Homework 11

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4 Let $P(n)$ be the statement that a postage of n cents can be formed using just 4-cent stamps and 7-cent stamps. The aprts of this exercise outline a strong induction proof that $P(n)$ is true for all integers $n \geq 18$.

(a) Show that the statements $P(18), P(19), P(20), P(21)$ are true, completing the basis step of a proof by strong induction.

Proof. For 18 cents of stamps, take two seven-cent stamps and one four-cent stamp.

For 19 cents of stamps, take one seven-cent stamp and three four-cent stamps.

For 20 cents of stamps, take five four-cent stamps.

For 21 cents of stamps, take three seven-cent stamps. □

(b) The inductive hypothesis of a proof by strong induction that $P(n)$ is true for all integers $n \geq 18$ is that $P(1), P(2), \ldots P(n-1)$ are all true.

(c) In the inductive step, we need to prove that for some $P(n)$, if the inductive hypothesis holds, then $P(n)$ is true.

(d) *Proof.* Suppose any amount 18 or more which is less than k can be made up of stamps of denominations of four and seven cents. Since $k \geq 21$, either $k = 21$ or $k \geq 22$. If $k = 21$, then k can be made up of 3 7-cent stamps. If $k \geq 22$, then $k - 4 \geq 18$, and so by the inductive hypothesis can be made into a combination of four and seven cent stamps. Then k an be made of the combination of $k - 4$ and an additional four-cent stamp. □

(e) Since the base cases were true, and all following cases were true if the base cases were true, then all cases greater than or equal to the base case must be true.

6 (a) The possible amounts, in cents, are 0, 3, 6, 9, 10, 12, 13, 15, 16, and $n \geq 18$.

(b) *Proof.* by mathematical induction: For small amounts, the following amounts can be formed, represented as $a : (b, c)$, where a is the amount of postage, and b and c are the number of 3- and 10-cent stamps respectively: $0 : (0, 0), 3 : (1, 0), 6 : (2, 0), 9 : (3, 0), 10 : (0, 1), 12 : (4, 0) : 13 : (1, 1), 15 : (5, 0), 16 : (2, 1)$.

For amounts $n \geq 18$: In the base case, $18 : (6, 0)$.

Now suppose that for some $k \geq 19$, k can be formed from stamps of 3 and 10 cents. This must consist of at least 3 3-cent stamps or at least 2 10-cent stamps, since the maximum amount that can be made from less than 3 3-cent and 2 10-cent stamps is 16 cents. If there are 3 3-cent stamps, remove them and replace with a 10-cent stamp, incrementing the value by 1. Otherwise, there must be 2 10-cent stamps. Remove these and replace with 7 3-cent stamps, also increasing the value by 1. Therefore $k + 1$ cents can be formed if k can.

Since the $k + 1$ case followed from the k, by the principle of mathematical induction, any number $n \geq 18$ can be formed from 3- and 10-cent stamps. □

(c) *Proof.* by strong induction:

For small amounts, the following amounts can be formed, represented as $a : (b, c)$, where a is the amount of postage, and b and c are the number of 3- and 10-cent stamps respectively: $0 : (0, 0), 3 : (1, 0), 6 : (2, 0), 9 : (3, 0), 10 : (0, 1), 12 : (4, 0) : 13 : (1, 1), 15 : (5, 0), 16 : (2, 1)$.

For amounts $n \geq 18$: In the base cases, $18 : (6, 0), 19 : (3, 1), 20 : (0, 2)$.

Now suppose that for some $k \geq 21$, all numbers $n | 18 \leq n < 21$ can be formed from three- and ten-cent stamps. *(Inductive Hypothesis)*

Since $k \geq 21$, then $k - 3 \geq 18$, and so can be formed from three- and ten-cent stamps. Then k can be formed from the combination making up $k - 3$ plus one three-cent stamp.

Since the kth case followed from the previous cases, by the principle of strong mathematical induction, any integer number of cents greater than or equal to 18 can be formed from three- and ten-cent stamps. □

12 *Proof.* by strong mathematical induction:

In the base case, $1 = 2^0$.

Now suppose that for some k, where for any $n | 1 \leq n \leq k$, n can be written as a sum of distinct powers of two. Then if $k+1$ is even, $\frac{k+1}{2}$ is an integer, and can be written as a sum of distinct powers of two $2^a + 2^b + 2^c \ldots$. Then $k + 1$ can be written as $2^{a+1} + 2^{b+1} + \cdots$, multiplying each power of two by two, which means that they are still distinct.

Now if $k + 1 isodd$, then k is even, and $k + 1$ can be written as $k + 2^0$, and since k is even, it cannot contain 2^0, so the sum is of distinct powers of two.

Since the $k+1$ case followed from the previous cases, by the principle of strong mathematical induction, any positive integer can be written as a sum of distinct powers of two. $\square$

30 The induction step assumes that both $a^j = 1$ and $k > 0$. No base case can be shown where both of these are true.