Quality Engineering
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/lqen20

IMPROVING THE PERFORMANCE OF THE T^2 CONTROL CHART

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Published online: 30 Mar 2007.

Quality Engineering, 5(4), 619–625 (1993)

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Key Words

Multivariate SPC; T^2 control chart; Mean Square Successive Differences (MSSD); Variance–covariance matrix.

Introduction

We were recently studying the particle size distribution for a plant in Europe. There were three sizes of particles being monitored to ensure that the "grit" was being manufactured in a consistent manner. The full data set from the plant is given in Table 2. The first few lines of the data are shown here.

Percent by Weight by Particle Size

LARGE (L)	MEDIUM (M)	SMALL (S)
5.4	93.6	1.0
3.2	92.6	4.2
5.2	91.7	3.1
3.5	86.9	9.6

Correlation among the three screen size weights is almost guaranteed because of the large percentage on screen size M. As the percentage on M goes up, the other two must go down. The correlation matrix shown below for the grit data indicates this is the case.

	L	M	S
L	1.00	−0.77	0.35
M	−0.77	1.00	−0.87
S	0.35	−0.87	1.00

The existence of this correlation means that the T^2 chart is the appropriate one for monitoring the grit size distribution. Using several single-variable charts in this environment, a common practice, produces erroneous signals relative to the state of control of the process. Points outside the control limits on the T^2 chart indicate that the particle size distribution has changed. Possible causes for this shift may be changes in raw material, the manufacturing process, or screen wearout.

The original work in T^2 control charts was done by Hotelling (1). These control charts have been shown to be useful for detecting out-of-control conditions in chemical plants by Jackson (2) and by Holmes and Zook (3) in monitoring particle size distributions for a refractory example. For more information on multivariate quality control charts, see, for example, the work of Alt (4).

In this article we discuss how the Mean Square Successive Difference (MSSD) concept can be extended to the variance-covariance matrix and demonstrate the improvement in response time of the T^2 chart.

Discussion

Suppose that there are n observations on each of p different, possibly correlated, variables. Let X be the $(p \times 1)$ vector representing an observation of the

p variables, $\bar{X}$ be the $(p \times 1)$ vector of the averages of the p variables, and S be the sample variance–covariance $(p \times p)$ matrix. The statistic T^2 is calculated using

$$T^2 = (X - \bar{X})'S^{-1}(X - \bar{X}) \tag{1}$$

where S^{-1} is the inverse of S.

The T^2 distribution is related to the F distribution by the relationship given below which may be used to set the upper control limit for T^2 which has a Type I risk level of α.

$$T^2_{p,n,\alpha} = \frac{p(n-1)}{n-p} F_{p,n-p,\alpha} \tag{2}$$

where n is the number of observations of the p variables.

In T^2 charts, the variance–covariance matrix is usually calculated for the "base" data with no subgrouping and is then applied to future observations. The effect of this is similar to no subgroups being used to set up the control limits for the usual control charts such as X-bar charts: The control limits may be too wide. The net result is that process changes are not detected as early as would be possible using the subgroup approach. Thus, where the univariate approach has the advantage of getting a "within subgroup" standard deviation to use for judging process performance, the multivariate usually does not.

This drawback can be overcome using the subgroup averages and the average of the subgroup variance–covariance matrices. The problem here is that the subgroups must be rather large to ensure adequate degrees of freedom in the subgroup matrix calculations. This, in turn, makes the T^2 chart somewhat slow to respond to process changes because when the subgroup size is large, the subgroups are too widely separated in time to achieve rapid response times.

To avoid the subgrouping problem and to make the T^2 chart more sensitive to process changes we propose that the Mean Square Successive Difference (MSSD) approach be used to calculate the variance–covariance matrix. The usual variance estimator for σ^2 is

$$s^2 = \frac{\sum_{i=1}^{n} (X_1 - \bar{X})^2}{n-1} \tag{3}$$

Hald (5) discusses the estimation of variance by using the MSSD approach, which is

$$q^2 = \frac{1}{2(n-1)} \sum_{i=1}^{n-1} (X_{i+1} - X_i)^2 \tag{4}$$

where q^2 is an unbiased estimate of σ^2. The q^2 estimate of σ^2 will be similar to the s^2 estimate of σ^2 if the process is random but will be quite different if there are trends, cycles, or other nonrandom patterns in the data.

The usual calculation for the covariance of two variables is

$$\mathrm{Cov}(X,Y) = \frac{\sum_{i=1}^{n} (X_i - \bar{X})(Y_i - \bar{Y})}{n - 1} \tag{5}$$

If we extend the MSSD approach from the variance to the variance–covariance matrix, the calculation would be as follows:

$$\mathrm{Cov}\ M(X,Y) = \frac{\sum_{i=2}^{n} (X_i - X_{i-1})(Y_i - Y_{i-1})}{2(n - 1)} \tag{6}$$

If the X and Y processes are random and have no cross-correlation other than at zero lag, these two estimates of the universe covariance will be similar. If these conditions are not met, then there will be a difference in the estimators.

To demonstrate the calculation of the variance–covariance matrix using the proposed MSSD approach, we will use the example provided in Jackson's 1980 article (2). The data on Jackson's two variables (methods) are listed in Table 1.

The variance-covariance matrix, S, using the regular method and using the proposed MSSD approach are given by

$$S = \begin{vmatrix} s_1 & s_{12} \\ s_{12} & s_2 \end{vmatrix} = \begin{cases} \text{Regular Method} \\ \begin{vmatrix} 0.7986 & 0.6793 \\ 0.6793 & 0.7343 \end{vmatrix} \\ \text{MSSD Method} \\ \begin{vmatrix} 0.8065 & 0.6954 \\ 0.6954 & 0.7345 \end{vmatrix} \end{cases}$$

Thus, it would appear that the data used for the calculation of the variance-covariance matrix meet the conditions stated above and that there were no non-random events during the time those 15 observations were obtained.

Example

Next, we deal with the example of the particle size distribution for a plant in Europe. The data on three different particle sizes are given in Table 2. Only the first two columns are used in the analysis since the total of the percentages is always 100 and the variance–covariance matrix will not invert under these conditions. The T^2 chart (with $\alpha = 0.003$) for the data (i.e., the first two columns)

Table 1. Jackson's T^2 Example

METHOD A	METHOD B
10.0	10.7
10.4	9.8
9.7	10.0
9.7	10.1
11.7	11.5
11.0	10.8
8.7	8.8
9.5	9.3
10.1	9.4
9.6	9.6
10.5	10.4
9.2	9.0
11.3	11.6
10.1	9.8
8.5	9.2

Source: Extracted with permission from Ref. (2).

Table 2. Percent by Weight by Particle Size

SAMPLE NUMBER	PARTICLE SIZE			SAMPLE NUMBER	PARTICLE SIZE		
	L	M	S		L	M	S
1	5.4	93.6	1.0	29	7.4	83.6	9.0
2	3.2	92.6	4.2	30	6.8	84.8	8.4
3	5.2	91.7	3.1	31	6.3	87.1	6.6
4	3.5	86.9	9.6	32	6.1	87.2	6.7
5	2.9	90.4	6.7	33	6.6	87.3	6.1
6	4.6	92.1	3.3	34	6.2	84.8	9.0
7	4.4	91.5	4.1	35	6.5	87.4	6.1
8	5.0	90.3	4.7	36	6.0	86.8	7.2
9	8.4	85.1	6.5	37	4.8	88.8	6.4
10	4.2	89.7	6.1	38	4.9	89.8	5.3
11	3.8	92.5	3.7	39	5.8	86.9	7.3
12	4.3	91.8	3.9	40	7.2	83.8	9.0
13	3.7	91.7	4.6	41	5.6	89.2	5.2
14	3.8	90.3	5.9	42	6.9	84.5	8.6
15	2.6	94.5	2.9	43	7.4	84.4	8.2
16	2.7	94.5	2.8	44	8.9	84.3	6.8
17	7.9	88.7	3.4	45	10.9	82.2	6.9
18	6.6	84.6	8.8	46	8.2	89.8	2.0
19	4.0	90.7	5.3	47	6.7	90.4	2.9
20	2.5	90.2	7.3	48	5.9	90.1	4.0
21	3.8	92.7	3.5	49	8.7	83.6	7.7
22	2.8	91.5	5.7	50	6.4	88.0	5.6
23	2.9	91.8	5.3	51	8.4	84.7	6.9
24	3.3	90.6	6.1	52	9.6	80.6	9.8
25	7.2	87.3	5.5	53	5.1	93.0	1.9
26	7.3	79.0	13.7	54	5.0	91.4	3.6
27	7.0	82.6	10.4	55	5.0	86.2	8.8
28	6.0	83.5	10.5	56	5.9	87.2	6.9

given in Table 2 is displayed in Figure 1. The points using the usual variance-covariance matrix are plotted with $*$. The control limit is calculated using Eq. (2). For this example, the number of observations (n) is 56, the number of variables (p) is 2, and we selected $\alpha = 0.003$ to agree with usual control chart procedures. Using these values in Eq. (2) the control limit for T^2 becomes 13.338. As you see, there is no out-of-control signal given by the chart.

However, when we recalculate the T^2 values using the proposed MSSD approach, we get different signals. For example, at data points 26 and 45, the T^2 values calculated using MSSD approach are 14.38 and 17.67, respectively. These two values have been superimposed on the chart in Figure 1 and circled to emphasize the out-of-control conditions. The causes for these out-of-control points were, in fact, identified and corrective actions were initiated to avoid the future occurrence of these problems.

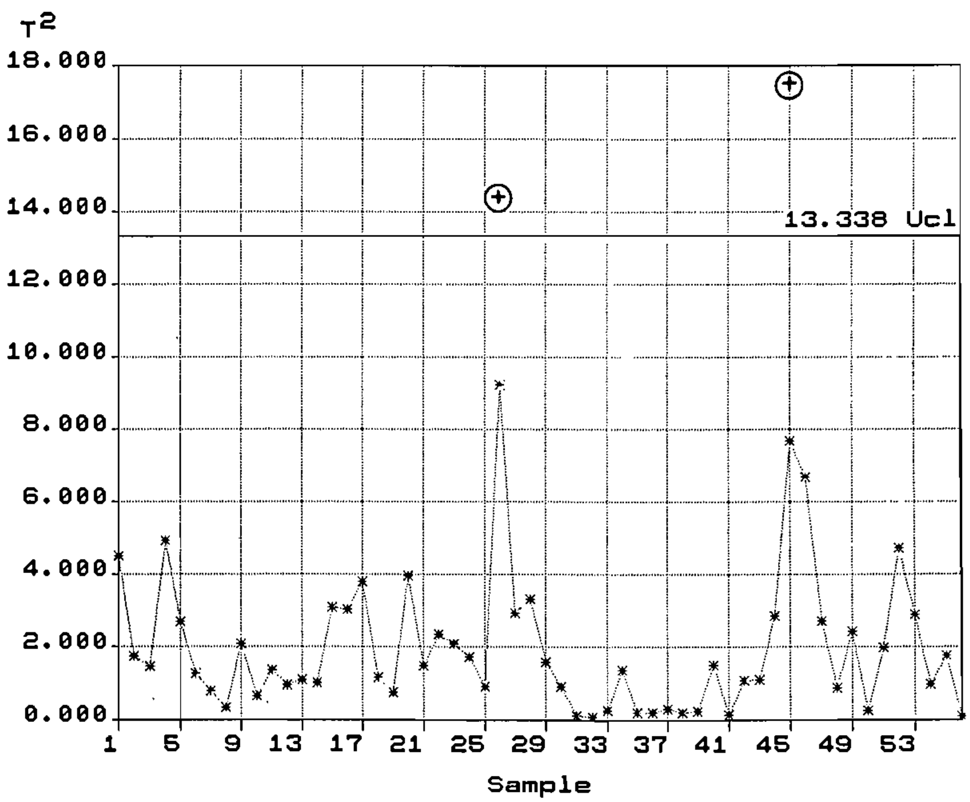

Figure 1. Particle size distribution T^2 chart.

Conclusion

The variance-covariance matrix determined through MSSD approach estimates what the real variance-covariance would have been for the process if there was no trend, cycle, etc., in the process. Thus, if the process is purely random, the variance-covariance matrix determined through the MSSD approach will be very close to the one calculated by the regular approach. If the process has some nonrandom patterns, the two approaches would differ. Thus, the MSSD approach will lead to quality improvements that might otherwise have been missed.

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