Summary of selected works

Chandra Nair

Introduction

In this document I provide summary of the five selected works (published in the last six calendar years) for review. The summary consists of the abstract as well as a small paragraph detailing what I think are the principal contributions of this work.

1. An Outer Bound to the Capacity Region of the Broadcast Channel [1]

   Abstract: An outer bound to the capacity region of the two-receiver discrete memoryless broadcast channel is given. The outer bound is tight for all cases where the capacity region is known. When specialized to the case of no common information, this outer bound is contained in the Körner-Marton outer bound. This containment is shown to be strict for the *binary skew-symmetric* broadcast channel. Thus, this outer bound is in general tighter than all other known outer bounds.

   Comments: This paper has two components: (i) the derivation of an outer bound (later came to be referred to as the Nair–El Gamal outer bound, see Chapter 8 in [2]) for the two-receiver broadcast channel with private and common messages, (ii) to show that the new outer bound strictly improves on the previously best-known outer bound due to Korner and Marton [3] (which was considered to be the best in the literature for about thirty years). The first part essentially mimics the arguments in [4] and is not very novel. The second part is indeed the more valuable contribution as it implies that Marton's inner bound[3] and Korner and Marton's outer bound do not necessarily coincide for the broadcast channel. Among the interesting new techniques introduced in this paper is the argument used in Lemma 3.2. The proof technique here has been used in several other papers [5, 6, 7] to show equivalence between various different appearing regions involving auxiliary random variables. The use of the binary skew-symmetric channel as an example where the bounds differ has spurned a great deal of interest in this particular channel. Further this work has rekindled the interest in discrete-memoryless broadcast channels, an important open problem, that remained dormant for a long time. Recently, in [8] we have shown that the outer bound derived in this paper is strictly sub-optimal for some classes of broadcast channels.

2. Proof of the local REM conjecture for number partitioning I: Constant energy scales [9] **and** Proof of the local REM conjecture for number partitioning II: Growing energy scales [10]

   Abstract - Part I: In this paper we consider the number partitioning problem (Npp) in the following probabilistic version: Given n numbers $X_1, \ldots, X_n$ drawn i.i.d. from some distribution, one is asked to find the partition into two subsets such that the sum of the numbers in one subset is as close as possible to the sum of the numbers in the other set. In this probabilistic version, the Npp is equivalent to a mean-field antiferromagnetic Ising spin glass, with spin configurations corresponding to partitions, and the energy of a spin configuration corresponding to the weight difference.

   Although the energy levels of this model are *a priori* highly correlated, a surprising recent conjecture of Bauke, Franz and Mertens asserts

that the energy spectrum of number partitioning is locally that of a random energy model (REM): the spacings between nearby energy levels are uncorrelated. More precisely, it was conjectured that the properly scaled energies converge to a Poisson process, and that the spin configurations corresponding to nearby energies are asymptotically uncorrelated. In this paper, we prove these two claims, collectively known as the local REM conjecture.

Abstract - Part II: We continue our analysis of the number partitioning problem with n weights chosen i.i.d. from some fixed probability distribution with density ρ. In Part I of this work, we established the so-called local REM conjecture of Bauke, Franz and Mertens. Namely, we showed that, as $n \to \infty$, the suitably rescaled energy spectrum above some *fixed* scale α tends to a Poisson process with density one, and the partitions corresponding to these energies become asymptotically uncorrelated. In this part, we analyze the number partitioning problem for energy scales α_n that *grow with* n, and show that the local REM conjecture holds as long as $n^{-1/4}\alpha_n \to 0$, and fails if α_n grows like $\kappa n^{1/4}$ with $\kappa > 0$.

We also consider the SK-spin glass model, and show that it has an analogous threshold: the local REM conjecture holds for energies of order $o(n)$, and fails if the energies grow like κn with $\kappa > 0$.

Comments: In these two papers we establish some mathematical properties of partitions that have small differences (or minimum load imbalances). In particular we show that they behave *as if* every partition were assigned a uniformly random load from a Gaussian distribution. The results imply that the space of partitions have a large number of local maxima and these local maxima are very well separated (in hamming distance). This mathematically justifies the Random Energy Model (REM) conjecture which yields a nice intuitive explanation as to why the performance of the best algorithms in this problem yield answers exponentially far away from the optimal answer. The technique for showing Poisson convergence is using convergence of factorial moments, and to do this we establish certain strong forms of local limit theorems which were not previously known.

3. The Capacity Region of a Class of 3-Receiver Broadcast Channels with Degraded Message Sets [11]

Abstract: Körner and Marton [12] established the capacity region for the 2-receiver broadcast channel with degraded message sets. Recent results and conjectures suggest that a straightforward extension of the Körner-Marton region to more than 2 receivers is optimal. This paper shows that this is not the case. We establish the capacity region for a class of 3-receiver broadcast channels with 2-degraded message sets and show that it can be strictly larger than the straightforward extension of the Körner-Marton region. The idea is to split the private message into two parts, superimpose one part onto the "cloud center" representing the common message, and superimpose the second part onto the resulting "satellite codeword". One of the receivers finds the common message directly by decoding the "cloud center," a second receiver finds it *indirectly* by decoding a satellite codeword, and a third receiver by jointly decoding the transmitted codeword. This idea is then used to establish new inner and outer bounds on the capacity region of the general 3-receiver broadcast channel with two and three degraded message sets. We show that these bounds are tight for some nontrivial cases. The results suggest that finding the capacity region of the 3-receiver broadcast channel with degraded message sets is as at least as hard finding as the capacity region of the general 2-receiver broadcast channel with common and private message.

Comments: In this paper we disprove a wide-held belief (for close to thirty years) and conjecture that superposition coding is optimal for a three receiver scenario consisting of two receiver who want a common message and a third receiver who wants the common and a private message. The idea of indirect decoding shows that the extension of capacity results from two-receivers to three-receivers is not straightforward. The other interesting part of the paper is in showing the equivalence of various different looking regions. Showing strict improvement needs exact evaluation of various regions, which is not usually done (except for Gaussian settings) and the techniques and the candidate channels used here can be exploited further. There has been a fair amount of follow-up work. Using this idea, one can also show[13] that certain other believed to be optimal extensions like the compound Gelfand-Pinsker are strictly sub-optimal, as well. The key results of this paper are presented in Chapter 8 of [2].

4. Capacity regions of two new classes of 2-receiver broadcast channels [14]

   Abstract: Motivated by a simple broadcast channel, we generalize the notions of a *less noisy* receiver and a *more capable* receiver to an *essentially less noisy receiver* and an *essentially more capable* receiver respectively. We establish the capacity regions of these classes by borrowing on existing techniques; however these new classes contain additional interesting classes of broadcast channels, including the BSC/BEC broadcast channel. We also establish the relationships between the new classes and the existing classes.

   Comments: In this paper two new classes of broadcast channels is defined and using these classes determine the capacity region of BSC/BEC broadcast channel. This is the first extension of the capacity results for classes of broadcast channels (discrete) since the 70s. There are some new elements of proof introduced here like the *symmetrization argument* used in the evaluation of the outer bounds. This has helped one bypass the use of Mrs. Gerber's lemma. More significantly, the symmetrization argument also helps one evaluate regions of more general classes of binary input symmetric output broadcast channels [15] which was not possible using Mrs. Gerber's lemma. This argument was the key behind an investigation which resulted in the results in [16] where a similar argument is used in the Gaussian setting, bypassing the use of entropy power inequality. This paper also shows very natural examples which show the strict inclusions between degraded, less noisy, and more capable broadcast channels. The results of this paper are reproduced (in parts) in Chapters 5 and 8 of the book [2].

5. The capacity region of the three receiver less noisy broadcast channel [17]

   Abstract: We determine the capacity region of a 3-receiver less noisy broadcast channel. The difficulty in extending the two-receiver result to three-receivers involves extending the Csiszar-sum lemma to three or more sequences, a standard difficulty in this area. In this work we bypass the difficulty by using a new information inequality, for less noisy receivers, that is employed in the converse. We also generalize our result to obtain the capacity region for a class of less noisy receivers.

   Comments: In this paper we establish the capacity region of the three-receiver broadcast channel, solving a problem open since the mid 70s. Contrary to the results in [11], in this work we show that the natural extension of the optimal coding strategy for two receivers remains optimal when going to three-receiver setting, however our analysis techniques does not allow us to go beyond three receivers. This demonstrates a deficiency in the current converse techniques as all intuition suggests that superposition coding remains optimal beyond three receivers. Note that in [18] we showed that for the more capable ordering superposition coding is strictly sub-optimal for three receivers. This result is presented as a guided exercise in Chapter 5 of [2].

Other significant results

In this section I will present the results in three other papers which I consider to be as significant or more than some of the results in the earlier section. However, though all of them have conference versions published, journal versions of one has just been accepted and the other two are being reviewed. The full versions can be accessed at my website http://chandra.ie.cuhk.edu.hk.

(a) An information inequality and evaluation of Marton's inner bound for binary input broadcast channels [19]

Abstract: We establish an information inequality that is intimately connected to the evaluation of the sum rate given by Marton's inner bound for two receiver broadcast channels with a binary input alphabet. The inequality implies that randomized time-division strategy indeed achieves the sum rate of Marton's inner bound for all binary input broadcast channels. To deduce this, we produce a new cardinality bound for evaluating the sum-rate for Marton's inner bound for all broadcast channels. Using these tools we explicitly evaluate the inner and outer bounds for the binary skew-symmetric broadcast channel and demonstrate a gap between the bounds; in the process correcting the evaluation of the outer bound to an earlier published result. This is the first example where the gaps between the best inner and outer bounds are explicitly determined.

Comments: In this paper we establish a new information equality for a quintuple of random variables where one of them has a binary cardinality. This information inequality is established by a careful perturbation analysis in the space of probability distributions to isolate the local maxima. This inequality allows one to represent Marton's inner bound (sum-rate point) for binary input broadcast channels using a much simpler representation. This again follows the theme of recent results [20, 14] establishing that the boundary of Marton's inner bound can be computed using simple choices of auxiliary random variables. This work has led us to search for potentially much simpler representations of Marto's inner bound and to utilize such representations to prove its optimality. This work has led to some new results [21] and some new conjectures [22].

(b) On Marton's inner bound and its optimality for classes of product broadcast channels [8] (conference version)

Abstract: Marton's inner bound is the tightest known inner bound on the capacity region of the broadcast channel. It is not known, however, if this bound is tight in general. One approach to settle this key open problem in network information theory is to investigate the multi-letter extension of Marton's bound, which is known to be tight in general. This approach has become feasible only recently through the development of a new method for bounding cardinalities of auxiliary random variables by Gohari and Anantharam. This paper undertakes this long overdue approach to establish several new results, including (i) establishing the optimality of Marton's bound for new classes of product broadcast channels, (ii) showing that the best known outer bound by Nair and El Gamal is not tight in general, and (iii) finding sufficient conditions for a global maximizer of Marton's bound that imply that the 2-letter extension does not increase the achievable rate. Motivated by the new capacity results, we establish a new outer bound on the capacity region of product broadcast channels in general.

Comments: In this paper we establish that the Nair–El Gamal outer bound is strictly suboptimal for the broadcast channel by showing a class of product channels for which Marton's inner bound is optimal and is strictly contained inside the outer bound. The paper also introduces several new ideas and techniques, most notable a new min-max interchange that allows one to focus on a quantity called the λ-sum-rate. We then develop an approach called factorization of concave envelopes over product channels which would yield the optimality (or not) of Marton's inner bound.

(c) The capacity region of the two-receiver vector Gaussian broadcast channel with private and common messages [16] (conference version)

Abstract: We develop a new method for showing the optimality of the Gaussian distribution in multiterminal information theory problems. As an application of this method we show that Marton's inner bound achieves the capacity of the two-receiver vector Gaussian broadcast channels with private and common messages, solving an open problem of considerable interest.

Comments: In this work we utilize a combination of ideas introduced in the discrete world such as factorization of concave envelopes [8], symmetrization argument [14] to derive at a new method for proving the optimality of the Gaussian distribution in multiterminal information theory problems. This technique bypasses the use of Entropy Power Inequality and can be applied to problems where Entropy Power Inequality could not be applied. Apart from solving the open problem (see open problem 9.3 in [2]), it also vastly simplifies the existing proofs and hence answer question 9.2 in [2]. In particular, this work also lays clear the importance of the various results and insights that we obtained in the discrete world.

References

[1] C. Nair and A. El Gamal, "An outer bound to the capacity region of the broadcast channel," *International Symposium on Information Theory*, pp. 2205–2209, 2006.

[2] A. El Gamal and Y.-H. Kim, *Network Information Theory*. Cambridge University Press, 2012.

[3] K. Marton, "A coding theorem for the discrete memoryless broadcast channel," *IEEE Trans. Info. Theory*, vol. IT-25, pp. 306–311, May, 1979.

[4] A. El Gamal, "The capacity of a class of broadcast channels," *IEEE Trans. Info. Theory*, vol. IT-25, pp. 166–169, March, 1979.

[5] C. Nair and Z. V. Wang, "On the inner and outer bounds for 2-receiver discrete memoryless broadcast channels," *Proceedings of the ITA Workshop*, 2008.

[6] A. Gohari, A. El Gamal, and V. Anantharam, "On an outer bound and an inner bound for the general broadcast channel," *International Symposium on Information Theory*, 2010.

[7] C. Nair, "A note on outer bounds for broadcast channel," *Presented at International Zurich Seminar*, 2010.

[8] Y. Geng, A. Gohari, C. Nair, and Y. Yu, "The capacity region of classes of product broadcast channels," *Proceedings of IEEE International Symposium on Information Theory*, pp. 1549–1553, 2011.

[9] C. Borgs, J. T. Chayes, S. Mertens, and C. Nair, "Proof of the local rem conjecture for number partitioning I: Constant energy scales," *Random Structures and Algorithms*, vol. 34(2), pp. 217–240, 2009.

[10] C. Borgs, J. T. Chayes, S. Mertens, and C. Nair, "Proof of the local rem conjecture for number partitioning II: Growing energy scales," *Random Structures and Algorithms*, vol. 34(2), pp. 241–284, 2009.

[11] C. Nair and A. El Gamal, "The capacity region of a class of three-receiver broadcast channels with degraded message sets," *Information Theory, IEEE Transactions on*, vol. 55, pp. 4479–4493, Oct. 2009.

[12] J. Körner and K. Marton, "General broadcast channels with degraded message sets," *IEEE Trans. Info. Theory*, vol. IT-23, pp. 60–64, Jan, 1977.

[13] C. Nair, A. El Gamal, and Y.-K. Chia, "An achievability scheme for the compound channel with state noncausally available at the encoder," *CoRR*, vol. abs/1004.3427, 2010.

[14] C. Nair, "Capacity regions of two new classes of two-receiver broadcast channels," *Information Theory, IEEE Transactions on*, vol. 56, pp. 4207–4214, sep. 2010.

[15] Y. Geng, C. Nair, S. Shamai, and Z. V. Wang, "On broadcast channels with binary inputs and symmetric outputs," *International Symposium on Information Theory*, 2010.

[16] Y. Geng and C. Nair, "The capacity region of the two-receiver vector gaussian broadcast channel with private and common messages," in *2012 IEEE International Symposium on Information Theory (ISIT'2012)*, (Cambridge, Massachusetts, USA), pp. 591–595, July 2012.

[17] C. Nair and Z. Wang, "The capacity region of the three receiver less noisy broadcast channel," *Information Theory, IEEE Transactions on*, vol. 57, pp. 4058 –4062, july 2011.

[18] C. Nair and L. Xia, "On Three-Receiver more capable channels," in *2012 IEEE International Symposium on Information Theory (ISIT'2012)*, (Cambridge, Massachusetts, USA), pp. 383–387, July 2012.

[19] Y. Geng, V. Jog, C. Nair, and Z. V. Wang, "An information inequality and evaluation of marton's inner bound for binary input broadcast channel," *IEEE Trans. Info. Theory (accepted)*.

[20] H. Weingarten, Y. Steinberg, and S. Shamai, "The capacity region of the gaussian multiple-input multiple-output broadcast channel," *Information Theory, IEEE Transactions on*, vol. 52, pp. 3936 –3964, sept. 2006.

[21] Y. Geng, A. Gohari, C. Nair, and Y. Yu, "On Marton's inner bound for two receiver broadcast channels," *Presented at ITA Workshop*, 2011.

[22] A. A. Gohari, C. Nair, and V. Anantharam, "On marton's inner bound for broadcast channels," in *2012 IEEE International Symposium on Information Theory (ISIT'2012)*, (Cambridge, Massachusetts, USA), pp. 586–590, July 2012.