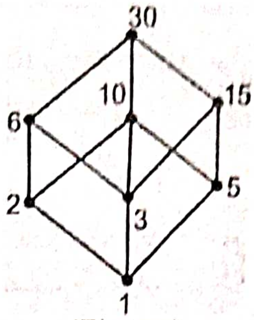
(2) For $a, a' \in A$ and $b, b' \in B$.
Let $(a, b) \leq (a', b')$ and $(a', b') \leq (a, b)$
($a \leq a'$ in $A$, $b \leq b'$ in $B$) and ($a' \leq a$ in $A$, $b' \leq b$ in $B$)
i.e. $a \leq a'$ and $a' \leq a$ in $A \rightarrow a = a'$
$b \leq b'$ and $b' \leq b$ in $B \rightarrow b = b'$
$\therefore \leq$ is antisymmetric in $A \times B$.

(3) For $(a, b) \leq (a', b')$ and $(a', b') \leq (a'', b'')$
whereas, $a', a'' \in A$ and $b, b', b'' \in B$.
$a \leq a'$ and $a' \leq a''$ in $A \rightarrow a \leq a''$
$b \leq b'$ and $b' \leq b''$ in $B \rightarrow b \leq b''$
Hence $(a,b)\leq(a'',b'')\rightarrow\leq$ is transitive in $A \times B$.
Thus $A \times B$ is a posed as $\leq$ is reflexive, ant symmetr and transitive.

**16. *Draw the Hasse diagram of $<D_{30},\backslash>$, where $D_{30}$ is the set of all the divisors of 30.*** **(2015-16)**

The Hasse diagram is given below :

$A = \{1, 2, 3, 5, 6, 10, 15\}$




(Figure)

| | |
|---|---|
| 17. ◆ | *Define lattice. Discuss properties of Lattices.* (2015-16) |
| ◆ | *What is a Lattice? Explain in term of POSET.* |

Let $(X, \leq)$ be a Poset and $A \subseteq X$ (i.e., $A$ is subset of $A$) then an element $y \in X$ said to be an upper bound of $A$ if for all $a \in A$, $a \leq u$ and similarly an element $v \in A$, $b \geq v$.

With lower and upper bound, given above, we may also have two elements $a'$ and $b'$ known as Least Upper Bound (or Supernum, generally refer as glb).

As name suggest lub is a least element of upper bound of a set and glb is a greatest element of lower bound of the set. It is not necessary for a set to have an upper bound or lower bound. If it has then it not necessary that it belong to '$A$'.

**Example :** If '$R$' be the set of real numbers then consider a subset $N$ as :

$N \subseteq R$ (where $N$ is set of natural numbers)

Obviously, lower bound of $N$ will be :

$R^{-} = \{0, -1, -2, -3, ...\}$ where $R^{-}$ is :

The set of all negative numbers also the glb of (i.e., greatest lower bound) $N$ will be 0. However there is no upper bound exist.

Now, the lattice is another class of Poset where every two element sub-set of $X$ has unique lub and glb. A lattice is given as $(L, \leq)$.

**Properties of Lattice**

A lattice have following properties :

(1) **Dual Nature :** In a given lattice $(L, \leq)$, the least upper bound (i.e., $V$) of $a$ and $b$ (as $a, b \in L$)

is equals to its greatest lower bound i.e. if $\leq$ is a partial order set then $\geq$ will also be partial order.

(2) Idempotent :

(a) $a \wedge a = a$

(b) $a \vee a = a$

(3) Commutative :

(a) $a \wedge b = b \wedge a$

(b) $a \vee b = b \vee a$

(4) Distributive :

(a) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(b) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(5) Associative :

(a) $a \vee (b \vee c) = (a \vee b) \vee c$

(b) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$

**Problems :**

**If $(L, \leq)$ is a lattice, then for $a, b, c \in L$ prove that :**

$$a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c.$$

Let, $a \leq c$, now from distributive property. We have,

$a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c) = (a \vee b) \wedge c$

Thus, $a \vee (b \wedge c) \leq (a \vee b) \wedge c$

Conversely if $a \vee (b \wedge c) \leq (a \vee b) \wedge c$ then

$(a \vee b) \wedge c \leq (a \vee b) \wedge (a \wedge c)$

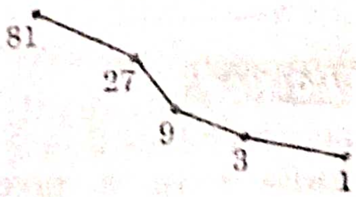
$(a \vee b) \vee c \leq (a \vee b) \vee (a \vee c)$

From Cancellation Law, we have :

$c = a \vee c \Leftrightarrow a \leq c$

**18.** ***Let D(81) be the set of all positive divisors of 81. Then show that D(81) under the binary relation 'divides' is a poset. Is the poset totally ordered.***

The Hasse diagram for the given Poset will be :




(Figure)

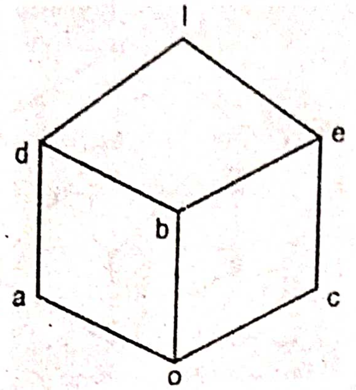
$\because \quad D(81) = (1, 3, 9, 27, 81)$

**Relation Set**

{(1,3), (1,9), (1, 27), (1, 81), (3, 9), (3, 27), (3, 81), (9, 27), (9, 81), (27, 81)}

Since, for every two element $x, y \in D(81)$, $x \leq y$. Thus, the given set is totally ordered.

**19.** ***Consider the lattice L given below :*** **(2015-16)**




(Figure)

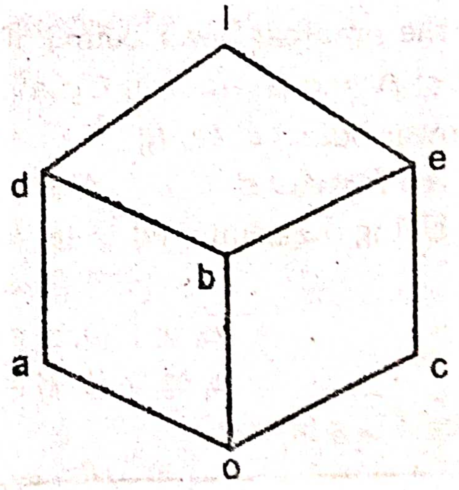
(1) ***Find all sub-lattices with 5 elements.***

(2) ***Find atoms.***

(3) ***Find complement of a and b if they exists.***

(4) ***Is L distributive?***

(5) ***Is L complemented?***




(Figure)

(1) All sub lattice with 5 elements *oabdl, oacdl, oadel, obcel, oacel, ocdel.*

(2) Atoms are *o* and *l*.

(3) *c* and *e* are complements of *a* and *b* has no complement.

(4) No.

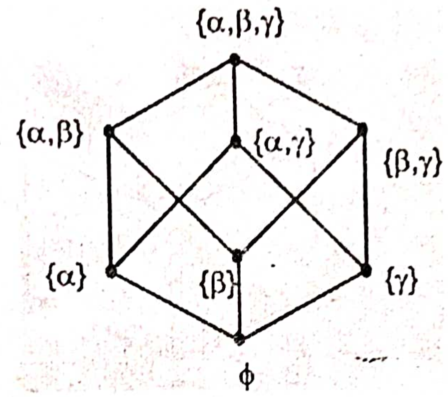
(5) No.

**20. *Draw the Hasse diagram of $(P(S), \subseteq)$ where $S = \{\alpha, \beta, \gamma\}$, and show that it is a lattice.* (2014-15)**

Here given $S = \{\alpha, \beta, \gamma\}$, then

$P(S) = \{\phi, \{\alpha\}, \{\beta\}, \{\gamma\}, \{\alpha, \beta\}, \{\alpha, \gamma\}, \{\beta, \gamma\}, \{\alpha, \beta, \gamma\}\}$

The Hasse diagram of $(P(S), \subseteq)$ is




(Figure)

If $A$, $B$, $\subseteq$, $S$, then an upper bound of $\{A, B\}$ is a subset of $S$ which contains both $A$ and $B$ and the least among them is $A \cup B \in P(S)$. To see this, note that

$A \subseteq A \cup B$ and $B \subseteq A \cup B$

And if $A \subseteq C$ and $B \subseteq C$ then it follows that $A \cup B \subseteq C$ similarly the greatest lower bound of $\{A, B\}$. On the other hand, if $C \subseteq$ A and $C \subseteq B$ then $C \subseteq A \cap B$. Then $A \cap B$ is the greatest lower bound of $\{A, B\}$

$A \vee B = A \cup B$; $\quad A \wedge B = A \cap B$

Using diagram ; $\{a\} \vee \{b\} = \{a, b\}$

$\{a\} \wedge \{b\} = \phi$

$\{a, b\} \wedge \{b, c\} = \{b\}$

$\{a, b\} \vee \{b, c\} = \{a, b, c\}$

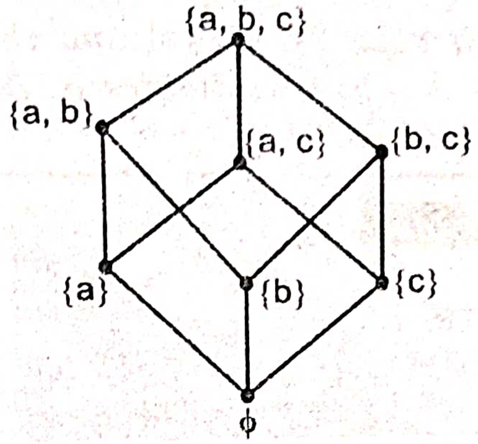
Hence $S$ is a lattice.

**21. *Draw the Hasse diagram for partial ordering of subset with powerset P (A) where set A = {1, 2, 3}. Is lattice or not? Explain it.* (2013-14)**

Let $A = \{1, 2, 3\}$, then

$P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{2, 3\}, \{3, 1\}, \{1, 2, 3\}\}$

The Hasse diagram of the poset $(P(S), \subseteq)$ is




(Figure)

The diagram of the poset $(P(S), \subseteq)$ is lattice. It is lattice because it has greatest lower bound and least upper bound.

**22.** ***Consider the Lattice $D_{20}$ = {1, 2, 4, 5, 10, 20} the divisor of 20 ordered by divisibility :*** **(2013-14)**

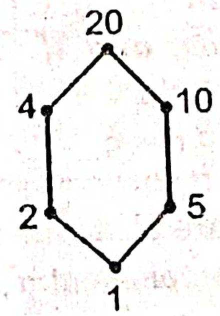
**(1)** ***Draw Hasse diagram.***

**(2)** ***Find the complements, if exist of 2, 4 and 10.***

**(3)** ***Is lattice complemented or not? Explain it.***

(1) Hasse diagram is :

$D_{20} = \{1, 2, 4, 5, 10, 20\}$




(Figure)

(2) Complements of 2 :

$2 \wedge 5 = 1$ & $2 \vee 5 = 20$

So, complements of 2 is 5.

Complements of 4 :

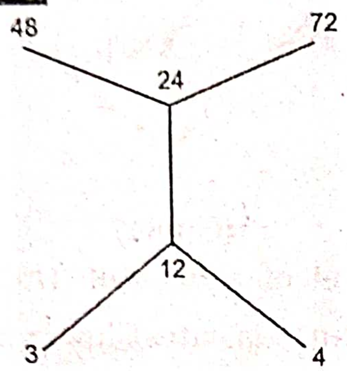
$4 \wedge 10 = 1$ & $4 \vee 10 = 20$

Similarly, a complement of 10 is 4.

(3) Yes, this lattice is complemented because every element in this lattice has complement.

**23. *Let $A = \{3, 4, 12, 24, 48, 72\}$ and the relation $\leq$ be such that $a \leq b$, if a divides b. Draw the Hasse diagram of $(A, \leq)$.*** **(2011-12)**

**Hasse Diagram**




(Figure)

**24. *Prove that $0' = 1$ and $1' = 0$.*** **(2011-12)**

By Boundedness law, we know

$0 + 1 = 1$ and $1.0 = 0$

Thus, by uniqueness of complement

$1 = 0'$

And by duality

$1' = 0$. **(Hence Proved)**

**25. *Prove the if 'a' and 'b' are elements in a bounded distributive lattice and if 'a' has complement a', then : $a \vee (a' \wedge b) = a \vee b$*** **(2011-12)**

***$a \wedge (a' \vee b) = a \wedge b$***

Since $a$ and $b$ are elements in a bounded distributive lattice and $a$ has a complement $a'$, therefore

$a \vee a' = 1$ ... (1)

$a \wedge a' = 0$ ... (2)

Where 1 and 0 are bounds of lattice.

Now $a \vee (a' \wedge b) = (a \vee a') \wedge (a \vee b)$ using distributive property

$= 1 \wedge (a \vee b)$ using (1) $= a \vee b$ **(Hence Proved)**

And

$a \wedge (a' \vee b) = (a \wedge a') \vee (a \wedge b)$ using distributive property

$= 0 \vee (a \wedge b)$ using (2)

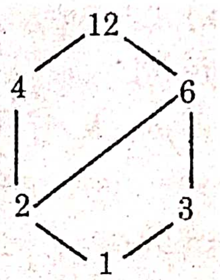
$= a \wedge b$ **(Hence Proved)**

**26.** ***Let X be the set of factors of 12 and let $\leq$ be the relation divider i.e., $x \leq y$ if and only if $x/y$. Draw the Hasse diagram of $(X, \leq)$.*** **(2010-11)**

Let us consider a Lattice $L=\{1, 2, 3, 4, 6, 12\}$ of factors of 12 under divisibility. Then $L' = \{1, 2\}$ and $L'' = \{1, 3\}$ are sublattices of $L$ as we show from the Hasse diagram $L' \cup L'' = \{1, 2, 3\}$ is not a sublattice

$\because 2, 3 \in L' \cup L''$

But $2 \cup 3 = 6 \in L' \cup L''$




**(Figure)**

Hence union of two sublattices $L'$ and $L''$ may not be sublattice.

**27.** ***Prove that the product of two lattices is a lattice.*** **(2010-11)**

Let us consider two Lattices $L$ and $L'$ and $L \times L' = \{(a, b) : a \in L, b \in L'\}$ is partially ordered set under the relation $\leq$ such that,

$< a_1, b_1 > \leq < a_2, b_2 > \Leftrightarrow a_1 \leq a_2$ in $L$ and $b_1 \leq b_2$ in $L'$

Now, we prove that $L \times L'$ forms a lattice.

If $<a_1, b_1>, <a_2, b_2> \in L \times L'$ then $a_1, a_2 \in L$ and $b_1, b_2 \in L'$

$\because$ $L$ and $L'$ are Lattices, hence sup and inf of $\{a_1, a_2\}$ and $\{b_1, b_2\}$ must exist in $L$ and $L'$ respectively.

Again consider $a_1 \wedge a_2 = \inf\{a_1, a_2\}$

and $b_1 \wedge b_2 = \inf\{b_1, b_2\}$

then $a_1 \wedge a_2 \leq a_1, a_1 \wedge a_2 \leq a_2$

and $b_1 \wedge b_2 \leq b_1, b_1 \wedge b_2 \leq b_2$

$\Rightarrow \quad < a_1 \wedge a_2, b_1 \wedge b_2 > \leq < a_1, b_1 >$

and $<a_1 \wedge a_2, b_1 \wedge b_2> \leq < a_1, b_2>$

$\Rightarrow <a_1 \wedge a_2, b_1 \wedge b_2> \leq <a_2, b_1> \;\&\; <a_1 \wedge a_2, b_1 \wedge b_1> \leq <a_2, b_2>$

$\Rightarrow \quad <a_1 \wedge a_2, b_1 \wedge b_2>$ is lower bound of

$\{<a_1, b_1>, <a_2, b_2>\}$ Now, suppose that $<a_3, b_3>$ is any lower bound of $\{<a_1, b_1>, <a_2, b_2>\}$

then $<a_3, b_3> \leq <a_1, b_1>$ and $<a_3, b_3> \leq <a_2 . b_2>$

$\Rightarrow \quad a_3 \leq a_1; a_3 \leq a_2; b_3 \leq b_1; b_3 \leq b_2$

$\Rightarrow \quad a_3$ is a lower bound of $<a_1, a_2>$ in $L$

and $b_3$ is a lower bound of $<b_1, b_2>$ in $L'$

$a_3 \leq a_1 \wedge a_2 = \inf\{a_1, a_2\}$

and $\quad b_3 \leq b_1 \wedge b_2 = \inf\{b_1, b_2\}$

$\Rightarrow \quad <a_3, b_3> \leq <a_1 \wedge a_2, b_1 \wedge b_2>$

$\Rightarrow \quad <a_1 \wedge a_2, b_1 \wedge b_2>$ is greatest lower bound $\{<a_1, b_1>, <a_2, b_2>\}$ similarly, by duality, we have $<a_1 v a_2, b_1 v b_2>$ is least upper bound of $\{<a_1, a_2>, <b_1, b_2>\}$ $LxL'$ is lattice.

**28.** ***Let A be the set of factors of a particular positive integer m and let $\leq$ be the relation divides i.e.,***

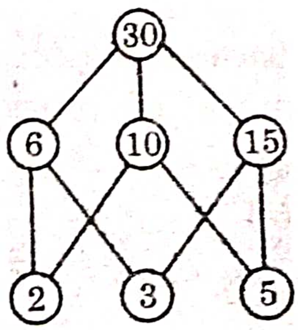
$$\leq = \{(x, y) | x \in A \text{ and } y \in A \text{ and } (x \text{ divides } y)\}$$

***Draw the Hasse diagram for :***

**(1)** $m = 30$
**(2)** $m = 12$
**(3)** $m = 45$

(1) $m = 30$

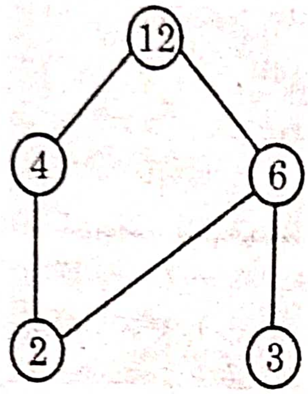
Set $m = \{2, 3, 5, 6, 10, 15, 30\}$




(Figure)

(2) $m = 12$

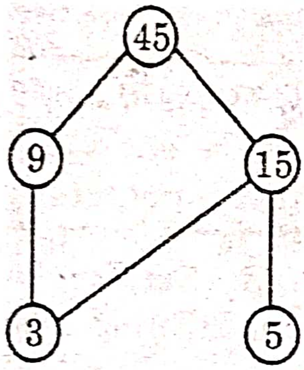
Set $m = \{2, 3, 4, 6, 12\}$




(Figure)

(3) $m = 45$

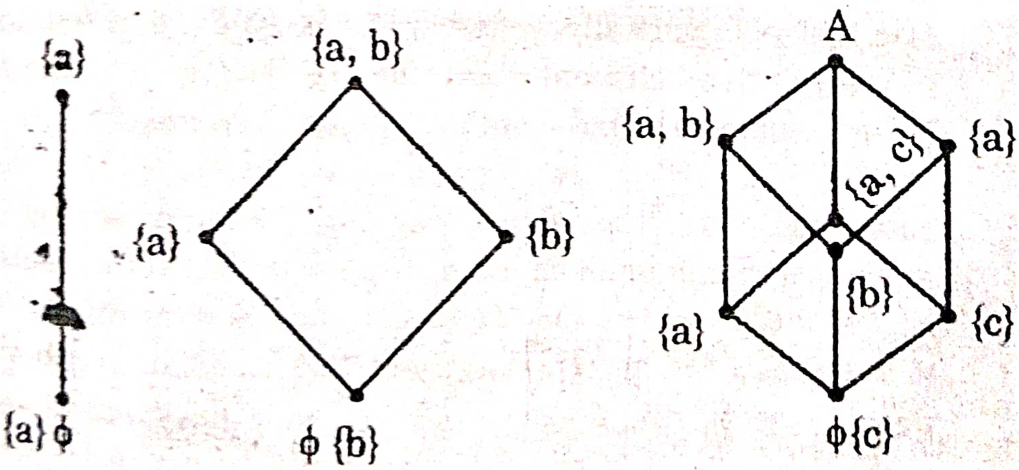
Set $m = \{3, 5, 9, 15, 45\}$




(Figure)

**29.** ***Let A be a given finite set and P(A) its power set. Let $\subseteq$ be the inclusion relation on the elements of P(A). Draw Hasse Diagram of $(P(A), \subseteq)$ For (a)$A = \{a\}$; (b)$A = \{a, b\}$; (c)$A = \{a, b, c\}$.***

The required Hasse diagrams are :




(Figure)

The following points may be noted about Hasse diagrams in general. For a given partially ordered set, a Hasse diagram is not unique, *aa* can be seen. From a Hasse diagram of $(P, \leq)$, the Hasse diagram of $(P, \leq)$, which is the dual of $(P, \leq)$, can be obtained by rotating the diagram through 180°. So that, the points at the top becomes the points at the bottom. Some Hasse diagrams have a unique point which is above all the order points, and similarly some diagram have a unique point which is above all the other points and similarly some hasse diagrams have a unique point which is below all other points.

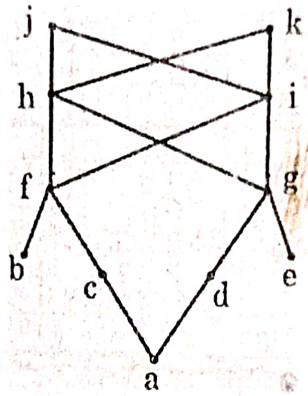
| | |
|---|---|
| **30.** | ***Define Partially Ordered Set, Totally Ordered Set and a Lattice. Give one example of a Partially Ordered Set which is not totally ordered.*** |

**Partially Ordered Set** : A binary relation is said to be a partial ordering relation if it is reflexive, anti-symmetric, and transitive. So if set $A$, together with a partial ordering relation $R$ and $A$, is called Partially Ordered Set and is denoted by $(A, R)$.

**Totally Ordered Set** : If $(A, \leq)$ be a partially ordered set, then $A$ is called a chain if every two elements in the subset are related. So a partially ordered set $(A, \leq)$ is called a totally ordered set if $A$ is a chain. In this case the binary relation $\leq$ called a totally ordering relation.

**Lattice** : A partially ordered set is said be a lattice if every two elements in the set have a unique least upper bound and unique greatest lower bound.

**Example** : In the above partially ordered set of $j$ and $k$ are maximal elements $a$, $b$, $e$ are minimal elements. An element $a$ is said to cover another elements $b$ if $b \leq a$ and for no other element $c$, $b \leq c \leq a$. In above figure $f$ covers, $b$, $f$ also covers $c$ but $f$ does not cover $a$. So, it is a partially ordered set but not a totally ordered set.




(Figure)

**31. *Find the dual of the Boolean : $f = x'yz' + x'y'z$.***
**(2018-19)**

$f = x'yz' + x'y'z$.

The dual of this

$(x' + y + z')(x' + y' + z)$ **(Ans.)**

**32.** ◆ ***Convert the following Boolean Function in DNF as well as CNF : $f(x, y, z) = xy' + xz + xy$.*** **(2018-19)**

◆ ***Express the expression $xy' + xz + xy$ in CN as well as DN form.*** **(2014-15)**

Expression is $xy' + xz + xy$

Let $f = xy' + xz + xy$

So $f(x,y,z) = xy' + xz + xy = xy' + xy + xz = x(y' + y) + xz$

$= x + xz = x(1 + z) = x$

So required disjunctive normal form is $= x$

$f(x, y, z) = x$

The required conjunctive normal form is :

$xy' + xz + xy = x(y + y') + xz = x + xz = x + 0$

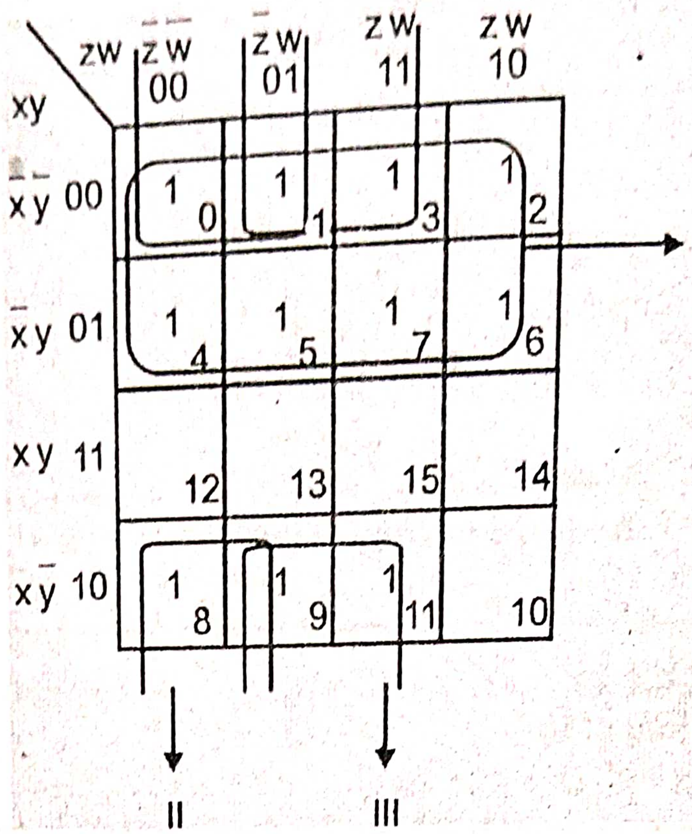
$x + yy' = (x + y)(x + y') = (x + y + zz')(x + y' + zz')$

$= (x + y + z)(x + y + z')(x + y' + z)(x + y' + z')$

**33. *Draw Karnaugh map (K-map) and simplify for the Boolean function :*** **(2018-19)**

$$f(x, y, z, w) = \sum(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11).$$

$$f(x, y, z, w) = \sum(0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 11)$$




(Figure)

Simplified Boolean expression from the K-map is.

$F = \text{I} + \text{II} + \text{III}$

$F = \bar{x} + \bar{y}\bar{z} + \bar{y}w$ **(Ans.)**

| | |
|---|---|
| **34.** | ◆ ***Express the following Boolean function in Sum of miniterms and Product of maxterm :*** $f(x,y,z) = x + y'z$ **(2018-19)** |
| | ◆ ***Define minterms and maxterms with examples. Express the Boolean function*** $f(x,y,z) = x + y'z$ ***as a sum of minterms.*** **(2017-18)** |

**Minterm :** Minterm is a product of all the literals (A boolean variable and its complement are called literals.

**Example :** Boolean variable $A$ and its complement $\sim A$ are literals).

**Example :** If we have two boolean variables $X$ and $Y$ then $X.(\sim Y)$ is a minterm.

We can express complement $\sim Y$ as $Y'$ so, the above minterm can be expressed as $XY'$

So, if we have two variables then the minterm will consists of product of both the variables.

**Maxterm :** Maxterm is a sum of all the literals (with or without complement).

**Example :** If we have two boolean variables $X$ and $Y$ then $X+(\sim Y)$ is a maxterm.

We can express complement $\sim Y$ as $Y'$ so, the above maxterm can be expressed as $X+Y'$

So, if we have two variables then the maxterm will consist of sum of both the variables.

**Boolean Function Solution :**

$$x+y'z = x(y+y')+y'z$$
$$= xy+xy'+y'z$$
$$= xy(z+z')+xy'(z+z')+y'z(x+x')$$
$$= xyz+xyz'+xy'z+xy'z'+xy'z+x'y'z$$
$$= xyz+xyz'+xy'z+xy'z'+x'y'z$$

---

**35.** ◆ ***Use a K-map to find a minimal sum for :***

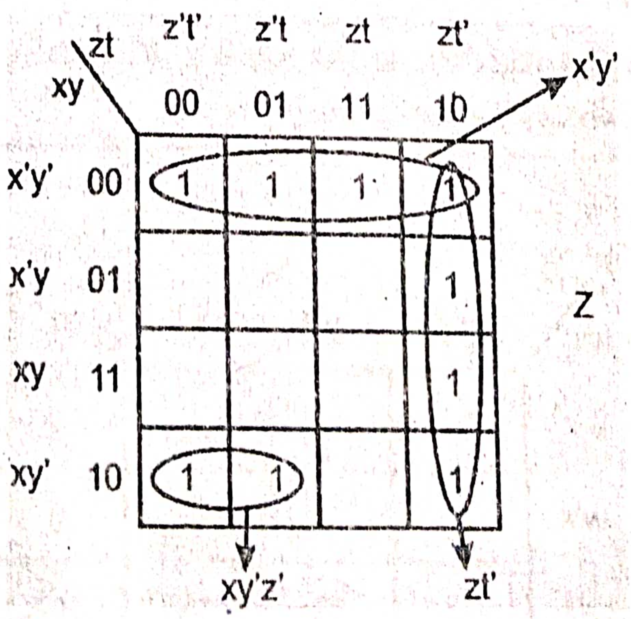
**(2017-18)**

$$E = y't'+y'z't+x'y'zt+yzt'$$

◆ ***Also draw the circuit diagram for the expression obtained.***

---

$y't'+y'z't+x'y'zt+yzt'$

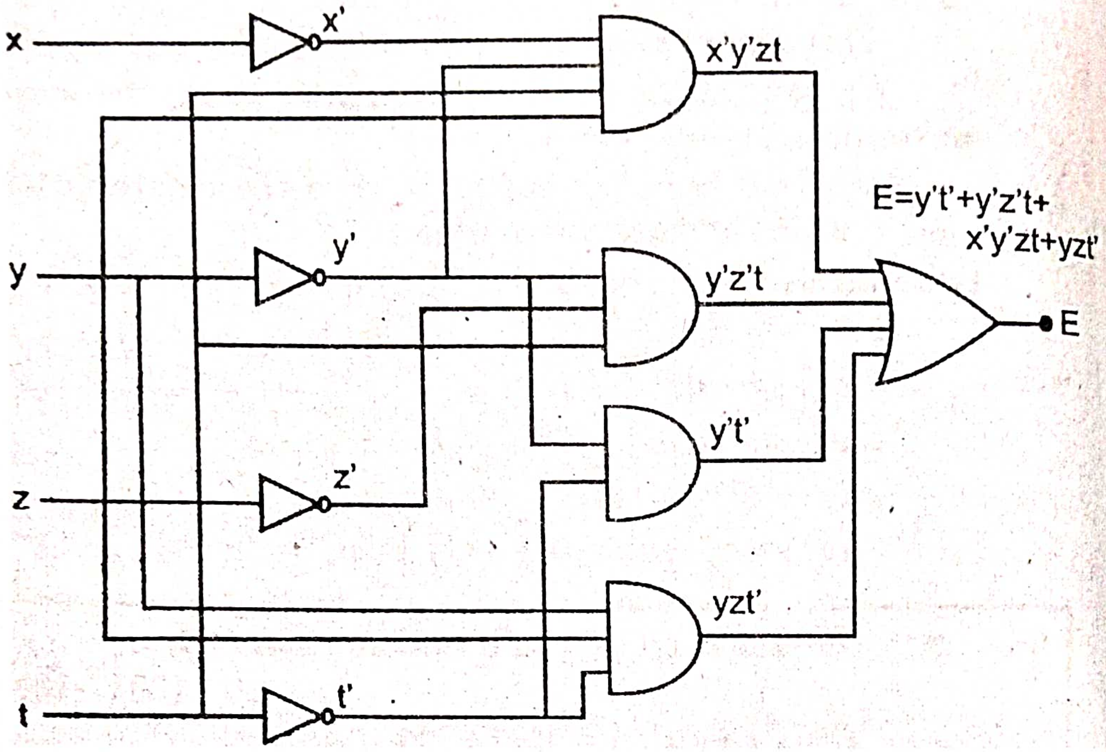



(Figure)

$$\sum m(x,y,z,t) = x'y'+xy'z'+zt'$$
$$= y'(x'+xz')+zt'$$

$$= y'(x'+z') + zt'$$

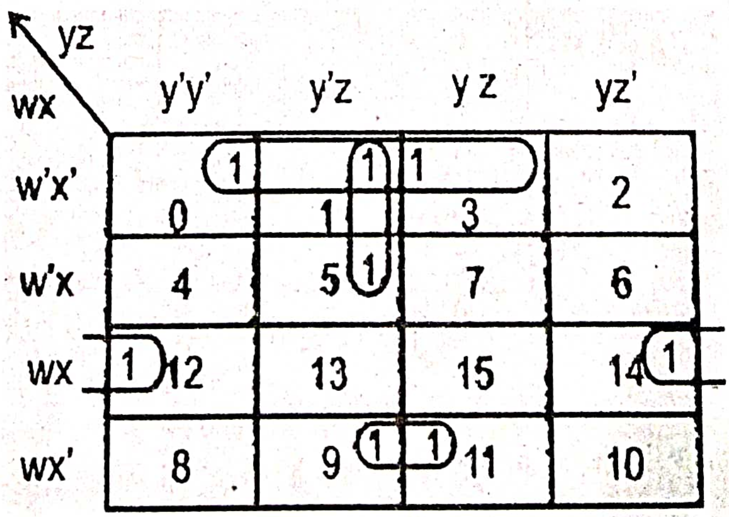
$$\sum m(x, y, z, t) = x'y'+z'y'+zt'$$



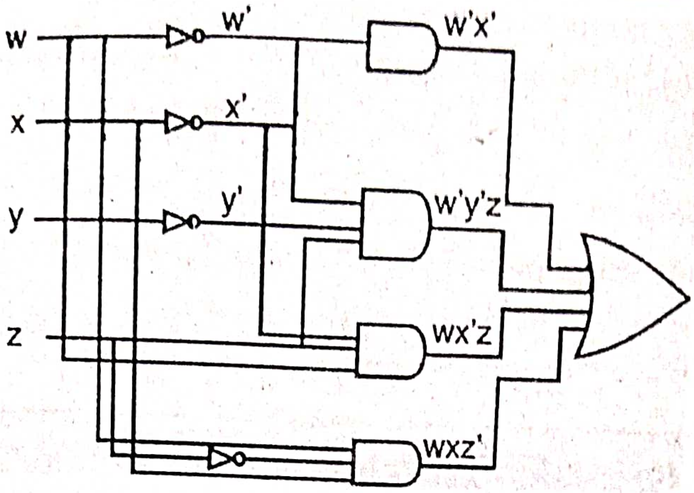

(Figure : Circuit Diagram)

> **36.** ***Simplify the Boolean expression $f(w, x, y, z) = \sum(0, 1, 3, 5, 9, 11, 12, 14)$ by using K-map. Also draw the logic and circuit diagram of the simplified expression.*** **(2016-17)**

$f(w, x, y, z) = \{0, 1, 3, 5, 9, 11, 12, 14\}$




(Figure)




(Figure)

**37. *Define Boolean algebra. If (B, +, •, ', 0, 1) is a Boolean algebra and $a, b \in B$ then prove that $(a+b)' = a' * b'$.*** **(2016-17)**

***Boolean Algebra***

Boolean algebra is a bit of an oddity in the math world. Boolean algebra uses,. as its fundamental values, the states of "true" and "false". To facilitate mathematics, we will turn "true" values into the number 1, and we will turn "false" values into the number 0. Then, we will go over some of the basic rules of boolean algebra :

**Addition :** $1 + 1 = 1$, $1 + 0 = 1$, $0 + 0 = 0$

There are only 2 values in boolean algebra, so there is no place to store the carry from an addition operation. The strangest part of boolean algebra is that $1 + 1 = 1$. It's strange, but it's important.

**Multiplication :** $1 \times 1 = 1$, $1 \times 0 = 0$, $0 \times 0 = 0$

These results seem pretty normal.

**Numerical :** The method of proof here is to show that $(a+b) + a'b' = 1$ and $(a+b).\, a'b' = 0$

This shows that $(a + b)$ and $a'\, b'$ are complements and by the theorem we establish the theorem.

Now $(a+b) + a'b' = [(a+b) + a'] \times [(a+b) + b'$ by distributive law

$= [a' + (a+b)] * [a + (b+b')]$ using associative law

$= [(a'+a) + b] * [a + (b+b')]$

$= (1+b) * (a+1)$ using complement law

$= 1.1$ using boundness law $= 1$

And $(a+b)*a'b' = a.(a'b') + b.(a'b')$ by distributive law

$= (aa')*b' + b*(b'a')$ using associative law

$= 0 \times b' + (bb')*a$ using complement and associative law

$= 0 + 0 * a'$ using boundedness and complementary law

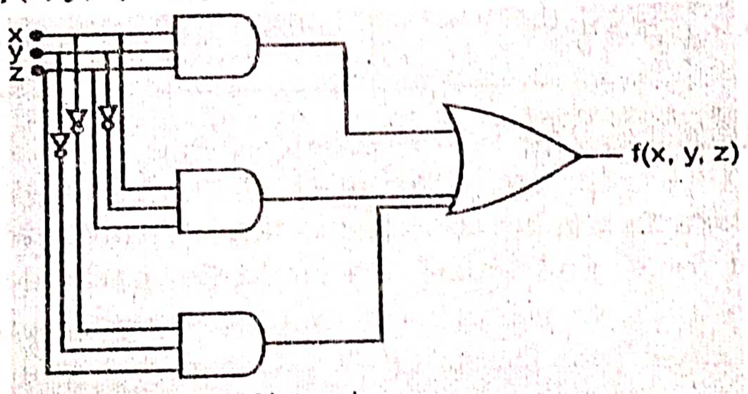
$= 0 + 0 = 0$

So $(a+b)' = a' * b'$

**38. *Draw the simplified network of $f(x, y, z) = x.y.z + x.y'.z + x'.y'.z$*** **(2015-16)**

Given that

$$f(x, y, z) = x.y.z + x.y'.z + x'.y'.z$$




Simplified Network

(Figure)

**39. *Following table gives the value of the function f(x, y, z). Find the corresponding function. Draw a simplified circuit diagram of the function. Also find the minterm normal form of f(x, y, z).*** **(2015-16)**

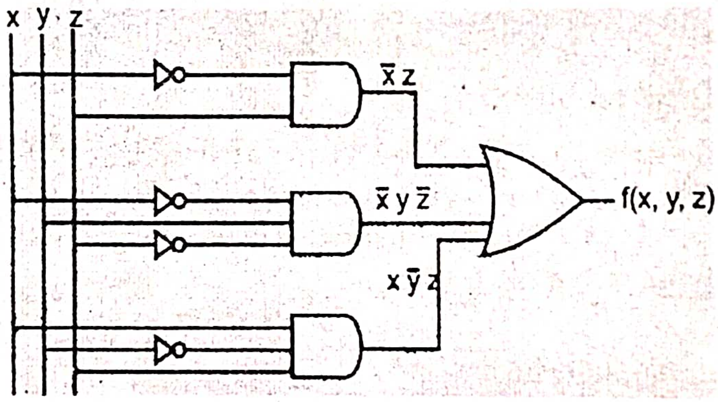
| x | y | z | f(x, y, z) |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 |

| x | y | z | f(x, y, z) |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | 1 | 1 |
| 0 | 1 | 0 | 1 |
| 0 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 |

Boolean expression is

$$f(x, y, z) = \bar{x}\,\bar{y}\,z + \bar{x}\,y\,\bar{z} + \bar{x}\,y\,z + x\,\bar{y}\,z$$
$$= \bar{x}\,z\,(y + \bar{y}) + \bar{x}\,y\,\bar{z} + x\,\bar{y}\,z$$
$$= \bar{x}\,z + \bar{x}\,y\,\bar{z} + x\,\bar{y}\,z$$

Simplified circuit diagram is




(Figure)

**40.** ***In a Boolean algebra, if a = b then prove that :***
***$a.b' + a'.b = 0$*** **(2015-16)**

Given that $a = b$

Now there may be two cases.

Either $a = T = b$

Or $a = F = b$

So,

L.H.S. $= ab' + a'b = T.T' + T'T$ if $a = T = 1$

$= T.F + F.T = F + F = F = 0$

$= R.H.S.$ **(Hence Proved)**

**41.** ◆ ***For every element a and b in a Boolean algebra show that***

(1) $(a.b)' = a' + b'$

(2) $(a + b)' = a'.b'$. (2014-15)

◆ ***For any x and y in a Boolean algebra, prove that***

$(x + y)' = x'.y'$. (2010-11)

**(1)** The method of proof here is, to show that

$(a + b) + a'b' = 1$ and $(a + b).\ a'b' = 0$

This shows that $(a + b)$ and $a'b'$ are complements and by the theorem we establish the theorem

Now, $(a + b) + a'b' = [(a + b) + a'].\ [(a + b) + b']$

(by distribution law)

$= [a' + (a + b)].\ [a + (b + b')]$

(using Associative law)

$= [(a' + a) + b].\ [a + (b + b')]$

$= (1 + b)\ (a + 1)$

(using complement law)

$= 1.1 = 1$ (using Boundless law)

And

$(a + b).a'b' = a.\ (a'\ b') + b.(a'b')$

(by distribution law)

$= (aa').\ b' + b.(b'\ a')$

(using Associative law )

$= 0.b' + (bb').a$

(Using Complement and Associative law)

$= 0 + 0.a' = 0 + 0 = 0$

So, $(a+b)' = a'.b'$

**(2)** $(x + y)' = x'\, y'$ ...(1)

From (1) we observe that $a'b'$ is complement of $a+b$. Therefore to prove, it is sufficient to show,

$(a+b) + a'b' = 1$

And $(a+b).\ (a'b') = 0$

Where 1 is multiplicative and 0 is additive identity is $B$. Now

$(a+b) + (a'b') = [(a+b) + a'].[(a+b) + b']$

$= [a + (a'+b)].[a + (b + b')]$

$= [a + (a'+b)].[a + (b + b')]$

$= [(a + a') + b].[a + (b + b')]$

$= (1 + b).(a + 1) = 1.1 = 1.$

Therefore

$$(a+b)+a'.b'=1 \quad \dots (2)$$

$$(a+b).(a'.b') = a(a'b')+b(a'b')$$

$$= (a.a').b'+(b.b').a' \quad = 0.b'+0.a' \; = 0+0$$

$$= 0$$

Therefore, $(a+b).(a'.b')=0$ ... (3)

Hence, from (2) and (3) we conclude that $a'.b'$ is complement of $a+b$, i.e.,

$$(a+b)'=a'.b'$$

**42.** ***From the input/output table given below form a corresponding Boolean expression and make its simplified circuit.*** **(2014-15)**

| Input $x_1$ | Input $x_2$ | Output f $(x_1, x_2)$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |

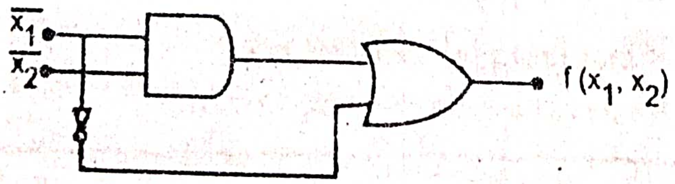
| Input $x_1$ | Input $x_2$ | Output $f(x_1, x_2)$ |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 1 |
| 0 | 1 | 0 |
| 0 | 0 | 1 |

So Boolean expression is

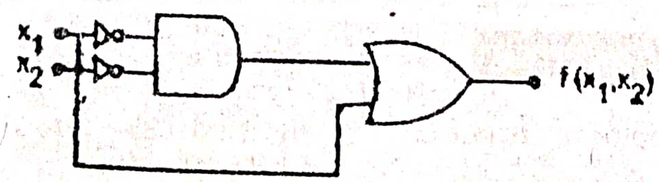
$$f(x_1,x_2) = x_1x_2 + x_1\bar{x}_2 + \bar{x}_1\bar{x}_2 \qquad \text{(using truth table)}$$

$$= x_1 + \bar{x}_1\bar{x}_2$$

And own circuit is




OR




**(Figure)**

**43. *Simplify using Karnaugh Map.***

$$F(A,B,C,D) = \Sigma(0, 2, 3, 5, 7, 8, 10, 11, 12, 15)$$

**(2013-14)**

| | C' D' | C' D | C D | |
|---|---|---|---|---|
| A' B' | 1 | | 1 | 1 |
| A' B | | 1 | 1 | |
| A B | 1 | | 1 | |
| A B' | 1 | | 1 | 1 |

**(Figure)**

For $F(A, B, C, D) = CD + A'BD + AC'D' + B'D$

$$= CD + B'D + A'BD + AC'D'.$$

**44. *Write the following Boolean expressions in an equivalent product of sums canonical form in three variables $x_1$, $x_2$ and $x_3$:*** **(2011-12)**

**(1)** $x_1 * x_2$ **(2)** $x_1 \oplus x_2$

**(1)** $x_1 * x_2$ can be expressed in three variables such as

$= x_1 x_2$

$= x_1x_2 (x_3+x_3')=x_1x_2x_3 + x_1x_2x_3'$

**(2)** $x_1 \oplus x_2$ can be expressed as in three variables

$= x_1x_2' + x_1'x_2$

$= x_1x_2'(x_3+x_3')+ x_1'x_2(x_3+x_3')$

$=x_1x_2'x_3+x_1x_2'x_3' + x_1x_2x_3+x_1x_2x_3$

**45. *Define a Boolean function.*** **(2010-11)**

***Boolean Function***

A Boolean function is defined as an Algebraic expression formed with constants and Boolean variables with finite applications of logical operator '+', '.' And '.' for example $xy'+z(xy+x'z')+0$ is a Boolean function. In other words let $(B, +, ., o, 1)$ be a Boolean Algebra, then a Boolean function of n variables defined as a function $f: B^n \to B$ which is associated with a Boolean expression of $n$ variables.

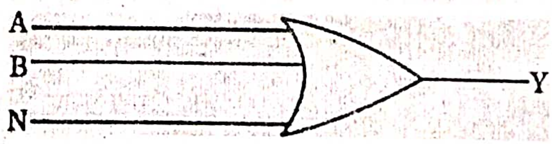
## 46. *What are Logic Circuits?*

**Logic Circuits**

Logic Circuits are structures which are built up from certain elementary circuits called logic gates. Each logic circuit may be viewed as a machine $L$ which contains one or more input devices and exactly one output device. Each input device in $L$ sends a signal specifically a bit i.e., binary digit 0 or 1 to the circuit $L$ and $L$ processes the set of bits to yield an output bit. Accordingly on *n-bit* sequence may be assigned to each input device and $L$ processes the input sequences one bit at a time to produce an n but output sequence.

**Logic Gates :** There are three basic large gates OR, AND and NOT Gate.

**OR GATE :** An OR gate has two or more inputs and a single output and it operates in accordance with the following definition.

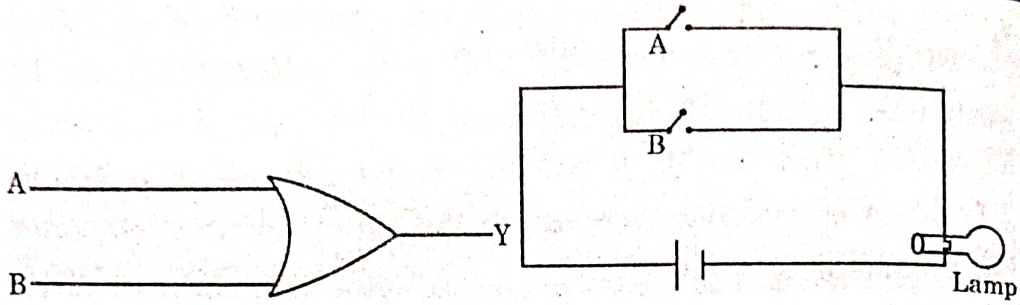



(Figure : Y = N + B + N ... + N)

'The output of an OR assumes the 1 state if one or more inputs assume the 1 state.

The $n$ input to a logic circuit are designated by $A$, $B$... $N$ and the output by $Y$ each of these symbols may be assumed one of the possible values either 0 or 1.

The OR gate only yields $O$ when all input bits are $O$. The electronic symbol of two inputs OR gate and its equivalent switching circuit is shown in figure :

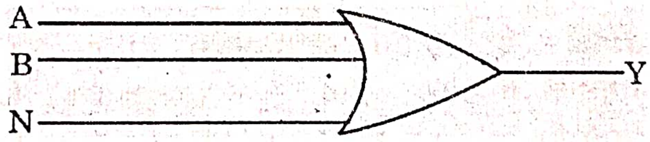



(Figure)

Truth table of OR gate is as follows :

| Input | | Output |
|---|---|---|
| A | B | Y |
| 0 | 0 | 0 |
| 0 | 1 | 1 |
| 1 | 0 | 1 |
| 1 | 1 | 1 |

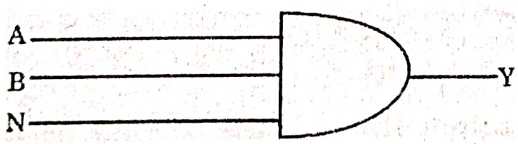
**And Gate :** An AND gate has two or more inputs and a single output and it operates in accordance with the following definition.



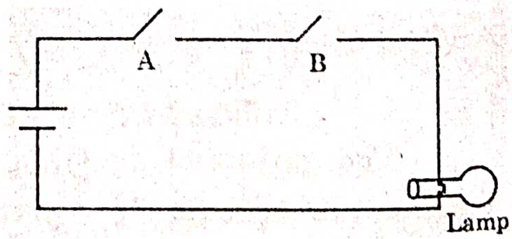

(Figure : A.B .... N)

'The output of an AND assumes the 1 state if and only if all the inputs assume the 1 state'.

The $n$ input to a logic circuit will be designated by $A$, $B$ $N$ and the output by $Y$ and each of these symbols may assume one of two possible values either 0 or 1. The electronic symbol of a 2 input AND gate and its equivalent switching circuit is shown in the figure :




(Figure : A . B = Y)



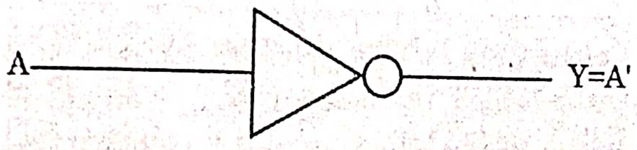

(Figure)

Truth table of AND gate is shown below :

| Input | | Output |
|---|---|---|
| A | B | Y |
| 0 | 0 | 0 |
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 1 | 1 | 1 |

**Not Gate :** The output of NOT circuit takes on the 1 state if and only if the input does not take on the 1 state. The NOT circuit has a single input and a single output and is so called because its output is NOT the same as its input.

The symbol for a NOT gate and the Boolean expression for negation is shown in the following figures :




(Figure)

| Input | Output |
|---|---|
| A | Y = A' |
| 1 | 0 |
| 0 | 1 |

| 47. | *What is Minimization of Gates? Explain the term : Minterm, Maxterm and K-map?* |
|---|---|

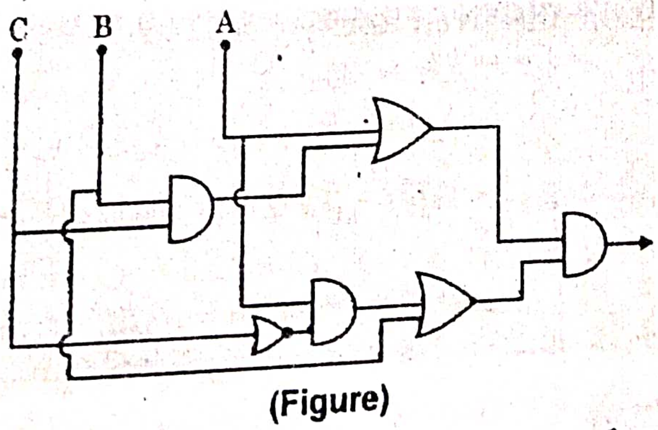
**Minimization of Gates**

In order require lesser element of hardware and therefore for reduction in cost, every boolean equation should be reduced to as simplest form. This simplified form will have small number of hardware i.e. Gates.

For example consider a function :

$$f = (A + BC)(B + A\overline{C})$$

Circuit diagram for the above function (i.e. gate network) will be shown as :




(Figure)

Now by algebraic simplification, the function can be converted in form of either Sum Of Product (SOP) or Product Of Sum (POS).

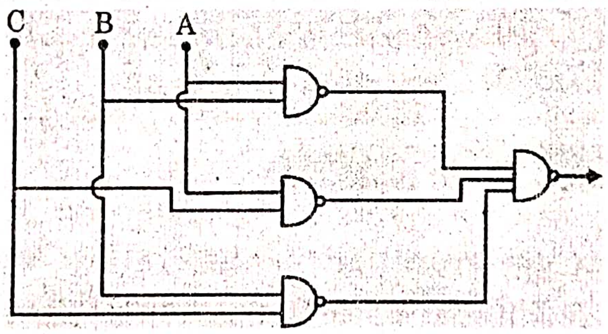
**Sum of Product :** For the given function $F$

$$f = (A + BC)(B + AC)$$
$$= AB + AAC + BBC + ABCC \qquad (\because CC = 0)$$
$$= AB + AC + BC + 0$$
$$f = AB + AC + BC$$

The given function, in form of SOP, can realize using NAND gate only where number of NAND gates are equal to the number of terms in (SOP + 1). Thus realization of function using NAND gate will be shown as :

$$Y = \overline{\overline{AB}.\overline{BC}.\overline{AC}}$$
$$= \overline{\overline{AB}} + \overline{\overline{BC}} + \overline{\overline{AC}}$$
$$= AB + BC + AC \qquad \text{(equals of } F\text{)}$$

Note the number of gates we use to realize the same function.




**(Figure)**

**Standard or Canonical form of SOP :** The standard form of SOP expression can be obtained by introducing every variable of that function in either true or complement form (in each term).

Thus, the given function $F$

$$f = AB + BC + AC$$
$$= AB(C+\overline{C}) + (A+\overline{A})BC + A(B+B)\overline{C}$$
$$= ABC + AB\overline{C} + ABC + \overline{A}BC + AB\overline{C} + A\overline{B}\overline{C}$$
$$= ABC + AB\overline{C} + \overline{A}BC + A\overline{B}\overline{C}$$
$$= 111 + 110 + 011 + 100$$

Each term in SOP of function is called min term of SOP of the function i.e. for the given function 111, 110, 011, 100 (3, 4, 6, 7) are the min terms of function and it is entered as '1' in $K$-map (defined later). This may be written as :

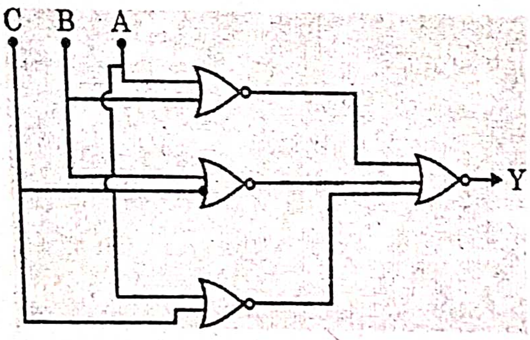
$$f = \Sigma(3,4,6,7)$$

**Product of Sum :** For the given function $F$ we have :

$$f = (A+BC)(B+A\overline{C})$$
$$= (A+B)(A+C)(B+A)(B+\overline{C})$$
$$= (A+B)(A+C)(B+\overline{C}) \qquad \text{(Product of sum)}$$

To realize a function, in form of POS we can use NOR gate only where number of required NOR gates = number of terms in POS + 1 example,

$$Y = \overline{\overline{(A+B)+(A+C)+(B+C)}}$$
$$= \overline{\overline{(A+B)}+\overline{(A+C)}+\overline{(B+C)}}$$
$$= (A+B)+(A+C)+(B+C)$$




**(Figure)**

Canonical or standard $POS$ :

$$(A+B)+(A+C)+(B+\overline{C})$$
$$= (A+B+C\overline{C})(A+B\overline{B}+C)(A\overline{A}+B+\overline{C})$$

$$= (A+B+C)(A+B+\overline{C})(A+B+C)(A\overline{B}+C)(A+B+\overline{C})$$
$$(\overline{A}+B+\overline{C})$$
$$= (A+B+C)(A+B+\overline{C})(A+\overline{B}+C)(\overline{A}+B+\overline{C})$$

Each term in canonical POS is called max term of the function and centered as '0' in $K$-map thus for the given function :

$$f = \Pi(0,1,2,5)$$

**Karnaugh Map :** $K$-map is a shortcut method to reduce a Boolean function into smallest form. To construct a $K$-map for the given truth table, we will consider following rules :

(1) Total number of cells in a $K$-map is given by $2^N$, where $N$ = number of variables.
(2) The $LSB$ of variable should be placed in right bottom corner of $K$-map, so that cell number will run from top to bottom.
(3) Whenever there is 3rd row or 3rd column take a jump.
(4) Try to make groups of maximum 1's or 0's in power of 2.
(5) Reusing of 1's and 0's is permitted.
(6) Folding of individual $K$-map is allowed.
(7) Groups from diagonal $K$-map is not allowed in case of variable $K$-map.

□□

# PROPOSITIONAL / PREDICATE LOGIC

1. *Form the conjunction of p and q for the following statements :*
   *p : It is cold* *q : It is raining* **(2020-21)**

Conjunction of $p$ and $q$ is $p \wedge q$

So : It is cold and It is raining is Conjunction of $p$ and $q$ .

2. *Show that $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradtction.*
   **(2020-21) (2020-21)**

A statement that is always false is called a contradiction. Here we use 0 for false & 1 for true value.

| p | q | $p \vee q$ | $\sim p$ | $\sim q$ | $\sim p \wedge \sim q$ | $(p \vee q) \wedge (\sim p \wedge \sim q)$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 0 |
| 0 | 1 | 1 | 1 | 0 | 0 | 0 |
| 1 | 0 | 1 | 0 | 1 | 0 | 0 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 |

So it shows contradiction

3. *If $X = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Determine the truth value of each of the following statement :*
   (1) $(\forall x \in X) x + 5 < 18$
   (2) $(\exists x \in X) x + 6 = 15$
   (3) $(\forall x \in X) x + 1 \leq 10$
   (4) $(\exists x \in X) x + 3 > 15$ **(2020-21)**

(1) Truth value of $(\forall x \in x) x + 5 < 18$ is true, last case $9 + 5 = 14 < 18$ so true.

(2) Truth value of $(\exists x \in x) x + 6 = 15$ is true for $x = 9$

(3) Truth value of $(\forall x \in x) x + 1 \leq 10$ is true, last case $a + 1 = 10$ which is $\leq 10$

(4) $(\exists x \in x) x + 3 < 15$ is true, if it is true for at least one value of $x$, then it is true

4. *Show that ~r is a valid conclusion from the premises :*
$p \to \sim q, r \to p, q$
(1) *With truth table.*
(2) *Without truth table.* (2020-21)

(1) **With Truth Table :** we have to prove
$(p \to \sim q) \wedge (r \to p) \wedge q \to \sim r$.

| p | q | r | $p \to \sim q$ | $r \to p$ | $(p \to \sim q) \wedge (r \to p) \wedge q$ | $(p \to \sim q) \wedge (r \to p) \wedge q \to \sim r$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 | 0 | 1 |
| 0 | 1 | 0 | 1 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 1 | 1 | 0 | 1 | 0 | 1 |

So it is a valid conclusion.

(2) **Without Truth Table :**
(a) $p \to \sim q$
(b) $r \to \sim p$
(c) $\sim q$
$\sim r$

From 1st & 3rd premises

$$\left.\begin{array}{l} p \to \sim q \\ \underline{\sim q} \\ \underline{\sim p} \end{array}\right\} \text{Modus Ponens}$$

From 2 nd Premises

$$\left.\begin{array}{l} r \to p \\ \sim p \end{array}\right\} \textit{Modus Tullent}$$
$$\overline{\underline{\sim r}}$$

So it is a valid statement.

5. *What do you mean by Negations of Quantified Statements?* (2020-21)
*Negate the following statements :*
(1) *All integers are greater than 8.*
(2) *For all real number x, if $x > 3$ then $x^2 > 9$*

**Negation of a Quantifier**

Negation of a quantifier is statement with universal quantifier will be converted to a statement with existential quantifier

(1) Negation of $(\forall\ integer\ x)\quad(x>8)$ is $(\to integer\, x)(x \le 8)$. There is an integer less than or equal to 8

(2) $\forall x\ (x>3) \to (x^2>9)$ negation of this is

$$\exists x[(x>3) \wedge (x^2 \le 9)]$$

Because. $\sim p \vee q$

$$\sim(\sim p \vee q)$$

$$(p \vee \sim q)$$

> 6. ◆ ***Define the term Tautology and Contradiction.*** **(2018-19)**
>
> ◆ ***Show that $(P \to (q \wedge r)) \to (\sim r \to \sim p)$ is a tautology.***

**Tautology**

A tautology is a formula which is "always true" - that is, it is true for every assignment of truth values to its simple components. You can think of a tautology as a rule of logic.

**Contradiction**

The opposite of a tautology is a contradiction, a formula which is "always false". In other words, a contradiction is false for every assignment of truth values to its simple components

**Numerical Solution :** $(P \to (q \wedge r)) \to (\sim r \to \sim p)$ is a tautology.

| p | q | r | $(q \wedge r)$ | $(p \to (q \wedge r))$ | $\sim r$ | $\sim p$ | $\sim r \to \sim p$ | $(p \to (q \wedge r)) \to (\sim r \to \sim p)$ |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | F | F | T | T |
| T | T | F | F | F | T | F | F | T |
| T | F | T | F | F | F | F | T | T |
| T | F | F | F | F | T | F | F | T |
| F | T | T | T | T | F | T | T | T |
| F | T | F | F | T | T | T | T | T |
| F | F | T | F | T | F | T | T | T |
| F | F | F | F | T | T | T | T | T |

Here, it is true for every assignment of truth values i.e., last column contains only T's.
So, it shows tautology.

**7.** ***Define the terms converse, contrapositive and inverse of a proposition. Show that $(p \to q) \wedge (r \to q) \equiv (p \vee r) \to q$*** **(2018-19)**

***Converse, Inverse and Contrapositive of*** $P \to Q$

**Converse :** $Q \to P$ (Change position)

**Inverse :** (Change sign)

**Contrapositive :** $\sim Q \to \sim P$ (Change both position and sign)

The implication is equivalent to its contrapositive.
The inverse is equivalent to the converse

**Numerical Solution :**

| p | q | r | $(p \to q)$ | $(r \to q)$ | $(p \vee r)$ | $(p \vee r) \to q$ | $(p \to q) \wedge (r \to q)$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | T | T | T | T |
| T | F | T | F | F | T | F | F |
| T | F | F | F | T | T | F | F |
| F | T | T | T | T | T | T | T |
| F | T | F | T | T | F | T | T |
| F | F | T | T | F | T | F | F |
| F | F | F | T | T | F | T | T |

Since, $(p \to q) \wedge (r \to q)$ and $(p \vee r) \to q$ both columns are identical, hence these are logically equivalent i.e., $(p \to q) \wedge (r \to q) \equiv (p \vee r) \to q$.

**(Hence Proved.)**

**8.** ◆ ***Construct the truth table $((P \Rightarrow q) \vee (q \Rightarrow p)) \Leftrightarrow p$.***

◆ ***Is the preposition : Tautology, Contradiction or Contingency?*** **(2018-19)**

◆ ***Construct truth table :***
$((p \to q) \vee (q \to p)) \leftrightarrow p$

$p \to q$ $q \to p$



◆ ***Is the preposition : tautology, contradiction or contingency?*** **(2013-14)**

| p | q | (p→q) | (q→p) | (p→q)v(q→p) | (p→q)v(q→p)↔p |
|---|---|---|---|---|---|
| F | F | T | T | T | F |
| F | T | T | F | T | F |
| T | F | F | T | T | T |
| T | T | T | T | T | T |

It is contingency because atleast one value is true and false from the truth table.

**9.** ***Write the following conditional statement in symbolic form. Also give the converse, inverse and contra-positive of the statement :***
***"If the flood destroys Mohan's house or the fire destroy Mohan's house, then Mohan's insurance company will pay him."*** **(2017-18)**

Let $P$ = the flood destroys Mohan's house
$Q$ = the fire destroys Mohan's house
$R$ = Insurance company will pay Mohan

The symbolic form of the statement is $(P \vee Q) \to R$

(1) Converse is $R \Rightarrow (P \vee Q)$
(2) Inverse is $\sim (P \vee Q) \Rightarrow \sim R$
(3) Contrapositive is $\sim R \Rightarrow \sim (P \vee Q)$

**10.** ◆ ***What do you mean by existential quantifiers and universal quantifiers? Explain with proper examples***
◆ ***Find a counter example, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.*** **(2017-18)**
(1) $\forall x (x_2 \geq x)$
(2) $\forall x (x > 0 V x < 0)$
(3) $\forall x (x = 1)$

**Existential Quantifier**

It is a symbol of symbolic logic which expresses that the statements within its scope are true for at least one instance of something.

The symbol $\exists$, which appears as a backwards "$E$", is used as the existential quantifier.

Existential quantifiers are normally used in logic in conjunction with predicate symbols, which say something about a variable or constant, in this case the variable being quantified.

The existential quantifier $\exists$ (which means "there exists"), differs from the universal quantifier $\forall$ (which means "for all").

For example, if the predicate symbol $Bx$ is taken to mean "$x$ is a ball", then we may formalize an expression using an existential quantifier :

$$\exists xBx$$

Translated back into English, this reads as "there is an $x$ such that $x$ is a ball", or more simply, "there is a ball".

We may formalize the expression "Some $P$ are $Q$" using an existential quantifier and a conjunction :

$$\exists x(Px \wedge Qx)$$

This reads as "There exists an $x$ such that $x$ is a $P$ and $x$ is a $Q$", which may be literally as "at least one $P$ is a $Q$", or more generally, "some $P$ are $Q$". (In logic, the word "some" is almost always taken to mean "at least one".

**Universal Quantifier :** It is a symbol of symbolic logic which expresses that the statements within its scope are true for everything, or every instance of a specific thing.

The symbol $\forall$, which appears as a vertically inverted "$A$", is used as the universal quantifier.

Universal quantifiers are normally used in logic in conjunction with predicate symbols, which say something about a variable or constant, in this case the variable being quantified.

The universal quantifier $\forall$ (which means "for all"), differs from the existential quantifier $\exists$ (which means "there exists", or in contrast to $\forall$, "for at least one").

For example, if the predicate symbol $Mx$ is taken to mean "$x$ is matter", then we may formalize an expression using a universal quantifier :

$$\forall xMx$$

Translated back into English, this reads as ""for every $x$, $x$ is matter", or more simply, "everything is matter".

If we wanted to say that everything is matter or energy, we could add the predicate symbol $Ex$, to refer to "$x$ is energy", and formalize :

$$\forall x(Mx \vee Ex)$$

The common logical phrase "All $P$ are $Q$" may be similarly formulated using a universal quantifier and a conditional statement :

$$\forall x(Px \rightarrow Qx)$$

This reads as "For every $x$, if $x$ is a $P$, then $x$ is a $Q$", which may be expressed simply as, "all $P$ are $Q$".

**Counter Example :**

(1) $\forall \times (x_2 \geq x)$ = No counter example

(2) $\forall \times (x > 0 V x < 0)$ = 0, since 0 is not less than or greater than 0, 0 is a counter example.

(3) $\forall \times (x = 1)$= 2, since 2 is not 1, so 2 is a counter example.

| |
|---|
| 11. ***Write the conjunctive normal form for the expression (y + z') of three variable x, y, z.*** **(2016-17)** |

We have $y + z' = y + z' + 0 = y + z' + xx'$
$= (y + z' + x)\ (y + z' + x')$ i.e. CNF.

| |
|---|
| 12. ***Write converse and inverse for the following statement "If $x + 3 = 8$ then $x = 6$ ".*** **(2016-17)** |

$r : x + 3 = 8$
$t : \quad x = 6$
$P : \quad r = t$
Converse of $p$ is $t \Rightarrow r$
Inverse of $p$ is $\sim r \Rightarrow t$

| |
|---|
| 13. ◆ ***Explain the quantifiers in details. Also write the following English language into symbolic statement.*** <br> ◆ ***"Every students of this university is either academician or sportsman".*** **(2016-17)** |

**Quantifiers**

Quantifiers are words that refer to quantities such as "some" or "all" and indicate how frequently a certain statement is true.

The phrase "for all" denoted by $\forall$ is called the universal quantifier. For example, consider the sentence "all human beings are mortal".

Let $P(x)$ denotes "$x$ is mortal". Then the above sentence can be written as $(\forall x \in S)P(x)$ or $\forall x P(x)$ ... (1)

Where $S$ denotes the set of all human beings.

The statement (1) is called a universal statement. The expression $P(x)$ by itself is an open sentence and therefore has no truth value.

**Numerical Solution :** For all $x$, if $x$ is every students of this university then $x$ is either academician or sportsman.

$A(x)$ : $x$ is a student

$B(x)$ : either academician or sportsman.

i.e.$(\forall x)\ (A(x) \Rightarrow B(x))$

**14. *Define Inference theory. Also explain the rules of inference with example.* (2016-17)**

In logical reasoning, a certain number of propositions are assumed to be true and based on that assumption some other proposition are inferred or derived. The propositions that are assumed to be true are called premises or hypotheses and the proposition derived by using the rules of inference is called a conclusion

**Rules of Inference**

Rules of inference are no more than valid arguments. The simplest yet most fundamental valid arguments are :

**Modus Ponens :** $p \rightarrow q,\ p,\ \therefore\ q$

**Modus Tollens :** $p \rightarrow q,\ \sim q,\ \therefore\ \sim p$

Latin phrases modus ponens and modus tollens carry the meaning of "method of affirming" and "method of denying" respectively. That they are valid can be easily established. Modus tollens, for instance, can be seen or derived by the following truth table shows the validity of the argument form.

| p | q | p→q | ~q | ~p |
|---|---|---|---|---|
| T | T | T | F | F |
| T | F | F | T | F |
| F | T | T | F | T |
| F | F | T | [illegible] | [illegible] |

The only critical row

**(Figure : Shows the Ability of the Argument form)**

(1) **Disjunctive Addition :** $p, \therefore p \vee q$.
(2) **Conjunctive Addition :** $p, q, \therefore p \wedge q$.
(3) **Conjunctive Simplification :** $p \wedge q, \therefore p$
(4) **Disjunctive Syllogism :** $p \vee q, \sim q, \therefore p$
(5) **Hypothetical Syllogism :** $p \rightarrow q, q \rightarrow r, \therefore p \rightarrow r$
(6) **Division Into Cases :** $p \vee q, p \rightarrow r, q \rightarrow r, \therefore r$
(7) **Rule of Contradiction :** $\sim p \rightarrow$ contradiction, $\therefore p$

The validity of the above argument forms can all be easily verified via truth tables. These rules may not mathematically look very familiar. But it is most likely that everyone has used them all, individually or jointly, at some stage subconsciously.

**15.** ***Define tautology. Prove that the statement $(p \wedge q) \rightarrow (p \vee q)$ is tautology.*** **(2016-17)**

**Tautology**

A proposition $P$ is a tautology if it is true under all circumstances, in other words, they are true for any truth values of their variables. It means it contains only $T$ in the final column of its truth table.

**$(p \wedge q) \rightarrow (p \vee q)$ is Tautology :**

| p | q | p∧q | p∨q | (p∧q) → (p∨q) |
|---|---|---|---|---|
| T | T | T | T | T |
| T | F | F | T | T |
| F | T | F | T | T |
| F | F | F | F | T |

Hence it is a tautology.

**16.** ***Show that $(p \rightarrow q) = \sim (p \vee q)$.*** **(2015-16)**

Given $(p \rightarrow q) = \sim (p \vee q)$

Let's verify this with the help of truth table

| p | q | p ∨ q | p → q | ~(p ∨ q) | ~p | ~p ∨ q |
|---|---|---|---|---|---|---|
| F | F | F | T | T | T | T |
| F | T | T | T | F | T | T |
| T | F | T | F | F | F | T |
| T | T | T | T | F | F | T |

It can be seen from the truth table that

$$(p \rightarrow q) \neq \sim (p \vee q)$$

But

$$p \rightarrow q = \sim p \vee q$$

**17.** ***Determine the validity of the following argument :***
***"If wages increase, there will be inflation. The cost of living will not increase, if there is no inflation. Wages will increase; therefore the cost of living will not increase".*** **(2015-16)**

Let $W$ stand for "wages will increase," $I$ stands for "there will be inflation," and $C$ stand for "cost of living will increase." Therefore the argument is : $W \to I$, $\neg I \to \neg C$, $W \Rightarrow C$. The argument is invalid. The easiest way to see this is through a truth table. Let $x$ be the conjunction of all premises.

| W | I | C | ¬I | ¬C | W→I | ¬I→¬C | x | x→C |
|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 1 |
| 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 |
| 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 |
| 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 |

**18.** ***Determine the truth value for each of the following statements. Assume x, y are elements of set of integers.***

$\forall x\, \exists y \quad$ **$x + y$ is even**

$\exists x\, \forall y \quad$ **$x + y$ is even** **(2015-16)**

Given that

$\forall x\, \exists y \quad x + y$ is even $\qquad x, y \in I$

$\exists x\, \forall y \quad x + y$ is even

Now the meaning is for all integer there is another integer which can be added to get an even integer.

Now both the statements are right here because of simple number system.

**19.** ***Show that*** $(p^\wedge(\sim p^\vee q))^\vee(q^\wedge \sim (p^\wedge q)) \equiv q.$ **(2015-16)**

This can be proved by many ways one is truth table method



We have

$(p^\wedge(\sim p \vee q)) \vee (q \wedge \sim (p \wedge q)) \equiv q.$

| p | q | ~p | (p∧q) | ~(p∧q) | (~p∨q) | p∧(~p∨q) ⇓ (A) let | (q∧~(p∧q)) ⇓ (B) let | A∨B |
|---|---|---|---|---|---|---|---|---|
| F | F | T | F | T | T | F | F | F |
| F | T | T | F | T | T | F | T | T |
| T | F | F | F | T | F | F | F | F |
| T | T | F | T | F | T | T | T | T |

Now it can be seen from truth table that $A \vee B = q$

**(Hence Proved)**

**20.** ***Write converse, inverse and contrapositive of the following statement :***
***If the teacher is absent, then some students do not complete their homework.*** **(2015-16)**

If teacher is absent, then some students do not complete their homework

$T(x) = x$ is teacher $\quad S(y) = y$ is students

$A(x) = x$ is absent $\quad H(y) = y$ completes homework

Now the statement can be written as

$\forall x[T(x) \to [A(x) \wedge \exists y[S(y) \wedge \sim H(y)]$

Since our statement is :

$\forall x[T(x) \to [A(x) \wedge \exists y[S(y) \wedge \sim H(y)]$

We know that

If $p \to q$

Then

Converse is $q \to p$

Contra positive is $\sim p \to \sim q$

Inverse is $\sim p \to \sim q$

So our statement become :

**Converse :**

$\forall x[A(x) \to T(x)] \wedge \exists y[H(y) \wedge \sim S(y)]\{(p \to q) \equiv p \wedge \sim q\}$

**Contra Positive :**

$\forall x[\sim A(x) \to \sim T(x)] \wedge \exists y[\sim H(y) \wedge \sim S(y)]$

**Inverse :**

$\forall x[\sim T(x) \to \sim A(x)] \wedge \exists y[S(y) \wedge H(y)]$

**21. Verify that the Proposition : $p \vee \neg(p \wedge q)$ is Tautology. (2013-14)**

| p | q | p∧q | ¬(p∧q) | p∨¬(p∧q) |
|---|---|---|---|---|
| F | F | F | T | T |
| F | T | F | T | T |
| T | F | F | T | T |
| T | T | T | T | T |

Since, the truth value of $p \vee \neg(p \wedge q)$ is true for all values of $p$ and $q$, Hence, the proposition is tautology.

**22. Determine the validity or fallacy of the following arguments :**

**(1) (premises) $p, p \to q \vdash \neg p$ (Conclusion)**

**$p \to q, q \vdash p$ using theorem of tautology. (2013-14)**

(1)

| p | q | ¬p | p→q | p∧(p→q) | p∧(p→q)→¬p |
|---|---|---|---|---|---|
| F | F | T | T | F | T |
| F | T | T | T | F | T |
| T | F | F | F | F | T |
| T | T | F | T | T | T |

From the truth table it is clear that argument is valid.

(2)

| p | q | p→q | (p→q)∧q | (p→q)∧q→p |
|---|---|---|---|---|
| F | F | T | F | T |
| F | T | T | T | F |
| T | F | F | F | T |
| T | T | T | T | T |

It is clear from the truth table that arguments show fallacy.

**23. Write the negation of the following statement. "If it is raining, then the game is cancelled". (2011-12)**

Let $p$ : it is raining. $q$ : the game is cancelled. The given statement can be written as $p \to q$ the negation of $p \to q$ is written as $\sim(p \to q) \equiv p \wedge \sim q$.



Hence, the negation of the given statement is it is raining and the game is not cancelled.

**24. Show that $(p \wedge q) \to p$ is tautology. (2011-12)**

| p | q | (p∧q) | (p∧q)⇒p |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | F | T |
| F | F | F | T |

**25. Prove that $(p \vee q) \Rightarrow (p \wedge q)$ is logically equivalent to $p \Leftrightarrow q$. (2011-12)**

$p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$

$\equiv [(\sim p \vee q) \wedge \sim q] \vee [(\sim p \vee q) \wedge p]$ By distributive law

$\equiv [\sim q \wedge (\sim p \vee q)] \vee [p \wedge (\sim p \vee q)]$ By commutative law

$\equiv [\sim q \wedge q) \vee (\sim q) \wedge \sim p)] \vee [(p \wedge q) \vee (p \wedge \sim q)]$

by distributive law

$\equiv [F \vee (\sim q \wedge \sim p)] \vee [(p \wedge q) \vee F]$ By complement law

$\equiv [(\sim q \wedge \sim p)] \vee [(p \wedge q)]$ By identity law

$\equiv [\sim (p \vee q)] \vee [(p \wedge q)]$ By De Morgan's law

$\equiv [(p \vee q)] \Rightarrow [(p \wedge q)]$ **(Hence Proved)**

**26. Compute the truth value of the statement : (2011-12)**

**$p \to q) \leftrightarrow (\sim q \to \sim p)$.**

| p | q | ~p | ~q | (p⇒q) | (~q⇒~p) | (p⇒q) (~q⇒~p) |
|---|---|---|---|---|---|---|
| T | T | F | F | T | T | T |
| T | F | F | T | F | F | T |
| F | T | T | F | T | T | T |
| F | F | T | T | F | T | F |

**27. Show that : (2011-12)**

**$[p \to (q \to r)] \to [(p \to q) \to (q \to r)]$ is a tautology.**

| p | q | r | q→r | p→q | p→r | p→(q→r) | (p→q)→(p→r) | [p→(q→r)]→ [(p→q) → (p→r)] |
|---|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T | T |
| T | T | F | F | T | F | F | F | T |
| T | F | T | T | F | T | T | T | T |
| T | F | F | T | F | F | T | T | T |
| F | T | T | T | T | T | T | T | T |
| F | T | F | F | T | T | T | T | T |
| F | F | T | T | T | T | T | T | T |
| F | F | F | T | T | T | T | T | T |

**28. *Construct the truth table for the following :*** **(2011-12)**

**(1)** $p \wedge (\neg q)$

**(2)** $(\neg p \vee q) \leftrightarrow (p \rightarrow q)$

**(1)** $p \wedge (\neg q)$

| p | q | ~q | p∧~q |
|---|---|---|---|
| T | T | F | F |
| T | F | T | T |
| F | T | F | F |
| F | F | T | F |

**(2)** $(\neg p \vee q) \leftrightarrow (p \rightarrow q)$

| p | q | ~p | ~p∨q | p⇒q | ~p∨q ⇔p⇒q |
|---|---|---|---|---|---|
| T | T | F | T | T | T |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | F | T | T | T | T |

**29. *Prove that following is a tautology :*** **(2010-11)**

$A \vee (\overline{B \wedge C}) = (A \vee \overline{B}) \vee \overline{C}$.

| A | B | C | $B \wedge C$ | $\overline{B \wedge C}$ | $A \wedge (\overline{B \wedge C})$ |
|---|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 1 | 1 |
| 0 | 0 | 1 | 0 | 1 | 1 |
| 0 | 1 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | 1 | 1 |
| 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 |
| 1 | 1 | 1 | 1 | 0 | 1 |



| A | B | C | $\overline{B}$ | $A \vee \overline{B}$ | $\overline{C}$ | $(A \vee \overline{B}) \vee (\overline{C})$ |
|---|---|---|---|---|---|---|
| 0 | 0 | 0 | 1 | 1 | 1 | 1 |
| 0 | 0 | 1 | 1 | 1 | 0 | 1 |
| 0 | 1 | 0 | 0 | 0 | 1 | 1 |
| 0 | 1 | 1 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 1 | 1 | 1 |
| 1 | 0 | 1 | 1 | 1 | 0 | 1 |
| 1 | 1 | 0 | 0 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 1 | 0 | 1 |

$A \vee (\overline{B \wedge C}) = (A \vee \overline{B}) \vee \overline{C}$, hence it is a tautology

**30. *Given the value of $p \rightarrow q$ is true. Determine the value of $\sim p \vee (p \leftrightarrow q)$.*** **(2010-11)**

| P | Q | P→Q |
|---|---|---|
| 1 | 1 | 1 |
| 1 | 0 | 0 |
| 0 | 1 | 1 |
| 0 | 0 | 1 |

| P | Q | ~p | P↔Q | ~P∨(P↔Q) |
|---|---|---|---|---|
| 1 | 1 | 0 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 |
| 0 | 1 | 1 | 0 | 1 |
| 0 | 0 | 1 | 1 | 1 |

P→Q is true and the value of ~pv (P→Q) is also true.

**31. *State the contrapositive and converse statement of the following statement :*** **(2011-12)**
***"If the triangle is equilateral, then it is equiangular."***

$p$ : triangle is equilateral

$q$ : triangle is equiangular

The implication $p \Rightarrow q$ states that

$r$ : if the triangle is equilateral then it is equiangular.

The converse of this implication, namely $q \Rightarrow p$ states that $s$ : if the triangle is equiangular, then there has been a equilateral triangle.

The contrapositive of the implication $p \Rightarrow q$, namely $\sim q \Rightarrow \sim p$ states that

$t$ : if the triangle is not equiangular, then there has been no equilateral triangle.

**32. *Negate the statement :*** (2010-11)
***For all real x, if "$x > 3$", then "$x^2 > 9$".***

Let $p : x > 3$
$q : x^2 > a$

The given statement can be written as $p \Rightarrow q$ the negation of $p \Rightarrow q$ is written as $\sim (p \Rightarrow q) \equiv p \wedge \sim q$

Hence the negation of the given statement is $x$ is greater than 3 and the $x^2$ is not greater than 9.

**33. *Construct the truth table for the following :*** (2010-11)

$$((P \rightarrow Q) \vee R) \vee (P \rightarrow Q \rightarrow R).$$

| P | Q | R | P→Q | P→Q→R | (P→Q)VR | ((P→Q)VR (P→Q→R) |
|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T |
| T | T | F | T | F | T | T |
| T | F | T | F | T | T | T |
| T | F | F | F | T | F | T |
| F | T | T | T | T | T | T |
| F | T | F | T | F | T | T |
| F | F | T | T | T | T | T |
| F | F | F | T | F | T | T |

**34. *What are the Connectives? Define each of them through example.***

A statement is declarative sentence which has one and only one of two possible values as True (T) and False (F). The sentences like :

(1) May God Bless you.
(2) Wish you a happy journey.
(3) Please Wait.

Are not the statement since they do not have any definite value i.e., either $T$ or $F$ but the sentence like :

(a) India is a country (T)
(b) Calcutta is an old city (T)



(c) 3 is an even number (F)

Are the statements having values T or F?

Now, it is possible to construct rather complicated statements from simpler statements by using certain connecting words known as "CONNECTIVES".

The three most fundamental connectives are :

(a) NOT Negation ($\neg$)
(b) AND Conjunction ($\wedge$)
(c) OR Disjunction ($\vee$)

(a) **Negation ($\neg$) :** Negation of a statement is formed by introducing a word "NOT" in it. Consider an atomic statement $P$ as :

$P$: Calcutta is a city.

Now negation of P denoted by $\neg P$ will be given as :

$\neg P$: Calcutta is not a city.

**Truth Table for Negation :**

| P | ¬P |
|---|---|
| T | F |
| F | T |

(b) **Conjunction ($\wedge$) :** The conjunction of two statements $P$ and $Q$, denoted by $(P \wedge Q)$ and read as '$P$ and $Q$'. The truth value of $(P \wedge Q)$ is $T$ if both $P$ and $Q$ have the truth value $T$ else it has truth value $F$.

| P | Q | P∧Q |
|---|---|---|
| F | F | F |
| F | T | F |
| T | F | F |
| T | T | T |

**Example :** Translate the statement "jack and Jill went up the hill" into symbolic form.

Since, the given statement is conjunction of two statement $P$ and $Q$ as :

**P :** Jack went up the hill
**Q :** Jill went up the hill.

Thus, it may be written as $(P \wedge Q)$.

(c) **Disjunctive ($\vee$) :** The disjunction of two statements $P$ and $Q$ is written as $(P \vee Q)$ and read as '$P$ or $Q$'. It $(P \vee Q)$ has a truth value $F$ if both $P$ and $Q$ has truth value F otherwise if any of $P$ and $Q$ has truth value $T$ then $P \vee Q$ will have truth value $T$ as :

| P | Q | P ∨ Q |
|---|---|---|
| F | F | F |
| F | T | T |
| T | F | T |
| T | T | T |

**Example :** Consider a statement "Mark is either rich or happy". The statement is equivalent to "Mark is rich or Mark is happy" thus we may write it as :

$(P \vee Q)$ where

$P$ : Mark is rich.

$Q$ : Mark is Happy.

The combination of such connectives (some more) yields a statement formula.

**Conditional(→) and Bi-conditional (↔) Connectives :** If $P$ and $Q$ are two statements then the statement $P \to Q$, read as if $P$ then $Q$, is a conditional statement. Here, $P \to Q$ has a truth value F when $P$ has truth value $T$ and $Q$ has truth value $F$ else it has truth value $T$. also, $P$ is called antecedent and $Q$ is called consequent.

Truth table for $P \to Q$ :

| P | Q | P → Q |
|---|---|---|
| F | F | T |
| F | T | T |
| T | F | F |
| T | T | T |

**Note :** Obviously, the statement formula ( $P \to Q$) is equivalent to ($\neg P \vee Q$) statement formula.

Similarly, if $P$ and $Q$ are two statements then the statement $P \leftrightarrow Q$, read as "$P$ if and only if $Q$" or "$P$ if $Q$", is a bi-conditional statement. $P \leftrightarrow Q$ has truth value $T$ if both $P$ and $Q$ have same truth value as :

| P | Q | P ↔ Q |
|---|---|---|
| F | F | T |
| F | T | F |
| T | F | F |
| T | T | T |

**Example :** Consider a statement "The crop will be destroyed if there is a Flood". The statement consists of two atomic statements as :



$P$ : If there is a Flood.

$Q$ : The crop will be destroyed.

Thus, the equivalent formula for the statement will be $(P \to Q) \vee (Q \to P)$.

**Statement Formula and Truth Table :** As we've known that statements are of two types : one which have no connectives called Atomic and others are compound or Molecular statements which have one or more atomic statements with some connectives. Example,

$\neg P, P \vee Q, P \wedge Q, (\neg P \vee Q), (\neg P \to Q)$ etc.

In these compound statements, $P$ and $Q$ are the statement variables also known as component of statements formula. Thus, a statement formula consists of one or more statement variable(s) and connective(s).

The truth table is a table which show the truth value of a statement formula for each possible combination of truth values of above defined component of statement formula i.e. statement variables.

For example, to draw the truth table of statement formula ( $P \vee \neg Q$) we first take the all possible combinations of these two statement variables and then perform the given operation. As :

| P | Q | ¬Q | P ∨ ¬Q |
|---|---|---|---|
| F | F | T | T |
| F | T | F | F |
| T | F | T | T |
| T | T | F | T |

Similarly, the truth table for $P \wedge \neg P$ will be :

| P | ¬P | P ∧ ¬ |
|---|---|---|
| F | F | F |
| T | T | F |

Again, for $(P \vee Q) \vee \neg P$ we will get :

| P | Q | (P ∨ Q) | ¬P | (P ∨ Q) ∨ ¬P |
|---|---|---|---|---|
| F | F | F | T | T |
| F | T | T | T | T |
| T | F | T | F | T |
| T | T | T | F | T |

**35.** ***Prove that the formula*** $B \vee (B \to C)$ ***is a tautology.***

| B | C | B → C | B ∨ (B → C) |
|---|---|---|---|
| T | T | T | T |
| T | F | F | T |
| F | T | T | T |
| F | F | T | T |

All true so it is a tautology.

**36. *Show that $B \to E$ is a valid conclusion drawn from the following premises :***

***$A \vee (B \to D) \sim C \to (D \to E)$, $A \to C$ and $\sim C$.***

**Given :** $A \vee (B \to D)$ ... (i)

$A \vee (B \to D) \equiv \sim A \to (B \to D)$ $\{p \to q \equiv \sim p \vee q\}$

**Given :** $A \to (C \wedge \sim C)$ ... (ii)

$A \to (C \wedge \sim C) \equiv (A \to C) \wedge (A \to \sim C)$

**Given :** $\sim C \to (D \to E)$ ... (iii)

From (ii) and (iii),

$(A \to C) \wedge (A \to \sim C)$

$\equiv (A \to C) \wedge (A \to (D \to E))$

$\equiv (A \to C) \wedge (A \to E)$

$\equiv C \to E$ ... (iv)

From (iii),

$\sim A \to \sim (C \wedge \sim C)$

$\sim A \to \sim C \vee C$

From (i),

$\therefore (B \to D) \to (\sim C \vee C)$

$\equiv (B \to D) \to C$

$\equiv B \to C$ ... (v)

From (iv) and (v) $B \to E$ [Hypothetical Syllogism]

**37. *Prove that following argument is valid using predicate logic :***

***(1) All dogs are barking.***

***(2) Some animals are dogs.***

***∴ Some animals are barking.***

Let,

$K(x)$ : $x$ is dog



$M(x)$ : $x$ is barking.

There exists an $x$ such that $x$ is dog and $x$ is barking.

$(\exists x)(K(x) \wedge M(x))$

**38. *Make a truth table for $(p \wedge \sim p) \vee (\sim (q \wedge r))$.***

**Truth Table for $(p \wedge \sim p) \vee (\sim (q \wedge r))$ :**

| p | q | r | ~p | (p ∧ ~p) | (q ∧ r) | ~(q ∧ r) | (p ∧ ~p) ∨ (~(q ∧ r)) |
|---|---|---|---|---|---|---|---|
| T | T | T | F | F | T | F | F |
| T | T | F | F | F | F | T | T |
| T | F | T | F | F | F | T | T |
| T | F | F | F | F | F | T | T |
| F | T | T | T | F | T | F | F |
| F | T | F | T | F | F | T | T |
| F | F | T | T | F | F | T | T |
| F | F | F | T | F | F | T | T |

**39. *Determine the truth value for each of the following statements. Assume x, y are elements of set of integers.***

***(1) $\forall x, y$ $x + y$ is even***

***(2) $\exists x$ $\forall y$ $x + y$ is even***

$x$ and $y$ is set of integers. So, $x$ (1, 2, 3, 4, 5, 6, 7, 8, 9 ......) and $y$ (1, 2, 3, 4, 5, 6, 7, 8, 9......)

**(1)** $\forall$ $x$ and $\forall$ $y$, $x + y$ is even

This statement is false because for some condition there exist some $x$ and some $y$ where this condition is not true like (2, 3) = 2 + 3 = 5 and 5 is not even but exist in $x$ and $y$.

**(2)** $\exists$ $x$ and $\forall$ $y$ $x + y$ is even

This statement is false because for some condition there exist some $y$ when this condition is not true because if we choose an even integer from $x$ then not for all $y$, $x + y$ become even, in case if $y$ is even then answer become even, and if we choose a odd integer from $x$ then if we choose a odd integer from $y$ then it become even so condition not hold for all $y$.

**40. *Obtain the Equivalent Conjunctive Normal Form of the formula :***

$$(P \Rightarrow Q) \vee (\neg P \vee \neg Q)$$

***where $\Rightarrow$, $\vee$ and $\rceil$ denote respectively biconditional, Disjunction and negative and P and Q denote prepositional variables.***

$(P \rightarrow Q) \vee (\rceil P \vee \rceil Q)$

$\Leftrightarrow \quad (\rceil P \vee Q) \vee (\rceil P \vee \rceil Q)$

Taking negation of whole :

$\Leftrightarrow \quad \rceil[(\rceil P \vee Q) \vee (\rceil P \vee \rceil Q)]$

$\Leftrightarrow \quad \rceil(\rceil P \vee Q) \wedge \rceil(\rceil P \vee \rceil Q)$ from D'Morgan's law

Which is the required form of Conjunctive Normal Form.

**41. *Write equivalent form of the following in P-edicate Calculus :***

***(1) All lions have tails.***

***(2) Some pet dogs are dangerous.***

***(3) A resident of Delhi is a resident of India.***

(1) $L(x)$ : $x$ is lion

$T(x)$ : $x$ has tail

$(\forall x)\ (Lx) \rightarrow T(x))$

(2) $P(x)$ : $x$ is pet dog

$D(x)$ : $x$ is dangerous.

$(\exists x)\ (Px) \wedge D(x))$

(3) $D(x)$ : $x$ is resident of Delhi

$I(x)$ : $x$ is resident of India

$(\forall x)\ (Dx) \rightarrow I(x))$

**42. *Write and prove 'Modus Ponens' rule of inference.* (2017-18)**

An argument is a sequence of statements. The last statement is the conclusion and all its preceding statements are called premises (or hypothesis). The symbol "$\therefore$", (read therefore) is placed before the conclusion. A valid argument is one where the conclusion follows from the truth values of the premises.



## *Modus Ponens*

If $P$ and $P \rightarrow Q$ are two premises, we can use Modus Ponens to derive $Q$.

$P \rightarrow Q$

$P$

............

$\therefore Q$

**Example :**

"If you have a password, then you can log on to facebook", $P \rightarrow Q$

"You have a password", $P$

Therefore, "You can log on to facebook"

**43. ◆ *Define the term Arguments.* (2018-19)**

**◆ *Prove the validity of the following argument "If I get the job and work hard, then I will get promoted. If I get promoted, then I will be happy. I will not be happy. Therefore, either I will not get the job or I will not work hard."* (2017-18, 2018-19)**

## *Arguments*

An argument is a sequence of statements called premises, plus a statement called the conclusion. A valid argument is an argument such that the conclusion is true whenever the premises are all true.

**Note :** An argument has the following form :

$$(P_1 \wedge P_2 \wedge ... \wedge P_n) \rightarrow C$$

**Numerical Solution :** Let $p$ : I get the job

$q$ : I work hard

$r$ : I get promoted

$s$ : I will be happy

given argument can be written in symbolic form as

$(p \wedge q) \Rightarrow r$

$r \Rightarrow s$

$\sim s$

$(p \wedge q) \Rightarrow r$ and $r \Rightarrow s$ gives $(p \wedge q) \Rightarrow s$ by hypothetical syllogism

$(p \wedge q) \Rightarrow s$ and $\sim s$ gives $\sim (p \wedge q)$ by modus tollens

So $\sim p \vee \sim q$

i.e. either I will not get the job or I will not work hard.
Hence the argument is valid.

# ALGEBRAIC STRUCTURES

**1. *Define the Order of an element of a group.* (2020-21)**

***Order of an Element of a Group***

By the order of the element $a \in G$ we mean that the least positive integer such that $a^n = e$, where $e$ is the identity element of a group.

**2. *Define the characteristic of a Ring.* (2020-21)**

***Characteristic of a Ring***

An algebraic system $(R, +, .)$ is Ring if it satisfies following properties.

(1) $(R, +)$ is an abelian group
(2) $(R, .)$ is semi group
(3) Operation '.' is distributed over '+'
E.g. $a.(b+c) = a.b + a.c$.

**3. *Show that the set of fourth roots of unit is an Abelian group with respect to multiplication.* (2020-21)**

Set of fourth Root of unity is $\{1, -1, i, -i\}$

$$\begin{array}{c|cccc} & 1 & -1 & i & -i \\ \hline 1 & 1 & -1 & i & -i \\ -1 & -1 & 1 & -i & i \\ i & i & -i & -1 & 1 \\ -i & -i & i & 1 & -1 \end{array}$$

**(1) Closure Property :** All entries are available in table so it is a closure property

**(2) Associative Property :** $(1 - i). - i = 1.(i. - i)$

**(3) Existence of Identity :** Here identity element is 1

**(4) Existence of Inverse :** $(1)^{-1} = 1$, $(-1)^{-1} = -1$, $(-i)^{-1} = i$, $(i)^{-1} = -i$

(5) **Commutative Law :** Multiplication of complex No. is always commutative.

**4.** ***Prove that the Inverse of every element of a group is Unique.*** **(2020-21)**

Let $a$ be any arbitrary element of a group $G$ and let $e$ be the identity element.
Let $b$ & $c$ are two inverse of $a$

$$ba = e = ab$$
$$ca = e = ac$$
$$b(ac) = be$$
$$= b$$
$$(ba)c = ec$$
$$= c$$
$$b(ac) = (ba)c$$
$$b = c$$

Therefore inverse of every element of a group is unique.

**5.** ***Show that set :*** $G = \{a + \sqrt{2}\, b : a, b \in Q\}$ ***is a group with respect to addition.*** **(2020-21)**

(1) **Closure Property :** Let $x$, $y$ be two element of $G$.
Then $x = a + \sqrt{2}\, b$

$$y = c + \sqrt{2} d$$
$$x + y = (a + c) + \sqrt{2}(b + d) \in G$$

(2) **Associative Law :** The element of $G$ are rational number & addition of rational number is associative

(3) **Existence of Identity :** We have $0 + \sqrt{2}.0 \in G$ since $0 \in O$

$$(0 + \sqrt{2}.0) + (a + \sqrt{2}b) = (0 + a) + \sqrt{2}(0 + b) = a + \sqrt{2}\, b$$

So $0 + \sqrt{2}.0$ is identity.

(4) **Existence of Inverse :**

$$(-a) + \sqrt{2}(-b) + a + \sqrt{2}\, b = 0 + \sqrt{2}\, 0$$

So $(-a) + \sqrt{2}(-b)$ is inverse of $a + \sqrt{2}\, b$
Hence $G$ is a group with respect to addition.



**6.** ***Write a note on Algebraic Structure.***

An algebraic structure is a set (called carrier set or underlying set) with one or more finitely operations defined on it that satisfies a list of axioms. Examples of algebraic structures include groups, rings, fields, and lattices.

An algebraic structure is a set equipped with an operation (or operations) that satisfy a standard set of algebraic laws. For example, the set could be the set of all real numbers and the operations could be addition and multiplication (and their inverses, subtraction and division). In this case the standard rules of algebra are :

(1) **Associative Law of Addition :** $(a + b) + c = a + (b + c)$

(2) **Commutative Law of Addition :** $a + b = b + a$

(3) **Additive Identity :** $a + 0 = a$

(4) **Additive Inverses :** $a + -a = 0$

(5) **Associative Law of Multiplication :** $(a \times b) \times c = a \times (b \times c)$

(6) **Commutative Law of Multiplication :** $a \times b = b \times a$

(7) **Multiplicative Identity :** $a \times 1 = a$

(8) **Multiplicative Inverses :** For $a \neq 0, a \times a^{-1} = 1$

(9) **Distributive Law :** $a \times (b + c) = (a \times b) + (a \times c)$

**7.** ***Define a group. Let*** $S = \{0, 1, 2, 3, 4, 5, 6, 7\}$ ***and * denote "multiplication module 8" that is*** $x * y = (xy)$ ***mod 8.***

A system consisting of a non-empty set $G$ of elements $a$, $b$, $c$ etc. with an operation $O$ is said to be a group provided. The following postulates are satisfied :

$P_1$ **: Closure Property :** $a, b \in G \Rightarrow aob \in G$ i.e. $G$ is closed under the operation.

$P_2$ **: Associativity :** $(aob)oc = ao(boc)$, for every $a, b, c \in G$.

$P_3$ **: Existence of Identity :** There exists an unique elements in G such that $eoa = a = aoe$, for every $a \in G$. This element $e$ is called the Identity.

$P_4$ : **Existence of Inverse :** For each $a \in G$; there exists an elements $a^{-1} \in G$, such that $aoa^{-1} = e = a^{-1}oa$.

**8. *Let $G_1$ and $G_2$ be sub group of a group G. Show that $G_1 \cap G_2$ is also a subgroup of G.***

Since $x \in G_1 \cap G_2$, $G_1 \cap G_2$ is non-void

Now $a, b \in G_1 \cap G_2$

$\Rightarrow$ $a, b \in G_1$ and $a, b \in G_2$

$\Rightarrow$ $ab^{-1} \in G_1$ and $ab^{-1} \in G_2$

$\Rightarrow$ $ab^{-1} \in G_1 \cap G_2$

Hence $G_1 \cap G_2$ is a subgroup of $G$

**9. *Define a group. Let S={0, 1, 2, 3, 4, 5, 6, 7} and * denote "multiplication modulo 8 i.e. x*y=(xy) mod 8. Write three distinct groups (G, *) where G⊂S and G has two elements.***

**Group**

Lets, (G, *) be algebraic structure where * is a binary operation, then (G, *) is called a group under this operation if the following condition are satisfied :

(1) **Closure law :** The binary * is closed operation i.e. $a*b \in G$ for all $a,b \in G$.

(2) **Associative law :** The binary operation * is an associative operation i.e. $a*(b*c)=(a*b)*c$ for all $a,b,c \in G$.

(3) **Identity Element :** There exists an identity element i.e for some $e \in S$, $e*a=a*e$, $a \in G$.

(4) **Inverse Element :** For each $a$ in $G$, there exists an element $a'$ (The inverse of $a$ ). In $G$ such that $a*a' = a'*a = e$.

**10. *What are Semi-group and Monoid.***

**Semi-Group**

Let (A, *) be an algebraic system where * is a binary operation on A. (A, *) is called a Semigroup, if the following conditions are satisfied :

(1) *is a closed operation.

(2) *is an associative operation.



**Monoid :** Let (A, *) be an algebraic system, where * is a binary operation on A. (A, *) is called a Monoid if the following conditions are satisfied.

(1) * is a closed operation.

(2) * is an associative operation.

(3) There is an identity.

**11. *Prove that quotient group of an abelian group is abelian.*** (2014-15)

Let $G$ be an abelian group. Then quotient group of $G$ is :

If $H$ be normal subgroup of $G$. The set of all cosets of $H$ in $G$ is known as quotient $G/H$.

i.e. $G/H = \{Ha, a \in G\}$

Since $a \in G$, Now it is clear that the quotient group is also an abelian group because we know that all cosets of a normal group is a group with respect to multiplication. So only commutative property is checked that clearly satisfied.

**12. *Explain Cosets in detail with example.***

By the definition of a subgroup it is clear that not every subset of a group is subgroup. The problem that we try to solve here is to find those subsets which can quality to become subgroups. An important relationship is explained by a theorem, known as Lagrange's theorem. This theorem has important applications in the development of efficient group codes required in the transmission of information. Such modules are constructed by joining various subgroup modules that do operation in subgroups.

Let, (G, *) be a group and (H, *) be a subgroup of (G, *). We shall define an equivalence relation on G called a left coset relation with respect to the subgroup (H, *) or a left coset, relation modulo H, denoted by the symbol ≡ such that for $a,b \in G$, $a \equiv b$ or more preciously $a \equiv b$ (mod H) if $b^{-1} * a \in H$. We now show that this relation is an equivalence relation.

Since, $H$ is a subset of $G$, $e_G \in H$. For any $a \in G$, $a^{-1} * a \in e_g \in H$, implying that if $a \equiv b \pmod{H}$, then $b \equiv a \pmod{H}$. Similarly one can see that if $b^{-1} * a \in H$ and $c^{-1} * b \in H$ then $c^{-1} * a = (c^{-1} * b) * (b^{-1} * a) \in H$, implying that from $a \equiv b \pmod{H}$ and $b \equiv c \pmod{H}$, we have $a \equiv c \pmod{H}$. Hence a left coset modulo $H$ relation is an equivalence relation on $G$.

**13. *Define and explain the Permutation Group by suitable examples.* (2013-14)**

**Permutation Group**

Suppose $S$ be a finite set having '$n$' distinct elements then one–one mapping of $S$ onto itself is called a permutation of degree '$n$' that is a function,

$F : S \to S$ is said to be permutation of $S$ if,

(1) $F$ is one-one

(2) $F$ is onto

The no. of distinct element in the finite set $S$ is known as the degree of permutation.

Example : if $S = \{1,2,3,4\}$ is finite set having 4 elements then

$$F = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{bmatrix} \text{etc}$$

are called permutation.

Where, $F(1) = 2 \quad F(3) = 1$

$F(2) = 4 \quad F(4) = 3$

Let $A$ be a set of degree $n$. Let $P_n$ be the set of all permutations of degree $n$ on $A$ then $(P_n, *)$ is a group, called a permutation group and the operation $*$ is the composition of permutations.

**14. ◆ *Discuss the Homomorphism and Isomorphism of Groups with examples.* (2013-14)**

**◆ *What do you mean by group homomorphism and group isomorphism? Explain with example.***



**Homomorphism and Isomorphism of Groups**

Let $(G, *)$ and $(G_1, *_1)$ be two groups and $F$ is a function from $G$ into $G_1$. Then $F$ is called a homomorphism of $G$ into $G_1$ if for all $a, b \in G$

$f(a * b) = f(a) * f(b)$

A homomorphism is called and eqimorphism if $f$ is onto $G_1$ and $f$ is called a monomorphism if $f$ is one-one. If there is an eqimorphism $f$ from $G$ onto $G_1$, then $G$ is called a homomorphic image.

A homomorphism $f$ of a group into a group $G_1$ is called an Isomorphism of $G$ onto $G$. If $f$ is one-one onto $G_1$, $G$ and $G_1$ are said to be Isomorphic and denoted by $G \cong G_1$ an Isomorphism of a group $G$ onto $G$ is called an automorphism.

**15. *What do you mean by isomorphism of semigroups? How does an isomorphism of semigroups differ from an isomorphism of posets?***

**Isomorphism of Semigroups**

Let $(S, \alpha)$ and $(T, \beta)$ be any two semigroups. A mapping $g : s \to T$ such that for any two elements $a, b \in s$.

$g(a \alpha b) = g(a) \beta g(b)$ is called a semigroup homomorphism.

A semigroup homomorphism is called a semigroup monomorphism, eqimorphism or isomorphism depending on whether the mapping is one-to-one, onto or one-to-one onto respectively. Two semigroup $(S, \alpha)$ and $(T, \beta)$ are said to be isomorphic. If there exists a semigroup isomorphism from S to T.

**An Isomorphism of Semigroups Differ from An Isomorphism of Posets :** This section sets out concept useful for understanding the structure of semigroup. Two semigroups S and $T$ are said to be isomorphic if there is a bijection $f : S \leftrightarrow T$ with the property that, for any elements $a$, $b$ in S, $f(ab) = f(a) \cdot f(b)$. In this case $T$ and S are also isomorphic, and for the purposes of semigroup theory, the two semigroups are identical. If $A$ and B are subsets of some semigroup then $AB$ denots the set $\{ab(a$ in $A$ and $b$ in $B)\}$. A subset $A$ of a semigroup $S$ is called a subsemigroup if it is closed under the semigroup operation, that is $AA$ is a subset of $A$.

Two posets are said to be isomorphic if their "structure' are entirely analogous. Formally, posets $P = (x, \leq)$ and $Q = (x', \leq')$ are isomorphic if there is a bijection $f$ from $x$ to $x'$ such that $x \leq x'$ precisely when $f(x) \leq' f(x')$.

## 16. What is sub-group?

**Definition**

If a subset $H$ of a group $G$ (which we write as $H \subseteq G$) is itself a group under the operation defined in $G$, we say that $H$ is a subgroup of $G$.

## 17. What are cyclic groups?

**Cyclic Groups**

If there exists a group element $g \in G$ such that $\langle g \rangle = G$, we call the group $G$ a cyclic group. We call the element that generates the whole group a generator of $G$. (A cyclic group may have more than one generator, and in certain cases, groups of infinite orders can be cyclic.) Examples will make this very clear.

**Example :**

Returning to $(Z_{10}, +)$, we saw that $\langle 2 \rangle$ generated a subgroup of $(Z_{10}, +)$ but did not generate the whole group. Let's consider a different element of this group. Observe that by repeatedly adding 7 to itself and reducing mod 10 we see that

$$\langle 7 \rangle = \{7, 4, 1, 8, 5, 2, 9, 6, 3, 0\} = Z_{10}$$

So indeed $(Z_{10}, +)$ is a cyclic group. We can say that $Z_{10}$ is a cyclic group generated by 7, but it is often easier to say 7 is a generator of $Z_{10}$. This implies that the group is cyclic.



## 18. Show that $G = \{0, 1, 2, 3, 4\}$ is a cyclic group under addition modulo 5. (2014-15)

Here, Given $G = \{0, 1, 2, 3, 4\}$

For $(G, +_5)$

$1^1 = 1,\ 1^2 = 1 +_6 1 = 2,\ 1^3 = 1 +_6 1^2 = 3$

$1^4 = 1 +_6 1^3 = 4,\ 1^5 = 0$

Thus $G = \{1^5, 1^1, 1^2, 1^3, 1^4\}$

Hence $G$ is a cyclic group and 1 is generator

Similarly, it can be shown that 4 is another generator.

## 19. Find product of two permutations

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{bmatrix} \text{and} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{bmatrix}$$

**Check whether it is odd or even. (2014-15)**

The two given permutations are :

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{bmatrix} \text{and} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{bmatrix}$$

Now $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 1 & 2 & 3 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 3 & 6 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 6 & 5 & 2 & 1 & 4 \end{bmatrix}$$

Now since we can write this like :

$$\begin{bmatrix} 1 & 3 & 5 \\ 3 & 5 & 1 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 \\ 6 & 4 & 2 \end{bmatrix}$$

Hence the product is even.

## 20. Show that in a group identity element is unique. (2014-15)

Let us assume that $a \in G$ and $e, e'$ be the two identities in G.

$e \in G \quad a \in G \Rightarrow ae = a$ ... (1)

$e' \in G \quad a \in G \Rightarrow ae'$ ... ... (2)

From (1) and (2), we get

$$ae = ae' \Rightarrow e = e'$$

Hence the identity element in a group is unique.

**21. *Show that Kernel is a normal subgroup in group isomorphism.*** **(2014-15)**

If $f$ be homomorphism of group $G$ into $G'$ then a set (Subset) $k$ is said to be kernel if it consists all those elements of G whose image is identity of $G'$.

Ker $f = k = \{a : f(\alpha) = e' \ \forall\ a \in G\}$

Let $f$ be a homomorphism of a group $G$ into a group $G'$. Let $e$ and $e'$ be the identities of $G$ and $G'$ respectively.

Since $f(e) = e'$ therefore atleast $e \in G \Rightarrow k \neq \phi$

$$a, b \in k \Rightarrow f(a) = e', f(b) = e'$$

$$f(ab^{-1}) = f(a)\ f(b^{-1}) = f(a)$$

$$[f(b)]^{-1} = e'\ (e')^{-1} = e'\ e' = e'$$

$$ab^{-1} \in k$$

Thus $a \in k, b \in k \Rightarrow ab^{-1} \in k$.

Therefore $k$ is subgroup of $G$. Now to prove $k$ is normal in $G$.

Let $g \in G$.

$$k_1 \in k \Rightarrow f(k) = e'$$

$$f(gk, g^{-1}) = f(g)\ f(k)\ f(g^{-1}) = f(g)\ e'\ [f(g)]^{-1}$$

$$= [f(g)]\ [f(g)]^{-1} = e'$$

$$g\ k_1 g^{-1} \in k.$$

Thus $g \in G, k \in k \Rightarrow g\ k_1 g^{-1} \in k$

Hence $k$ is normal subgroup of $G$.

**22. *Which property in a group implies :***
***(1) Uniqueness of the inverse of every element? Prove this fact.***
***(2) The left inverse of an element is also a right inverse of the element? Prove this fact.***

(1) Suppose that both $b$ and $c$ are inverse of $a$. That is :

$b\,c = e$ and $c\,a = e$



It follows that :

$(b\ a)\ b) = (c\ a)\ b$

Or $b(a\ b) = c\ (a\ b)$

Or $b = c$

This is proved.

(2) Let $b$ be a left inverse of $a$ and $c$ be a left inverse of $b$. let $e$ denote the identify. Since :

$$(b\ a)\ b = e\ b = b$$

We have :

$$c\ (b\ a)\ b) = c\ b = e$$

From,

$$c\ ((b\ a)\ b) = ((c\ b)\ a)\ b$$
$$= (e\ a)\ b$$
$$= a\ b$$

We have : $a\ b = e$

Thus, $b$ is also a right inverse of $a$. This is proved.

**23. *Let (G, *) be a finite cyclic group generated by an element $a \in G$. If G is of order n, that is, $|G| = n$, then $a^n = e$, so that : $G = (a, a^2, a^3, ... a^n = e)$ Furthermore, n is the least positive integer for which $a^n = e$. Prove this statement.***

Let us assume that for some positive integer $m < n$, $a^m = e$. Since, G is a cyclic group, any element of G can be written as $a^k$ for some $k \in I$. Now from Euclid's algorithm, we can be write $k = mq + r$, where, $q$ is some integer and $0 \leq r < m$. This means $a^k = a^{mq+r} = (a^m)^q * a^r = a^r$. So that every element of G can be expressed as $a^r$ for some $0 \leq r < m$ this implying that G has at most $m$ distinct elements, that is, $|G| = m < n$ which is a contradiction. Hence $a^m = e$ for $m < n$ is not possible. As a next step we show that elements, $a, a^2, a^3, ... a^n$ are all distinct where $a^m = e$. Assume to the contrary that $a^i = a^j$ for $i < j \leq n$. This means that $a^{j-i} = e$ where $j - i < n$, which is again a contradiction.

**24. Prove that every finite group of order $n$ is isomorphic to a permutation group of degree $n$.**

Let, (G, *) be a group of order $n$. we know that every row and column in the composition table of (G, *) represents a permutation of the element of G. Corresponding to an element $a \in G$. We denote by $p_a$ the permutation given by the column under $a$ in the composition table.

Thus $p_a(c)\ c * a$ for any $c \in a$. For every column, we can define permutation $G$. Let the set of permutation be denoted by $P$. Obviously, $P$ has $n$ elements. We shall now show that $(P, \Diamond)$ is a group, where $\Diamond$ denote the right composition of the permutation of $P$. Note that since $e \in G, p_e \in P$ and :

$p_e \Diamond p_a = p_a \Diamond p_e = p_a$ for any $a \in G$

Also for any $a \in G$, $p_{a^{-1}} \Diamond p_a = p_e$

Also for $a, b \in G$, $p_b \Diamond p_e = p_{a*b}$ the above equation follows from the fact that for any element $c \in G$ $p_a(c) = c * a$ so that $(p_a \Diamond p_b)(c) = (c*a)*b = c*(a*b) = p_{a*b}(c)$. Hence $(P, \Diamond)$ is a group. The last step is sufficient to guarantee that $(P, \Diamond)$ is a group, because it shows that $(P, \Diamond)$ is isomorphic to (G, *). Consider a mapping $f : G \to P$ given by $f(a) = p_a$ for any $a \in G$. Naturally $f$ is one to one onto. This can be written as :

$f(a * b) = f(a) \Diamond f(b)$

Showing that $f$ is an isomorphism.

**25. Let S = {1, 3, 7, 9} and G = (S, multiplication mod 10). Determine all left and right cosets of the subgroup {1, 9}. (2008-09)**

$g^H = \{gh : h$ an element of $H\}$ is a left coset of $H$ in G.

$H_g = \{hg : h$ an element of $H\}$ is a right coset of $H$ in G.



A coset is a left or right coset of some subgroup in G. Since, $H_g = g(g^{-1}H_g)$, the right coset $H_g$ (of $H$) and the left coset

$g(g^{-1}H_g)$ are the same.

$S = \{1, 3, 7, 9\}$

Subgroup $H = \{1, 9\}$

$G = (S$, multiplication mod 10 )

{ 1, 1}{3, 1} {7, 1} {9, 1}

{1, 9} {3, 9} {7, 9} {9, 9}

Left cosets Right cosets

{1, 3, 7, 9} {9, 7, 3, 1}

**26. How does a field differ from a ring? Explain with example.**

**Differences between Field and Ring**

**Let (R,+,·) be a Ring :** If $1 \in R$ such that $a \cdot 1 = 1 \cdot a = a$ for every $a \in R$, then we say that $R$ is a ring with identity element. If $a \cdot b = b \cdot a$ for all $ab \in R$ then $R$ is said to be a commutative ring.

**Example :**

**(1)** (z (set of integers), +, •) is a commutative ring with identity.

**(2)** (Q (set of rationals), +, •) is a commutative ring with identity.

**Field :** An algebraic system (F, +, •) where $F$ is a non-empty set + and • are two binary operations on $F$ is said to be a field if it satisfies the following three conditions.

**(a)** (F, +) is an abelian group.

**(b)** (F*, •) where $F^* = F/\{o\}$ and $O$ is additive identity, is a commutative group.

**(c)** $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$ for all $a,b,c = F$.

**Example :**

**(a)** Q the set of rational numbers is a field with respect to usual addition and multiplication of rational numbers.

(b) $R$, the set of real numbers, is a field with respect to usual addition and multiplication of real numbers.

**27. *If $(R, +, \cdot)$ is a ring with unity, then show that, for all $a \in R$.***
**(1) $(-1).a = -a$**
**(2) $(-1).(-1) = 1$**

(1) We have $(-1 + a).a = (-1.a) + (1.a)$ left distributive law

$0.a = (-1).a + (1).a$

$0 = (-1).a + (1).a$

$(-1).a = -a$ **(Proved)**

(2) $(-1).(-1) = -[(-1).(1)] = [-(1.1)] = 1.1 = 1$

$\because$ $(-a).(-b) = a.b$ **(Proved)**

**28. *Explain the following with suitable example :***
**(1) *Ring***
**(2) *Field***

(1) **Ring :** An algebraic system $(A, +, .)$ is called a ring, if the following conditions are satisfied :

(a) $(A, +)$ is an abelian group

(b) $(A, .)$ is a semigroup

(c) The operation is distributive over the operation $+$.

Example : Let $Z_n$ be the set of integers $\{0,1,2,.....n-1\}$. Let $\oplus$ be a binary operation on $Z_n$ such that :

$$a \oplus b = \begin{cases} a+b & \text{if } a+b < n \\ a+b-n & \text{if } a+b \geq n \end{cases}$$

Let $\odot$ be a binary operation on $Z_n$ such that :

$a \odot b =$ the remainder of $ab$ divided by $n$.

$(Z_n, \oplus)$ is an abelian group. Furthermore, it is not difficult to see the $\odot$ is a closed and an associated operation. Consequently $(Z_n, \odot)$ is a semi group. Thus to conclude the $(Z, \oplus, \odot)$ is a ring.



**Field :** Let $(A, +, .)$ be an algebraic system with two binary operations. $(A, +, .)$ is called a field if,

(1) $(A, +)$ is an abelian group.

(2) $(A - \{0\}, .)$ is an abelian group.

(3) The operation is distributed over the operation $+$.

**Example :** Fields are $(R, +, .)$ where $R$ is the set of all real numbers and (+) and (.) are the ordinary addition and multiplication operations of real numbers.

**29. *Consider the Set Q of rational numbers defined by $a * b = a + b - ab$. Is $(Q, *)$ a Semi Group?*** **(2013-14)**

Here $a*b = a + b - ab$, Now we check for closure property. Since addition, subtraction and multiplication operation of two rational number give a rational number so $a*b = a + b - ab$ is closed. If we check associative property then we find, it is associative hence, this is a semi group.

**30. *Prove that $G = \{1, w, w^2\}$ is a group under multiplication where 1, $w$, $w^2$ are cube root of unity.*** **(2013-14)**

Since $w$ is the cube root of unity thus $w^3 = 1$

The composition table for multiplication (X) is as follows :

| X | 1 | w | $w^2$ |
|---|---|---|---|
| 1 | 1 | w | $w^2$ |
| w | w | $w^2$ | $w^3$ |
| $w^2$ | $w^2$ | $w^3$ | $w^4$ |

| X | 1 | w | $w^2$ |
|---|---|---|---|
| 1 | 1 | w | $w^2$ |
| w | w | $w^2$ | 1 |
| $w^2$ | $w^2$ | 1 | w |

We know that $w^3 = 1$ hence $w^4 = w^3 \times w = w$. To prove $G = \{1, w, w^2\}$ as a group under multiplication we have to check following properties

(1) **Closure Property :** Since all entries in the table belongs to set G, hence closure property is satisfied.

(2) **Associative Property :** Since, multiplication is associative under number and G is set of complex number, So multiplication is associative on G.

(3) **Existence of Identity :** Since row headed by is same as the initial row, 1 is the identity element.

(4) **Existence of Inverse :** It is clear the $(1)^{-1} = 1$, $(w)^{-1} = w^2$, $(w^2)^{-1} = = w$ hence there exist an inverse for each element in $G$.

(5) **Commutative Property :** Since rows coincide with columns respectively, so multiplication is commutative on $S$.

Since, all the properties of group are satisfied on $G$, so $G$ is a group.

**31. *Let G be a group with identity e, show that if $a^2 = e$ for all $a \in G$, then G is abelian Group.*** **(2013-14)**

Given $a^2 = e$ for all $a \in G$

i.e., $a * a = e$

Pre-multiplying both the sides by $a^{-1}$.

i.e., $(a^{-1}) * (a * a) = a^{-1} * e$

i.e., $(a^{-1} * a) = a^{-1}$ (By associativity and definition of identity)

i.e., $e * a = a^{-1}$ (By def. of Inverse)

$a = a^{-1}$ (By def. of Identity)

$a = a^{-1}$ ... (1)

i.e., every element is inverse of itself

Now $a, b, \in G$

$$a * b = a^{-1} * b^{-1}$$
$$= (b * a)^{-1}$$
$$= b * a$$

Thus, $G$ is abelian group.

**32. *Consider the binary operation * on $Q_1$ of all rational numbers other than 1, defined by $a * b = a + b - ab$. Determine the identity of the binary operation *, if exists.*** **(2011-12)**

Identity means $a^* e = e^* a = a$

Given : $a^* b = a + b - ab$

Taking L.H.S.

$$e^* a = e + a - ea$$



$$a = e + a - ea$$

We get $e(1 - a) = 0$ implies $e = 0$

Similarly R.H.S.

$$a^* e = a + e - ae = a$$

We get $e(1 - a) = 0$ implies $e = 0$

**33. *Define the order of the finite group.*** **(2011-12)**

The order of an element $g$ in a group $G$ is the smallest positive integers $n$ such that $g^n = e$. The order of an element $g$ is denoted by $O(g)$

**Example :** $G = \{1, -1\}$ 1 is the identity element in $G$ so

$1^1 = 1 \Rightarrow O(1) = 1$

Similarly $(-1)^2 = 1$, $(-1)^n \neq 1$ for any positive integer $n < 2$.

Hence $O(-1) = 2$

**34. *Let G be an abelian group and N is a subgroup of G. Prove that G/N is an abelian group.*** **(2011-12)**

Let $X, Y \in G/N$ be arbitrary, then

$X = Na$, $Y = Nb$ for some $a, b \in G$. we have

$XY = NaNb = Nab = Nba$, Since $G$ is abelian, $NbNa = YX$

Hence $G/N$ is abelian **(Hence Proved)**

**35. *Let G = {1, –1, i, –i} with binary operation multiplication be an algebraic structure, where $i = \sqrt{-1}$ :*** **(2011-12)**

***(1) Determine whether G is an abelian.***

***(2) If G is a cyclic group, then determine the generators of G.***

(1) Let G = {1, –1, i, –i} we form a composite table

| x | 1 | –1 | i | –i |
|---|---|---|---|---|
| 1 | 1 | –1 | | –i |
| –1 | –1 | 1 | –i | i |
| i | i | –i | –1 | 1 |
| –i | –i | i | 1 | –1 |

**Closure Property :** Since all the entries in the table are the elements of $G$ and hence $G$ is closed with respect to multiplication.

**Associative Law :** $a(bc) = (ab)c$ for all $a$, $b$ in $G$.

**Commutative Law :** $ab = ba$ for all $a$, $b$ in $G$. From the composition table it is clear that elements in each row are the same as elements in the corresponding column so that $ab = ba$.

**Identity Element :** $1 \in G$ is identity element as $1.a = a.1 = a$. It can be seen from the first row and first column of the table.

**Inverses :** Inverses of $1, -1, i, -i$ are $1, -1, i, -i$ respectively and all those belong to $G$.

Hence, it follows that $G$ is an abelian multiplicative group.

(2) A group $G$ is called cyclic if for some $a \in G$, every element $x \in G$ is of the form $a^n$, where $n$ is some integer. The element '$a$' is then called a generator of $G$.

The multiplicative group $G = \{1, -1, i, -i\}$ is cyclic.

We can write $g = \{i, i^2, i^3, i^4\}$

Thus $G$ is a cyclic group and $i$ is a generator.

We can also write $G = \{-i, (-i)^2, (-i)^3, (-i)^4\}$

Thus $-i$ is also generator of $G$. So there may be more than one generator of a cyclic group.

**36. *Let G be that set of all non-zero real numbers and let $a * b = \dfrac{ab}{2}$.* (2011-12)**

***Show that (G, *) is an abelian group.***

We have to show that $(G,*)$ is a group under the composition

$a*b = ab/2$

**Closure Property :** Since for every element $a, b \in G$, $ab/2$ is also in $G$, therefore $G$ is closed with respect to operation $*$.

**Associative Property :** For $a, b \in G$ we have

$$(a*b)*c = (ab/2)*c$$
$$= a*(bc/2)$$
$$= a*(b*c)$$

**Commutative Law :** For $a, b \in G$, we have $a*b = (ab/2) = (ba/2) = b*a$

**Identity Element :** Let $e$ be the identity element in $G$ such that $e*a = a = a*e$

Now $e*a = a \rightarrow (ea)/2 = a \rightarrow (a/2)(e-2) = 0$

It implies that $e = 2$ since $a \in G \rightarrow a > 0$.

But $2 \in G$ and we have $2*a = (2a)/2 = a = a*2$ for all $a \in G$.

**Inverses :** Let $a$ be any element of $G$. If the number $b$ is to be the inverse of a, then we must have

$$B*a = e = 2 \rightarrow \frac{ba}{2} = 2 \rightarrow b = \frac{4}{a} \in G$$

We have $(4/a)*a = 4a/2a = 2 = a*(4/a)$

Therefore $4/a$ is the inverse of $a$. Thus each element of $G$ is invertible.

Hence $(G,*)$ is an abelian group.

**37. *Let $(G,*)$ be an group and $a \in G$. Let $f : G \rightarrow G$ be given by $f(x) = a * x * z * a^{-1}$ for every $x$ in $G$, prove that $f$ is an isomorphism of $G$ onto $G$.* (2011-12)**

A mapping $f : G \rightarrow G'$ where $(G,*)$ and $(G',*)$ are groups, is an isomorphism if

(1) F is one to one that is distinct elements in G have distinct $f$ image in G'.

(2) F is onto and

(3) F is homomorphism

The isomorphism is a special type of homomorphism. If $f$ is a homomorphism of G into G' and F is also one to one, then $f$ is an isomorphism of G into G'.

If G is a group then the mapping $f : G \rightarrow G$ defined as $f(x) = a*x*a^{-1}$ for each $x \in G$ is a homomorphism.

Let $x, y \in G$ so that $xy \in G$. we have

$$f(xy) = (a*x*a^{-1})(a*y*a_{-1})$$
$$= f(x)f(y)$$

Hence $f$ is a homomorphism .

And $f$ is one to one i.e. $f(x) = f(y) \rightarrow x = y;\ x, y \in G$.

So $f$ is an isomorphism of $G$ into $G$.

**38. *Consider a ring (R,+,*) defined by a * a = a. Determine whether the ring is commutative or not.* (2011-12)**

We have given the ring $(R, +, *)$ satisfy the following properties :

**(1)** $(R, +)$ is an abelian group.

**(2)** $(R,*)$ is a semi group.

**(3)** The operation * distributes over +.

Also we have $a*a = a$ for every $a$ in $(R, +, *)$.

To prove that the ring $(R, +, *)$ is commutative we have to prove that there exists an identity element and $(R, *)$ is a commutative monoid.

Let us assume that $a, b \in R$

Also let $c = a + b$, since + is a closed operation. Hence $c \in R$

Then we have $c*c = c*(a+b)$

We know that the operation * distributes over +.

So

$$c*c = (a + b)*(a + b)$$
$$= a*a + a*b + b*a + b*b$$
$$C = a + b + a*b + b*a$$
$$C = c + (a*b + b*a)$$

Therefore $(a*b + b*a) = c$ is an identity for operation + also $a*b$ is inverse $b*a$.

Again let us assume $a + a = b$

$$a*(a + a) = a*b$$
$$a*a + a*a = a*b$$
$$a + a = a*b \qquad \ldots (1)$$

Similarly

$$(a + a)*a = b*a$$
$$a*a + a*a = b*a$$
$$a + a = b*a \qquad \ldots (2)$$

From equation (1) and (2) we get $a*b = b*a$

Therefore the operation * is commutative.

Since $(R,*)$ is a commutative monoid. So the ring is commutative.



**39. *How many generators are there of the Cyclic group of order 8?* (2010-11)**

Let $G = [a]$, then $0(a_3) = 8$

$\therefore \quad G = \{a, a^2, a^3, a^4, a^5, a^6, a^7, a^8 = e\}$

Now we have $0(a) = 0(G) = 8$

Now the other generators of $G$ are elements of the form $a^n$ where $n$ is relatively prime to 8. Since 3 is prime to $8 \Rightarrow a^3$ is also generator of $G$.

Again 7 is prime to $8 \Rightarrow a^7$ is a generator of $G$. Thus $G$ has four generators $a, a^3, a^5, a^7$

**40. *Define a commutative ring with unity.* (2010-11)**

An algebraic structure $(R,+,.)$ where $R$ is a non empty set and + and-two binary operations knows as addition and multiplication respectively, is called a ring if the postulates are satisfied.

**(1)** $(R,+)$ is an abelian group

**(2)** $(R,.)$ is a semi group.

**(3)** Multiplication is a distributive with respect to addition. A ring $(R,+,.)$ is called a commutative ring if

$$a.b = b.a \ \forall a, b \in R$$

A ring $(R, +,.)$ is said to be ring with unity if $R$ has the identity element for its multiplicative composition if there exist $e \in R$ such that

$$a.e = a = e.a \ \forall a \in R$$

**41. *Show that if a, b are arbitrary elements of a group G, then (ab)² = a² b² if and only if G is abelian.* (2010-11)**

Suppose G is abelian then we have to show that

$$(ab)^2 = a^2b^2$$

Now $\quad (ab)^2 = (ab)(ab)$

$= a(ba)b$ [by associativity]

$= a(ab)b$ [G is abelian]

$= (aa)(bb)$ [by associativity]

$= a^2b^2$

Conversely, let $(ab)^2 = a^2b^2, \forall a,b \in G$, then we have to show that $G$ is abelian.

Now $(ab)^2 = a^2b^2$

$\Rightarrow \quad (ab)(ab) = (aa)(bb)$

$\Rightarrow \quad a(ba)b = a(ab)$

$\Rightarrow \quad (ba)b = (ab)b$ by associativity

$\Rightarrow \quad ba = ab$

$\Rightarrow$ $G$ is an abelian group

**42. *Show that the set of rational numbers Q forms a group under the binary operation * defined by $a * b = a + b - ab, \forall a, b \in Q$ is this group abelian?*** **(2010-11)**

(1) Since $a, b \in Q$ therefore $a * b = a+b-ab \in Q$

(2) **Associativity**, for $a,b,c$

$(a*b)*C=(a+b-ab)*c= a+b-ab+c-\{ac+bc-abc\}$

$= a+b-ab+c-ac-bc+abc$

$a*(b*c)=a*(b+c-ab) = a+b+c-bc-\{ab+ac-abc\}$

$= a+b+c-bc-ab-ac+abc$

Hence, $(a*b)*c=a*(b*c)$

(3) **Identity Element :** Let $e \in Q$ be an idenlity element such that $a*e=a=e*a$, for all $a \in Q$

$\therefore \quad a+e-ae=a$

$\Rightarrow \quad e(1-a) = 0 \Rightarrow e = 0$

Therefore $0 \in Q$ is the identify element for *.

(4) **Existence of Inverse :**

$a+b-ab = a *b$ of $a*b=e$ where e is the identity elements

Then $a+b-ab=e$ $\{e=0\}$

$a+b-ab=0$

$\Rightarrow \quad b(1-a)=-a \Rightarrow b = a/a-1$ so inverse of a is $\dfrac{a}{a-1}$.

Since, a is an arbitrary elements of $Q$ therefore every elements of $Q$ in invertible



(5) $a*b = a + a - ab$ because addition and multiplication are commutative composition is $Q$ therefore $a*b = b + a - ab = b * a$

So it is a abelian group.

**43. *Let G be an Abelian group and N is a subgroup of G. Prove that G/N is an Abelian group.***

Let the collection of left cosets of $N$ in $G$ be denoted by $G/N$.

(1) Let, $X = aN$ and $Y = bN$ for some $a, b, \in G$, and $X, Y \in G/N$.

Then, $\quad XY = (aN)(bN)$

$= abN \in G/N$

So, coset multiplication is a binary operation of G/N.

(2) Assume that $aN, bN, cN \in G/N, (a,b,c \in G)$

Then,

$(aN)[(bN)(cN)] = (aN)(bcN)$

$= a(bcN)$ ...(i)

And, $\quad [(aN)(bN)](cN) = (abN)(cN)$

$= (ab)cN$ ...(ii)

But, $a(bc) = (ab)c$ {Multiplication is associative in G}

Therefore, from (i) and (ii) the coset multiplication is also associative.

(3) The element $N = eN \in G/N$

for some $a \in G, aN \in G/N$.

and $\quad (eN)(aN) = (eaN) = aN = aeN$

$= (aN)(eN)$

So, $eN$ is the identity element for $G/N$.

(4) Assume that $aN \in G/N$ for some $a \in G$.

$(aN)(a^{-1}N) = (aa^{-1})N = eN = (a^{-1}a)N$

$= (a^{-1}N)(aN)$

Therefore, $a^{-1}N$ is the inverse of $aN$.

(5) Assume that $aN, bN \in G/N, (a, b, \in, G)$ ...(i)

Then, $(aN)(bN) = (ab)N$

$$(bN)(aN) = (ba)N \quad \text{...(ii)}$$

But, $ab = ba$ {multiplication is commutative is G} therefore, from (i) and (ii) the coset multiplication is also commutative. Hence, $G/N$ is an abelian group under coset multiplication.

□□

# RECURRENCE RELATIONS / COMBINATORICS

**1. *Write down the properties of Generating function.***
**(2020-21)**

## Properties of Generating Function A (z)

(1) $a_r = 1$ $\quad A(z) = \dfrac{1}{1-z}$

(2) $a_r = r$ $\quad A(z) = \dfrac{2}{(1-z)^2}$

(3) $a_r = r^2$ $\quad A(z) = \dfrac{2(z+1)}{(1-z)^3}$

(4) $a_r = r(r+1)$ $\quad A(z) = \dfrac{2z}{(1-z)^3}$

(5) $a_r = \alpha^r$ $\quad A(z) = \dfrac{1}{1-az}$

**2.** ◆ ***State the "Pigeonhole Principle".***
**(2013-14, 2018-19, 2020-21)**

◆ ***Write short note on Pigeon hole principle.***
**(2017-18)**

◆ ***Define pigeon hole principle. Find the minimum number of boys born in the same minute out of 3000 boys on a day.*** **(2016-17)**

## Pigeonhole Principle

A pigeonhole principle, also known as the shoe box arguments or Dirichlet drawer principle. In an informal way the pigeonhole principle says that if there are many pigeons and a few pigeonholes, there must be some pigeonhole occupied by two or more pigeons. Formally, let $D$ and $R$ be finite sets. If $|D|>|R|$, then for any function. $f$ from $D$ to $R$, there exist $d_1, d_2 \in D$ such that $f(d_1) = f(d_2)$. Some trivial applications of the pigeonhole principle are :

Among 13 people, there are at least 2 of them who were born in same month. Here, 13 people are the pigeons, and the 12 months are the pigeonholes. Also, if 11 shoes are selected from 10 pairs of shoes there must be a pair of matched shoes among the selection. There the 11 shoes are the pigeons and the 10 pairs are the pigeonholes. The pigeonhole principle can be stated in a slightly more general form : For any function $f$ from $D$ or $R$, there exist $i$ elements $d_1, d_2 \ldots, d_i$ in $D, i = [|D|/|R|]$, such that $f(d_1) = f(d_2) = \ldots = f(d_i)$.

$$f(30) = 17.107 + 0.25\Delta f(29) + (-0.09375)\Delta^2 f(28)$$
$$+(-0.234375)\Delta^3 f(28) + (0.410156)\Delta^4 f(27)$$
$$= 17.107 + 0.25 \times (-0.76) + 0.09375)(0.05)$$
$$+(0.234375) \times (0.02)$$
$$= 17.107 - 0.19 + 0.0046875 + 0.0046875$$
$$= 17.107 - 0.19 + 0.009375 = 16.927635$$

**Numerical Solution :** We may assign each pigeon to the minutes of the day on which he was born then 3000 boys are to be assigned to 1440 mins (in a day)

Here $m = 1440$ and $n = 3000$ and

The formula is $\left\lfloor \frac{(n-1)}{m} \right\rfloor + 1 = \left\lfloor \frac{(3000-1)}{1440} \right\rfloor + 1 = 2$

So, minimum number of boys born in the same minute are 2.

**3. *Prove by Mathematical Induction that : $6^{n+2} + 7^{2n+1}$ is divisible by 43 for each positive integer n.*** **(2020-21)**

Let, $P(n) = 6^{n+2} + 7^{2n+1}$ is divisible by 43 for $n = 1$

$P(1) = 6^3 + 7^3 = 559$ which is divisible by 43

Assume $P(k)$ is true i.e. $P(k) = 6^{k+2} + 7^{2k+1}$ is divisible by 43

Now for $n = k+1$

$$P(k+1) = 6^{k+1+2} + 7^{2(k+1)+1}$$
$$= 6^{k+3} + 7^{(2k+3)}$$
$$= 6 \times 6^{k+2} + 7^2 \times 7^{(2k+1)}$$
$$= 6 \times 6^{k+2} + (6+43) \times 7^{(2k+1)}$$



$$= 6 \times 6^{k+2} + 6 \times 7^{2k+1} + 43 \times 7^{2k+1}$$
$$= 6\left(6^{k+2} + 7^{2k+1}\right) + 43 \times 7^{2k+1}$$
$$= 6 \times p(k) + 43 \times 7^{2k+1}$$

Since, each component of this sum is divisible by 43 so it is the entire sum & formula holds for $k+1$.

**4. *Solve the following Recurrence Relation :*** **(2020-21)** ***$a_n = 4(a_{n-1} - a_{n-2})$ with initial conditions $a_0 = a_1 = 1$***

Put $a_n = A\alpha^n$

$$A\alpha^n - 4A\alpha^{n-1} - 4 \quad 4A\alpha^{n-2}$$
$$A\alpha^n - 4A\alpha^{n-1} + 4A\alpha^{n-2} = 0$$
$$A\alpha^n\left(1 - \frac{4}{\alpha} + \frac{4}{\alpha^2}\right) = 0$$
$$A\alpha^n\left(\frac{\alpha^2 - 4\alpha + 4}{\alpha^2}\right) = 0$$
$$\alpha^2 - 4\alpha + 4 = 0$$
$$\alpha^2 - 2\alpha - 2\alpha + 4 = 0$$
$$\alpha(\alpha - 2) - 2(\alpha - 2) = 0$$
$$(\alpha - 2)^2 = 0$$
$$\alpha = 2, 2$$

Which is repeated foot.
Therefore solution is

$$a_r = (A_1 + A_2 r)\, 2^r$$

But $a_0 = 1, a_1 = 1$

$$a_0 = A_1 = 1$$
$$a_1 = (A_1 + A_2)^2$$
$$1 = (1 + A_2)^2$$
$$A_2 = -\frac{1}{2}$$

Hence $a_r = \left(1 - \frac{1}{2}r\right)2^r$.

**5. Solve the following Recurrence Relations Using Generating Functions :**

$$a_n - 9a_{n-1} + 20a_{n-2} = 0$$

*with initial conditions* $a_0 = -3$ *and* $a_1 = -10$

(2020-21)

We have

$$a_n - 9a_{n-1} + 20a_{n-2} = 0$$

Multiply $2^n$ on both sides of equation then take $r = 2$ to $\infty$ we get

$$\sum_{n=2}^{\infty} a_n z^n - 9\sum_{n=2}^{\infty} a_{n-1} z^n + 20\sum_{n=2}^{\infty} a_{n-2} z^n = 0$$

$$\left(a_2 z^2 + a_3 z^3 ---\right) - 9\left(az^2 + a_2 z^3 ---\right) + 20\left(a_0 z^2 + a_1 z^3 ---\right)$$

$$\left(A(z) - a_0 - a_1 z\right) - 9z\left(A(z) - a_0\right) + 20z^2\left(A(z)\right) = 0$$

$$a_0 = -3, a_1 = -10$$

$$A(z) - 3 + 10z - 9z\,A(z) - 9z \times 3 + 20z^2\,A(z) = 0$$

$$A(z)\left(1 - 9z + 20z^2\right) = 17z + 3$$

$$A(z) = \frac{17z+3}{20z^2 - 9z + 1} = \frac{17z+3}{20z^2 - 5z - 4z + 1}$$

$$= \frac{17z-3}{5z(4z-1) - 1(4z-1)} = \frac{17z-3}{(5z-1)(4z-1)}$$

$$= \frac{A}{(5z-1)} + \frac{B}{(4z-1)}$$

$$= \frac{A(4z-1) + B(5z-1)}{(5z-1)(4z-1)}$$

$$= \frac{4Az - A + 5Bz - B}{(5z-1)(4z-1)}$$

$$= \frac{(4A+5B)z - (A+B)}{(5z-1)(4z-1)}$$

On comparing the co-efficient

$$4A + 5B = 17 \quad \text{...(1)}$$

$$5(A + B = 3)$$

$$5A + 5B = 15 \quad \text{...(2)}$$

$$-A = 2$$



$$A = -2$$

$$B = 5$$

$$A(z) = -\frac{2}{(5z-1)} + \frac{5}{(4z-1)}$$

$$A(z) = \frac{2}{1-5z} - \frac{5}{1-4z}$$

$$\sum_{r=0}^{\infty} a_n z^n = \sum_{r=0}^{\infty} 2(5)^n z^n - \sum_{r=0}^{\infty} 5(4)^r z^n$$

$$a_n = 2(5)^n - 5(4)^n$$

By comparing the co-efficient of $2^r$ on both sides.

**6. What are Natural numbers?**

**Natural Numbers**

Natural numbers are a part of the number system which includes all the positive integers from 1 till infinity and are also used for counting purpose. It does not include zero (0). In fact, 1,2,3,4,5,6,7,8,9...., are also called counting numbers.

Natural numbers are part of real numbers, that include only the positive integers i.e. 1, 2, 3, 4, 5, 6, ......... excluding zero, fractions, decimals and negative numbers.

**Note :** Natural numbers do not include negative numbers or zero.

**Natural Number Definition :** Natural numbers are the numbers which are positive integers and includes numbers from 1 till infinity(∞). These numbers are countable and are generally used for calculation purpose. The set of natural numbers is represented by the letter "$N$".

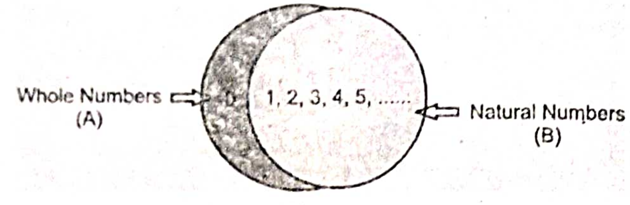
$$N = \{1,2,3,4,5,6,7,8,9,10......\}$$

**Natural Numbers and Whole Numbers :** Natural numbers include all the whole numbers excluding the number 0. In other words, all natural numbers are whole numbers, but all whole numbers are not natural numbers.

**(1)** Natural Numbers = {1,2,3,4,5,6,7,8,9,.....}

**(2)** Whole Numbers = {0,1,2,3,4,5,7,8,9,....}

Check out the difference between natural and whole numbers to know more about the differentiating properties of these two sets of numbers.




(Figure)

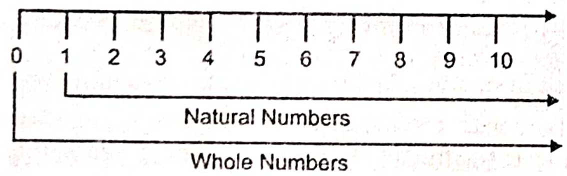
The above representation of sets shows two regions. A ∩ B i.e. intersection of natural numbers and whole numbers (1, 2, 3, 4, 5, 6, ........) and the green region showing A-B, i.e. part of the whole number (0).

Thus, a whole number is "a part of Integers consisting of all the natural number including 0."

**Is '0' a Natural Number?**

The answer to this question is 'No'. As we know already, natural numbers start with 1 to infinity and are positive integers. But when we combine 0 with a positive integer such as 10, 20, etc. it becomes a natural number. In fact, 0 is a whole number which has a null value.

**Representing Natural Numbers on a Number Line :** Natural numbers representation on a number line is as follows :




(Figure)

The above number line represents natural numbers and whole numbers. All the integers on the right-hand side of 0 represent the natural numbers, thus forming an infinite set of numbers. When 0 is included, these numbers become whole numbers which are also an infinite set of numbers.

**7.** ***State the Principle of Mathematical Induction. And show that $8^n - 3^n$ is divisible by 5 for $n \geq 1$.*** **(2018-19)**



## *Principle of Mathematical Induction*

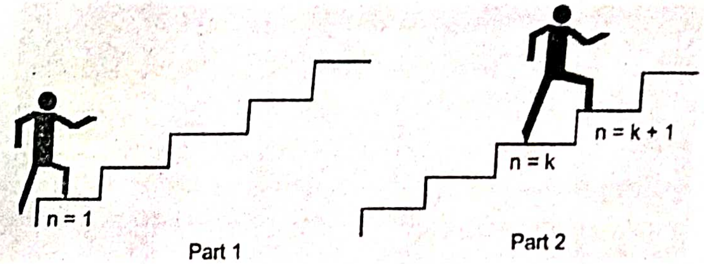
*Mathematical induction is used to prove that the given statement is true or not. It uses 2 steps to prove it.*

**First Principle of Mathematical Induction :**

(1) **Base Case :** *The given statement is correct for first natural number that is, for $n = 1$, $p$ (1) is true.*

(2) **Inductive Step :** *If the given statement is true for any natural number like $n = k$ then it will be correct for $n = k + 1$ also that is, if $p(k)$ is true then $p(k+1)$ will also be true.*

*The first principle of mathematical induction says that if both the above steps are proven then $p(n)$ is true for all natural numbers.*




(Figure)

**Numerical Solution :**

**Base Case :** $n = 1$

Therefore, $8^n - 3^n = 8^1 - 3^1 = 5$ which is divisible by 5

**Inductive Hypothesis :** $n = k \geq 1$, $8^k - 3^k$ is divisible by 5

i.e., $8^k - 3^k = 5p$

**Inductive Step :** $n = k + 1$

$$8(k+1) - 3(k+1) = 8.8^k - 3.3^k$$
$$= 3.(8^k - 3^k) + 5.8^k$$
$$= 3.5p + 5.8^k$$
$$= 5.(3p + 8^k)$$

Therefore, $8(k+1) - 3(k+1)$ is divisible by 5. **(Ans.)**

**8.** ***Using mathematical induction, show that $11^{n+1} + 122^{n-1}$ is divisible by 133 for all $n \geq 1$.*** **(2017-18)**

For $n = 1$

$$11^{n+1} + 12^{2n-1} = 11^2 + 12 = 121 + 12 = 133$$

Which is divisible by 133

So, its true for $n = 1$

Let us assume it is true for $n = k$

$$11^{k+1} + 12^{2k-1} = 133s$$

For some integer $s$

$$\Rightarrow 12^{2k-1} = 133s - 11^{k+1}$$

$$\Rightarrow 12^{2k+1} = 144\left(133s - 11^{k+1}\right)$$

Now for $n = k+1$

$$11^{k+2} + 12^{2k+1}$$

$$= 11^{k+2} + 144\left(133s - 11^{k+1}\right)$$

$$= 11^{k+2} + 144 \times 133s - 144 \times 11^{k+1}$$

$$= 11^{k+1}(11 - 144) + 133 \times 144s$$

$$= 11^{k+1}(-133) + 133 \times 144s$$

$$= 133\left(144s - 11^{k+1}\right)$$

Which is a multiple of 133, so its true for $n = k+1$

Thus, by mathematical induction it is true for all $n \in N$

**9. *Explain Piano Axioms with examples also explain Mathematical induction.***

The set $N = \{0, 1, 2, 3, \ldots\}$ of natural numbers (including zero) can be generated by starting with a null set $\phi$ and the notation of a successor set. A successor set of a set $A$ is denoted by $A^+$ and defined to be the set $A^+ = A \cup \{A\}$.

Let $\phi$ be the empty set, and obtain the successor set $\phi^+, (\phi^+)^+$ .... these sets are $\phi$, $\{\phi\}$, $\{\phi, \{\phi\}\}$, $\{\phi, \{\phi\}\}$, $\{\phi, \{\phi\}\}\}$.

If we rename the $\phi$, as 0 (zero) then :

$\phi^+ = 0^+ = \{\phi\} = 1, 1^+ = \{\phi, \{\phi\}\} = (0,1) = 2, \ldots$ and we get the set $\{0, 1, 2, 3 \ldots\}$ in which each elements is a successor set of the previous element except for the element 0 which is assumed to be present. This discussion can be summarized by saying that the set of natural numbers can be obtained from the following axioms known as Piano Axioms.

**(1)** $0 \in N$ (where, $0 = \phi$)

**(2)** If $n \in N$, then $n^+ \in N$ where, $n^+ = n \cup \{n\}$

**(3)** If a sub set $S \subset N$ passes the properties.

**(a)** $0 \in S$, and

**(b)** If $n \in S$, then $n^+ \in S$ then $S = N$.

Property 3 is also the basic of the Principle of Mathematical Induction which is frequently employed in proofs. We shall put axiom 3 in an equivalent form.

If $P(n)$ is any property defined over the set of natural numbers and :

**(a)** If $P(0)$ is true.

**(b)** If $P(m) \Rightarrow P(m^+)$ for any $m \in N$, then $P(n)$ holds for all $n \in N$.

**Example :** Show that $n < 2^n$.

**Solution :** Let $P(n) : n < 2^n$

**(a)** For $n = 0$, $P(0) : 0 < 2^0 = 1$, so that $P(0)$ is true.

**(b)** For some arbitrary choose of $m \in N$, assume that $P(m)$ holds, that is, $P(m) : m < 2^m$. From this, by adding 1 to both sides, we get :

$$m + 1 < 2^m + 1 < 2^m + 2^m = 2^m \times 2 = 2^{m+1}$$

Which is exactly $P(m + 1)$. So, $P(m) \Rightarrow P(m+1)$.

Hence from the mathematical induction, $P(n)$ is true for all $n \in N$.

**10. *Everybody in a room shakes hands with everybody else. The total number of handshakes is 66. How many people are there in the room?*** **(2018-19)**

If there are $n$ people in the room. Then there are $n(n-1)/2$ handshakes.

$$\frac{n(n-1)}{2} = 66$$

$$n^2 - n = 132$$

$$n^2 - n - 132 = 0$$

$$n^2 - 12n + 11n - 132 = 0$$
$$n(n-12) + 11(n-12) = 0$$
$$(n-12)(n+11) = 0$$
$$n = 12, -11$$

Since the number of people cannot be negative.
So, $n = 12$.
Total number of persons = 12 **(Ans.)**

**11.** ***Determine $S^2a$ and $S^{-2}a$ for the following numeric function :*** **(2018-19)**

$$a_r = \begin{cases} 2 & , 0 \le r \le 3 \\ 2^{-r} & , r \ge 4 \end{cases}$$

$$a_r = \begin{cases} 2 & , 0 \le r \le 3 \\ 2^{-r} & , r \ge 4 \end{cases}$$

$$a_r = \begin{cases} 2 & , 0 \le r \le 3 \\ 2^{-r} & , r \ge 4 \end{cases}$$

We know that

$$b = S^i a = b_r = \begin{cases} 0 & 0 \le r \le i-1 \\ a_{r-i} & r \ge i \end{cases}$$

$0 \le r \le 1$ , $0$

$r \ge 2$ , $a_{r-2}$

$r = 2$ , $a_0 = 2$

$r = 3$ , $a_1 = 2$

$r = 5$ , $a_3 = 2$

$r = 6$ , $a_4 = 2^{-r}$

Now, for, $a_r = \begin{cases} 2 & , 0 \le r \le 3 \\ 2^{-r} & , r \ge 4 \end{cases}$

$$b = S^2a : b_r = \begin{cases} 0 & , 0 \le r \le 1 \\ 2 & , 2 \le r \le 5 \\ 2^{-r} & , r \ge 6 \end{cases}$$ **(Ans.)**

Now, $S^{-2}a : b_r = \begin{cases} 2 & , 0 \le r \le 1 \\ 2^{-r} & , r \ge 2 \end{cases}$ **(Ans.)**



$$b_r = a_{r+i} = a_{r+2}$$
$$r = 0,\ a_2 = 2$$
$$r = 1,\ a_3 = 2$$
$$r = 2,\ a_4 = 2^{-r}$$

**12.** ***Solve the following recurrence relation :*** **(2018-19)**
**$a_n + 6a_{n-1} + 9a_{n-2} = 3$ Given that : $a_0 = 0$ and $a_1 = 1$**

The characteristic equation is :

$$\alpha^2 + 6\alpha + 9 = 0$$
$$\alpha^2 + 3\alpha + 3\alpha + 9 = 0$$
$$\alpha(\alpha+3) + 3(\alpha+3) = 0$$
$$(\alpha+3)(\alpha+3) = 0$$
$$\alpha = -3, -3$$

Hence, the general solution is

$$a_n = A_1(-3)^n + A_2(-3)^n.n \quad \ldots(1)$$

Here, $a_0 = 0, a_1 = 1$

$$0 = A_1(-3)^0 + A_2(-3)^0 \times 0$$
$$0 = A_1 + A_2 \times 0$$
$$A_1 = 0$$

Now,

$$a_1 = 1$$
$$1 = A_1(-3)^1 + A_2(-3)^1 \times 1$$
$$1 = -3A_1 - 3A_2 \Rightarrow 3A_1 + 3A_2 = -1$$

On putting value of

$A_1 = 0$

$3A_2 = -1$

$A_2 = -1/3$

We have $A_1 = 0, A_2 = -1/3$

Hence, the solution is

$$a_n = A_1(-3)^n + A_2.n(-3)^n$$
$$a_n = 0 + \left(\frac{-1}{3}\right).n(-3)^n$$
$$a_n = -\frac{n(-3)^n}{3} \Rightarrow \frac{n(-3)^n}{(-3)}$$

$a_n = n(-3)^n.(-3)^{-1}$

$a_n = n(-3)^{n-1}$ (Ans.)

**13. *Solve the following recurrence relations :*** **(2017-18)**

(1) $f_n = 5f_{n-1} + 6f_{n-2}$

(2) $d_n = 2d_{n-1} - d_{n-2}$

(1) $f_n = 5f_{n-1} + 6f_{n-2}$

The characteristics equation is $r^2 - 5r - 6 = 0$

After solving we get $r = -1, 6$ so solution is

$A_1(-1)^n + A_2(6)^n$

(2) $d_n = 2d_{n-1} - d_{n-2}$

The associated equation for this linear homogeneous solution is $x^2 - 2x + 1 = 0$

This equation has one multiple root i.e. 1

Thus $d_n = u(1)^n + vn(1)^n$

**14. *Solve the following linear non-homogeneous recurrence relation with constant coefficient :*** **(2017-18)**

***$a_n = 5a_{n-1} + 6a_{n-2} + 3.5^n$ where $a_0 = 4$ and $a_1 = 7$ Verify your answer for $a_2$.***

The given recurrence relation is

$$a_n = 5a_{n-1} + 6a_{n-2} + 3.5^n \qquad ...(1)$$

$$a_n - 5a_{n-1} - 6a_{n-2} = 3.5^n$$

The characteristic equation corresponding to this, is

$$\alpha^2 - 5\alpha - 6 = 0$$

We get $\alpha = -1, 6$

Since 5 is not characteristic root,

We assume that the general form of particular solution is $Q.(5)^n$

Substituting in equation (1)

$$Q.(5)^n - 5.Q.(5)^{n-1} - 6Q.(5)^{n-2} = 3(5)^n$$

$$\rightarrow Q.(5)^{n-2}[25 - 25 - 6] = 3(5)^n$$



$$\rightarrow Q.(5)^{n-2}[-6] = 3(5)^n$$

$$\rightarrow (-2)Q.(5)^{n-2} = (5)^n$$

$$\rightarrow -2Q = 25$$

$$\rightarrow Q = -\frac{25}{2}$$

Hence particular solution is

$$-\frac{25}{2}.(5)^n = -\frac{1}{2}.(5)^{n+2}$$

**15. *Write short note on Polya's Counting Theorem.*** **(2017-18)**

The Polya enumeration theorem (PET), also known as Redfield-Polya's Theorem, is a theorem in combinatorics, generalizing Burnside's lemma about number of orbits. This theorem was first discovered and published by John Howard Redfield in 1927 but its importance was overlooked and Redfield's publication was not noticed by most of mathematical community. Independently the same result was proved in 1973 by Gorge Polya, who also demonstrated a number of its applications, in particular to enumeration of chemical compounds.

The PET gave rise to symbolic operators and symbolic methods in enumerative combinatorics and was generalized to the fundamental theorem of combinatorial enumeration.

**Informal PET Statement :** Suppose you have a set of n slots and a set of objects being distributed into these slots and a generating function $f(a, b, ...)$ of the objects by weight. Furthermore there is a permutation group $A$ acting on the slots that created equivalence classes of filled slot configurations (two configurations are equivalent if one may be obtained from the other by a permutation from $A$). Then the generating function of the equivalence classes by weight, where the weight of a configuration is the sum of the weights of the objects in the slots, is obtained by evaluating the cycle index $Z(A)$ of $A$.

**Polya's Theorem :** Problem which amount to colouring with restrictions, such as that a particular colour should appear a particular number of time. Moreover, even in those cases where Burnside's theorem could be applied, the computation of the number of fixed points of a group element had often to be done 'by hand' so to speak. This can be cumbersome and prone to mistakes. It is therefore natural to inquire whether these drawbacks can somehow be removed. A generalization of Burnside's theorem, due to Polya, does this job quite well. It is also remarkable because of the elegant use of algebra it makes, so that the desired answers appear as various coefficients in suitable polynomials.

Before presenting this theory, let us revisit problem in which the three vertices, say, $A$, $B$ and $C$ of an equilateral triangle are to be painted with four colours, say $a$, $b$, $c$ and $d$. Each possible coloring triangle are to be painted with a monomial in $a$, $b$, $c$, $d$. i.e., an expression of the form $a^p b^q c^r d^s$ where $p$, $q$, $r$, $s$ denote the numbers of vertices painted with colours $a$, $b$, $c$, $d$ respectively. (Since there are 3 vertices, $p + q + r + s$ will always be 3). For example, if the vertex $A$ is coloured with $c$ and $B$, $C$ each with a the corresponding monomial will be $a^2c$.

**16.** ***Given an example of homogenous and non-homogeneous recurrence relation of order 4 and degree 3.*** **(2016-17)**

Homogeneous recurrence relation of order 4 and degree 3 is

$$a_n^3 + 2a_{n-1}^2 + a_{n-4} = 0$$

Non-Homogeneous recurrence relation of order 4 and degree 3 is $a_n^3 + 2a_{n-1}^2 + a_{n-4} = 1$

**17.** ***Find complete solution of the recurrence relation***

$$u_n - 4u_{n-1} + 3u_{n-2} = 5^n + n \qquad \textbf{(2016-17)}$$

The characteristic equation corresponding to the given recurrence relation is

$$x^2 - 4x + 3 = 0 \Rightarrow (x-3)(x-1) \Rightarrow x = 1, 3$$

Since roots are distinct, therefore the solution is

$$u_n = A_1 1^n + A_2 3^n = A_1 + A_2 3^n$$



Here $f(n) = 5^n$ so a reasonable trial solution is

$$u_n^p = p.5^n$$

Substituting the terms of this sequence into the recurrence relation implies that

$$p.5^n = 4p.5^{n-1} - 3p5^{n-2} + 5^n = \frac{5}{2}$$

Hence the particular solution is $u_n^p = \left(\frac{5}{2}\right).5^n$

**18.** ***In a MCA class of 40 students 5 are weak. Determine how many ways we can make a group of students.***

***(1) Five good students*** **(2016-17)**

***(2) Five students in which exactly three are weak.***

**(1) Five Good Students :** Total students 40 and 5 are weak so remaining good students are 35 so we have to choose 5 good student from 35 i.e.

$$35_{C_5} = \frac{35!}{5!\,30!} = 324632$$

**(2)** Five Students in which exactly three are week. i.e.

$$35_{C_2}.5_{C_3} = \frac{35!}{2!\,33!}.\frac{5!}{3!\,2!} = 5950$$

**19.** ***Solve the following recurrence relation :***

$$\boldsymbol{a_n - 4a_{n-1} + 4a_{n-2} = n + 1;\ a_0 = 0,\ a_1 = 1,\ n \geq 2}$$

**(2015-16)**

We assume that the general form of the particular solution is

$$a_n^{(p)} = P_1 n + P_2$$

Putting this into the L.H.S. of given recurrence relation, we get

$$(P_1 n + P_2) - 4(P_1(n-1) + P_2) + 4(P_1(n-2) + P_2) = n + 1$$

$$nP_1 + P_2 - 4nP_1 + 4P_1 - 4P_2 + 4nP_1 - 8P_1 + 4P_2 = n + 1$$

$$nP_1 - 4P_1 + P_2 = n + 1$$

Comparing the coefficient of $n$ both the side we get

$$P_1 = 1$$
$$P_2 - 4 = 1$$
$$P_2 = 5$$

Hence the particular solution is

$$a_n^{(p)} = n + 5$$

**20.** ***In a shipment of 50 CDs 10 are defective. Determine***
**(1)** ***In how many ways we can select 35 CDs.***
**(2)** ***In how many ways we can select 35 non – defective CDs.***
**(3)** ***In how many ways we can select 35 CDs containing exactly 5 defective CDs.***
**(4)** ***In how many ways we can select 35 CDs containing at least 5 defective CDs.*** **(2015-16)**

Total CDs = 50
Total defective = 10

(1) No. of ways to select 35 CDs
$= {}^{50}C_{35}$

(2) No. of ways to select 35 non-defective CDs
$= {}^{40}C_{35}$

(3) No. of ways to select 35 CDs containing exactly 5 defective
$= {}^{45}C_{35}$

(4) No. of ways to select 35 CDs containing at least 5 defective CDS
$= {}^{45}C_{35} + {}^{46}C_{35} + {}^{47}C_{35} + {}^{48}C_{35} + {}^{49}C_{35} + {}^{50}C_{35}$

**21.** ***Solve the recurrence relation T (1) = 1, T (n) = 3T $\left(\frac{n}{3}\right)$ + n.*** **(2014-15)**

Given $T(n) = 3T(n/3) + n$
Using substitution

$$T(n) = 3T\left(\frac{n}{3}\right) + n$$



$$T(n) = 3\left[3T\left(\frac{n}{3^2}\right) + \frac{n}{3}\right] + n$$
$$= 3^2T\left[\frac{n}{3^2}\right] + 3\frac{n}{3} + n$$
$$T(n) = 3^3T\left(\frac{n}{3^3}\right) + 3^2.\frac{n}{3} + 3.\frac{n}{3} + n$$
$$= 3^kT\left(\frac{n}{3^k}\right) + 3^k\frac{n}{3} + \ldots\ldots + n$$
$$= 3^kT\left(\frac{n}{3^k}\right) + \frac{n}{3}\left(3^k + 3^{k-1} + \ldots + 3 + 1\right)$$

Let $\frac{n}{3^k} = 1$, then $k = \log_3(n)$

So $T(n) = 3^kT(1) + \frac{n}{3}\left[\frac{3^{k-1}}{3-1}\right]$

$$= 3^k + \frac{n}{6}\left[3^{k-1}\right] \qquad \{\because T(1) = 1\}$$

Now put $k = \log_3(n)$

$$T(n) = 3^{\log_3(n)} + \frac{n}{6}\left[3^{\log_3(n)} - 1\right]$$

Further solving we have :

$$T(n) = 3^{\log_3(n)} + \frac{n}{6}\left[3^{\log_3(n)} - 1\right] = n + \frac{n}{6}[n-1]$$

So $\quad T(n) = n + \frac{n}{6}[n-1]$

**22.** ***In a shipment there are 40 floppy disks of which 5 are defective. Determine in how many ways we can select :*** **(2014-15)**
**(1)** ***Five non-defective floppy disks***
**(2)** ***Five floppy disks in which exactly three are defective.***

Total Floppy disks = 40
Since 5 disks are defective, hence
Fair disks are 35.

So Total No. of non defective disks can be taken by $^{35}C_5$ ways

And when we have to select exactly three defective then $^{35}C_2 \times {}^5C_3$ {Because 3 are defective and 2 are fair}

**23. *Write note on the Generating function.*** **(2014-15)**

## Generating Function

Generating functions are one of the most surprising and useful inventions in discrete mathematics. Roughly speaking, generating functions transform problems about sequences into problems about functions. The ordinary generating function for the infinite sequence $<a_0, a_1, a_2, \ldots>$ is the power series :

$$A(z) = \sum_{n=0}^{\infty} a_n z^n = a_0 + a_1 z + a_2 z^2 + \ldots + a_r z^r + a_r z^n + \ldots$$

Where $z$ is a variable

A generating function is a "formal" power series in the sense that we usually regard '$z$' as a place holder rather than a number. Generating functions are important for solving counting functions

For example :

$$<0, 0, 0\ldots> \leftrightarrow 0 + 0.z + 0.z^2 + 0.z^3 + \ldots = 0$$

$$<1, 0, 0, 0\ldots> \leftrightarrow 1 + 0.z + 0.z^2 + \ldots = 1$$

$$<2, 3, 1, 0\ldots> \leftrightarrow 2 + 3.z + 1.z^2 + 0.z^3 + 0.z^4 = 2 + 3z + z^2$$

**24. *Write Generating function of the following series :*** **1,−1,1,−1,1,−1 ...is** **(2013-14)**

Generating functions of the sequence

1, -1, 1, -1, 1, -1,..... is

$$A(z) = 1 + (-1)z + (1)z^2 + (-1)z^3 + (1)z^4 + \ldots\ldots$$

$$= 1 - z + z^2 - z^3 + z^4 - z^5$$

$$= \frac{1}{(1+z)}$$

**25 *Determine the discrete numeric function corresponding to the following generating function :***



$$A(z) = \frac{1}{5 - 6z + z^2}$$ **(2013-14)**

Here given $A(z) = \dfrac{1}{5 - 6z + z^2}$

$$= \frac{1}{(1-z)(5-z)}$$

$$= \frac{1}{4}\left[\frac{1}{(1-z)} - \frac{1}{(5-z)}\right]$$

$$a_r = \frac{1}{4}\left[(-1)^r - \frac{1}{5}\left(\frac{1}{5}\right)^r\right]$$

$$= \frac{1}{4}\left[1 - \left(\frac{1}{5}\right)^{r+1}\right]$$

This is the required numeric function.

**26. *Solve the recurrence relation $a_{n+2} - 5a_{n+1} + 6a_n = 2$, where $a_0 = 1$, $a_1 = 2$ by the method of Generating Function.*** **(2013-14)**

Let $A(z) = \sum_{n=0}^{\infty} a_n z^n$ be the generating function and the given recurrence relation is

$$a_{n+2} - 5a_{n+1} + 6a_n = 2 \qquad \ldots(i)$$

Multiplying (i) by $z^n$ and summing up from $n = 0$ to $\infty$, we get

$$\sum_{n=0}^{\infty} a_{n+2} z^n - 5\sum_{n=0}^{\infty} a_{n+1} z^n + 6\sum_{n=0}^{\infty} a_n z^n = 2\sum_{n=0}^{\infty} z^n$$

Putting the value $a_1 = 1$, $a_1 = 2$, we get

$$\frac{1}{z^2}[A(z) - 1 - 2z] - \frac{5}{z}[A(z) - 3] + 6A(z) = \frac{2}{1-z}$$

$$A[z][z^{-2} - 5z^{-1} + 6] = \frac{2}{1-z} + z^{-2} + 2z^{-1} - 15z^{-1}$$

$$A[z][z^{-2} - 5z^{-1} + 6] = \frac{2}{1-z} + z^{-2} + (-13)z^{-1}$$

$$A[z]\left[z^{-2}-5z^{-1}+6\right]=\frac{15z^2-14z+1}{z^2(1-z)}$$

$$A[z]=\frac{15z^2-14z+1}{z^2(1-z)}\times\frac{z^2}{(1-5z+6z^2)}$$

$$=\frac{15z^2-14z+1}{(1-z)(3z-1)(2z-1)}$$

$$A(z)=\frac{15z^2-14z+1}{(1-z)(1-3z)(1-2z)}=\frac{A_1}{(1-z)}+\frac{A_2}{(1-3z)}+\frac{A_3}{(1-2z)}$$

$$a_n=A_1+A_2(3)^n+A_3(2^n)$$

**27.** ***Consider the recurrence relation :***

$$a_n-6a_{n-1}+8a_{n-2}=3^n \text{ with } a_0=3 \text{ and } a_1=7$$

***Find :***

(1) ***Homogeneous solution***
(2) ***Particular solution***
(3) ***Total solution.*** **(2013-14)**

$a_n-6a_{n-1}+8a_{n-2}=3^n$ with $a_0=3$, $a_1=7$

The Characteristic equation is :

$$\alpha^2-6\alpha+8=0$$
$$(\alpha-4)(\alpha-2)=0$$
$$\alpha=2,\ 4$$

Homogeneous equation is :

$$a_n^{(h)}=A_1 2^n+A_2 4^n \quad \text{....(i)}$$

Here $A_1$ and $A_2$ are the constants whose values can be determined using the condition

$$a_0=3,\ a_1=7$$

Put $n=0$ in equation (i), we get

$$a_0=2A_1+A_2$$
$$2A_1+A_2=3 \quad \text{...(a)}$$

Putting $n=1$ we get

$$a_1=2A_1+4A_2$$
$$7=2A_1+4A_2 \quad \text{...(b)}$$

On solving equation (a) & (b) we get

$$3A_2=4 \text{ so } A_2=4/3 \ \ \& \ \ A_1=5/6$$



$$a_n^{(h)}=\left(\frac{5}{6}\right)2^n+\left(\frac{4}{3}\right)4^n$$

The general form of particular solution is

$$P8^n=P3^n$$

Putting this in equation (i) we get

$$a_n^{(P)}=P.3^n$$

$$P3^n-6P\,3^{n-1}+8P3^{n-2}=3^n$$

$$P-\frac{6P}{3}+\frac{8P}{9}=1$$

$$P-2P+\frac{8P}{9}=1$$

$$\frac{8P}{9}-P=1$$

$$P=-9$$

Hence $a_n^{(P)}=(-9)3^n$

Total solution = Homogeneous solution + particular solution

$$\left(\frac{5}{6}\right)2^n+\left(\frac{4}{3}\right)4^n+(-9)3^n$$

**28.** ***Solve the following problems :***

(1) ***Suppose 15 staffs in an office. Find minimum number of staffs that can have their joining in the same months.***
(2) ***What should be the minimum number of staffs in the office so that at least 3 staffs have joining in the same months?*** **(2013-14)**

We know that total no. of the months are 12. So, there would be minimum 2 staffs who have their joining in same month.

Total number of months = 12

So, minimum number of staffs in the office so that at least 3 staffs have joining in same month = 12 × 3
= 36 staffs.

**29.** ◆ ***Find the first five terms of the following recurrence relation :*** **(2011-12)**

◆ For $a_k = a_{k-1} + 3a_{k-2}$ *for all integers* $k \geq 2$, $a_0 = 1$, $a_1 = 2$.

$a_k = a_{k-1} + 3a_{k-2}$

$a_0 = 1$, $a_1 = 2$, $a_2 = a_1 + 3a_0 = 5$, $a_3 = a_2 + 3a_1 = 11$,

$a_4 = a_3 + 3a_2 = 26$ **(Ans.)**

**30.** ***Suppose that two distinguishable dice are rolled. In how many ways we get a sum of 6 or 8?*** **(2011-12)**

Two dice can be thrown in $6 \times 6$ ways = 36 ways

So $n(S) = 36$ where $S$ is the sample space.

Let $E_1$ be the event of getting a sum of 6 so $E_1 = \{(1, 5), (5, 1), (2, 4), (4, 2), (3, 3)\}$ and $n(E_1) = 5$

So required probability $\frac{n(E_1)}{n(S)} = \frac{5}{36}$

Again let $E2$ be the event of getting a sum of 8 so $E_2 = \{(2, 6), (6, 2), (3, 5), (5, 3), (4, 4)\} = 5$

So required probability $\frac{n(E2)}{n(S)} = \frac{5}{36}$

So, total ways $\frac{5}{36} + \frac{5}{36} = \frac{10}{36}$ **(Ans.)**

**31.** ***Explain the extended pigeonhole principle.*** **(2011-12)**

If $n$ pigeons are assigned to $m$ pigeonholes, then one of the pigeonholes must contain at least $(n\text{-}1)/m + 1$ pigeons.

Let us assume that none of the pigeonholes contain more pigeons. Then there are at most pigeons. This contradicts our assumption that there are $n$ pigeons. Therefore, one of the pigeonholes must contain at least pigeons.

**32.** ***Discuss Pascals triangle.*** **(2011-12)**

Pascal's triangle is a triangular array of the binomial coefficient in a triangle.

The rows of Pascal's triangle are conventionally enumerated starting with row $n = 0$ at the top. The entries



in each row are numbered from the left beginning with $k = 0$ and are usually staggered relative to the numbers in the adjacent rows.

A simple construction of the triangle proceeds in the following manner.

On row 0, write only the number 1. Then, to construct the elements of following rows, add the number above and to the left with the number above and to the right to find the new value. If either the number to the right or left is not present, substitute a zero in its place. For example, the first number in the first row is $0 + 1 = 1$, whereas the numbers 1 and 3 in the third row are added to produce the number 4 in the fourth row.

**33.** ***In a shipment, there are 40 floppy disks of which 5 are defective. Determine :*** **(2011-12)**

**(1)** ***In how many ways can we select floppy disks?***

**(2)** ***In how many ways can we select five floppy disks containing exactly three defective floppy disks?***

**(3)** ***In how many ways can we select five floppy disks containing at least one defective floppy disk?***

**(1)** The ways we can select five floppy disks $= {}^{40}C_5$

**(2)** The ways that we can select five floppy disks containing exactly three defective floppy disks

$$= {}^5C_3 \times {}^{35}C_2$$

**(3)** The ways that we can select five floppy disks containing atleast one defective floppy disks

$$= {}^5C_1 \times {}^{35}C_4 + {}^5C_2 \times {}^{35}C_3 + {}^5C_3 \times {}^{35}C_2 + {}^5C_4 \times {}^{35}C_1 + {}^5C_5 \times {}^{35}C_0$$

**34.** ***If the sequence $< a_n >$, where $0 \leq n \leq \infty$, has generating function $f(x)$. Then :*** **(2011-12)**

**(1)** ***What sequence is generated by the function $g(x) = (1+x)\, f(x)$?, and***

**(2) *What sequence is generated by the function***

$$h(x) = \frac{f(x)}{1+x}?$$

(1) Here $f(x) = a_0 + a_1x + a_2x^2 + ..... + a_rx^r + ....$

$g(x) = (1+x)f(x) = (1+x)(a_0 + a_1x + a_2x^2 + ..... + a_rx^r)$

$= a_0 + (a_0 + a_1)x + (a_1 + a_2)x^2 + ....$

Thus, sequence corresponding to $g(x)$ say, $b_n$ is

$$b_n = \begin{cases} a_0, & n = 0 \\ a_{n-1} + a_n, & n \geq 1 \end{cases}$$

(2) $h(x) = \dfrac{f(x)}{(1+x)} = (a_0 + a_1x + a_2x^2 + ....)$.

$(1 + (-1)x + (-1)^2x^2 + ....(-1)^rx^r + ....)$

$= a_0 + (a_1 + (-1)a_0)x + (a_2 + (-1)a_1 + (-1)^2a_0)x^2 + ........$

$+ ar + a_{r-1}(-1) + a_{r-2}(-1)^2 + ... + a_1(-1)^r + a_0(-1)^rx^n + ...$

Hence, if $C_n$ be the sequence corresponding to the generating function $h(x)$ then

$C_n = a_r + a_{r-1}(-1) + a_{r-2}(-1)^2 + ..... a_1(-1)^{r-1} + a_0(-1)^r$

**35. *Using generating function, solved :* (2011-12)**

$$a_{k+2} - 5a_{k+1} + 6a_k = 2,\ k \geq 0,\ a_0 = 3,\ a_1 = 7.$$

The associated homogeneous recurrence relation is

$a_{n+2} - 5a_{n+1} + 6a_n = 0$ ...(1)

Let $a_n = r^n$ be a solution of (1). The characteristic equation is

$r^2 - 5r + 6 = 0$ so $r = 3, 2$.

So, the solution of equation (1) is $a_n^{(h)} = C_13^n + C_22^n$

To find the particular solution of the given equation, let $a_n^{(p)} = A$ Substituting in the given equation

$A - 5A + 6 = 2$ we get $A = 1$

$a_n^{(p)} = 1$ Which is a particular solution?

Hence, the general solution is $a_n = a_n^{(h)} + a_n^{(p)}$

$= C_13^n + C_22^n + 1$ ... (2)

To find $C_1$ and $C_2$, put $n = 0$ and $n = 1$ in (2)



$A_0 = c_1 + c_2 + 1$

$3 = c_1 + c_2 + 1$

$c_1 + c_2 = 2$

Again $a_1 = 3c_1 + 2c_2 + 1$

$7 = 3c_1 + 2c_2 + 1$

$3c_1 + 2c_2 = 6$

After solving we get $c_1 = 2$ and $c_2 = 0$

Putting the values of $c_1$ and $c_2$ in equation (2), the required solution is

$a_n = 2.3^n + 1$ (Ans.)

**36. *How many integer solutions are there to the equation?* (2011-12)**

$$x_1 + x_2 + x_3 + x_4 = 13,\ 0 \leq x_i \leq 5.$$

Each solution of the given equation is equivalent to selecting 13 items from the set $\{x_1, x_2, x_3, x_4\}$, repetition allowed.

Hence the required number of solutions

$= C(13 + 4 - 1, 13)$

$= C(16,3) = 560$ (Ans.)

**37. *Find the generating function of the following numeric function :* (2010-11)**

$$a_n = \frac{1}{(n+1)!}, n \geq 0.$$

$a_n = \dfrac{1}{(n+1)!}, n \geq 0$

$G(x) = \dfrac{1}{1!} + \dfrac{1}{2!}x + \dfrac{1}{3!}x^2 + ......$

$= \dfrac{1}{x}\left[-1 + 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + .....\right]$

$= \dfrac{1}{x}\left[-1 + e^x\right] = \dfrac{e^x - 1}{x}$ (Ans.)

**38. *Determine the numeric function for the corresponding generating function* (2010-11)**

$$G(x) = \frac{10}{1-x} + \frac{12}{2-x}$$

$$G(x) = \frac{10}{1-x} + \frac{12}{2-x}$$

$$\sum_{n=0}^{\infty} a_n x^n = \frac{10}{1-x} + \frac{6}{1-\frac{x}{2}}$$

$$= 10\sum_{n=0}^{\infty} x^n - 6\sum_{n=0}^{\infty} \frac{x}{2}$$

$$a_n = 10 - 6\left(\frac{1}{2}\right)^n ; n \geq 0$$

**39. *Use generating function to solve the recurrence relation $a_{n+2} - 2a_{n+1} + a_n = 2^n$, With conditions $a_0 = 2, a_1 = 1$.*** **(2010-11)**

The given recurrence relation is :

$$a_{n+2} - 2a_{n+1} + a_n = 2^n$$

Let $G(x) = \sum_{n=0}^{\infty} ax^n$ where $G(x)$ is the generating function for the sequence $\{a_n\}$. Multiplying each term in the given recurrence relation by $x^n$ and summing from o to $\infty$. We get

$$\sum_{n=0}^{\infty} a_{n+2}x^n - 2\sum_{n=2}^{\infty} a_{n+1}x^n + \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} 2^n x^n$$

Or $$\frac{G(x) - a_0 - a_1 x}{x^2} - 2\left(\frac{G(x) - a_0}{x}\right) + G(x) = \frac{1}{1-2x}$$

Or $$\frac{G(x) - 2 - x}{x^2} - 2\left(\frac{G(x) - 2}{x}\right) + G(x) = \frac{1}{1-2x}$$

Or $$(x^2 - 2x + 1)G(x) = 2 + 3x + \frac{x^2}{1-2x}$$

Or $$G(x) = \frac{2}{(1-x)^2} + \frac{3x}{(1-x)^2} + \frac{x^2}{(1-2x)(1-x)^2}$$

By Partial fraction $$\frac{x^2}{(1-2x)(1-x^2)} = \frac{1}{1-2x} - \frac{1}{(1-x)^2}$$

$$\therefore \quad G(x) = \frac{1}{(1-x)^2} + \frac{3x}{(1-x)^2} + \frac{1}{1-2x}$$



$$\therefore \quad \sum a_n x^n = \sum (n+1)x^n + 3\sum nx^n + \sum 2^n x^n$$

$$a_n = (n+1) + 3n + 2^n = 1 + 4n + 2^n$$

**40. *The generating function of a sequence $a_0, a_1, a_2, \ldots$ is the expression $A(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots$ using generating functions, solve the recurrence relation $a_n + 3a_{n-1} - 10a_{n-2} = 0$ for $n \geq 2$ and $a_0 = 1$, $a_1 = 4$.***

**Recurrence Relation**

Given, $a_n + 3a_{n-1} - 10a_{n-2} = 0$ for $n \geq 2$

$$a_0 = 1,\ a_1 = 4$$

$$a_n + 3a_{n-1} - 10a_{n-2} = 0$$

$$m^2 + 3m - 10 = 0$$

$$m^2 + (5m - 2m) - 10 = 0$$

$$m^2 + 5m - 2m - 10 = 0$$

$$m(m+5) - 2(m+5) = 0$$

$$(m+5) \quad (m-2)$$

$$m = -5, 2$$

$$a_n = c_1 2^n + c_2(-5)^n$$

$$a_0 = c_1 + c_2$$

$$a_1 = 2c_1 - 5c_2$$

$$c_1 + c_2 = 1 \quad \ldots(1)$$

$$2c_1 - 5c_2 = 4 \quad \ldots(2)$$

Equation (1) multiplied by 5 and divided by equation (2)

$$5c_1 + 5c_2 = 5$$

$$2c_1 - 5c_2 = 4$$

$$7c_1 \quad = 9$$

$$c_1 = \frac{9}{7}$$

Put the value of $c_1$ in equation (1),

$$\frac{9}{7} + c_2 = 1$$

$$c_2 = 1 - \frac{9}{7}$$

$$\frac{7+9}{7} = \frac{16}{7}$$

$$c_2 = \frac{-2}{7}$$

Put the value of $c_1$ and $c_2$,

$$a_n = c_1 2^n + c_2(-5)^n$$

$$a_n = \frac{9}{7} 2^n + \frac{-2}{7}(-5)^n$$

□□