Homework 7

Chandler Swift

March 7, 2019

12 Determine whether each of these functions from $\mathbb{Z}$ to $\mathbb{Z}$ is one-to-one:

(a) $f(n) = n - 1$ is one-to-one:

Proof. Let $a, b \in \mathbb{Z}$, where $f(a) = f(b)$. Since $f(a) = f(b)$, $a - 1 = b - 1$. Adding 1 to both sides, $a = b$, so the elements are the same, and the sets are one-to-one. $\square$

(b) $f(n) = n^2 + 1$ is not one-to-one, as both $f(1) = f(-1) = 2$.

(c) $f(n) = n^3$ is one-to-one:

Proof. Let $a, b \in \mathbb{Z}$, where $f(a) = f(b)$. Since $f(a) = f(b)$, $a^3 = b^3$. Taking the cube root of each side, $a = b$, so the elements are the same, and therefore the sets are one-to-one. $\square$

(d) $f(n) = \lceil n/2 \rceil$ is not one-to-one, as $f(1) = f(2) = 1$.

20 Give an example of a function from $\mathbb{N}$ to $\mathbb{N}$ which is

(a) one-to-one but not onto: $f(n) = n^2$

(b) onto but not one-to-one: $f(n) = \lceil n/2 \rceil$

(c) both onto and one-to-one (but diff erent from the identity function):

$$f(n) = \begin{cases} f(n) = n + 1 & n \text{ is even} \\ f(n) = n - 1 & n \text{ is odd} \end{cases}$$

(d) neither one-to-one nor onto: $f(n) = 0$

22b Determine whether $f(x) = -3x^2 + 7$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$.
This function is not a bijection, as $f(1) = f(-1) = 4$.

22c Determine whether $f(x) = (x + 1)/(x + 2)$ is a bijection from $\mathbb{R}$ to $\mathbb{R}$.
This is not a function as $f(-2)$ is undefined, which means it is also not a bijection. Additionally, there is no $x \in \mathbb{R}$ such that $f(x) = \frac{1}{2}$.

30 Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if

(a) $f(x) = 1$: $\{1, 1, 1, 1, 1\} = \{1\}$

(b) $f(x) = 2x + 1$: $\{-1, 1, 5, 9, 15\}$
(c) $f(x) = \lceil x/5 \rceil$: $\{0, 0, 1, 1, 2\} = \{0, 1, 2\}$
(d) $f(x) = \lfloor (x^2 + 1)/3 \rfloor$: $\{0, 0, 1, 5, 16\} = \{0, 1, 5, 16\}$

36 If f and $f \circ g$ are one-to-one, does it follow that g is one-to-one? Justify your answer.
Yes (assuming both have the same domain and codomain)

Proof. Suppose g is not one-to-one. Then there exists some x, y where $x \neq y$ in the domain of g where $g(x) = (g)y$. Because functions produce equal output for equal input, since $g(x) = g(y)$, then $f(g(x)) = f(g(y))$, despite $x \neq y$. But we said $f \circ g$ is one-to-one: a contradiction! Therefore g must be one-to-one. □

42 Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that

(a) $f(S \cup T) = f(S) \cup f(T)$

Proof. Let $x \in f(S) \cup f(T)$. Since $x \in f(S \cup T)$, then either $x \in f(S)$ or $x \in f(T)$. If $x \in f(S)$, then there exists some y in A such that $f(y) = x$. Since $y \in S \cup T$, either $y \in S$ or $y \in T$, so $y \in S \cup T$, and then $f(y) \in f(S \cup T)$. Thus, $f(S) \cup f(T) \subset f(S \cup T)$.
Similarly, let $x \in f(S \cup T)$. So there exists some $y \in A$ such that $f(y) = x$, so y is in $S \cup T$. This means that either $y \in S$ or $y \in T$, so $f(y) \in f(S)$ or $f(y) inf(T)$. Thus, $f(S \cup T) \subset f(S) \cup f(T)$.
Since both are subsets of the other, $f(S \cup T) = f(S) \cup f(T)$. □

(b) $F(S \cap T) \subset f(S) \cap f(T)$

Proof. Let $x \in f(S \cap T)$. Then $\exists y \in A$ such that $f(y) = x$. So $y \in S \cap T$, meaning that $y \in S$ and $y \in T$. Then $f(y) \in f(S) \wedge f(y) \in f(T)$, making $F(S \cap T) \subset f(S) \cap f(T)$. □

76 Prove or disprove each of these statements:

(a) True

Proof. Let the integer n be the ceiling of x. Since n is an integer, the floor of n is $n = \lceil x \rceil$. □

(b) False: $x = y = 1.5$
(c) True

Proof. Let the ceiling of $x/4$ be n, so $n - 1 < x/4 \leq n$, and $2n - 2 < 2/x \leq 2n$. □

(d) False: $x = 3.5$
(e) True