PROBABILITY THEORY: CLASS NOTES 2

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Disclaimer: These notes are created from a variety of great references in the field, most notably, Varadhan's lecture notes and Dembo's lecture notes.

2. Random variables and integration

Definition 2.1. A *random variable* or a *measurable function* is a map $f : (\Omega, \Sigma) \mapsto (\mathbb{R}, \mathcal{F}_B)$ (really a mapping $\Omega \to \mathbb{R}$ but with the sigma-algebras specified) such that $\forall B \in \mathcal{F}_B$, $f^{-1}(B) = \{w : f(w) \in B\} \in \Sigma$.

In the above definition $\mathcal{F}_B$ denotes the Borel σ-algebra generated by the open intervals on the real line.

Exercise 2.1. For a class of sets $\mathcal{A} \subset \mathcal{F}_B$, and a mapping $f : (\Omega, \Sigma) \mapsto (\mathbb{R}, \mathcal{F}_B)$, suppose it holds that $f^{-1}(A) \in \Sigma$ for all $A \in \mathcal{A}$, then show that $f^{-1}(C) \in \Sigma$ for all $C \in \sigma(\mathcal{A})$, where $\sigma(\mathcal{A})$ is the smallest σ-algebra containing $\mathcal{A}$.

Remark 2.1. The above exercise show that to verify that a function is measurable, it suffices to consider the inverse images of any collection of sets that generate $\mathcal{F}_B$.

Some facts about random variables:

(1) If $A \in \Sigma$ then
$$\mathbf{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$$
is bounded and measurable.

(2) Sums, products, limits, etc of measurable functions are measurable.

Exercise 2.2. Show that if f_1, f_2 are measurable, then $f_1 f_2$ (their pointwise product) is measurable.

(3) if $\{A_j : 1 \leq j \leq n\}$ is a finite disjoint partition of Ω into measurable sets, then the function (called a *simple function*)
$$f(w) = \sum_{j=1}^{n} c_j \mathbf{1}_{A_j}(\omega)$$
is bounded and measurable.

Lemma 2.1. *Any bounded measurable function is the uniform limit of simple functions.*

Proof. Suppose $|f(\omega)| < M$, then divide the interval $(-M, M]$ into n disjoint, $\{I_i\}$, intervals of length $\frac{2M}{n}$. Let c_i denote the midpoint of the intervals. Then define

Date: September 12, 2019.

$A_i = w : f(w) \in I_i$. Clealry $\{A_i\}$'s are measurable and disjoint. Consider the simple function

$$f_n(\omega) = \sum_{i=1}^{n} c_i \mathbf{1}_{A_i}(\omega).$$

Clearly $|f(\omega) - f_n(\omega)| \leq \frac{M}{n}, \forall \omega$. Hence the convergence is uniform. $\square$

2.1. **Definition of integrals.**

(1) For a simple function f defined on the probability space $(\Omega, \Sigma, \mathrm{P})$ we define the *integral* with respect to the probability measure as

$$\int f dP = \int \left(\sum_{i=1}^{n} c_i \mathbf{1}_{A_i} \right) dP = \sum_{i=1}^{n} c_i P(A_i).$$

(2) If f is a bounded, measurable function and f_n be any sequence of simple functions that converge to f uniformly, then we define

$$\int f dP = \lim_n \int f_n dP.$$

(Why does this limit exist, and why is it independent of the particular sequence f_n?)

The limit exists because the sequence $\int f_n dP$ is Cauchy. It is also independent of the particular sequence f_n because the difference of two such sequences $f_n - g_n$ is bounded and decreases to 0 pointwise (uniformly).

Exercise 2.3. Complete the details of the argument and show that $\int f dP$ is well defined, when f is bounded measurable function.

Definition 2.2. A sequence of functions f_n is said to converge to f pointwise (or everywhere) if

$$\lim_{n \to \infty} f_n(\omega) = f(\omega), \ \forall \omega \in \Omega.$$

Definition 2.3. A sequence of measurable functions f_n is said to converge to a measurable function f *almost everywhere* (or *almost surely*) if $\exists\ N \subset \Omega, \mathrm{P}(N) = 1$ such that

$$\lim_{n \to \infty} f_n(\omega) = f(\omega), \ \forall \omega \in N.$$

Definition 2.4. A sequence of measurable functions f_n is said to converge to a measurable function f *in measure* (or *in probability*) if $\forall \epsilon > 0$,

$$\lim_{n \to \infty} \mathrm{P}(\{w : |f_n(\omega) - f(\omega)| > \epsilon\}) = 0.$$

Convergence in measure is a weaker notion than almost sure convergence.

Lemma 2.2. *If a sequence of measurable functions f_n converge to f almost everywhere then the sequence of measurable functions also converge to f in measure.*

Proof. For any $\epsilon > 0$, define the sets

$$A_n^\epsilon = \{\omega : |f_n(\omega) - f(\omega)| > \epsilon\}, \quad B_n^\epsilon = \cup_{m=n}^\infty A_n^\epsilon.$$

Let $B_n^\epsilon \downarrow B^\epsilon$. Since $f_n(\omega) \to f(\omega)$ for $\omega \in N$, then $B^\epsilon \subseteq N^c$, and hence $\mathrm{P}(B^\epsilon) = 0$ implying that $\mathrm{P}(B_n^\epsilon) \downarrow 0$; and since $\mathrm{P}(A_n^\epsilon) \leq \mathrm{P}(B_n^\epsilon)$ we have $\mathrm{P}(A_n^\epsilon) \to 0$ as desired, establishing convergence in measure. $\square$

Let $\Omega = [0,1]$ and the probability measure induced by the Lebesgue measure on this set. Consider the following sequence of real valued functions from $\mathbb{R}_+ \to [0,1]$ define by

$$r_n(x) = \begin{cases} 1 & x \in (H_n, H_{n+1}] \\ 0 & \text{otherwise} \end{cases} .$$

Here $H_n = \sum_{i=1}^n \frac{1}{i}$ is the harmonic sum. Use the above sequence of functions to define measurable functions $f_n(\omega)$ according to

$$f_n(\omega) = \sum_{i=1}^{\infty} r_n(\omega + i).$$

Exercise 2.4. Show that $f_n(\omega) \to 0$ in measure, while $\lim_n f_n(\omega)$ does not exist almost surely, thus there is no convergence almost surely.

On the other hand, convergence in measure does imply almost sure convergence on a sub-sequence as demonstrated by the following lemma.

Lemma 2.3. *If a sequence of measurable functions f_n converge to f in measure, then there is a subsequence, n_i such that, the sequence of measurable functions f_{n_i} converge to f almost everywhere.*

Proof. For any $k \in \mathbb{N}$, define the set

$$A_n^k = \{\omega : |f_n(\omega) - f(\omega)| > \frac{1}{k}\}.$$

From convergence in measure, we know that $\mathrm{P}(A_n^k) \to 0$ as $n \to \infty$. Let n_k be such that $\mathrm{P}(A_{n_k}^k) \leq \frac{1}{2^k}$. Define

$$B_k = \cup_{m=k}^{\infty} A_{n_m}^m.$$

Thus $\mathrm{P}(B_k) \leq \frac{1}{2^{k-1}}$ and $B_k \downarrow B$ with $\mathrm{P}(B) = 0$. Note that

$$B = \{\omega : \limsup_k |f_{n_k}(\omega) - f(\omega)| > 0\}$$

and this completes the argument. □

Remark 2.2. We will only be dealing with measurable functions; so unless explicitly stated please assume that all functions are measurable.

Theorem 2.4. *(Bounded Convergence Theorem) If a sequence $\{f_n(\omega)\}$ of uniformly bounded functions converge to a bounded function $f(\omega)$ in measure then*

$$\int f_n dP \to \int f dP.$$

Proof. First note that (argue why using definition)

$$\int f_n dP - \int f dP = \int (f_n - f) dP.$$

Again argue that

$$\left| \int f_n dP - \int f dP \right| = \left| \int (f_n - f) dP \right| \leq \int |f_n - f| dP.$$

As before, define

$$A_n^{\epsilon} = \{\omega : |f_n(\omega) - f(\omega)| > \epsilon\}.$$

Using this we have

$$\left|\int f_n dP - \int f dP\right| \leq \int |f_n - f| dP \leq \epsilon(1 - \mathrm{P}(A_n^\epsilon)) + 2M\mathrm{P}(A_n^\epsilon).$$

Since $\mathrm{P}(A_n^\epsilon) \to 0$ as $n \to \infty$ we get

$$\limsup \left|\int f_n dP - \int f dP\right| \leq \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we are done. $\square$

Definition 2.5. For an non-negative measurable function $f(\omega)$ we define

$$\int f(\omega) dP = \{\sup \int g dP : g \text{ is bounded}, 0 \leq g \leq f\}.$$

Lemma 2.5 (Fatou)**.** *Let $f_n \geq 0$ converge in measure to f (also assumed non-negative) as $n \to \infty$ then*

$$\int f dP \leq \liminf_{n\to\infty} \int f_n dP.$$

Proof. Consider any $0 \leq g \leq f$ such that g is bounded. Define $h_n = \min\{f_n, g\}$. The observe that $\{h_n\}$ is uniformly bounded and $h_n \to g$ in measure. Thus from bounded convergence theorem, we have

$$\int g dP = \lim_{n\to\infty} \int h_n dP \leq \liminf_{n\to\infty} \int f_n dP.$$

Since g is an arbitrary bounded function such that $0 \leq g \leq f$, taking *sup* over the class of such g yields the desired result. $\square$

An alternate version of Fatou's lemma that is often used is the following:

Lemma 2.6 (Fatou (alternate))**.** *Let $f_n \geq 0$ then*

$$\int \liminf_{n\to\infty} f_n dP \leq \liminf_{n\to\infty} \int f_n dP.$$

Proof. First assume that $g = \liminf_{n\to\infty} f_n$ is finite *almost everywhere*. Define $g_n = \inf_{m\geq n} f_m$ and observe that $g_n \uparrow g = \liminf_{n\to\infty} f_n$ pointwise (and in measure (why?)). Thus from the former version we have that

$$\int \liminf_{n\to\infty} f_n dP = \int g dP \leq \liminf_{n\to\infty} \int g_n dP \leq \liminf_{n\to\infty} \int f_n dP,$$

where the last inequality is a consequence of $0 \leq g_n \leq f_n$.

If $g = +\infty$ on A with $P(A) > 0$, then for any $M > 0$ observe that the earlier part yields

$$MP(A) \leq \int \liminf_{n\to\infty} \{f_n \wedge M\} dP \leq \liminf_{n\to\infty} \int \{f_n \wedge M\} dP \leq \liminf_{n\to\infty} \int f_n dP.$$

Taking $M \to \infty$ implies that both integrals of interest tend to infinity. $\square$

Corollary 2.7 (Monotone Convergence Theorem)**.** *If a sequence of non-negative functions $f_n \uparrow f$ then*

$$\lim_{n\to\infty} \int f_n dP \to \int f dP.$$

Proof. $0 \le f_n \le f$ implies that (why?)

$$\int f_n dP \le \int f dP$$

and taking lim sup yields

$$\limsup_{n\to\infty} \int f_n dP \le \int f dP.$$

The other half follows from Fatou's lemma. □

Definition 2.6. A non-negative measurable function $f(\omega)$ is said to be *integrable* if

$$\int f dP < \infty.$$

Definition 2.7. A measurable function $f(\omega)$ is said to be *integrable* if

$$\int |f| dP < \infty.$$

For integrable functions f we define $f_+ = \max\{f, 0\}$ and $f_- = \max\{-f, 0\}$. Thus $f = f_+ - f_-$, where f_+ and f_- are non-negative measurable functions. We now define

$$\int f dP = \int f_+ dP - \int f_- dP.$$

Exercise 2.5. Show that the integral satisfies the following properties.

a) If f, g are integrable, the for any $a, b \in \mathbb{R}$ the function $af + bg$ is also integrable.
b) If $f = 0$ almost everywhere, then f is integrable and $\int f dP = 0$. As a consequence, two integrable functions that agree almost everywhere has the same integral.

Lemma 2.8 (Jensen's inequality)**.** *If $\Phi(x)$ is a convex function, and $f(\omega)$ and $\Phi(f(\omega))$ are integrable, then*

$$\int \Phi(f(\omega)) dP \ge \Phi\left(\int f(\omega) dP\right).$$

Proof. The proof of Jensen's inequality in this case, as well as in some other cases, will use the fact that any convex function can be written as the pointwise supremum of supporting hyperplanes, i.e.

$$\Phi(x) = \sup_{(a,b)\in\mathcal{E}} ax + b.$$

Hence for any $(a, b) \in \mathcal{E}$

$$\Phi(f(\omega)) \ge af(\omega) + b,$$

yielding

$$\int \Phi(f(\omega)) dP \ge a \int f(\omega) dP + b.$$

Taking supremum of $(a, b) \in \mathcal{E}$ yields the result. □

Theorem 2.9 (Dominated Convergence Theorem)**.** *If an sequence $\{f_n\}$ converge to f in measure and $|f_n| \le g$, where g is an integrable function, then*

$$\lim_{n\to\infty} \int f_n dP \to \int f dP.$$

Proof. A simple application of Fatou's lemma yields that f is integrable. The proof follows using further applications of Fatou's lemma. Observe that the two non-negative sequence of functions $g - f_n$ and $g + f_n$ converge in measure to $g - f$ and $g + f$ respectively (why?). Now argue that

$$\liminf_{n\to\infty} \int (g - f_n) dP = \int g dP - \limsup_{n\to\infty} \int f_n dP,$$

$$\liminf_{n\to\infty} \int (g + f_n) dP = \int g dP + \liminf_{n\to\infty} \int f_n dP.$$

Applying Fatou's Lemma yields (justify the second relations)

$$\int (g - f) dP \le \liminf_{n\to\infty} \int (g - f_n) dP = \int g dP - \limsup_{n\to\infty} \int f_n dP$$

$$\implies \limsup_{n\to\infty} \int f_n dP \le \int f dP,$$

and

$$\int (g + f) dP \le \liminf_{n\to\infty} \int (g + f_n) dP = \int g dP + \liminf_{n\to\infty} \int f_n dP$$

$$\implies \liminf_{n\to\infty} \int f_n dP \ge \int f dP.$$

□

Exercise 2.6. Suppose an integrable f satisfies that

$$\int f(\omega) \mathbf{1}_A(\omega) dP = 0, \ \forall A \in \mathcal{F}.$$

Show that $f = 0$ *almost everywhere.*

2.2. **Transformations.**

Definition 2.8. A *measurable transformation* $T : (\Omega_1, \mathcal{F}_1) \mapsto (\Omega_2, \mathcal{F}_2)$ is a mapping that satisfies

$$T^{-1}(B) = \{\omega_1 : T(\omega_1) \in B\} \in \mathcal{F}_1$$

for all $B \in \mathcal{F}_2$.

If the space $(\Omega_1, \mathcal{F}_1)$ was endowed with a probability measure P then a measurable mapping T induces a probability measure on $(\Omega_2, \mathcal{F}_2)$ according to

$$\mathrm{Q}(B) = \mathrm{P}(T^{-1}(B)), \forall B \in \mathcal{F}_2.$$

Exercise 2.7. Verify that the measure Q defined above is indeed a countably additive probability measure (assuming P is one).

Theorem 2.10. *Let $f(\omega_2)$ be a measurable mapping (random variable) on $(\Omega_2, \mathcal{F}_2)$ to $(\mathbb{R}, \mathcal{B}_R)$ and T be a measurable transformation from $(\Omega_1, \mathcal{F}_1) \mapsto (\Omega_2, \mathcal{F}_2)$; then the mapping $g(\omega_1) := f(T(\omega_1))$ is measurable. Further $g(\omega_1)$ is integrable with respect to P if and only if $f(\omega_2)$ is integrable with respect to $\mathrm{Q} = PT^{-1}$ and*

$$\int_{\Omega_2} f(\omega_2) dQ = \int_{\Omega_1} g(\omega_1) dP.$$

Proof. For any $B \in \mathcal{B}_{\mathbb{R}}$ observe that

$$\{\omega_1 : g(\omega_1) \in B\} = T^{-1}(f^{-1}(B)).$$

Since $f^{-1}(B) \in \mathcal{F}_2$ (measurability of $f(\omega_2)$); by measurability of T we have $T^{-1}(f^{-1}(B)) \in \mathcal{F}_1$, establishing the measurability of $g(\omega_1)$. The second part follows by the *standard-machine argument*, i.e. verify it (using previous parts) when

1) $f(\omega_2)$ is indicator function (use definition of P and Q).
2) $f(\omega_2)$ is a simple function (use linearity)
3) $f(\omega_2)$ is a bounded non-negative function (use bounded convergence theorem)
4) $f(\omega_2)$ is a non-negative function (use monotone convergence theorem by considering $f_n = \min\{f, n\}$)
5) Finally, $f(\omega_2)$ is an integrable function (use positive and negative parts).

□

2.3. **Product spaces.** Consider two probability spaces $(\Omega_1, \mathcal{F}_1, \mathrm{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathrm{P}_2)$. The goal of this section is to work with the product space $\Omega_1 \times \Omega_2$. A *natural σ-field* that one can define on the product space is the σ-field, $\mathcal{F}$, generated by the *measurable rectangles*, i.e. sets of the form $A_1 \times A_2$ with $A_1 \in \mathcal{F}_1, A_2 \in \mathcal{F}_2$. Our next goal is to define a countably additive probability measure on $(\Omega_1 \times \Omega_2, \mathcal{F})$ that naturally extends P_1 and P_2.

For a measurable rectangle, a natural candidate is

$$\mathrm{P}(A_1 \times A_2) = \mathrm{P}_1(A_1)\mathrm{P}_2(A_2).$$

For finite disjoint union of measurable rectangles one can define

$$\mathrm{P}(\sqcup_{i=1}^n A_{1i} \times A_{2i}) = \sum_{i=1}^n \mathrm{P}_1(A_{1i})\mathrm{P}_2(A_{2i}).$$

Exercise 2.8. Show that

i) Finite disjoint union of measurable rectangles is an algebra $\mathcal{A}$.
ii) P is well defined, i.e. if $\sqcup_{i=1}^n A_{1i} \times A_{2i} = \sqcup_{j=1}^m B_{1j} \times B_{2j}$ then

$$\sum_{i=1}^n \mathrm{P}_1(A_{1i})\mathrm{P}_2(A_{2i}) = \sum_{j=1}^m \mathrm{P}_1(B_{1j})\mathrm{P}_2(B_{2j}).$$

Thus P is a finitely additive probability measure on $\mathcal{A}$.

Lemma 2.11. *P is a countably additive probability measure on $\mathcal{A}$.*

Proof. Let $E_n \downarrow \varnothing$, $E_n \in \mathcal{A}$. Define the set

$$E_{n,\omega_2} = \{\omega_1 : (\omega_1, \omega_2) \in E_n\}.$$

Now define $f_n(\omega_2) = \mathrm{P}_1(E_{n,\omega_2})$ (See (show) that $f_n(\omega_2)$ is a simple function, i.e. it takes only finitely many distinct values.) Note that

$$\mathrm{P}(E_n) = \int f_n(\omega_2) d\mathrm{P}_2.$$

Now $0 \le f_n(\omega_2) \le 1$ and since $E_{n,\omega_2} \downarrow \varnothing$ implies that $\mathrm{P}(E_{n,\omega_2}) \downarrow 0$ (by countable additivity of P_1). Hence $f_n(\omega_2) \downarrow 0$ and by using bounded convergence theorem we get that

$$\int f_n(\omega_2) d\mathrm{P}_2 \downarrow 0 \implies \mathrm{P}(E_n) \downarrow 0.$$

□

By Caratheodory's extension theorem, we can extend P to a countably additive probability measure on $\mathcal{F} = \sigma(\mathcal{A})$ on $\Omega_1 \times \Omega_2$ and this measure is called the *product measure.*

2.3.1. *Iterated integrals and Fubini's theorem.* In this section we establish an oft-invoked theorem for justifying exchange of integrals. The proofs are basically a consequence of the *standard-machine* argument. As in the earlier section we consider two probability spaces $(\Omega_1, \mathcal{F}_1, \mathrm{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathrm{P}_2)$ and its natural extension (as defined in the previous section) of $(\Omega_1 \times \Omega_2, \mathcal{F}, \mathrm{P})$. We begin by establishing the following lemma.

Lemma 2.12. *For any $A \in \mathcal{F}$ denote by A_{ω_1} and A_{ω_2} the sets (sections)*

$$A_{\omega_1} = \{\omega_2 : (\omega_1, \omega_2) \in A\}, \quad \text{and} \;\; A_{\omega_2} = \{\omega_1 : (\omega_1, \omega_2) \in A\}.$$

Then

a) For each ω_1 the $A_{\omega_1} \in \mathcal{F}_2$ and for each ω_2 the $A_{\omega_2} \in \mathcal{F}_1$.

b) the functions $P_2(A_{\omega_1})$ and $P_1(A_{\omega_2})$ are measurable and

$$P(A) = \int_{\Omega_1} P_2(A_{\omega_1}) dP_1 = \int_{\Omega_2} P_1(A_{\omega_2}) dP_2.$$

Outline of Proof. The first part claims that sections of sets in product sigma-algebra belong to the individual sigma-algebras. Observe that this part is immediate if the original set is a rectangle. Then observe that the collection of all sets for which the part holds is a sigma-algebra, thus contains the sigma-algebra generated by the rectangles.

The second assertion is again immediate if $A = A_1 \times A_2$, A is a measurable rectangle. From linearity, the assertion follows for finite disjoint union of rectangles (why?). Now consider the class, $\mathcal{C}$, of all sets for which the assertion is valid. Show that $\mathcal{C}$ is a monotone class and hence contains $\mathcal{F}$. □

Theorem 2.13 (Fubini)**.** *Let $f(\omega) = f(\omega_1, \omega_2)$ be a measurable function on $(\Omega, \mathcal{B})$. For each fixed ω_1 consider $g_{\omega_1}(\omega_2) := f(\omega_1, \omega_2)$ as a mapping from $\Omega_2 \to \mathbb{R}$ and for each fixed ω_2 consider $h_{\omega_2}(\omega_1) := f(\omega_1, \omega_2)$ as a mapping from $\Omega_1 \to \mathbb{R}$. Then*

a) For each ω_1 the function $g_{\omega_1}(\omega_2)$ is measurable (similarly for each fixed ω_2, the function $f_{\omega_2}(\omega_1)$ is measurable).

b) If f is integrable, then for almost all ω_1, the function $g_{\omega_1}(\omega_2)$ is integrable. (Similarly for almost all ω_2, the function $f_{\omega_2}(\omega_1)$ is integrable) and further the functions

$$G(\omega_1) := \int_{\Omega_2} g_{\omega_1}(\omega_2) dP_2, \quad \text{and} \quad H(\omega_2) := \int_{\Omega_1} f_{\omega_2}(\omega_1) dP_2$$

are integrable. Finally, we also have

$$\int_{\Omega} f(\omega_1, \omega_2) = \int_{\Omega_1} G(\omega_1) dP_1 = \int_{\Omega_2} H(\omega_1) dP_2.$$

Proof. The first part is an immediate consequence of the first part in Lemma 2.12 that sections of sets in product sigma-algebra belong to the individual sigma-algebras.

The proof of the second part uses the *standard machine* approach. For indicator functions, the theorem reduces to second part of Lemma 2.12. Simple functions follows by linearity and uniform limits imply the result for bounded measurable functions. Now monotone convergence theorem implies the result for non-negative functions and by taking f_+ and f_- the result follows for integrable functions. $\square$

Exercise 2.9. Consider $\Omega = \mathbb{N} \times \mathbb{N}$ and $\mathrm{P}(i,j) = \frac{1}{2^{i+j}}$, $i,j \geq 1$. For $i,j \geq 1$

$$f(i,j) = \begin{cases} 2^{i+j} & j = i+1 \\ -2^{i+j} & j = i-1, i \geq 2 \\ 0 & o.w. \end{cases}$$

Here $\mathrm{P}_1(i) = \mathrm{P}_2(i) = \frac{1}{2^i}, i \geq 1$. Compute the functions

$$G(i) := \sum_j f(i,j)\mathrm{P}_2(j) \quad \text{and} \quad H(j) := \sum_i f(i,j)\mathrm{P}_1(i).$$

What are the sums $\sum_{i\geq 1} G(i)\mathrm{P}_1(i)$ and $\sum_{j \geq 1} H(j)p_2(j)$. (Note that f is not integrable).

2.4. **Borel-Cantelli Lemma 1.**

Lemma 2.14. *Let $(\Omega, \mathcal{F}, P)$ be a probability space. Consider a collection of sets $\{A_n\}, A_n \in \mathcal{F}$ such that $\sum_{n=1}^{\infty} P(A_n) < \infty$. Define the set $\bar{A} := \cap_n \cup_{m \geq n} A_m$. Then $P(\bar{A}) = 0$.*

Proof. Define $B_n = \cup_{m\geq n} A_m$. Then $\mathrm{P}(B_n) \leq \sum_{m \geq n} \mathrm{P}(A_m)$; hence $\mathrm{P}(B_n) \to 0$ as $n \to \infty$, since $\sum_{n=1}^{\infty} \mathrm{P}(A_n) < \infty$. Now $B_n \downarrow \bar{A}$ and the result is immediate. $\square$

Remark 2.3. This is an often used lemma to deduce sub-sequential almost-sure convergence. Let us use this to show that convergence in measure implies that there is a sub-sequential almost-sure convergence.

Given $k \in \mathbb{N}$, define

$$A_n := \{w : |f_n(\omega) - f(\omega)| > \frac{1}{k}\}.$$

Since $P(A_n) \to 0$, define n_k to be the smallest n such that $P(A_{n_k}) \leq \frac{1}{2^k}$. Clearly $\sum_{k=1}^{\infty} P(A_{n_k}) \leq 1 < \infty$, hence (by the Borel-Cantelli lemma) $P(\bar{A}) = 0$ where $\bar{A} = \cap_k \cup_{m \geq k} A_{n_m}$. Note that $\bar{A}$ coincides with the set $\{\omega : \limsup_k |f_{n_k}(\omega) - f(\omega)| > 0\}$.