Homework 4

Chandler Swift

February 13, 2019

10 Use a direct proof to show that the product of two rational numbers is rational.

Proof. Assume x and y are rational numbers. Then there exist integers h, j, k, and l such that $x = \frac{h}{j}$ and $y = \frac{k}{l}$. The product of x and y can then be written as $x \cdot y = \frac{h}{j} \cdot \frac{k}{l} = \frac{h \cdot k}{j \cdot l}$, which is rational. □

12 Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.
The product is irrational. A proof by contradiction:

Proof. Suppose that a rational number x and an irrational number y multiply to give us a rational number z. Then x can be represented as the fraction $\frac{h}{j}$ and z as the fraction $\frac{k}{l}$, with h, j, k, l being integers, such that $\frac{h}{j} \cdot y = \frac{k}{l}$. Dividing, we have $y = \frac{(\frac{k}{l})}{(\frac{h}{j})} = \frac{kj}{hl}$. Since y is represented as the ratio of two integers, it is rational—a contradiction! Therefore the product must be irrational. □

16 Prove that if x, y, and z are integers and $x + y + z$ is odd, then at least one of x, y, and z is odd.

Proof. A proof by contradiction: Suppose that x, y, z are even integers. Then there exists integers p, q, r such that $x = 2p$, $y = 2q$, and $z = 2r$. The sum of these numbers is $x + y + z = 2p + 2q + 2r = 2(p + q + r)$, and so p+q+r is even—a contradiction! Thus at least one of x, y, and z must be odd. □

18 Prove that if m and n are integers and mn is even, then m is even or n is even.

Proof. A proof by contradiction: Suppose that m and n are integers, mn is even, and neither m nor n is even (that is, both are odd). Then there exist integers x and y such that $m = 2x + 1$ and $n = 2y + 1$. Then $mn = (2x+1)(2y+1) = 4xy+2x+2y+1 = 2(2xy+x+y)+1$, and since

$2xy + x + y$ is an integer, the product is even—a contradiction! Thus, at least one of m or n must be even. $\square$

28 Prove that if n is a positive integer, then n is even iff $7n + 4$ is even.

Proof. In the forward direction, if n is even, then there exists an integer k such that $n = 2k$. We can write $7n+4$ as $7(2k)+4 = 14k+4 = 2(7k+2)$, which is even.

In the other direction, if $7n + 4$ is even, then there exists an integer k such that $7n + 4 = 2k$. Because $7n = 2k - 4 = 2(k - 2)$, $7n$ is also even. By problem 18, since $7n$ is even, then either 7 or n must be even. Since 7 is not even (it is odd, as $7 = 2 \cdot 3 + 1$), the n must be the even multiplicand. $\square$