Statement of Research

Chandra Nair

Introduction

My research addresses fundamental problems in areas that use probability, combinatorics, and random structures. In particular, I have contributed to several problems in combinatorial optimization, information theory, algorithms, and networks.

My research contributions as a doctoral student at Stanford and later as a post-doctoral fellow at Microsoft research have been to resolve several conjectures in combinatorial optimization made by statistical physicists [1, 2, 3]. I have also worked on the design and analysis of algorithms arising in various network scenarios [4, 5]. Over the past five years at CUHK I have been mainly working in a completely different area: on problems in multiuser information theory, in particular the broadcast channel, which is one of the key open problems in the field. Here I have contributed several key results and insights that have opened up a multitude of promising and exciting avenues to explore in the near future.

Research Details

I will describe my research contributions in some detail. The names of my coauthors on these contributions can be found in the list of references appended at the end.

Discrete Optimization: In my doctoral research I resolved a set of conjectures related to the *assignment problem* [6, 1] which were put forth by statistical physicists based on heuristics. The assignment problem studies a question regarding properties of matchings. Indeed, matchings turn up in many application areas across a variety of engineering disciplines; for example, minimum cost assignment of jobs to machines, maximum weight matchings in network switches, or selecting non-interfering links in a wireless network for simultaneous communication.

Of central interest is the following question: Given N jobs and N machines and (exponentially distributed) independent random cost values of executing each job i on each machine j, what is the average value of the minimum cost assignment? This question has attracted a lot attention since the 1960s in various communities, and a number of attempts were made to solve it. In 1985, Mezard and Parisi used the non-rigorous *replica method* of statistical physics to claim that the exact answer is $\pi^2/6$, in the limit as N tends to infinity.

The rigorous techniques had yielded decreasing upper bounds of 3(Walkup '79), 2(Karp '84), 1.98(Coppersmith-Sorkin '98) and increasing lower bounds of $1+\frac{1}{e}$(Lazarus '79), 1.441(Goemans-Kodialam '93). The exact limit was established rigorously by Aldous in 2001.

Moreover Parisi had conjectured more strikingly that, in fact, for every N the average minimum cost ought to be $\sum_{i=1}^{N}\frac{1}{i^2}$. Parisi's conjecture was later generalized by Coppersmith and Sorkin. My doctoral thesis resolved these (reasonably notorious) conjectures for every finite N and involved the development of new probabilistic and combinatorial arguments.

Subsequently, I looked at the *number partitioning problem*, a classical problem concerning the equal division of a set of n randomly chosen numbers into two bins. This problem is one of the six basic NP-complete problems and is used in various ways to model load balancing and makespan problems. Conjectures were made in the partition problem regarding the structure of the solution space using methods from statistical physics. In general, the physicists believe that there is a correlation between the structure of the solution space of a problem and the hardness of approximation. So establishing the structural claims by the physicists is a first step towards a mathematical underpinning of this relationship.

During my post-doctoral research, we resolved the conjectures related to the minimal "energies" in the partitioning problem. The conjectures were related to the Poisson convergence of an ordered sequence of load imbalances corresponding to the various possible partitions. In [2] we established this Poisson convergence for the ordered spectrum of load imbalances (energy values in the language of statistical physics) at levels that did not scale with the number of jobs in the system. In [3], extending the methods of [2], we were able to show Poisson convergence at energy levels that scaled with n, the number of jobs in the system. We disproved the belief that Poisson convergence occurred for all energy values up to $o(\sqrt{n})$; however we showed that Poisson convergence did occur for values until $o(n^{0.25})$ and failed at $\Theta(n^{0.25})$. We also demonstrated that our techniques could establish Poisson convergence in other spin glass models such as the Sherrington-Kirkpatrick model. In the process of establishing these results we had to prove generalizations of local limit theorems which could be of independent interest.

Information Theory: My interest in information theory began from two class-projects during my final year at Stanford. In [7] we computed the entropy rate of a hidden Markov model in a rare transitions regime using martingale techniques.

The second class project led to a longer term interest in the *broadcast channel*; a problem of one transmitter trying to reliably communicate different messages to multiple receivers. The capacity region for this channel is a key open problem and a large portion of my research in this area has been devoted to the evaluation and comparison of the existing inner and outer bounds as well as developing some new outer bounds. To put my work into perspective, for roughly thirty years, there was no serious effort to compare the best known inner bound (Marton '79) and the commonly used outer bound (Korner-Marton '79) to the capacity region of the broadcast channel. This was primarily because of the difficulty in explicit evaluation of the inner and outer bounds that were expressed as unions over auxiliary random variables. Indeed the celebrated case of the MIMO Gaussian broadcast channel [8] can be regarded as an exercise (a tedious and non-trivial one) of showing that the inner and outer bounds coincided for this class. In particular, it was not even known if the bounds evaluated to different regions.

In [9] we improved on the Korner-Marton outer bound by showing that it is *strictly* inferior to UV-outer bound (more commonly referred to as the Nair–El Gamal outer bound [10]), thereby showing that the inner bound and the outer bound did not match in general and that Korner-Marton outer bound was weak. In [11], we identified a channel for which the best inner bound and the UV-outer bound may potentially differ and conjectured an information inequality, which if proved, would quantify the gap between the bounds. Subsequent to this work, Gohari and Anantharam[12] used perturbation ideas to convert Marton's inner bound to a finite dimensional optimization problem, and using the same approach showed that the inner and outer bounds differed for our candidate channel. In [13] we extended the perturbation ideas and established the conjectured information inequality thus quantifying the gap between the bounds for the candidate channel. In [14] we established a vast generalization of the conjectured information inequality that allowed one to compute the sum-rate of Marton's inner bound for every binary input broadcast channel in an explicit manner. Building on this work, in [15] we identified more classes of broadcast channels where the best inner and outer bounds differed. The next natural step was to identify which of the bounds was strictly sub-optimal (possibly both). In [16, 17] we showed that the UV-outer bound is strictly sub-optimal using a novel approach.

In a serendipitous occurrence, the techniques developed in [17] along with the intuition from a symmetrization argument [18], led to a novel way of establishing the optimality of Gaussian distributions in certain key optimization problems involving auxiliary random variables in multiuser information theory settings. This new technique immediately resulted in settling a central open problem: determining the capacity region of a MIMO Gaussian broadcast channel with private and common messages [19]. This technique also vastly simplifies current proofs of existing results and has immense potential in new situations involving channels with Gaussian noise.

Another collection of problems that I have been involved with has to do with the generalization of existing results from two to three or more receivers. The normal belief (often justified) is that the inner bounds (achievable strategies) are optimal and it is the techniques for devising outer bounds that form the main bottleneck. In [20] we showed an example where a conjectured-to-be-optimal achievable region for a degraded message setting can be strictly improved using a idea called indirect decoding. This idea has also been applied to other similar scenarios; for instance in a compound Gelfand-Pinsker channel setting[21]. In [22] we established the capacity region for a sequence of three less noisy receivers, a question that was unresolved since the two-receiver region was established in '76. In [18] I established the capacity region of some new classes of two receiver broadcast channels where the main idea was to focus on the extremal auxiliary distributions and use their properties. In [23], we showed a strict gap between the best known inner and outer bounds for a three receiver degraded message setting, and then showed that in a particular example the inner bound was tight and the outer bound was strictly loose. The importance of this work lies in the use of a new object, quasi-codebooks, to fashion a converse.

While all of the above results focus primarily on the broadcast channel, the techniques and ideas developed have been applied to many other scenarios such as communication with secrecy constraints. In particular we have applied some of the ideas to interference channels [24] and we are working on a few other extensions as well.

Networks: I have been very interested in problems and algorithms that arise in wireless and data networks. In [4], I collaborated on the fluid model analysis for the performance prediction of an algorithm designed for controlling congestion at internet routers. In [5], I demonstrated the optimality of an algorithm, MoveRight, for the energy-efficient scheduling of the transmission of packets in wireless networks. In general this has been an area in which I have kept up a more-than-passing interest.

Future Research

The approach I used to understand the broadcast channel illustrates my research philosophy, which is to break a problem into suitably chosen special cases; instances that necessitate a novel reasoning yet remain in the realm of tractability. By working on and solving these special cases, I hope to build a better understanding of the landscape. This is also the manner in which my doctoral dissertation was done; where we identified many new properties of matchings and then strung together a subset of them to derive at the final conclusion.

In the short-term the tricks and techniques developed to evaluate bounds for the broadcast channel can be used to make progress in other fundamental open problems. Indeed the techniques developed to evaluate bounds have exposed several new branches of low-hanging-fruits in related network information theory problems. While not all of them may be interesting, they are a few that will induce a refinement of certain key pieces of intuition.

In the medium-term horizon I would like to make progress using some broader insights and results that I have obtained using my study of the broadcast channel problem. First, a better understanding of the extremal auxiliary distributions is required in the following sense: on several different instances of computing the extremal distributions, after tedious manipulations, we arrive at a simple structure. Intuitively, it is still unclear why only these simple structures arise. Answering this will go a long way in getting a better understanding of the landscape. We have also devised a different representation of the inner and outer bounds using concave envelopes of certain functions and a natural factorization property of these concave envelopes is required to show the optimality of the inner bounds. It is using these novel ideas and approaches that we were able to fashion a new and fundamentally different technique[19] to establish the optimality of Gaussian distributions in certain settings. The only previously known-technique of establishing the optimality of Gaussian involved applications of entropy power inequality or some of the key lemmas that led to the entropy power inequality. However it could not solve some important scenarios despite serious effort by researchers for a significant period of time. The power of our new method can be understood from the fact that it could solve long-standing open problems and make the current proofs of the known cases much simpler.

The aforementioned factorization properties of the concave envelopes is an idea that seems to be fundamental in proving optimality of various coding strategies. Indeed the concave envelope representations, that capture the essence of the properties of the extremal distributions, along with the factorization ideas seem to open up some of the hard and long-standing problems. Hence it is very important to understand these "factorization properties" of the concave envelopes which seem to be at the heart of all converse arguments. All of these directions go well beyond the broadcast channel in the same way that auxiliary random variables, first introduced by Cover[25] for the broadcast channel, found widespread use.

Working in two different fields have allowed me to observe very high level similarities between certain elements in both fields. One of the most curious observations is this occurrence of seemingly simple structures as solutions to non-trivial optimization problems. In the combinatorial optimization world this is the key intuition behind the heuristic 'replica method'. In information theory we do not have such a general approach, yet often after tedious calculations simple structures pop out as solutions. Although currently I am focusing on what I can prove formally in the information theory landscape, it is highly plausible that there is a very general theory that links up solutions of certain new families of optimization problems. Indeed the possibility that there is a deep link between the key issues in these two areas has been growing with time as one collects more evidence to the same.

There is lot to be explored and discovered regarding the structural properties of solutions that have wide-ranging and fundamental contributions in many areas and this would be the focus of my long-term research. My short term research has been and will be influenced by tractable yet carefully chosen problems.

Much of my work has been done with collaborators and students; and they are equally responsible for these results. I do like to get into the trenches and do a lot of dirty work myself as most of the time it is the dirty work that leads to the development of new intuition. I am a very hands-on adviser to my students and I enjoy working with them. I am a believer that every wrong approach is as useful as a right approach because there is a lesson to be learned from each approach. Finally, thinking about fundamental problems where I can say something new and useful is really what I am passionate about and it defines the researcher in me.

Publications

[1] C. Nair, B. Prabhakar, and M. Sharma, "Proofs of the Parisi and Coppersmith-Sorkin random assignment conjectures," *Random Structures and Algorithms*, vol. 27(4), pp. 413–444, 2005.

[2] C. Borgs, J. T. Chayes, S. Mertens, and C. Nair, "Proof of the local rem conjecture for number partitioning I: Constant energy scales," *Random Structures and Algorithms*, vol. 34(2), pp. 217–240, 2009.

[3] C. Borgs, J. T. Chayes, S. Mertens, and C. Nair, "Proof of the local rem conjecture for number partitioning II: Growing energy scales," *Random Structures and Algorithms*, vol. 34(2), pp. 241–284, 2009.

[4] R. Pan, C. Nair, B. Prabhakar, and B. Yang, "Packet dropping schemes: some examples and analysis,," *Proceedings of the 39th Annual Allerton Conference on Communication, Control and Computing*, pp. 563–572, 2001.

[5] A. El Gamal, C. Nair, B. Prabhakar, E. Uysal, and S. Zahedi, "Energy-efficient scheduling of packet transmissions over wireless networks," *Proceedings of the IEEE Infocom Conference*, vol. 3, pp. 1773–1782, June, 2002.

[6] C. Nair, B. Prabhakar, and M. Sharma, "Proofs of the Parisi and Coppersmith-Sorkin conjectures for the finite random assignment problem," *IEEE Foundations of Computer Science (FOCS)*, pp. 168–178, 2003.

[7] C. Nair, E. Ordentlich, and T. Weissman, "Asymptotic filtering and entropy rate of a hidden Markov process in the rare transitions regime," *International Symposium on Information Theory*, pp. 1838–1842, 2005.

[8] H. Weingarten, Y. Steinberg, and S. Shamai, "The capacity region of the gaussian multiple-input multiple-output broadcast channel," *Information Theory, IEEE Transactions on*, vol. 52, pp. 3936 –3964, sept. 2006.

[9] C. Nair and A. El Gamal, "An outer bound to the capacity region of the broadcast channel," *IEEE Trans. Info. Theory*, vol. IT-53, pp. 350–355, January, 2007.

[10] A. El Gamal and Y.-H. Kim, *Network Information Theory*. Cambridge University Press, 2012.

[11] C. Nair and Z. V. Wang, "On the inner and outer bounds for 2-receiver discrete memoryless broadcast channels," *Proceedings of the ITA Workshop*, 2008.

[12] A. Gohari and V. Anantharam, "Evaluation of Marton's inner bound for the general broadcast channel," *International Symposium on Information Theory*, pp. 2462–2466, 2009.

[13] V. Jog and C. Nair, "An information inequality for the bssc channel," *Proceedings of the ITA Workshop*, 2010.

[14] C. Nair, Z. V. Wang, and Y. Geng, "An information inequality and evaluation of Marton's inner bound for binary input broadcast channels," *International Symposium on Information Theory*, 2010.

[15] Y. Geng, C. Nair, S. Shamai, and Z. V. Wang, "On broadcast channels with binary inputs and symmetric outputs," *International Symposium on Information Theory*, 2010.

[16] Y. Geng, A. Gohari, C. Nair, and Y. Yu, "On Marton's inner bound for two receiver broadcast channels," *Presented at ITA Workshop*, 2011.

[17] Y. Geng, A. Gohari, C. Nair, and Y. Yu, "The capacity region of classes of product broadcast channels," *Proceedings of IEEE International Symposium on Information Theory*, pp. 1549–1553, 2011.

[18] C. Nair, "Capacity regions of two new classes of two-receiver broadcast channels," *Information Theory, IEEE Transactions on*, vol. 56, pp. 4207–4214, sep. 2010.

[19] Y. Geng and C. Nair, "The capacity region of the two-receiver vector gaussian broadcast channel with private and common messages," in *2012 IEEE International Symposium on Information Theory (ISIT'2012)*, (Cambridge, Massachusetts, USA), pp. 591–595, July 2012.

[20] C. Nair and A. El Gamal, "The capacity region of a class of three-receiver broadcast channels with degraded message sets," *Information Theory, IEEE Transactions on*, vol. 55, pp. 4479–4493, Oct. 2009.

[21] C. Nair, A. El Gamal, and Y.-K. Chia, "An achievability scheme for the compound channel with state noncausally available at the encoder," *CoRR*, vol. abs/1004.3427, 2010.

[22] C. Nair and Z. Wang, "The capacity region of the three receiver less noisy broadcast channel," *Information Theory, IEEE Transactions on*, vol. 57, pp. 4058 –4062, july 2011.

[23] C. Nair and Z. V. Wang, "On the inner and outer bounds of 3-receiver broadcast channels with 2-degraded message sets," *International Symposium on Information Theory*, pp. 1844–1848, 2009.

[24] M. H. M. Costa and C. Nair, "On the achievable rate sum for symmetric gaussian interference channels," *ITA Workshop*, 2012.

[25] T. Cover, "Broadcast channels," *IEEE Trans. Info. Theory*, vol. IT-18, pp. 2–14, January, 1972.