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Linear-fractional programming

In mathematical optimization, **linear-fractional programming** (**LFP**) is a generalization of linear programming (LP). Whereas the objective function in a linear program is a linear function, the objective function in a linear-fractional program is a ratio of two linear functions. A linear program can be regarded as a special case of a linear-fractional program in which the denominator is the constant function one.

Contents

Relation to linear programming

Both linear programming and linear-fractional programming represent optimization problems using linear equations and linear inequalities, which for each problem-instance define a feasible set. Fractional linear programs have a richer set of objective functions. Informally, linear programming computes a policy delivering the best outcome, such as maximum profit or lowest cost. In contrast, a linear-fractional programming is used to achieve the highest *ratio* of outcome to cost, the ratio representing the highest efficiency. For example, in the context of LP we maximize the objective function **profit = income − cost** and might obtain maximum profit of \$100 (= \$1100 of income − \$1000 of cost). Thus, in LP we have an efficiency of \$100/\$1000 = 0.1. Using LFP we might obtain an efficiency of \$10/\$50 = 0.2 with a profit of only \$10, but only requiring \$50 of investment.

Definition

Formally, a linear-fractional program is defined as the problem of maximizing (or minimizing) a ratio of affine functions over a polyhedron,

$$\begin{aligned} \text{maximize} \quad & \frac{\mathbf{c}^T \mathbf{x} + \alpha}{\mathbf{d}^T \mathbf{x} + \beta} \\ \text{subject to} \quad & A\mathbf{x} \leq \mathbf{b}, \end{aligned}$$

where $\mathbf{x} \in \mathbb{R}^n$ represents the vector of variables to be determined, $\mathbf{c}, \mathbf{d} \in \mathbb{R}^n$ and $\mathbf{b} \in \mathbb{R}^m$ are vectors of (known) coefficients, $A \in \mathbb{R}^{m \times n}$ is a (known) matrix of coefficients and $\alpha, \beta \in \mathbb{R}$ are constants. The constraints have to restrict the feasible region to $\{\mathbf{x} | \mathbf{d}^T \mathbf{x} + \beta > 0\}$, i.e. the region on which the denominator is positive.[1][2] Alternatively, the denominator of the objective function has to be strictly negative in the entire feasible region.

Transformation to a linear program

Under the assumption that the feasible region is non-empty and bounded, the Charnes-Cooper transformation[1]

$$\mathbf{y} = \frac{1}{\mathbf{d}^T \mathbf{x} + \beta} \cdot \mathbf{x} \, ; \quad t = \frac{1}{\mathbf{d}^T \mathbf{x} + \beta}$$

translates the linear-fractional program above to the equivalent linear program:

$$\begin{aligned} \text{maximize} \quad & \mathbf{c}^T \mathbf{y} + \alpha t \\ \text{subject to} \quad & A\mathbf{y} \leq \mathbf{b} t \\ & \mathbf{d}^T \mathbf{y} + \beta t = 1 \\ & t \geq 0. \end{aligned}$$

Then the solution for $\mathbf{y}$ and t yields the solution of the original problem as

$$\mathbf{x} = \frac{1}{t} \mathbf{y}.$$

Duality

Let the dual variables associated with the constraints $A\mathbf{y} - \mathbf{b}t \leq \mathbf{0}$ and $\mathbf{d}^T \mathbf{y} + \beta t - 1 = 0$ be denoted by $\mathbf{u}$ and λ, respectively. Then the dual of the LFP above is [3][4]

$$\begin{aligned} \text{minimize} \quad & \lambda \\ \text{subject to} \quad & A^T \mathbf{u} + \lambda \mathbf{d} = \mathbf{c} \\ & -\mathbf{b}^T \mathbf{u} + \lambda \beta \geq \alpha \\ & \mathbf{u} \in \mathbb{R}_+^m, \lambda \in \mathbb{R}, \end{aligned}$$

which is an LP and which coincides with the dual of the equivalent linear program resulting from the Charnes-Cooper transformation.

Properties and algorithms

The objective function in a linear-fractional problem is both quasiconcave and quasiconvex (hence quasilinear) with a monotone property, pseudoconvexity, which is a stronger property than quasiconvexity. A linear-fractional objective function is both pseudoconvex and pseudoconcave, hence pseudolinear. Since an LFP can be transformed to an LP, it can be solved using any LP solution method, such as the simplex algorithm (of George B. Dantzig),[5][6][7][8] the criss-cross algorithm,[9] or interior-point methods.

Notes

1. Charnes, A.; Cooper, W. W. (1962). "Programming with Linear Fractional Functionals". *Naval Research Logistics Quarterly*. **9** (3–4): 181–186. doi:10.1002/nav.3800090303 (https://doi.org/10.1002%2Fnav.3800090303). MR 0152370 (https://www.ams.org/mathscinet-getitem?mr=0152370).
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3. Schaible, Siegfried (1974). "Parameter-free Convex Equivalent and Dual Programs". *Zeitschrift für Operations Research*. **18** (5): 187–196. doi:10.1007/BF02026600 (https://doi.org/10.1007%2FBF02026600). MR 0351464 (https://www.ams.org/mathscinet-getitem?mr=0351464). S2CID 28885670 (https://api.semanticscholar.org/CorpusID:28885670).
4. Schaible, Siegfried (1976). "Fractional programming I: Duality". *Management Science*. **22** (8): 858–867. doi:10.1287/mnsc.22.8.858 (https://doi.org/10.1287%2Fmnsc.22.8.858). JSTOR 2630017 (https://www.jstor.org/stable/2630017). MR 0421679 (https://www.ams.org/mathscinet-getitem?mr=0421679).
5. Chapter five: Craven, B. D. (1988). *Fractional programming*. Sigma Series in Applied Mathematics. Vol. 4. Berlin: Heldermann Verlag. p. 145. ISBN 978-3-88538-404-5. MR 0949209 (https://www.ams.org/mathscinet-getitem?mr=0949209).
6. Kruk, Serge; Wolkowicz, Henry (1999). "Pseudolinear programming". *SIAM Review*. **41** (4): 795–805. CiteSeerX 10.1.1.53.7355 (https://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.53.7355). doi:10.1137/S0036144598335259 (https://doi.org/10.1137%2FS0036144598335259). JSTOR 2653207 (https://www.jstor.org/stable/2653207). MR 1723002 (https://www.ams.org/mathscinet-getitem?mr=1723002).
7. Mathis, Frank H.; Mathis, Lenora Jane (1995). "A nonlinear programming algorithm for hospital management". *SIAM Review*. **37** (2): 230–234. doi:10.1137/1037046 (https://doi.org/10.1137%2F1037046). JSTOR 2132826 (https://www.jstor.org/stable/2132826). MR 1343214 (https://www.ams.org/mathscinet-getitem?mr=1343214).
8. Murty (1983, Chapter 3.20 (pp. 160–164) and pp. 168 and 179)
9. Illés, Tibor; Szirmai, Ákos; Terlaky, Tamás (1999). "The finite criss-cross method for hyperbolic programming". *European Journal of Operational Research*. **114** (1): 198–214. CiteSeerX 10.1.1.36.7090 (https://citeseerx.ist.psu.edu/viewdoc/summary?doi=10.1.1.36.7090). doi:10.1016/S0377-2217(98)00049-6 (https://doi.org/10.1016%2FS0377-2217%2898%2900049-6). Zbl 0953.90055 (https://zbmath.org/?format=complete&q=an:0953.90055). Postscript preprint (http://www.cas.mcmaster.ca/~terlaky/files/dut-twi-96-103.ps.gz).

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Further reading

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Software

- WinGULF (http://zeus.nyf.hu/~bajalinov/WinGulf/wingulf.html) – educational interactive linear and linear-fractional programming solver with a lot of special options (pivoting, pricing, branching variables, etc.)
- JOptimizer – Java convex optimization library (open source)