IERG 6154: Network Information Theory
Homework 3

Due: February 4, 2019

1. Determine the capacity and the capacity achieving distribution of the Z channel depicted below

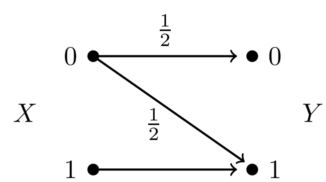

Figure 1: Z Channel

2. Independently generated codebooks: Consider a joint distribution $p(x, y)$, and let $p_X(x)$ and $p_Y(y)$ be its marginal distributions. Generate two codebooks $\mathcal{C}_1 = \{X_1^n, ..., X_{2^{nR_1}}^n\}$, and $\mathcal{C}_2 = \{Y_1^n, ..., Y_{2^{nR_2}}^n\}$ respectively by generating each X^n sequence independently according to $\prod_{i=1}^n p_X(x_i)$ and each Y^n sequence independently according to $\prod_{i=1}^n p_Y(y_i)$. Define the set

$$\mathcal{C} = \{(x^n, y^n) \in \mathcal{C}_1 \times \mathcal{C}_2 \text{ such that } (x^n, y^n) \in \mathcal{T}_\epsilon^{(n)}(X, Y)\}.$$

Show that the expected size of $\mathcal{C}$ satisfies

$$\lim_{n\to\infty} \frac{1}{n} \log_2 \mathrm{E}(|\mathcal{C}|) = R_1 + R_2 - I(X; Y).$$

3. Erasure degradation: Consider a DMC $(\mathcal{X}, p(y|x), \mathcal{Y})$ and let Z be a noisy version of Y, $\mathcal{Z} = \mathcal{Y} \cup \{E\}$ (where $E \notin \mathcal{Y}$ is an erasure symbol), such that $\mathrm{P}(Z = y|Y = y) = 1 - e, \mathrm{P}(Z = E|Y = y) = e, \forall y$. Further assume that $X \to Y \to Z$ is Markov. Show that for all $p(x)$, we have

$$I(X; Z) = (1 - e)I(X; Y).$$

4. Capacity of product channel: Consider two DMCs $(\mathcal{X}_1, p_1(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p_2(y_2|x_2), \mathcal{Y}_2)$ with capacities C_1 and C_2 respectively. A new channel

is $(\mathcal{X}_1 \times \mathcal{X}_2, p_1(y_1|x_1)p_2(y_2|x_2), \mathcal{Y}_1 \times \mathcal{Y}_2)$ is formed in which a symbol $x_1 \in \mathcal{X}_1$ and $x_2 \in \mathcal{X}_2$ is transmitted resulting in outputs y_1, y_2 respectively. Find the capacity of this product channel.

5. Capacity of sum channel: Consider two DMCs $(\mathcal{X}_1, p_1(y_1|x_1), \mathcal{Y}_1)$ and $(\mathcal{X}_2, p_2(y_2|x_2), \mathcal{Y}_2)$ with capacities C_1 and C_2 respectively. Let $\mathcal{X}_1 \cup \mathcal{X}_2 = \emptyset, \mathcal{Y}_1 \cup \mathcal{Y}_2 = \emptyset$. A new channel is $(\mathcal{X}_1 \cup \mathcal{X}_2, p(y|x), \mathcal{Y}_1 \cup \mathcal{Y}_2)$ is formed in which a symbol $x_1 \in \mathcal{X}_1$ or $x_2 \in \mathcal{X}_2$ is transmitted resulting in outputs $y_1 \in \mathcal{Y}_1$, $y_2 \in \mathcal{Y}_2$ respectively. Hence $p(y|x) = p_1(y_1|x_1)$ if $x_1 \in \mathcal{X}_1$ and $p(y|x) = p_2(y_2|x_2)$ if $x_2 \in \mathcal{X}_2$. Find the capacity of this sum channel.

6. Consider two DMCs, where the first channel is a $BSC(p)$ (binary symmetric channel) and the second channel is a $BEC(e)$. Let the output of the first channel be called Y and the output of the second channel be called Z. Assume both channels to have the same input X. (Therefore Y and Z are two different noisy versions of X.) Let $\mathrm{P}(X = 0) = x, 0 \leq x \leq 1$. Define the function $D(x) = I(X; Z) - I(X; Y)$. Show that

   (a) $D(x)$ is concave in x if and only if $e \leq 4p(1-p)$.

   (b) $D(x) \geq 0, \forall x \in [0, 1]$ if and only if $e \leq H(p) = -p \log_2 p - (1 - p) \log_2(1-p)$.

   Convexity of $H(p * H^{-1}(u))$ in u may be useful.

7. Let $U, X, Y \sim p(u, x, y)$ be discrete random variables. Let $0 < \delta_n < \epsilon_n$ such that $\epsilon_n \to 0$ and $\delta_n \sqrt{n}, (\epsilon_n - \delta_n)\sqrt{n} \to \infty$. For any n, consider a fixed $u_0^n \in T_{\delta_n}(U)$. Let $\mathcal{A}_n = \{x^n : (u_0^n, x^n) \in T_{\epsilon_n}(U, X)\}$ and let $\mathcal{B}_n = \{y^n : (u_0^n, y^n) \in T_{\epsilon_n}(U, Y)\}$. Let $\mathcal{C}_n \subseteq \mathcal{A}_n \times \mathcal{B}_n$ be defined by $\mathcal{C}_n = \{(x^n, y^n) : (x^n, y^n) \in T_{\epsilon_n}(X, Y)\}$. Show that

   (a) If X^n is generated according to $\prod_{i=1}^n p(x_i|u_{0i})$, then $\mathrm{P}(X^n \in \mathcal{A}_n) \to 1$. Similarly if Y^n is generated according to $\prod_{i=1}^n p(y_i|u_{0i})$, then $\mathrm{P}(Y^n \in \mathcal{B}_n) \to 1$.

   (b) Show that

   $$\frac{1}{n} \log \mathrm{P}\big((X^n, Y^n) \in \mathcal{C}_n | \{X^n \in \mathcal{A}_n, Y^n \in \mathcal{B}_n\}\big) \to - \min_{p(u,x,y) \in Q} I(X; Y|U),$$

   where Q is the collection of distributions on (U, X, Y) such that $q(u, x) = p(u, x), q(u, y) = p(u, y)$, and $q(x, y) = p(x, y)$.