20654

VI Semester B.Sc. Examination, May/June 2015
(Semester Scheme)
Mathematics (Paper – X)
Complex Analysis

Time : 3 Hours | Max. Marks : 80

Instructions : *1) Section* ***A*** *is* ***compulsory****.*
2) ***Five full*** *questions to be answered from Sections* ***B*** *and* ***C*** *choosing atleast* ***two*** *from* ***each*** *Section.*
3) ***All*** *questions in Sections* ***B*** *and* ***C*** *carry* ***equal*** *marks.*

SECTION – A

1. Answer **any ten** questions. **Each** question carries **two** marks.

a) Find the modulus and amplitude of $-2 + 5i$.

b) If z_1 and z_2 are two complex numbers then prove that $|z_1 + z_2| \leq |z_1| + |z_2|$.

c) What point set is represented by the equation $z^2 + (\bar{z})^2 = 2$?

d) Find the equation of the line through the points $z_1 = 2 + i$ and $z_2 = 3 - 2i$.

e) Find the area of the parallelogram whose two adjacent sides are z_1 and z_2.

f) Evaluate $\lim_{z \to 2+i} \left(\dfrac{z^2 - 2iz}{z^2 + 4} \right)$.

g) Show that the function $f(z) = e^{\bar{z}}$ is not analytic.

h) Show that an analytic function with constant imaginary part is constant.

i) Verify the function $u(x, y) = e^x \cos y$ is harmonic or not.

j) Evaluate $\int_0^{3+i} z^2 \, dz$ along the line $3y = x$.

k) Find $\int_C \dfrac{e^z}{z+1} dz$ where $C : |z| = 2$.

P.T.O.

l) State Liouville's theorem.

m) Find the Jacobian of the transformation $w = z^2$.

n) Find the fixed points of the transformation $w = \dfrac{2z-5}{z+4}$.

o) Determine the poles and its order of $f(z) = \dfrac{1}{z^3\,(z-1)}$.

SECTION – B

2. a) If the sum and product of two complex numbers are real then prove that either both complex numbers are real or one is the conjugate of other.

b) Show that the equation $|z + 3| + |z - 3| = 10$ represents an ellipse. Find its foci.

c) Find the equation of a circle which is described on the line joining the two points z_1 and z_2 as diameter.

3. a) Discuss the continuity of the function $f(z) = \begin{cases} \dfrac{\operatorname{Re} z^2}{|z|^2} & \text{for } z \neq 0 \\ 0 & \text{for } z = 0 \end{cases}$ at $z = 0$.

b) Find the derivative of $f(z) = \dfrac{2z-i}{z+2i}$ at $z = -i$ using the definition of derivative.

c) Derive Cauchy-Riemann equations in polar form.

4. a) If f(z) is analytic function then prove that

$$\left[\frac{\partial}{\partial x}|f(z)|\right]^2 + \left[\frac{\partial}{\partial y}|f(z)|\right]^2 = |f'(z)|^2.$$

b) If u and v are harmonic functions show that $(u_y - v_x) + i\,(u_x + v_y)$ is analytic.

c) Find the analytic function f(z) whose imaginary part is sinhx .siny and hence find the real part.

5. a) Evaluate $\int_{(0,3)}^{(2,4)} \left[(2y + x^2)\,dx + (3x - y)\,dy\right]$ along the curve $x = 2t$ and $y = t^2 + 3$.

b) Prove that $\int_C \frac{1}{(z-a)^n}\,dz = \begin{cases} 2\pi i & \text{for } n = 1 \\ 0 & \text{for } n = 2, 3, 4, \ldots \end{cases}$ where C is the circle $|z - a| = r$.

c) State and prove Cauchy's integral formula for an analytic function.

SECTION – C

6. a) Evaluate $\int_C \frac{\sin(\pi z^2) + \cos(\pi z^2)}{(z-1)(z-2)}\,dz$ where C is $|z| = 3$.

b) Evaluate $\int_C \frac{e^{2z}}{(z-2)^3}\,dz$ where $C : |z| = 4$.

c) State and prove Cauchy's inequality.

7. a) Prove that the equation of a circle with z_1 and z_2 as inverse points is $\left|\frac{z - z_1}{z - z_2}\right| = K$ where $K \neq 1$.

b) Show that the composition of two bilinear transformation is a bilinear transformation.

c) Show by means of the transformation $w = \frac{1}{z}$ the circle $|z - 3| = 5$ is mapped onto the circle $\left|w + \frac{3}{16}\right| = \frac{5}{16}$.

8. a) Prove that the transformation $w = \frac{(z - i)}{(z + i)}$ maps the upper half of the z-plane into the interior of the unit circle in w-plane.

b) Find the bilinear transformation which maps 0, –i, –1 onto i, 1, 0.

c) Discuss the transformation $w = \frac{1}{2}\left(z + \frac{1}{z}\right)$.

9. a) If f(z) has a simple pole (is pole of order 1) at z = a then prove that $\text{Re } f(a) = \underset{z \to a}{\text{Lt}} \left[(z - a)\, f(z)\right]$.

b) Evaluate $\int_C \frac{z^2 + 5}{(z - 2)(z - 3)}\, dz$ using Residue theorem where $C : |z| = 4$.

c) Using the calculus of residue, evaluate $\int_0^{2\pi} \frac{d\theta}{5 + 3\cos\theta}$.