Representation of Quivers

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Homological Algebra

2.1 Chain Complexes

Definition 2.1.1. A *chain complex* $C_\bullet$ consists of a sequence of $\mathbb{R}$-modules C_i ($i \in \mathbb{Z}$) and morphisms of the form,

$$C: \qquad \ldots \to C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} C_{-2} \xrightarrow{\delta_{-2}} \ldots$$

such that $\delta_{n-1}\delta_n = 0$ for all n, i.e. the composition of any two consecutive maps is zero. The maps δ_n are called the *differentials* of C.

Remark 2.1.2. It is convention that the map δ_n starts at C_n.

Example 2.1.3. If we have a field K then we can create the following chain complex:

$$C: \qquad \ldots \to 0 \to K^2 \xrightarrow{\left(\begin{smallmatrix}1&3&0\\2&0&0\end{smallmatrix}\right)} K^3 \xrightarrow{\left(\begin{smallmatrix}0\\0\\1\end{smallmatrix}\right)} K^1 \to 0 \to \ldots$$

We can clearly see that the maps uphold the $\delta^2 = 0$ condition as,

$$\begin{pmatrix}1&3&0\\2&0&0\end{pmatrix}\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0&0\end{pmatrix}.$$

Definition 2.1.4. If C is a chain complex then its *homology* is defined to be,

$$H_n(C) = \frac{Ker(\delta_n : C_n \to C_{n-1})}{Im(\delta_{n+1} : C_{n+1} \to C_n)} = \frac{Z_n(C)}{B_n(C)}.$$

This becomes an $\mathbb{R}$-module and, since δ^2, it follows that $B_n(C) \subseteq Z_n(C)$.

Examples 2.1.5 and 2.1.8 are taken from [1] and are included here because they are felt to be the clearest at demonstrating a chain complex and homology, however, the more general statement of the second example is presented as Proposition because it is an interesting result.

Example 2.1.5. If we take a module M then we can make a chain complex;

$$C' : \qquad \ldots \to 0 \to M \to 0 \to \ldots$$

where M is at degree n. Then the homology will be:

$$H_i(C') = \begin{cases} \frac{Ker(M \to 0)}{Im(0 \to M)} = M & i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.1.6. *If we have a module homomorphism between R-modules, $f : M \to N$, the we get the chain complex,*

$$C : \qquad \underset{deg}{\ldots} \to \underset{n+2}{0} \to \underset{n+1}{M} \xrightarrow{f} \underset{n}{N} \to \underset{n-1}{0} \to \ldots,$$

and the homology becomes,

$$H_i(C) = \begin{cases} \frac{N}{Im(f)} = Coker(f) & i = n \\ Ker(f) & i = n+1 \\ 0 & otherwise. \end{cases}$$

Proof. Firstly, at degree n we have that,

$$H_n(C) = \frac{Ker(N \to 0)}{Im(M \xrightarrow{f} N)} = \frac{N}{Im(f)} = Coker(f).$$

Then at degree $n+1$ we have that,

$$H_{n+1}(C) = \frac{Ker(M \xrightarrow{f} N)}{Im(0 \to M)} = Ker(f).$$

Finally, it is clear that everywhere else there is no homology. $\square$

Notation 2.1.7. Here,

$$Coker(f) = \frac{\text{Codomain of } f}{\text{Image of } f},$$

is the *cokernel* of the map f.

Example 2.1.8. We can have a chain complex of $\mathbb{Z}$-modules,

$$C'' : \qquad \underset{\text{deg}}{\ldots} \to \underset{2}{0} \to \underset{1}{\mathbb{Z}} \xrightarrow{a} \underset{0}{\mathbb{Z}} \to \underset{-1}{0} \to \ldots$$

where the map a is right multiplication by some $a \in \mathbb{Z}$. The homology is,

$$H_i(C'') = \begin{cases} \frac{Ker((Z) \to 0)}{Im(\mathbb{Z} \xrightarrow{a} \mathbb{Z})} = \frac{\mathbb{Z}}{a\mathbb{Z}} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that,

$$H_0(C'') = \frac{\mathbb{Z}}{a\mathbb{Z}} = \frac{\text{Codomain of } f}{\text{Image of } f} = Coker(a).$$

Also,

$$H_1(C'') = \frac{Ker(\mathbb{Z} \xrightarrow{a} \mathbb{Z})}{Im(0 \to \mathbb{Z})} = Ker(a) = 0,$$

because $Ker(a)$ is empty.

Definition 2.1.9.

- The elements of $B_n(C)$ are called $n-$*boundaries*.
- The elements of $Z_n(C)$ are called $n-$*cycles*.

Remark 2.1.10. If $x \in Z_n(C)$ then its image in $H_n(C)$ is usually written as $[x]$.

Definition 2.1.11. A chain complex C is said to be:

- *acyclic* if $H_n(C) = 0$ for all n.
- *bounded above* if there exists some $n \in \mathbb{N}$, $C_k = 0$ for all $k > n$.
- *bounded below* if for some $n \in \mathbb{N}$, $C_k = 0$ for all $k < n$.
- *bounded* if it is bounded above and below.
- *non-negative* if $C_n = 0$ for $n < 0$.

Definition 2.1.12. A *cochain complex* $C^\bullet$ consists of a sequence of $\mathbb{R}$-modules C^i $(i \in \mathbb{Z})$ and morphisms of the form,

$$C: \qquad \ldots \to C^{-2} \xrightarrow{\delta^{-2}} C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \ldots$$

such that $\delta^{n-1}\delta^n = 0$ for all n, i.e. the composition of any two consecutive maps is zero.

Bibliography

[1] *Cohomology and Central Simple Algebras.* [Online-PDF file]. [Accessed October 2014]. Available from: http://www1.maths.leeds.ac.uk/ pmtwc/cohom.pdf

[2] *Representation of Quivers.* [Online-PDF file]. [Accessed October 2014]. Available from: http://www1.maths.leeds.ac.uk/ pmtwc/quivlecs.pdf