Notations : b : impact parameter

p : Periastron distance

$x = \frac{r_s}{r} = r_s u$

$\xi = \frac{r_s}{b}$

$x_p = \frac{r_s}{p}$

- For Uncorrected Metric : $\left(\frac{du}{d\phi}\right)^2 = r_s\left(u^3 - \frac{u^2}{r_s} + \frac{1}{r_s b^2}\right) \Rightarrow \boxed{\left(\frac{dx}{d\phi}\right)^2 = x^3 - x^2 + \xi^2}$

Now, where $x = x_p = \frac{r_s}{p}$, $\frac{dx}{d\phi} = 0 \Rightarrow 0 = x_p^3 - x_p^2 + \xi^2 \Rightarrow \boxed{\xi^2 = x_p^2 - x_p^3}$

this result is the same as : $\boxed{b^2 = \frac{p^3}{p - r_s}}$

Part A

- For Corrected Metric :

$$\boxed{\left(\frac{dx}{d\phi}\right)^2 = G_0^N(x) + \tilde{t} H^N(x)}$$

where

$$G_0^N = x^3 - x^2 + \xi^2$$

$$h_{tt}^N = -2x^3\left(x^2 - \coth(1/x^2)\right)$$

$$h_{rr}^N = \frac{-2x^2}{(1-x)^2}\left(x^2 - \coth(1/x^2)\right)$$

$$h_{\phi\phi}^N = -\frac{2}{x}\left(x - \coth\left(\frac{1}{x}\right)\right)$$

$$H^N(x) = G_0^N\left[(1-x)h_{rr}^N - x^2 h_{\phi\phi}^N\right] + \xi^2\left[x^2 h_{\phi\phi}^N + \frac{h_{tt}}{1-x}\right]$$

Again at $x = x_p = \frac{r_s}{p}$, $\left(\frac{dx}{d\phi}\right) = 0$. Thus,

$$G_0^N(x_p) + \tilde{t} H^N(x_p) = 0$$

$$\Rightarrow (x^3 - x^2 + \xi^2) + \tilde{t}(x^3 - x^2 + \xi^2)\left[(1-x)h_{rr}^N - x^2 h_{\phi\phi}^N\right] + \tilde{t}\xi^2\left[x^2 h_{\phi\phi}^N + \frac{h_{tt}^N}{1-x}\right] = 0$$

$$\Rightarrow \xi^2\left[1 + \tilde{t}R(x) + \tilde{t}S(x)\right] = (x^2 - x^3)\left[1 + \tilde{t}R(x)\right]$$

, where $R(x) = (1-x)h_{rr}^N - x^2 h_{\phi\phi}^N$

$S(x) = x^2 h_{\phi\phi}^N + \frac{h_{tt}^N}{1-x}$

$$\Rightarrow \boxed{\xi^2 = (x^2 - x^3)\left[\frac{1 + \tilde{t}R(x)}{1 + \tilde{t}R(x) + \tilde{t}S(x)}\right]}$$

Expanding 'ξ' upto first order in $\tilde{t}$:

$$\xi = \sqrt{x^2 - x^3}\,(1 + \tilde{t}R)^{1/2}(1 + \tilde{t}R + \tilde{t}S)^{-1/2} = \sqrt{x^2 - x^3}\left(1 + \frac{\tilde{t}R}{2} + O(\tilde{t}^2)\right)\left(1 - \frac{\tilde{t}R}{2} - \frac{\tilde{t}S}{2} + O(\tilde{t}^2)\right)$$

$$= \sqrt{x^2 - x^3}\left[1 + \frac{\tilde{t}R}{2} - \frac{\tilde{t}R}{2} - \frac{\tilde{t}S}{2} + O(\tilde{t}^2)\right] = \sqrt{x^2 - x^3}\left(1 - \frac{\tilde{t}S}{2}\right) + O(\tilde{t}^2)$$

Thus, $\boxed{\xi = \sqrt{x_p^2 - x_p^3}\left[1 - \frac{\tilde{t}S(x_p)}{2}\right]}$ upto first order in '$\tilde{t}$'.

$\hookrightarrow$ This expression can directly be used in the codes

$$\Rightarrow \frac{r_s}{b} = \sqrt{\frac{r_s^2}{p^2} - \frac{r_s^3}{p^3}}\left[1 - \frac{\tilde{t}}{2} S(x_p)\right] \Rightarrow \boxed{b = \sqrt{\frac{p^3}{p - r_s}}\left[1 - \frac{\tilde{t}}{2} S(x_p)\right]^{-1}}$$

Expression for impact parameter for corrected metric

Part B

□ Estimate of b_c: (for corrected metric)

- let $\boxed{P_c = \frac{3}{2} r_s [1 + \alpha \tilde{t}]}$ $\Rightarrow b = \left[\frac{\frac{27}{8} r_s^3 (1+\alpha\tilde{t})^3}{\frac{r_s}{2}(1+3\alpha\tilde{t})}\right]^{1/2} \left(1 - \frac{\tilde{t}}{2} S(x_p)\right)^{-1}$

$$\Rightarrow b = \frac{3\sqrt{3}}{2} r_s \underbrace{(1+\alpha\tilde{t})^{3/2}(1+3\alpha\tilde{t})^{-1/2}}_{(1 + O(\tilde{t}^2))} \left(1 - \frac{\tilde{t}}{2} S\left(\frac{r_s}{P_c}\right)\right)^{-1} = \frac{3\sqrt{3}}{2} r_s \left[1 - \frac{\tilde{t}}{2} S\left(\frac{r_s}{P_c}\right)\right]^{-1} + O(\tilde{t}^2)$$

Here, $S(x) = x^2 h^N_{\phi\phi}(x) + \frac{h^N_{tt}}{1-x} = -2x\left[x - \coth\left(\frac{1}{x}\right)\right] - \frac{2x^3}{1-x}\left[x^2 - \coth\left(\frac{1}{x^2}\right)\right]$

$$h^N_{tt} = -2x^3\left(x^2 - \coth(1/x^2)\right)$$
$$h^N_{rr} = \frac{-2x^2}{(1-x)^2}\left(x^2 - \coth(1/x^2)\right)$$
$$h^N_{\phi\phi} = -\frac{2}{x}\left(x - \coth\left(\frac{1}{x}\right)\right)$$

$$\Rightarrow \boxed{S(x) = -2x^2 + 2x\coth\left(\frac{1}{x}\right) - \frac{2x^5}{1-x} + \frac{2x^3}{1-x}\coth\left(\frac{1}{x^2}\right)}$$

Now, $x_{p,c} = \frac{r_s}{P_c} = \frac{2r_s}{3r_s(1+\alpha\tilde{t})} = \frac{2}{3(1+\alpha\tilde{t})}$

$$\Rightarrow S(x_{p,c}) = -\frac{8}{9}(1+\alpha\tilde{t})^{-2} + \frac{4}{3}(1+\alpha\tilde{t})^{-1}\coth\left(\frac{3}{2}(1+\alpha\tilde{t})\right) - \frac{64}{243}(1+\alpha\tilde{t})^{-5}\left[1 - \frac{2}{3(1+\alpha\tilde{t})}\right]^{-1}$$
$$+ \frac{16}{27}(1+\alpha\tilde{t})^{-3}\left[1 - \frac{2}{3(1+\alpha\tilde{t})}\right]^{-1}\coth\left[\frac{9}{4}(1+\alpha\tilde{t})^2\right]$$

$$\Rightarrow S(x_{p,c}) = \left(-\frac{8}{9}\right) + \frac{4}{3}\coth\left(\frac{3}{2}\right) - \frac{64}{243} + \frac{16}{27}\coth\left(\frac{9}{4}\right) + O(\tilde{t}) = 0.9267 + O(\tilde{t})$$

Thus, $b_c = \frac{3\sqrt{3}}{2} r_s \left(1 + \frac{\tilde{t}}{2} S\right) \Rightarrow \boxed{b_c \approx \frac{3\sqrt{3}}{2} r_s \left(1 + 0.46335\,\tilde{t}\right)}$

Remarks: ① This term is independent of 'α'!

② This expression doesn't match the one given in Swayamsiddha's paper: $b_c = 3\sqrt{3} M [1 + 0.8066\,\tilde{t}]$