1) Given height is assumed to be Normal distribution with μ (unknown) and σ (known).

<u>Prior distribution</u> of μ: Normal distribution with μ_{pr} and σ_{pr} parameters.

$$P(\mu) = \frac{1}{\sqrt{2\pi}\,\sigma_{pr}} e^{-\left\{\frac{(\mu-\mu_{pr})^2}{2\sigma_{pr}^2}\right\}} \quad \text{---- } (1)$$

<u>Posterior distribution of μ</u>:

$$P(\mu|x) = \frac{P(\mu)\cdot P(X|\mu)}{P(x)}$$

$P(x)$ is constant for a given μ $\Rightarrow P(\mu|x) \propto P(\mu)\cdot P(x|\mu)$

$$\Rightarrow P(\mu|X) \propto P(\mu)\cdot P(x|\mu)$$

$$= \frac{1}{\sqrt{2\pi}\,\sigma_{pr}} e^{-\left\{\frac{(\mu-\mu_{pr})^2}{2\sigma_{pr}^2}\right\}} \cdot \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left\{\frac{(x_i-\mu)^2}{2\sigma^2}\right\}}$$

$$= \frac{1}{(2\pi)^{(n+1)/2}\,\sigma_{pr}\,\sigma^n} \exp\left\{\frac{-\mu^2+2\mu\mu_{pr}-\mu_{pr}^2}{2\sigma_{pr}^2} - \sum_{i=1}^{n}\frac{x_i^2-2\mu x_i+\mu^2}{2\sigma^2}\right\}$$

$$\doteq c\cdot\exp\left\{\left(-\mu^2(\sigma^2+n\sigma_{pr}^2)+2\mu(\mu_{pr}\sigma^2+\sigma_{pr}^2x_1+\cdots+\sigma_{pr}^2x_n) - (\mu_{pr}^2\sigma^2+\sigma_{pr}^2x_1^2+\cdots+\sigma_{pr}^2x_n^2)\right)\Big/2\sigma_{pr}^2\sigma^2\right\}$$

$$= C_1 \cdot \exp\left\{\left(-\mu^2 + 2\mu \frac{\left(\mu_{pr}\sigma^2 + \sum_{i=1}^{n}\sigma_{pr}^2 x_i\right)}{\sigma^2 + n\sigma_{pr}^2} - \left(\frac{\mu_{pr}\sigma^2 + \sum_{i=1}^{n}\sigma_{pr}^2 x_i}{\sigma^2 + n\sigma_{pr}^2}\right)^2\right) \Big/ 2\left(\frac{\sigma_{pr}^2\,\sigma^2}{\sigma^2 + n\sigma_{pr}^2}\right)\right\}$$

$$= C_1 \exp\left\{-\frac{\left(\mu - \left(\frac{\mu_{pr}\sigma^2 + \sum_{i=1}^{n}\sigma_{pr}^2 x_i}{\sigma^2 + n\sigma_{pr}^2}\right)\right)^2}{2\left(\frac{\sigma_{pr}^2\,\sigma^2}{\sigma^2 + n\sigma_{pr}^2}\right)}\right\}$$

$\Rightarrow$ It is a normal distribution with parameters

$N(\mu_1, \sigma_1^2)$

where $\sigma_1^2 = \dfrac{\sigma^2\,\sigma_{pr}^2}{\sigma^2 + n\sigma_{pr}^2} = \dfrac{1}{\sigma_{pr}^{-2} + n\sigma^{-2}}$

$$\mu_1 = \frac{\mu_{pr}\sigma^2 + \sum_{i=1}^{n}\sigma_{pr}^2 x_i}{\sigma^2 + n\sigma_{pr}^2} = \frac{\mu_{pr}\sigma_{pr}^{-2} + \sum_{i=1}^{n} x_i\,\sigma^{-2}}{\sigma_{pr}^{-2} + n\sigma^{-2}}$$

$$= \sigma_1^2\left(\mu_{pr}\sigma_{pr}^{-2} + \sum_{i=1}^{n} x_i\,\sigma^{-2}\right)$$

Maximum - a - Posteriori :

$\Rightarrow \text{argmax}_{\mu} [\text{Posterior of } \mu] = \text{argmax}_{\mu} [\log(P(\mu|x))]$

$= \text{argmax}_{\mu} [\log(P(\mu|x))]$

$= \text{argmax}_{\mu} \left[\log c + \log\left(\exp \frac{(\mu-\mu_1)^2}{2\sigma_1^2}\right)\right]$

$= \text{argmax}_{\mu} \left[(\mu - \mu_1)^2 / 2\sigma_1^2\right]$

where μ_1 & σ_1 are from previous derivations.

$$\text{max value} = \frac{2(\mu-\mu_1)}{2\sigma_1^2} = 0$$

$$\Rightarrow \boxed{\mu = \mu_1 = \frac{\mu_{pr}\,\sigma_{pr}^{-2} + \sum_{i=1}^{n} x_i \sigma^{-2}}{\sigma_{pr}^{-2} + n\,\sigma^{-2}}}$$

Expected - a - Posteriori

$$\Rightarrow E\left[P(\mu|x)\right] = \frac{\int_{-\infty}^{\infty} \mu \cdot e^{-\frac{(\mu-\mu_{Pr})^2}{2\sigma_{Pr}^2}} \cdot e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2} \cdot d\mu}{\int_{-\infty}^{\infty} e^{\frac{(\mu-\mu_{Pr})^2}{2\sigma_P^2}} \cdot e^{-\frac{1}{2\sigma^2}\sum(x_i-\mu)^2} d\mu}$$

$$= \frac{\int_{-\infty}^{\infty} \mu \cdot \exp\left[-\frac{(\mu-\mu_{Pr})^2}{2\sigma_{Pr}^2} - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right] \cdot d\mu}{\int_{-\infty}^{\infty} \exp\left[-\frac{(\mu-\mu_{Pr})^2}{2\sigma_{Pr}^2} - \frac{\sum_{i=1}^{n}(x_i-\mu)^2}{2\sigma^2}\right] d\mu}$$

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