# **APPENDIX-A**

The assumptions and derivations of multi-linear crack hinge model are discussed in this section. Based on these derivations a MATLAB code was developed and also used to propose the stress-crack width relationships of different UHPFRC beams. In continuation of the traction-separation model section, the following subsections comprise the derivations and assumptions in detail.

## A.1 Derivations for the Neutral Axis Depth *yo* and *y\**

In Phase I, the compressive and tensile stresses are equal and symmetrical about the mid-depth of the hinge. Thus, in the pre-cracking zone of loading, the tensile stress does not exceed *ft* anywhere in the UHPFRC as shown in Figure 26. Here,

|  |  |
| --- | --- |
|  | (42) |

where *yo* is the distance of the neutral axis from the top surface of the notched beam and *y\** is the distance of the layer with the peak tensile stress from the neutral axis, as shown in Figure 26.

|  |
| --- |
|  |
| Figure 26. Stress distribution in compression and tension of notched beams in different phases of loading |

After the crack initiation at the end of Phase I, at the location of the crack tip, the crack width and tensile stress at any given time are *w*(*ycr*) = *0* and *σw*(*w*(*ycr*))= *ft*, and the location of a crack tip can be given by *ycr* = *h – d*. Substituting these values in **Equation (16)** yields

|  |  |
| --- | --- |
|  | (43) |

By using the normalized rotation θ (**Equation (25)**), the above equations can be rewritten as

|  |  |
| --- | --- |
|  | (44) |

By using the normalized crack depth formulation as detailed in Equation(26), Equation (71) can be rewritten as

|  |  |
| --- | --- |
|  | (45) |

Also, from Figure 26 it can be seen that

|  |  |
| --- | --- |
|  | |
|  | (46) | |

## A.2 Concrete in compression (fc)

Assumed linear stress strain plays an important role as condition given below contribution in strain distribution resulting from θ. By using similar triangle principle in phase-I of Figure 26. Following equations are obtained.

|  |  |
| --- | --- |
|  | (47) |
| = | (48) |
| == = | (49) |

## A.3 Derivation of Load *P(θ)* from the Moment Equation

Given that the beam under consideration is simply supported in the three-point bending test, the maximum moment on such a beam at the mid-span is given by:

|  |  |
| --- | --- |
|  | (50) |

where *M* is the bending moment, *P* is the applied vertical load at mid-span and *L* is the length of the beam between the supports. Both *M* and *P* depend upon the rotation *θ* of the beam as shown in Figure 10. By using the normalized moment **Equation (21)** , the above load equation can be rewritten as:

|  |  |
| --- | --- |
|  | (51) |

Above Equation can be written as

|  |  |
| --- | --- |
| = | (52) |

|  |  |
| --- | --- |
|  | (53) |

## A.4 Derivation for CMOD in pre-cracking and post cracking:

The crack mouth opening displacement (CMOD), where measurement is carried out at the end of the notch (bottom most fiber). Where y=H and no fiber across notch for bridging action and hence no elastic deformation on either side of notch is considered. Thus, the Equation for the CMOD can be obtained from **Equation (11)** as:

|  |  |  |  |
| --- | --- | --- | --- |
|  | (54) | | |
|  | (55) | | |
|  | (56) | | |
|  | (57) | | |
|  | (58) | | |
|  | (59) | | |
|  | | (60) |
| For pre cracking  For post cracking θ | | (61) |
| for θ≥1 post cracking | | (62) |
| for θ≤1 pre-cracking | | (63) |

where CMOD is the crack mouth opening displacement, *θ* is the normalized rotation of the beam, *α* is the normalized crack depth, *a1* is the notch width, *h* is the height of the hinge which is measured from the top surface of the beam to the notch tip, *s* is the hinge width, *ft* is the tensile strength of concrete, and *E* is the elastic modulus of concrete.

## A.5 Derivation of the CTOD in the Pre-Cracking and Post-Cracking Stages

Similarly, the crack tip opening displacement (CTOD) is the total deformation between the notch faces at the location of the tip of the notch for *y = h*. This deformation can be obtained by deducting the elastic deformations of the FRC strips on the sides of the notch from the total deformation of that layer of the hinge. Here, *bb* is the stress coefficient for the bottom-most layer of the FRC in the hinge, and its derivation is given in the subsection for determining *bb* below. Here,

|  |  |
| --- | --- |
|  | (64) |

Now, by using the values of *yo* from **Equations (42)** and **(46)**, and putting *bb* = *θ* in the pre-cracking state, the Equation for the CTOD can be obtained as:

|  |  |  |
| --- | --- | --- |
|  |  | (65) |
|  |  | (66) |

## A.6 Deflection of the Beams

The deflection of the beam, *∆*, is derived from the basic geometrical relationships as the function of the rotation, hinge width and length of the beam as shown in **Equation (67)** below:

|  |  |
| --- | --- |
|  | (67) |

## A.7 Derivation of the Effective Depth Coefficient *ki*

The effective depth coefficient *ki* for *ith* line segment in the tension zone of the hinge is the ratio of the depth of the hinge under the *ith* segment of the *σ-w* relationship to the depth of the hinge in the elastic tensile zone, i.e.

|  |  |
| --- | --- |
|  | (68) |

For the *ith* segment in the tension zone, from **Equation (16)**, the following equations can be obtained:

|  |  |  |
| --- | --- | --- |
|  |  | (69) |
|  |  | (70) |

Here,  and  are the depths of the start and end points of the *ith* segment of the *σ-w* relationship from the top layer of the beam as shown in Figure 16. Subtracting **Equation (69)** from **Equation (70)** gives:





|  |  |
| --- | --- |
|  | (71) |

Using the normalized rotation *θ* from **Equations** (26) and (46), the following Equation can be obtained as:

|  |  |
| --- | --- |
|  | (72) |

## A.8 Derivation of the Transition Rotation *θi*

For finding out the transition rotation *θi*, each representing an end of the *ith* segment and the start of *(i+1)th* segment at the bottom-most layer of the hinge, considering the force equilibrium  yields:

|  |  |
| --- | --- |
|  | (73) |

By applying **Equations** (46) and (49), **Equation** (73) becomes:



Dividing the above Equation by 

|  |  |
| --- | --- |
|  | (74) |

i.e. 





The obtained Equation above is in the quadratic form . Solving the above equation yields:

|  |  |
| --- | --- |
|  | (75) |

At the cracking point, *i = 0* and  becomes unit as stated earlier in **Equation** (25**)** for the crack initiation.

## A.9 Determination of the Parameter *bb*

In the case where the stress in the bottom-most layer of the hinge is not the specified stress condition in the *σ-w* model, i.e. , the force equilibrium gives

|  |  |
| --- | --- |
|  | (76) |

where *i ϵ 1* to *n*, and *n* is the total number of line segments in the *σ-w* model.

For a value of *θ* between the two transition values of *θi*, the stress in the bottom-most layer is not known directly. However, from the *σ-w* model using **Equation (18)**, the value of *wb* can be obtained using linear interpolation as:

|  |  |
| --- | --- |
|  | (77) |

Thus, where *y* = *h*, *w(y)* = ** and *b* = *bb*, the stress at the bottom-most layer can be given by **Equation (16)** as



Substituting the Equation for the normalized rotation  into the above equation yields:



Substituting the  value from **Equation (45)** to the above equation yields:



Where:

|  |  |
| --- | --- |
|  | (78) |
|  | (79) |

Hence,

|  |  |
| --- | --- |
|  | (80) |

Thus, the value from the above Equation can be put into the equilibrium equation to get all the unknowns in *y* terms.

## A.10 Derivations for Parameters *µ* and *α*

## (i) Phase I

As the location of the neutral axis is known beforehand, depending on the stress distribution over the depth of the uncracked hinge section, the moment carrying capacity of the hinge in the elastic zone can be easily calculated as

|  |  |
| --- | --- |
|  | (81) |

## (ii) Phase II

From the stress distribution diagram as shown in **Figure 16** for Phase I, similarly to **Equation (74)**, the Equation from the force equilibrium condition can be derived as



|  |  |
| --- | --- |
|  |  |

The above equation is in the quadratic form *aα2* + *bα* + *c* = *0*, and the solution is,

|  |  |
| --- | --- |
|  | (82) |

where:

|  |  |
| --- | --- |
|  | (83) |

Here *i* is the number of line segments, from the *σ-w* model, present in the tension zone of the beam in bending. This means that the crack width at the bottom-most layer, *wb*, lies in the *ith* segment of the model where *wi-1 < wb ≤ wi*. The moment about the tip of the crack is:

|  |  |
| --- | --- |
|  | (84) |

The relationship between the normalized moment and the normalized rotation is obtained from the moment equilibrium **Equation (87)** by using the  and  values from **Equations (25)** and **(26)** as:

|  |  |
| --- | --- |
|  | (85) |

The Equation is again simplified to make it easier for computational algorithms, see below:

|  |  |
| --- | --- |
|  | (86) |

## (iii) Phase III

In Phase III, the stress at the bottom of the hinge is zero as the crack width is greater than the critical crack width, as shown **Figure 16**. Therefore, the Equation for  becomes:



In this case, *bb* = 0. The equation is in the quadratic form *aα2* + *bα* + *c* = *0*, and the solution is,

|  |  |
| --- | --- |
|  | (87) |

From the moment equilibrium, it gets:

|  |  |
| --- | --- |
|  | (88) |

This Equation is again simplified to make it easier for computational algorithms, see below:

|  |  |
| --- | --- |
|  | (89) |