Representation of Quivers

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Chapter 1

Introduction

This project aims to cover interesting topics from the area of quiver representations. The first chapter will give a brief introduction to the topics in Homological Algebra needed for bulk of the report. It will cover chain complexes, exact sequences, short exact sequences, and extensions, most importantly Ext^1. The the second chapter will introduce quivers and look at some interesting results about path algebras before moving onto representations of quivers and their relationship with the modules of the corresponding path algebra. Next we will cover Dynkin and Euclidean diagrams before finishing with a proof of Gabriel's Theorem. The third chapter will cover Auslander-Reiten quivers.

Chapter 2

Homological Algebra

2.1 Chain Complexes

Definition 2.1.1. A *chain complex* $\mathbf{C}_\bullet$ consists of a sequence of $\mathbb{R}$-modules C_i ($i \in \mathbb{Z}$) and morphisms of the form,

$$\mathbf{C}: \qquad \ldots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} C_{-2} \xrightarrow{\delta_{-2}} \ldots$$

such that $\delta_{n-1}\delta_n = 0$ for all n, i.e. the composition of any two consecutive maps is zero. The maps δ_n are called the *differentials* of C.

Remark 2.1.2. It is convention that the map δ_n starts at C_n.

Example 2.1.3. If we have a field K then we can create the following chain complex:

$$\mathbf{C}: \qquad \ldots \to 0 \to K^2 \xrightarrow{\left(\begin{smallmatrix}1&2\\3&0\\0&0\end{smallmatrix}\right)} K^3 \xrightarrow{(0\ 0\ 1)} K \to 0 \to \ldots$$

We can clearly see that the maps uphold the $\delta^2 = 0$ condition as,

$$\begin{pmatrix}0 & 0 & 1\end{pmatrix}\begin{pmatrix}1 & 2\\ 3 & 0\\ 0 & 0\end{pmatrix} = \begin{pmatrix}0 & 0\end{pmatrix}.$$

Example 2.1.4. If we consider the sequence,

$$\ldots \to 0 \to K \xrightarrow{\delta_2 = \left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} K^2 \xrightarrow{\delta_1 = (1\ 0)} K \to 0 \to \ldots$$

however,

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = 1 \neq 0.$$

Hence, $\delta_1\delta_2 \neq 0$, and so the sequence is not a chain complex. However, if we change δ_2 slightly we obtain the chain complex,

$$\mathbf{C}: \qquad \ldots \to 0 \to K \xrightarrow{\left(\begin{smallmatrix}1\\0\end{smallmatrix}\right)} K^2 \xrightarrow{(0\ 1)} K \to 0 \to \ldots$$

since,

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 1 \neq 0.$$

Definition 2.1.5. If $\mathbf{C}$ is a chain complex then its *homology* is defined to be,

$$H_n(\mathbf{C}) = \frac{Ker(\delta_n : C_n \to C_{n-1})}{Im(\delta_{n+1} : C_{n+1} \to C_n)} = \frac{Z_n(\mathbf{C})}{B_n(\mathbf{C})}.$$

This becomes an $\mathbb{R}$-module and, since δ^2, it follows that $B_n(\mathbf{C}) \subseteq Z_n(\mathbf{C})$.

The following Lemma is the solution to Exercise 6.1 in [3].

Lemma 2.1.6. *If $\mathbf{C}$ is a chain complex with $C_n = 0$ for some n then $H_n(\mathbf{C}) = 0$.*

Proof. Well suppose we have such a chain complex,

$$\mathbf{C}: \qquad \ldots \to C_{n+1} \xrightarrow{\delta_{n+1}} 0 \xrightarrow{\delta_n} C_{n-1} \to \ldots$$

the the homology is,

$$H_n(\mathbf{C}) = \frac{Ker(\delta_n : 0 \to C_{n-1}}{Im(\delta_{n+1} : C_{n+1} \to 0)},$$

as the only element in C_n is the zero element, and so, $H_n(\mathbf{C}) = 0$, as required. $\square$

Examples 2.1.7 and 2.1.10 are taken from [1] and are included here because they are felt to be the clearest at demonstrating a chain complex and homology, however, the more general statement of the second example is presented as Proposition 2.1.8 because it is an interesting result.

Example 2.1.7. If we take a module M then we can make a chain complex;

$$\mathbf{C}: \qquad \ldots \to 0 \to M \to 0 \to \ldots$$

where M is at degree n. Then the homology will be:

$$H_i(\mathbf{C}) = \begin{cases} \frac{Ker(M \to 0)}{Im(0 \to M)} = M & i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.1.8. *If we have a module homomorphism between R-modules, $f : M \to N$, the we get the chain complex,*

$$\mathbf{C}: \qquad \underset{deg}{\ldots} \to \underset{n+2}{0} \to \underset{n+1}{M} \xrightarrow{f} \underset{n}{N} \to \underset{n-1}{0} \to \ldots,$$

and the homology becomes,

$$H_i(\mathbf{C}) = \begin{cases} \frac{N}{Im(f)} = Coker(f) & i = n \\ Ker(f) & i = n+1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Firstly, at degree n we have that,

$$H_n(\mathbf{C}) = \frac{Ker(N \to 0)}{Im(M \xrightarrow{f} N)} = \frac{N}{Im(f)} = Coker(f).$$

Then at degree $n+1$ we have that,

$$H_{n+1}(\mathbf{C}) = \frac{Ker(M \xrightarrow{f} N)}{Im(0 \to M)} = Ker(f).$$

Finally, it is clear that everywhere else there is no homology. $\square$

Notation 2.1.9. Here,

$$Coker(f) = \frac{\text{Codomain of } f}{\text{Image of } f},$$

is the *cokernel* of the map f.

Example 2.1.10. We can have a chain complex of $\mathbb{Z}$-modules,

$$\mathbf{C}: \qquad \underset{\deg}{\ldots} \to \underset{2}{0} \to \underset{1}{\mathbb{Z}} \xrightarrow{a} \underset{0}{\mathbb{Z}} \to \underset{-1}{0} \to \ldots$$

where the map a is right multiplication by some $a \in \mathbb{Z}$. The homology is,

$$H_i(\mathbf{C}) = \begin{cases} \frac{Ker(\mathbb{Z} \to 0)}{Im(\mathbb{Z} \xrightarrow{a} \mathbb{Z})} = \frac{\mathbb{Z}}{a\mathbb{Z}} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that,

$$H_0(C) = \frac{\mathbb{Z}}{a\mathbb{Z}} = \frac{\text{Codomain of } f}{\text{Image of } f} = Coker(a).$$

Also,

$$H_1(C) = \frac{Ker(\mathbb{Z} \xrightarrow{a} \mathbb{Z})}{Im(0 \to \mathbb{Z})} = Ker(a) = 0,$$

because $Ker(a)$ is empty.

Definition 2.1.11.
- The elements of $B_n(\mathbf{C})$ are called $n-$*boundaries*.
- The elements of $Z_n(\mathbf{C})$ are called $n-$*cycles*.

Remark 2.1.12. If $x \in Z_n(\mathbf{C})$ then its image in $H_n(\mathbf{C})$ is usually written as $[x]$.

Definition 2.1.13. A chain complex $\mathbf{C}$ is said to be:

- *acyclic* if $H_n(\mathbf{C}) = 0$ for all n.
- *bounded above* if there exists some $n \in \mathbb{N}$, $C_k = 0$ for all $k > n$.
- *bounded below* if for some $n \in \mathbb{N}$, $C_k = 0$ for all $k < n$.
- *bounded* if it is bounded above and below.
- *non-negative* if $C_n = 0$ for $n < 0$.

Example 2.1.14. All the chain complexes in the previous examples are bounded both above and below, however, neither is acyclic as they both have instances where the homology is non-zero. The chain complex in Example 2.1.10 is non-negative because $C_n \neq 0$ only when $n = 0, 1$.

Example 2.1.15. If we take another look at the chain complex in Example 2.1.4,

$$\mathbf{C}: \qquad \underset{\deg}{\dots} \to 0 \to \underset{1}{K} \xrightarrow{\binom{1}{0}} \underset{0}{K^2} \xrightarrow{(0\ 1)} \underset{-1}{K} \to 0 \to \dots$$

the homologies are,

$$H_1(\mathbf{C}) = \frac{Ker(K \xrightarrow{\binom{1}{0}} K^2)}{Im(0 \to K)} \cong \frac{K}{K} \cong 0,$$

$$H_0(\mathbf{C}) = \frac{Ker(K^2 \xrightarrow{(1\ 0)} K)}{Im(K \xrightarrow{\binom{1}{0}} K^2)} \cong \frac{K}{K} \cong 0,$$

$$H_{-1}(\mathbf{C}) = \frac{Ker(K \to 0)}{Im(K^2 \xrightarrow{(1\ 0)} K))} \cong \frac{K}{K} \cong 0.$$

Thus $H_n(\mathbf{C}) = 0$ for all n and so $\mathbf{C}$ is an acyclic chain complex. Later in the report, we will see that $\mathbf{C}$ is in fact a short exact sequence.

Example 2.1.16. The chain complex,

$$\mathbf{C}: \qquad \underset{\deg}{\dots} \xrightarrow{3} \underset{1}{\frac{\mathbb{Z}}{9\mathbb{Z}}} \xrightarrow{3} \underset{0}{\frac{\mathbb{Z}}{9\mathbb{Z}}} \xrightarrow{3} \underset{-1}{\frac{\mathbb{Z}}{9\mathbb{Z}}} \to \dots$$

where the differentials are the maps,

$$\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \to \frac{\mathbb{Z}}{9\mathbb{Z}},\ z + 9\mathbb{Z} \mapsto 3z + 9\mathbb{Z},$$

is unbounded. It is also acyclic, since the homology is,

$$H_n(\mathbf{C}) = \frac{Ker(\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \to \frac{\mathbb{Z}}{9\mathbb{Z}})}{Im(\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \to \frac{\mathbb{Z}}{9\mathbb{Z}})} \cong \frac{\mathbb{Z}/3\mathbb{Z}}{\mathbb{Z}/3\mathbb{Z}} \cong 0,$$

for all n.

Definition 2.1.17. A *cochain complex* $\mathbf{C}^\bullet$ consists of a sequence of $\mathbb{R}$-modules C^i $(i \in \mathbb{Z})$ and morphisms of the form,

$$\mathbf{C}: \qquad \dots \xrightarrow{\delta^{-3}} C^{-2} \xrightarrow{\delta^{-2}} C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \dots$$

such that $\delta^{n-1}\delta^n = 0$ for all n, i.e. the composition of any two consecutive maps is zero.

Remark 2.1.18. Chain and cochain complexes can be thought of as almost identical constructs with the only difference being thenumbering of the chain. The degree of a chain complex *decreases* from left to right, whereas, the degree of a cochain complex *increses* from left to right. So, we can compute one from the other by setting $C^{-n} = C_n$, or equivalently $C^n = C_{-n}$; this is called *renumbering*.

Definition 2.1.19. If $\mathbf{C}$ is a cochain complex then its *cohomology* is defined to be,

$$H^n(\mathbf{C}) = \frac{Ker(\delta^n : C^n \to C^{n+1})}{Im(\delta^{n-1} : C^{n-1} \to C^n)} = \frac{Z^n(\mathbf{C})}{B^n(\mathbf{C})}.$$

- The elements of $B_n(\mathbf{C})$ are called $n-$*coboundaries*.
- The elements of $Z_n(\mathbf{C})$ are called $n-$*cocycles*.

Example 2.1.20. We can renumber the chain complex in Example 2.1.10 to get the cochain complex,

$$\mathbf{C}: \qquad \underset{\text{deg}}{\ldots} \to \underset{-2}{0} \to \underset{-1}{\mathbb{Z}} \xrightarrow{a} \underset{0}{\mathbb{Z}} \to \underset{1}{0} \to \ldots$$

Its cohomology is,

$$H^i(\mathbf{C}) = \begin{cases} \frac{Ker(\mathbb{Z} \to 0)}{Im(\mathbb{Z} \xrightarrow{a} \mathbb{Z})} = \frac{\mathbb{Z}}{a\mathbb{Z}} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.1.21. Let $\mathbf{C}$ be a chain complex of left R-modules. If M is a left R-module then $Hom(\mathbf{C}, M)$ is the cochain complex where,

$$Hom(\mathbf{C}, M)^n = Hom(C_n, M),$$

and the differentials,

$$\delta^n : Hom(\mathbf{C}, M)^n \to Hom(C, M)^{n+1},$$

are induced by the differentials of $\mathbf{C}$, $\delta_n : C_{n+1} \to C_n$. The cohomology of this cochain complex is denoted $H^n(\mathbf{C}, M)$.

The following example is a generalised version of one found in [1].

Example 2.1.22. Consider the acyclic chain complex,

$$\mathbf{C}: \qquad \underset{\text{deg}}{\ldots} \to 0 \to \underset{1}{\mathbb{Z}} \xrightarrow{n} \underset{0}{\mathbb{Z}} \xrightarrow{nat} \underset{-1}{\frac{\mathbb{Z}}{n\mathbb{Z}}} \to \ldots$$

So applying $Hom(-, \mathbb{Z})$ we gives the cochain complex,

$$\mathbf{C}': \qquad \underset{\text{deg}}{\ldots} \to 0 \to \underset{1}{\mathbb{Z}} \xrightarrow{n} \underset{0}{\mathbb{Z}} \xrightarrow{nat} \underset{-1}{\frac{\mathbb{Z}}{n\mathbb{Z}}} \to \ldots$$

which has cohomology,

$$H^i(\mathbf{C}', \mathbb{Z}) = \begin{cases} \frac{Ker(\mathbb{Z} \to 0)}{Im(\mathbb{Z} \xrightarrow{n} \mathbb{Z})} \cong \frac{\mathbb{Z}}{n\mathbb{Z}} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that despite the chain complex being acyclic, its cohomology induced by $Hom(-, \mathbb{Z})$ is not zero everywhere.

Definition 2.1.23. A *chain map* $f : \mathbf{C} \to \mathbf{D}$, with $\mathbf{C}$ and $\mathbf{D}$ chain complexes, is given by a homomorphism $f_n : \mathbf{C}_n \to \mathbf{D}_n$ for each n, such that each square in the following diagram commutes,

$$\begin{tikzcd}
\cdots \arrow[r, "\gamma_{n+2}"] & \mathbf{C}_{n+1} \arrow[r, "\gamma_{n+1}"] \arrow[d, "f_{n+1}"'] & \mathbf{C}_n \arrow[r, "\gamma_n"] \arrow[d, "f_n"'] & \mathbf{C}_{n-1} \arrow[r, "\gamma_{n-1}"] \arrow[d, "f_{n-1}"'] & \cdots \\
\cdots \arrow[r, "\delta_{n+2}"] & \mathbf{D}_{n+1} \arrow[r, "\delta_{n+1}"] & \mathbf{D}_n \arrow[r, "\delta_n"] & \mathbf{D}_{n-1} \arrow[r, "\delta_{n-1}"] & \cdots
\end{tikzcd}$$

Note that there is an equivalent notion of a *cochain map* of cochain complexes.

Example 2.1.24. If we have the following diagram,

$$\begin{tikzcd}
\mathbf{C}: & \cdots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r, "n"] \arrow[d, "f_1"', "nat"] & \mathbb{Z} \arrow[r, "nat"] \arrow[d, "f_0"', "nat"] & \frac{\mathbb{Z}}{n\mathbb{Z}} \arrow[r] \arrow[d, equal, "f_{-1}"'] & 0 \arrow[r] \arrow[d, equal] & \cdots \\
\mathbf{D}: & \cdots \arrow[r] & 0 \arrow[r] & \frac{\mathbb{Z}}{n\mathbb{Z}} \arrow[r, "n"] & \frac{\mathbb{Z}}{n^2\mathbb{Z}} \arrow[r, "nat"] & \frac{\mathbb{Z}}{n\mathbb{Z}} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

then we can see that our non trivial chain maps are,

$$\begin{aligned}
f_1 &: \mathbb{Z} \to \frac{\mathbb{Z}}{n\mathbb{Z}}, \quad z \mapsto z + n\mathbb{Z}, \\
f_0 &: \mathbb{Z} \to \frac{\mathbb{Z}}{n^2\mathbb{Z}}, \quad z \mapsto z + n^2\mathbb{Z}, \\
f_1 &: \frac{\mathbb{Z}}{n\mathbb{Z}} \to \frac{\mathbb{Z}}{n\mathbb{Z}}, \quad z + n\mathbb{Z} \mapsto z + n\mathbb{Z},
\end{aligned}$$

and they satisfy,

$$f_0(\gamma_1(z)) = f_0(nz) = nz + n\mathbb{Z} = 0 \quad \& \quad \delta_1(f_1(z)) = \delta_1(z + n\mathbb{Z}) = nz + n\mathbb{Z} = 0.$$

Simmilarly, we can see that the other chain maps make the diagram commute.

2.2 Exact Sequences

Definition 2.2.1. A finite or infinite sequence of R-morphisms and left R-modules,

$$\ldots \to L \xrightarrow{f} M \xrightarrow{g} N \to \ldots$$

is said to be *exact* at M if $imf = kerg$. A finite or infinite sequence of R-morphisms and left R-modules,

$$\ldots \xrightarrow{f_{n+2}} M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} \ldots$$

is said to be an *exact sequence* if it is exact at every M_i, i.e. $imf_i = kerf_{i+1}$.

Example 2.2.2. The finite sequence,

$$0 \to K \xrightarrow{f} K^2 \xrightarrow{g} K^2 \xrightarrow{h} K^2 \to 0,$$

where $f(x) = (x, 0)$, $g(x, y) = (0, y)$, and $h(x, y) = (x, 0)$ is an exact sequence since $im(f) = \{(x, 0) : x \in K\} = ker(g)$ and $im(g) = \{(0, y) : u \in K\} = ker(h)$. However, if we instead consider the finite sequence,

$$0 \to K \xrightarrow{f} K^2 \xrightarrow{g} K^2 \xrightarrow{h'} K^2 \to 0,$$

where f and g are the same as above, but $h'(x, y) = (x, y)$. Obviously, once again we have that $im(f) = ker(g)$ and $im(g) = \{(0, y) : y \in K\}$, however, $ker(h') = \{(0, 0)\}$ and so $im(g) \neq ker(h')$. Hence this is not an exact sequence, because it is not exact everywhere.

Definition 2.2.3. A *short exact sequence* is an exact sequence of the form,

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0.$$

Example 2.2.4. The sequence,

$$0 \to K \xrightarrow{f} K^2 \xrightarrow{g} K \to 0,$$

where $f(x) = (x, 0)$ and $g(x, y) = y$ is a short exact sequence, since $im(f) = \{(x, 0) : x \in K\} = ker(g)$.

Definition 2.2.5. If we have a map $fi : A \to B$ then we will denote the cokernel to be

$$coker(f) = \frac{B}{im(f)}$$

Lemma 2.2.6. *The sequence,*

$$0 \to ker(f) \xrightarrow{\iota} M \xrightarrow{f} N \xrightarrow{\eta} coker(f) \to 0,$$

where ι, η are the inclusion, and natural maps respectively, is exact. In addition the sequence,

$$0 \to ker(f) \xrightarrow{(} \iota)M \xrightarrow{\eta} \frac{M}{im(f)} \to 0,$$

is a short exact sequence.

Proof. It is clear here that we have $im(\iota) = ker(f)$ as ι is the inclusion map and since $coker(f) = \frac{N}{im(f)}$ we also have that $im(f) = ker(\eta)$. Hence the sequence is exact. The second result follows a similar argument. □

Definition 2.2.7. Two short exact sequences,

$$\mathbf{E}: \quad 0 \to L \to M \to N \to 0 \qquad \& \qquad \mathbf{E}': \quad 0 \to L \to M' \to N \to 0,$$

are *equivalent* if there is a map $f : M \to M'$ such that the diagram,

$$\begin{tikzcd}
\mathbf{E}: \arrow[r] & 0 \arrow[r] \arrow[d, equal] & L \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "f"'] & N \arrow[r] \arrow[d, equal] & 0 \arrow[d, equal] \\
\mathbf{E}': \arrow[r] & 0 \arrow[r] & L \arrow[r] & M' \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

commutes.

2.3 Extensions

Chapter 3

Representation of Quivers

3.1 Quivers and Path Algebras

Definition 3.1.1. A *quiver* is defined as the tuple of sets and functions, $\mathbf{Q} = (Q_0,\, Q_1,\, s,\, t : Q_1 \to Q_0)$ such that:

- Q_0 is the set of vertices, which we will set to be the finite set $\{1, 2, \ldots, n\}$.
- Q_1 is the set of arrows, which we will also set to be finite.
- Functions s, t such that an arrow $\rho \in Q_1$ *starts* at the vertex $s(\rho) \in Q_0$ and *terminates* at the vertex $t(\rho) \in Q_0$, i.e. $\rho : s(\rho) \to t(\rho)$.

Example 3.1.2. A quiver $Q = (Q_0, Q_1, s, t : Q_1 \to Q_0)$ where $Q_0 = \{1, 2, 3, 4\}$, $Q_1 = \{\alpha, \beta\}$, and s, t are defined such that;

$$\begin{aligned} s : & Q_1 \to Q_0, \quad \alpha \mapsto 1,\, \beta \mapsto 2,\, \gamma \mapsto 4 \\ t : & Q_1 \to Q_0, \quad \alpha \mapsto 2,\, \beta \mapsto 3,\, \gamma \mapsto 3, \end{aligned}$$

looks like,

$$\mathbf{Q} : \quad \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"'] & 3 & 4 \arrow[l, "\gamma"'] \end{tikzcd}$$

Definition 3.1.3. A *non-trivial path*, p, in a quiver is a sequence of arrows $\rho_1, \ldots, \rho_n$ which satisfies $t(\rho_{i+1}) = s(\rho_i)$ for all $1 \leq i < n$, i.e. the start of an arrow is where the previous arrow terminated. The starting and terminatinating vertex of a path p are denoted $s(p)$ and $t(p)$, respectively.

Notation 3.1.4. In this report the arrows in a path will be ordered the same way as the composition of functions, as in [2], however, be aware that other publications may order the arrows the opposite way.

Definition 3.1.5. The *trivial path* is the path which contains no arrows, i.e. it is a single vertex, and is denoted e_i where the vertex is i.

Example 3.1.6. The paths of the quiver in Example 3.1.2 are:

$$p_1 = e_1, \quad p_2 = e_2, \quad p_3 = e_3, \quad p_4 = e_4, \quad p_5 = \alpha, \quad p_6 = \beta, \quad p_7 = \gamma, \quad p_8 = \beta\alpha.$$

However, $\gamma\beta\alpha$ is not a path because $t(\gamma) = 3 \neq s(\beta) = 2$.

Definition 3.1.7. A *path algebra* kQ is the k-alegbra which has the basis all the paths in Q, and the product of two paths p, q is defined as,

$$pq = \begin{cases} \text{obvious composition} & \text{if } t(q) = s(p), \\ 0 & \text{otherwise.} \end{cases}$$

This multiplication is assosciative.

Example 3.1.8. If we once again use the quiver Q from Example 3.1.2, then, from Example 3.1.6, we know the basis of the path algebra kQ will be,

$$\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \beta\alpha\}.$$

The product of β and α is $\beta\alpha$, but the product of α and β is zero, and the product of e_2 and α is just α.

Example 3.1.9. If Q is the following quiver,

$$\mathbf{Q}: \quad \begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

forms the path algebra $kQ \cong k[X, Y]$, the free, assosciative algebra on two letters. In fact, if we have a quiver with a single vertex and n loops, then this can be associated with the free, assosciated algebra on n letters.

Example 3.1.10. If we have the quiver,

$$\mathbf{Q}: \quad \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & 3 \arrow[r, "\gamma"] & 4 \end{tikzcd}$$

the the path algebra, $kQ \cong UT_4(k)$ by the isomorphism,

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 \alpha + \\ \lambda_6 \beta + \lambda_7 \gamma + \lambda_8 \beta\alpha + \lambda_9 \gamma\beta\alpha + \lambda_{10} \gamma\beta \mapsto \begin{pmatrix} \lambda_4 & \lambda_7 & \lambda_{10} & \lambda_9 \\ 0 & \lambda_3 & \lambda_6 & \lambda_8 \\ 0 & 0 & \lambda_2 & \lambda_5 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

Generally, a quiver of the form,

$$\mathbf{Q'}: \quad \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & \cdots \arrow[r, "\gamma"] & n \end{tikzcd}$$

induces a path alegbra $kQ' \cong UT_n(k)$ for any n.

Example 3.1.11. In fact, we find that if Q is the same quiver as above in Example 3.1.10, then $kQ \cong LT_4(k)$ as well, through the isomorphism,

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 \alpha + \\ \lambda_6 \beta + \lambda_7 \gamma + \lambda_8 \beta\alpha + \lambda_9 \gamma\beta\alpha + \lambda_{10} \gamma\beta \mapsto \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ \lambda_5 & \lambda_2 & 0 & 0 \\ \lambda_8 & \lambda_6 & \lambda_3 & 0 \\ \lambda_9 & \lambda_{10} & \lambda_7 & \lambda_4 \end{pmatrix}.$$

Once again, this extends to the general case that, $kQ' \cong LT_n(k)$.

Example 3.1.12. In a more general case, as long as there is only one path between any two vertices, we can identify kQ with the following subalgebra of $M_n(k)$,

$$\{M \in M_n(k) : M_ij = 0 \text{ if there is no path from } j \text{ to } i\}.$$

For instance, if we have the quiver,

Q :

$$\begin{tikzcd}
& & & 5 \arrow[r, "\epsilon"] \arrow[dl, "\delta"'] & 6 \\
1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & 3 \arrow[dr, "\gamma"] & & \\
& & & 4 &
\end{tikzcd}$$

the we can see that the path algebra kQ is isomorphic to the subalgebra with matrices of the form,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Remark 3.1.13. Note that the idea here has some parallels to similar results for directed graphs in graph theory.

The following results about the idempotents of path alegebras are from [2], however, we have either given a proof or expanded upon the one given. For the following results we set $A = kQ$ and the e_i are the trivial paths of Q.

Lemma 3.1.14. *The e_i are orthogonal, idempotents in A, i.e. $e_ie_i = e_i$ and $e_ie_j = 0$ where $i \neq j$. Thus $\sum_{i=1}^{n} e_i = 1_A$.*

Proof. Well, obviously, $e_ie_j = 0$ when$i \neq j$ because $t(e_j) \neq s(e_i)$ because they are the trivial paths at different vertices, i and j. Similarly, if we have the product e_ie_i then the composition makes sense here because $t(e_i) = s(e_i)$, but the composition is just e_i becuase if we travel along the trivial path e_i twice, then this is just the same as travelling the trivial path.
If we have some path p of the quiver, then bear in mind that,

$$e_ip = \begin{cases} e_ip = p & \text{if } i = t(p), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the sum of the trivial paths acting on our path p becomes,

$$\big(\sum_{i=1}^{n} e_i\big)p = (e_1 + e_2 + \cdots + e_n)p = e_ip = p = 1_Ap,$$

because only one of the e_i in the sum will satisfy $i = t(p)$ and all the rest will give zero. Similary, we can show that $p\big(\sum_{i=1}^{n} e_i\big) = p1_A$. $\square$

Lemma 3.1.15. *The bases of the spaces Ae_i and e_jA are all the paths starting at i and all the paths terminating at j, respectively. It then follows that the space e_jAe_i has as a basis the paths that start at i and terminate at j.*

Proof. For some $i \in Q_0$, and A has the basis $\{p_1, p_2, \dots, p_n\}$, then as Ae_i is a subspace of A the basis of Ae_i must be a subset of the basis of A, so,

$$Ae_i = \{ae_i : a \in A\} = span\{p_1e_i + p_2e_i + \dots + p_ne_i\},$$

and we know that,

$$p_re_i = \begin{cases} p_r & \text{if } s(p_r) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we can see that the basis of Ae_i must be all the paths starting at i. Similarly, as e_jA is a subspace of A snd so following a similar argument we can see that its basis must be all the paths terminating at j. The result for e_jAe_i follows from these as it simply the intersection of the two spaces. □

Lemma 3.1.16. *$A = \oplus_{i=1}^n Ae_i$, so each Ae_i is a projective left A-module.*

Proof. From Lemma 3.1.15 we know that for each i, the basis of Ae_i are all the paths starting at i and so $\oplus_{i=1}^n Ae_i$ must have as a basis all the paths starting at every vertex in Q, hence all the paths in Q. Thus $A = \oplus_{i=1}^n Ae_i$. Also, each Ae_i is obviously a left A-module with the action defined as multiplication by e_i. Hence, each Ae_i is a projective, left A-module. □

Remark 3.1.17. Similarly, $A = \oplus_{i=1}^n e_jA$, so each e_jA is a projective, right A-module.

Lemma 3.1.18. *If X is a left A-module, then $Hom_A(Ae_i, X) \cong e_iX$.*

Proof. Well, we want to show that there are some maps f and g such that they satisfy,

$$Hom_A(Ae_i, X) \overset{f}{\underset{g}{\leftrightarrow}} e_iX$$

Well, first consider the map $\theta : Ae_i \to X$ and so $\theta \in Hom_A(Ae_i, X)$, then we can have f such that $f(\theta) = \theta(e_i) = \theta(e_i^2) = e_i\theta(e_i) \in e_iX$. Hence, f maps a homomorphism from $Hom_A(Ae_i, X)$ to an element in e_iX. Now consider the map $g(x) : Ae_i \to X$, $g(x)(a) = ax$. We can check this is an A-module homomorphism as,

$$\begin{aligned} g(x)(a+b) &= (a+b)x = ax + bx = g(x)(a) + g(x)(b) && a, b \in Ae_i, \\ g(x)(\lambda a) &= \lambda ax = \lambda g(x)(a) && \lambda \in A, a \in Ae_i. \end{aligned}$$

Hence, $g \in Hom_A(Ae_i, X)$. However, we can also have that $g : eX \to Hom_A(Ae_i, X)$ through $x \mapsto g(x)(r)$.
So now we need to show that f and g are inverse contructions of one another. We have for some $\theta \in Hom_A(Ae_i, X)$,

$$\begin{aligned} \theta &\xrightarrow{f} \theta(e_i) \\ g(\theta(e_i))(a) = a\theta(e_i) = \theta(ae_i) &\xleftarrow[g]{} \theta(e_i) \end{aligned}$$

but $a \in Ae_i$ and so $a = \lambda e_i$ for some $\lambda \in A$, hence $ae_i = \lambda e_i^2 = \lambda e_i = a$. Thus, $g(\theta(e_i))(a) = \theta(ae_i) = \theta(a)$ so $g(\theta(e_i)) = \theta$ and f and g are inverses. □

Lemma 3.1.19. *If $0 \neq a \in Ae_i$ and $0 \neq b \in e_iA$ then $ab \neq 0$.*

Proof. We know that a and b must have the forms, $a = \lambda_1 p_1 + \dots \lambda_n p_n$ and $b = \mu_1 q_1 + \dots \mu_m q_m$ where the p are paths starting at vertex i, the q are paths termination at vertex i and p_r, q_r are paths of length r. Then the longest path in the product ab must be $\lambda_n \mu_m p_n q_m$ and so $\lambda_n \mu_m \neq 0$ and $p_n q_m \neq 0$, so the product $ab \neq 0$. □

Lemma 3.1.20. *The e_i are primitive idempotents, i.e. Ae_i is an indecomposable module.*

Proof. From Lemma 3.1.18 we know that $End_A(Ae_i) \cong e_iAe_i$ and if this contains an idempotent ϵ, then $\epsilon^2 = \epsilon = \epsilon e_i$, so $\epsilon(e_i - \epsilon) = 0$, but from Lemma 3.1.19 we know that this can not happen if $\epsilon, (e_i - \epsilon) \neq 0$, thus we must have that either $\epsilon = 0$ or $\epsilon = e_i$ and the result follows. □

Lemma 3.1.21. *If $e_i \in Ae_jA$ then $i = j$.*

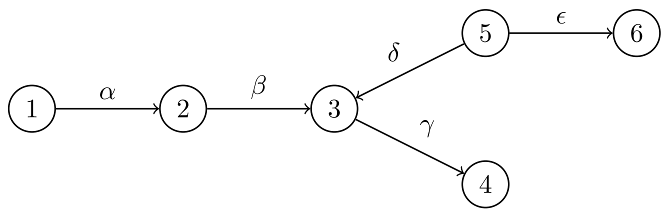
Proof. For similar reasoning as in Lemma **??**, we can see that Ae_jA has as a basis all the paths in A which pass through the vertex j and, by the definition of the trivial path, e_i cannot pass through the vertex j unless $i = j$ and so if $e_i \in Ae_jA$ we must have that $i = j$. □

Lemma 3.1.22. *The e_i are inequivalent, i.e. $Ae_i \ncong Ae_j$ for $i \neq j$.*

Proof. Two idempotent elements e_i, e_j are said to be equivalent iff there exists some $u \in e_iAe_j$ and $v \in e_jAe_i$ such that $uv = e_i$ and $vu = e_j$. From Lemma 3.1.18 we can see that $Hom(Ae_i, Ae_j) \cong e_iAe_j$ and $Hom(Ae_j, Ae_i) \cong e_jAe_i$, and so inverse isomorphisms give elements u and v as described above with $uv = e_i$ and $vu = e_j$. However, this means that the path e_i must pass through the vertex j at some point, and vice versa, which contradicts Lemma . □

Example 3.1.23. Consider the quiver from Example 3.1.12,

Q :



then we have have the following examples displaying the previous results for the idempotents of kQ. Let $A = kQ$.

1. We can see that $e_1e_1 = e_1$ and $e_1e_2 = 0$. Also, $1_A = e_1 + e_2 + e_3 + e_4 + e_5 + e_6$ and so

$$\begin{aligned}
1_A\gamma\beta &= (e_1 + e_2 + e_3 + e_4 + e_5 + e_6)\gamma\beta, \\
&= e_1\gamma\beta + e_2\gamma\beta + e_3\gamma\beta + e_4\gamma\beta + e_5\gamma\beta + e_6\gamma\beta, \\
&= e_4\gamma\beta, \\
&= \gamma\beta.
\end{aligned}$$

2. Now we can see that Ae_1 is spanned by all the paths starting at vertex 1, as,

$$\begin{aligned}
Ae_1 &= \{ae_1 : a \in A\}, \\
&= span_k\{\sum pe_1 : p \text{ is in the basis of } A\}, \\
&= span_k\{e_1e_1 + e_2e_1 + e_3e_1 + e_4e_1 + e_5e_1 + e_6e_1 + \alpha e_1 + \\
&\qquad \beta e_1 + \gamma e_1 + \delta e_1 + \epsilon e_1 + \beta\alpha e_1 + \gamma\beta e_1 + \gamma\delta e_1 + \gamma\beta\alpha e_1\}, \\
&= span_k\{e_1 + \alpha + \beta\alpha + \gamma\beta\alpha\}, \\
&= span_k\{\text{all paths starting at vertex 1}\}.
\end{aligned}$$

Similarly, we can see that,

$$\begin{aligned}
e_3A &= \{e_3a : a \in A\}, \\
&= span_k\{e_3 + \beta + \delta + \beta\alpha\}, \\
&= span_k\{\text{all the paths terminating at vertex 3}\}.
\end{aligned}$$

Then we can see that,

$$\begin{aligned}
e_3Ae_1 &= \{e_3ae_1 : a \in A\}, \\
&= span_k\{\beta\alpha\}, \\
&= span_k\{\text{all paths starting at vertex 1 and terminating at vertex 3}\}.
\end{aligned}$$

3. We can see that,

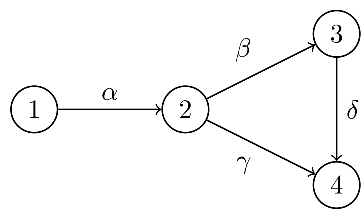
$$\begin{aligned}
\oplus_{i=1}^{6} Ae_i &= Ae_1 \oplus Ae_2 \oplus Ae_3 \oplus Ae_4 \oplus e_5 \oplus e_6, \\
&= span_k\{e_1 + \alpha + \beta\alpha + \gamma\beta\alpha\} \oplus span_k\{e_2 + \beta + \gamma\beta\} \oplus span_k\{e_3 + \gamma\} \oplus span_k\{e_4\} \oplus span_k\{e_5 + \delta \\
&= A.
\end{aligned}$$

The following properties of path algebras are from [2], where they are given as exercises. In this report we will present them with proofs. Once again $A = kQ$.

Lemma 3.1.24. *A is finite dimensional if and only if Q has no oriented cycles.*

Proof. $\Rightarrow$: If Q has an oriented cycle then it will have a infinite number of paths as you can keep going round the cycle. This means that the basis of A, which is all the paths in Q, will be infinite, and so A will not be finite dimensional.
$\Leftarrow$: If Q has no oriented cycles then it must have a finite number of paths and so the basis of A will be finite and hence A will be finite dimensional. $\square$

Example 3.1.25. If we consider the quiver Q,

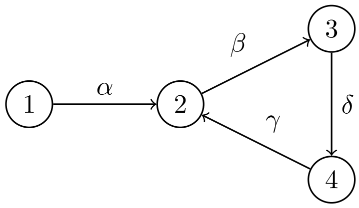


Q :

and we can see that Q has the paths $e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \beta\alpha, \gamma\alpha, \beta\delta, \delta\beta\alpha$, and so the basis of A is $\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \beta\alpha, \gamma\alpha, \beta\delta, \delta\beta\alpha\}$, which is finite, and

so A is finite dimensional.
However, if we have the quiver Q',

$\mathbf{Q'}$:



it has paths $e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \beta\alpha, \delta\beta, \gamma\delta, \delta\beta\alpha, \gamma\delta\beta\alpha, \beta\gamma\delta\beta\alpha, \delta\beta\gamma\delta\beta\alpha, \ldots$, and so on, an infinte number of paths, meaning that the basis of A is infinte and hence A is not finite dimensional.

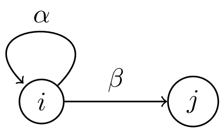
Lemma 3.1.26. *A is prime, i.e. $IJ \neq 0$ for two sided ideals $I, J \neq 0$ if and only if for all $i, j \in Q_0$ there exists a path i to j.*

Proof. *Need to include proof here* $\square$

Lemma 3.1.27. *A is left noetherian if and only if, if there is an oriented cycle through the vertex i, then only one arrow starts at the vertex i.*

Proof. Let Ap represent the left-ideal whose basis are all the paths starting with p for any path p.
Suppose we have a quiver Q of the form,

$\mathbf{Q}$:



assuming $i \neq j$ and where α represents an orientated cycle, then we can see that,

$$A\beta + A\beta\alpha + \cdots + A\beta\alpha^n \subseteq A\beta + A\beta\alpha + \cdots + A\beta\alpha^{n+1}.$$

However, since $e_j\beta\alpha^{n+1} \notin A\beta + A\beta\alpha + \cdots + A\beta\alpha^n$ the inclusion is strict. Hence, if Q is of the form above, there is an ascending chain of left ideals,

$$A\beta \subset A\beta + A\beta\alpha \subset A\beta + A\beta\alpha + A\beta\alpha^2 \subset \ldots$$

which does not terminate and so A is not noetherian.
Suppose that for all $i \in Q_0$, if α an oriented cycle where $s(\alpha) = i$, then for all $\rho \in Q_1$, $\rho = \alpha_1$ or $s(\rho) \neq i$; where $\alpha = \alpha_1\alpha_2 \ldots \alpha_m$ for some $\alpha_1, \ldots, \alpha_m \in Q_1$. This means there are some $\rho_1, \ldots, \rho_n$ paths such that given any path $\beta \in A$ we have that $\beta = q\rho_j$ for some $j \in \{1, \ldots, n\}$ and some q a path in an oriented cycle. Now if we let Q' be the quiver, and A' the corresponding path algebra, where,

$$Q'_0 = Q_0 \ \& \ Q_1 = Q'_1 \cup \{a \in Q_1 : a \text{ is not in an oriented cycle of } Q\}.$$

So, for any basis element β of A, $\beta = qp_j$ where q is a path in A'. Following the previous two results we can see that,

$$A = A'\rho_1 + A'\rho_2 + \cdots + A'p_n \tag{3.1}$$

Claim. *If S is a subring of a ring R and R is finitely generated as a left S-module, then, if S is a noetherian ring, so is R.*

Proof. From the result that a finitely generated module over a noetherian ring is noetherian, gives us that R is noetherian as a left S-module. Let $I_1 \subseteq I_2 \subseteq \dots$ be an ascending chain of left ideals in R, and each I_k is a left S-submodule of R; so by the above, this chain terminates. $\square$

Using the above Claim and the earlier result that A is finitely generated as a left A'-module, we can see that our problem of proving that A is left noetherian reduces to proving that A' is left noetherian.
Now let $\alpha_j^{(l)}$ be the j^{th} arrow of the l^{th} oriented cycle in Q' and then,

$$Q_1' = \{\alpha_j^{(l)} : j \in \{1, \dots, q_l\}, l \in \{1, \dots, r\}\}.$$

Consider the endomorphism,

$$\begin{aligned} X_l :& A' \to A', \\ & e_t(p) \mapsto \alpha_{k-1}^{(l)}\alpha_{k-2}^{(l)} \dots \alpha_1^{(l)}\alpha_{q_l}^{(l)} \dots \alpha_k^{(l)} p, \text{ where } t(p) = t = s(\alpha_k^{(l)}), \\ & \text{otherwise } \mapsto 0. \end{aligned}$$

We can adapt Hilbert's Basis Theorem to show that $k[X_1, \dots, X_r]$ is a noetherian ring and we can see that A' is a finitely generated $k[X_1, \dots, X_r]$-module. Hence, A' is left noetherian, and so A is left noetherian. $\square$

Definition 3.1.28. We can define the *length* of any path p by the following: Let,

$$length(p) = length(\sum_{\text{arrows } \rho} \lambda_\rho \rho) := max\{length(\rho) : \lambda_\rho \neq 0\},$$

where $\lambda_\rho \neq 0$ for some ρ.

Lemma 3.1.29. *The basis of $J(A)$ is $\{$path i to j : there is no path from j to $i\}$.*

Proof. Firstly, lets prove that,

$$J'(A) := span_k\{\text{paths } p : s(p) = i, t(p) = j \text{with no paths from } j \text{ to } i\} \subseteq J(A).$$

Given any $w \in J'(A)$ we have that,

1. $(wp)^2 = 0$ for any path p.

   Proof. There are no paths p such that $t(p) = s(w)$ and $s(p) = t(w)$, so $wpw = 0$, so $wpwp = (wp)^2 = 0$. $\square$

2. $(w(p + p'))^2 = 0$ for any path p and $\lambda \in k$.

   Proof.

   $$(w(p + p'))^2 = (wp + wp')^2 = (wp)^2 + wpwp' + wp'wp + (wp')^2$$

   As wpw and $wp'w$ are both equal to zero by the above, we have that, $(w(p + p'))^2 = 0$. $\square$

3. $(w(\lambda p))^2 = \lambda^2 (wp)^2 = 0$

   Proof. By item 1. $\square$

Now let $w \in J'(A)$ and $z \in A$, then,

$$wz = u(\sum_{\text{paths } p} \lambda_p p) = w(\sum_{\substack{\text{paths } p: \\ t(p)=s(w)}} \lambda_p p) := \tau$$

and from above we can see that $\tau^2 = 0$, meaning $(wz)^2 = 0$, so, $(1 + wz)(1 - wz) = 1$, hence, $1 + wz \in U(A)$ for all $z \in A$. Thus $w \in J(A)$.
Now we want to prove that $J(A) \subseteq J'(A)$. Let w be a basis element of $J(A)$ coming from the basis of A. Suppose also that $length(w) = 0$, so $w = e_i$ for some $i \in Q_0$. Let,

$$M_i := \sum_{\substack{\text{paths } p: \\ p \neq e_i}} Ap,$$

then $M_i \trianglelefteq A$ and if $M_i \subsetneq N \trianglelefteq A$, then, $e_i \in N$, so, $N = A$; hence, M_i is a maximal left ideal, but $w \notin M_i$ contradicting that $w \in J(A)$.Hence, $length(w) > 0$.
Now suppose $w \notin J'(A)$, so there's a path p such that $s(w) = t(p)$ and $s(p) = t(w)$. Since $w \in J(A)$ and $p \in A$, $1 + wp \in U(A)$, so there exists some $z \in A$ such that $(1 + wp)z = 1$, with,

$$z = \sum_{l=0}^{m} \sum_{\substack{\text{paths } p: \\ length(p_l)=l}} \lambda_{p_l} p_l.$$

Since, $length(1) = length(l_1 + \cdots + l_n) = 0$, we have that,

$$\begin{aligned} 0 &= length((1 + wp)z), \\ &= length(z + wpz), \\ &= max(length(z), length(wpz)), \end{aligned}$$

so $wpz = 0$ as otherwise $length(wpz) = 0$, contradicting that $length(w) > 0$. So as $(1 + wp)z = z + wpz = 1$, $z = 1$, so $wp = 0$, a contradiction. $\square$

Lemma 3.1.30. *The centre of A is $k \times k \times \dots k[T] \times k[T] \times \dots$, with one factor for each connected component C of Q, and that the factor is $k[T]$ if and only if C is an oriented cycle.*

Proof. Firstly, we know that if our quiver Q is composed of n connected components $C_1, \dots, C_n$ then we our path algebra looks like $kQ = kC_1 \times kC_2 \times \cdots \times kC_n$.

Claim. *Where $Z(A)$ represents the centre of A, we have that,*

$$Z(A \times B) = Z(A) \times Z(B).$$

Proof. Suppose we have $(a, b) \in Z(A \times B)$, then we have that,

$$\begin{aligned} &(a, b)(a', b') = (a', b')(a, b) \forall (a', b') \in A \times B, \\ \Leftrightarrow &(aa', bb') = (a'a, b'b) \forall a' \in A, b' \in B, \\ \Leftrightarrow &aa' = a'a \ \& \ bb' = b'b \forall a' \in A, b' \in B, \\ \Leftrightarrow &a \in Z(A) \ \& \ b \in Z(B). \end{aligned}$$

Hence, $Z(A \times B) \cong Z(A) \times Z(B)$. $\square$

Now, using the claim, we can see that,

$$Z(kQ) \cong Z(kC_1) \times Z(kC_2) \times \cdots \times Z(kC_n),$$

Now, assume the connected component C_i is not an oriented cycle, and consider $a \in Z(kC_i)$, which is a linear combination of paths. Let us choose a path of maximal length $\rho_1 \ldots \rho_m$, and say there exists some σ such that we can have $\sigma\rho_1 \ldots \rho_m$ but with $t(\sigma) \neq s(\rho_m)$, but since $\rho_1 \ldots \rho_m$ was maximal this path is involved $\sigma a = a\sigma$ as $a \in Z(A)$. So, $a\sigma$ is also a linear combination of paths, i.e. $\tau_1 \ldots \tau_m \sigma$, wheere $\tau_1 \ldots \tau_m$ is a path in a.This causes,

$$\sigma\rho_1 \ldots \rho_m = \tau_1 \ldots \tau_m \sigma,$$

which implies that $\sigma = \rho_m$, which causes an oriented cycle, hence a contradiction. Thus, $Z(kC_i) \cong k$.
Need to complete the case for oriented cycle connected component.

□

Example 3.1.31. Consider the quiver,

$$\mathbf{Q}: \quad \begin{tikzcd} & & & & 5 \arrow[dr, "\delta"] & \\ 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & 3 & 4 \arrow[ur, "\gamma"] & & 6 \arrow[ll, "\epsilon"] \end{tikzcd}$$

and so, $Z(kQ) = Z(kC_1) \times Z(kC_2)$, where C_2 is the oriented cycle component. Then,

$$\begin{aligned} Z(kC_1) =& \{a \in kC_1 : ac = ca \forall c \in kC_1\}, \\ &= \{\lambda(e_1 + e_2 + e_3) : \lambda \in k\}, \\ &\cong k, \end{aligned}$$

since none of the elements of kC_1 are commutative, apart from the identity. Also,

$$\begin{aligned} Z(kC_2) &= \{a \in kC_2 : ac = ca \forall c \in kC_2\}, \\ &= \{\lambda(\gamma\delta\epsilon + \delta\epsilon\gamma + \epsilon\gamma\delta) : \lambda \in k\}, \\ &\cong k[T], \text{ where } T = \gamma\delta\epsilon + \delta\epsilon\gamma + \epsilon\gamma\delta, \end{aligned}$$

because none of the elements of kC_2 are commutative other than the identity element. Thus, $Z(kQ) \cong k \times k[T]$.

3.2 Representations of Quivers

Definition 3.2.1. A *representation* X of a quiver Q is given by considering each vertex $i \in Q_0$ as a vector space X_i, and each arrow $\rho \in Q_1$ as a linear map $X_\rho : X_{s(\rho)} \to X_{t(\rho)}$.

Example 3.2.2. Let Q be the quiver,

$$\mathbf{Q}: \quad \begin{tikzcd} 1 & 2 \arrow[l] \arrow[r] & 3 \end{tikzcd}$$

then we can have representations X and Y,

$$\mathbf{X}: \quad \begin{tikzcd} k & k \arrow[l, "\alpha"', "1"] \arrow[r, "\beta", "1"'] & k \end{tikzcd} \qquad \& \qquad \mathbf{Y}: \quad \begin{tikzcd} k & k \arrow[l, "\gamma"', "1"] \arrow[r, "\delta"] & 0 \end{tikzcd}$$

Definition 3.2.3. A *morphism* $\theta : X \to X'$ between representationsis given by linear maps $\theta_i : X_i \to X'_i$ for each $i \in Q_0$ satisfy $X'_\rho \theta_{s(\rho)} = \theta_{t(\rho)} X_\rho$ for each $\rho \in Q_1$. The *composition* of morphisms, $\theta : X \to X'$ with $\phi : X' \to X''$ is given by $(\phi\theta)_i = \phi_i \theta_i$.

Example 3.2.4.

3.3 Standard Resolution

3.4 Bricks

3.5 Dynkin and Euclidean diagrams

3.6 Gabriel's Theorem

Chapter 4

Auslander-Reiten Quivers

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