* Scalar :-

→ A Scalar is a Quantity that has only magnitude (size) and no direction.

→ It is just a numberical value that represents some measurement.

Ex:-

* Temperature : 30°c
* Mass : 5kg
* Speed : 60 km/h (without specifying direction)
* Time : 10 sec

* Scalars can be added, subtracted, multiplied & divided like regular numbers.

* Vector :-

→ A vector is a Quantity that has both magnitude (size) and direction.

→ It is Represented by an arrow in geometry or as an ordered list of numbers in algebra.

Ex:-

* Velocity : 60 km/h East (both speed and direction)
* Force : 10 Newtons at 30° angle
* Displacement : Moving 5 meters North.
* Acceleration : 9.8 m/s^2 downward

$V=(x,y)$ or $V=(x,y,z)$ in 2D or 3D Space

Types of Vectors:-

* Zero vector (0): A vector where all components are zero.

  Ex:- $(0,0)$.

* Unit Vector : A vector with magnitude 1, often used to indicate direction.

* Column Vector : $v = \begin{bmatrix} x \\ y \end{bmatrix}$

* Row vector : $v = [x \quad y]$

* position vector : Represents a point in space relative to the origin.

* Equal vectors : vectors with the same magnitude & direct

Vector Operations:-

1. Vector Addition:-

   If $a = (a_1, a_2)$ and $b = (b_1, b_2)$, then

   $$a+b = (a_1+b_1, a_2+b_2)$$

2. Scalar Multiplication:-

   Multiplying a vector by a scalar c scales its magnitude

   $$cv = (cv_1, cv_2)$$

3. Dot product (Scalar Product):-

The dot product of two vectors a & b is,

$$a \cdot b = a_1 b_1 + a_2 b_2 + \dots + a_n b_n$$

It gives a scalar result.

4. Cross product (Only in 3D):-

For vectors in 3D:

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

It results in a vector perpendicular to both a & b.

Vector properties:-

* Commutative : $a+b = b+a$
* Associative : $(a+b)+c = a+(b+c)$
* Distributive : $c(a+b) = ca+cb$.

Applications of vectors in Linear Algebra:-

* Computer Graphics : Vectors represent movement, Scaling & transformations.
* Machine Learning : Feature vectors store numerical data for analysis.
* Physics & Engineering : Vectors describe forces, velocity & acceleration
* Navigation : GPS Systems use vectors to determine position and direction:

Role of Scalars in Linear Algebra:-

1. Scalar Multiplication of a Vector

if a vector $v = (v_1, v_2)$ is multiplied by a scalar c,

then:

$$cv = (cv_1, cv_2)$$

Ex:-

$$2(3, 4) = (6, 8)$$

2. Scalar Multiplication of a Matrix

if A is a matrix and c is a scalar, then:

$$cA = c\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} \\ ca_{21} & ca_{22} \end{bmatrix}$$

Ex:-

$$3\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 \\ 9 & 12 \end{bmatrix}$$

3. Scalar in Inner (Dot) product

The dot product of two vectors results in a scalar;

$$a \cdot b = a_1 b_1 + a_2 b_2 + \dots + a_n b_n.$$

Ex:- $(2, 3)(4, 5) = 2(4) + 3(5) = 8 + 15 = 23.$

Scalar properties in Linear Algebra

* Associative property : $(ab)v = a(bv)$

* Distributive property : $a(v+w) = av + aw$

* Multiplicative Identity : $1 \cdot v = v$

* Multiplicative zero : $0 \cdot v = 0$.

Applications of Scalar in Linear Algebra :-

* Scaling vectors & matrices in graphics & physics.

* Eigean values (scalars) in Eigenvector problems.

* Determinants of matrices are Scalar values.

* Machine Learning : Scalars adjust weights & biases in neural networks.

Matrix in Linear Algebra:-

→ A matrix is a rectangular array of number, Symbols, or Expressions arranged in rows & columns.

→ Matrices are fundamental in Linear Algebra and are widely used in Computer Science, physics, Engineering, data science & Machine learning.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

where a_{ij} represents the Elements at the i^{th} row and j^{th} column.

Ex:- 3x3 matrix.

$$B: \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}_{3\times 3}$$

Types of Matrices:-

1. Row matrix :- $[1, 2, 3]_{1\times 3}$

2. column Matrix :- $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times 1}$

3. Square Matrix:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$$ Here m=n.

4. Diagonal matrix!

A Square matrix where all non-diagonal Elements are zero.

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

5. Identity matrix:-

A Square matrix with 1's on the diagonal Elements and 0's Else where.

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. zero matrix:-

all Elements in matrix are zero.

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

7. Upper & lower Triangular matrices:-

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix}$$

Upper triangle

$$\begin{bmatrix} 1 & 0 & 0 \\ 4 & 5 & 0 \\ 7 & 8 & 9 \end{bmatrix}$$

Lower Triangle

Matrix Operations:-

→ **Addition and Subtraction:-**

Matrices of the same size can be added or Substracted Element-wise.

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

→ **Scalar Multiplication:**

Multiply Each Element of matrix by a Scalar.

$$2 \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

→ **Matrix Multiplication:-**

For two matrices $A_{m \times n}$ & $B_{n \times p}$.

there product $C = A \times B$ is an $m \times p$ matrix.

$$C_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

Ex:-

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1(5) + 2(7) & 1(6) + 2(8) \\ 3(5) + 4(7) & 3(6) + 4(8) \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

→ Transpose Matrix (A^T)

~~Att~~ Flipping a matrix over its diagonal.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \; ; \; A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

→ Determinant of a Square Matrix :

2×2

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\det(A) = ad - bc.$$

3×3

$$\det(A) = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

→ Inverse of a Matrix :

for 2×2 matrix,

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\det(A) \neq 0$ (Non-Singular matrix)

Applications:-

1. Computer Graphics: Transformations like rotation, scaling & translation.

2. Machine learning & Data Science: Representing datasets, linear regression, and neural networks.

3. Physics & Engineering: Solving Equations in mechanics, circuits & Quantum mechanics.

4. Cryptography: Encoding and decoding information

5. Network Theory: Representing graphs & connections.

Row Echelon Method (Gaussian Elimination) :-

It is also known as Gaussian Elimination, is a step-by-step approach to Solving Systems of linear Equations by Converting a matrix into Row Echelon ~~method~~ form (REF) using row operations.

Row Echelon form (REF)

* Leading Entries (pivot) are 1, and Each leading 1 appears to the right of the leading 1 in the row above
* rows with all zeros are at the bottom of the matrix
* Below a leading 1, all Elements are zero.

Ex :-

$$\begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & 4 & -2 \\ 0 & 0 & 2 & 5 \end{bmatrix}$$

Applications :-

* Solving Systems of linear Equations
* finding the rank of matrix
* computing determinants
* Linear transformations in computer graphics
* Machine learning (Solving normal Equations in regression models).

Ex:-

$$x + 2y - z = 3$$
$$2x + 3y + z = 9$$
$$-3x + y + 2z = 2$$

Convert it into an augmented matrix.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2 & 3 & 1 & 9 \\ -3 & 1 & 2 & 2 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 + 3R_1$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 2-2 & 3-4 & 1+2 & 9-6 \\ -3+3 & 1+6 & 2-3 & 2+9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -1 & 3 & 3 \\ 0 & 7 & -1 & 11 \end{bmatrix}$$

$R_2 \to -1 \times R_2$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -3 & -3 \\ 0 & 7 & -1 & 11 \end{bmatrix}$$

$R_3 \to R_3 - 7R_2$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 20 & 32 \end{bmatrix}$$

$R_3 \to \frac{1}{20} R_3$

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 1.6 \end{bmatrix}$$

$$x + 2y - z = 3$$
$$y - 3z = -3$$
$$z = 1.6$$

$\Rightarrow y - 3(1.6) = -3$

$\Rightarrow y = 1.8$

$\Rightarrow x + 2(1.8) - (1.6) = 3$

$\Rightarrow x + 3.6 - 1.6 = 3$

$\Rightarrow x = 1$

$\therefore x = 1$

$y = 1.8$

$z = 1.6$

$(x, y, z) = (1, 1.8, 1.6)$

Inverse Method for Solving Linear Equation :-

The inverse method is a technique for solving a system of linear equations using the inverse of a matrix.

This method is applicable when the coefficient matrix is square $(n \times n)$ and invertible $(\det(A) \neq 0)$.

General form, $AX = B$ → Column vector of constants $(n \times 1)$

A → Coefficient matrix $(n \times n)$

X → vector of variables $(n \times 1)$

Ex :-

$2x + 3y = 8$

$5x + 7y = 19$

$$\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 19 \end{bmatrix}$$

$$A \quad X = B.$$

Steps :-

1. Write the system as $AX = B$
2. Compute A^{-1} if $\det(A) \neq 0$
3. Multiply $A^{-1}B$ to find X.
4. Extract the values of the unknowns.

When to use

Good for:

- Small systems ($n \le 3$)
- Theoretical applications in linear algebra & machine learning.

Not good for:

- Large systems ($n > 3$), as computing A^{-1} is computational[ly] learning.
- When A is singular ($\det(A) = 0$), means it has no inverse.

For large systems, Gaussian Elimination (Row Echelon method) or LU Decomposition is preferred.

Applications:-

- Engineering :- ~~Circutan~~ Circuit analysis, Structural analysis.
- Economics :- Solving Input output models.
- Machine Learning :- Linear regression solution using the normal Equation.
- Computer Graphics Transformation & 3D rendering.

Ex:-

$$2x + 3y = 8$$
$$5x + 7y = 19$$

Step-1

$$AX = B$$

$$\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 19 \end{bmatrix}$$

Step-2

$$X = A^{-1} \cdot B$$

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\det(A) = (a \cdot d) - (b \cdot c)$$

$$A = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$$

$$\det(A) = (2 \times 7) - (3 \times 5) = -1$$

$$A^{-1} = \frac{1}{-1} \begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix}$$

$$A^{-1} = \begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix}$$

Step-3

$$X = A^{-1} \cdot B$$

$$X = \begin{bmatrix} 7 & -3 \\ -5 & 2 \end{bmatrix} \begin{bmatrix} 8 \\ 19 \end{bmatrix}$$

$$X = \begin{bmatrix} (-7 \times 8) + (3 \times 19) \\ (5 \times 8) + (-2 \times 19) \end{bmatrix}$$

$$= \begin{bmatrix} -56 + 57 \\ 40 - 38 \end{bmatrix}$$

$$X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Step-4 :-

$$x = 1, \quad y = 2$$