Final Exam: IERG 6300

Total points: 50

The exam is due by 12:00 **noon** on Monday December 23rd, 2019

Even if you opt for this course to be Pass/Fail you need to answer this exam

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This exam contains 2 pages (including this cover page) and 6 questions.

Important Notes

- Use of internet is not allowed.
- You may use the lecture notes and quote relevant results. You can also use standard theorems from analysis.

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1. (a) (2 points) Let X be a non-negative random variable such that $\mathrm{E}(X^2) < \infty$. Let $0 \leq a < \mathrm{E}(X)$. Show that

$$\mathrm{P}(X > a) \geq \frac{(\mathrm{E}(X) - a)^2}{\mathrm{E}(X^2)}.$$

Hint: Apply Cauchy-Schwartz inequality to $X 1_{X>a}$.

(b) (3 points) Let $\{A_n\}_{n\geq 1} \in \mathcal{F}$ such that $\sum_{n=1}^{\infty} \mathrm{P}(A_n) = \infty$. Further let

$$\limsup_{n\to\infty} \frac{\left(\sum_{k=1}^{n} P(A_k)\right)^2}{\left(\sum_{1\leq j,k\leq n} \mathrm{P}(A_j \cap A_k)\right)} = \alpha > 0.$$

Show that $\mathrm{P}(A_n \ i.o.) \geq \alpha$.
Hint: Use previous part with $X_n = \sum_{k=1}^{n} 1_{A_k}$ and $a_n = \lambda \, \mathrm{E}(X_n)$.

2. (6 points) Suppose X and Y are $[0,1]$-valued random variables such that $\mathrm{E}(X^n) = \mathrm{E}(Y^n)$ for $n = 1, 2, ...$ Show that X has the same distribution as Y.
Hint: Recall (Weierstrass approximation theorem) that if $h(x)$ is continuous in $[0,1]$ then there exists a sequence of polynomials $p_n(x)$ such that $\sup_{x\in[0,1]} |h(x) - p_n(x)| \to 0$ as $n \to \infty$.

3. Let X be an $\mathcal{F}$-measurable random variable satisfying $\mathrm{E}(X^2) < \infty$. For any σ-algebra $\mathcal{G} \subseteq \mathcal{F}$ define

$$\mathrm{Var}(X|\mathcal{G}) = \mathrm{E}((X - \mathrm{E}(X|\mathcal{G}))^2|\mathcal{G}).$$

(a) (3 points) Consider σ-algebras satisfying $\mathcal{G}_2 \subseteq \mathcal{G}_1 \subseteq \mathcal{F}$. Show that $\mathrm{E}(\mathrm{Var}(X|\mathcal{G}_2)) \geq \mathrm{E}(\mathrm{Var}(X|\mathcal{G}_1))$

(b) (3 points) Show that for any σ-algebra $\mathcal{G} \subseteq \mathcal{F}$

$$\mathrm{E}(X^2) = \mathrm{E}(\mathrm{Var}(X)|\mathcal{G}) + \mathrm{Var}(\mathrm{E}(X|\mathcal{G}))$$

4. Let $s > 0$ and let $\{Z_n\}_{n\geq 1}$ be a sequence of independent random variables such that $\mathrm{P}(Z_n = 1) = \mathrm{P}(Z_n = -1) = \frac{1}{2n^s}$, $\mathrm{P}(Z_n = 0) = 1 - \frac{1}{n^s}$, $n \geq 1$. Let $Y_0 = 0$ and for $n \geq 1$, define

$$Y_n = n^s Y_{n-1}|Z_n| + Z_n 1_{\{Y_{n-1}=0\}}.$$

(a) (2 points) Show that $\{Y_n\}$ is a martingale w.r.t. to the canonical filtration $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_i = \sigma(Z_1, .., Z_i), i \geq 1$.

(b) (4 points) Show that

$$\mathrm{P}(\max_{1 \leq k \leq n} Y_k \geq x) \leq \frac{1}{2x}\left(1 + \sum_{k=2}^{n} \frac{1}{k^s}\left(1 - \frac{1}{(k-1)^s}\right)\right)$$

(Hint: Use Doob's inequality. Theorem 1 in lecture notes 6)

(c) Show that

i. (2 points) $Y_n \to 0$ in probability.

ii. (3 points) $Y_n \to 0$ almost surely *if and only if* $s > 1$

iii. (1 point) For no value of $s > 0$ does $\mathrm{E}(|Y_n|) \to 0$.

5. (6 points) For $n \geq 1$, let X_n, Y_n be non-negative integrable random variables adapted to the filtration $\mathcal{F}_n$. Assume that $\mathrm{E}(X_{n+1}|\mathcal{F}_n) \leq Y_n + X_n(1 + Y_n)$. Further let $\sum_n Y_n < \infty$ almost surely. Show that X_n converges almost surely.
Hint: Construct a suitable non-negative supermartingale, i.e. one whose convergence implies the convergence of X_n, and use the Martingale Convergence Theorem for non-negative supermartingales in the lecture notes.

6. Let $\{X_i\}_{i \geq 1}$ denote a sequence of independent and identically distributed Bernoulli random variables with

$$P(X_i = 1) = P(X_i = -1) = \frac{1}{2}.$$

Let $S_0 = 0$ and let $S_n = \sum_{i=1}^{n} X_i$. Further let $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F}_i = \sigma(X_1, .., X_i), i \geq 1$ denote the canonical filtration.

(a) (2 points) Show that $\mathrm{P}(S_n - S_k \geq 0) \geq \frac{1}{2}$ for $k = 1, .., n$.

(b) (3 points) Fix $a > 0$. Let $\tau_a = \inf\{k \geq 1 : S_k > a\}$. Show that

$$\mathrm{P}(S_n > a) \geq \sum_{k=1}^{n} \mathrm{P}(\tau_a = k, S_n - S_k \geq 0) \geq \frac{1}{2}\,\mathrm{P}(\tau_a = k).$$

(c) (1 point) Deduce that for any $n \geq 1$ and $a > 0$

$$\mathrm{P}(\max_{1 \leq k \leq n} S_k > a) \leq 2P(S_n > a).$$

(d) (3 points) For any $p, n \in \mathbb{N}$, show that

$$\mathrm{P}(\max_{1 \leq k \leq n} S_k \geq p) = 2P(S_n \geq p) - \mathrm{P}(S_n = p).$$

(e) (3 points) Let Z_n denote the number of **strict** sign changes within $\{S_0, S_1, .., S_n\}$. Show that

$$\mathrm{P}(Z_{2n+1} \geq p | S_1 = -1) = \mathrm{P}(\max_{1 \leq k \leq 2n+1} S_k \geq 2p - 1 | S_1 = -1).$$

Hint: flip the signs of $\{X_i\}$'s between the odd and even strict sign changes of S_k.

(f) (3 points) Use the above parts to show that Z_{2n+1} has the same distribution as $\frac{|S_{2n+1}| - 1}{2}$.