Midterm

Due date: October 14, 2019.

Rules of the game

- Use of internet is not allowed.
- You may use Lecture Notes and quote the relevant results. You can also use standard theorems from analysis.
- Logarithms are taken to the natural base.

Problems

1. Show the following:

   (a) Let $X_n \to X$ in measure (probability) and let $f : \mathbb{R} \mapsto \mathbb{R}$ be a continuous function. Then show that $f(X_n) \to f(X)$ in measure.

   (b) If $\mathrm{E}(|X_n - X|) \to 0$ as $n \to \infty$, show that $X_n \to X$ in measure.

2. Let $\mathcal{A}$ be an algebra or a field. Let $B \in \sigma(\mathcal{A})$. Show that there is a countable collection, $\mathcal{C}_B$, of sets in $\mathcal{A}$ such that $B \in \sigma(\mathcal{C}_B)$.

3. Show the following:

   (a) Prove that if Z is a Gaussian random variable with mean zero and variance 1 (standard Normal), show that for $x > 0$

   $$P(Z > x) \leq \frac{1}{x\sqrt{2\pi}} e^{-\frac{x^2}{2}}.$$

   (b) Let $X_1, .., X_n, ..$ be a sequence of independent and identically distributed random variables according to the distribution of Z (standard normal). Define

   $$L = \limsup_n \frac{X_n}{\sqrt{2 \log n}}$$

   Show that $\mathrm{P}(L \leq 1) = 1$.

4. Let $X_1, ..., X_n$ be a sequence of independent and identically distributed random variables with the density given by

$$f(x) = \frac{c}{(1+x^2)\log(2+x^2)}.$$

Let $U_n = \frac{X_1+\cdots+X_n}{n}$. Show that $\mathrm{E}(e^{itU_n}) \to 1$ as $n \to \infty$ for every finite t. Hence argue that $U_n \to 0$ in measure.
Note that X_n does not even have a well-defined mean