Homework 1: IERG 6300

Due date: Friday Sep 13, 2019.

Exercises

1. A probability measure is said to be *finitely additive* if $P(AUB) = P(A)+P(B)$ whenever $A \cap B = \emptyset$. A probability measure is said to be *countably additive* if $P(\cup_i A_i) = \sum_i P(A_i)$ whenever A_i's are pairwise disjoint. Show that a finitely additive probability measure on a σ-field $\mathcal{B}$ is countably additive *if and only if* it satisfies either of the following teo equivalent conditions:

   - If A_n is any non-increasing sequence of sets in $\mathcal{B}$ and $A = \cap_n A_n = \lim_n A_n$, then
   $$P(A) = \lim_n P(A_n).$$
   - If A_n is any non-decreasing sequence of sets in $\mathcal{B}$ and $A = \cup_n A_n = \lim_n A_n$, then
   $$P(A) = \lim_n P(A_n).$$

2. If P is a countably additive probability measure defined on σ-field $\mathcal{B}$, then for any $A_n \in \mathcal{B}$ we have $P(\cup_n A_n) \leq \sum_n P(A_n)$.

3. Prove: If $\{\omega_n : n \geq 1\}$ are distinct points in Ω and $p_n \geq 0$ are non-negative numbers with $\sum_n p_n = 1$, then
$$P(A) = \sum_{n:\omega_n \in A} p_n$$
defines a countably additive probability measure on the σ-field of all subsets of Ω.

4. Consider events $\{A_n\}$ in a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ that are almost pairwise disjoint, i.e. $\mathrm{P}(A_n \cap A_m) = 0$ whenever $n \neq m$. Show that
$$\mathrm{P}(\cup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mathrm{P}(A_n).$$

5. Define a probability space $(\Omega, \mathcal{F}, \mathrm{P})$ to be *non-atomic* whenever $\mathrm{P}(A) > 0, A \in \mathcal{F}$ implies that there exists $B \in \mathcal{F}, B \subset A$ such that $0 < \mathrm{P}(B) < \mathrm{P}(A)$. Suppose the space is *non-atomic*, $A \in \mathcal{F}$ with $\mathrm{P}(A) > 0$. Then show that:

   (a) For every $\epsilon > 0$ there exists $B \in \mathcal{F}, B \subseteq A$ such that $0 < \mathrm{P}(B) < \epsilon$.

   (b) If $0 < a < P(A)$ there exists $B \in \mathcal{F}, B \subset A$ such that $P(B) = a$. (Hint: Fix $\epsilon_n \downarrow 0$ and inductively define x_n and sets $G_n \in \mathcal{F}$ with $H_0 = \emptyset$, $H_n = \cup_{k<n} G_k, x_n = \sup\{\mathrm{P}(G) : G \subseteq A \setminus H_n, \mathrm{P}(H_n \cup G) \leq a\}$ and $G_n \subseteq A \setminus H_n$ such that $\mathrm{P}(H_n \cup G_n) \leq a$ and $\mathrm{P}(G_n) \geq (1 - \epsilon_n) x_n$. Consider $B = \cup_k G_k$.)

6. Given a collection of sets $\mathcal{A}$, let $\sigma(\mathcal{A})$ denote the smallest σ-field containing the elements of $\mathcal{A}$. Verify the following alternate definitions for Borel σ-field $\mathcal{B}_R$ of reals (i.e. show that all the following σ-fields are identical):

   - $\sigma(\{(a, b) : a < b \in \mathbb{R}\})$
   - $\sigma(\{[a, b] : a < b \in \mathbb{R}\})$
   - $\sigma(\{(-\infty, b) : b \in \mathbb{R}\}$
   - $\sigma(\{(-\infty, b) : b \in \mathbb{Q}\})$
   - $\sigma(\{\mathcal{O} \subset \mathbb{R} \text{ is open}\})$.

7. A σ-field $\mathcal{B}$ is said to be *countably generated* if there exists a countable collection of sets $\{A_i\}$ that generates $\mathcal{B}$. For any space Ω, let $\mathcal{G}$ consists of all $A \subset \Omega$ such that either A is a countable set or A^c is a countable set. Show that

   - $\mathcal{G}$ is a σ-field.
   - $\mathcal{G}$ is *countably generated* if and only if Ω is a countable set.