CS 215 ASSIGNMENT 2

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Q-1

$Y_1 = \max(X_1, X_2, X_3 \cdots X_n)$

$Y_2 = \max(X_1, X_2, X_3 \cdots X_n)$

CDF of $Y_1 = P(Y_1 \le x)$

$= P(\max(X_1, X_2 \cdots X_n) \le x)$ [all $x_i \le x$ since $\max \le x$]

$= P((X_1 \le x) \wedge (x_2 \le x) \wedge (\cdots\cdots (x_n \le x))$.

$= P(X_1 \le x) \cdot P(x_2 \le x) \cdots P(X_n \le x)$

$= (F_X(x))^n$.

PDF of $Y_1 = (\text{CDF of } Y_1)' = n \cdot (F_X(x))^{n-1} \cdot f_X(x)$

CDF of $Y_2 = P(Y_2 \le x)$

$= P(\min(X_1, X_2 \cdots X_n) \le x)$.

$= 1 - P(\min(X_1, X_2 \cdots X_n) \ge x)$

$= 1 - P((X_1 \ge x) \wedge (x_2 \ge x) \cdots (x_n \ge x))$

$= 1 - P(X_1 \ge x) \cdot P(x_2 \ge x) \cdots P(X_n \ge x)$

$= 1 - (1 - P(X_1 \le x)) \cdot (1 - P(X_2 \le x)) \cdots (1 - P(X_n \le x))$

$= 1 - (1 - F_X(x))^n$

PDF of $Y_2 = (\text{CDF of } Y_2)' = -n(1 - F_X(x))^{n-1} \cdot (-f_X(x))$

$= n f_X(x) (1 - F_X(x))^{n-1}$

Q.2 for the first part.

to find $E(X)$, $Var(X)$, MGF of X.

where X belong to the GMM. i.e $X \sim \sum_{i=1}^{k} p_i \mathcal{N}(\mu_i, \sigma_i^2)$

$$f_X(x) = \sum_{i=1}^{k} p_i \mathcal{N}(\mu_i, \sigma_i^2)$$

$$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x) \cdot dx = \int_{-\infty}^{+\infty} x \cdot \sum_{i=1}^{k} p_i \mathcal{N}(\mu_i, \sigma_i^2) \cdot dx$$

$$= \sum_{i=1}^{k} p_i \int_{-\infty}^{+\infty} x \, \mathcal{N}(\mu_i, \sigma_i^2) \cdot dx$$

$$\boxed{E(X) = \sum_{i=1}^{k} p_i \mu_i}$$

$$\cancel{V}E(X^2) = \int_{-\infty}^{\infty} x^2 f_X(x) \cdot dx = \int_{-\infty}^{+\infty} x^2 \sum_{i=1}^{k} p_i \mathcal{N}(\mu_i, \sigma^2) dx$$

$$= \sum_{i=1}^{k} p_i \underbrace{\int_{-\infty}^{+\infty} x^2 \mathcal{N}(\mu_i, \sigma^2) \cdot dx}_{E(X^2) \text{ of a gaussian}}$$

for a gaussian $\mathcal{N}(\mu_i, \sigma_i^2)$

$$Var(X) = E(X^2) - (E(X))^2$$

$$\sigma_i^2 = E(X^2) - \mu_i^2 \rightarrow E(X^2) = \sigma_i^2 + \mu_i^2$$

~~so $Var(X) = E(X^2) - (E(X))^2$~~

$$E(X^2) = \sum_{i=1}^{k} p_i (\sigma_i^2 + \mu_i^2)$$

$$\text{so } Var(X) = E(X^2) - (E(X))^2$$

$$\boxed{Var(X) = \sum_{i=1}^{k} p_i (\sigma_i^2 + \mu_i^2) - \left(\sum_{i=1}^{k} p_i \mu_i\right)^2}$$

$$MGF(X) = \cancel{\phi_X(t)} \cancel{\sum_{i=1} p_i \mathcal{N}(\mu_i, \sigma_i^2)} = \int_{-\infty}^{\infty} e^{tx} \sum_{i=1}^{k} p_i \mathcal{N}(\mu_i, \sigma_i^2) \cdot dx.$$

$$MGF(x) = \sum_{i=1}^{k} p_i \int_{-\infty}^{+\infty} e^{tx} \cdot N(\mu_i, \sigma_i^2) \cdot dx$$

$$\boxed{\phi_x(t) = \sum_{i=1}^{k} p_i \, e^{\mu_i t + \frac{1}{2}\sigma_i^2 t^2}}$$

for second part.

given $z = \sum_{i=1}^{k} p_i x_i$ where $x_i \sim N(\mu_i, \sigma_i^2)$

and $x_1, x_2, x_3 \ldots x_k \rightarrow$ independent

$$E(z) = \cancel{\Sigma} \, E\left(\sum_{i=1}^{k} p_i x_i\right)$$

$$\boxed{E(z) = \sum_{i=1}^{k} p_i E(x_i) = \sum_{i=1}^{k} p_i \mu_i}$$

$$Var(z) = Var\left(\sum_{i=1}^{k} p_i x_i\right)$$

$$= \sum_{i=1}^{k} p_i^2 \, Var(x_i) \quad \text{(since all } x_i \text{ independent)}$$

$$\boxed{Var(z) = \sum_{i=1}^{k} p_i^2 \sigma_i^2}$$

$$MGF(z) = \phi_z(t) = \phi_{\sum p_i x_i}(t)$$

$$= \phi_{p_1 x_1}(t) \cdot \phi_{p_2 x_2}(t) \ldots \ldots \phi_{p_k x_k}(t)$$

$$= \cancel{p_1 \cdot e^{\mu_1 t + \frac{1}{2}\sigma_1^2 t^2} \cdot p_2 e^{\mu_2 t + \frac{1}{2}\sigma_2^2 t^2}} \ldots$$

$$= \boxed{\left(e^{p_1 \mu_1 t + \frac{1}{2} p_1^2 \sigma_1^2 t^2}\right) \cdot \left(e^{p_2 \mu_2 t + \frac{1}{2} p_2^2 \sigma_2^2 t^2}\right) \ldots \left(e^{p_k \mu_k t + \frac{1}{2} p_k^2 \sigma_k^2 t^2}\right)}$$

$$\cancel{MGF(z) = (p_1 \cdot p_2 \ldots p_k) \, e^{(\mu_1 + \mu_2 \ldots \mu_k) t + \frac{1}{2}(\sigma_1^2 + \sigma_2^2 \ldots \sigma_k^2) t^2}}$$

$$\boxed{MGF(z) = e^{(p_1 \mu_1 + p_2 \mu_2 \ldots p_k \mu_k) \cdot t + \frac{1}{2}(p_1^2 \sigma_1^2 + p_2^2 \sigma_2^2 \ldots p_k^2 \sigma_k^2) \cdot t^2}}$$

Given MGF(Z). we can say that Z is gaussian distributed. as MGF is of form

$$e^{\mu' t + \frac{1}{2}\sigma'^2 t^2}$$

so. PDF of Z is

$$\frac{1}{\sigma\sqrt{2\pi}} \; e^{-\frac{1}{2}\left(\frac{x-\mu'}{\sigma'}\right)^2}$$

where, $\mu' = p_1\mu_1 + p_2\mu_2 + \dots p_k\mu_k$

$\sigma' = p_1^2\sigma_1^2 + p_2^2\sigma_2^2 - - - \cdot \; p_k^2\sigma_k^2$

Q.3 To prove : (i) $P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$ $\tau > 0$

(ii) $P(x-\mu \geq \tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}$ $\tau < 0$

$\rightarrow$ (i) $P((X-\mu+t) \geq (\tau+t)) \leq \frac{E((X-\mu+t)^2)}{(\tau+t)^2}$

$\because \left[P((X-\mu+t) \geq (\tau+t)) \leq P((X-\mu+t)^2 \geq (\tau+t)^2) \leq \frac{E((X-\mu+t)^2)}{(\tau+t)^2} \right]$

$\rightarrow \frac{E((X-\mu)^2 + t^2 + 2(X-\mu))}{(\tau+t)^2} = \frac{\sigma^2+t^2}{(\tau+t)^2}$

Now we maximize this expression :

$$\frac{d}{dt}\left(\frac{\sigma^2+t^2}{(\tau+t)^2}\right) = 0 \rightarrow \frac{(\tau+t)^2(2t) - (\sigma^2+t^2)(2(\tau+t))}{(\tau+t)^2} = 0$$

$$\rightarrow t = \sigma^2/\tau$$

So $\frac{\sigma^2+t^2}{(\tau+t)^2} \leq \frac{\sigma^2+\frac{\sigma^4}{\tau^2}}{\left(\tau+\frac{\sigma^2}{\tau}\right)^2} = \frac{\sigma^2}{\sigma^2+\tau^2}$

So $\boxed{P(X-\mu \geq \tau) \leq \frac{\sigma^2}{\sigma^2+\tau^2}}$ $\because P(X-\mu \geq t) = P(X-\mu+t \geq \tau+t)$

for $\tau > 0$

(ii) $P(x-\mu \geq t) = 1 - P(x-\mu \leq \tau)$

$= 1 - P((\mu - x) \geq |\tau|)$

$\downarrow$ same procedure can be done as previous ques

so $P((\mu-x) \geq |\tau|) \leq \frac{\sigma^2}{\sigma^2+\tau^2}$

so $1 - P((\mu-x) \geq |\tau|) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}$

so. $\boxed{P(x-\mu \geq \tau) \geq 1 - \frac{\sigma^2}{\sigma^2+\tau^2}}$ for $\tau<0$

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Q.4

Sol→ $t>0$: $P(X \geq x) = P(Xt \geq xt) = P(e^{Xt} \geq e^{xt}) \leq \frac{E(e^{Xt})}{e^{xt}}$

so $P(X \geq x) \leq e^{-xt} \cdot E(e^{Xt}) = e^{-xt} \phi_X(t)$

$t<0$: $P(X \leq x) = P(Xt \geq xt) = P(e^{Xt} \geq e^{xt}) \leq \frac{E(e^{Xt})}{e^{xt}}$

so $P(X \leq x) \leq e^{-tx} \cdot E(e^{Xt}) = e^{-xt} \cdot \phi_X(t)$

Now Given $X = x_1 + x_2 \cdots x_n$ where $x_i \rightarrow$ Bernoulli random var.

$E(X_i) = P_i$ and $\sum P_i = \mu$

$P(X > (1+\delta)\mu) = P(e^{Xt} > e^{(1+\delta)t\mu}) \leq \frac{E(e^{Xt})}{e^{(1+\delta)t\mu}}$

$\frac{\phi_X(t)}{e^{(1+\delta)t\mu}}$

$\phi_X(t) = \phi_{x_1}(t) \cdot \phi_{x_2}(t) \cdots \phi_{x_n}(t)$

$= (1-P_1+P_1e^t)(1-P_2+P_2e^t)\cdots(1-P_n+P_ne^t)$

$= (1+(P_1)(e^t-1))\cdot(1+P_2(e^t-1))\cdots(1+P_n(e^t-1))$

$\leq e^{P_1(e^t-1)} \cdot e^{P_2(e^t-1)} \cdots e^{P_n(e^t-1)}$

$\leq e^{(P_1+P_2+P_3 \cdots P_n)(e^t-1)}$

$\leq e^{\mu(e^t-1)}$

→ $\boxed{\text{so } P(X > (1+\delta)\mu) \leq \frac{e^{\mu(e^t-1)}}{e^{t\mu(1+\delta)}}}$

⇒ To tighten the bound: diff. wrt. t

$\frac{d}{dt}\left(e^{\mu(e^t-1)-t\mu(1+\delta)}\right) = e^{\mu(e^t-1)-t\mu(1+\delta)} \cdot (\mu(e^t) - \mu(1+\delta)) = 0$

$\Rightarrow t = \ln(1+\delta)$

→ feeding this we get: $P(x > (1+\delta)\mu) \leq \frac{e^{\mu\delta}}{e^{\ln(1+\delta)\mu(1+\delta)}}$

Q.6 → For non-inverted case:

The positive correlation coefficent tells us that the pictures are not inverted of each other.

→ For inverted case:

The minimum correlation coefficent would appear when the two images overlap (i.e shift = 0). as can be seen in the figure. Furthermore the graphs tell us of the inverse relationship between QMI and corelation coefficent. The lack of symmetry of the curves tell us that the image itself isn't symmetric about Y axis

Also, QMI is higher when the joint histogram is more concentrated into a smaller number of bins than when the data is more spread out.

Q.7 we know that MGF of multinomial is given by

$$\phi_x(t) = (p_1 e^{t_1} + p_2 e^{t_2} + \dots + p_k e^{t_k})^n$$

① calculating $(i,j)^{th}$ entry of covariance matrix for $i \neq j$

$$c(i,j) = cov(X_i, X_j) = \left.\frac{\partial^2 \phi_x(t)}{\partial t_i \partial t_j}\right|_{t=0} - \left(\left.\frac{\partial \phi_x(t)}{\partial t_i}\right)\right|_{t=0} \left(\left.\frac{\partial \phi_x(t)}{\partial t_j}\right)\right|_{t=0}$$

* $$\boxed{\frac{\partial \phi_x}{\partial t_i} = n(p_1 e^{t_1} + p_2 e^{t_2} \dots p_k e^{t_k})^{n-1} \cdot p_i e^{t_i}}$$ ~~$\rightarrow E(x_i) = p_i n$~~ ~~$E(x_j) = p_j n$~~

* $$\boxed{\frac{\partial^2 \phi_x}{\partial t_i \partial_j} = n(n-1)(p_1 e^{t_1} + p_2 e^{t_2} \dots p_k e^{t_k}) \cdot p_i p_j e^{t_i} e^{t_j}}$$

* $$\boxed{\text{so } E(x_i) = \left.\frac{\partial \phi_x}{\partial t_i}\right|_{t=0} = n(p_1 + p_2 \dots p_k) \cdot p_i = np_i}$$

* $$\boxed{\text{similarly } E(x_j) = np_j}$$

$$E(x_i x_j) = \left.\frac{\partial^2 \phi_x}{\partial t_i \partial t_j}\right|_{t=0} = (n)(n-1)(p_1 + p_2 \dots p_k) \cdot p_i p_j = n(n-1) p_i p_j$$

$$c(i,j) = cov(x_i, x_j) = E(x_i x_j) - E(x_i) \cdot E(x_j)$$
$$= n(n-1) \cdot p_i p_j - (np_i)(np_j)$$
$$\boxed{c(i,j) = -n p_i p_j} \quad \text{for } i \neq j$$

② calculating $(i,j)^{th}$ entry of covariance matrix for $i = j$

$$\frac{\partial^2 \phi_x(t)}{\partial t_i^2} = \frac{\partial}{\partial t_i}\left[n(p_1 e^{t_1} + p_2 e^{t_2} \dots p_k e^{t_k})^{n-1} \cdot p_i e^{t_i}\right]$$

$$= n(p_i) \cdot (p_1 e^{t_1} + p_2 e^{t_2} \dots p_k e^{t_k})^{n-1} \cdot (e^{t_i})$$
$$+ np_i e^{t_i} (n-1) \cdot p_i e^{t_i} (p_1 e^{t_1} + p_2 e^{t_2} \dots p_k e^{t_k})$$

$$E(x_i^2) = \left.\frac{\partial^2 \phi_x(t)}{\partial t_i^2}\right|_{t=0}$$

$$= n p_i [1 + (n-1) p_i]$$

$$c(i,i) = E(x_i^2) - (E(x_i))^2$$

$$= n p_i [1 + (n-1) p_i] - (n p_i)^2$$

$$\boxed{c(i,i) = n p_i (1 - p_i)}$$

<u>So</u>, $c(i,j) = \begin{cases} -n p_i p_j & \text{for } i \neq j \\ n p_i (1 - p_i) & \text{for } i = j \end{cases}$