6

QUADRATIC EQUATIONS

INTRODUCTION

We already know that $ax^2 + bx + c$, where a, b, c are real numbers and $a \neq 0$, is called a quadratic (or second degree) polynomial in the variable x with real coefficients. Also an equation of the type $ax^2 + bx + c = 0$, $a \neq 0$, is called a **quadratic equation** in the variable x.

The equation $ax^2 + bx + c = 0$, where a, b, c are real numbers and $a \neq 0$, is called the ***general*** *(or* ***standard) quadratic equation*** *in the variable x with real coefficients. The equation of the type $ax^2 + c = 0$, $a \neq 0$, is called a pure quadratic equation.*

For example : $2x^2 - 5x + 3 = 0$, $\sqrt{2}x^2 - x - 7 = 0$ are general quadratic equations in the variable x with real coefficients; the equations $2x^2 + 3 = 0$, $7x^2 - 5 = 0$ are pure quadratic equations.

A number α (real or complex) is called a **root** *(or* **solution***) of the quadratic equation*

$$ax^2 + bx + c = 0 \text{ if and only if } a\alpha^2 + b\alpha + c = 0.$$

Set of roots is called the **solution set.**

In previous classes, we have already solved some quadratic equations in the system of real numbers in the cases where discriminant ≥ 0 *i.e.* $b^2 - 4ac \geq 0$. In this chapter, we shall also solve the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, b, c are real numbers and the discriminant is negative *i.e.* $b^2 - 4ac < 0$. We know that the square roots of negative real numbers exist in the system of complex numbers, therefore, the solutions of the above equation are available in the system of complex numbers.

Fundamental theorem of algebra

You may be interested to know as to how many roots does an equation have? In this regard, we have

"A polynomial equation has atleast one root".

From this result, we can derive a very important theorem known as *Fundamental theorem of algebra* which we are stating below (without proof) :

"A polynomial equation of degree n has exactly n roots".

6.1 SOLVING QUADRATIC EQUATIONS BY FORMULA

6.1.1 A general method of solving a quadratic equation :

Let the general equation be $ax^2 + bx + c = 0$, $a \neq 0$, $a, b, c \in \mathbf{R}$

$$\Rightarrow \quad ax^2 + bx = -c \Rightarrow x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

To make left hand side a perfect square, adding $\left(\frac{b}{2a}\right)^2$ to both sides, we get

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \left(\frac{b}{2a}\right)^2 - \frac{c}{a}$$

$$\Rightarrow \quad \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\Rightarrow \quad x + \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a} \qquad \text{(taking square roots)}$$

$$\Rightarrow \quad x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a} \Rightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Hence, the roots of the equation $ax^2 + bx + c = 0$ $(a \neq 0)$ *are*

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

6.1.2 Solving pure quadratic equation

Let the pure quadratic equation be $ax^2 + c = 0$, $a \neq 0$

$$\Rightarrow \quad ax^2 = -c \Rightarrow x^2 = -\frac{c}{a} \Rightarrow x = \pm\sqrt{-\frac{c}{a}}.$$

Hence, the roots are $\sqrt{-\frac{c}{a}}, -\sqrt{-\frac{c}{a}}$.

6.1.3 A quadratic equation has exactly two roots : equal or distinct; real or imaginary.

We have already seen that roots of the quadratic equation

$$ax^2 + bx + c = 0 \ (a \neq 0) \text{ are } x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

When $b^2 - 4ac \geq 0$, the roots are real ; when $b^2 - 4ac < 0$, the roots are imaginary (complex). Also when $b^2 - 4ac = 0$, the two roots are equal.

Thus the quadratic equation has atleast two roots.

Now let us assume that if possible, α, β, γ be three roots of the given quadratic equation $ax^2 + bx + c = 0$.

Then, by factor theorem, $x - \alpha$, $x - \beta$ and $x - \gamma$ are all factors of $ax^2 + bx + c$

$\Rightarrow$ $(x - \alpha)(x - \beta)(x - \gamma)$ divides $ax^2 + bx + c$

$\Rightarrow$ $x^3 - (\alpha + \beta + \gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$ divides $ax^2 + bx + c$

$\Rightarrow$ a cubic polynomial divides a quadratic polynomial, which is not possible.

Hence, our assumption that a quadratic equation can have three roots is wrong.

Thus every quadratic equation has exactly two roots.

REMARKS

1. If the coefficients of a quadratic equation are rational, and one of the roots is irrational, then the other root is its irrational conjugate *i.e.* if $p + \sqrt{q}$ is one root of the quadratic equation then the other root is $p - \sqrt{q}$ where $\sqrt{q}$ is irrational.

2. If a quadratic equation has real coefficients and if its roots are complex numbers, they occur in conjugate pair.

ILLUSTRATIVE EXAMPLES

Example 1. *Solve the following quadratic equations :*

(*i*) $3x^2 - 4x - 4 = 0$ (*ii*) $4x^2 - 2x + \frac{1}{4} = 0$ (*iii*) $3x^2 - 4x + \frac{20}{3} = 0.$

Solution. (*i*) The given quadratic equation is $3x^2 - 4x - 4 = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get $a = 3$, $b = -4$, $c = -4$.

$\therefore$ Discriminant $= b^2 - 4ac = (-4)^2 - 4 \times 3 \times (-4) = 64$.

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-4) \pm \sqrt{64}}{2 \times 3} = \frac{4 \pm 8}{6} = \frac{4+8}{6}, \frac{4-8}{6}$$

$$= \frac{12}{6}, \frac{-4}{6} = 2, -\frac{2}{3}.$$

Hence, the roots of the given equation are $2, -\frac{2}{3}$.

(*ii*) Given quadratic equation is $4x^2 - 2x + \frac{1}{4} = 0$.

It can be written as $16x^2 - 8x + 1 = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 16$, $b = -8$, $c = 1$.

$\therefore$ Discriminant $= b^2 - 4ac = (-8)^2 - 4 \times 16 \times 1 = 0$.

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-8) \pm \sqrt{0}}{2 \times 16} = \frac{8 \pm 0}{16}$$

$$= \frac{8+0}{32}, \frac{8-0}{32} = \frac{8}{32}, \frac{8}{32} = \frac{1}{4}, \frac{1}{4}.$$

Hence, the given equation has two equal roots $\frac{1}{4}, \frac{1}{4}$.

(*iii*) Given equation is $3x^2 - 4x + \frac{20}{3} = 0$.

It can be written as $9x^2 - 12x + 20 = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 9$, $b = -12$, $c = 20$.

Discriminant $= b^2 - 4ac = (-12)^2 - 4 \times 9 \times 20 = 144 - 720 = -576$.

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-12) \pm \sqrt{-576}}{2 \times 9} = \frac{12 \pm 24i}{18} = \frac{2 \pm 4i}{3}.$$

Hence, the roots of the given equation are $\frac{2 \pm 4i}{3}$.

Example 2. *Solve the following equations :*

(*i*) $x^2 + 3x + 9 = 0$ (*ii*) $27x^2 - 10x + 1 = 0$.

Solution. (*i*) Given equation is $x^2 + 3x + 9 = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 1$, $b = 3$, $c = 9$.

Discriminant $= b^2 - 4ac = 3^2 - 4 \times 1 \times 9 = 9 - 36 = -27.$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-3 \pm \sqrt{-27}}{2 \times 1} = \frac{-3 \pm 3\sqrt{3}\,i}{2}.$$

Hence, the roots of the given equation are $\frac{-3 \pm 3\sqrt{3}\,i}{2}$.

(*ii*) Given equation is $27x^2 - 10x + 1 = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 27, b = -10, c = 1.$

Discriminant $= b^2 - 4ac = (-10)^2 - 4 \times 27 \times 1 = 100 - 108 = -8.$

$$\therefore \quad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-10) \pm \sqrt{-8}}{2 \times 27} = \frac{10 \pm 2\sqrt{2}i}{54} = \frac{5 \pm \sqrt{2}i}{27}.$$

Hence, the roots of the given equation are $\frac{5 \pm \sqrt{2}i}{27}$.

Example 3. *Solve the following equations :*

(*i*) $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$ (*ii*) $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$

Solution. (*i*) Given equation is $\sqrt{3}x^2 - \sqrt{2}x + 3\sqrt{3} = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get

$a = \sqrt{3}, b = -\sqrt{2}, c = 3\sqrt{3}.$

Discriminant $= b^2 - 4ac = (-\sqrt{2})^2 - 4 \times \sqrt{3} \times 3\sqrt{3}$

$= 2 - 36 = -34.$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-\sqrt{2}) \pm \sqrt{-34}}{2 \times \sqrt{3}} = \frac{\sqrt{2} \pm \sqrt{34}\,i}{2\sqrt{3}}.$$

Hence, the roots of the given equation are $\frac{\sqrt{2} \pm \sqrt{34}\,i}{2\sqrt{3}}$.

(*ii*) Given equation is $x^2 + \frac{x}{\sqrt{2}} + 1 = 0$.

It can be written as $\sqrt{2}x^2 + x + \sqrt{2} = 0$.

Comparing it with $ax^2 + bx + c = 0$, we get

$a = \sqrt{2}, b = 1, c = \sqrt{2}.$

Discriminant $= b^2 - 4ac = 1^2 - 4 \times \sqrt{2} \times \sqrt{2} = 1 - 8 = -7.$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-1 \pm \sqrt{-7}}{2 \times \sqrt{2}} = \frac{-1 \pm \sqrt{7}\,i}{2\sqrt{2}}.$$

Hence, the roots of the given equation are $\frac{-1 \pm \sqrt{7}\,i}{2\sqrt{2}}$.

Example 4. *Find the roots of the equation* $x^2 + x - (a + 2)(a + 1) = 0$.

Solution. Comparing the given equation with $Ax^2 + Bx + C = 0$, we get

$A = 1, B = 1, C = -(a + 2)(a + 1) = -(a^2 + 3a + 2).$

$\therefore \quad x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-a}}{2 \times 1}$

$= \frac{-1 \pm \sqrt{1 + 4a^2 + 12a + 8}}{2} = \frac{-1 \pm \sqrt{4a^2 + 12a + 9}}{2}$

$= \frac{-1 \pm \sqrt{(2a+3)^2}}{2} = \frac{-1 \pm (2a+3)}{2}$

$= \frac{-1 + 2a + 3}{2}, \frac{-1 - 2a - 3}{2} = a + 1, -a - 2.$

Hence, the roots of the given equation are $a + 1, -a - 2$.

Example 5. *Solve the equation $2x^2 - 5x + 2 = 0$ by the method of completing the squares.*

Solution. The given quadratic equation is $2x^2 - 5x + 2 = 0$

$\Rightarrow \quad x^2 - \frac{5}{2}x + 1 = 0$ (dividing throughout by 2)

$\Rightarrow \quad x^2 - \frac{5}{2}x = -1$

Adding $\left(\frac{1}{2} \times \frac{-5}{2}\right)^2 = \frac{25}{16}$ to both sides, we get

$x^2 - \frac{5}{2}x + \frac{25}{16} = \frac{25}{16} - 1 = \frac{9}{16}$

$\Rightarrow \quad \left(x - \frac{5}{4}\right)^2 = \frac{9}{16} = \left(\frac{3}{4}\right)^2$

$\Rightarrow \quad x - \frac{5}{4} = \pm\frac{3}{4}$

$\Rightarrow \quad x = \frac{5}{4} \pm \frac{3}{4} = \frac{5+3}{4}, \frac{5-3}{4} = 2, \frac{1}{2}.$

Hence, the roots of the given equation are $2, \frac{1}{2}$.

Example 6. *Find two numbers such that their sum is 6 and the product is 14.*

Solution. Let the two numbers be α and β, then according to given,

$\alpha + \beta = 6$ ...(*i*)

and $\alpha\beta = 14$ ...(*ii*)

Substituting the value of β from (*i*) in (*ii*), we get

$\alpha(6 - \alpha) = 14$

$\Rightarrow \quad 6\alpha - \alpha^2 - 14 = 0$

$\Rightarrow \quad \alpha^2 - 6\alpha + 14 = 0$ ...(*iii*)

Comparing it with $ax^2 + bx + c = 0$, we get

$a = 1, b = -6, c = 14.$

Discriminant $= b^2 - 4ac = (-6)^2 - 4 \times 1 \times 14 = -20.$

$\therefore \quad \alpha = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-6) \pm \sqrt{-20}}{2 \times 1} = \frac{6 \pm 2\sqrt{5}i}{2}$

$= 3 \pm \sqrt{5}\,i.$

From (*i*), when $\alpha = 3 + \sqrt{5}\,i$, $\beta = 6 - (3 + \sqrt{5}\,i) = 3 - \sqrt{5}\,i$;

when $\alpha = 3 - \sqrt{5}\,i$, $\beta = 6 - (3 - \sqrt{5}\,i) = 3 + \sqrt{5}\,i$.

Hence, the two numbers are $3 + \sqrt{5}\,i$ and $3 - \sqrt{5}\,i$.

Example 7. *Three consecutive natural numbers are such that the square of the middle number exceeds the difference of the squares of the other two by 60. Assume the middle number to be x and form a quadratic equation satisfying the above statement. Hence, find the three numbers.*

Solution. Since the middle number of the three consecutive numbers is x, the other two numbers are $x - 1$ and $x + 1$. According to the given condition, we have

$$x^2 = [(x + 1)^2 - (x - 1)^2] + 60$$

$$\Rightarrow \quad x^2 = (x^2 + 2x + 1) - (x^2 - 2x + 1) + 60 = 4x + 60$$

$$\Rightarrow \quad x^2 - 4x - 60 = 0 \Rightarrow (x - 10)(x + 6) = 0$$

$$\Rightarrow \quad x = 10 \text{ or } -6.$$

Since x is a natural number, therefore, we get $x = 10$.

Hence, the three numbers are 9, 10, 11.

Example 8. *A two digit number is four times the sum and twice the product of its digits. Find the number.*

Solution. Let x be the digit in ten's place and y be the digit in unit's place, then the number is $10x + y$.

According to given conditions, we have

$$10x + y = 4(x + y) \Rightarrow 6x = 3y \Rightarrow 2x = y \quad \ldots(i)$$

$$10x + y = 2xy \quad \ldots(ii)$$

Substituting the value of y from (i) in (ii), we get

$$10x + 2x = 2x.2x \Rightarrow 12x = 4x^2$$

$$\Rightarrow \quad x^2 = 3x \Rightarrow x^2 - 3x = 0$$

$$\Rightarrow \quad x(x - 3) = 0 \Rightarrow x = 0, x = 3.$$

If $x = 0$, from (i), $y = 0$ and we find that the number in this case does not have two digits. Rejecting this value of x, we get $x = 3$ and from (i), we get $y = 2 \times 3 = 6$.

Hence, the required number is 36.

Example 9. *A factory kept increasing its output by the same percentage every year. If its output doubled in the last two years, find the percentage increase every year.*

Solution. Let the number of articles produced by the factory in a certain year be x and let the percentage increase in its output every year be $p\%$.

The number of articles produced by the factory after two years

$$= x\left(1 + \frac{p}{100}\right)^2.$$

According to given, $x\left(1 + \frac{p}{100}\right)^2 = 2x$

$$\Rightarrow \quad \left(1 + \frac{p}{100}\right)^2 = 2 \quad (\because x \neq 0)$$

$$\Rightarrow \quad 1 + \frac{p}{100} = \sqrt{2} \quad \text{(take only positive value)}$$

$$\Rightarrow \quad p = 100(\sqrt{2} - 1).$$

Hence, the percentage increase in the output every year $= 100(\sqrt{2} - 1)\%$.

EXERCISE 6.1

Solve the following (1 to 7) quadratic equations :

1. (i) $x^2 + 2 = 0$ $\quad$ (ii) $4x^2 + 5 = 0$.
2. (i) $x^2 - x - 12 = 0$ $\quad$ (ii) $25x^2 + 30x + 7 = 0$.
3. (i) $\sqrt{3}x^2 - 4x + \sqrt{3} = 0$ $\quad$ (ii) $15x^2 + 15 = 34x$.

4. (i) $x^2 + x + 1 = 0$ (ii) $x^2 - x + 2 = 0$.
5. (i) $x^2 + 3x + 5 = 0$ (ii) $2x^2 + x + 1 = 0$.
6. (i) $\sqrt{2}x^2 + x + \sqrt{2} = 0$ (ii) $\sqrt{5}x^2 + x + \sqrt{5} = 0$.
7. (i) $21x^2 - 28x + 10 = 0$ (ii) $x^2 - 2x + \frac{3}{2} = 0$.
8. Find the values of x if $p + 7 = 0$, $q - 12 = 0$ and $x^2 + px + q = 0$.
9. Divide 39 into 2 parts whose product is 338.
10. The sum of two numbers is 9 and the sum of their squares is 41. Taking one number as x, form an equation in x and solve it to find the numbers.
11. The sum of the squares of two consecutive positive even integers is 340. Find the integers.
12. A two digit number is four times the sum and three times the product of its digits. Find the number.
13. The hypotenuse of a right angled triangle is 17 cm long and the difference between the other two sides is 7 cm. Find the lengths of two unknown sides.
14. Originally a rectangle was twice as long as it is wide. When 4 m were added to its length and 3 m subtracted from its width, the resulting rectangle had an area of 600 m^2. Find the dimensions of the new rectangle.
15. A rectangular garden 10 m by 16 m is to be surrounded by a concrete walk of uniform width. Given that area of the walk is 120 square metres, find its width.
16. Two years ago, a man's age was three times the square of his son's age. In three years time, his age will be four times his son's age. Find their present ages.
17. A piece of cloth costs ₹ 175. If the piece were 4 m longer and each metre costs ₹ 5 less, the cost would have remained unchanged. How long is the piece?

6.2 EQUATIONS REDUCIBLE TO QUADRATIC FORM

Some algebraic equations which are not quadratic can be reduced to quadratic equations by simplification, squaring, using substitution etc. and hence can be solved. In case of squaring, some *extraneous roots* may creep in, so it is necessary to verify that the values obtained satisfy the original equation.

ILLUSTRATIVE EXAMPLES

Example 1. *Find the roots of the equation* $\frac{1}{x+1} + \frac{2}{x+2} = \frac{4}{x+4}$.

Solution. The given equation is $\frac{1}{x+1} + \frac{2}{x+2} = \frac{4}{x+4}$

$$\Rightarrow \frac{1(x+2)+2(x+1)}{(x+1)(x+2)} = \frac{4}{x+4}$$

$$\Rightarrow (3x+4)(x+4) = 4(x+1)(x+2)$$

$$\Rightarrow 3x^2 + 12x + 4x + 16 = 4x^2 + 12x + 8$$

$$\Rightarrow -x^2 + 4x + 8 = 0 \Rightarrow x^2 - 4x - 8 = 0.$$

Comparing it with $ax^2 + bx + c = 0$, we get, $a = 1$, $b = -4$, $c = -8$.

$$\therefore \quad x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(1)(-8)}}{2(1)} = \frac{4 \pm \sqrt{16+32}}{2}$$

$$= \frac{4 \pm \sqrt{48}}{2} = \frac{4 \pm 4\sqrt{3}}{2} = 2 \pm 2\sqrt{3}.$$

Hence, the roots of the given equation are $2 + 2\sqrt{3}$, $2 - 2\sqrt{3}$.

Example 2. *Solve* $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-5}{x-6}-\frac{x-6}{x-7}$.

Solution. Dividing each numerator by denominator, we see that $\frac{x-1}{x-2}=1+\frac{1}{x-2}$ etc. Thus, the given equation reduces to

$$\left(1+\frac{1}{x-2}\right)-\left(1+\frac{1}{x-3}\right)=\left(1+\frac{1}{x-6}\right)-\left(1+\frac{1}{x-7}\right)$$

$$\Rightarrow \quad \frac{1}{x-2}-\frac{1}{x-3}=\frac{1}{x-6}-\frac{1}{x-7}$$

$$\Rightarrow \quad \frac{(x-3)-(x-2)}{(x-2)(x-3)}=\frac{(x-7)-(x-6)}{(x-6)(x-7)} \Rightarrow \frac{-1}{(x-2)(x-3)}=\frac{-1}{(x-6)(x-7)}$$

$$\Rightarrow \quad (x-6)(x-7)=(x-2)(x-3)$$

$$\Rightarrow \quad x^2-13x+42=x^2-5x+6$$

$$\Rightarrow \quad 8x=36 \Rightarrow x=\frac{36}{8}=\frac{9}{2}=4\frac{1}{2}.$$

Hence, the only root is $x=4\frac{1}{2}$.

Example 3. *Solve* : $\frac{a}{ax+1}+\frac{b}{bx+1}=a+b,\ ab\neq 0,\ a+b\neq 0.$

Solution. Given $\frac{a}{ax+1}+\frac{b}{bx+1}=a+b$

$$\Rightarrow \quad \left(\frac{a}{ax+1}-a\right)+\left(\frac{b}{bx+1}-b\right)=0$$

$$\Rightarrow \quad \frac{a-a^2x-a}{ax+1}+\frac{b-b^2x-b}{bx+1}=0$$

$$\Rightarrow \quad -\frac{a^2x}{ax+1}-\frac{b^2x}{bx+1}=0 \Rightarrow x\left(\frac{a^2}{ax+1}+\frac{b^2}{bx+1}\right)=0$$

$$\Rightarrow \quad x=0 \text{ or } \frac{a^2}{ax+1}+\frac{b^2}{bx+1}=0$$

$$\Rightarrow \quad x=0 \text{ or } \frac{a^2bx+a^2+ab^2x+b^2}{(ax+1)(bx+1)}=0$$

$$\Rightarrow \quad x=0 \text{ or } ab(a+b)x+(a^2+b^2)=0$$

$$\Rightarrow \quad x=0 \text{ or } x=-\frac{a^2+b^2}{ab(a+b)}.$$

Hence, the roots are 0, $-\frac{a^2+b^2}{ab(a+b)}$.

Example 4. *Solve* : $2^{2x+8}+1=32\,.\,2^x$.

Solution. Given $2^{2x+8}+1=32\,.\,2^x \Rightarrow 2^8\,.\,2^{2x}+1=32\,.\,2^x$

$\Rightarrow \quad 256\,.\,(2^x)^2-32\,.\,2^x+1=0.$

Putting $2^x=t$, we get, $256t^2-32t+1=0$

$$\Rightarrow \quad t=\frac{-(-32)\pm\sqrt{(-32)^2-4(256)(1)}}{2(256)}=\frac{32\pm 0}{512}=\frac{1}{16},\frac{1}{16}.$$

But $t=2^x$, therefore,

$$2^x=\frac{1}{16},\frac{1}{16}=2^{-4},2^{-4}$$

$\Rightarrow \quad x=-4,-4.$

Hence, the solution of the given equation is $x=-4$.

Example 5. *Solve* $(x^2 - 5x)^2 - 30(x^2 - 5x) - 216 = 0$.

Solution. Given $(x^2 - 5x)^2 - 30(x^2 - 5x) - 216 = 0$, which is a *biquadratic i.e.* equation of degree 4.

Putting $x^2 - 5x = t$, we get, $t^2 - 30t - 216 = 0$

$$\Rightarrow \quad t = \frac{-(-30) \pm \sqrt{(-30)^2 - 4(1)(-216)}}{2 \times 1} = \frac{30 \pm \sqrt{1764}}{2}$$

$$= \frac{30 \pm 42}{2} = 36, -6.$$

But $t = x^2 - 5x$, therefore,

either $x^2 - 5x = 36$

$\Rightarrow \quad x^2 - 5x - 36 = 0$

$$\Rightarrow \quad x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(-36)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{169}}{2} = \frac{5 \pm 13}{2}$$

$= 9, -4$

or $\quad x^2 - 5x = -6$

$\Rightarrow \quad x^2 - 5x + 6 = 0$

$$\Rightarrow \quad x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)}$$

$$= \frac{5 \pm \sqrt{1}}{2} = \frac{5 \pm 1}{2}$$

$= 3, 2.$

Hence, the roots of the given equation are $9, -4, 3, 2$.

Example 6. *Solve* $(x + 1)(x + 2)(x + 3)(x + 4) = 120$.

Solution. *If the problem has four factors $(x + a)(x + b)(x + c)(x + d)$, and the sum of two of the numbers a, b, c, d is equal to the sum of the other two, then we make the substitution as under :*

$(x + 1)(x + 2)(x + 3)(x + 4) = 120$

$\Rightarrow \quad [(x + 1)(x + 4)][(x + 2)(x + 3)] = 120$

$\Rightarrow \quad (x^2 + 5x + 4)(x^2 + 5x + 6) = 120.$

Putting $y = x^2 + 5x$, we get

$(y + 4)(y + 6) = 120 \Rightarrow y^2 + 10y + 24 = 120$

$$\Rightarrow \quad y^2 + 10y - 96 = 0 \Rightarrow y = \frac{-10 \pm \sqrt{(10)^2 - 4(1)(-96)}}{2(1)}$$

$$\Rightarrow \quad y = \frac{-10 \pm \sqrt{484}}{2} = \frac{-10 \pm 22}{2} = 6, -16.$$

When $y = 6$, we get $x^2 + 5x = 6 \Rightarrow x^2 + 5x - 6 = 0 \Rightarrow (x - 1)(x + 6) = 0$

$\Rightarrow \quad x = 1, -6.$

When $y = -16$, we get $x^2 + 5x = -16 \Rightarrow x^2 + 5x + 16 = 0$

$$\Rightarrow \quad x = \frac{-5 \pm \sqrt{(5)^2 - 4(1)(16)}}{2(1)} = \frac{-5 \pm \sqrt{-39}}{2} = \frac{-5 \pm i\sqrt{39}}{2}.$$

Hence, the solutions are $1, -6, \dfrac{-5 \pm i\sqrt{39}}{2}$.

Example 7. *Solve :* $x^{2/3} + x^{1/3} = 2$.

Solution. Putting $x^{1/3} = y$, we get $y^2 + y = 2$

$\Rightarrow \quad y^2 + y - 2 = 0 \Rightarrow (y + 2)(y - 1) = 0 \Rightarrow y = 1, -2.$

When $y = 1$, $x^{1/3} = y = 1 \Rightarrow x = (1)^3 = 1.$

When $y = -2$, $x^{1/3} = y = -2 \Rightarrow x = (-2)^3 = -8.$

Hence, the solutions are $1, -8$.

Example 8. *Solve the equation* $8\sqrt{\dfrac{x}{x+3}} - \sqrt{\dfrac{x+3}{x}} = 2.$

Solution. Given equation is $8\sqrt{\dfrac{x}{x+3}} - \sqrt{\dfrac{x+3}{x}} = 2.$

Putting $t = \sqrt{\dfrac{x}{x+3}}$, we get $8t - \dfrac{1}{t} = 2 \Rightarrow \dfrac{8t^2 - 1}{t} = 2$

$$\Rightarrow \quad 8t^2 - 2t - 1 = 0 \Rightarrow t = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(8)(-1)}}{2(8)} = \frac{2 \pm \sqrt{36}}{16}$$

$$\Rightarrow \quad t = \frac{2 \pm 6}{16} = \frac{1}{2}, -\frac{1}{4} \Rightarrow \sqrt{\frac{x}{x+3}} = \frac{1}{2}, -\frac{1}{4}.$$

Since square root of a +ve real number is non-negative, we ignore $-\dfrac{1}{4}$.

Now $\sqrt{\dfrac{x}{x+3}} = \dfrac{1}{2} \Rightarrow \dfrac{x}{x+3} = \dfrac{1}{4} \Rightarrow 4x = x + 3$

$\Rightarrow \quad 3x = 3 \Rightarrow x = 1.$

Hence, the only solution of the given equation is $x = 1$.

Example 9. (*i*) *Evaluate* $\sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots \text{ to } \infty}}}$.

(*ii*) *If* $3 = \sqrt{x + \sqrt{x + \sqrt{x + \ldots \text{ to } \infty}}}$, *find* x.

Solution. (*i*) Let $x = \sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots \text{ to } \infty}}}$

Squaring both sides, we get

$$x^2 = 6 + \sqrt{6 + \sqrt{6 + \sqrt{6 + \ldots \text{ to } \infty}}} = 6 + x$$

$\Rightarrow \quad x^2 - x - 6 = 0 \Rightarrow (x - 3)(x + 2) = 0 \Rightarrow x = 3, -2.$

As the value of given expression should be positive, we ignore $x = -2$.

Hence, the value of the given expression is 3.

(*ii*) We are given that $3 = \sqrt{x + \sqrt{x + \sqrt{x + \ldots \text{ to } \infty}}}$

Squaring both sides, we get

$$9 = x + \sqrt{x + \sqrt{x + \sqrt{x + \ldots \text{ to } \infty}}} = x + 3 \Rightarrow x = 6.$$

Example 10. *Solve the equation* $\dfrac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = \dfrac{a}{x}, x \neq 0.$

Solution. Given $\dfrac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} = \dfrac{a}{x}.$

Rationalising the denominator of L.H.S., we get

$$\frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} - \sqrt{a-x}} \times \frac{\sqrt{a+x} + \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}} = \frac{a}{x}$$

$$\Rightarrow \quad \frac{\left(\sqrt{a+x} + \sqrt{a-x}\right)^2}{\left(\sqrt{a+x}\right)^2 - \left(\sqrt{a-x}\right)^2} = \frac{a}{x}$$

$$\Rightarrow \quad \frac{(a+x) + (a-x) + 2\sqrt{(a+x)(a-x)}}{(a+x) - (a-x)} = \frac{a}{x}$$

$\Rightarrow \quad \dfrac{2a + 2\sqrt{a^2 - x^2}}{2x} = \dfrac{a}{x}$

$\Rightarrow \quad a + \sqrt{a^2 - x^2} = a \Rightarrow \sqrt{a^2 - x^2} = 0$

$\Rightarrow \quad a^2 - x^2 = 0 \Rightarrow x^2 = a^2$

$\Rightarrow \quad x = a, -a.$

We can verify that $x = a$ or $-a$ both satisfy the given equation.

Hence, the roots are $a, -a$.

Example 11. *Solve* $\sqrt{x} = x - 2$.

Solution. Given $\sqrt{x} = x - 2$.

Squaring both sides, we get

$x = x^2 - 4x + 4 \Rightarrow x^2 - 5x + 4 = 0$

$\Rightarrow \quad (x - 4)(x - 1) = 0 \Rightarrow x = 4, 1.$

We note that $x = 4$ satisfies the given equation but $x = 1$ does not satisfy the given equation.

Check : For $x = 1$,

$\sqrt{x} = x - 2 \Rightarrow \sqrt{1} = 1 - 2 \Rightarrow 1 = -1$, which is wrong.

Hence, the only solution of the given equation is $x = 4$.

Example 12. *Solve the equation* $\sqrt{x^2 + 3x + 32} + \sqrt{x^2 + 3x + 5} = 9$.

Solution. Given equation is $\sqrt{x^2 + 3x + 32} + \sqrt{x^2 + 3x + 5} = 9$

$\Rightarrow \quad \sqrt{x^2 + 3x + 32} = 9 - \sqrt{x^2 + 3x + 5}.$

Squaring both sides, we get

$x^2 + 3x + 32 = 81 + x^2 + 3x + 5 - 18\sqrt{x^2 + 3x + 5}$

$\Rightarrow \quad 18\sqrt{x^2 + 3x + 5} = 54 \Rightarrow \sqrt{x^2 + 3x + 5} = 3.$

Again on squaring, we get $x^2 + 3x + 5 = 9 \Rightarrow x^2 + 3x - 4 = 0$

$\Rightarrow \quad (x + 4)(x - 1) = 0 \Rightarrow x = -4, 1.$

We can verify that for $x = -4$ or 1, both expressions $x^2 + 3x + 32$ and $x^2 + 3x + 5$ are positive and satisfy the given equation.

Hence, the solutions of the given equation are $-4, 1$.

Example 13. *Solve the equation* $\sqrt{x + 5} + \sqrt{x + 21} = \sqrt{6x + 40}$.

Solution. Given $\sqrt{x + 5} + \sqrt{x + 21} = \sqrt{6x + 40}$.

Squaring both sides, we get

$x + 5 + x + 21 + 2\sqrt{x + 5}\sqrt{x + 21} = 6x + 40$

$\Rightarrow \quad 2\sqrt{x + 5}\sqrt{x + 21} = 4x + 14$

$\Rightarrow \quad \sqrt{x + 5}\sqrt{x + 21} = 2x + 7$ (squaring both sides)

$\Rightarrow \quad (x + 5)(x + 21) = (2x + 7)^2$

$\Rightarrow \quad x^2 + 26x + 105 = 4x^2 + 28x + 49$

$\Rightarrow \quad 3x^2 + 2x - 56 = 0$

$\Rightarrow \quad (3x + 14)(x - 4) = 0 \Rightarrow x = -\dfrac{14}{3}, 4.$

We note that $x = 4$ satisfies the given equation but $x = -\dfrac{14}{3}$ does not satisfy the given equation.

Check : For $x = -\frac{14}{3}$, we get

$$\sqrt{-\frac{14}{3}+5}+\sqrt{-\frac{14}{3}+21} = \sqrt{6.\left(-\frac{14}{3}\right)+40} \Rightarrow \frac{1}{\sqrt{3}}+\frac{7}{\sqrt{3}} = 2\sqrt{3}$$

$\Rightarrow \quad 1 + 7 = 6 \Rightarrow 8 = 6$, which is wrong.

Hence, the only solution of the given equation is $x = 4$

Example 14. *Solve the equation* $\left(x+\frac{1}{x}\right)^2 = 4+\frac{3}{2}\left(x-\frac{1}{x}\right)$.

Solution. Putting $x - \frac{1}{x} = t$, we get

$$\left(x+\frac{1}{x}\right)^2 = \left(x-\frac{1}{x}\right)^2 + 4 = t^2 + 4.$$

So the given equation becomes

$$t^2 + 4 = 4 + \frac{3}{2}t \Rightarrow t^2 = \frac{3}{2}t \Rightarrow t^2 - \frac{3}{2}t = 0$$

$$\Rightarrow \quad t\left(t-\frac{3}{2}\right) = 0 \Rightarrow t = 0, \frac{3}{2}.$$

Now $t = 0 \Rightarrow x - \frac{1}{x} = 0 \Rightarrow x = \frac{1}{x} \Rightarrow x^2 = 1 \Rightarrow x = \pm 1.$

And $t = \frac{3}{2} \Rightarrow x - \frac{1}{x} = \frac{3}{2} \Rightarrow \frac{x^2-1}{x} = \frac{3}{2} \Rightarrow 2x^2 - 3x - 2 = 0$

$$\Rightarrow \quad (2x+1)(x-2) = 0 \Rightarrow x = -\frac{1}{2}, 2.$$

Hence, the solutions of the given equation are $-\frac{1}{2}, -1, 1, 2.$

Example 15. *Solve* $4x^4 - 16x^3 + 7x^2 + 16x + 4 = 0.$

Solution. Dividing throughout by x^2, we obtain

$$4x^2 - 16x + 7 + \frac{16}{x} + \frac{4}{x^2} = 0$$

$$\Rightarrow \quad 4\left(x^2+\frac{1}{x^2}\right) - 16\left(x-\frac{1}{x}\right) + 7 = 0$$

Putting $x - \frac{1}{x} = t$, we get

$$x^2 + \frac{1}{x^2} = \left(x-\frac{1}{x}\right)^2 + 2 = t^2 + 2.$$

Hence, the given equation becomes

$$4(t^2+2) - 16t + 7 = 0 \Rightarrow 4t^2 - 16t + 15 = 0$$

$$\Rightarrow \quad t = \frac{-(-16)\pm\sqrt{(-16)^2-4\times4\times15}}{2\times4} = \frac{16\pm4}{8} = \frac{5}{2}, \frac{3}{2}.$$

Now $t = \frac{5}{2} \Rightarrow x - \frac{1}{x} = \frac{5}{2} \Rightarrow 2x^2 - 5x - 2 = 0$

$$\Rightarrow \quad x = \frac{-(-5)\pm\sqrt{(-5)^2-4\times2\times(-2)}}{2\times2} = \frac{5\pm\sqrt{41}}{4}.$$

And $t = \frac{3}{2} \Rightarrow x - \frac{1}{x} = \frac{3}{2} \Rightarrow 2x^2 - 3x - 2 = 0$

$$\Rightarrow \quad x = \frac{-(-3) \pm \sqrt{(-3)^2 - 4 \times 2 \times (-2)}}{2 \times 2} = \frac{3 \pm 5}{4} = 2, -\frac{1}{2}.$$

Hence, the roots of the given equation are $2, -\frac{1}{2}, \frac{5 \pm \sqrt{41}}{4}$.

Example 16. *Solve the equations* $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{10}{3}$, $x + y = 10$ *simultaneously for x and y.*

Solution. $\sqrt{\frac{x}{y}} + \sqrt{\frac{y}{x}} = \frac{10}{3} \Rightarrow \frac{x + y}{\sqrt{xy}} = \frac{10}{3} \Rightarrow \frac{10}{\sqrt{xy}} = \frac{10}{3} \Rightarrow \sqrt{xy} = 3$

$\Rightarrow \quad xy = 9.$

Now we can easily solve $x + y = 10$, $xy = 9$.

$$xy = 9 \Rightarrow y = \frac{9}{x}.$$

So $x + y = 10 \Rightarrow x + \frac{9}{x} = 10 \Rightarrow x^2 - 10x + 9 = 0$

$\Rightarrow \quad (x - 9)(x - 1) = 0$

$\Rightarrow \quad x = 9, 1.$

When $x = 9$, $y = \frac{9}{x} = \frac{9}{9} = 1.$

When $x = 1$, $y = \frac{9}{x} = \frac{9}{1} = 9.$

Hence, the two solutions are $x = 9$, $y = 1$, and $x = 1$, $y = 9$.

Example 17. *A train covers a distance of 90 km at a uniform speed. Had the speed been 15 km/h more, it would have taken 30 minutes less for the journey. Find the original speed of the train.*

Solution. Let the original speed of the train be x km/h.

$\therefore$ Time taken to cover a distance of 90 km $= \frac{90}{x}$ hours.

New speed of the train $= (x + 15)$ km/h.

$\therefore$ New time taken to cover 90 km $= \frac{90}{x + 15}$ hours.

By given condition, $\frac{90}{x} - \frac{90}{x + 15} = \frac{1}{2}$

$$\Rightarrow \quad \frac{90(x + 15) - 90x}{x(x + 15)} = \frac{1}{2} \Rightarrow \frac{1350}{x^2 + 15x} = \frac{1}{2}$$

$\Rightarrow \quad x^2 + 15x = 2700 \Rightarrow x^2 + 15x - 2700 = 0$

$\Rightarrow \quad (x + 60)(x - 45) = 0 \Rightarrow x = -60, 45.$

As speed of a train cannot be negative, therefore, the original speed of the train = 45 km/h.

Example 18. *A swimming pool is fitted with three pipes with uniform flow. The first two pipes operating simultaneously, fill the pool in the same time during which the pool is filled by the third pipe alone. The second pipe fills the pool five hours faster than the first pipe and four hours slower than the third pipe. Find the time required by each pipe to fill the pool individually.*

Solution. Let V be the volume of the swimming pool and let x hours be the time taken by the second pipe alone to fill the pool. Then the time taken by the first and third pipes individually to fill the pool is $(x + 5)$ hours and $(x - 4)$ hours respectively.

$\therefore$ The parts of the pool filled by the first, second and third pipes individually in one hour are respectively

$$\frac{V}{x+5}, \frac{V}{x} \text{ and } \frac{V}{x-4}.$$

According to the question, we have

$$\frac{V}{x+5} + \frac{V}{x} = \frac{V}{x-4} \qquad (\because V \neq 0)$$

$$\Rightarrow \quad \frac{1}{x+5} + \frac{1}{x} = \frac{1}{x-4}$$

$$\Rightarrow \quad \frac{x+(x+5)}{x(x+5)} = \frac{1}{x-4} \Rightarrow (2x+5)(x-4) = x(x+5)$$

$$\Rightarrow \quad 2x^2 - 8x + 5x - 20 = x^2 + 5x$$

$$\Rightarrow \quad x^2 - 8x - 20 = 0 \Rightarrow (x-10)(x+2) = 0$$

$$\Rightarrow \quad x = 10, -2$$

But x, being the number of hours, cannot be negative. Therefore, $x = 10$.

Hence the time taken by the first, second and third pipes to fill the pool individually is 15 hours, 10 hours and 6 hours respectively.

EXERCISE 6.2

Solve the following (1 to 11) equations :

1. (i) $\frac{x-2}{x+2} + \frac{x+2}{x-2} = 4$ (ii) $\frac{x-1}{x-2} + \frac{x-3}{x-4} = 3\frac{1}{3}$.

2. $\frac{1}{x+1} - \frac{2}{x+2} = \frac{3}{x+3} - \frac{4}{x+4}$.

3. (i) $3^x + 3^{-x} = 2$ (ii) $2^{x+1} + 4^x = 8$.

4. $5^{x+1} + 5^{2-x} = 5^3 + 1$.

5. $x^{2/3} - x^{1/3} - 2 = 0$.

6. (i) $(x^2 + 3x + 2)^2 - 8(x^2 + 3x) - 4 = 0$ (ii) $(x^2 - 5x + 7)^2 - (x-2)(x-3) = 1$.

7. $x(x+2)(x+3)(x+5) = 72$

8. $x(x+2)(x^2-1) = -1$.

Hint. $x(x+2)(x^2-1) = -1 \Rightarrow x(x+2)(x+1)(x-1) = -1$

$\Rightarrow [x(x+1)][(x+2)(x-1)] = -1$

$\Rightarrow (x^2+x)(x^2+x-2) = -1$

$\Rightarrow y(y-2) = -1$ where $y = x^2 + x$

$\Rightarrow y^2 - 2y + 1 = 0 \Rightarrow y = 1, 1$

$\Rightarrow x^2 + x = 1, x^2 + x = 1.$

9. $(x+1)(2x+3)(2x+5)(x+3) = 945$.

Hint. L.H.S. $= 4[(x+1)(x+3)]\left[\left(x+\frac{3}{2}\right)\left(x+\frac{5}{2}\right)\right]$.

10. $(x^2 + x - 6)(x^2 - 3x - 4) = 24$.

Hint. L.H.S. $= (x+3)(x-4)(x-2)(x+1)$.

11. (i) $\sqrt{\frac{x}{1-x}} + \sqrt{\frac{1-x}{x}} = \frac{13}{6}$ (ii) $\sqrt{\frac{4x-1}{4x+1}} - \sqrt{\frac{4x+1}{4x-1}} = \frac{8}{3}, x \neq \pm\frac{1}{4}$.

12. Evaluate (i) $\sqrt{3+\sqrt{3+\sqrt{3+\ldots \text{ to } \infty}}}$ (ii) $\sqrt{2+\sqrt{2+\sqrt{2+\ldots \text{ to } \infty}}}$.

13. Solve $5 = \sqrt{x + \sqrt{x + \sqrt{x + \ldots \text{ to } \infty}}}$.

Solve the following (14 to 22) equations :

14. (*i*) $\sqrt{3x+4} = x$ (*ii*) $\sqrt{3x+1} - \sqrt{x-1} = 2$.

15. (*i*) $\sqrt{x+1} + \sqrt{x-1} = 1$ (*ii*) $\sqrt{x+3} + \sqrt{x-3} = 6$.

16. (*i*) $\sqrt{3x^2 - 4x + 34} - \sqrt{3x^2 - 4x - 11} = 5$

(*ii*) $\sqrt{3x^2 - 7x - 30} - \sqrt{2x^2 - 7x - 5} = x - 5$.

Hint. (*ii*) Given equation is $\sqrt{3x^2 - 7x - 30} = (x-5) + \sqrt{2x^2 - 7x - 5}$.

Square both sides.

$$3x^2 - 7x - 30 = (x-5)^2 + (2x^2 - 7x - 5) + 2(x-5)\sqrt{2x^2 - 7x - 5}$$

$$\Rightarrow 10x - 50 - 2(x-5)\sqrt{2x^2 - 7x - 5} = 0$$

$$\Rightarrow 2(x-5)(5 - \sqrt{2x^2 - 7x - 5}) = 0.$$

17. $12 + 9\sqrt{(x-1)(3x+2)} = 3x^2 - x$. **18.** $\dfrac{\sqrt{x^2+1} - \sqrt{x^2-1}}{\sqrt{x^2+1} + \sqrt{x^2-1}} = \dfrac{1}{3}$.

19. $\dfrac{\sqrt{a+x} - \sqrt{a-x}}{\sqrt{a+x} + \sqrt{a-x}} = \dfrac{x}{a}$, $a \neq 0$.

20. (*i*) $\left(x + \dfrac{1}{x}\right)^2 - 2\left(x - \dfrac{1}{x} + 4\right) = 11$ (*ii*) $12x^4 - 56x^3 + 89x^2 - 56x + 12 = 0$.

21. Solve $(x^2 + 2)^2 + 8x^2 = 6x(x^2 + 2)$

Hint. Divide throughout by x^2.

22. Solve the following equations simultaneously for x and y :

(*i*) $\sqrt{\dfrac{x}{y}} + \sqrt{\dfrac{y}{x}} = 2\dfrac{1}{2}$, $x + y = 10$ (*ii*) $x^2 + y^2 = 10$, $x + y = 4$.

23. An express train makes a run of 240 km at a certain speed. Another train, whose speed is 12 km/h less takes an hour longer to make the same trip. Find the speed of the express train.

24. A trader bought a number of articles for ₹ 1200. Ten were damaged and he sold each of the rest at ₹ 2 more than what he paid for, thus clearing a profit of ₹ 60 on the whole transaction. Taking the number of articles he bought as x, form an equation in x and solve it.

6.3 NATURE OF ROOTS OF A QUADRATIC EQUATION

The roots of the equation $ax^2 + bx + c = 0$, $a \neq 0$, are

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \text{ where } a, b, c \in \mathbf{R}.$$

Thus the nature of the roots depends on the quantity under the square root sign *i.e.* $b^2 - 4ac$. This quantity is called ***discriminant*** of the equation and is usually denoted by Δ.

$\therefore$ **Discriminant** $\boldsymbol{\Delta = b^2 - 4ac}$.

Case I. When a, b, c are real numbers, $a \neq 0$.

(*i*) If $\Delta = b^2 - 4ac = 0$, then roots are **equal** (and real).

(*ii*) If $\Delta = b^2 - 4ac > 0$, then roots are **real and unequal.**

(*iii*) If $\Delta = b^2 - 4ac < 0$, then roots are **complex.** It is easy to see that roots are a **pair of complex conjugates.**

Case II. When a, b, c are rational numbers, $a \neq 0$.

(*i*) If $\Delta = b^2 - 4ac = 0$, then roots are **rational and equal.**

(*ii*) If $\Delta = b^2 - 4ac > 0$, and Δ is a perfect square of a rational number, then roots are **rational and unequal.**

(*iii*) If $\Delta = b^2 - 4ac > 0$ but Δ is not a square of rational number, then roots are **irrational and unequal.** They form a pair of irrational conjugates $p + \sqrt{q}$, $p - \sqrt{q}$ where p, $q \in \mathbf{Q}$, $q > 0$.

(*iv*) If $\Delta = b^2 - 4ac < 0$, then roots are a pair of complex conjugates.

ILLUSTRATIVE EXAMPLES

Example 1. *Discuss the nature of the roots of the following equations :*

(*i*) $4x^2 - 12x + 9 = 0$ (*ii*) $3x^2 - 10x + 3 = 0$

(*iii*) $9x^2 - 2 = 0$ (*iv*) $x^2 + x + 1 = 0$.

Solution. (*i*) Here the coefficients are rational, and

discriminant $\Delta = b^2 - 4ac = (-12)^2 - 4(4)(9) = 144 - 144 = 0$.

Hence, the roots are rational and equal.

(*ii*) Here the coefficients are rational, and

discriminant $\Delta = b^2 - 4ac = (-10)^2 - 4(3)(3) = 100 - 36 = 64$.

Now $\Delta = 64 > 0$, and 64 is a perfect square of a rational number.

Hence, the roots are rational and unequal.

(*iii*) Here the coefficients are rational, and

discriminant $\Delta = b^2 - 4ac = (0)^2 - 4(9)(-2) = 72$.

Now $\Delta = 72 > 0$ but is not a perfect square of a rational number.

Hence, the roots are irrational and unequal.

(*iv*) Here the coefficients are rational, and

discriminant $\Delta = b^2 - 4ac = (1)^2 - 4(1)(1) = -3 < 0$.

Hence, the roots are a pair of complex conjugates.

Example 2. *Discuss the nature of the roots of the following equations:*

(*i*) $4x^2 - 4\sqrt{3}x + 3 = 0$ (*ii*) $2\sqrt{3}x^2 - 5x + \sqrt{3} = 0$.

Solution. (*i*) Here the coefficients are real, and

discriminant $\Delta = b^2 - 4ac = (4\sqrt{3})^2 - 4(4)(3) = 48 - 48 = 0$.

Hence, the roots are real and equal.

(*ii*) Here the coefficients are real, and discriminant

$\Delta = b^2 - 4ac = (-5)^2 - 4(2\sqrt{3})(\sqrt{3}) = 25 - 24 = 1 > 0$.

Hence, the roots are real and unequal.

Example 3. *If a, b, $h \in$ **R**, show that the equation $(x - a)(x - b) = h^2$ has real roots.*

Solution. The given equation can be written as

$x^2 - (a + b)x + (ab - h^2) = 0$, which is a quadratic equation with real coefficients.

$$\begin{aligned}\text{Its discriminant} &= (-(a+b))^2 - 4 \cdot 1 \cdot (ab - h^2)\\ &= (a+b)^2 - 4ab + 4h^2\\ &= (a-b)^2 + (2h)^2 \geq 0\end{aligned}$$

$\Rightarrow$ the given equation has real roots.

Example 4. *If the roots of* $ax^2 + x + b = 0$ *are real and distinct, show that the roots of the equation* $\frac{x^2+1}{x} = 4\sqrt{ab}$ *are imaginary.*

Solution. Given roots of the equation $ax^2 + x + b = 0$ are real and distinct

$\Rightarrow$ discriminant $> 0 \Rightarrow (1)^2 - 4ab > 0$

$\Rightarrow$ $4ab - 1 < 0$ ...(i)

Now $\frac{x^2+1}{x} = 4\sqrt{ab} \Rightarrow x^2 + 1 = 4\sqrt{ab}\,x$

$\Rightarrow$ $x^2 - 4\sqrt{ab}\,x + 1 = 0$ ...(ii)

Its discriminant $= (-4\sqrt{ab})^2 - 4 \times 1 \times 1 = 16ab - 4$

$= 4(4ab - 1) < 0$ (using (i))

$\Rightarrow$ the roots of equation (ii) are imaginary (complex).

Hence, the roots of the equation $\frac{x^2+1}{x} = 4\sqrt{ab}$ are imaginary.

Example 5. *Given that a, b, c* $\in$ **Q**, *show that the roots of the equation*

$$(b-c)x^2 + (c-a)x + (a-b) = 0$$

are rational. Find the condition for the roots to be equal.

Solution. Here coefficients are rational and

discriminant $\Delta = (c-a)^2 - 4(b-c)(a-b)$

$= c^2 + a^2 - 2ac - 4(ab - b^2 - ac + bc)$

$= (c^2 + a^2 + 2ac) + 4b^2 - 4b(a+c)$

$= (c+a)^2 - 4b(a+c) + (2b)^2 = (c + a - 2b)^2.$

Thus Δ is the square of a rational number.

Hence, the roots of the given equation are rational.

For roots to be equal, $\Delta = (c + a - 2b)^2 = 0$

$\Rightarrow$ $c + a - 2b = 0 \Rightarrow b = \frac{a+c}{2}$, which is the required condition.

Example 6. *Discuss the nature of the roots of the equation*

$$(m+6)x^2 + (m+6)x + 2 = 0.$$

Solution. Discriminant $\Delta = (m+6)^2 - 4(m+6)(2)$

$= m^2 + 12m + 36 - 8m - 48$

$= m^2 + 4m - 12 = (m+6)(m-2).$

Now (i) Roots are real and equal if $\Delta = (m+6)(m-2) = 0$

i.e. if $m = 2$ or -6.

(ii) Roots are real and unequal when $\Delta = (m+6)(m-2) > 0$

i.e. when $m < -6$ or when $m > 2$.

(iii) Roots are a pair of complex conjugates when

$\Delta = (m+6)(m-2) < 0$

i.e. when $-6 < m < 2$.

Example 7. *For what value of k does the expression* $x^2 - 2x(1 + 3k) + 7(3 + 2k)$ *become a perfect square? Also write the expression as perfect square in that case.*

Solution. The given expression will be a perfect square iff the equation

$x^2 - 2x(1 + 3k) + 7(3 + 2k) = 0$ has equal roots

i.e. iff the discriminant of this equation $= 0$ i.e. iff $\Delta = 0$

$\Leftrightarrow$ $(-2(1 + 3k))^2 - 4(1)(7(3 + 2k)) = 0$

$\Leftrightarrow \quad 9k^2 - 8k - 20 = 0$

$\Leftrightarrow \quad k = \dfrac{-(-8) \pm \sqrt{(-8)^2 - 4(9)(-20)}}{2(9)} = \dfrac{8 \pm 28}{18} = 2, -\dfrac{10}{9}.$

When $k = 2$, the given expression

$= x^2 - 2x(1 + 3 \times 2) + 7(3 + 2 \times 2)$

$= x^2 - 14x + 49 = (x - 7)^2.$

When $k = -\dfrac{10}{9}$, the given expression

$= x^2 - 2x\left[1+3\left(-\dfrac{10}{9}\right)\right]+7\left[3+2\left(-\dfrac{10}{9}\right)\right]$

$= x^2 + \dfrac{14}{3}x + \dfrac{49}{9} = \left(x + \dfrac{7}{3}\right)^2.$

Example 8. *Determine a positive real value of k such that both the equations $x^2 + kx + 64 = 0$ and $x^2 - 8x + k = 0$ may have real roots.*

Solution. The given equations are

$x^2 + kx + 64 = 0$ and $x^2 - 8x + k = 0.$

As both the equations have real roots, the discriminant of each ≥ 0

$\Rightarrow \quad k^2 - 4 \times 1 \times 64 \geq 0$ and $(-8)^2 - 4 \times 1 \times k \geq 0$

$\Rightarrow \quad k^2 - 256 \geq 0$ and $64 - 4k \geq 0$

$\Rightarrow \quad (k + 16)(k - 16) \geq 0$ and $16 - k \geq 0$

$\Rightarrow \quad (k \leq -16$ or $k \geq 16)$ and $k \leq 16$ but k is positive real number

$\Rightarrow \quad k \geq 16$ and $k \leq 16 \Rightarrow k = 16.$

Hence, the positive real value of k for which both the given equations have real roots is 16.

Example 9. *Show that if p, q, r, s are real numbers and $pr = 2(q + s)$ then atleast one of the equations $x^2 + px + q = 0$ and $x^2 + rx + s = 0$ has real roots.*

Solution. If possible, let both the equations have complex roots. Then, for both, discriminant < 0

$\Rightarrow \quad p^2 - 4q < 0, r^2 - 4s < 0$

$\Rightarrow \quad (p^2 - 4q) + (r^2 - 4s) < 0 \Rightarrow p^2 + r^2 - 4(q + s) < 0$

$\Rightarrow \quad p^2 + r^2 - 2pr < 0$ (putting $2(q + s) = pr$)

$\Rightarrow \quad (p - r)^2 < 0$, which is impossible, as both p, r are real numbers

$\Rightarrow \quad$ our assumption is wrong

$\Rightarrow \quad$ atleast one of the given equations has real roots.

EXERCISE 6.3

1. Find the nature of roots of the following equations without solving them :

   (i) $x^2 + 9 = 0$ (ii) $4x^2 - 24x + 35 = 0$

   (iii) $x^2 - 2\sqrt{2}x + 1 = 0$ (iv) $2x^2 - 2\sqrt{5}x + 3 = 0.$

2. (i) Prove that the roots of the equation $x^2 - 2ax + a^2 - b^2 - c^2 = 0$ are always real, $a, b, c \in \mathbf{R}$.

   (ii) Show that roots of the equation $(x - a)(x - b) = abx^2$; $a, b \in \mathbf{R}$ are always real. When are they equal?

   Hint. (ii) $\Delta = (a - b)^2 + (2ab)^2$.

3. Show that the roots of the equation $(x - a)(x - b) + (x - b)(x - c) + (x - c)(x - a) = 0$, where $a, b, c \in \mathbf{R}$ are always real. Find the condition that the roots may be equal. What are the roots when this condition is satisfied?

   Hint. $\Delta = 2((a - b)^2 + (b - c)^2 + (c - a)^2)$.

4. Discuss the nature of roots of the following equations :

(i) $\sqrt{3}x^2 - 2x - \sqrt{3} = 0$ (ii) $x^2 - (p+1)x + p = 0$ (iii) $(x-a)(x-b) = ab$.

It is given that $p \in \mathbf{Q}$, and $a, b \in \mathbf{R}$.

5. Find m so that roots of the equation $(4+m)x^2 + (m+1)x + 1 = 0$ may be equal.

6. Show that the roots of the equation $x^2 + 2(3a+5)x + 2(9a^2+25) = 0$ are complex unless $a = -\frac{5}{3}$.

7. If $a, b, c, d \in \mathbf{R}$ show that the roots of the equation

$$(a^2+c^2)x^2 + 2(ab+cd)x + (b^2+d^2) = 0$$

cannot be real unless they are equal.

8. If $a, b, c \in \mathbf{Q}$ and $a+b+c=0$, show that roots of the equation $ax^2+bx+c=0$ are rational. Is the converse of above true?

9. Prove that the equation $x^2+px-1=0$ has real and distinct roots for all real values of p.

10. Show that the roots of the equation $x^2 - 2\left(m+\frac{1}{m}\right)x + 3 = 0$ are real for all (non-zero) real values of m.

11. If the roots of the equation $a(b-c)x^2 + b(c-a)x + c(a-b) = 0$ are equal, show that $\frac{2}{b} = \frac{1}{a} + \frac{1}{c}$.

12. For what value of k, $(4-k)x^2 + 2(k+2)x + (8k+1)$, is a perfect square.

13. If the roots of the equation $x^2 - 8x + m(m-6) = 0$ are real and distinct, then find all possible values of m.

6.4 RELATIONS BETWEEN ROOTS AND COEFFICIENTS

Sum and product of roots

Let the roots of the equation $ax^2+bx+c=0$, $a \neq 0$, be

$$\alpha = \frac{-b+\sqrt{b^2-4ac}}{2a}, \quad \beta = \frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Then, sum of roots $= \alpha + \beta$

$$= \frac{-b+\sqrt{b^2-4ac}}{2a} + \frac{-b-\sqrt{b^2-4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}.$$

$$\therefore \textit{ Sum of roots} = \alpha+\beta = -\frac{b}{a} = -\frac{\text{coefficient of } x}{\text{coefficient of } x^2}.$$

$$\text{Product of roots} = \alpha.\beta = \frac{-b+\sqrt{b^2-4ac}}{2a} \times \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{(-b)^2 - \left(\sqrt{b^2-4ac}\right)^2}{(2a)^2} = \frac{b^2-(b^2-4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

$$\therefore \textit{ Product of roots} = \alpha\beta = \frac{c}{a} = \frac{\text{constant term}}{\text{coefficient of } x^2}.$$

Symmetric functions of roots

If α, β are the roots of the quadratic equation $ax^2+bx+c=0$, $a \neq 0$, then the expressions of the form $\alpha+\beta$, $\alpha\beta$, $\alpha^2 \pm \beta^2$, $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ etc. are called **functions** of the roots α and β.

If an expression doesn't change on interchanging α and β, then it is called **symmetric**. Thus $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ is a symmetric function while $\alpha^2 - \beta^2$ is not a symmetric function. The expressions $\alpha + \beta$ and $\alpha\beta$ are called **elementary symmetric functions.**

We have already seen that for the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, the value of $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. With this information, values of other functions of α and β can be calculated :

$$\begin{aligned}
(\alpha - \beta)^2 &= (\alpha + \beta)^2 - 4\,\alpha\,\beta,\\
\alpha^2 + \beta^2 &= (\alpha + \beta)^2 - 2\,\alpha\beta,\\
\alpha^3 + \beta^3 &= (\alpha + \beta)^3 - 3\,\alpha\beta\,(\alpha + \beta),\\
\alpha^4 + \beta^4 &= [(\alpha + \beta)^2 - 2\,\alpha\beta]^2 - 2\,(\alpha\beta)^2 \text{ and}\\
\alpha^2 - \beta^2 &= (\alpha + \beta)\,(\alpha - \beta) \text{ etc.}
\end{aligned}$$

6.4.1 Formation of a quadratic equation with given roots

Let the given roots be α and β. Suppose that the equation whose roots are α, β is $ax^2 + bx + c = 0$, $a \neq 0$.

Then $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$.

Now $ax^2 + bx + c = 0 \Leftrightarrow x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ $(\because a \neq 0)$

$\Leftrightarrow \quad x^2 - \left(-\frac{b}{a}\right)x + \left(\frac{c}{a}\right) = 0 \Leftrightarrow x^2 - (\alpha + \beta)\,x + \alpha\beta = 0$

$\Leftrightarrow \quad$ ***x^2 – (sum of roots) x + product of roots = 0***

$\Leftrightarrow \quad$ ***$x^2 - Sx + P = 0$, where S = sum of roots and P = product of roots.***

6.4.2 To find conditions when roots are connected by given relations

Sometimes the relation between roots of a quadratic equation is given and we are asked to find the condition *i.e.* relation between the coefficients a, b, c of quadratic equation. This is easily done using the formula $\alpha + \beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$. This will be clear when you go through illustrative examples.

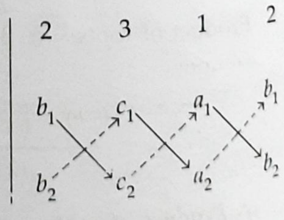
One common root

Suppose the equations $a_1x^2 + b_1x + c_1 = 0$ $(a_1 \neq 0)$ and $a_2x^2 + b_2x + c_2 = 0$ $(a_2 \neq 0)$ have a common root, say α.

Then $a_1\alpha^2 + b_1\alpha + c_1 = 0$ and $a_2\alpha^2 + b_2\alpha + c_2 = 0$.

Solving by cross multiplication rule,

$$\frac{\alpha^2}{b_1c_2 - b_2c_1} = \frac{\alpha}{c_1a_2 - c_2a_1} = \frac{1}{a_1b_2 - a_2b_1}$$



From first two, $\alpha = \frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1}$ and

from last two, $\alpha = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$.

Thus $\frac{b_1c_2 - b_2c_1}{c_1a_2 - c_2a_1} = \frac{c_1a_2 - c_2a_1}{a_1b_2 - a_2b_1}$

i.e. $(c_1a_2 - c_2a_1)^2 = (b_1c_2 - b_2c_1)\,(a_1b_2 - a_2b_1)$, which is the required condition for the two equations to have a common root.

Both roots common

Suppose the equations $a_1x^2 + b_1x + c_1 = 0$ and $a_2x^2 + b_2x + c_2 = 0$ have both roots common. Let those roots be α, β.

Then, as α, β are roots of $a_1x^2 + b_1x + c_1 = 0$, $\alpha + \beta = -\frac{b_1}{a_1}$, $\alpha\beta = \frac{c_1}{a_1}$.

Similarly $\alpha + \beta = -\frac{b_2}{a_2}$, $\alpha\beta = \frac{c_2}{a_2}$

$\Rightarrow \quad -\frac{b_1}{a_1} = -\frac{b_2}{a_2}, \frac{c_1}{a_1} = \frac{c_2}{a_2} \Rightarrow \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$, which is the required condition for the two equations to have both roots common.

ILLUSTRATIVE EXAMPLES

Example 1. *If α, β are roots of the equation $x^2 - 4x + 2 = 0$, find the values of*

(*i*) $\alpha^2 + \beta^2$ (*ii*) $\alpha^2 - \beta^2$ (*iii*) $\alpha^3 + \beta^3$ (*iv*) $\frac{1}{\alpha} + \frac{1}{\beta}$.

Solution. As α, β are roots of the equation $x^2 - 4x + 2 = 0$,

$$\alpha + \beta = -\frac{(-4)}{1} = 4, \ \alpha\beta = \frac{2}{1} = 2.$$

(*i*) $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = (4)^2 - 2 \times 2 = 12.$

(*ii*) $\alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta)$.

Now $(\alpha - \beta)^2 = (\alpha + \beta)^2 - 4\alpha\beta = (4)^2 - 4 \times 2 = 8$

$\Rightarrow \alpha - \beta = \pm\sqrt{8} = \pm 2\sqrt{2}$

$\therefore \ \alpha^2 - \beta^2 = (\alpha + \beta)(\alpha - \beta) = 4 \times (\pm 2\sqrt{2}) = \pm 8\sqrt{2}$.

(*iii*) $\alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta) = (4)^3 - 3 \times 2 \times 4 = 64 - 24 = 40.$

(*iv*) $\frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha + \beta}{\alpha\beta} = \frac{4}{2} = 2.$

Example 2. *If α, β are the roots of the equation $ax^2 + bx + c = 0$, then show that $ax^2 + bx + c = a(x - \alpha)(x - \beta)$.*

Solution. Since α, β are roots of the equation $ax^2 + bx + c = 0$,

$$\alpha + \beta = -\frac{b}{a}, \ \alpha\beta = \frac{c}{a}.$$

Now $ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a\left[x^2 - \left(-\frac{b}{a}\right)x + \frac{c}{a}\right]$

$$= a[x^2 - (\alpha + \beta)x + \alpha\beta] = a(x - \alpha)(x - \beta).$$

Example 3. *For the quadratic equation $(k - 1)x^2 = kx - 1$, $k \neq 1$, find k so that*

(*i*) *one root is* -3 (*ii*) *the sum of the roots is* 2

(*iii*) *the product of the roots is* -3 (*iv*) *the roots are equal*

(*v*) *the roots are numerically equal but opposite in sign.*

Solution. The given quadratic equation is $(k - 1)x^2 - kx + 1 = 0$.

(*i*) Since -3 is a root of the given equation, it must satisfy the equation.

$\therefore \ (k - 1)(-3)^2 - k(-3) + 1 = 0$

$\Rightarrow 9k - 9 + 3k + 1 = 0 \Rightarrow 12k = 8$

$\Rightarrow k = \frac{8}{12} = \frac{2}{3}$.

(*ii*) Given the sum of the roots = 2

$\Rightarrow -\frac{-k}{k-1} = 2 \Rightarrow 2k - 2 = k \Rightarrow k = 2.$

(*iii*) Given the product of the roots = −3

$\Rightarrow \frac{1}{k-1} = -3 \Rightarrow 1 = -3k + 3 \Rightarrow 3k = 2 \Rightarrow k = \frac{2}{3}.$

(*iv*) Given the roots are equal ⇒ discriminant $\Delta = 0$

$\Rightarrow (-k)^2 - 4.(k-1).1 = 0 \Rightarrow k^2 - 4k + 4 = 0$

$\Rightarrow (k-2)^2 = 0 \Rightarrow k - 2 = 0 \Rightarrow k = 2.$

(*v*) Given the roots are numerically equal but opposite in sign, the roots may be taken as $\alpha, -\alpha$ so that the sum of roots is zero

$\Rightarrow -\frac{k}{k-1} = 0 \Rightarrow \frac{k}{k-1} = 0 \Rightarrow k = 0.$

Example 4. *If one of the roots of $3x^2 - 7px + 10p = 0$ be 5, find p and also the other root.*

Solution. Let the roots be α, 5.

Then $\alpha + 5 = -\frac{-7p}{3} = \frac{7p}{3}$ and $\alpha \times 5 = \frac{10p}{3}$ *i.e.* $\alpha = \frac{2p}{3}$

$\Rightarrow \quad \frac{2p}{3} + 5 = \frac{7p}{3} \Rightarrow 2p + 15 = 7p$

$\Rightarrow \quad -5p = -15 \Rightarrow p = 3.$

$\therefore \quad \alpha = \frac{2 \times 3}{3} = 2$

Hence, $p = 3$, and the other root is 2.

Example 5. *The sum of the roots of the equation $\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$ is zero. Prove that the product of the roots is $-\frac{1}{2}(a^2 + b^2)$.*

Solution. The given equation is $\frac{1}{x+a} + \frac{1}{x+b} = \frac{1}{c}$

$\Rightarrow \quad \frac{x+b+x+a}{(x+a)(x+b)} = \frac{1}{c}$

$\Rightarrow \quad (x+a)(x+b) = c(2x+a+b)$

$\Rightarrow \quad x^2 + (a+b)x + ab - 2cx - ca - bc = 0$

$\Rightarrow \quad x^2 + (a+b-2c)x + (ab - bc - ca) = 0.$

Sum of roots = 0 (given) $\Rightarrow -\frac{a+b-2c}{1} = 0$

$\Rightarrow \quad a + b - 2c = 0 \Rightarrow c = \frac{a+b}{2}$ ...(*i*)

Product of roots $= \frac{ab - bc - ca}{1} = ab - (a+b)c$

$= ab - (a+b) \cdot \frac{a+b}{2}$ (using (*i*))

$= \frac{2ab - (a+b)^2}{2} = -\frac{a^2 + b^2}{2}.$

Example 6. *If α, β be the roots of the equation $ax^2 + bx + c = 0$, evaluate*

(*i*) $\alpha^2\beta + \alpha\beta^2$ (*ii*) $\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}$ (*iii*) $(a\alpha + b)^{-2} + (a\beta + b)^{-2}$.

Solution. Since α, β are roots of $ax^2 + bx + c = 0$,

$$\alpha + \beta = -\frac{b}{a},\ \alpha\beta = \frac{c}{a}.$$

(*i*) $\alpha^3\beta + \alpha\beta^3 = \alpha\beta(\alpha^2 + \beta^2) = \alpha\beta[(\alpha+\beta)^2 - 2\alpha\beta]$

$$= \frac{c}{a}\cdot\left[\left(-\frac{b}{a}\right)^2 - 2.\frac{c}{a}\right] = \frac{c}{a}\cdot\frac{b^2 - 2ac}{a^2} = \frac{c(b^2 - 2ac)}{a^3}.$$

(*ii*) $$\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha} = \frac{\alpha^3 + \beta^3}{\alpha\beta} = \frac{(\alpha+\beta)^3 - 3\alpha\beta(\alpha+\beta)}{\alpha\beta} = \frac{\left(-\frac{b}{a}\right)^3 - 3\left(\frac{c}{a}\right)\left(-\frac{b}{a}\right)}{\frac{c}{a}}$$

$$= \frac{-b^3 + 3abc}{a^3}\cdot\frac{a}{c} = \frac{3abc - b^3}{a^2c}.$$

(*iii*) Since α is root of $ax^2 + bx + c = 0$, we have $a\alpha^2 + b\alpha + c = 0$

$$\Rightarrow \alpha(a\alpha + b) = -c \Rightarrow a\alpha + b = -\frac{c}{\alpha}.$$

Similarly, $a\beta + b = -\frac{c}{\beta}$.

$$\therefore\ (a\alpha + b)^{-2} + (a\beta + b)^{-2} = \left(-\frac{c}{\alpha}\right)^{-2} + \left(-\frac{c}{\beta}\right)^{-2} = \left(\frac{\alpha}{c}\right)^2 + \left(\frac{\beta}{c}\right)^2 = \frac{\alpha^2 + \beta^2}{c^2}$$

$$= \frac{(\alpha+\beta)^2 - 2\alpha\beta}{c^2} = \frac{\left(-\frac{b}{a}\right)^2 - 2\left(\frac{c}{a}\right)}{c^2} = \frac{b^2 - 2ac}{a^2c^2}.$$

Example 7. (*i*) *Form an equation whose roots are* $2, -\frac{1}{2}$.

(*ii*) *Form an equation with rational coefficients, one of whose roots is* $\frac{1}{3 + 2\sqrt{2}}$.

(*iii*) *Form an equation with real coefficients one of whose roots is* $-2 + i$.

Solution. (*i*) Since roots are $2, -\frac{1}{2}$; sum of roots $S = 2 + \left(-\frac{1}{2}\right) = \frac{3}{2}$

and product of roots $P = 2.\left(-\frac{1}{2}\right) = -1$.

The required equation is $x^2 - Sx + P = 0$ *i.e.* $x^2 - \frac{3}{2}x - 1 = 0$

i.e. $2x^2 - 3x - 2 = 0$.

(*ii*) One of the roots is $\frac{1}{3 + 2\sqrt{2}} = \frac{1}{3 + 2\sqrt{2}}\cdot\frac{3 - 2\sqrt{2}}{3 - 2\sqrt{2}} = \frac{3 - 2\sqrt{2}}{9 - 8} = 3 - 2\sqrt{2}$.

Since the equation has rational coefficients, the other root is $3 + 2\sqrt{2}$.

$\therefore$ Sum of roots $S = (3 - 2\sqrt{2}) + (3 + 2\sqrt{2}) = 6$.

Product of roots $P = (3 - 2\sqrt{2})(3 + 2\sqrt{2}) = 9 - 8 = 1$.

The required equation is $x^2 - Sx + P = 0$ *i.e.* $x^2 - 6x + 1 = 0$.

(*iii*) Since one of the roots is $-2 + i$, and the equation has real coefficients, the other root must be $-2 - i$.

$\therefore$ Sum of roots $S = (-2 + i) + (-2 - i) = -4$.

Product of roots $P = (-2 + i)(-2 - i) = 4 - i^2 = 4 + 1 = 5$.

$\therefore$ Required equation is $x^2 - Sx + P = 0$ *i.e.* $x^2 + 4x + 5 = 0$.

Example 8. *If α, β are roots of the quadratic equation $ax^2 + bx + c = 0$, form an equation whose roots are*

(i) $-\alpha, -\beta$ (ii) $\frac{1}{\alpha}, \frac{1}{\beta}$ (iii) α^2, β^2 (iv) α^3, β^3.

Solution. Since α, β are roots of $ax^2 + bx + c = 0$,

$$\alpha + \beta = -\frac{b}{a},\ \alpha\beta = \frac{c}{a}.$$

(i) Here, $S = (-\alpha) + (-\beta) = -(\alpha + \beta) = -\left(-\frac{b}{a}\right) = \frac{b}{a}$ and

$$P = (-\alpha)(-\beta) = \alpha\beta = \frac{c}{a}.$$

The required equation is $x^2 - Sx + P = 0$

i.e. $x^2 - \frac{b}{a}x + \frac{c}{a} = 0$ i.e. $ax^2 - bx + c = 0$.

(ii) Here, $S = \frac{1}{\alpha} + \frac{1}{\beta} = \frac{\alpha+\beta}{\alpha\beta} = \frac{-b/a}{c/a} = -\frac{b}{c}$ and

$$P = \frac{1}{\alpha}\cdot\frac{1}{\beta} = \frac{1}{\alpha\beta} = \frac{1}{c/a} = \frac{a}{c}.$$

The required equation is $x^2 - Sx + P = 0$

i.e. $x^2 - \left(-\frac{b}{c}\right)x + \frac{a}{c} = 0$ i.e. $cx^2 + bx + a = 0$.

(iii) Here, $S = \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = \left(-\frac{b}{a}\right)^2 - 2.\frac{c}{a} = \frac{b^2 - 2ac}{a^2}$ and

$$P = \alpha^2 . \beta^2 = (\alpha\beta)^2 = \left(\frac{c}{a}\right)^2 = \frac{c^2}{a^2}.$$

The required equation is $x^2 - Sx + P = 0$

i.e. $x^2 - \frac{b^2 - 2ac}{a^2}x + \frac{c^2}{a^2} = 0$

i.e. $a^2x^2 - (b^2 - 2ac)x + c^2 = 0$.

(iv) Here, $S = \alpha^3 + \beta^3 = (\alpha + \beta)^3 - 3\alpha\beta(\alpha + \beta)$

$$= \left(-\frac{b}{a}\right)^3 - 3.\frac{c}{a}.\left(-\frac{b}{a}\right) = -\frac{b^3}{a^3} + \frac{3bc}{a^2} = \frac{3abc - b^3}{a^3} \text{ and}$$

$$P = \alpha^3 . \beta^3 = (\alpha\beta)^3 = \left(\frac{c}{a}\right)^3 = \frac{c^3}{a^3}.$$

The required equation is $x^2 - Sx + P = 0$

i.e. $x^2 - \frac{3abc - b^3}{a^3}x + \frac{c^3}{a^3} = 0$

i.e. $a^3x^2 - (3abc - b^3)x + c^3 = 0$.

Example 9. *Find the equation whose roots are n times the roots of the equation $ax^2 + bx + c = 0$.*

Solution. Let α, β be roots of the equation $ax^2 + bx + c = 0$, then

$$\alpha + \beta = -\frac{b}{a} \quad \text{...(i)} \quad \text{and} \quad \alpha\beta = \frac{c}{a} \quad \text{...(ii)}$$

Since the roots of the required equation are n times the roots of the given equation, roots of the required equation are $n\alpha$, $n\beta$.

Sum of the roots $S = n\alpha + n\beta = n(\alpha + \beta)$

$$= n\left(-\frac{b}{a}\right) \quad \text{(using (i))}$$

and product of the roots $P = n\alpha . n\beta = n^2 . \alpha\beta = n^2 . \frac{c}{a}$ (using (*ii*))

$\therefore$ The required equation is $x^2 - Sx + P = 0$

i.e. $x^2 - \left(-\frac{nb}{a}\right)x + n^2\frac{c}{a} = 0$ or $ax^2 + nbx + n^2c = 0$.

Example 10. *Let p, q be roots of* $3x^2 + 6x + 2 = 0$. *Form an equation whose roots are* $\frac{-p^2}{q}, \frac{-q^2}{p}$.

Solution. Since p, q are roots of $3x^2 + 6x + 2 = 0$, $p + q = -\frac{6}{3} = -2$, $pq = \frac{2}{3}$.

Here, $S = -\frac{p^2}{q} - \frac{q^2}{p} = -\frac{p^3 + q^3}{pq} = -\frac{(p+q)^3 - 3pq(p+q)}{pq} = -\frac{(-2)^3 - 3\left(\frac{2}{3}\right)(-2)}{\frac{2}{3}}$

$= -(-8 + 4) . \frac{3}{2} = 6$ and

$P = \left(-\frac{p^2}{q}\right)\left(-\frac{q^2}{p}\right) = pq = \frac{2}{3}$.

$\therefore$ Required equation is $x^2 - Sx + P = 0$ *i.e.* $x^2 - 6x + \frac{2}{3} = 0$

i.e. $3x^2 - 18x + 2 = 0$.

Example 11. (*i*) *If the roots of the equation* $x^2 - lx + m = 0$ *differ by 1, then prove that* $l^2 = 4m + 1$.

(*ii*) *Find the condition that roots of the equation* $ax^2 + bx + c = 0$ *may be in the ratio* $p : q$.

Solution. (*i*) Let the roots be α and $\alpha + 1$.

Then $\alpha + (\alpha + 1) = -\frac{(-l)}{1} = l$, $\alpha(\alpha + 1) = \frac{m}{1} = m$.

From first equation, we get $\alpha = \frac{l-1}{2}$. Putting this value in second,

$$\frac{l-1}{2}\left(\frac{l-1}{2} + 1\right) = m \Rightarrow \frac{l-1}{2} \cdot \frac{l+1}{2} = m \Rightarrow l^2 - 1 = 4m$$

$\Rightarrow l^2 = 4m + 1$.

(*ii*) Since the roots of the equation $ax^2 + bx + c = 0$ are in ratio $p : q$, let the roots be $p\alpha, q\alpha$.

Then $p\alpha + q\alpha = -\frac{b}{a}$, $p\alpha . q\alpha = \frac{c}{a}$.

From first equation, we get $\alpha = \frac{-b}{a(p+q)}$. Putting this value in second,

$$pq\left(\frac{-b}{a(p+q)}\right)^2 = \frac{c}{a} \Rightarrow pqb^2 = ca(p+q)^2,$$ which is the required condition.

Example 12. *If* α, β *are the roots of* $x^2 + ax + b = 0$, *then prove that* $\frac{\alpha}{\beta}$ *is a root of the equation* $bx^2 + (2b - a^2)x + b = 0$.

Solution. As α, β are roots of the equation $x^2 + ax + b = 0$,

$\alpha + \beta = -a$ and $\alpha\beta = b$ ...(*i*)

Now $\frac{\alpha}{\beta}$ is a root of the equation $bx^2 + (2b - a)x + b = 0$

if $\quad b\left(\frac{\alpha}{\beta}\right)^2 + (2b - a^2) \cdot \frac{\alpha}{\beta} + b = 0$ is true

i.e. if $b(\alpha^2 + \beta^2) + (2b - a^2)\,\alpha\beta = 0$ is true

i.e. if $b((\alpha + \beta)^2 - 2\alpha\beta) + (2b - a^2)\alpha\beta = 0$ is true

i.e. if $b((-a)^2 - 2b) + (2b - a^2)b = 0$ is true (using (*i*))

i.e. if $b(a^2 - 2b) + (2b - a^2)b = 0$ is true

i.e. if $0 = 0$ is true, which is true.

Example 13. *If the roots of the equation $ax^2 + bx + b = 0$, $b^2 \geq 4ab$, are in ratio $p : q$, prove that* $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} = \sqrt{\frac{b}{a}}$.

Solution. Since $b^2 \geq 4ab$, the roots are real. Let the roots be α, β.

Then $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{b}{a}$. Also $\frac{\alpha}{\beta} = \frac{p}{q}$ (given).

Now $\left(\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}}\right)^2 = \frac{p}{q} + \frac{q}{p} + 2 = \frac{\alpha}{\beta} + \frac{\beta}{\alpha} + 2 = \frac{\alpha^2 + \beta^2 + 2\alpha\beta}{\alpha\beta} = \frac{(\alpha+\beta)^2}{\alpha\beta} = \frac{(-b/a)^2}{b/a} = \frac{b}{a}$.

Taking square roots of both sides, we get $\sqrt{\frac{p}{q}} + \sqrt{\frac{q}{p}} = \sqrt{\frac{b}{a}}$. $\quad (\because$ L.H.S. $\geq 0)$

Example 14. *The roots of the equation $px^2 - 2(p + 2)x + 3p = 0$ differ by 2. Find p and also the roots.*

Solution. Let the roots be α, $\alpha + 2$.

Then $\alpha + \alpha + 2 = \frac{2(p+2)}{p}$, $\alpha(\alpha + 2) = \frac{3p}{p} = 3$.

Now $\alpha + \alpha + 2 = \frac{2p + 4}{p} = 2 + \frac{4}{p} \Rightarrow 2\alpha = \frac{4}{p} \Rightarrow \alpha = \frac{2}{p}$.

$\therefore \quad \alpha(\alpha + 2) = 3 \Rightarrow \frac{2}{p}\left(\frac{2}{p} + 2\right) = 3 \Rightarrow 3p^2 - 4p - 4 = 0$

$\Rightarrow \quad (3p + 2)(p - 2) = 0 \Rightarrow p = 2, -\frac{2}{3}$.

When $p = 2$, then $\alpha = \frac{2}{p} = 1$; so roots are 1, 3.

When $p = -\frac{2}{3}$, then $\alpha = \frac{2}{p} = -3$; so roots are -3, -1.

Example 15. *If the difference of the roots of the equation $x^2 + px + q = 0$ be the same as that of the roots of $x^2 + qx + p = 0$, then prove that $p + q + 4 = 0$; $p \neq q$.*

Solution. Let α, β be the roots of $x^2 + px + q = 0$ and

γ, δ be the roots of $x^2 + qx + p = 0$, then

$\alpha + \beta = -p$, $\alpha\beta = q$ ...(*i*)

and $\gamma + \delta = -q$, $\gamma\delta = p$ ...(*ii*)

Given $\alpha - \beta = \gamma - \delta \Rightarrow (\alpha - \beta)^2 = (\gamma - \delta)^2$

$\Rightarrow \quad (\alpha + \beta)^2 - 4\alpha\beta = (\gamma + \delta)^2 - 4\gamma\delta$

$\Rightarrow \quad (-p)^2 - 4q = (-q)^2 - 4p$ (using (*i*) and (*ii*))

$\Rightarrow \quad p^2 - 4q = q^2 - 4p$

$\Rightarrow \quad p^2 - q^2 + 4p - 4q = 0$

$\Rightarrow \quad (p - q)(p + q) + 4(p - q) = 0$

$\Rightarrow \quad (p - q)(p + q + 4) = 0$ but $p \neq q$ *i.e.* $p - q \neq 0$

$\Rightarrow \quad p + q + 4 = 0.$

Example 16. *The coefficient of x in the equation $x^2 + px + q = 0$ was taken as 17 in place of 13 and thus its roots were found to be -2 and -15. Find the roots of the original equation.*

Solution. By given conditions, -2 and -15 are roots of the equation $x^2 + 17x + q = 0$.

The product of roots $= (-2)(-15) = \frac{q}{1} \Rightarrow q = 30.$

Therefore, the original equation is $x^2 + 13x + 30 = 0$

$\Rightarrow \quad (x + 10)(x + 3) = 0 \Rightarrow x = -3, -10.$

Hence, the roots of the original equation are $-3, -10$.

Example 17. *If one root of the equation $ax^2 + bx + c = 0$ is the square of the other, prove that $b^3 + ac(c + a) = 3abc$.*

Solution. Let the roots of the given equation $ax^2 + bx + c = 0$ be α and α^2.

Then $\alpha + \alpha^2 = -\frac{b}{a}$ ...(*i*) and $\alpha.\alpha^2 = \frac{c}{a} \Rightarrow \alpha^3 = \frac{c}{a}$ ...(*ii*)

Cubing both sides of (*i*), we get

$$\alpha^3 + \alpha^6 + 3.\alpha.\alpha^2(\alpha + \alpha^2) = -\frac{b^3}{a^3}$$

$$\Rightarrow \quad \alpha^3 + (\alpha^3)^2 + 3\alpha^3(\alpha + \alpha^2) = -\frac{b^3}{a^3}$$

$$\Rightarrow \quad \frac{c}{a} + \frac{c^2}{a^2} + 3.\frac{c}{a}\left(-\frac{b}{a}\right) = -\frac{b^3}{a^3} \qquad \text{(using } (i) \text{ and } (ii))$$

$$\Rightarrow \quad a^2c + ac^2 - 3abc = -b^3 \qquad \text{(multiplying by } a^3)$$

$\Rightarrow \quad b^3 + ac(c + a) = 3abc$, as required.

Example 18. *If one root of the quadratic equation $ax^2 + bx + c = 0$ is equal to the nth power of the other root, then show that $(ac^n)^{\frac{1}{n+1}} + (a^nc)^{\frac{1}{n+1}} + b = 0$.*

Solution. Let the roots of the given equation $ax^2 + bx + c = 0$ be α and α^n.

Then $\alpha \,.\, \alpha^n = \frac{c}{a} \Rightarrow \alpha^{n+1} = \frac{c}{a} \Rightarrow \alpha = \left(\frac{c}{a}\right)^{\frac{1}{n+1}}$ ...(*i*)

Since α is a root of the given equation $ax^2 + bx + c = 0$,

$$a\alpha^2 + b\alpha + c = 0 \Rightarrow a\alpha + b + \frac{c}{\alpha} = 0$$

$$\Rightarrow \quad a\left(\frac{c}{a}\right)^{\frac{1}{n+1}} + b + c\,.\left(\frac{a}{c}\right)^{\frac{1}{n+1}} = 0 \qquad \text{(using } (i))$$

$$\Rightarrow \quad \left(a^{n+1}.\frac{c}{a}\right)^{\frac{1}{n+1}} + b + \left(c^{n+1}.\frac{a}{c}\right)^{\frac{1}{n+1}} = 0$$

$$\Rightarrow \quad (a^nc)^{\frac{1}{n+1}} + b + (c^na)^{\frac{1}{n+1}} = 0.$$

Example 19. *If the equations $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$ have a common root and $a \neq b$, then prove that $a + b + 1 = 0$.*

Solution. Let α be the common root of $x^2 + ax + b = 0$ and $x^2 + bx + a = 0$, then

$$\alpha^2 + a\alpha + b = 0 \quad \text{...(i)}$$

and $$\alpha^2 + b\alpha + a = 0 \quad \text{...(ii)}$$

Subtracting (ii) from (i), we get

$$(a - b)\alpha + (b - a) = 0$$

$$\Rightarrow \quad (a - b)(\alpha - 1) = 0 \Rightarrow \alpha = 1 \quad (\because a \neq b)$$

Substituting $\alpha = 1$ in (i), we get

$$1^2 + a \times 1 + b = 0 \Rightarrow a + b + 1 = 0$$

Example 20. *If the equations $3x^2 + px + 1 = 0$ and $2x^2 + qx + 1 = 0$ have a common root, show that $2p^2 + 3q^2 - 5pq + 1 = 0$.*

Solution. Let α be the common root of $3x^2 + px + 1 = 0$ and $2x^2 + qx + 1 = 0$, then

$$3\alpha^2 + p\alpha + 1 = 0 \quad \text{...(i)}$$

and $$2\alpha^2 + q\alpha + 1 = 0 \quad \text{...(ii)}$$

Substracting (ii) from (i), we get

$$\alpha^2 + (p - q)\alpha = 0 \Rightarrow \alpha + p - q = 0 \quad (\because \alpha \neq 0)$$

$$\Rightarrow \quad \alpha = q - p.$$

Substituting this value of α in (i), we get

$$3(q - p)^2 + p(q - p) + 1 = 0$$

$$\Rightarrow \quad 3(q^2 + p^2 - 2pq) + pq - p^2 + 1 = 0$$

$$\Rightarrow \quad 2p^2 + 3q^2 - 5pq + 1 = 0.$$

Example 21. *Find the value(s) of k so that the equations $x^2 - kx - 21 = 0$ and $x^2 - 3kx + 35 = 0$ may have one common root.*

Solution. Let α be the common root of the given equations, then

$$\alpha^2 - k\alpha - 21 = 0 \quad \text{...(i)}$$

$$\alpha^2 - 3k\alpha + 35 = 0 \quad \text{...(ii)}$$

Subtracting (ii) from (i), we get

$$2k\alpha - 56 = 0 \Rightarrow \alpha = \frac{28}{k}.$$

Putting this value of α in (i), we get

$$\left(\frac{28}{k}\right)^2 - k.\frac{28}{k} - 21 = 0 \Rightarrow \left(\frac{28}{k}\right)^2 - 49 = 0$$

$$\Rightarrow 16 - k^2 = 0 \Rightarrow k^2 = 16 \Rightarrow k = 4, -4.$$

Hence, the values of k are 4, -4.

Example 22. *If the equations $x^2 - ax + b = 0$ and $x^2 - cx + d = 0$ have one root in common and the second equation has equal roots, then prove that $ac = 2(b + d)$.*

Solution. As the equation $x^2 - cx + d = 0$ ...(i) has equal roots, we have

$$(-c)^2 - 4.1.d = 0 \quad \text{(discriminant = 0)}$$

$$\Rightarrow c^2 = 4d \Rightarrow d = \frac{c^2}{4} \quad \text{...(ii)}$$

Substituting this value of d in (i), we get

$$x^2 - cx + \frac{c^2}{4} = 0 \Rightarrow \left(x - \frac{c}{2}\right)^2 = 0 \Rightarrow x = \frac{c}{2}.$$

So $\frac{c}{2}$ is the common root, therefore, $\frac{c}{2}$ is also the root of $x^2 - ax + b = 0$

$\Rightarrow \left(\frac{c}{2}\right)^2 - a.\frac{c}{2} + b = 0$

$\Rightarrow c^2 - 2ac + 4b = 0 \Rightarrow 2ac = c^2 + 4b$

$\Rightarrow 2ac = 4d + 4b$ (using (*ii*))

$\Rightarrow ac = 2(b + d).$

Example 23. *If the roots of the equation $2x^2 + (k+1)x + (k^2 - 5k + 6) = 0$ are of opposite signs then show that $2 < k < 3$.*

Solution. Since the roots are of opposite signs, roots are real and distinct.

So discriminant > 0 and product of roots < 0.

$\therefore \quad (k+1)^2 - 4.2.(k^2 - 5k + 6) > 0$ and $\frac{k^2 - 5k + 6}{2} < 0.$

First condition is always true when second holds. $(\because (k+1)^2 \geq 0)$

Therefore, $\frac{k^2 - 5k + 6}{2} < 0 \Rightarrow k^2 - 5k + 6 < 0 \Rightarrow (k-2)(k-3) < 0$

$\Rightarrow \quad 2 < k < 3.$

Example 24. *Both roots of equation $x^2 - (a+1)x + a + 4 = 0$ are negative. Calculate the values of a.*

Solution. As both roots of equation $x^2 - (a+1)x + a + 4 = 0$ are negative

$\Rightarrow$ sum of roots is negative and product of roots is positive

$\Rightarrow a + 1 < 0$ and $a + 4 > 0 \Rightarrow a < -1$ and $a > -4$

$\Rightarrow -4 < a < -1$ ...(*i*)

Since the roots of the given equation are negative, so the roots are real,

$\therefore$ discriminant ≥ 0

$\Rightarrow (-(a+1))^2 - 4.1.(a+4) \geq 0 \Rightarrow a^2 + 2a + 1 - 4a - 16 \geq 0$

$\Rightarrow a^2 - 2a - 15 \geq 0 \Rightarrow (a+3)(a-5) \geq 0$

$\Rightarrow$ either $a \leq -3$ or $a \geq 5$ ...(*ii*)

Combining (*i*) and (*ii*), we get $-4 < a \leq -3$.

EXERCISE 6.4

1. Write the sum and product of the roots of the following equations :

   (*i*) $x^2 + 9 = 0$ (*ii*) $\sqrt{2}\,x^2 - \sqrt{2}\,x + 4 = 0.$

2. If α, β are the roots of the equation $3x^2 - 6x + 4 = 0$,

   evaluate $\left(\frac{\alpha^2}{\beta} + \frac{\beta^2}{\alpha}\right) + \left(\frac{\alpha}{\beta} + \frac{\beta}{\alpha}\right) + 2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) + 3\alpha\beta.$

3. If the roots of the equation $x^2 + px + 7 = 0$ are denoted by α and β, and $\alpha^2 + \beta^2 = 22$, find the possible values of p.

4. If α, β be the roots of the equation $ax^2 + bx + c = 0$, find the value of

   (*i*) $\alpha^2 + \beta^2$ (*ii*) $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$ (*iii*) $\alpha^3 + \beta^3$ (*iv*) $|\alpha - \beta|$

   (*v*) $\alpha^2 - \beta^2$ (*vi*) $\alpha^6 + \beta^6$.

5. If α, β are roots of $x^2 + kx + 12 = 0$ and $\alpha - \beta = 1$, find k.

6. If one root of the equation $px^2 - (2p+3)x - (p+1) = 0$ be 3, find p and also the other root.

7. Two candidates attempt to solve a quadratic equation of the form $x^2 + px + q = 0$. One starts with a wrong value of p and finds the roots to be 2 and 6. The other starts with a wrong value of q and finds the roots to be 2 and -9. Find the correct roots and the equation.

8. Given that α and β are the roots of the equation $x^2 = 7x + 4$.

   (i) Show that $\alpha^3 = 53\alpha + 28$ (ii) Find the value of $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$.

9. Form an equation whose roots are

   (i) $3, -\frac{1}{3}$ (ii) $\sqrt{3}, \frac{1}{\sqrt{3}}$ (iii) 0, 0.

10. Form an equation with rational coefficients one of whose roots is $\frac{\sqrt{3}+1}{\sqrt{3}-1}$.

11. Form an equation with real coefficients one of whose roots is

    (i) $-1 - 2i$ (ii) $-2 - \sqrt{-3}$ (iii) $\frac{1}{2+\sqrt{-2}}$.

12. If α, β are the roots of the equation $ax^2 + bx + c = 0$, then form an equation whose roots are :

    (i) $2\alpha, 2\beta$ (ii) $\frac{\alpha}{2}, \frac{\beta}{2}$ (iii) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (iv) $(\alpha+\beta)^2, (\alpha-\beta)^2$

    (v) $\alpha + \frac{1}{\beta}, \beta + \frac{1}{\alpha}$ (vi) $\alpha + k, \beta + k$.

13. If α, β be the roots of the equation $2x^2 - 3x + 1 = 0$, find an equation whose roots are $\frac{\alpha}{2\beta+3}, \frac{\beta}{2\alpha+3}$.

14. If α, β be the roots of the equation $3x^2 + 2x + 1 = 0$, form an equation whose roots are $\frac{1-\alpha}{1+\alpha}, \frac{1-\beta}{1+\beta}$.

15. If α, β be the roots of the equation $2x^2 - x + 3 = 0$, form an equation whose roots are $\alpha - 2, \beta - 2$.

16. The ratio of the roots of the equation $x^2 + \alpha x + \alpha + 2 = 0$ is 2. Find the value of the parameter α.

17. If r is the ratio of the roots of the quadratic equation $ax^2 + bx + c = 0$, show that $(r+1)^2 ac = b^2 r$.

18. Find k so that one root of the equation $2kx^2 - 20x + 21 = 0$ exceeds the other by 2.

19. Find the condition that one of roots of $ax^2 + bx + c = 0$ may be

    (i) reciprocal of the other (ii) negative of the other (iii) twice the other.

20. Find the condition that

    (i) one of the roots of the equation $ax^2 + bx + c = 0$ may be unity.

    (ii) one of the roots of the equation $ax^2 + bx + c = 0$ may be zero.

    (iii) exactly one of the roots of the equation $ax^2 + bx + c = 0$ may be zero.

    (iv) both the roots of the equation $ax^2 + bx + c = 0$ may be zero.

    (v) one of the roots of the equation $ax^2 + bx + c = 0$ may be positive and the other negative.

21. (i) For what value of a is one of the roots of the equation $x^2 + (2a+1)x + a^2 + 2 = 0$ is double the other.

    (ii) Determine k if one of the roots of the equation $k(x-1)^2 = 5x - 7$ is double the other.

(*iii*) Find the value of k for which the roots α, β of the equation $x^2 - 6x + k = 0$ satisfy the relation $2\alpha + 3\beta = 20$.

(*iv*) Solve the equation $x^2 + px + 45 = 0$, given that the square of the difference of the roots is equal to 144.

22. Find the value of a if one root of the equation $8x^2 - 6x + a = 0$ is the square of the other.

23. Find the condition for the equations $ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$ to have reciprocal roots.

24. Find the value(s) of k so that the equations $2x^2 + kx - 5 = 0$ and $x^2 - 3x - 4 = 0$ may have one root in common. Also determine the common root.

25. Find the value(s) of k so that the equations $x^2 - 11x + k = 0$ and $x^2 - 14x + 2k = 0$ may have a common root.

26. If the equations $ax^2 + bx + c = 0$ and $cx^2 + bx + a = 0$ have one root in common, prove that $a + b + c = 0$ or $a - b + c = 0$.

6.5 QUADRATIC FUNCTIONS

Let a, b, c be real numbers. Then $f(x) = ax^2 + bx + c$, $a \neq 0$, is known as ***quadratic polynomial*** *or* ***quadratic function.***

Graph of quadratic function

$$f(x) = ax^2 + bx + c = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

$$= a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{b^2 - 4ac}{4a^2}\right].$$

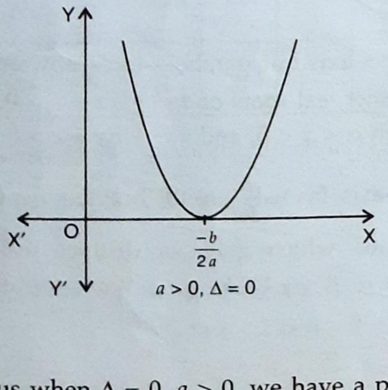
Now $\quad b^2 - 4ac = \text{discriminant} = \Delta$

$$\therefore \quad f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right] \quad \text{i.e. } y = a\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a}$$

or $\quad y + \dfrac{\Delta}{4a} = a\left(x + \dfrac{b}{2a}\right)^2$ which represents a **parabola** with **vertex** at $\left(-\dfrac{b}{2a}, -\dfrac{\Delta}{4a}\right)$, and face upwards if $a > 0$ and face downwards if $a < 0$.

Case I. When $\Delta = 0$, the equation $ax^2 + bx + c = 0$ has two equal real roots $x = -\dfrac{b}{2a}, -\dfrac{b}{2a}$,

and $f(x) = a\left(x + \dfrac{b}{2a}\right)^2$, which is a parabola touching x-axis.



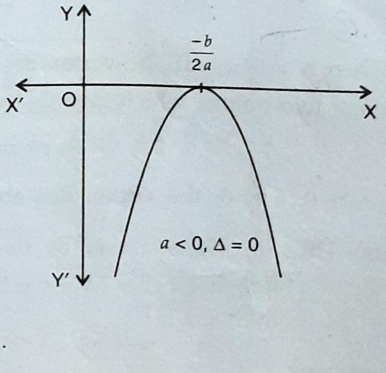


Fig. 6.1.

Thus when $\Delta = 0$, $a > 0$, we have a parabola touching x-axis and face upwards.

When $\Delta = 0$, $a < 0$, we have a parabola touching x-axis and face downwards.

Case II. When $\Delta < 0$, the equation $ax^2 + bx + c = 0$ has no real roots, and

$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$, which represents a parabola with vertex at $\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$.

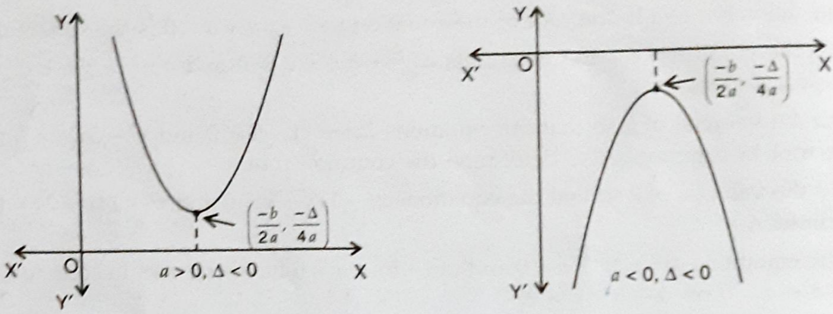


Fig. 6.2.

Thus, when $\Delta < 0$, $a > 0$, we have a parabola fully lying above x-axis and face upwards. Since it doesn't meet x-axis, there is no real x such that $f(x) = 0$ *i.e.* $ax^2 + bx + c = 0$ has no real roots. Thus $ax^2 + bx + c > 0$ in this case, for all real x.

When $\Delta < 0$, $a < 0$, we have a parabola lying fully below x-axis and face downwards. Thus $ax^2 + bx + c < 0$ in this case, for all real x.

Case III. When $\Delta > 0$, the equation has two distinct real roots, say $\alpha, \beta\,(\alpha < \beta)$. Then

$f(x) = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$ represents a parabola with vertex at $\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$.

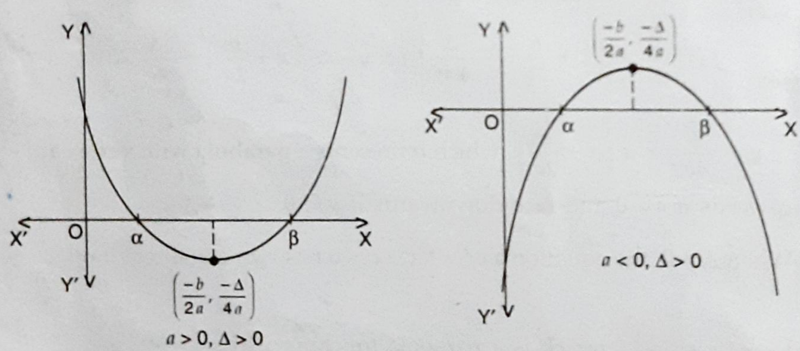


Fig. 6.3.

Thus when $\Delta > 0$, $a > 0$, the vertex lies below x-axis but parabola faces upwards. Thus it meets x-axis at two points, which give us two distinct real roots of $ax^2 + bx + c = 0$. Note that $ax^2 + bx + c = 0$ at $x = \alpha, \beta$; $ax^2 + bx + c < 0$ when $\alpha < x < \beta$, and $ax^2 + bx + c > 0$ otherwise.

When $\Delta > 0$, $a < 0$, the vertex lies above x-axis (as $-\frac{\Delta}{4a} > 0$), but the parabola faces downwards. Thus it meets x-axis at two points, which give us distinct real roots of $ax^2 + bx + c = 0$. Note that $ax^2 + bx + c = 0$ at $x = \alpha, \beta$; $ax^2 + bx + c > 0$ when $\alpha < x < \beta$ and $ax^2 + bx + c < 0$ otherwise.

Sign of expression $ax^2 + bx + c$

From the above discussion, we see that :

(i) When $\Delta = 0$: $ax^2 + bx + c > 0$ if $a > 0$; $ax^2 + bx + c < 0$ if $a < 0$, and $ax^2 + bx + c = 0$ at $x = -\dfrac{b}{2a}$. Thus $ax^2 + bx + c$ has the same sign as a (except at $x = -\dfrac{b}{2a}$ where it is zero).

(ii) When $\Delta < 0$: $ax^2 + bx + c > 0$ if $a > 0$; $ax^2 + bx + c < 0$ if $a < 0$. Thus $ax^2 + bx + c$ has the same sign as a.

(iii) When $\Delta > 0$: $ax^2 + bx + c = 0$ at $x = \alpha, \beta$; $ax^2 + bx + c$ has sign opposite to a when $\alpha < x < \beta$, otherwise it has same sign as a.

In short, $ax^2 + bx + c$ has same sign as a except when $ax^2 + bx + c = 0$ has two real distinct roots α, β and $\alpha < x < \beta$.

Corollary 1. $ax^2 + bx + c > 0$ for all $x \in \mathbf{R}$,
iff $\Delta = b^2 - 4ac < 0$, $a > 0$.

Corollary 2. $ax^2 + bx + c < 0$ for all $x \in \mathbf{R}$ iff $a < 0$, $\Delta < 0$.

Corollary 3. $ax^2 + bx + c \geq 0$ for all real x iff $a > 0$, $\Delta \leq 0$.

Corollary 4. $ax^2 + bx + c \leq 0$ for all real x iff $a < 0$, $\Delta \leq 0$.

Maximum/Minimum value of $ax^2 + bx + c$

From previous graphs, we see that maxima / minima occurs at $x = -\dfrac{b}{2a}$, and its value is $-\dfrac{\Delta}{4a}$. When $a > 0$, we have a *minima* ; when $a < 0$, we have *maxima.*

ILLUSTRATIVE EXAMPLES

Example 1. *Draw the graph of $f(x) = x^2 - 5x + 6$. Hence, discuss the sign of $x^2 - 5x + 6$.*

Solution. Here $\Delta = b^2 - 4ac = (-5)^2 - 4 \times 1 \times 6 = 25 - 24 = 1 > 0$. So, the roots are real and distinct. Also $f(x) = x^2 - 5x + 6 = (x-2)(x-3)$. Hence, the roots are $x = 2, 3$. The graph is a parabola with vertex at $\left(-\dfrac{b}{2a}, -\dfrac{\Delta}{4a}\right)$ i.e. $\left(\dfrac{-(-5)}{2\times 1}, -\dfrac{1}{4\times 1}\right) = \left(\dfrac{5}{2}, -\dfrac{1}{4}\right)$.

Also, as the graph is a curve, we should get a few more points :

x	0	1	2	$\frac{5}{2}$	3	4	5
$f(x)$	6	2	0	$-\frac{1}{4}$	0	2	6

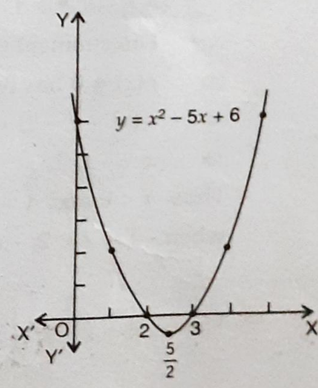


Fig. 6.4.

The graph of the quadratic function $f(x)$ is shown in fig. 6.4.

From the graph, it is clear that

$$x^2 - 5x + 6 \begin{cases} = 0, \text{ when } x = 2, 3 \\ < 0, \text{ when } 2 < x < 3 \\ > 0, \text{ when } x < 2 \text{ or } x > 3. \end{cases}$$

Example 2. *Discuss the sign of the following quadratic functions:*
(i) $x^2 + x + 1$ (ii) $2x^2 - 8x + 8$ (iii) $2 + x - x^2$.

Solution. (i) $f(x) = x^2 + x + 1$.

Comparing it with $ax^2 + bx + c$, we get $a = 1 > 0$, $b = 1$, $c = 1$,

$\Delta = b^2 - 4ac = (1)^2 - 4 \times 1 \times 1 = -3 < 0$.

Thus $a > 0$, $\Delta < 0$. Hence, $x^2 + x + 1 > 0$ for all real x.

Alternatively, $f(x) = x^2 + x + 1 = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4}$.

Now $\left(x + \frac{1}{2}\right)^2 \geq 0$ for all real x

$\Rightarrow \quad f(x) = \left(x + \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}$.

This shows that $x^2 + x + 1$ is always positive.

(ii) Given $f(x) = 2x^2 - 8x + 8$.

Comparing it with $ax^2 + bx + c$, we get

$a = 2$, $b = -8$, $c = 8$.

$\therefore$ Discriminant $\Delta = b^2 - 4ac = (-8)^2 - 4 \times 2 \times 8 = 0$.

Thus $a > 0$ and $\Delta = 0$. Therefore, $2x^2 - 8x + 8 > 0$ for all real x except

when $x = -\frac{b}{2a} = -\frac{-8}{2 \times 2} = 2$ and at $x = 2$, $2x^2 - 8x + 8 = 0$.

Hence, $2x^2 - 8x + 8 > 0$ for all x except at $x = 2$ and at $x = 2$, $2x^2 - 8x + 8 = 0$.

Alternatively,

$$f(x) = 2x^2 - 8x + 8 = 2(x^2 - 4x + 4) = 2(x - 2)^2.$$

As x is real, $(x - 2)^2 \geq 0 \Rightarrow 2(x - 2)^2 \geq 0$

$\Rightarrow \quad f(x) = 2x^2 - 8x + 8 \geq 0$ for all real x.

The equality occurs when $x - 2 = 0$ *i.e.* when $x = 2$.

Hence, $2x^2 - 8x + 8 > 0$ for all real x except at $x = 2$

and at $x = 2$, $2x^2 - 8x + 8 = 0$.

(iii) Given $f(x) = 2 + x - x^2 = -x^2 + x + 2$.

Comparing it with $ax^2 + bx + c$, we get

$a = -1$, $b = 1$, $c = 2$.

$\therefore$ Discriminant $\Delta = b^2 - 4ac = 1^2 - 4(-1) \times 2 = 9 > 0$.

So $f(x) = 0$ has two real and distinct roots. The roots are given by $-x^2 + x + 2 = 0$

$\Rightarrow \quad -(x^2 - x - 2) = 0 \Rightarrow (x + 1)(x - 2) = 0$

$\Rightarrow \quad x = -1, 2$.

Thus $a < 0$ and $\Delta > 0$. Therefore, $-x^2 + x + 2 = 0$ at $x = -1, 2$; $-x^2 + x + 2 > 0$ when $-1 < x < 2$; $-x^2 + x + 2 < 0$ when $x < -1$ or $x > 2$.

$$\text{Hence, } 2 + x - x^2 \begin{cases} = 0 \text{ when } x = -1, 2 \\ > 0 \text{ when } -1 < x < 2 \\ < 0 \text{ when } x < -1 \text{ or } x > 2 \end{cases}$$

Alternatively,

$$f(x) = 2 + x - x^2 = -(x^2 - x - 2) = -(x + 1)(x - 2).$$

Now $f(x) = 0$ when $x = -1, 2$.

Also, we know that $(x + 1)(x - 2) < 0$ when $-1 < x < 2$

$\Rightarrow \quad -(x + 1)(x - 2) > 0$ when $-1 < x < 2$

$\Rightarrow \quad 2 + x - x^2 > 0$ when $-1 < x < 2$.

And $\quad (x + 1)(x - 2) > 0$ when $x < -1$ or $x > 2$

$\Rightarrow \quad -(x + 1)(x - 2) < 0$ when $x < -1$ or $x > 2$

$\Rightarrow \quad 2 + x - x^2 < 0$ when $x < -1$ or $x > 2$.

$$\text{Hence, } 2 + x - x^2 \begin{cases} = 0 \text{ when } x = -1, 2 \\ > 0 \text{ when } -1 < x < 2 \\ < 0 \text{ when } x < -1 \text{ or } x > 2 \end{cases}$$

Example 3. *Determine the values of a so that the expression $x^2 - 2(a + 1)x + 4$ is always positive.*

Solution. Given $x^2 - 2(a + 1)x + 4 > 0$ for all real x.

Here coefficient of $x^2 = 1 > 0$, and

discriminant $\Delta = [-2(a + 1)]^2 - 4.1.4 = 4(a^2 + 2a - 3)$.

For the given expression to be always positive, $\Delta < 0$

$\Rightarrow \quad 4(a^2 + 2a - 3) < 0 \Rightarrow (a + 3)(a - 1) < 0 \Rightarrow -3 < a < 1$.

EXERCISE 6.5

1. Draw the graph of $f(x) = x^2 - 3x - 4$. Hence, discuss the sign of $x^2 - 3x - 4$.
2. Discuss the signs of the following quadratic functions :

   (*i*) $2x^2 + 3x + 5$ (*ii*) $3x^2 - 6x + 3$ (*iii*) $2 - x - x^2$
3. Find the values of a so that expression $x^2 - (a + 2)x + 4$ is always positive.

TYPICAL ILLUSTRATIVE EXAMPLES

Example 1. *Solve* $\left(\sqrt{3} + \sqrt{2}\right)^x + \left(\sqrt{3} - \sqrt{2}\right)^x = 10$.

Solution. Note that $\dfrac{1}{\sqrt{3} + \sqrt{2}} = \dfrac{1}{\sqrt{3} + \sqrt{2}} \cdot \dfrac{\sqrt{3} - \sqrt{2}}{\sqrt{3} - \sqrt{2}} = \dfrac{\sqrt{3} - \sqrt{2}}{3 - 2} = \sqrt{3} - \sqrt{2}$.

Putting $\left(\sqrt{3} + \sqrt{2}\right)^x = y$, we get $\left(\sqrt{3} - \sqrt{2}\right)^x = \left(\dfrac{1}{\sqrt{3} + \sqrt{2}}\right)^x = \dfrac{1}{\left(\sqrt{3} + \sqrt{2}\right)^x} = \dfrac{1}{y}$.

So given equation $\Rightarrow y + \dfrac{1}{y} = 10 \Rightarrow y^2 + 1 = 10y \Rightarrow y^2 - 10y + 1 = 0$

$$\Rightarrow y = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(1)}}{2(1)} = \frac{10 \pm \sqrt{96}}{2} = \frac{10 \pm 4\sqrt{6}}{2} = 5 \pm 2\sqrt{6}.$$

Now we notice that $5 + 2\sqrt{6} = 3 + 2\sqrt{3}\sqrt{2} + 2 = \left(\sqrt{3} + \sqrt{2}\right)^2$

and $5 - 2\sqrt{6} = 3 - 2\sqrt{3}\sqrt{2} + 2 = \left(\sqrt{3} - \sqrt{2}\right)^2$.

Hence $y = 5 + 2\sqrt{6} \Rightarrow \left(\sqrt{3} + \sqrt{2}\right)^x = 5 + 2\sqrt{6} = \left(\sqrt{3} + \sqrt{2}\right)^2 \Rightarrow x = 2$.

Also $y = 5 - 2\sqrt{6} \Rightarrow \left(\sqrt{3} + \sqrt{2}\right)^x = 5 - 2\sqrt{6} = \left(\sqrt{3} - \sqrt{2}\right)^2 = \left(\dfrac{1}{\sqrt{3} + \sqrt{2}}\right)^2 = \left(\sqrt{3} + \sqrt{2}\right)^{-2}$

$\Rightarrow x = -2$.

Hence, the solutions are 2, – 2.

Example 2. *Determine all values of x for which* $(x^2 - 5x + 5)^{x^2-2x-48} = 1$.

Solution. We know that $a^b = 1$ if $b = 0$, $a \neq 0$ or if $a = 1$.

Given that $(x^2 - 5x + 5)^{x^2-2x-48} = 1$.

Now $x^2 - 2x - 48 = 0 \Rightarrow (x - 8)(x + 6) = 0 \Rightarrow x = 8, -6$.

Also for $x = 8$ or -6, base $= x^2 - 5x + 5 \neq 0$.

$$x^2 - 5x + 5 = 1 \Rightarrow x^2 - 5x + 4 = 0 \Rightarrow (x - 1)(x - 4) = 0 \Rightarrow x = 1, 4.$$

Hence the solutions of the given equation are $x = -6, 1, 4, 8$.

Example 3. *Find p, q if p and q are roots of the equation* $x^2 + px + q = 0$.

Solution. Since p, q are roots of $x^2 + px + q = 0$, $p + q = -p$ and $pq = q$.

Now $pq = q \Rightarrow pq - q = 0 \Rightarrow q(p - 1) = 0 \Rightarrow q = 0$ or $p = 1$.

When $q = 0$ then $p + q = -p \Rightarrow 2p = -q = 0 \Rightarrow p = 0$.

When $p = 1$, then $p + q = -p \Rightarrow q = -2p = -2$.

Hence, $p = q = 0$ or $p = 1, q = -2$.

Example 4. *Prove that the roots of* $(a - b)^2 x^2 + 2(a + b - 2c)x + 1 = 0$, $a < b$, *are real or imaginary according as c does not or does lie between a and b.*

Solution. Given equation is $(a - b)^2 x^2 + 2(a + b - 2c)x + 1 = 0$.

$$\text{Discriminant } \Delta = (2(a + b - 2c))^2 - 4(a - b)^2 \times 1$$
$$= 4(a + b - 2c + \overline{a - b})(a + b - 2c - \overline{a - b})$$
$$= 4(2a - 2c)(2b - 2c) = 16(c - a)(c - b).$$

For real roots, $\Delta \geq 0 \Rightarrow 16(c - a)(c - b) \geq 0$

$\Rightarrow \quad (c - a)(c - b) \geq 0 \Rightarrow c \leq a$ or $c \geq b$

$\Rightarrow \quad c$ does not lie between a and b.

For imaginary roots, $\Delta < 0 \Rightarrow 16(c - a)(c - b) < 0$

$\Rightarrow \quad (c - a)(c - b) < 0 \Rightarrow a < c < b$

$\Rightarrow \quad c$ lies between a and b.

Example 5. *If* α, β *are roots of* $x^2 - p(x + 1) - c = 0$, $c \neq 1$, *then show that*

(*i*) $(\alpha + 1)(\beta + 1) = 1 - c$ (*ii*) $\dfrac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + c} + \dfrac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + c} = 1$.

Solution. (*i*) The given equation can be written as $x^2 - px - (p + c) = 0$.

Since α, β are roots of the given equation, we have

$$\alpha + \beta = p \text{ and } \alpha\beta = -(p + c).$$

$$\text{Now } (\alpha + 1)(\beta + 1) = \alpha\beta + (\alpha + \beta) + 1 = -(p + c) + p + 1$$
$$= 1 - c.$$

(*ii*)
$$\frac{\alpha^2 + 2\alpha + 1}{\alpha^2 + 2\alpha + c} + \frac{\beta^2 + 2\beta + 1}{\beta^2 + 2\beta + c} = \frac{(\alpha + 1)^2}{(\alpha + 1)^2 - (1 - c)} + \frac{(\beta + 1)^2}{(\beta + 1)^2 - (1 - c)}$$
$$= \frac{(\alpha + 1)^2}{(\alpha + 1)^2 - (\alpha + 1)(\beta + 1)} + \frac{(\beta + 1)^2}{(\beta + 1)^2 - (\alpha + 1)(\beta + 1)} \quad \text{(using part (i))}$$
$$= \frac{\alpha + 1}{(\alpha + 1) - (\beta + 1)} + \frac{\beta + 1}{(\beta + 1) - (\alpha + 1)}$$
$$= \frac{\alpha + 1}{\alpha - \beta} + \frac{\beta + 1}{\beta - \alpha} = \frac{(\alpha + 1) - (\beta + 1)}{\alpha - \beta}$$
$$= \frac{\alpha - \beta}{\alpha - \beta} = 1.$$

Example 6. *If α, β are roots of $a(x^2 + m^2) + amx + bm^2x^2 = 0$, then show that*
$a(\alpha^2 + \beta^2) + a\,\alpha\beta + b\,\alpha^2\beta^2 = 0$.

Solution. The given equation can be written as

$$(a + bm^2)\, x^2 + a\,mx + a\,m^2 = 0.$$

Since α, β, are roots of the given equation, we have

$$\alpha + \beta = -\frac{am}{a + bm^2} \quad \ldots(i) \qquad \text{and} \qquad \alpha\beta = \frac{am^2}{a + bm^2} \quad \ldots(ii)$$

Now $a(\alpha^2 + \beta^2) + a\,\alpha\beta + b\,\alpha^2\beta^2 = a((\alpha + \beta)^2 - 2\alpha\beta) + a\,\alpha\beta + b\,\alpha^2\beta^2$

$$= a(\alpha + \beta)^2 - a\,\alpha\beta + b(\alpha\beta)^2$$

$$= a\left(-\frac{am}{a + bm^2}\right)^2 - a.\frac{am^2}{a + bm^2} + b\left(\frac{am^2}{a + bm^2}\right)^2$$

$$= a\,.\frac{a^2m^2}{(a + bm^2)^2} - \frac{a^2m^2}{a + bm^2} + b.\frac{a^2m^4}{(a + bm^2)^2}$$

$$= \frac{a^3m^2 - a^2m^2(a + bm^2) + a^2bm^4}{(a + bm^2)^2}$$

$$= \frac{a^3m^2 - a^3m^2 - a^2bm^4 + a^2bm^4}{(a + bm^2)^2} = \frac{0}{(a + bm^2)^2} = 0.$$

Example 7. *If a, b are roots of the equation $x^2 + px + 1 = 0$ and c, d are roots of $x^2 + qx + 1 = 0$, prove that $(a - c)(b - c) + (a - d)(b - d) = q^2 - pq$.*

Solution. Since a, b are roots of $x^2 + px + 1 = 0$, $a + b = -p$, $ab = 1$.

Also, as c, d are roots of $x^2 + qx + 1 = 0$, $c + d = -q$, $cd = 1$.

Now $(a - c)(b - c) + (a - d)(b - d) = ab - ac - bc + c^2 + ab - ad - bd + d^2$

$= c^2 + d^2 + 2 - ac - bc - ad - bd$ (putting $ab = 1$)

$= c^2 + d^2 + 2cd - (a + b)\,c - (a + b)\,d$ (putting $1 = cd$)

$= (c + d)^2 - (a + b)(c + d) = (-q)^2 - (-p)(-q) = q^2 - pq.$

Example 8. *If α is a root of $4x^2 + 2x - 1 = 0$, prove that $4\alpha^3 - 3\alpha$ is the other root.*

Solution. Let β be the other root of the equation $4x^2 + 2x - 1 = 0$, then

$$\alpha + \beta = -\frac{2}{4} \quad \Rightarrow \beta = -\alpha - \frac{1}{2}.$$

We will try to prove that $4\alpha^3 - 3\alpha = -\alpha - \frac{1}{2}$.

Since α is a root of $4x^2 + 2x - 1 = 0$, so $4\alpha^2 + 2\alpha - 1 = 0$ ...(i)

Now $\quad 4\alpha^3 - 3\alpha = \alpha(4\alpha^2 + 2\alpha - 1) - 2\alpha^2 - 2\alpha$

$= \alpha.0 - \frac{1}{2}(4\alpha^2 + 2\alpha - 1) - \alpha - \frac{1}{2}$ (using (i))

$= 0 - \frac{1}{2}.0 - \alpha - \frac{1}{2}$ (using (i))

$= -\alpha - \frac{1}{2}.$

Hence, $4\alpha^3 - 3\alpha$ is the other root.

Example 9. *α, β are roots of the equation $p(x^2 - x) + x + 5 = 0$. If p_1, p_2 are the values of p for which α, β are connected by $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5}$, find the value of $\frac{p_1}{p_2^2} + \frac{p_2}{p_1^2}$.*

Solution. Given α, β are roots of the equation $px^2 - (p-1)x + 5 = 0$,

$\therefore \quad \alpha + \beta = \frac{p-1}{p}$ and $\alpha\beta = \frac{5}{p}$ ...(i)

Now $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{4}{5} \Rightarrow \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{4}{5}$

$\Rightarrow \quad 5(\alpha^2 + \beta^2) = 4\alpha\beta \Rightarrow 5((\alpha + \beta)^2 - 2\alpha\beta) = 4\alpha\beta$

$\Rightarrow \quad 5(\alpha + \beta)^2 = 14\alpha\beta$

$\Rightarrow \quad 5\left(\frac{p-1}{p}\right)^2 = 14 \times \frac{5}{p}$ (using (i))

$\Rightarrow \quad (p-1)^2 = 14p \Rightarrow p^2 - 2p + 1 - 14p = 0$

$\Rightarrow \quad p^2 - 16p + 1 = 0$

As p_1, p_2 are the values of p,

$\therefore \quad p_1 + p_2 = 16$ and $p_1p_2 = 1$. ...(ii)

$$\therefore \quad \frac{p_1}{p_2^2} + \frac{p_2}{p_1^2} = \frac{p_1^3 + p_2^3}{(p_1p_2)^2} = \frac{(p_1+p_2)^3 - 3p_1p_2(p_1+p_2)}{(p_1p_2)^2}$$

$$= \frac{(16)^3 - 3 \times 1 \times 16}{1^2} \quad \text{(using (ii))}$$

$$= 4096 - 48 = 4048.$$

Example 10. *If each pair of the three equations $x^2 + a_1x + b_1 = 0$, $x^2 + a_2x + b_2 = 0$ and $x^2 + a_3x + b_3x = 0$ have a common root, then prove that*

$$a_1^2 + a_2^2 + a_3^2 + 4(b_1 + b_2 + b_3) = 2(a_1a_2 + a_2a_3 + a_3a_1).$$

Solution. The given equations are

$x^2 + a_1x + b_1 = 0$ ...(i)

$x^2 + a_2x + b_2 = 0$ ...(ii)

$x^2 + a_3x + b_3 = 0$ ...(iii)

Since each pair of given equations has a common root, let the roots of the given equations (i), (ii) and (iii) be α, β; β, γ and γ, α respectively. Then

$\alpha + \beta = -a_1, \quad \alpha\beta = b_1$ ...(iv)

$\beta + \gamma = -a_2, \quad \beta\gamma = b_2$ ...(v)

$\gamma + \alpha = -a_3, \quad \gamma\alpha = b_3$ ...(vi)

Now $(\alpha + \beta) - (\beta + \gamma) = (-a_1) - (-a_2) \Rightarrow \alpha - \gamma = a_2 - a_1$

$\Rightarrow (\alpha - \gamma)^2 = (a_2 - a_1)^2$.

We know that $(\alpha + \gamma)^2 - 4\gamma\alpha = (\alpha - \gamma)^2$

$\Rightarrow (-a_3)^2 - 4b_3 = (a_2 - a_1)^2$

$\Rightarrow (a_2 - a_1)^2 = a_3^2 - 4b_3$ ...(vii)

Similarly $(a_3 - a_2)^2 = a_1^2 - 4b_1$ ...(viii)

and $(a_1 - a_3)^2 = a_2^2 - 4b_2$ ...(ix)

On adding (vii), (viii) and (ix), we get

$(a_2 - a_1)^2 + (a_3 - a_2)^2 + (a_1 - a_3)^2 = a_3^2 + a_1^2 + a_2^2 - 4(b_3 + b_1 + b_2)$

$\Rightarrow a_1^2 + a_2^2 + a_3^2 + 4(b_1 + b_2 + b_3) = 2(a_1a_2 + a_2a_3 + a_3a_1)$.

Exampel 11. *Find the extreme (minimum/maximum) value of each of the following expressions for real values of x. Also find the value(s) of x for which this extreme is obtained.*

(i) $3 - 20x - 25x^2$ (ii) $(x+2)(x-5) + 13$ (iii) $(2x-5)(3-2x)$.

Solution. We know that vertex of parabola (*i.e.* graph of $ax^2 + bx + c$) is $\left(-\frac{b}{2a}, -\frac{\Delta}{4a}\right)$. Also if $a > 0$, we have minimum, if $a < 0$, we have maximum.

(*i*) Given expression $f(x) = 3 - 20x - 25x^2$.
Comparing with $ax^2 + bx + c$; $a = -25 < 0$,
and discriminant $\Delta = b^2 - 4ac = (-20)^2 - 4(-25)(3) = 700$.

Hence, the *greatest* value occurs at $x = \frac{-b}{2a} = \frac{-(-20)}{2(-25)} = -\frac{2}{5}$,

and value obtained is $\frac{-\Delta}{4a} = \frac{-700}{4(-25)} = 7$.

(*ii*) $f(x) = (x+2)(x-5) + 13 = x^2 - 3x - 10 + 13 = x^2 - 3x + 3$.
Comparing with $ax^2 + bx + c$, $a = 1 > 0$ and
discriminant $\Delta = b^2 - 4ac = (-3)^2 - 4(1)(3) = 9 - 12 = -3$.

Hence, *minimum* value occurs at $x = \frac{-b}{2a} = \frac{-(-3)}{2(1)} = \frac{3}{2}$,

and value is $\frac{-\Delta}{4a} = \frac{-(-3)}{4(1)} = \frac{3}{4}$.

(*iii*) $f(x) = (2x-5)(3-2x) = -4x^2 + 16x - 15$.
Comparing with $ax^2 + bx + c$, $a = -4 < 0$ and
discriminant $\Delta = b^2 - 4ac = (16)^2 - 4(-4)(-15) = 256 - 240 = 16$

$\therefore$ *Maximum* value occurs at $x = \frac{-b}{2a} = \frac{-16}{2(-4)} = 2$, and

value is $\frac{-\Delta}{4a} = \frac{-16}{4(-4)} = 1$.

Example 12. *For what values of a is the inequality* $\frac{x^2 + ax - 2}{x^2 - x + 1} < 2$ *satisfied for all real values of x?*

Solution. $\frac{x^2 + ax - 2}{x^2 - x + 1} < 2 \Rightarrow \frac{x^2 + ax - 2}{x^2 - x + 1} - 2 < 0$

$\Rightarrow \quad \frac{x^2 + ax - 2 - 2x^2 + 2x - 2}{x^2 - x + 1} < 0 \Rightarrow \frac{-x^2 + x(a+2) - 4}{x^2 - x + 1} < 0$

$\Rightarrow \quad -x^2 + x(a+2) - 4 < 0 \qquad \left(\because x^2 - x + 1 = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4} > 0 \text{ for all real } x\right)$

$\Rightarrow \quad x^2 - x(a+2) + 4 > 0$

Now the expression $x^2 - x(a+2) + 4$ is positive for all real x if $\Delta < 0$
$(\because \text{ coeff. of } x^2 = 1 > 0)$

$\Rightarrow \quad [-(a+2)]^2 - 4.1.4 < 0 \Rightarrow a^2 + 4a + 4 - 16 < 0$

$\Rightarrow \quad a^2 + 4a - 12 < 0 \Rightarrow (a+6)(a-2) < 0$

$\Rightarrow \quad -6 < a < 2$, which is the required range for a.

EXERCISE 6.6 — TYPICAL PROBLEMS

1. Solve : $(5 - 2\sqrt{6})^x + (5 + 2\sqrt{6})^x = 98$.
2. Solve : $(5 + 2\sqrt{6})^{x^2-3} + (5 - 2\sqrt{6})^{x^2-3} = 10$.
3. Solve : $(x^2 - 2x + 1)^{x^2-1} = 1$.
4. If $P(x) = ax^2 + bx + c$ and $Q(x) = -ax^2 + dx + c$ where $ac \neq 0$, then prove that $P(x).Q(x) = 0$ has at least two real roots.

   Hint. If $P(x) = 0$ and $Q(x) = 0$ both have complex roots, then $b^2 - 4ac < 0$ and $d^2 + 4ac < 0 \Rightarrow b^2 + d^2 < 0$, which is wrong.

5. If p, q are roots of $4x^2 - (5a + 1)x + 5a = 0$ and $q = 1 + p$. Find the values of a, p and q.
6. If α and β are roots of $x^2 + px + q = 0$ and α^4, β^4 are roots of $x^2 - rx + s = 0$, then prove that the equation $x^2 - 4qx + 2q^2 - r = 0$ has distinct and real roots.

Hint. $\alpha + \beta = -p$, $\alpha\beta = q \Rightarrow \alpha^4 + \beta^4 = p^4 + 2q^2 - 4p^2q$ and $\alpha^4\beta^4 = q^4$.
Also $\alpha^4 + \beta^4 = r$ and $\alpha^4\beta^4 = s \Rightarrow p^4 + 2q^2 - 4p^2q = r$ and $q^4 = s$.
Disc. of $x^2 - 4qx + (2q^2 - r) = 0$ is $(-4q)^2 - 4.1.(2q^2 - r) = 8q^2 + 4r$
$= 8q^2 + 4(p^4 + 2q^2 - 4p^2q) = (2p^2 - 4q)^2$.

7. The equations $ax^4 + bx^3 + c = 0$ and $cx^4 + bx^3 + a = 0$ have a root in common. Find all possible values of b, if $a + c = 100$. Also find the common root.
8. If each pair of three equations $x^2 + px + qr = 0$, $x^2 + qx + pr = 0$ and $x^2 + rx + pq = 0$ have a common root, find the sum and the product of the common roots.

Hint. Let the roots of the given equations be $\alpha, \beta; \beta, \gamma; \gamma, \alpha$.

9. Find the values of p if the roots of the equation $(p - 3)x^2 - 2px + 5p = 0$ are positive.

Hint. Since the roots are positive, so the roots are real $\Rightarrow \Delta \geq 0$. Also sum of roots > 0 and product of roots > 0.

CHAPTER TEST

1. Solve (*i*) $a(x^2 + 1) = (a^2 + 1)x$, $a \neq 0$ (*ii*) $4x^2 - 4ax + (a^2 - b^2) = 0$.
2. Solve $2^{2x+2} - 6^x - 2.3^{2x+2} = 0$.

Hint. Divide by 6^x; put $\left(\frac{2}{3}\right)^x = y$.

3. Solve $(x + 2)(3x + 4)(3x + 7)(x + 3) = 2600$.
4. Solve $(x^2 - 3x - 10)(x^2 - 5x - 6) = 144$.
5. Solve $\sqrt{x+5} + \sqrt{x+12} = \sqrt{2x+41}$.
6. Solve $\dfrac{1}{x+a} + \dfrac{1}{x+2a} + \dfrac{1}{x+3a} = \dfrac{3}{x}$.

Hint. $\left(\dfrac{1}{x+a} - \dfrac{1}{x}\right) + \ldots = 0$.

7. Solve : $\sqrt{\dfrac{2x^2+1}{x^2-1}} + 6\sqrt{\dfrac{x^2-1}{2x^2+1}} = 5$.
8. An aeroplane flying with a wind of 30 km / h takes 40 minutes less to fly 3600 km than what it would have taken to fly against the same wind. Find the plane's speed of flying in still air.

Hint. Form the equation $\dfrac{3600}{x-30} - \dfrac{3600}{x+30} = \dfrac{2}{3}$ and solve it.

9. Two pipes running together can fill a tank in $11\frac{1}{9}$ minutes. If one pipe takes 5 minutes more than the other to fill the tank, find the time in which each pipe would fill the tank.

Hint. Let the time taken be x minutes and y minutes; then $x = y + 5$ and $\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{9}{100}$.

10. If the roots of the equation $(c^2 - ab)x^2 - 2(a^2 - bc)x + (b^2 - ac) = 0$ are equal then prove that either $a = 0$ or $a^3 + b^3 + c^3 = 3abc$.
11. Form a quadratic equation with rational coefficients one of whose roots is $\dfrac{3+\sqrt{5}}{2-\sqrt{5}}$.

Hint. $\dfrac{3+\sqrt{5}}{2-\sqrt{5}} = \dfrac{(3+\sqrt{5})(2+\sqrt{5})}{(2-\sqrt{5})(2+\sqrt{5})} = \dfrac{11+5\sqrt{5}}{4-5} = -11 - 5\sqrt{5}$.
As the coefficients are rational, other root $= -11 + 5\sqrt{5}$.

12. If α, β are the roots of the equation $x^2 - px + q = 0$, form an equation whose roots are $\frac{1}{p\alpha - q}, \frac{1}{p\beta - q}$.
13. Given that α and β are the roots of the equation $2x^2 - 3x + 4 = 0$, find an equation whose roots are $\alpha + \frac{1}{\alpha}, \beta + \frac{1}{\beta}$.
14. If the sum of the roots of the equation $x^2 - px + q = 0$ be m times their difference, prove that $p^2(m^2 - 1) = 4m^2 q$.
15. If the roots of the equation $(x - a)(x - b) - k = 0$ be c and d, then prove that the roots of the equation $(x - c)(x - d) + k = 0$ are a and b.
16. Determine the values of a so that $x^2 - 2(a + 1)x + 4 = 0$ has
    (*i*) real and distinct roots (*ii*) equal roots.
17. Find the values of m for which the quadratic equation $x^2 - m(2x - 8) - 15 = 0$ has
    (*i*) equal roots (*ii*) both roots positive.
18. Find the values of k for which the graph of $y = x^2 + kx - x + 9$ lies above the x-axis.
    Hint. Find the values of k for which $x^2 + kx - x + 9 > 0$ for all real values of x.

ANSWERS

EXERCISE 6.1

1. (*i*) $\pm\sqrt{2}\,i$ (*ii*) $\pm\frac{\sqrt{5}}{2}i$ **2.** (*i*) 4, −3 (*ii*) $\frac{-3 \pm \sqrt{2}}{5}$

3. (*i*) $\sqrt{3}, \frac{1}{\sqrt{3}}$ (*ii*) $\frac{3}{5}, \frac{5}{3}$ **4.** (*i*) $\frac{-1 \pm \sqrt{3}\,i}{2}$ (*ii*) $\frac{1 \pm \sqrt{7}\,i}{2}$

5. (*i*) $\frac{-3 \pm \sqrt{11}\,i}{2}$ (*ii*) $\frac{-1 \pm \sqrt{7}\,i}{4}$ **6.** (*i*) $\frac{-1 \pm \sqrt{19}\,i}{2\sqrt{5}}$ (*ii*) $\frac{-1 \pm \sqrt{7}\,i}{2\sqrt{2}}$

7. (*i*) $\frac{14 \pm \sqrt{14}\,i}{21}$ (*ii*) $\frac{2 \pm \sqrt{2}\,i}{2}$ **8.** 3, 4 **9.** 26, 13

10. $x^2 + (9 - x)^2 = 41$; the numbers are 4, 5 **11.** 12, 14 **12.** 24

13. 8 cm, 15 cm **14.** length = 40 m, width = 15 m **17.** 10 metres

15. 2 metres **16.** 29 years, 5 years

EXERCISE 6.2

1. (*i*) $2\sqrt{3}, -2\sqrt{3}$ (*ii*) 5, $\frac{5}{2}$ **2.** 0, $-\frac{5}{2}$ **3.** (*i*) 0 (*ii*) 1

4. −1, 2 **5.** −1, 8 **6.** (*i*) −4, −3, 0, 1 (*ii*) 2, 3, $\frac{5 \pm \sqrt{3}\,i}{2}$

7. 1, −6, $\frac{-5 \pm i\sqrt{23}}{2}$ **8.** $\frac{-1 \pm \sqrt{5}}{2}, \frac{-1 \pm \sqrt{5}}{2}$

9. 2, −6, $\frac{-4 \pm i\sqrt{59}}{2}$ **10.** 0, 1, $\frac{1 \pm \sqrt{57}}{2}$ **11.** (*i*) $\frac{9}{13}, \frac{4}{13}$ (*ii*) $-\frac{5}{16}$

12. (*i*) $\frac{1 + \sqrt{13}}{2}$ (*ii*) 2 **13.** 20 **14.** (*i*) 4 (*ii*) 1, 5

15. (*i*) No solution (*ii*) $\frac{37}{4}$ **16.** (*i*) 3, $-\frac{5}{3}$ (*ii*) 5, 6

17. $6, -\frac{17}{3}$ **18.** $\pm\sqrt{\frac{5}{3}}$ **19.** $0, a, -a$

20. (*i*) $\frac{5 \pm \sqrt{29}}{2}, \frac{-3 \pm \sqrt{13}}{2}$ (*ii*) $\frac{1}{2}, \frac{2}{3}, \frac{3}{2}, 2$ **21.** $1 \pm i, 2 \pm \sqrt{2}$

22. (*i*) $x = 8, y = 2$ or $x = 2, y = 8$ (*ii*) $x = 3, y = 1$ or $x = 1, y = 3$

23. 60 km/h **24.** $(x - 10)\left(\frac{1200}{x} + 2\right) = 1200 + 60; x = 100$

EXERCISE 6.3

1. (*i*) pair of complex conjugates (*ii*) rational and unequal
(*iii*) real and unequal (*iv*) pair of complex conjugates

2. (*ii*) Roots are real and equal when $a = b = 0$

3. Roots are equal when $a = b = c$. Then roots are a, a

4. (*i*) real and distinct
(*ii*) rational and distinct when $p \neq 1$; when $p = 1$, roots are rational and equal
(*iii*) real and distinct when $a + b \neq 0$; when $a + b = 0$, roots are both 0

5. 5, −3

8. Converse is not true. For example, $x^2 - x - 2 = 0$ has rational roots 2, −1 but sum of its coefficients ≠ 0

12. 0, 3 **13.** $-2 < m < 8$

EXERCISE 6.4

1. (*i*) 0, 9 (*ii*) $1, 2\sqrt{2}$ **2.** 8 **3.** ± 6

4. (*i*) $\frac{b^2 - 2ac}{a^2}$ (*ii*) $\frac{b^2 - 2ac}{ac}$ (*iii*) $\frac{3abc - b^3}{a^3}$ (*iv*) $\frac{\sqrt{b^2 - 4ac}}{|a|}$

(*v*) $\pm\frac{b\sqrt{b^2 - 4ac}}{a^2}$ (*vi*) $\frac{b^6 + 9a^2b^2c^2 - 6ab^4c - 2c^3a^3}{a^6}$

5. ±7 **6.** $5; -\frac{2}{5}$ **7.** $-3, -4; x^2 + 7x + 12 = 0$

8. (*ii*) $-\frac{57}{4}$

9. (*i*) $3x^2 - 8x - 3 = 0$ (*ii*) $\sqrt{3}x^2 - 4x + \sqrt{3} = 0$ (*iii*) $x^2 = 0$

10. $x^2 - 4x + 1 = 0$

11. (*i*) $x^2 + 2x + 5 = 0$ (*ii*) $x^2 + 4x + 7 = 0$ (*iii*) $6x^2 - 4x + 1 = 0$

12. (*i*) $ax^2 + 2bx + 4c = 0$ (*ii*) $4ax^2 + 2bx + c = 0$
(*iii*) $acx^2 - (b^2 - 2ac)x + ac = 0$
(*iv*) $a^4x^2 - 2a^2(b^2 - 2ac)x + b^2(b^2 - 4ac) = 0$
(*v*) $acx^2 + b(a + c)x + (c + a)^2 = 0$
(*vi*) $ax^2 - (2ak - b)x + (ak^2 - bk + c) = 0$

13. $40x^2 - 14x + 1 = 0$ **14.** $x^2 - 2x + 3 = 0$ **15.** $2x^2 + 7x + 9 = 0$

16. $6, -\frac{3}{2}$ **18.** $2, -\frac{25}{2}$

19. (*i*) $c = a \neq 0$ (*ii*) $b = 0$ (*iii*) $2b^2 = 9ac$

20. (*i*) $a + b + c = 0$ (*ii*) $c = 0$ (*iii*) $c = 0, b \neq 0$
(*iv*) $b = c = 0$ (*v*) c and a are of opposite signs

21. (*i*) 4 (*ii*) 2, −25 (*iii*) − 16 (*iv*) 3 and 15 or −3 and −15

22. 1, -27

23. $\frac{b}{b^2} = \frac{c}{a^2} = \frac{a}{c^2}$

24. $k = -3$, common root is -1; $k = -\frac{27}{4}$, common root is 4

25. 0, 24

EXERCISE 6.5

1. Graph is an upward parabola with vertex at $\left(\frac{3}{2}, -\frac{25}{4}\right)$ and passing through points $(-1, 0)$, $(4, 0)$;

$$8x^2 - 3x - 4 \begin{cases} = 0 \text{ when } x = -1, 4 \\ < 0 \text{ when } -1 < x < 4 \\ > 0 \text{ when } x < -1 \text{ or } x > 4 \end{cases}$$

2. (i) $2x^2 + 3x + 5 > 0$ for all real x

(ii) $3x^2 - 6x + 3 > 0$ for all real x except at $x = 1$ and at $x = 1$, $3x^2 - 6x + 3 = 0$

(iii) $2 - x - x^2 = \begin{cases} = 0 \text{ when } x = -2, 1 \\ > 0 \text{ when } -2 < x < 1 \\ < 0 \text{ when } x < -2 \text{ or } x > 1 \end{cases}$

3. $-6 < a < 2$

EXERCISE 6.6

1. 2, -2

2. $\pm 2, \pm\sqrt{2}$

3. $-1, 0, 2$

5. $a = 3, p = \frac{3}{2}, q = \frac{5}{2}$; $a = -\frac{1}{5}, p = -\frac{1}{2}, q = \frac{1}{2}$

7. $b = \pm 100$. If $b = 100$, common root is -1; if $b = -100$, common root is 1

8. $-\frac{1}{2}(p + q + r)$; pqr

9. $p \in \left(3, \frac{15}{4}\right]$

CHAPTER TEST

1. (i) $a, \frac{1}{a}$ (ii) $\frac{a+b}{2}, \frac{a-b}{2}$

2. -2

3. $2, -\frac{19}{3}, \frac{-13 \pm i\sqrt{599}}{6}$

4. 7, -3, 2, 2

5. 4

6. $\frac{a}{6}\left(-11 \pm \sqrt{13}\right)$

7. $\pm\sqrt{\frac{5}{2}}, \pm\sqrt{\frac{10}{7}}$

8. 570 km/h

9. 20 minutes and 25 minutes

11. $x^2 + 22x - 4 = 0$

12. $q^2x^2 - (p^2 - 2q)x + 1 = 0$

13. $8x^2 - 18x + 13 = 0$

16. (i) $a \in (-\infty, -3) \cup (1, \infty)$ (ii) $a = -3, 1$

17. (i) $m = 3, 5$ (ii) $m \in \left(\frac{15}{8}, 3\right] \cup [5, \infty)$

18. $k \in (-5, 7)$