THE CHINESE UNIVERSITY OF HONG KONG

Course Examination, 2nd Term 2018-2019

Course Code and & Title: IERG 6154, *Network Information Theory*

Time Allowed: 1 week (take-home)

Student I.D.: ____________________ **Seat No.**: ___________

Question 1 [10 points]: Inequality

Let X be a random variable with $\mathbf{E}(X^2) < \infty$. Let $Z_1, \ldots, Z_k$ be a set of mutually independent Gaussian random variables with finite variances that are also independent of X. Prove that

$$h\left(\frac{X + Z_1 + Z_2 + \cdots + Z_k}{\sqrt{k+1}}\right) \geq \frac{1}{k+1}\left(h\left(\frac{\sum_j Z_j}{\sqrt{k}}\right) + \sum_{i=1}^{k} h\left(\frac{X + \sum_{j,j\neq i} Z_j}{\sqrt{k}}\right)\right)$$

In the above $h(X)$ denotes the differential entropy. (Hint: Use the Entropy Power Inequality)

Question 2 [30 points]: Broadcast Channel

Consider a broadcast channel depicted by the following figure:

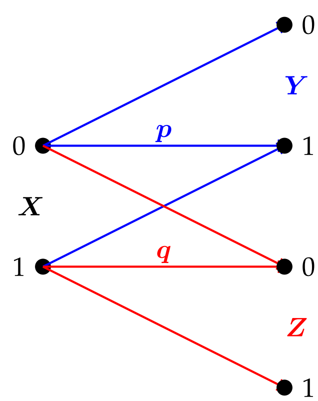

Figure 1: Skew-symmetric-Z-broadcast channel

(a) [20 points] Show that if (p, q) satisfy

$$q \geq \frac{p}{p^2 + (1-p)^2 e^{\frac{H(p)}{1-p}}}.$$

then the sum-capacity of this broadcast channel is given by $C_1 = \max_{p(x)} I(X;Y)$. In the above $H(p) = -p\log(p) - (1-p)\log(1-p)$ is the binary entropy function and logarithms are taken to their natural base.
Hint: Consider the upper bound on the sum-rate given by

$$R_1 + R_2 \leq \max_{p(u,x)} I(U;Y) + I(X;Z|U).$$

(b) [10 points] Show that the above condition is necessary as well for the sum-capacity of this broadcast channel to be given by $C_1 = \max_{p(x)} I(X;Y)$.

Hint: Consider the lower bound, valid when $\beta \geq 1$, on the achievable weighted sum-rate given by

$$\beta R_1 + R_2 = \min_{\alpha \in [0,1]} \max_{p(x)} (\beta - \alpha) I(X;Y) + \alpha I(X;Z) + \mathcal{C}_{\mu_X} \left[-(\beta-\alpha) I(X;Y) - \alpha I(X;Z) + \max_{p(u,v|x)} \{\beta I(U;Y) + I(V;Z) - I(U;V)\} \right].$$

Further it is known that it suffices to restrict to $(U,V) : |U| + |V| \leq |X| + 1$. Hence in this case, the inner maximization reduces to $\max\{\beta I(X;Y), I(X;Z)\}$. The notation $\mathcal{C}_{\mu_X}$ denotes the upper concave envelope when the channels are fixed and the inner functional is determined by μ_X.

Question 3 [10 points]: Inequality

For both parts please assume that X and Z are independent.

(a) [5 points] Let $X \sim \mathcal{N}(0,P)$ and $Z \sim \mathcal{N}(0,N)$. Then show that

$$\sup_{U: U \to X \to X+Z} \frac{I(U;X+Z)}{I(U;X)} = \frac{P}{P+N}.$$

Hint: Use the Entropy Power Inequality

(b) [5 points] Let $X \sim \mathcal{N}(0,P)$ and $Z \sim \mathcal{N}(0,N)$. Then show that

$$\sup_{U: U \to X+Z \to X} \frac{I(U;X)}{I(U;X+Z)} = \frac{P}{P+N}.$$

Question 4 [10 points]: Interference Channel

Consider an interference channel obtained as follows. Let X_1, X_2 be binary random variables taking values in $\{0,1\}$. Let $\oplus$ denote the sum (modulo 2). Let $\hat{Y}_1 = X_1 \oplus X_2$ and $\hat{Y}_2 = X_2$. The real receivers Y_1 and Y_2 are obtained by passing $\hat{Y}_1$ and $\hat{Y}_2$ through symmetric BECs, with erasure probabilities ε_1 and ε_2 respectively. Find the capacity region of this interference channel.