Exercise Set

Simplicial Homology

1. (10 pts) Show that an n-simplex has $\binom{n+1}{m+1}$ m-faces. (Hint: find a recursive formula and use induction)
2. (5 pts) Let σ be a 8-simplex. How many vertices does σ have? How many total faces does σ have? How many of them are proper faces? How many facets does σ have? What is the total number of faces of the facets of σ?
3. (3 pts) Let $A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, [1,2], [1,4], [2,3], [2,4], [2,5], [3,5], [4,5], [4,6], [7,8], [8,9], [2,4,5], [7,8,9]\}$. Draw the underlying graph of A. Is A a simplicial complex?
4. (10 pts) Show $\partial_{k-1} \circ \partial_k = 0$. (Hint: use the definition)
5. (6 pts) Explain the intuition behind the definition of **boundary map, cycle,** and **hole**. Feel free to use visual examples.
6. (36 pts) For the following simplicial complexes, draw their visual representations and find H_0 and H_1 without any computation. Then verify your answers algebraically by finding the simplicial k-chains, kernel and image of the boundary maps, and taking the quotient.
   (i) $\{\emptyset, \{1\}, \{2\}, \{3\}, [1,2]\}$
   (ii) $\{\emptyset, \{1\}, \{2\}, \{3\}, [1,2], [1,3], [2,3]\}$
   (iii) $\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, [1,2], [1,3], [2,3], [2,4], [3,4], [2,3,4]\}$
   (iv) $\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, [a,b], [c,d]\}$
   (v) $\{\emptyset, \{a\}, \{b\}, \{c\}, \{d\}, [a,b], [b,c], [c,d]\}$
   (vi) $\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, \{6\}, \{7\}, \{8\}, \{9\}, [1,2], [1,4], [2,3], [2,4], [2,5], [3,5], [4,5], [4,6], [7,8], [8,9], [2,4,5]\}$
7. (20 pts) Find H_0, H_1, H_2, H_3 for the following simplicial complex in any way you want. Clearly explain your reasoning if you choose to do it non-algebraically.

   $\{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{5\}, [1,2], [1,3], [1,4], [1,5], [2,3], [2,4], [2,5], [3,4], [3,5], [4,5], [1,2,3], [1,2,4],$

   $[1,2,5], [1,3,4], [1,3,5], [2,3,4], [2,3,5], [1,2,3,4]\}$
8. (10 pts) Find the only simplicial complex that has $rank(H_0) = 0$. Find a simplicial complex with rank sequence $5, 4, 2, 0, 0, \ldots..$