Proofs of the Parisi and Coppersmith-Sorkin Conjectures for the Finite Random Assignment Problem*

Chandra Nair Balaji Prabhakar Mayank Sharma
Stanford University.
mchandra,balaji,msharma@stanford.edu

Abstract

Suppose that there are n jobs and n machines and it costs c_{ij} to execute job i on machine j. The assignment problem concerns the determination of a one-to-one assignment of jobs onto machines so as to minimize the cost of executing all the jobs. The average case analysis of the classical random assignment problem has received a lot of interest in the recent literature, mainly due to the following pleasing conjecture of Parisi: The average value of the minimum-cost permutation in an $n \times n$ matrix with i.i.d. $\exp(1)$ *entries equals $\sum_{i=1}^{n} \frac{1}{i^2}$. Coppersmith and Sorkin (1999) have generalized Parisi's conjecture to the average value of the smallest k-assignment when there are n jobs and m machines. We prove both conjectures based on a common set of combinatorial and probabilistic arguments.*

1. Introduction

Suppose there are n jobs and n machines and it costs c_{ij} to execute job i on machine j. An assignment (or a matching) is a one-to-one mapping, $\pi : \{1, ..., n\} \to \{1, ..., n\}$, of jobs onto machines. The cost of the assignment π equals $\sum_{i=1}^{n} c_{i,\pi(i)}$, and the assignment problem is about finding the minimum cost assignment. Let $A_n = \min_\pi \sum_{i=1}^{n} c_{i,\pi(i)}$ represent the cost of the minimizing assignment. In the *random* assignment problem the c_{ij}'s are random variables drawn from some distribution, and the quantity of interest is the expected minimum cost, $\mathbb{E}(A_n)$.

When the c_{ij} are i.i.d. $\exp(1)$ variables, Parisi [21] has made the following conjecture:

$$\mathbb{E}(A_n) = \sum_{i=1}^{n} \frac{1}{i^2}.$$

Coppersmith and Sorkin [6] have proposed a larger class of conjectures which state that the expected cost of the minimum k-assignment in an $m \times n$ matrix of i.i.d. exp(1) entries is:

$$F(k, m, n) = \sum_{i,j \geq 0, i+j<k} \frac{1}{(m-i)(n-j)}.$$

By definition, $F(n, n, n) = \mathbb{E}(A_n)$ and their expression coincides with Parisi's conjecture.

In this paper, we prove Parisi's conjecture by two different but related strategies. Both involve establishing the exponentiality of the increments of the weights of matchings. The first builds on the work of Sharma and Prabhakar [22] and establishes Parisi's conjecture by showing that certain increments of weights of matchings are exponentially distributed with a given rate and are independent. The second method builds on the work of Nair [18] and establishes the Coppersmith-Sorkin conjectures. It does this by showing that certain other increments are exponentials with given rates; the increments are not required to be (and, in fact, are not) independent.

The two methods mentioned above use a common set of combinatorial and probabilistic arguments. For ease of exposition, we choose to present the proof of the conjectures in [22] first. We then show how the arguments also resolve the conjectures in [18]. Before surveying prior work, it is important to mention that simultaneously and independently of our work Linusson and Wästlund [15] have also announced a proof of the Parisi and Coppersmith-Sorkin conjectures based on a quite different approach.

1.1. Background and related work

There has been a lot of work on determining bounds for the expected minimum cost, A_n, and calculating its asymp-

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totic value. Assuming, for now, that $\lim_n \mathbb{E}(A_n)$ exists, let us denote it by A^*. We survey some of the work; more details can be found in [24, 6]. Early work uses feasible solutions to the dual linear programming (LP) formulation of the assignment problem for obtaining the following lower bounds for A^*: $(1 + 1/e)$ by Lazarus [13], 1.441 by Goemans and Kodialam [8], and 1.51 by Olin [20]. The first upper bound of 3 was given by Walkup [26], who thus demonstrated that $\limsup_n E(A_n)$ is finite. Walkup's argument was later made constructive by Karp *et al* [12]. Karp [10, 11] made a subtle use of LP duality to obtain a better upper bound of 2. Coppersmith and Sorkin [6] have further improved the bound to 1.94.

Meanwhile, it had been observed through simulations that for large n, $E(A_n) \approx 1.642$ [4]. Mézard and Parisi [16] used the *replica method* [17] of statistical physics to argue that $A^* = \frac{\pi^2}{6}$. (Thus, Parisi's conjecture for the finite random assignment problem is an elegant restriction, for i.i.d. exp(1) costs, of the asymptotic result to the first n terms in the expansion: $\frac{\pi^2}{6} = \sum_{i=1}^{\infty} \frac{1}{i^2}$.) More interestingly, their method allowed them to determine the density of the edge-weight distribution of the limiting optimal matching. These sharp (but non-rigorous) asymptotic results, and others of a similar flavor that they obtained in several combinatorial optimization problems, sparked interest in the replica method and in the random assignment problem.

Aldous [1] proved that A^* exists by identifying the limit as the value of a minimum-cost matching problem on a certain random weighted infinite tree. In the same work he also established that the distribution of c_{ij} affects A^* only through the value of its density function at 0 (provided it exists and is strictly positive). Thus, as far as the value of A^* is concerned, the distributions $U[0, 1]$ and $\exp(1)$ are equivalent. More recently, Aldous [2] has established that $A^* = \pi^2/6$, and obtained the same limiting optimal edge-weight distribution as in [16]. He also obtains a number of other interesting results such as the asymptotic essential uniqueness (AEU) property—which roughly states that almost-optimal matchings have almost all their edges equal to those of the optimal matching.

Another notable paper on the infinite random assignment problem is due to Talagrand [25]. He considers a version of the assignment problem in the "very high temperature" regime and rigorously establishes that the structure of the solution is indeed as predicted by the replica method. This work constitutes a part of a larger program Talagrand has initiated on rigorizing the replica method for combinatorial optimization problems.

Generalizations of Parisi's finite conjecture have also been made in other ways. Linusson and Wästlund [14] conjecture an expression for the expected cost of the minimum k-assignment in an $m \times n$ matrix consisting of zeroes at some specified positions and $\exp(1)$ entries at all other places. Indeed, it is by establishing this conjecture in their recent work [15] that they obtain proofs of the Parisi and Coppersmith-Sorkin conjectures. Buck, Chan and Robbins [5] generalize the Coppersmith-Sorkin conjecture for matrices $M \equiv [m_{ij}]$ with $m_{ij} \sim \exp(r_i c_j)$ for $r_i, c_j > 0$.

Alm and Sorkin [3] verify the Coppersmith-Sorkin conjecture when $k \leq 4$, $k = m = 5$ and $k = m = n = 6$; and Coppersmith and Sorkin [7] study the expected incremental cost of going from the smallest $(m-1)$-assignment in an $(m-1) \times n$ matrix to the smallest m-assignment in an $m \times n$ matrix.

2. Preliminaries

We introduce some notation that will be used in the paper. For an $(n-j) \times n$ matrix, L_{n-j}, of i.i.d. $\exp(1)$ entries define T_1^{n-j} to be the weight of the smallest matching of size $n - j$.

Let K_1 be the matrix formed by the $n - j$ columns used by T_1^{n-j}. Consider the set of all matchings in L_{n-j} of size $n - j$ which use exactly $n - j - 1$ columns of K_1. Define T_2^{n-j} to be the weight of the smallest matching in this set.

Now let K_{12} be the matrix of size $(n-j) \times (n-j-1)$ consisting of the columns used by *both* T_1^{n-j} and T_2^{n-j}. Consider the set of all matchings in L_{n-j} of size $n - j$ which use exactly $n - j - 2$ columns of K_{12}. Let T_3^{n-j} be the weight of the smallest matching in this set.

In this way we can recursively define $K_{12\ldots i}$, $i = 3, \ldots, n-j$ to be the matrix of size $(n-j) \times (n-j-i+1)$ consisting of the columns present in all the matchings $\{T_1^{n-j}, T_2^{n-j}, \ldots, T_i^{n-j}\}$, and obtain T_4^{n-j}, ..., T_{n-j+1}^{n-j} as above. We shall refer to the matchings $\{T_1^{n-j}, ..., T_{n-j+1}^{n-j}\}$ as the *T-matchings* of the matrix L_{n-j}.[1]

The *T-matchings* for a 2×3 matrix are illustrated below.

L_2:

3	6	11
9	2	20

[1]Conventions: (1) If the minimum weight permutation is not unique, we shall consistently break the tie in favor of one of these permutations. (2) To avoid an explosion of notation, we shall use the same symbol for both the *name* of a matching and for its *weight*. For example, the T_1 defined above might refer to the smallest matching as well as to its weight.

3	6		6	**11**		3	**11**
9	**2**		**2**	20		**9**	20

$T_1^2 = 5$ $\quad T_2^2 = 13$ $\quad T_3^2 = 20$

Conjecture 1 *[22] For $j = 1, \ldots, n-1$, $T_{j+1}^{n-1} - T_j^{n-1} \sim \exp(j(n-j))$ and these increments are independent of each other.*[2]

Remarks: It was shown in [22] that Conjecture 1 implies Parisi's conjecture: $E(A_n) = \sum_{i=1}^{n} \frac{1}{i^2}$. Further, it was established that $T_2^{n-1} - T_1^{n-1} \sim \exp(n-1)$ and that $T_3^{n-1} - T_2^{n-1} \perp\!\!\!\perp T_2^{n-1} - T_1^{n-1}$. As noted in [22], a natural generalization of their argument for showing $T_2^{n-1} - T_1^{n-1} \sim \exp(n-1)$ fails for higher increments. However, we use a different generalization in this paper, involving a subtle randomization step, to prove Conjecture 1.

2.1. A Sketch of the Proof

Our proof of Conjecture 1 is inductive, and follows the steps below.

INDUCTIVE HYPOTHESIS: Start with a matrix L_{n-j} of size $(n-j) \times n$ containing i.i.d. $\exp(1)$ random variables such that $T_{k+1}^{n-1} - T_k^{n-1} = T_{k-j+2}^{n-j} - T_{k-j+1}^{n-j}, k \geq j$.

INDUCTION STEP:

Step 1: Obtain L_{n-j-1}, a matrix of size $(n-j-1) \times n$, from L_{n-j} such that

$$T_{k+1}^{n-1} - T_k^{n-1} = T_{k-j+1}^{n-j-1} - T_{k-j}^{n-j-1}, k \geq j+1.$$

Step 2: Establish that the entries of L_{n-j-1} are i.i.d. $\exp(1)$ random variables.

This completes the induction step since L_{n-j-1} satisfies the induction hypothesis for the next iteration.

In Step 2 we also show the following:

1. $T_2^{n-j} - T_1^{n-j} \sim \exp(j(n-j))$ and hence conclude that $T_{j+1}^{n-1} - T_j^{n-1} \sim \exp(j(n-j))$.
2. $T_{j+1}^{n-1} - T_j^{n-1}$ is independent of L_{n-j-1}. Observe that the higher increments $T_{k+1}^{n-1} - T_k^{n-1}$, for $k > j$, are functions of L_{n-j-1}. This will allow us to conclude that $T_{j+1}^{n-1} - T_j^{n-1}$ is independent of $T_{k+1}^{n-1} - T_k^{n-1}$ for $k > j$.

[2] The symbol '$\sim$' stands for 'is distributed as' and the symbol '$\perp\!\!\!\perp$' stands for 'is independent of'.

Remark: The randomization procedure alluded to earlier is used in obtaining L_{n-j-1} from L_{n-j} and ensures that L_{n-j-1} has i.i.d. $\exp(1)$ entries.

We state some combinatorial properties regarding matchings in Section 3 that will be useful for the rest of the paper. Section 4 establishes the above induction, thus completing the proof of Conjecture 1. We extend the method used in Section 4 to prove the Coppersmith-Sorkin conjecture in Section 5. Due to page limitations the proofs of the combinatorial properties and some lemmas in Section 4 have been removed. The interested reader can find them in the full version of the paper, [19].

3. Some combinatorial properties of matchings

Lemma 1 *Consider an $n \times (n+l)$ matrix M and let K_1 be the set of columns used by T_1^n. Then for any column $\mathbf{c} \in K_1$, the smallest matching of size n that doesn't use $\mathbf{c}$, denoted by $\mathcal{M}_{\mathbf{c}}$, contains exactly one element outside the columns in K_1.*

Lemma 2 *Consider an $n \times (n+l)$ matrix M in which T_1^n is the smallest matching of size n. Suppose there exists a collection of n columns in M, denoted by K, with the property that the smallest matching $\mathcal{M}$ of size n in K is lighter than $\mathcal{M}'$: the smallest among all matchings of size n that have exactly one element outside K. Then $\mathcal{M} = T_1^n$.*

Lemma 3 *Let $\mathcal{M}$ be the smallest matching of size n in an $n \times (n+l)$ matrix M. Let M_{ext} be any $(n+r) \times (n+l)$ matrix ($r \leq l$) whose first n rows equal M (i.e. M_{ext} is an extension of M). Let $\mathcal{M}'$ denote the smallest matching of size $n+r$ in M_{ext}. Then the set of columns in which the elements of $\mathcal{M}$ lie is a subset of the columns in which the elements of $\mathcal{M}'$ lie.*

The following is a useful alternate description of T_i^{n-j}, $i \geq 2$. Let S_i^{n-j} be the weight of the smallest matching in L_{n-j} after removing its i^{th} column. Note that for all $i \notin K_1$, $S_i^{n-j} = T_1^{n-j}$. Consider S_i^{n-j} for $i \in K_1$ and arrange them in increasing order to get $\{R_i^{n-j}, i = 1, ..., n-j\}$.

Lemma 4 $T_{i+1}^{n-j} = R_i^{n-j}, \quad i = 1, ..., n-j.$

Corollary 1 *The matchings T_i^{n-j}, $i = 2, ..., n-j+1$ contain exactly one element outside K_1.*

Corollary 2 *Given a matrix L_{n-j}, arranging the S_i^{n-j} for $i \in K_{12\ldots k}$ in increasing order gives the sequence $T_{k+1}^{n-j}, T_{k+2}^{n-j}, \ldots, T_{n-j+1}^{n-j}$.*

Lemma 5 *[23] Consider all the elements of an $n \times (n+1)$ matrix M that participate in at least one of the matchings T_i (or, equivalently, in one of the S_j). The total number of such elements is $2n$ and exactly 2 such elements are present in every row of M.*

Lemma 6 *Consider an $n \times (n+l)$ matrix M and mark all the elements that participate in any one of the matchings $T_1^n, T_2^n, \ldots, T_{n+1}^n$ defined in Section 1. Then exactly two elements will be marked in each row.*

4. Proof of Conjecture 1

We shall execute the two steps mentioned earlier. Thus, we shall begin with an $(n-j) \times n$ matrix, L_{n-j}, which has i.i.d. $\exp(1)$ entries.

Now consider the general step j. Suppose we have that L_{n-j} contains i.i.d. $\exp(1)$ entries and $T_{k+1}^{n-1} - T_k^{n-1} = T_{k-j+2}^{n-j} - T_{k-j+1}^{n-j}$ for $k \geq j$, where the T_i^{n-j} are as defined in Section 1.

4.1. Step 1: Obtaining L_{n-j-1} from L_{n-j}

The matrix L_{n-j-1} is obtained from L_{n-j} by applying the series of operations Φ, Λ and Π, as depicted below

$$L_{n-j} \xrightarrow{\Phi} L_{n-j}^* \xrightarrow{\Lambda} \tilde{L}_{n-j} \xrightarrow{\Pi} L_{n-j-1}.$$

It would be natural to represent the *T-matchings* corresponding to each of the intermediate matrices L_{n-j}^* and $\tilde{L}_{n-j}$ using appropriate superscripts. For example, the *T-matchings* of L_{n-j}^* can be denoted $\{T_i^{*(n-j)}\}$. However, this creates a needless clutter of symbols. We will instead denote L_{n-j}^* and $\{T_i^{*(n-j)}\}$ simply as L^* and $\{T_i^*\}$, respectively. Table 1 summarizes the notation we shall use, with the matrix, its dimensions and notation for its *T-matchings* given in the first, second and third rows respectively.

L_{n-j}	$L^* = L_{n-j}^*$	$\tilde{L} = \tilde{L}_{n-j}$	L_{n-j-1}
$(n-j) \times n$	$(n-j) \times n$	$(n-j) \times n$	$(n-j-1) \times n$
$\{T_i^{n-j}\}$	$\{T_i^*\}$	$\{\tilde{T}_i\}$	$\{T_i^{n-j-1}\}$

Table 1. Operations to transform L_{n-j} to L_{n-j-1}.

We now specify the operations Φ, Λ and Π.

Φ : In the matrix L_{n-j}, subtract the value $T_2^{n-j} - T_1^{n-j}$ from each entry not in the sub-matrix K_1. (Recall from Section 1 that K_1 denotes the $(n-j) \times (n-j)$ sub-matrix of L_{n-j} whose columns contain the entries used by T_1^{n-j}.) Let the resultant matrix of size $(n-j) \times n$ be L^*.

Λ : Consider the matrix L^*. Generate a random variable X with $\mathbb{P}(X=1) = \mathbb{P}(X=2) = \frac{1}{2}$. Define Y so that $Y=1$ when $X=2$ and $Y=2$ when $X=1$. Denote by e_1 the unique entry of the matching T_X^* outside the matrix K_{12}. Denote by e_2 the unique entry of the matching T_Y^* outside the matrix K_{12}. Now remove the row of L^* in which e_1 is present and append it to the bottom of the matrix.[3] Call the resultant matrix $\tilde{L}$. Note that $\tilde{L}$ is a row permutation of L^*.

Π : Remove the last row from $\tilde{L}$ (i.e., the row containing e_1 and appended to the bottom in operation Λ) to obtain the matrix L_{n-j-1}.

Lemma 7 *The following statements hold:*
(i) $T_2^* = T_1^* = T_1^{n-j}$.
(ii) *For* $i \geq 2$, $T_{i+1}^* - T_i^* = T_{i+1}^{n-j} - T_i^{n-j}$.
(iii) *For* $i \geq 1$, $\tilde{T}_i = T_i^*$.

Proof Since T_1^{n-j} is entirely contained in the sub-matrix K_1, its weight remains the same in L^*. Let $\mathcal{S}$ be the set of all matchings of size $n-j$ in L^* that contain exactly one element outside the columns of K_1. From the definition of L^*, it is clear that every matching in $\mathcal{S}$ is lighter by exactly $T_2^{n-j} - T_1^{n-j}$ compared to its weight in L_{n-j}. From the definition of T_2^{n-j} we know that every matching in $\mathcal{S}$ had a weight larger than (or equal to) T_2^{n-j} in the matrix L_{n-j}. Therefore, every matching in $\mathcal{S}$ has a weight larger than (or equal to) $T_2^{n-j} - (T_2^{n-j} - T_1^{n-j})$ in L^*. So every matching that has exactly one element outside the columns of K_1 in L^* has a weight larger than (or equal to) T_1^{n-j}. Therefore, from Lemma 2 it follows that T_1^{n-j} is the smallest matching in L^*. Thus, we have $T_1^* = T_1^{n-j}$.

From Corollary 1 we know that T_i^* has exactly one element outside the columns of K_1 for $i \geq 2$. Since every matching in $\mathcal{S}$ is lighter by $T_2^{n-j} - T_1^{n-j}$ from its weight in L_{n-j}, it follows that $T_i^* = T_i^{n-j} - (T_2^{n-j} - T_1^{n-j})$ for $i \geq 2$. Substituting $i = 2$, we obtain $T_2^* = T_1^{n-j}$. This proves part (i). And considering the differences $T_{i+1}^* - T_i^*$ establishes part (ii).

Since the values of *T-matchings* are invariant under row and column permutations, part (iii) follows from the fact that $\tilde{L}$ is a row permutation of L^*. ■

To complete Step 2 of the induction we need to establish that L_{n-j-1} has the following properties.

[3] The random variable X is used to break the tie between the two matchings T_1^* and T_2^*, both of which have the same weight (see Lemma 7). This randomized tie-breaking is essential for ensuring that L_{n-j-1} has i.i.d. $\exp(1)$ entries; indeed, if we were to choose the entry in T_1^* (or, for that matter, in T_2^*) with probability 1, then the corresponding L_{n-j-1} will not have i.i.d. $\exp(1)$ entries.

Lemma 8 $T_{k+1} - T_k = T_{k-j+1}^{n-j-1} - T_{k-j}^{n-j-1}$, $j+1 \le k \le n-1$.

Proof The proof of the lemma consists of establishing the following: For $k = j+1, \ldots, n-1$

$$\begin{aligned} T_{k+1} - T_k &\stackrel{(a)}{=} T_{k-j+2}^{n-j} - T_{k-j+1}^{n-j} \\ &\stackrel{(b)}{=} \tilde{T}_{k-j+2} - \tilde{T}_{k-j+1} \\ &\stackrel{(c)}{=} T_{k-j+1}^{n-j-1} - T_{k-j}^{n-j-1}, \end{aligned}$$

where (a) follows from the induction hypothesis on $T_{i+1}^{n-j} - T_i^{n-j}$, and (b) follows from Lemma 7. We shall establish (c) by showing that

$$\tilde{T}_i = T_{i-1}^{n-j-1} + v,\ i = 2, \ldots, n-j+1 \qquad (1)$$

for some appropriately defined constant v.

Two cases arise: For e_1 and e_2 as defined in the operation Λ, Case 1 is when e_1 and e_2 are present in the last row of $\tilde{L}$, and Case 2 is when e_1 is present in the last row of $\tilde{L}$ and e_2 is present in a different row.

Case 1: We claim that the values of e_1 and e_2 are equal, say to v. This is because e_1 and e_2 choose the same matching of size $n-j-1$ from the matrix K_{12} (call this matching $\mathcal{M}$) to form the matchings $\tilde{T}_1$ and $\tilde{T}_2$. But $\tilde{T}_1 = \tilde{T}_2 = T_1^{n-j}$, from Lemma 7. Therefore $e_1 = e_2$.

Consider all matchings of size $n-j-1$ in L_{n-j-1} that have exactly one entry outside the columns defined by K_{12}. Clearly, one (or possibly both) of the entries e_1 and e_2 could have chosen these matchings to form candidates for $\tilde{T}_1$. The fact that the weight of these candidates is larger than $\tilde{T}_1$ indicates that the weight of these size $n-j-1$ matchings are larger than the weight of $\mathcal{M}$. Thus, from Lemma 2, we have that $\mathcal{M}$ equals T_1^{n-j-1}: the smallest matching of size $n-j-1$ in L_{n-j-1}. Therefore, $T_1^{n-j} = \tilde{T}_1 = \tilde{T}_2 = T_1^{n-j-1} + v$.

Now consider $\tilde{S}_i$, the smallest matching in $\tilde{L}$ obtained by deleting the i^{th} column in K_{12}. Since this is $\tilde{T}_k$ for some $k \ge 3$, $\tilde{S}_i$ must use one of the entries e_1 or e_2, according to Lemma 6. Therefore, $\tilde{S}_i \ge S_i^{n-j-1} + v$, where S_i^{n-j-1} is the weight of the best matching in L_{n-j-1} that doesn't use the i^{th} column in K_{12}.

However, from Corollary 1 applied to L_{n-j-1}, we have that S_i^{n-j-1} has exactly one element outside K_{12} since K_{12} defines the columns in which the smallest matching of L_{n-j-1} is present. Therefore S_i^{n-j-1} can pick at least one of the two entries e_1 or e_2 to form a candidate for $\tilde{S}_i^{n-j}$ that has weight $S_i^{n-j-1} + v$. This implies $\tilde{S}_i \le S_i^{n-j-1} + v$.

This shows $\tilde{S}_i = S_i^{n-j-1} + v$.

But, arranging $\tilde{S}_i$, for all $i \in K_{12}$, in increasing order gives us $\tilde{T}_3, \ldots, \tilde{T}_{n-j+1}$. And arranging S_i^{n-j-1} in increasing order gives us $T_2^{n-j-1}, \ldots, T_{n-j}^{n-j-1}$. These observations follow from Corollary 2. This verifies equation (1) and completes Case 1.

Case 2: Please refer to [19] for the proof of this case.

Corollary 3 *Let $\mathcal{M}$ be the smallest matching of size $n-j-1$ in $\tilde{L}$, contained in the columns of K_{12}, that e_1 goes with to form $\tilde{T}_1 = T_1^{n-j}$. Then $\mathcal{M} = T_1^{n-j-1}$, the smallest matching of size $n-j-1$ in L_{n-j-1}.*

Proof Letting v be the weight of e_1, we note that $\tilde{T}_1$ is formed by e_1 and $\mathcal{M}$. From Equation (1), $\tilde{T}_1$ has a weight equal to $v + T_1^{n-j-1}$. Hence the weight of $\mathcal{M}$ equals T_1^{n-j-1}. ■

4.2. Step 2: L_{n-j-1} has i.i.d. $\exp(1)$ entries

We compute the joint distribution of the entries of L_{n-j-1} and verify that they are i.i.d. $\exp(1)$ variables. To do this, we identify the set, $\mathcal{D}$, of all $(n-j) \times n$ matrices, L_{n-j}, that have a non-zero probability of mapping to a particular realization of L_{n-j-1} under the operations Φ, Λ and Π. By induction, the entries of L_{n-j} are i.i.d $\exp(1)$ random variables. We integrate L_{n-j} over $\mathcal{D}$ to obtain the joint distribution of the entries of L_{n-j-1}.

Accordingly, fix a realization of $L_{n-j-1} \in \mathbb{R}_+^{(n-j-1)\times n}$ and represent it as below

$$L_{n-j-1} = \begin{bmatrix} l_{1,1} & \cdot & \cdot & l_{1,n-1} & l_{1,n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ l_{n-j-1,1} & \cdot & \cdot & l_{n-j-1,n-1} & l_{n-j-1,n} \end{bmatrix}. \qquad (2)$$

Let $\mathcal{D}_\Pi = \Pi^{-1}(L_{n-j-1})$ be the pre-image of L_{n-j-1} under Π, and let $\mathcal{D}_\Lambda = \Lambda^{-1}\Pi^{-1}(L_{n-j-1})$. Now Λ is a random map, whose action depends on the value taken by X. In turn, this is related to whether e_1 and e_2 are on the same row or not. Therefore we may write $\mathcal{D}_\Lambda$ as the disjoint union of the sets $\mathcal{D}_\Lambda^s$ and $\mathcal{D}_\Lambda^d$, which respectively correspond to e_1 and e_2 belonging to the same and to different rows. Finally, $\mathcal{D} = \Phi^{-1}\Lambda^{-1}\Pi^{-1}(L_{n-j-1})$, the set we're trying to determine.

Consider an $M' \in \mathbb{R}_+^{(n-j)\times n}$, such as represented below

$$M' = \begin{bmatrix} & & & L_{n-j-1} & & & \\ r_1 & r_2 & \cdot\ \cdot\ \cdot & r_{n-j-1} & x_1 & \cdot\ \cdot & x_{j+1} \end{bmatrix}.$$

In the above representation $r_1, ..., r_{n-j-1}$ denote the elements in the columns of K_1^{n-j-1}, the sub-matrix of L_{n-j-1} which contains T_1^{n-j-1}.

We shall show that $\mathcal{D}_\Pi$ is precisely the set of all such M' for which the vector $(r_1, ..., r_{n-j-1}, x_1, ..., x_{j+1})$ satisfies the conditions stated in Lemma 9.

Consider an element d outside the columns of K_1^{n-j-1} in L_{n-j-1}. Let Δ_d be the cost of the smallest matching, say $\mathcal{M}_d$, of size $n-j-1$ with the following property: The entries of $\mathcal{M}_d$ are in $K_1^{n-j-1} \cup (r_1, .., r_{n-j-1})$ and no entry is present in the same row as d. Clearly $d \cup \mathcal{M}_d$ is a matching of size $n-j$ in the matrix M'. Let d_o be that entry outside the columns of K_1^{n-j-1} in L_{n-j-1} which minimizes $d + \Delta_d$. Let $J = d_o + \Delta_{d_o}$, and let j_o denote the column in which d_o occurs.

Given any $\vec{r} = (r_1, ..., r_{n-j-1}) \in \mathbb{R}_+^{n-j-1}$, the following lemma stipulates conditions that $(x_1, ..., x_{j+1})$ must satisfy so that M' is in $\mathcal{D}_\Pi$.

Lemma 9 *For any $\vec{r} \in \mathbb{R}_+^{n-j-1}$, let $\mathcal{S}_\Pi(\vec{r})$ be the collection of all M' such that one of the following two conditions hold:*

(i) *There exist i and k such that $x_i = x_k$, $x_i + T_1^{n-j-1} < J$ and $x_m > x_i$ for all $m \neq i, k$.*

(ii) *There exists $i \neq j_o$ such that $x_m > x_i$ for all $m \neq i$ and $x_i + T_1^{n-j-1} = J$.*

Then $\mathcal{D}_\Pi = \mathcal{S}_\Pi \stackrel{\triangle}{=} \bigcup_{\vec{r} \in \mathbb{R}_+^{n-j-1}} \mathcal{S}_\Pi(\vec{r})$.

Proof Please refer to [19] for a proof. ■

Let $\mathcal{D}_\Pi^s$ and $\mathcal{D}_\Pi^d$ be the subsets of matrices in $\mathcal{D}_\Pi$ that satisfy Conditions (i) and (ii) of Lemma 9, respectively. Clearly $\mathcal{D}_\Pi$ is the disjoint union of $\mathcal{D}_\Pi^s$ and $\mathcal{D}_\Pi^d$. (The superscripts s and d are mnemonic for whether e_1 and e_2 occurred in the same row or in different rows.)

Now that we have identified $\mathcal{D}_\Pi$ explicitly in Lemma 9, $\mathcal{D}_\Lambda$ can be identified following manner: Pick a matrix $M_1 \in \mathcal{D}_\Pi$ and form $n-j$ matrices by removing the last row of M_1 and placing it back as the ith row, for $i = 1, ..., n-j$. Call this collection of $n-j$ matrices $\mathcal{S}_\Lambda(M_1)$. Define

$$\mathcal{S}_\Lambda = \bigcup_{M_1 \in \mathcal{D}_\Pi} \mathcal{S}_\Lambda(M_1).$$

Lemma 10 $\mathcal{S}_\Lambda = \mathcal{D}_\Lambda$.

Proof Please refer to [19] for a proof. ■

Partition $\mathcal{D}_\Lambda$ into $\mathcal{D}_\Lambda^s = \mathcal{S}_\Lambda(\mathcal{D}_\Pi^s)$ and $\mathcal{D}_\Lambda^d = \mathcal{S}_\Lambda(\mathcal{D}_\Pi^d)$; these partitions correspond to the cases where e_1 and e_2 are in the same row and in different rows. Also observe that when $M \in \mathcal{D}_\Lambda^s$ we have $\Pi\Lambda(M) = L_{n-j-1}$ with probability one and when $M \in \mathcal{D}_\Lambda^d$ we have $\Pi\Lambda(M) = L_{n-j-1}$ with probability $\frac{1}{2}$.

We are finally ready to characterize $\mathcal{D}$, the set of L_{n-j}'s that map to a particular realization of L_{n-j-1} with non-zero probability.

Consider any $M \in \mathcal{D}_\Lambda$. Let θ be any positive number. Consider the column, say $\mathbf{c}_1$ in M which contains x_i. (Recall, from Lemma 9, that x_i is the smallest of the x_m's in the last row deleted by Π.) Let K be the union of the columns of K_1^{n-j-1} and $\mathbf{c}_1$. Add θ to every entry in M outside the columns K. Denote the resultant matrix by $S_1(\theta, M)$. Let

$$\mathcal{S}_1 = \bigcup_{\theta > 0, M \in \mathcal{D}_\Lambda} S_1(\theta, M).$$

Now consider the column, say $\mathbf{c}_2$ in M where the entry x_k or d_o (depending on whether $\Lambda(M)$ satisfies condition (i) or condition (ii) of Lemma 9) is present. Let K' be the union of columns of K_1^{n-j-1} and $\mathbf{c}_2$. Now add θ to every entry in M outside the columns K'. Call this matrix $S_2(\theta, M)$. Let

$$\mathcal{S}_2 = \bigcup_{\theta > 0, M \in \mathcal{D}_\Lambda} S_2(\theta, M),$$

and note that $\mathcal{S}_1$ and $\mathcal{S}_2$ are disjoint since $\mathbf{c}_1 \neq \mathbf{c}_2$.
Remark: Note that θ is added to precisely $j(n-j)$ entries in M in each of the two cases above.

Lemma 11 $\mathcal{D} = \mathcal{S}_1 \cup \mathcal{S}_2$.

Proof Please refer to [19] for a proof. ■

Remark: Note that the variable θ used in the characterization of $\mathcal{D}$ precisely equals the value of $T_2^M - T_1^M$, as shown in the proof of Lemma 11.

Continuing, we can partition $\mathcal{D}$ into the two sets $\mathcal{D}^s$ and $\mathcal{D}^d$ as below:

$$\begin{aligned} \mathcal{D}^s &= S_1(\mathbb{R}_+, \mathcal{D}_\Lambda^s) \cup S_2(\mathbb{R}_+, \mathcal{D}_\Lambda^s) \\ \mathcal{D}^d &= S_1(\mathbb{R}_+, \mathcal{D}_\Lambda^d) \cup S_2(\mathbb{R}_+, \mathcal{D}_\Lambda^d). \end{aligned} \tag{3}$$

Observe that whenever $M \in \mathcal{D}^s$, we have $\Phi(M) \in \mathcal{D}_\Lambda^s$ and hence $\Pi\Lambda\Phi(M) = L_{n-j-1}$ with probability 1. For $M \in \mathcal{D}^d$, $\Phi(M) \in \mathcal{D}_\Lambda^d$. Hence $\Pi\Lambda\Phi(M) = L_{n-j-1}$ with probability $\frac{1}{2}$.

Now that we have characterized $\mathcal{D}$, we "integrate out the marginals" $(r_1, ..., r_{n-j-1}, x_1, ..., x_{j+1})$ and θ by setting

$$\vec{v} = (L_{n-j-1}, \vec{r}, \theta) \quad \text{and} \quad \vec{w} = (\vec{v}, \vec{x}),$$

where $L_{n-j-1} \in \mathbb{R}_+^{(n-j-1)\times n}$ is as defined at equation (2). We will evaluate $f_v(\vec{v}) = \int_{\mathcal{R}_1} f_w(\vec{v},\vec{x})d\vec{x} + \int_{\mathcal{R}_2} f_w(\vec{v},\vec{x})d\vec{x}$, to obtain the marginal density of $\vec{v}$. The regions $\mathcal{R}_1$ and $\mathcal{R}_2$ are defined by the set of all $\vec{x}$s that satisfy (i) and (ii) of Lemma 9, respectively.

On $\mathcal{R}_1$, we have that $x_i = x_k < J - T_1^{n-j-1}$ for J as in Lemma 9. We shall set $U = J - T_1^{n-j-1}$, and $u_m = x_m - x_i$ for $m \neq i, k$. Finally, define

$$s_v = l_{1,1} + ... + l_{n-j-1,n} + r_1 + ... + r_{n-j-1} + j(n-j)\theta.$$

Thus, s_v denotes the sum of all of the entries of L_{n-j} except those in $\vec{x}$. As noted in the remark preceding Lemma 11, the value θ was added to precisely $j(n-j)$ entries. We have

$$\begin{aligned} & \int_{\mathcal{R}_1} f_w(\vec{v},\vec{x})d\vec{x} \\ \stackrel{(a)}{=} \quad & 2(n-j)\binom{j+1}{2}\int_0^U \int_0^\infty \int \cdots \int_0^\infty \\ & e^{-(s_v+(j+1)x_i+u_1+\ldots+u_{j-1})}\, du_1 \ldots du_{j-1}\, dx_i \\ = \quad & j(n-j)e^{-s_v}\left(1-e^{-(j+1)U}\right). \end{aligned} \tag{4}$$

The factor $\binom{j+1}{2}$ at equality (a) comes from the possible choices for i and k from 1,...,$j+1$, the factor $(n-j)$ comes from the row choices available to e_1 as in Lemma 10, and the factor 2 corresponds to the partition, $\mathcal{S}_1$ or $\mathcal{S}_2$, that L_{n-j} belongs to.

Similarly, on $\mathcal{R}_2$, we have that $x_i = J - T_1^{n-j-1} \triangleq U$ and we shall set $u_m = x_m - x_i$ for $m \neq i$ to obtain

$$\begin{aligned} & \int_{\mathcal{R}_2} f_w(\vec{v},\vec{x})d\vec{x} \\ \stackrel{(b)}{=} \quad & \frac{1}{2}[2(n-j)j\int_0^\infty \int \cdots \int_0^\infty \\ & e^{-(s_v+(j+1)U+u_1+\ldots+u_j)}\, du_1 \ldots du_j] \\ = \quad & j(n-j)e^{-s_v}e^{-(j+1)U}. \end{aligned} \tag{5}$$

In equality (b) above, the factors j, $(n-j)$ and 2 come, respectively, from the choice of positions available to x_i,[4] the row choices available to e_1 and the partition, $\mathcal{S}_1$ or $\mathcal{S}_2$, that L_{n-j} belongs to. The factor $\frac{1}{2}$ comes from the fact that on $\mathcal{R}_2$, e_1 and e_2 occur on different rows. Therefore, L_{n-j} is in $\mathcal{D}^d$ and will map to the desired L_{n-j-1} with probability $\frac{1}{2}$.

Putting (4) and (5) together, we obtain

$$\begin{aligned} f_v(\vec{v}) &= j(n-j)e^{-s_v} \\ &= e^{-(l_{1,1}+\ldots+l_{n-j-1,n})} \times j(n-j)e^{-j(n-j)\theta} \\ &\quad \times e^{-(r_1+\ldots+r_{n-j-1})}. \end{aligned}$$

[4]Note that there are only j choices available to x_i since it has to occur in a column other than the one in which d_o occurs.

We summarize the above in the following lemma.

Lemma 12 *The following hold:*
(i) L_{n-j-1} *consists of i.i.d.* $\exp(1)$ *variables.*
(ii) $\theta = T_2^{n-j} - T_1^{n-j}$ *is an* $\exp(j(n-j))$ *random variable.*
(iii) $\vec{r}$ *consists of i.i.d.* $\exp(1)$ *variables.*
(iv) L_{n-j-1}, $(T_2^{n-j} - T_1^{n-j})$, *and* $\vec{r}$ *are independent.*

From Lemma 8 we know that the increments $\{T_{k+1} - T_k, k > j\}$ are a function of the entries of L_{n-j-1}. Given this and the independence of L_{n-j-1} and $T_{j+1} - T_j$ from the above lemma, we get the following corollary.

Corollary 4 $T_{j+1} - T_j$ *is independent of* $T_{k+1} - T_k$ *for* $k > j$.

In conjunction with Lemma 12, Corollary 4 completes the proof of Conjecture 1. It has been shown in [22] that establishing Conjecture 1 proves Parisi's conjecture.

5. The Coppersmith-Sorkin Conjecture

As mentioned in the introduction, Coppersmith and Sorkin [6] have conjectured that the expected cost of the minimum k-assignment in an $m \times n$ matrix, P, of i.i.d. exp(1) entries is:

$$F(k,m,n) = \sum_{i,j\geq 0, i+j<k} \frac{1}{(m-i)(n-j)}. \tag{6}$$

Consider the matrix P and w.l.o.g. assume that $m \leq n$. For $1 \leq p \leq m$, let V_1^p be the weight of the smallest matching of size p and denote the columns it occupies by K_1^p. Next, for $1 \leq p \leq m-1$, define $V_2^p, \ldots, V_{p+1}^p$ to be the ordered sequence (increasingly by weight) of the smallest matching obtained by deleting one column of K_1^p at a time. We shall refer to the set $V_1^m \bigcup \left(\bigcup_{p=1}^{m-1}\{V_1^p, \ldots, V_{p+1}^p\}\right)$ as the *V-matchings* of the matrix P. Nair [18] has made the following distributional conjectures regarding the increments of the *V-matchings*.

Conjecture 2 *[18] The following hold for* $1 \leq i \leq m-1$*:*

$$\begin{aligned} V_2^i - V_1^i &\sim \exp(m(n-i)) \\ & \;\;\; . \quad . \quad . \\ V_{p+1}^i - V_p^i &\sim \exp((m-p+1) \\ &\qquad \times (n-i+p-1)) \\ & \;\;\; . \quad . \quad . \\ V_{i+1}^i - V_i^i &\sim \exp((m-i+1)(n-1)) \\ V_1^{i+1} - V_{i+1}^i &\sim \exp((m-i)n) \end{aligned}$$

We have grouped the increments according to size. That is, the i^{th} group consists of the differences in the weights: $V_1^i, V_2^i, \ldots, V_{i+1}^i, V_1^{i+1}$, and V_1^i is the matching of size i, V_2^i is the second smallest of size i, etc, until V_1^{i+1}—the smallest matching of size $i+1$. Note that according to Conjecture 2 the telescopic sum $V_1^{i+1} - V_1^i$ has expected value $F(i+1,m,n) - F(i,m,n)$. Note also that $V_1^1 \sim \exp(mn)$, being the smallest of $m \times n$ independent $\exp(1)$ variables.

The rest of the Section is devoted to establishing Conjecture 2 for the i^{th} group.

Proof of Conjecture 2

We will establish the conjectures for the i^{th} group inductively, as in Section 4. Consider a matrix, P_{m-j+1}, of size $(m-j+1) \times n$ and let $\{V_p^{q,m-j+1}\}$ denote[5] its *V-matchings*. The induction consists of the following two steps:

Inductive Hypothesis:

- Assume the increments satisfy the following combinatorial identities

$$\begin{array}{rcl}
V_2^{i-j+1,m-j+1} - V_1^{i-j+1,m-j+1} & = & V_{j+1}^i - V_j^i \quad (7) \\
V_3^{i-j+1,m-j+1} - V_2^{i-j+1,m-j+1} & = & V_{j+2}^i - V_{j+1}^i \\
\cdots & \cdots & \cdots \\
V_{i-j+2}^{i-j+1,m-j+1} - V_{i-j+1}^{i-j+1,m-j+1} & = & V_{i+1}^i - V_i^i \\
V_1^{i-j+2,m-j+1} - V_{i-j+2}^{i-j+1,m-j+1} & = & V_1^{i+1} - V_{i+1}^i.
\end{array}$$

- The entries of P_{m-j+1} are i.i.d. $\exp(1)$ random variables.

Induction Step:

Step 1: From P_{m-j+1}, form a matrix P_{m-j} of size $(m-j) \times n$ with the property that

$$\begin{array}{rcl}
V_2^{i-j,m-j} - V_1^{i-j,m-j} & = & V_{j+2}^i - V_{j+1}^i \\
V_3^{i-j,m-j} - V_2^{i-j,m-j} & = & V_{j+3}^i - V_{j+2}^i \\
\cdots & \cdots & \cdots \\
V_{i-j+1}^{i-j,m-j} - V_{i-j}^{i-j,m-j} & = & V_{i+1}^i - V_i^i \\
V_1^{i-j+1,m-j} - V_{i-j+1}^{i-j,m-j} & = & V_1^{i+1} - V_{i+1}^i.
\end{array}$$

Step 2: Establish that the entries of P_{m-j} are i.i.d. $\exp(1)$ random variables.

This completes the induction step since P_{m-j} satisfies the induction hypothesis for the next iteration.

[5]We regret the cumbersome notation; but we must keep track of three indices: one for the number of rows in the matrix (of size $m-j+1 \times n$), one for the size of the matching, q, and one for the rank of the matching, p, among matchings of size q.

In Step 2 we also show that $V_2^{i-j+1,m-j+1} - V_1^{i-j+1,m-j+1} \sim \exp((m-j+1)(n-i+j-1))$ and hence conclude from equation (7) that $V_{j+1}^i - V_j^i \sim \exp((m-j+1)(n-i+j-1))$.

The induction starts at $j=1$ and terminates at $j=i-1$. Observe that the matrix P satisfies the inductive hypothesis for $j=1$ trivially.

Proof of the Induction:

Step 1: Form the matrix $\bar{P}_{m-j+1}$ of size $(m-j+1) \times (m-i+n)$ by adding $m-i$ columns of zeroes to the left of P_{m-j+1} as below

$$\bar{P}_{m-j+1} = [\mathbf{0}\,|P_{m-j+1}\,].$$

Let $\bar{T}_1^{m-j+1}, \ldots, \bar{T}_{m-j+2}^{m-j+1}$ denote the weight of the *T-matchings* of the matrix $\bar{P}_{m-j+1}$. Then we make the following claim.

Claim 1

$$\begin{array}{rcl}
\bar{T}_1^{m-j+1} & = & V_1^{i-j+1,m-j+1} \\
\bar{T}_2^{m-j+1} & = & V_2^{i-j+1,m-j+1} \\
 & \cdots & \\
\bar{T}_{i-j+2}^{m-j+1} & = & V_{i-j+2}^{i-j+1,m-j+1}
\end{array}$$

and

$$\bar{T}_{i-j+3}^{m-j+1} = \bar{T}_{i-j+4}^{m-j+1} = \cdots = \bar{T}_{m-j+2}^{m-j+1} = V_1^{i-j+2,m-j+1}.$$

Proof Note that any matching of size $m-j+1$ in $\bar{P}_{m-j+1}$ can have at most $m-i$ zeroes. Therefore, it is clear that the smallest matching of size $m-j+1$ in $\bar{P}_{m-j+1}$ is formed by picking $m-i$ zeroes along with the smallest matching of size $i-j+1$ in P_{m-j+1}. Thus, $\bar{T}_1^{m-j+1} = V_1^{i-j+1,m-j+1}$.

If we remove any of the columns containing zeroes we get the smallest matching of size $m-j+1$ in the rest of the matrix $\bar{P}_{m-j+1}$ by combining $m-i-1$ zeroes with the smallest matching of size $i-j+2$ in P_{m-j+1}. Hence $m-i$ of the $\bar{T}$'s, corresponding to each column of zeroes, have weight equal to $V_1^{i-j+2,m-j+1}$.

If we remove any column containing $V_1^{i-j+1,m-j+1}$, then the smallest matching of size $m-j+1$ in $\bar{P}_{m-j+1}$ is obtained by $m-i$ zeroes and the smallest matching of size $i-j+1$ in P_{m-j+1} that avoids this column. Hence they have weights $V_p^{i-j+1,m-j+1}$ for $p \in \{2,3,...,i-j+2\}$.

We claim that $V_1^{i-j+2,m-j+1}$ is larger than $V_p^{i-j+1,m-j+1}$ for p in $\{1,2,3,...,i-j+2\}$. Clearly $V_1^{i-j+2,m-j+1} > V_1^{i-j+1,m-j+1}$. Further, for $p \geq 2$, we have a matching of size $i-j+1$ in $V_1^{i-j+2,m-j+1}$

that avoids the same column that $V_p^{i-j+1,m-j+1}$ avoids in $V_1^{i-j+1,m-j+1}$. But $V_p^{i-j+1,m-j+1}$ is the smallest matching of size $i-j+1$ that avoids this column. Hence we conclude that $V_1^{i-j+2,m-j+1} > V_p^{i-j+1,m-j+1}$.

Hence arranging the weights (in increasing order) of the smallest matchings of size $m-j+1$ in $\bar{P}_{m-j+1}$, obtained by removing one column of $\bar{T}_1^{m-j+1}$ at a time, establishes the claim. ■

From the above it is clear that the matchings $\bar{T}_1^{m-j+1}$ and $\bar{T}_2^{m-j+1}$ are formed by $m-i$ zeroes and the matchings $V_1^{i-j+1,m-j+1}$ and $V_2^{i-j+1,m-j+1}$ respectively. Hence, as in Section 4, we have two elements, one each of $\bar{T}_1^{m-j+1}$ and $\bar{T}_2^{m-j+1}$ that lie outside the common columns of $\bar{T}_1^{m-j+1}$ and $\bar{T}_2^{m-j+1}$. Let $\bar{K}_{12}$ denote these common columns. (Note that $\bar{K}_{12}$ necessarily includes the $m-i$ columns of zeros).

We now proceed to perform the procedure outlined in Section 4 for obtaining P_{m-j} from P_{m-j+1} by working through the matrix $\bar{P}_{m-j+1}$ which is an extension of P_{m-j+1}.

Accordingly, form the matrix $\bar{P}^*$ by removing the value $\bar{T}_2^{m-j+1} - \bar{T}_1^{m-j+1}$ from all the entries in $\bar{P}_{m-j+1}$ that lie outside $\bar{K}_{12}$. Generate a random variable X, as before, with $\mathbb{P}(X=1) = \mathbb{P}(X=2) = \frac{1}{2}$. Let $Y=1$ when $X=2$ and let $Y=2$ when $X=1$. Denote by e_1 the unique entry of the matching $\bar{T}_X^*$ outside the matrix $\bar{K}_{12}$. Denote by e_2 the unique entry of the matching $\bar{T}_Y^*$ outside the matrix $\bar{K}_{12}$. Remove the row containing e_1 and call this matrix $\bar{P}_{m-j}$. Now remove the $m-i$ columns of zeroes to obtain the matrix P_{m-j} of size $(m-j)\times n$.

Let $\bar{T}_1^{m-j}, ..., \bar{T}_{m-j+1}^{m-j}$ denote the weight of the *T-matchings* of the matrix $\bar{P}_{m-j}$ and $V_p^{q,m-j}$ denote the *V-matchings* of the matrix P_{m-j}. We make the following claim.

Claim 2

$$
\begin{aligned}
\bar{T}_1^{m-j} &= V_1^{i-j,m-j} \\
\bar{T}_2^{m-j} &= V_2^{i-j,m-j} \\
&\cdots \\
\bar{T}_{i-j+1}^{m-j} &= V_{i-j+1}^{i-j,m-j}
\end{aligned}
$$

and

$$\bar{T}_{i-j+2}^{m-j} = \bar{T}_{i-j+3}^{m-j} = \cdots = \bar{T}_{m-j+1}^{m-j} = V_1^{i-j+1,m-j}.$$

Proof The proof is identical to that of Claim 1. ■

Now from Lemma 8 in Section 4 we know that

$$\bar{T}_{p+2}^{m-j+1} - \bar{T}_{p+1}^{m-j+1} = \bar{T}_{p+1}^{m-j} - \bar{T}_p^{m-j} \text{ for } 1 \le p \le m-j. \tag{8}$$

Finally, combining Equation (8), Claim 1, Claim 2 and the inductive hypothesis on P_{m-j+1} we obtain:

$$
\begin{aligned}
V_2^{i-j,m-j} - V_1^{i-j,m-j} &= V_{j+2}^i - V_{j+1}^i \\
V_3^{i-j,m-j} - V_2^{i-j,m-j} &= V_{j+3}^i - V_{j+2}^i \\
\cdots \quad &\cdots \quad \cdots \\
V_{i-j+1}^{i-j,m-j} - V_{i-j}^{i-j,m-j} &= V_{i+1}^i - V_i^i \\
V_1^{i-j+1,m-j} - V_{i-j+1}^{i-j,m-j} &= V_1^{i+1} - V_{i+1}^i.
\end{aligned}
$$

This completes Step 1 of the induction.

Step 2: Again we reduce the problem to the one in Section 4 by working with the matrices $\bar{P}_{m-j+1}$ and $\bar{P}_{m-j}$ instead of the matrices P_{m-j+1} and P_{m-j}. (Note that the necessary and sufficient conditions for a P_{m-j+1} to be in the pre-image of a particular realization of P_{m-j} is exactly same as the necessary and sufficient conditions for a $\bar{P}_{m-j+1}$ to be in the pre-image of a particular realization of $\bar{P}_{m-j}$.)

Let $\mathcal{R}_1$ denote all matrices $\bar{P}_{m-j+1}$, of size $m-j+1$, that map to a particular realization of $\bar{P}_{m-j}$ with e_1 and e_2 in the same row. Let $\mathcal{R}_2$ denote all matrices $\bar{P}_{m-j+1}$ that map to a particular realization of $\bar{P}_{m-j}$ with e_1 and e_2 in different rows. Observe that in $\mathcal{R}_2$, $\bar{P}_{m-j+1}$ will map to the particular realization of $\bar{P}_{m-j}$ with probability $\frac{1}{2}$ as in Section 4. We borrow the notation from Section 4 for the rest of the proof.

Remarks: Before proceeding, it helps to relate the quantities in this Section to their counterparts in Section 4. The matrix L_{n-j} had dimensions $(n-j)\times n$; its counterpart $\bar{P}_{m-j+1}$ has dimensions $(m-j+1)\times(m-i+n)$. The number of columns in L_{n-j} outside the columns of T_1^{n-j} equalled j; now the number of columns of $\bar{P}_{m-j+1}$ outside the columns of $\bar{T}_1^{m-j+1}$ equals $n-i+j-1$. This implies that the value $\theta = \bar{T}_2^{m-j+1} - \bar{T}_1^{m-j+1}$ will be subtracted from precisely $(m-j+1)(n-i+j-1)$ elements of $\bar{P}_{m-j+1}$. Note also that the vector $\vec{r}$, of length $m-j$, has exactly $m-i$ zeroes and $i-j$ non-zero elements.

Let $P_{m-j} = [p_{k,l}]$ denote a particular realization of P_{m-j}. We proceed by setting, as in Section 4,

$$\vec{v} = (P_{m-j}, \vec{r}, \theta) \quad \text{and} \quad \vec{w} = (\vec{v}, \vec{x}).$$

To obtain the marginal density of $\vec{v}$, we will evaluate $f_v(\vec{v}) = \int_{\mathcal{R}_1} f_w(\vec{v},\vec{x})d\vec{x} + \int_{\mathcal{R}_2} f_w(\vec{v},\vec{x})d\vec{x}$.

On $\mathcal{R}_1$, we have that $x_a = x_b < U$ for U as in Section 4. (The counterparts of x_a and x_b in Section 4 were x_i and x_k, and these were defined according to Lemma 9.) We shall set $u_l = x_l - x_a$ for $l \neq a, b$. Finally, define

$$
\begin{aligned}
s_v &= p_{1,1} + ... + p_{m-j,n} + r_1 + ... + r_{i-j} \\
&\quad + (m-j+1)(n-i+j-1)\theta.
\end{aligned}
$$

Thus, s_v denotes the sum of all of the entries of P_{m-j+1} except those in $\vec{x}$. We have

$$\begin{aligned}
&\int_{\mathcal{R}_1} f_w(\vec{v},\vec{x})d\vec{x} \\
&\overset{(a)}{=} \quad 2(m-j+1)\binom{n-i+j}{2}\int_0^U \int_0^\infty \cdots \int \\
&\qquad e^{-(s_v+(n-i+j)x_a+u_1+\ldots+u_{n-i+j-2})}\, du_1 \ldots du_{n-i+j-2}\, dx_a \\
&= (m-j+1)(n-i+j-1)e^{-s_v}\left(1-e^{-(n-i+j)U}\right). \quad (9)
\end{aligned}$$

The factor $\binom{n-i+j}{2}$ at equality (a) comes from the possible choices for a,b from the set $\{1,\ldots,n-i+j\}$, the factor $(m-j+1)$ comes from the row choices available to e_1 as in Section 4, and the factor 2 corresponds to the partition, $\mathcal{S}_1$ or $\mathcal{S}_2$ (defined likewise), that P_{m-j+1} belongs to.

Similarly, on $\mathcal{R}_2$, we have that $x_a = U$ and we shall set $u_l = x_l - x_a$ for $l \neq a$ to obtain

$$\begin{aligned}
&\int_{\mathcal{R}_2} f_w(\vec{v},\vec{x})d\vec{x} \\
&\overset{(b)}{=} \quad \frac{1}{2}[2(m-j+1)(n-i+j-1)\int_0^\infty \cdots \int \\
&\qquad e^{-(s_v+(n-i+j)U+u_1+\ldots+u_{n-i+j-1})}\, du_1 \ldots du_{n-i+j-1}] \\
&= \quad (m-j+1)(n-i+j-1)e^{-s_v}e^{-(n-i+j)U}. \qquad (10)
\end{aligned}$$

In equality (b) above, the factor $(n-i+j-1)$ comes from the choice of positions available to x_a (note that x_a cannot occur in same column as the entry d_o which was defined in Lemma 9). The factor $(m-j+1)$ comes from the row choices available to e_1, and the factor 2 is due to the partition, $\mathcal{S}_1$ or $\mathcal{S}_2$, that P_{m-j} belongs to. Finally, the factor $\frac{1}{2}$ comes from the fact that on $\mathcal{R}_2$, e_1 and e_2 occur on different rows. Therefore, P_{m-j+1} will map to the desired P_{m-j} with probability $\frac{1}{2}$.

Putting (9) and (10) together, we obtain

$$\begin{aligned}
&f_v(\vec{v}) = e^{-s_v} \\
&= \quad e^{-(p_{1,1}+\ldots+p_{m-j,n})} \times e^{-(r_1+\ldots+r_{i-j})} \\
&\quad \times (m-j+1)(n-i+j-1)e^{-(m-j+1)(n-i+j-1)\theta}.
\end{aligned}$$

We summarize the above in the following lemma.

Lemma 13 *The following hold:*
(i) P_{m-j} consists of i.i.d. $\exp(1)$ variables.
(ii) $\theta = \bar{T}_2^{m-j+1} - \bar{T}_1^{m-j+1}$ is an $\exp((m-j+1)(n-i+j-1))$ random variable.
(iii) $\vec{r}$ consists of i.i.d. $\exp(1)$ variables and $m-i$ zeroes.
(iv) P_{m-j}, $(\bar{T}_2^{m-j} - \bar{T}_1^{m-j})$, and $\vec{r}$ are independent.

This completes Step 2 of the induction. ■

From Claim 1, we have that $\bar{T}_2^{m-j+1} - \bar{T}_1^{m-j+1} = V_2^{i-j+1,m-j+1} - V_1^{i-j+1,m-j+1}$, and from the inductive hypothesis we have $V_2^{i-j+1,m-j+1} - V_1^{i-j+1,m-j+1} = V_{j+1}^i - V_j^i$. Hence we have the following corollary.

Corollary 5 $V_{j+1}^i - V_j^i \sim \exp((m-j+1)(n-i+j-1))$ *for* $j = 1,2,..,i-1$.

To complete the proof of Conjecture 2 we need to compute the distribution of the two increments $V_{i+1}^i - V_i^i$ and $V_1^{i+1} - V_{i+1}^i$. At the last step of the induction, i.e. $j = i-1$, we have a matrix P_{m-i+1} consisting of i.i.d. $\exp(1)$ random variables and satisfying the following properties: $V_2^{1,m-i+1} - V_1^{1,m-i+1} = V_{i+1}^i - V_i^i$ and $V_1^{2,m-i+1} - V_2^{1,m-i+1} = V_1^{i+1} - V_{i+1}^i$. The following lemma completes the proof of Conjecture 2.

Lemma 14 *The following identities hold:*

(i) $V_2^{1,m-i+1} - V_1^{1,m-i+1} \sim \exp((m-i+1)(n-1))$.
(ii) $V_1^{2,m-i+1} - V_2^{1,m-i+1} \sim \exp((m-i)n)$.

Proof This can be easily deduced from the memoryless property of the exponential distribution; equally, one can refer to Lemma 1 in [18] for the argument. (Remark: There is a row and column interchange in the definitions of the *V-matchings* in [18].) ■

Thus, we have fully established Conjecture 2 and obtain

Theorem 1

$$F(k,m,n) = \sum_{i,j\geq 0, i+j<k} \frac{1}{(m-i)(n-j)}.$$

Proof

$$\begin{aligned}
V_1^k \quad &= \quad (V_1^k - V_k^{k-1}) + (V_k^{k-1} - V_{k-1}^{k-1}) \\
&\qquad + \cdots + (V_2^1 - V_1^1) + V_1^1 \\
\Rightarrow \quad \mathbb{E}(V_1^k) \quad &= \quad \sum_{i,j\geq 0, i+j<k} \frac{1}{(m-i)(n-j)}, \\
\text{and} \quad F(k,m,n) \quad &\triangleq \quad \mathbb{E}(V_1^k).
\end{aligned}$$

■

This gives an alternate proof to Parisi's conjecture since [6] shows that $E_n = F(n,n,n) = \sum_{i=1}^n \frac{1}{i^2}$.

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