3.(a)

Given Prior distribution : $P(\lambda) = \dfrac{\lambda^3 e^{-\lambda}}{6}$

Posterior distribution : $P(\lambda|x)$

$$P(\lambda|x) = \frac{P(\lambda) \cdot P(x|\lambda)}{P(x)}$$

$$= \frac{\left(\frac{\lambda^3 \cdot e^{-\lambda}}{6}\right)\left(\prod_{i=1}^{n} \lambda e^{-\lambda x_i}\right)}{\int_0^\infty \frac{\lambda^3 e^{-\lambda}}{6}\left(\prod_{i=1}^{n} \lambda e^{-\lambda x_i}\right) \cdot d\lambda}$$

$$= \frac{\lambda^{3+n} \cdot e^{-\lambda(1+x_1+x_2+\cdots x_n)}}{\int_0^\infty \lambda^{3+n} e^{-\lambda(1+x_1+x_2+\cdots x_n)} d\lambda}$$

$\Rightarrow$ let $\bar{x} = \dfrac{\sum_{i=1}^{n} x_i}{n} \Rightarrow \sum x_i = n\bar{x}$

$$\Rightarrow = \frac{\lambda^{3+n} e^{-\lambda(1+n\bar{x})}}{\int_0^\infty \lambda^{3+n} e^{-\lambda(1+n\bar{x})} \cdot d\lambda}$$

let $\lambda(1+n\bar{x}) = t$

$$\Rightarrow \quad d\lambda = \frac{dt}{(1+n\bar{x})} \quad \text{---} \; ①$$

let $(1+n\bar{x}) = \alpha$ ... ② | $(3+n) = \beta$ ... ③

substituting ①, ②, & ③

$$= \frac{\lambda^{\beta} e^{-\lambda\alpha}}{\int_0^{\infty} \left(\frac{t}{\alpha}\right)^{\beta} e^{-t} \cdot \frac{dt}{\alpha}} = \frac{\alpha^{\beta+1} \cdot \lambda^{\beta} \cdot e^{-\lambda\alpha}}{\int_0^{\infty} t^{\beta} e^{-t} \cdot dt}$$

$$= \frac{\alpha^{\beta+1} \cdot \lambda^{\beta} e^{-\lambda\alpha}}{\Gamma(\beta)} \quad \text{---} \; \text{Posterior distribution}$$

BAP of (λ)

$$= E\left[P(\lambda|x)\right] = \int_0^{\infty} \lambda \cdot \frac{\alpha^{\beta+1} \lambda^{\beta} e^{-\lambda\alpha}}{\Gamma(\beta)} \cdot d\lambda$$

$$= \frac{\alpha^{\beta+1}}{\Gamma(\beta)} \cdot \int_0^{\infty} \lambda^{\beta+1} \cdot e^{-\lambda\alpha} \cdot d\lambda$$

let $\lambda\alpha = t_0$

$$\Rightarrow \lambda = \frac{t_0}{\alpha} \Rightarrow d\lambda = \frac{1}{\alpha} dt_0$$

$$\Rightarrow \frac{\alpha^{\beta+1}}{\Gamma(\beta)} \cdot \int_0^\infty \left(\frac{t_0}{\alpha}\right)^{\beta+1} \cdot e^{-t_0} \cdot \frac{dt_0}{\alpha}$$

$$= \frac{\alpha^{\beta+1}}{\Gamma(\beta)} \frac{1}{\alpha^{\beta+2}} \int_0^\infty t_0^{\beta+1} e^{-t_0} . dt_0$$

$$\Rightarrow \frac{\Gamma(\beta+1)}{\alpha}$$

$$\boxed{E\lambda P = \frac{\Gamma(\beta+1)}{\alpha}}$$