Homework 9

Chandler Swift

March 27, 2019

4 Let $p(n)$ be the statement $1^3+2^3+\cdots+n^3 = (n(n+1)/2)^2$ for the positive integer n.

(a) $P(1)$ says that $1^3 = (1(1+1)/2)^2$.

(b) $(1(1+1)/2)^2 = (1(1))^2 = 1 = 1^3$.

(c) $P(n)$ is true for some positive integer n.

(d) If $P(n)$ is true, then $P(n+1)$ is true.

(e) Suppose $1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2$ (inductive hypothesis). Then

$$\begin{aligned}
1^3 + 2^3 + \cdots + n^3 + (n+1)^3 &= (n(n+1)/2)^2 + (n+1)^3 \\
&= (\frac{1}{2}n^2 + \frac{1}{2}n)^2 + (n^3 + 3n^2 + 3n + 1) \\
&= \frac{1}{4}n^4 + \frac{1}{2}n^3 + \frac{1}{4}n^2 + n^3 + 3n^2 + 3n + 1 \\
&= \frac{1}{4}n^4 + \frac{3}{2}n^3 + \frac{13}{4}n^2 + 3n + 1 \\
&= (\frac{1}{2}n^2 + \frac{3}{2}n + 1)^2 \\
&= ((n+1)(n+2)/2)^2 \\
&= ((n+1)((n+1)+1)/2)^2
\end{aligned}$$

and so $P(n+1)$ is true.

(f) Since we have shown that the propositional function is true for $n+1$ if it is true for n, and $P(1)$ has been shown to be true, we can also say that $P(2)$ is true. Now this means that $P(3)$ is true, which means that $P(4)$ is true, which can be repeated n times to demonstrate that $P(n)$ is true.

6 *Proof.* by way of mathematical induction:
When $n = 1$, $1 \cdot 1! = 1 = (1+1)! - 1$, which is true.

Now suppose that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$ for some n.

$$\begin{aligned}
1 \cdot 1! + \cdots + n \cdot n! + (n+1)(n+1)! &= (n+1)! - 1 + (n+1)(n+1)! \\
&= 1(n+1)! + (n+1)(n+1)! - 1 \\
&= (n+2)(n+1)! - 1 \\
&= (n+2)! - 1 \\
&= ((n+1)+1)! - 1
\end{aligned}$$

Since the $n+1$th case followed from the nth case, by the principle of mathematical induction, $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1$. $\square$

10 $f(n) = \frac{n}{n+q}$

Proof. by way of mathematical induction:
In the base case, $n = 1$, $\frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{n}{n+1}$, which is true.

Now suppose that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$ for some n.

$$\begin{aligned}
\frac{1}{1 \cdot 2} + \cdots + \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} &= \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} \\
&= \frac{n(n+2)+1}{(n+1)(n+2)} \\
&= \frac{n^2+2n+1}{(n+1)(n+2)} \\
&= \frac{(n+1)^2}{(n+1)(n+2)} \\
&= \frac{(n+1)}{(n+1)+1}
\end{aligned}$$

Since the $n+1$th case followed from the nth case, by the principle of mathematical induction, $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$. $\square$

12 (attached)

20 *Proof.* by way of mathematical induction:
In the base case, $n = 7$, $3^7 = 2187 < 5040 = 7!$.

Now suppose that $3^n < n!$ for some $n > 6$. Then $3^{n+1} = 3 \cdot 3^n$ and $(n+1)! = (n+1)n!$. $3^n < n!$, so $3 \cdot 3^n < (n+1)n!$ when $n+1 > 3$ which is given.

Since the $n+1$th case followed from the nth case, by the principle of mathematical induction, $3^n < n!$ for $n > 6$. $\square$

28 *Proof.* by mathematical induction:
In the base case, where $n = 3$, $n^2 - 7n + 12 = 9 - 21 + 12 = 0$, which is nonnegative.

Now suppose that $n^2 - 7n + 12$ is nonnegative for some $n \geq 3$. Since $n \geq 3$, $2n - 6 \geq 0$. Then $(n+1)^2 - 7(n+1) + 12 - (n^2 - 7n + 12) = 2n - 6 \geq 0$, so $(n+1)^2 - 7(n+1) + 12 \geq n^2 - 7n + 12)$. Because $f(n+1) \geq f(n)$ and $f(n) \geq 0$, $f(n+1) \geq 0$.

Since the $n + 1$th case followed from the nth case, by the principle of mathematical induction, $n^2 - 7n + 12$ is nonnegative. $\square$

32 *Proof.* by way of mathematical induction:
In the base case, $n = 1$, $n^3 + 2n = 3$, which is divisible by 3.

Now suppose that $n^3 + 2n$ is divisible by 3 for some positive int n. Then $\exists k \in \mathbb{N} \in$ such that $3k = n^3 + 2n$. Then

$$\begin{aligned}(n+1)^3 + 2(n+1) &= n^3 + 3N^2 + 3n + 1 + 2n + 2 \\ &= n^3 + 2n + 3(n^2 + n + 1) \\ &= 3(k + n^2 + n + 1)\end{aligned}$$

which has a factor of 3.

Since the $n + 1$th case followed from the nth case, by the principle of mathematical induction, $n^3 + 2n$ is divisible by 3. $\square$