Homework 10

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22 Claim: $n^2 \leq n!$ for all n except $n = 2, 3$.

Proof. When $n = 0$, $n^2 = 0$ and $n! = 1$, so $n^2 \leq n!$.
When $n = 1$, $n^2 = 1$ and $n! = 1$, so $n^2 \leq n!$.
When $n = 2$, $n^2 = 4$ and $n! = 2$, so $n^2 > n!$.
When $n = 3$, $n^2 = 9$ and $n! = 6$, so $n^2 > n!$.
When $n = 4$, $n^2 = 16$ and $n! = 24$, so $n^2 \leq n!$.

Now suppose that $n^2 \leq n!$ for some $n > 4$. ...? $\square$

24 *Proof.* by way of mathematical induction:
In the base case, $n = 1$, $1/2 \leq 1/2$, which is true.

Now suppose that $1/(2n) \leq [1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)]/(2 \cdot 4 \cdot \cdots \cdot 2n$ for some positive n.

Then $\frac{1}{2n} \leq \frac{1\cdot 3\cdot \cdots \cdot(2n-1)}{2\cdot 4\cdot \cdots \cdot 2n} \equiv \frac{1}{2n}\frac{2n}{2n+2} \leq \frac{1\cdot 3\cdot \cdots \cdot(2n-1)}{2\cdot 4\cdot \cdots \cdot 2n}\frac{2n}{2n+2}$ (given that $\frac{2n}{2n+2}$ is positive, which it is as n is positive), which is less than $\frac{1\cdot 3\cdot \cdots \cdot(2(n+1)-1)}{2\cdot 4\cdot \cdots \cdot 2(n+2)}$.
By the transitivity of inequality, then, $1/(2(n+1)) \leq \frac{1\cdot 3\cdot \cdots \cdot(2(n+1)-1)}{2\cdot 4\cdot \cdots \cdot(2(n+1))}$.

Since the $n+1$ case followed from the n case, by the principle of mathematical induction, the inequality is true when n is a positive integer. $\square$

30 *Proof.* by way of mathematical induction:
In the base case, $n = 1$, $H_1 = 2H_1 - 1 = 1$, which is true.

Now suppose that $H_1 + H_2 + \cdots + H_n = (n+1)H_n - n$ for some n. Then

$$\begin{aligned}
H + 1 + H_2 + \cdots + H_n + H_n + 1 &= (n+1)H_n - n + H_n + 1 \\
&= (n+1)H_n + \frac{n+1}{n+1} - \frac{n+1}{n+1} - n + H_n + 1 \\
&= (n+1)H_{n+1} - n - 1 + H_n + 1 \\
&= (n+2)H_{n+1} - n - 1 \\
&= ((n+1)+1)H_{n+1} - (n+1)
\end{aligned}$$

Since the $n+1$th case followed from the nth case, by the principle of mathematical induction, $H_1 + H_2 + \cdots + H_n = (n+1)H_n - n$ $\square$

36 *Proof.* by induction:
In the base case, $n = 1$, $4^2 + 5^1 = 21$, which is divisible by 21.

Now suppose that $4^{n+1} + 5^{2n-1}$ is divisible by 21. Then there exists some integer k such that $21k = 4^{n+1} + 5^{2n-1}$. Then

$$\begin{aligned} 4^{(n+1)+1} + 5^{2(n+1)-1} &= 4 \cdot 4^{n+1} + 25 \cdot 5^{2n-1} \\ &= 4(4^{n+1} + 5^{2n-1}) + 21 \cdot 5^{2n-1} \\ &= 4(21k) + 21 \cdot 5^{2n-1} \\ &= 21(4k + 5^2 n - 1) \end{aligned}$$

which is divisible by 21.

Since the $n + 1$th case followed from the nth case, by the principle of mathematical induction, 21 divides $4^{n+1} + 5^{2n-1}$ whenever n is a positive integer. □

46 *Proof.* In the base case, a set of 3 elements has exactly 1 subset of three elements, the set itself.

Now suppose that a set of n elements has $n(n-1)(n-2)/6$ elements for some n. Then adding 1 element to this set allows us to create $n(n-1)/2$ new subsets where one element is the new element, and the other two are chosen from the existing set. So the new set of all subsets of length 3 contains $n(n-1)(n-2)/6 + n(n-1)/2 = \frac{n(n-1)(n-2)+3n(n-1)}{6} = \frac{n^3-3n^2+2n+3n^2-3n}{6} = \frac{n^3-n}{6} = \frac{(n+1)(n)(n-1)}{6} = \frac{(n+1)((n+1)-1)((n+1)-2)}{6}$ elements.

Since the $n + 1$th case followed from the nth case, by the principle of mathematical induction, a set with n elements has $n(n-1)(n-2)/6$ subsets containing exactly 3 elements, $n \geq 3$. □