On the Scalar Gaussian Interference Channel

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Question

Does Han-Kobayashi achievable region with Gaussian signaling exhaust the capacity region of the scalar Gaussian interference channel?

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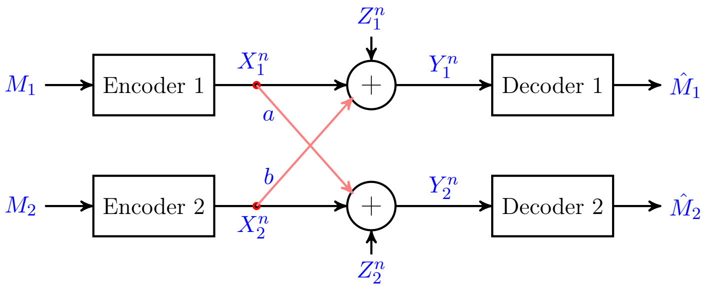
This talk

Perhaps it may

- We establish some evidence towards this end
- Conjecture an information inequality, which if true, would establish the optimality for the Z-interference channel

Scalar Gaussian Interference Channel



(Some) known results about the capacity region

- Determined $a \geq 1, b \geq 1$ (Sato '79)
- *Corner Points* (Sato '81, Costa '85, Sason '02, Polyanskiy-Wu '15)
- Maximum rate-sum $a(1 + b^2 P_2) + b(1 + a^2 P_1) \leq 1$ (3 groups '09)
- Han–Kobayashi region within 0.5 bits per dimension (Etkin, Tse, Wang '07)

As a side note

Investigations on this problem have led to

- Costa's discovery: concavity of entropy power
- Use of HWI to establish converses (Polyanskiy-Wu '15)
- Use of "genies"
  - To establish converses/bounds (Kramer, Etkin-Tse-Wang, ...)
  - As a tool for proving sub-additivity/tensorization

On the Han–Kobayashi achievable region

Background

- 1981: Han and Kobayashi proposed an achievable region (HK-IB) for memoryless interference channels
- 2015: HK-IB was shown to be strictly sub-optimal for some channels (with: Xia, Yazdanpanah)
  - Result: 2-letter extension of HK-IB outperformed HK-IB
  - Difficulty: Evaluating HK-IB (1-letter and 2-letter)
  - Channels: Clean Z-interference channels

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Natural Questions

How about if one restricts to the special case: scalar Gaussian interference channels?

- Is HK-IB (with Gaussian signaling) optimal?
- Or does k-letter extensions (with Gaussian signaling), or in other words do correlated Gaussian input vectors improve the region?
  - **Remark**: There is a paper (2016) that claims such an improvement but it ignores the role of "power control" (which was known to improve on naive region since 1985; see also Costa - ITA 2010)

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  - **Remark**: There is a paper (2016) that claims such an improvement but it ignores the role of "power control" (which was known to improve on naive region since 1985; see also Costa - ITA 2010)
- **Main Result**: No improvement in going to correlated Gaussians
  - Cheng and Verdu had such a result for $\alpha I(X_1^k; Y_1^k) + I(X_2^k; Y_2^k)$ (1993)
  - We had a similar result for Z-interference ($b = 0$) last year.

H–K IB with Gaussian signaling (k-letter)

Non-negative rate pairs R_1, R_2 satisfying

$$R_1 \leq \frac{1}{2k}\,\mathrm{E}_Q\left(\log \frac{\left|I + (K_{U_1}^Q + K_{V_1}^Q) + b^2 K_{V_2}^Q\right|}{\left|I + b^2 K_{V_2}^Q\right|}\right)$$

$$R_2 \leq \frac{1}{2k}\,\mathrm{E}_Q\left(\log \frac{\left|I + (K_{U_2}^Q + K_{V_2}^Q) + a^2 K_{V_1}^Q\right|}{\left|I + a^2 K_{V_1}^Q\right|}\right)$$

$$R_1 + R_2 \leq \frac{1}{2k}\,\mathrm{E}_Q\left(\log \frac{\left|I + (K_{U_1}^Q + K_{V_1}^Q) + b^2 (K_{U_2}^Q + K_{V_2}^Q)\right|}{\left|I + b^2 K_{V_2}^Q\right|} + \log \frac{\left|I + K_{V_2}^Q + a^2 K_{V_1}^Q\right|}{\left|I + a^2 K_{V_1}^Q\right|}\right)$$

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$$2R_1 + R_2 \leq \frac{1}{2k}\,\mathrm{E}_Q\left(\log \frac{\left|I + (K_{U_1}^Q + K_{V_1}^Q) + b^2 (K_{U_2}^Q + K_{V_2}^Q)\right|}{\left|I + b^2 K_{V_2}^Q\right|} + \log \frac{\left|I + K_{V_1}^Q + b^2 K_{V_2}^Q\right|}{\left|I + b^2 K_{V_2}^Q\right|} + \log \frac{\left|I + K_{V_2}^Q + a^2 (K_{U_1}^Q + K_{V_1}^Q)\right|}{\left|I + a^2 K_{V_1}^Q\right|}\right)$$

$$R_1 + 2R_2 \leq \frac{1}{2k}\,\mathrm{E}_Q\left(\log \frac{\left|I + (K_{U_2}^Q + K_{V_2}^Q) + a^2 (K_{U_1}^Q + K_{V_1}^Q)\right|}{\left|I + a^2 K_{V_1}^Q\right|} + \log \frac{\left|I + K_{V_2}^Q + a^2 K_{V_1}^Q\right|}{\left|I + a^2 K_{V_1}^Q\right|} + \log \frac{\left|I + K_{V_1}^Q + b^2 (K_{U_2}^Q + K_{V_2}^Q)\right|}{\left|I + b^2 K_{V_2}^Q\right|}\right)$$

for some $K_{U_1}^q, K_{V_1}^q, K_{U_2}^q, K_{V_2}^q \succeq 0$ satisfying $\mathrm{E}_Q\left(\mathrm{tr}\left(K_{U_1}^Q + K_{V_1}^Q\right)\right) \leq kP_1$ and $\mathrm{E}_Q\left(\mathrm{tr}\left(K_{U_2}^Q + K_{V_2}^Q\right)\right) \leq kP_2$, and some "time-sharing" variable Q.

Result: k-letter region is identical to 1-letter region

Note: Dealing with optimizers of a **non-convex** optimization problem

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Proof:

Define

$$\begin{aligned}
\hat{K}^q_{V_1} &:= \operatorname{diag}\left(\{\lambda_i(K^q_{V_1})\}\right) \\
\hat{K}^q_{U_1} &:= \operatorname{diag}\left(\{\lambda_i(K^q_{U_1} + K^q_{V_1}) - \lambda_i(K^q_{V_1})\}\right) \\
\hat{K}^q_{V_2} &:= \operatorname{diag}\left(\{\lambda_{n+1-i}(K^q_{V_2})\}\right) \\
\hat{K}^q_{U_2} &:= \operatorname{diag}\left(\{\lambda_{n+1-i}(K^q_{U_2} + K^q_{V_2}) - \lambda_{n+1-i}(K^q_{V_1})\}\right).
\end{aligned}$$

where $\lambda_1(A) \leq \cdots \leq \lambda_k(A)$ denote the eigenvalues of a $k \times k$ Hermitian matrix A, and $\operatorname{diag}(\{a_i\})$ indicates a diagonal matrix with diagonal entries $a_1, .., a_k$.

These choices dominate the inequalities term-by-term.
This "observation" and feasibility of these choices relies on two well-known results.

Difficulty: Making this guess (came after a few months of failed other approaches)
There were multiple solutions to KKT conditions, for instance.

Two results

Theorem (Courant-Fischer min-max theorem)

Let A be a $k \times k$ Hermitian matrix. Then we have

$$\lambda_i(A) = \inf_{\substack{V \subseteq \mathbb{R}^k \\ \dim V = i}} \sup_{\substack{x \in V \\ \|x\|=1}} x^T A x = \sup_{\substack{V \subseteq \mathbb{R}^k \\ \dim V = n-i+1}} \inf_{\substack{x \in V \\ \|x\|=1}} x^T A x,$$

where V denotes subspaces of the indicated dimension.

Corollary

Let A, B be $k \times k$ Hermitian matrices with $B \succeq 0$. Then $\lambda_i(A+B) \geq \lambda_i(A)$ for $i = 1, \cdots, k$.

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Theorem (Fiedler '71)

Let A, B be $k \times k$ Hermitian matrices. Suppose $\lambda_k(A) + \lambda_k(B) \geq 0$. Then

$$\prod_{i=1}^{k} (\lambda_i(A) + \lambda_i(B)) \leq |A+B| \leq \prod_{i=1}^{k} (\lambda_i(A) + \lambda_{k+1-i}(B))$$

What next?

Obvious: Do Gaussian inputs optimize HK-IB?

Observations

- Timesharing variable Q is a cause of trouble
- Without Q, there are P_1, P_2 for which non-Gaussian distributions outperform Gaussian distribution
  - Using perturbations based on Hermite Polynomials (Abbe-Zhang 09)

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Observations

- Timesharing variable Q is a cause of trouble
- Without Q, there are P_1, P_2 for which non-Gaussian distributions outperform Gaussian distribution
  - Using perturbations based on Hermite Polynomials (Abbe-Zhang 09)
- What is Q doing?
  - Answer: Q is used to compute the upper concave envelope of a functional defined on (P_1, P_2)
  - **Observation:** Since the dual of the dual (in the sense of Fenchel) yields the concave envelope, we just need to check that Gaussians optimize the dual functional

A conjecture

Let $\alpha, \beta \geq 0$, and $\lambda \geq 1$ be constants.

Conjecture (main)

The maximum of

$$(\lambda - 1)h(\mathbf{X}_2 + a\mathbf{X}_1 + \mathbf{Z}) + h(\mathbf{X}_1 + \mathbf{Z}) - \lambda h(a\mathbf{X}_1 + \mathbf{Z}) - \alpha \, \mathrm{E}(\|\mathbf{X}_1\|^2) - \beta \, \mathrm{E}(\|\mathbf{X}_2\|^2)$$

over independent variables $\mathbf{X}_1$ and $\mathbf{X}_2$ taking values in $\mathbb{R}^k$ is attained by Gaussians $\mathbf{X}_1 \sim \mathcal{N}(0, a\mathbf{I}), \mathbf{X}_2 \sim \mathcal{N}(0, b\mathbf{I})$.

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Why should one care about this

- If true, this establishes the capacity region of the Gaussian Z-interference channel
- Let $\alpha = 0$. Suppose you show that, $\forall \beta > 0, \exists \lambda^* < \infty$ such that the conjecture is true $\forall \lambda \geq \lambda^*$, then we improve on the outer bound obtained using HWI
- Flavor of a reverse-entropy-power inequality

Remarks

Easy regimes

The conjecture is true when either

- $\beta \geq \frac{\lambda-1}{2}$
- $\alpha \geq \frac{1-a^2}{2}$

Proof: Consequence of data-processing and Entropy-Power-Inequality (or doubling trick)

Observation (numerical)

Appears that Stam's path (O-U semigroup) may work

Let X_1, X_2 be independent random variables. Suppose Q_1^*, Q_2^* maximizes

$$\frac{\lambda - 1}{2}\log(1 + a^2 Q_1 + Q_2) + \frac{1}{2}\log(1 + Q_1) - \frac{\lambda}{2}\log(1 + a^2 Q_1) - \alpha Q_1 - \beta Q_2.$$

For $t \in [0, 1]$ define

$$f(t) := (\lambda - 1)h(X_{2t} + aX_{1t} + Z) + h(X_{1t} + Z) - \lambda h(aX_{1t} + Z) - \alpha\,\mathrm{E}(X_{1t}^2) - \beta\,\mathrm{E}(X_{2t}^2)$$

where

$$X_{1t} := \sqrt{1-t}X_1 + \sqrt{t}\mathcal{N}(0, Q_1^*)$$
$$X_{2t} := \sqrt{1-t}X_2 + \sqrt{t}\mathcal{N}(0, Q_2^*).$$

Then $f(t)$ is increasing and concave.

Increasing along the path

Conjecture

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Then

$$\begin{aligned}&(\lambda-1)(Q_2^*+a^2Q_1^*+1)I(X_2+aX_1+Z)+(Q_1^*+1)I(X_1+Z)\\&\qquad-\lambda(a^2Q_1^*+1)I(aX_1+Z)-2\alpha(Q_1^*-\mathrm{E}\left(X_1^2\right))-2\beta(Q_2^*-\mathrm{E}\left(X_2^2\right))\\&\geq 0,\end{aligned}$$

where $I(X)$ is the Fisher information of X.

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Then

$$\begin{aligned}
&(\lambda - 1)(Q_2^* + a^2 Q_1^* + 1) I(X_2 + aX_1 + Z) + (Q_1^* + 1) I(X_1 + Z) \\
&\qquad - \lambda(a^2 Q_1^* + 1) I(aX_1 + Z) - 2\alpha(Q_1^* - \mathrm{E}\left(X_1^2\right)) - 2\beta(Q_2^* - \mathrm{E}\left(X_2^2\right)) \\
&\geq 0,
\end{aligned}$$

where $I(X)$ is the Fisher information of X.

Can establish this for some subset of the parameter space (involving $E(X_1^2), E(X_2^2)$).

Remarks about doubling trick

Does the "doubling trick" work to show Gaussian optimality?

Remarks

- There are interference channels for which

$$\mathcal{C}_{X_1 \perp X_2}[(\lambda - 1)H(Y_2) + H(Y_1) - \lambda H(Y_2|X_2)]$$

  is not sub-additive
- To make this approach work, one needs to show the sub-additivity of the above functional for Gaussian interference channel
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- To make this approach work, one needs to show the sub-additivity of the above functional for Gaussian interference channel
  - The proof of subadditivity needs to use the channel structure
- Of course, there are various arguments in literature that does rely on channel structure
  - Genie based converses
  - Injective deterministic interference channel

Recap

- Correlated Gaussians do not improve the HK-IB

- Conjectured an entropy-variance inequality
  - Motivated by considering the Fenchel dual form of a functional arising in the H–K region for the Z-interference channel
  - Presented possible attack strategies

Recap

- Correlated Gaussians do not improve the HK-IB

- Conjectured an entropy-variance inequality
  - Motivated by considering the Fenchel dual form of a functional arising in the H–K region for the Z-interference channel
  - Presented possible attack strategies
  - Hope it will be resolved one-way or the other soon by someone (perhaps one of you)

Thank You