Homework 5

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2 Use a proof by cases to show that 10 is not the square of a positive integer.

Proof. Case 1: Suppose $1 \leq x \leq 3$. None of the squares 1, 2, and 3 (1, 4, and 9) are equal to 10. Case 2: Suppose $x \geq 4$. All values of x square to more than 10, with the smallest, 4, squaring to 16. $\square$

4 Prove that there are no positive perfect cubes less than 1000 that are the sum of the cubes of two positive integers.

Proof. Proof by exhaustion:
For two positive numbers to sum to a thousand, they must both be smaller than 1000. There are nine positive perfect cubes less than a thousand: $1^3 = 1, 2^3 = 8, 3^3 = 27, 4^3 = 64, 5^3 = 125, 6^3 = 218, 7^3 = 343, 8^3 = 512, 9^3 = 729$. If one cube is $1^3 = 1$, then the other cube must be $1000 - 1 = 999$, which is not a perfect cube. If one cube is 8, then then the other must be 992, which is not a perfect cube. Similarly, the other perfect cubes would require complements of 963, 936, 875, 782, 657, 488, and 271; none of which are perfect cubes. All other cubes would produce numbers greater than a thousand. $\square$

12 Prove that either $2 \cdot 10^{500}$ or $2 \cdot 10^{500}$ is not a perfect square; that is, for big numbers, n and n+1 aren't perfect squares.

Proof. Let $x = 14142135623730950488016887242096980785696718753769$
$480731766797379907324784621070388503875343276415727350138462309$
$122970249248360558507372126441214970999358314132226659275055927$
$557999505011527820605714701095599716059702745345968620147285174$
186408891986.

Then $x^2 = 19999999999999999999999999999999999999999999999999999$
$999999999999999999999999999999999999999999999999999999999999999$
$999999999999999999999999999999999999999999999999999999999999999$
$999999999999999999999999999999999999999999999999999999999999999$
$999999997298193289999228009518895947571376931932785474994324356$
$082085924607018660267610379324142405528110343824662274772915135$

086945882872259248936883583381612245882659082060547470527297162
1183655209655497591595552553981101339907371766857358706139448215
024196, which is less than $2 \cdot 10^{500} + 15$.

Similarly, $(x+1)^2 =$ 20000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000
000000000000000000000000000000000000000000000000000000000000000
000000000000000002558246453746112898555267043176533850332622298
253328581961568068442166822950975115633189309208094179852493919
939116107558544260398927399318976601332361096251092353537909758
238241316112838857660676118856174375341322074539806370409765327
6487821032808169, which is more than $2 \cdot 10^{500} + 16$. Thus neither is a perfect square. $\square$

(This is a nonconstructive proof.)

16 Disprove: 2 and $\frac{1}{2}$ are both rational numbers, but $\sqrt{2}$ is not rational.

18 Show that if a, b, and c are real numbers and $a \neq 0$, then there is a unique solution of the equation $ax + b = c$.

Proof. Suppose a, b, and c are real numbers, with $a \neq 0$. Rearranging, we have $x = \frac{c-b}{a}$, a solution.

Suppose there were two unique solutions, x and y. Then $ax + b = c$ and $ay + b = c$, so $ax + b = ay + b$. Subtracting b from both quantities leaves $ax = ay$, and dividing by a leaves $x = y$, which means the solutions are not unique. Therefore there is eqactly one solution. $\square$

24 Show that if x is a nonzero real number, then $x^2 + 1/x^2 \geq 2$.

Proof. Since $(x - 1/x)^2 \geq 0$ (given), we can write this as $(x - 1/x)^2 = (x - 1/x)(x - 1/x) \geq 0$. This, in turn, can be written as $x^2 - 2\cancel{\frac{x}{x}} + 1/x^2 \geq 0$, which is equivalent to $x^2 + 1/x^2 \geq 2$. $\square$

26 Conjecture: The quadratic mean is greater than or equal to the arithmetic mean when both numbers are positive.

Proof. Suppose there is a pair of numbers where the quadratic mean is less than the arithmetic mean. Then for x, y, $\sqrt{(x^2+y^2)/2} < (x+y)/2$.

$$\sqrt{x^2+y^2} < (x+y)$$

$$\sqrt{x^2+y^2} < \sqrt{x^2+2xy+y^2}$$

Since x and y are positive, then 2xy is positive (proven previously), and the second root is greater than the first—a contradiction! Therefore, the quadric mean is greater than or equal to the arithmetic mean for a pair of positive numbers. $\square$

32 *Proof.* Because both addends are the product of a positive number and a squared integer, they must both be zero or positive. For two positive integers to sum to 14 they must be less than or equal to 14. If $|x| > 2$ or $|y| > 1$, then that addend will be greater than 14 and not satisfy the equation. If $y = 0$, then no $|x|$ in $0, 1, 2$ satisfies the equation. Otherwise, if $|y| = 1$, then no $|x|$ in $0, 1, 2$ satisfies the equation. If $|y| \geq 2$, then $5y^2 > 14$, so no solution can exist. Therefore, there is no solution. $\square$

36 Prove that $\sqrt[3]{2}$ is irrational.

Proof. A proof by contradiction: Suppose that the cube root of two. Then there exist some natural numbers x, y for which $\frac{x}{y} = \sqrt[3]{2}$, and where the fraction is in its simplest form. Then $2 = \frac{x^3}{y^3} \rightarrow 2y^3 = x^3$. So x^3 (and therefore x) are divisible by 2, so $x = 2k$ for some natural number k. By the same logic, y must also be divisible by 2, and so they numbers have a common factor. However, we supposed they were in reduced form. Contradiction! Therefore, $\sqrt[3]{2}$ must be irrational. $\square$