Representation of Quivers

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Contents

Chapter 1

Introduction

Chapter 2

Homological Algebra

2.1 Chain Complexes

Definition 2.1.1. A *chain complex* $\mathbf{C}_\bullet$ consists of a sequence of $\mathbb{R}$-modules C_i ($i \in \mathbb{Z}$) and morphisms of the form,

$$\mathbf{C}: \qquad \ldots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} C_{-2} \xrightarrow{\delta_{-2}} \ldots$$

such that $\delta_{n-1}\delta_n = 0$ for all n, i.e. the composition of any two consecutive maps is zero. The maps δ_n are called the *differentials* of C.

Remark 2.1.2. It is convention that the map δ_n starts at C_n.

Example 2.1.3. If we have a field K then we can create the following chain complex:

$$\mathbf{C}: \qquad \ldots \to 0 \to K^2 \xrightarrow{\left(\begin{smallmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 0 \end{smallmatrix}\right)} K^3 \xrightarrow{(0\ 0\ 1)} K \to 0 \to \ldots$$

We can clearly see that the maps uphold the $\delta^2 = 0$ condition as,

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

Example 2.1.4. If we consider the sequence,

$$\ldots \to 0 \to K \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} K^2 \xrightarrow{(1\ 0)} K \to 0 \to \ldots$$

however,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \neq 0.$$

Hence, $\delta^2 \neq 0$ and so the sequence is not a chain complex. However, if we change the second map slightly we obtain the chain complex,

$$\mathbf{C}: \qquad \ldots \to 0 \to K \xrightarrow{\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)} K^2 \xrightarrow{(0\ 1)} K \to 0 \to \ldots$$

since,

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 1 \neq 0.$$

Definition 2.1.5. If $\mathbf{C}$ is a chain complex then its *homology* is defined to be,

$$H_n(\mathbf{C}) = \frac{Ker(\delta_n : C_n \to C_{n-1})}{Im(\delta_{n+1} : C_{n+1} \to C_n)} = \frac{Z_n(\mathbf{C})}{B_n(\mathbf{C})}.$$

This becomes an $\mathbb{R}$-module and, since δ^2, it follows that $B_n(\mathbf{C}) \subseteq Z_n(\mathbf{C})$.

Examples 2.1.6 and 2.1.9 are taken from [1] and are included here because they are felt to be the clearest at demonstrating a chain complex and homology, however, the more general statement of the second example is presented as Proposition 2.1.7 because it is an interesting result.

Example 2.1.6. If we take a module M then we can make a chain complex;

$$\mathbf{C}: \qquad \ldots \to 0 \to M \to 0 \to \ldots$$

where M is at degree n. Then the homology will be:

$$H_i(\mathbf{C}) = \begin{cases} \frac{Ker(M \to 0)}{Im(0 \to M)} = M & i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.1.7. *If we have a module homomorphism between R-modules, $f : M \to N$, the we get the chain complex,*

$$\mathbf{C}: \qquad \underset{deg}{\ldots} \to \underset{n+2}{0} \to \underset{n+1}{M} \xrightarrow{f} \underset{n}{N} \to \underset{n-1}{0} \to \ldots,$$

and the homology becomes,

$$H_i(\mathbf{C}) = \begin{cases} \frac{N}{Im(f)} = Coker(f) & i = n \\ Ker(f) & i = n+1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Firstly, at degree n we have that,

$$H_n(\mathbf{C}) = \frac{Ker(N \to 0)}{Im(M \xrightarrow{f} N)} = \frac{N}{Im(f)} = Coker(f).$$

Then at degree $n+1$ we have that,

$$H_{n+1}(\mathbf{C}) = \frac{Ker(M \xrightarrow{f} N)}{Im(0 \to M)} = Ker(f).$$

Finally, it is clear that everywhere else there is no homology. □

Notation 2.1.8. Here,

$$Coker(f) = \frac{\text{Codomain of } f}{\text{Image of } f},$$

is the *cokernel* of the map f.

Example 2.1.9. We can have a chain complex of $\mathbb{Z}$-modules,

$$\mathbf{C}: \qquad \underset{\text{deg}}{\ldots} \to \underset{2}{0} \to \underset{1}{\mathbb{Z}} \xrightarrow{a} \underset{0}{\mathbb{Z}} \to \underset{-1}{0} \to \ldots$$

where the map a is right multiplication by some $a \in \mathbb{Z}$. The homology is,

$$H_i(\mathbf{C}) = \begin{cases} \frac{Ker(\mathbb{Z} \to 0)}{Im(\mathbb{Z} \xrightarrow{a} \mathbb{Z})} = \frac{\mathbb{Z}}{a\mathbb{Z}} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that,

$$H_0(C) = \frac{\mathbb{Z}}{a\mathbb{Z}} = \frac{\text{Codomain of } f}{\text{Image of } f} = Coker(a).$$

Also,

$$H_1(C) = \frac{Ker(\mathbb{Z} \xrightarrow{a} \mathbb{Z})}{Im(0 \to \mathbb{Z})} = Ker(a) = 0,$$

because $Ker(a)$ is empty.

Definition 2.1.10.
- The elements of $B_n(\mathbf{C})$ are called $n-$*boundaries*.
- The elements of $Z_n(\mathbf{C})$ are called $n-$*cycles*.

Remark 2.1.11. If $x \in Z_n(\mathbf{C})$ then its image in $H_n(\mathbf{C})$ is usually written as $[x]$.

Definition 2.1.12. A chain complex $\mathbf{C}$ is said to be:

- *acyclic* if $H_n(\mathbf{C}) = 0$ for all n.
- *bounded above* if there exists some $n \in \mathbb{N}$, $C_k = 0$ for all $k > n$.
- *bounded below* if for some $n \in \mathbb{N}$, $C_k = 0$ for all $k < n$.
- *bounded* if it is bounded above and below.
- *non-negative* if $C_n = 0$ for $n < 0$.

Example 2.1.13. All the chain complexes in the previous examples are bounded both above and below, however, neither is acyclic as they both have instances where the homology is non-zero. The chain complex in Example 2.1.9 is non-negative because $C_n \neq 0$ only when $n = 0, 1$.

Example 2.1.14. If we take another look at the chain complex in Example 2.1.4,

$$\mathbf{C}: \qquad \underset{\text{deg}}{\ldots} \to 0 \to \underset{1}{K} \xrightarrow{\binom{1}{0}} \underset{0}{K^2} \xrightarrow{(0\ 1)} \underset{-1}{K} \to 0 \to \ldots$$

the homologies are,

$$H_1(\mathbf{C}) = \frac{Ker(K \xrightarrow{\binom{1}{0}} K^2)}{Im(0 \to K)} \cong \frac{K}{K} \cong 0,$$

$$H_0(\mathbf{C}) = \frac{Ker(K^2 \xrightarrow{(1\ 0)} K)}{Im(K \xrightarrow{\binom{1}{0}} K^2)} \cong \frac{K}{K} \cong 0,$$

$$H_{-1}(\mathbf{C}) = \frac{Ker(K \to 0)}{Im(K^2 \xrightarrow{(1\ 0)} K))} \cong \frac{K}{K} \cong 0.$$

Thus $H_n(\mathbf{C}) = 0$ for all n and so $\mathbf{C}$ is an acyclic chain complex. Later in the report, we will see that $\mathbf{C}$ is in fact a short exact sequence.

Example 2.1.15. The chain complex,

$$\mathbf{C}: \qquad \underset{\text{deg}}{\dots} \xrightarrow{3} \underset{1}{\frac{\mathbb{Z}}{9\mathbb{Z}}} \xrightarrow{3} \underset{0}{\frac{\mathbb{Z}}{9\mathbb{Z}}} \xrightarrow{3} \underset{-1}{\frac{\mathbb{Z}}{9\mathbb{Z}}} \to \dots$$

where the differentials are the maps,

$$\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \to \frac{\mathbb{Z}}{9\mathbb{Z}},\ z + 9\mathbb{Z} \mapsto 3z + 9\mathbb{Z},$$

is unbounded. It is also acyclic, since the homology is,

$$H_n(\mathbf{C}) = \frac{Ker(\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \to \frac{\mathbb{Z}}{9\mathbb{Z}})}{Im(\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \to \frac{\mathbb{Z}}{9\mathbb{Z}})} \cong \frac{\mathbb{Z}/3\mathbb{Z}}{\mathbb{Z}/3\mathbb{Z}} \cong 0,$$

for all n.

Definition 2.1.16. A *cochain complex* $\mathbf{C}^\bullet$ consists of a sequence of $\mathbb{R}$-modules C^i $(i \in \mathbb{Z})$ and morphisms of the form,

$$\mathbf{C}: \qquad \dots \xrightarrow{\delta^{-3}} C^{-2} \xrightarrow{\delta^{-2}} C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \dots$$

such that $\delta^{n-1}\delta^n = 0$ for all n, i.e. the composition of any two consecutive maps is zero.

Remark 2.1.17. Chain and cochain complexes can be thought of as almost identical constructs with the only difference being thenumbering of the chain. The degree of a chain complex *decreases* from left to right, whereas, the degree of a cochain complex *increses* from left to right. So, we can compute one from the other by setting $C^{-n} = C_n$, or equivalently $C^n = C_{-n}$; this is called *renumbering*.

Definition 2.1.18. If $\mathbf{C}$ is a cochain complex then its *cohomology* is defined to be,

$$H^n(\mathbf{C}) = \frac{Ker(\delta^n : C^n \to C^{n+1})}{Im(\delta^{n-1} : C^{n-1} \to C^n)} = \frac{Z^n(\mathbf{C})}{B^n(\mathbf{C})}.$$

- The elements of $B_n(\mathbf{C})$ are called $n-$*coboundaries*.
- The elements of $Z_n(\mathbf{C})$ are called $n-$*cocycles*.

Example 2.1.19. We can renumber the chain complex in Example 2.1.9 to get the cochain complex,

$$\mathbf{C}: \qquad \underset{\deg}{\dots} \to \underset{-2}{0} \to \underset{-1}{\mathbb{Z}} \xrightarrow{a} \underset{0}{\mathbb{Z}} \to \underset{1}{0} \to \dots$$

Its cohomology is,

$$H^i(\mathbf{C}) = \begin{cases} \frac{Ker(\mathbb{Z} \longrightarrow 0)}{Im(\mathbb{Z} \xrightarrow{a} \mathbb{Z})} = \frac{\mathbb{Z}}{a\mathbb{Z}} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Chapter 3

Representation of Quivers

3.1 Quivers and Path Algebras

Definition 3.1.1. A *quiver* is defined as the tuple of sets and functions, $\mathbf{Q} = (Q_0,\, Q_1,\, s,\, t : Q_1 \to Q_0)$ such that:

- Q_0 is the set of vertices, which we will set to be the finite set $\{1, 2, \ldots, n\}$.
- Q_1 is the set of arrows, which we will also set to be finite.
- Functions s, t such that an arrow $\rho \in Q_1$ *starts* at the vertex $s(\rho) \in Q_0$ and *terminates* at the vertex $t(\rho) \in Q_0$, i.e. $\rho : s(\rho) \to t(\rho)$.

Bibliography

[1] *Cohomology and Central Simple Algebras*. [Online-PDF file]. [Accessed October 2014]. Available from: http://www1.maths.leeds.ac.uk/ pmtwc/cohom.pdf

[2] *Representation of Quivers*. [Online-PDF file]. [Accessed October 2014]. Available from: http://www1.maths.leeds.ac.uk/ pmtwc/quivlecs.pdf