

# Representation of Quivers

Chanelle Lee

Student ID: 200646370

Supervisor: William Crawley-Boevey

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# Chapter 1

## Introduction

This project aims to cover interesting topics from the area of quiver representations. The first chapter will give a brief introduction to the topics in Homological Algebra needed for bulk of the report. It will cover chain complexes, exact sequences, short exact sequences, and extensions, most importantly  $Ext^1$ . The the second chapter will introduce quivers and look at some interesting results about path algebras before moving onto representations of quivers and their relationship with the modules of the corresponding path algebra. Next we will cover Dynkin and Euclidean diagrams before finishing with a proof of Gabriel's Theorem. The third chapter will cover Auslander-Reiten quivers.

## Chapter 2

# Homological Algebra

### 2.1 Chain Complexes

**Definition 2.1.1.** A *chain complex*  $\mathbf{C}_\bullet$  consists of a sequence of  $\mathbb{R}$ -modules  $C_i$  ( $i \in \mathbb{Z}$ ) and morphisms of the form,

$$\mathbf{C} : \quad \dots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} C_{-2} \xrightarrow{\delta_{-2}} \dots$$

such that  $\delta_{n-1}\delta_n = 0$  for all  $n$ , i.e. the composition of any two consecutive maps is zero. The maps  $\delta_n$  are called the *differentials* of  $C$ .

**Remark 2.1.2.** It is convention that the map  $\delta_n$  starts at  $C_n$ .

**Example 2.1.3.** If we have a field  $K$  then we can create the following chain complex:

$$\mathbf{C} : \quad \dots \rightarrow 0 \rightarrow K^2 \xrightarrow{\begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 0 \end{pmatrix}} K^3 \xrightarrow{(0 \ 0 \ 1)} K \rightarrow 0 \rightarrow \dots$$

We can clearly see that the maps uphold the  $\delta^2 = 0$  condition as,

$$(0 \ 0 \ 1) \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 0 \end{pmatrix} = (0 \ 0).$$

**Example 2.1.4.** If we consider the sequence,

$$\dots \rightarrow 0 \rightarrow K \xrightarrow{\delta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} K^2 \xrightarrow{\delta_1 = (1 \ 0)} K \rightarrow 0 \rightarrow \dots$$

however,

$$(1 \ 0) \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \neq 0.$$

Hence,  $\delta_1\delta_2 \neq 0$ , and so the sequence is not a chain complex. However, if we change  $\delta_2$  slightly we obtain the chain complex,

$$\mathbf{C} : \quad \dots \rightarrow 0 \rightarrow K \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} K^2 \xrightarrow{(0 \ 1)} K \rightarrow 0 \rightarrow \dots$$

since,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \neq 0.$$

**Definition 2.1.5.** For a chain complex,

$$\mathbf{C} : \quad \dots \rightarrow C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \rightarrow \dots$$

we define:

$$Z_n(\mathbf{C}) = \ker(\delta_n) \quad \& \quad B_n(\mathbf{C}) = \operatorname{im}(\delta_{n+1}).$$

To explore the relationship between  $Z_n(\mathbf{C})$  and  $B_n(\mathbf{C})$ , the following is a solution to Exercise 2.14 in [3]. (Note the example will not be provided here as exactness has not yet been covered.)

**Lemma 2.1.6.** *Let,*

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

*be a sequence of module maps. Then  $gf = 0$  if and only if  $\operatorname{im}(f) \subseteq \operatorname{im}(g)$ .*

*Proof.* Suppose  $gf = 0$ , i.e., for all  $a \in A$ ,  $g(f(a)) = 0$  which implies that  $f(a) \in \ker(g)$  for all  $a \in A$ . Hence,  $\operatorname{im}(f) \subseteq \ker(g)$ .

Conversely, suppose that  $\operatorname{im}(f) \subseteq \ker(g)$ . Every element  $b \in \operatorname{im}(f)$  is also an element of  $\ker(g)$ , thus  $gf = 0$ .  $\square$

**Definition 2.1.7.** If  $\mathbf{C}$  is a chain complex then its *homology* is defined to be,

$$H_n(\mathbf{C}) = \frac{\ker(\delta_n : C_n \rightarrow C_{n-1})}{\operatorname{im}(\delta_{n+1} : C_{n+1} \rightarrow C_n)} = \frac{Z_n(\mathbf{C})}{B_n(\mathbf{C})},$$

which is an  $\mathbb{R}$ -module.

The subsequent proposition is the solution to Exercise 6.1 in [3].

**Proposition 2.1.8.** *If  $\mathbf{C}$  is a chain complex with  $C_n = 0$  for some  $n$  then  $H_n(\mathbf{C}) = 0$ .*

*Proof.* Well suppose we have such a chain complex,

$$\mathbf{C} : \quad \dots \rightarrow C_{n+1} \xrightarrow{\delta_{n+1}} 0 \xrightarrow{\delta_n} C_{n-1} \rightarrow \dots$$

the the homology is,

$$H_n(\mathbf{C}) = \frac{\ker(\delta_n : 0 \rightarrow C_{n-1})}{\operatorname{im}(\delta_{n+1} : C_{n+1} \rightarrow 0)},$$

as the only element in  $C_n$  is the zero element, and so,  $H_n(\mathbf{C}) = 0$ , as required.  $\square$

Examples 2.1.9 and 2.1.12 are taken from [1] and are included here because they are felt to be the clearest at demonstrating a chain complex and homology, however, the more general statement of the second example is presented as Proposition 2.1.10 because it is an interesting result.

**Example 2.1.9.** If we take a module  $M$  then we can make a chain complex;

$$\mathbf{C} : \quad \dots \rightarrow 0 \rightarrow M \rightarrow 0 \rightarrow \dots$$

where  $M$  is at degree  $n$ . Then the homology will be:

$$H_i(\mathbf{C}) = \begin{cases} \frac{\ker(M \rightarrow 0)}{\operatorname{im}(0 \rightarrow M)} = M & i = n, \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 2.1.10.** If we have a module homomorphism between  $R$ -modules,  $f : M \rightarrow N$ , then we get the chain complex,

$$\mathbf{C} : \quad \dots \xrightarrow{\deg} 0_{n+2} \rightarrow M_{n+1} \xrightarrow{f} N_n \rightarrow 0_{n-1} \rightarrow \dots,$$

and the homology becomes,

$$H_i(\mathbf{C}) = \begin{cases} \frac{N}{\operatorname{im}(f)} = \operatorname{coker}(f) & i = n \\ \ker(f) & i = n + 1 \\ 0 & \text{otherwise.} \end{cases}$$

*Proof.* Firstly, at degree  $n$  we have that,

$$H_n(\mathbf{C}) = \frac{\ker(N \rightarrow 0)}{\operatorname{im}(M \xrightarrow{f} N)} = \frac{N}{\operatorname{im}(f)} = \operatorname{coker}(f).$$

Then at degree  $n + 1$  we have that,

$$H_{n+1}(\mathbf{C}) = \frac{\ker(M \xrightarrow{f} N)}{\operatorname{im}(0 \rightarrow M)} = \ker(f).$$

Finally, it is clear that everywhere else there is no homology. □

**Notation 2.1.11.** Here,

$$\operatorname{Coker}(f) = \frac{\operatorname{Codomain of } f}{\operatorname{Image of } f},$$

is the *cokernel* of the map  $f$ .

**Example 2.1.12.** We can have a chain complex of  $\mathbb{Z}$ -modules,

$$\mathbf{C} : \quad \dots \xrightarrow{\deg} 0_2 \rightarrow \mathbb{Z}_1 \xrightarrow{a} \mathbb{Z}_0 \rightarrow 0_{-1} \rightarrow \dots$$

where the map  $a$  is right multiplication by some  $a \in \mathbb{Z}$ . The homology is,

$$H_i(\mathbf{C}) = \begin{cases} \frac{\ker(\mathbb{Z} \rightarrow 0)}{\operatorname{im}(\mathbb{Z} \xrightarrow{a} \mathbb{Z})} = \frac{\mathbb{Z}}{a\mathbb{Z}} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that,

$$H_0(C) = \frac{\mathbb{Z}}{a\mathbb{Z}} = \frac{\operatorname{Codomain of } f}{\operatorname{Image of } f} = \operatorname{coker}(a).$$

Also,

$$H_1(C) = \frac{\ker(\mathbb{Z} \xrightarrow{a} \mathbb{Z})}{\operatorname{im}(0 \rightarrow \mathbb{Z})} = \ker(a) = 0,$$

because  $\ker(a)$  is empty.

**Definition 2.1.13.** • The elements of  $B_n(\mathbf{C})$  are called  $n$ -boundaries.

- The elements of  $Z_n(\mathbf{C})$  are called  $n$ -cycles.

**Remark 2.1.14.** If  $x \in Z_n(\mathbf{C})$  then its image in  $H_n(\mathbf{C})$  is usually written as  $[x]$ .

**Definition 2.1.15.** A chain complex  $\mathbf{C}$  is said to be:

- *acyclic* if  $H_n(\mathbf{C}) = 0$  for all  $n$ .
- *bounded above* if there exists some  $n \in \mathbb{N}$ ,  $C_k = 0$  for all  $k > n$ .
- *bounded below* if for some  $n \in \mathbb{N}$ ,  $C_k = 0$  for all  $k < n$ .
- *bounded* if it is bounded above and below.
- *non-negative* if  $C_n = 0$  for  $n < 0$ .

**Example 2.1.16.** All the chain complexes in the previous examples are bounded both above and below, however, neither is acyclic as they both have instances where the homology is non-zero. The chain complex in Example 2.1.12 is non-negative because  $C_n \neq 0$  only when  $n = 0, 1$ .

**Example 2.1.17.** If we take another look at the chain complex in Example 2.1.4,

$$\mathbf{C} : \quad \underset{\text{deg}}{\dots} \rightarrow 0 \rightarrow K_1 \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} K_0^2 \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} K_{-1} \rightarrow 0 \rightarrow \dots$$

the homologies are,

$$\begin{aligned} H_1(\mathbf{C}) &= \frac{\ker(K \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} K^2)}{\text{im}(0 \rightarrow K)} \cong \frac{K}{K} \cong 0, \\ H_0(\mathbf{C}) &= \frac{\ker(K^2 \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} K)}{\text{im}(K \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} K^2)} \cong \frac{K}{K} \cong 0, \\ H_{-1}(\mathbf{C}) &= \frac{\ker(K \rightarrow 0)}{\text{im}(K^2 \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} K)} \cong \frac{K}{K} \cong 0. \end{aligned}$$

Thus  $H_n(\mathbf{C}) = 0$  for all  $n$  and so  $\mathbf{C}$  is an acyclic chain complex. Later in the report, we will see that  $\mathbf{C}$  is in fact a short exact sequence.

**Example 2.1.18.** The chain complex,

$$\mathbf{C} : \quad \underset{\text{deg}}{\dots} \xrightarrow{3} \frac{\mathbb{Z}}{9\mathbb{Z}} \xrightarrow{3} \frac{\mathbb{Z}}{9\mathbb{Z}} \xrightarrow{3} \frac{\mathbb{Z}}{9\mathbb{Z}} \rightarrow \dots$$

$\quad \quad \quad 1 \quad \quad \quad 0 \quad \quad \quad -1$

where the differentials are the maps,

$$\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \rightarrow \frac{\mathbb{Z}}{9\mathbb{Z}}, \quad z + 9\mathbb{Z} \mapsto 3z + 9\mathbb{Z},$$

is unbounded. It is also acyclic, since the homology is,

$$H_n(\mathbf{C}) = \frac{\ker(\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \rightarrow \frac{\mathbb{Z}}{9\mathbb{Z}})}{\operatorname{im}(\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \rightarrow \frac{\mathbb{Z}}{9\mathbb{Z}})} \cong \frac{\mathbb{Z}/3\mathbb{Z}}{\mathbb{Z}/3\mathbb{Z}} \cong 0,$$

for all  $n$ .

**Definition 2.1.19.** A *cochain complex*  $\mathbf{C}^\bullet$  consists of a sequence of  $\mathbb{R}$ -modules  $C^i$  ( $i \in \mathbb{Z}$ ) and morphisms of the form,

$$\mathbf{C} : \quad \dots \xrightarrow{\delta^{-3}} C^{-2} \xrightarrow{\delta^{-2}} C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \dots$$

such that  $\delta^{n-1}\delta^n = 0$  for all  $n$ , i.e. the composition of any two consecutive maps is zero.

**Remark 2.1.20.** Chain and cochain complexes can be thought of as almost identical constructs with the only difference being the numbering of the chain. The degree of a chain complex *decreases* from left to right, whereas, the degree of a cochain complex *increases* from left to right. So, we can compute one from the other by setting  $C^{-n} = C_n$ , or equivalently  $C^n = C_{-n}$ ; this is called *renumbering*.

**Definition 2.1.21.** If  $\mathbf{C}$  is a cochain complex then its *cohomology* is defined to be,

$$H^n(\mathbf{C}) = \frac{\ker(\delta^n : C^n \rightarrow C^{n+1})}{\operatorname{im}(\delta^{n-1} : C^{n-1} \rightarrow C^n)} = \frac{Z^n(\mathbf{C})}{B^n(\mathbf{C})}.$$

- The elements of  $B_n(\mathbf{C})$  are called *n-coboundaries*.
- The elements of  $Z_n(\mathbf{C})$  are called *n-cocycles*.

**Example 2.1.22.** We can renumber the chain complex in Example 2.1.12 to get the cochain complex,

$$\mathbf{C} : \quad \dots \xrightarrow{\deg} 0 \xrightarrow{-2} \mathbb{Z} \xrightarrow{-1} \mathbb{Z} \xrightarrow{0} 0 \xrightarrow{1} \dots$$

Its cohomology is,

$$H^i(\mathbf{C}) = \begin{cases} \frac{\ker(\mathbb{Z} \rightarrow 0)}{\operatorname{im}(\mathbb{Z} \xrightarrow{a} \mathbb{Z})} = \frac{\mathbb{Z}}{a\mathbb{Z}} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

**Definition 2.1.23.** Let  $\mathbf{C}$  be a chain complex of left  $R$ -modules. If  $M$  is a left  $R$ -module then  $\operatorname{Hom}(\mathbf{C}, M)$  is the cochain complex where,

$$\operatorname{Hom}(\mathbf{C}, M)^n = \operatorname{Hom}(C_n, M),$$

and the differentials,

$$\delta^n : \operatorname{Hom}(\mathbf{C}, M)^n \rightarrow \operatorname{Hom}(\mathbf{C}, M)^{n+1},$$

are induced by the differentials of  $\mathbf{C}$ ,  $\delta_n : C_{n+1} \rightarrow C_n$ . The cohomology of this cochain complex is denoted  $H^n(\mathbf{C}, M)$ .

The following example is a generalised version of one found in [1].



**Example 2.1.24.** Consider the acyclic chain complex,

$$\mathbf{C} : \quad \begin{array}{ccccccc} \dots & \rightarrow & 0 & \rightarrow & \mathbb{Z} & \xrightarrow{n} & \mathbb{Z} & \xrightarrow{nat} & \frac{\mathbb{Z}}{n\mathbb{Z}} & \rightarrow & \dots \\ & & \text{deg} & & 1 & & 0 & & -1 & & \end{array}$$

So applying  $Hom(-, \mathbb{Z})$  we gives the cochain complex,

$$\mathbf{C}' : \quad \begin{array}{ccccccc} \dots & \rightarrow & 0 & \rightarrow & \mathbb{Z} & \xrightarrow{n} & \mathbb{Z} & \xrightarrow{nat} & \frac{\mathbb{Z}}{n\mathbb{Z}} & \rightarrow & \dots \\ & & \text{deg} & & 1 & & 0 & & -1 & & \end{array}$$

which has cohomology,

$$H^i(\mathbf{C}', \mathbb{Z}) = \begin{cases} \frac{\ker(\mathbb{Z} \rightarrow 0)}{\text{im}(\mathbb{Z} \xrightarrow{n} \mathbb{Z})} \cong \frac{\mathbb{Z}}{n\mathbb{Z}} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that despite the chain complex being acyclic, its cohomology induced by  $Hom(-, \mathbb{Z})$  is not zero everywhere.

**Definition 2.1.25.** A *chain map*  $f : \mathbf{C} \rightarrow \mathbf{D}$ , with  $\mathbf{C}$  and  $\mathbf{D}$  chain complexes, is given by a homomorphism  $f_n : \mathbf{C}_n \rightarrow \mathbf{D}_n$  for each  $n$ , such that each square in the following diagram commutes,

$$\begin{array}{ccccccc} \dots & \xrightarrow{\gamma_{n+2}} & \mathbf{C}_{n+1} & \xrightarrow{\gamma_{n+1}} & \mathbf{C}_n & \xrightarrow{\gamma_n} & \mathbf{C}_{n-1} & \xrightarrow{\gamma_{n-1}} & \dots \\ & & \downarrow f_{n+1} & & \downarrow f_n & & \downarrow f_{n-1} & & \\ \dots & \xrightarrow{\delta_{n+2}} & \mathbf{D}_{n+1} & \xrightarrow{\delta_{n+1}} & \mathbf{D}_n & \xrightarrow{\delta_n} & \mathbf{D}_{n-1} & \xrightarrow{\delta_{n-1}} & \dots \end{array}$$

Note that there is an equivalent notion of a *cochain map* of cochain complexes.

**Example 2.1.26.** If we have the following diagram,

$$\begin{array}{ccccccccccc} \mathbf{C}: & \dots & \longrightarrow & 0 & \longrightarrow & \mathbb{Z} & \xrightarrow{n} & \mathbb{Z} & \xrightarrow{nat} & \frac{\mathbb{Z}}{n\mathbb{Z}} & \longrightarrow & 0 & \longrightarrow & \dots \\ & & & \parallel & & \downarrow f_1 & nat & \downarrow f_0 & nat & \downarrow f_{-1} & \parallel & & \parallel & \\ \mathbf{D}: & \dots & \longrightarrow & 0 & \longrightarrow & \frac{\mathbb{Z}}{n\mathbb{Z}} & \xrightarrow{n} & \frac{\mathbb{Z}}{n^2\mathbb{Z}} & \xrightarrow{nat} & \frac{\mathbb{Z}}{n\mathbb{Z}} & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

then we can see that our non trivial chain maps are,

$$\begin{aligned} f_1 : \mathbb{Z} &\rightarrow \frac{\mathbb{Z}}{n\mathbb{Z}}, & z &\mapsto z + n\mathbb{Z}, \\ f_0 : \mathbb{Z} &\rightarrow \frac{\mathbb{Z}}{n^2\mathbb{Z}}, & z &\mapsto z + n^2\mathbb{Z}, \\ f_{-1} : \frac{\mathbb{Z}}{n\mathbb{Z}} &\rightarrow \frac{\mathbb{Z}}{n\mathbb{Z}}, & z + n\mathbb{Z} &\mapsto z + n\mathbb{Z}, \end{aligned}$$

and they satisfy,

$$f_0(\gamma_1(z)) = f_0(nz) = nz + n\mathbb{Z} = 0 \quad \& \quad \delta_1(f_1(z)) = \delta_1(z + n\mathbb{Z}) = nz + n\mathbb{Z} = 0.$$

Simmilarly, we can see that the other chain maps make the diagram commute.

## 2.2 Exact Sequences

**Definition 2.2.1.** A finite or infinite sequence of  $R$ -morphisms and left  $R$ -modules,

$$\dots \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow \dots$$

is said to be *exact* at  $M$  if  $\text{im} f = \ker g$ . A finite or infinite sequence of  $R$ -morphisms and left  $R$ -modules,

$$\dots \xrightarrow{f_{n+2}} M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} \dots$$

is said to be an *exact sequence* if it is exact at every  $M_i$ , i.e.  $\text{im} f_i = \ker f_{i+1}$ .

**Example 2.2.2.** The finite sequence,

$$0 \rightarrow K \xrightarrow{f} K^2 \xrightarrow{g} K^2 \xrightarrow{h} K^2 \rightarrow 0,$$

where  $f(x) = (x, 0)$ ,  $g(x, y) = (0, y)$ , and  $h(x, y) = (x, 0)$  is an exact sequence since  $\text{im}(f) = \{(x, 0) : x \in K\} = \ker(g)$  and  $\text{im}(g) = \{(0, y) : y \in K\} = \ker(h)$ . However, if we instead consider the finite sequence,

$$0 \rightarrow K \xrightarrow{f} K^2 \xrightarrow{g} K^2 \xrightarrow{h'} K^2 \rightarrow 0,$$

where  $f$  and  $g$  are the same as above, but  $h'(x, y) = (x, y)$ . Obviously, once again we have that  $\text{im}(f) = \ker(g)$  and  $\text{im}(g) = \{(0, y) : y \in K\}$ , however,  $\ker(h') = \{(0, 0)\}$  and so  $\text{im}(g) \neq \ker(h')$ . Hence this is not an exact sequence, because it is not exact everywhere.

**Definition 2.2.3.** A *short exact sequence* is an exact sequence of the form,

$$0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0.$$

**Example 2.2.4.** The sequence,

$$0 \rightarrow K \xrightarrow{f} K^2 \xrightarrow{g} K \rightarrow 0,$$

where  $f(x) = (x, 0)$  and  $g(x, y) = y$  is a short exact sequence, since  $\text{im}(f) = \{(x, 0) : x \in K\} = \ker(g)$ .

**Example 2.2.5.** Notice that the sequence,

$$0 \rightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{\eta} \frac{\mathbb{Z}}{3\mathbb{Z}} \rightarrow 0,$$

has  $\ker(\eta) = \{z : z + 3\mathbb{Z} = 0 \cong 3\mathbb{Z}\}$  and  $\text{im}(f) = \{6z : z \in \mathbb{Z}\} \cong 6\mathbb{Z}$ . Hence, as  $6\mathbb{Z} \subsetneq 3\mathbb{Z}$ ,  $\text{im}(f) \neq \ker(\eta)$  and the sequence is not exact.

Notice that you can obtain a chain complex from an exact sequence simply by deciding the degree of any term of the sequence. To consider the result when a chain complex is an exact sequence, the following is in response to Exercise 1.1.5 from [4]. (Note that the third condition has been omitted as we are not consider quasi-isomerism in this report.)

**Theorem 2.2.6.** *The following are equivalent for every chain complex  $\mathbf{C}$ :*

1.  $\mathbf{C}$  is an exact sequence, that is, exact at every  $C_n$ .

2.  $\mathbf{C}$  is acyclic, that is,  $H_n(\mathbf{C}) = 0$  for all  $n$ .

*Proof.* Suppose the chain complex,

$$\mathbf{C} : \quad \dots \rightarrow C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \dots$$

is an exact sequence, so we have that  $\text{im}(\delta_{n+1}) = \ker(\delta_n)$  for all  $n$ , i.e.  $B_n(\mathbf{C}) = Z_n(\mathbf{C})$  for all  $n$ . Hence, when we compute the homology of  $\mathbf{C}$ , we find that,

$$H_n(\mathbf{C}) = \frac{Z_n(\mathbf{C})}{B_n(\mathbf{C})} = 0 \quad \forall n.$$

Thus,  $\mathbf{C}$  is acyclic.

Conversely, suppose  $\mathbf{C}$  is acyclic, then we have that  $H_n(\mathbf{C}) = 0$  for all  $n$ . Therefore,  $Z_n(\mathbf{C})/B_n(\mathbf{C}) = 0$  for all  $n$ , by definition, but this is only possibly if either  $Z_n(\mathbf{C}) \subset B_n(\mathbf{C})$ , which is impossible from Definition 2.1.6, or  $Z_n(\mathbf{C}) = B_n(\mathbf{C})$ . Thus,  $\ker(\delta_n) = \text{im}(\delta_{n+1})$  and so  $\mathbf{C}$  is exact.  $\square$

**Remark 2.2.7.** The above theorem makes it clear that exact sequences can be thought of as acyclic chain complexes. By the same token, homology can be thought of as a chain complex's deviation from exactness.

The following is provided as an example in [6], however, it was felt too useful as a method of constructing exact sequences to be simply an example and so is presented on this report as a lemma.

**Proposition 2.2.8.** *The sequence,*

$$0 \rightarrow \ker(f) \xrightarrow{\iota} M \xrightarrow{f} N \xrightarrow{\eta} \text{coker}(f) \rightarrow 0,$$

where  $\iota, \eta$  are the inclusion, and natural maps respectively, is exact. In addition the sequence,

$$0 \rightarrow \ker(f) \xrightarrow{\iota} M \xrightarrow{\eta} \frac{M}{\text{im}(f)} \rightarrow 0,$$

is a short exact sequence.

*Proof.* It is clear here that we have  $\text{im}(\iota) = \ker(f)$  as  $\iota$  is the inclusion map and since  $\text{coker}(f) = N/\text{im}(f)$  we also have that  $\text{im}(f) = \ker(\eta)$ . Hence the sequence is exact. The second result follows a similar argument.  $\square$

**Example 2.2.9.** Consider the map  $f : \mathbb{Z}^2 \rightarrow \mathbb{Z}$ ,  $(x, y) \mapsto 2x$ . Then  $\ker(f) = \{(0, y) : y \in \mathbb{Z}\} \cong \mathbb{Z}$ , and  $\text{im}(f) = 2\mathbb{Z}$  giving that  $\text{coker}(f) = \frac{\mathbb{Z}}{2\mathbb{Z}}$ . It follows from Lemma 2.2.8 that the sequence,

$$0 \rightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z} \xrightarrow{\eta} \frac{\mathbb{Z}}{2\mathbb{Z}} \rightarrow 0,$$

is exact. In addition, the following is a short exact equation,

$$0 \rightarrow \mathbb{Z} \xrightarrow{\iota} \mathbb{Z}^2 \xrightarrow{\eta} \frac{\mathbb{Z}}{2\mathbb{Z}} \rightarrow 0.$$

Here are some simple properties of exact sequences as detailed by Rotman in Proposition 2.18 of [3]. Also included is the solution to Exercise 2.16 (ii) from [3]; it caught our attention as an interesting property, with such a simple proof, which shows the rigidity of the exactness condition.

**Proposition 2.2.10.** (i) *A sequence  $0 \rightarrow L \xrightarrow{f} M$  is exact if and only if  $f$  is injective.*

(ii) *A sequence  $M \xrightarrow{g} N \rightarrow 0$  is exact if and only if  $g$  is surjective.*

(iii) *A sequence  $0 \rightarrow L \xrightarrow{h} M \rightarrow 0$  is exact if and only if  $h$  is an isomorphism.*

(iv) *If  $L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} L'$  then  $f$  is surjective if and only if  $h$  is injective.*

*Proof.* (i) As the image of  $0 \rightarrow L$  is  $\{0\}$ , exactness gives  $\ker(f) = \{0\}$ , and so  $f$  is injective. Conversely, given some map  $f : L \rightarrow M$ , there is an exact sequence  $\ker(f) \xrightarrow{\iota} L \rightarrow fM$ , where  $\iota$  is the inclusion map, (see Proposition 2.2.8). If  $f$  is injective, then  $\ker(f) = \{0\}$ .

(ii) The kernel of  $N \rightarrow 0$  is  $N$ , so the exactness gives  $\operatorname{im}(g) = N$ , and so  $g$  is surjective. Conversely, given  $g : M \rightarrow N$ , there is an exact sequence  $M \xrightarrow{g} N \xrightarrow{\pi} \operatorname{coker}(g)$ , where  $\pi$  is the natural map, (see Proposition 2.2.8). If  $g$  is surjective, then  $N = \operatorname{im}(g)$  and  $\operatorname{coker}(g) = \{0\}$ .

(iii) Part (i) shows that the sequence is exact at  $L$  if and only if  $h$  is injective, and part (ii) shows that the sequence is exact at  $M$  if and only if  $h$  is surjective. Therefore, the sequence is exact everywhere if and only if  $h$  is an isomorphism.

(iv) Suppose  $f$  is surjective, then  $\operatorname{im}(f) = M$ , and exactness at  $M$  gives  $\ker(g) = M$ . Hence,  $\operatorname{im}(g) = \{0\}$ , and exactness at  $N$  gives  $\ker(h) = \{0\}$  and so  $h$  is injective. The converse is similar. □

**Definition 2.2.11.** Two short exact sequences,

$$\mathbf{E} : 0 \rightarrow L \rightarrow M \rightarrow N \rightarrow 0 \quad \& \quad \mathbf{E}' : 0 \rightarrow L \rightarrow M' \rightarrow N \rightarrow 0,$$

are *equivalent* if there is a map  $\xi : M \rightarrow M'$  such that the diagram,

$$\begin{array}{ccccccccc} \mathbf{E}: & 0 & \longrightarrow & L & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & \parallel & & \parallel & & \downarrow \xi & & \parallel & & \parallel \\ \mathbf{E}': & 0 & \longrightarrow & L & \longrightarrow & M' & \longrightarrow & N & \longrightarrow & 0 \end{array}$$

commutes.

**Example 2.2.12.** Consider the two short exact sequences,

$$\mathbf{E} : 0 \rightarrow K \xrightarrow{f} K^2 \xrightarrow{g} K \rightarrow 0 \quad \& \quad \mathbf{E}' : 0 \rightarrow K \xrightarrow{f'} K^2 \xrightarrow{g'} K \rightarrow 0,$$

where  $f : x \mapsto (x, 0)$ ,  $g : (x, y) \mapsto y$ ,  $f' : x \mapsto (0, x)$  and  $g' : (x, y) \mapsto x$ . Then we can construct the diagram:

$$\begin{array}{c}
 \mathbf{E}: \quad 0 \longrightarrow K \longrightarrow K^2 \longrightarrow K \longrightarrow 0 \\
 \quad \quad \parallel \quad \quad \parallel \quad \quad \downarrow \xi \quad \quad \parallel \quad \quad \parallel \\
 \mathbf{E}': \quad 0 \longrightarrow K \longrightarrow K^2 \longrightarrow K \longrightarrow 0
 \end{array}$$

This diagram commutes when  $\xi$  is the map  $\xi : K^2 \rightarrow K^2$ ,  $(x, y) \mapsto (y, x)$ , hence these two short exact equations are equivalent.

For a nonexample we turn to Exercise III 1.1 in [5].

**Example 2.2.13.** Consider the two short exact sequences,

$$\mathbf{E}: \quad 0 \rightarrow \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \frac{\mathbb{Z}}{3\mathbb{Z}} \rightarrow 0 \quad \& \quad \mathbf{E}': \quad 0 \rightarrow \mathbb{Z} \xrightarrow{f'} \mathbb{Z} \xrightarrow{g'} \frac{\mathbb{Z}}{3\mathbb{Z}} \rightarrow 0,$$

where  $f, f' : x \mapsto 3x$ ,  $g : x \mapsto x(\bmod 3)$  and  $g' : x \mapsto 2x(\bmod 3)$ . Then we can construct the diagram:

$$\begin{array}{c}
 \mathbf{E}: \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \frac{\mathbb{Z}}{3\mathbb{Z}} \longrightarrow 0 \\
 \quad \quad \parallel \quad \quad \parallel \quad \quad \downarrow \xi \quad \quad \parallel \quad \quad \parallel \\
 \mathbf{E}': \quad 0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \longrightarrow \frac{\mathbb{Z}}{3\mathbb{Z}} \longrightarrow 0
 \end{array}$$

Assume  $\mathbf{E}$  and  $\mathbf{E}'$  are equivalent, so we there must exist some  $\xi$  such that the diagram commutes. The second square in the diagram will commute if and only if  $\xi = id_{\mathbb{Z}}$ , since  $f = f'$ , however, this then means the third square will never commute, as  $g \neq g'$ . Thus no such  $\xi$  exists and the two short exact sequences are not equivalent.

## 2.3 Extensions

## Chapter 3

# Representation of Quivers

### 3.1 Quivers and Path Algebras

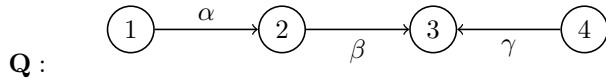
**Definition 3.1.1.** A *quiver* is defined as the tuple of sets and functions,  $\mathbf{Q} = (Q_0, Q_1, s, t : Q_1 \rightarrow Q_0)$  such that:

- $Q_0$  is the set of vertices, which we will set to be the finite set  $\{1, 2, \dots, n\}$ .
- $Q_1$  is the set of arrows, which we will also set to be finite.
- Functions  $s, t$  such that an arrow  $\rho \in Q_1$  *starts* at the vertex  $s(\rho) \in Q_0$  and *terminates* at the vertex  $t(\rho) \in Q_0$ , i.e.  $\rho : s(\rho) \rightarrow t(\rho)$ .

**Example 3.1.2.** A quiver  $Q = (Q_0, Q_1, s, t : Q_1 \rightarrow Q_0)$  where  $Q_0 = \{1, 2, 3, 4\}$ ,  $Q_1 = \{\alpha, \beta\}$ , and  $s, t$  are defined such that;

$$\begin{aligned} s : Q_1 &\rightarrow Q_0, & \alpha &\mapsto 1, \beta \mapsto 2, \gamma \mapsto 4 \\ t : Q_1 &\rightarrow Q_0, & \alpha &\mapsto 2, \beta \mapsto 3, \gamma \mapsto 3, \end{aligned}$$

looks like,



**Definition 3.1.3.** A *non-trivial path*,  $p$ , in a quiver is a sequence of arrows  $\rho_1, \dots, \rho_n$  which satisfies  $t(\rho_{i+1}) = s(\rho_i)$  for all  $1 \leq i < n$ , i.e. the start of an arrow is where the previous arrow terminated. The starting and terminating vertex of a path  $p$  are denoted  $s(p)$  and  $t(p)$ , respectively.

**Notation 3.1.4.** In this report the arrows in a path will be ordered the same way as the composition of functions, as in [2], however, be aware that other publications may order the arrows the opposite way.

**Definition 3.1.5.** The *trivial path* is the path which contains no arrows, i.e. it is a single vertex, and is denoted  $e_i$  where the vertex is  $i$ .

**Example 3.1.6.** The paths of the quiver in Example 3.1.2 are:

$$p_1 = e_1, \quad p_2 = e_2, \quad p_3 = e_3, \quad p_4 = e_4, \quad p_5 = \alpha, \quad p_6 = \beta, \quad p_7 = \gamma, \quad p_8 = \beta\alpha.$$

However,  $\gamma\beta\alpha$  is not a path because  $t(\gamma) = 3 \neq s(\beta) = 2$ .

**Definition 3.1.7.** A *path algebra*  $kQ$  is the  $k$ -algebra which has the basis all the paths in  $Q$ , and the product of two paths  $p, q$  is defined as,

$$pq = \begin{cases} \text{obvious composition} & \text{if } t(q) = s(p), \\ 0 & \text{otherwise.} \end{cases}$$

This multiplication is associative.

**Example 3.1.8.** If we once again use the quiver  $Q$  from Example 3.1.2, then, from Example 3.1.6, we know the basis of the path algebra  $kQ$  will be,

$$\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \beta\alpha\}.$$

The product of  $\beta$  and  $\alpha$  is  $\beta\alpha$ , but the product of  $\alpha$  and  $\beta$  is zero, and the product of  $e_2$  and  $\alpha$  is just  $\alpha$ .

**Example 3.1.9.** If  $Q$  is the following quiver,

$$Q: \quad \alpha \curvearrowright (1) \curvearrowleft \beta$$

forms the path algebra  $kQ \cong k[X, Y]$ , the free, associative algebra on two letters. In fact, if we have a quiver with a single vertex and  $n$  loops, then this can be associated with the free, associative algebra on  $n$  letters.

**Example 3.1.10.** If we have the quiver,

$$Q: \quad (1) \xrightarrow{\alpha} (2) \xrightarrow{\beta} (3) \xrightarrow{\gamma} (4)$$

the the path algebra,  $kQ \cong UT_4(k)$  by the isomorphism,

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 \alpha + \lambda_6 \beta + \lambda_7 \gamma + \lambda_8 \beta\alpha + \lambda_9 \gamma\beta\alpha + \lambda_{10} \gamma\beta \mapsto \begin{pmatrix} \lambda_4 & \lambda_7 & \lambda_{10} & \lambda_9 \\ 0 & \lambda_3 & \lambda_6 & \lambda_8 \\ 0 & 0 & \lambda_2 & \lambda_5 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

Generally, a quiver of the form,

$$Q': \quad (1) \xrightarrow{\alpha} (2) \xrightarrow{\beta} \cdots \xrightarrow{\gamma} (n)$$

induces a path algebra  $kQ' \cong UT_n(k)$  for any  $n$ .

**Example 3.1.11.** In fact, we find that if  $Q$  is the same quiver as above in Example 3.1.10, then  $kQ \cong LT_4(k)$  as well, through the isomorphism,

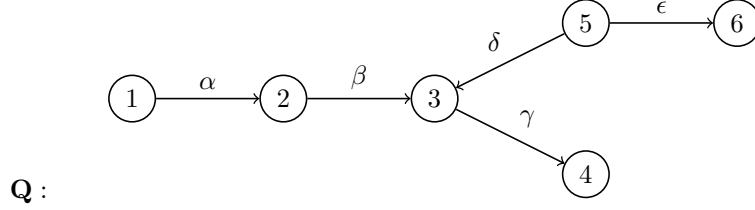
$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 \alpha + \lambda_6 \beta + \lambda_7 \gamma + \lambda_8 \beta\alpha + \lambda_9 \gamma\beta\alpha + \lambda_{10} \gamma\beta \mapsto \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ \lambda_5 & \lambda_2 & 0 & 0 \\ \lambda_8 & \lambda_6 & \lambda_3 & 0 \\ \lambda_9 & \lambda_{10} & \lambda_7 & \lambda_4 \end{pmatrix}.$$

Once again, this extends to the general case that,  $kQ' \cong LT_n(k)$ .

**Example 3.1.12.** In a more general case, as long as there is only one path between any two vertices, we can identify  $kQ$  with the following subalgebra of  $M_n(k)$ ,

$$\{M \in M_n(k) : M_{ij} = 0 \text{ if there is no path from } j \text{ to } i\}.$$

For instance, if we have the quiver,



the we can see that the path algebra  $kQ$  is isomorphic to the subalgebra with matrices of the form,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

**Remark 3.1.13.** Note that the idea here has some parallels to similar results for directed graphs in graph theory.

The following results about the idempotents of path algebras are from [2], however, we have either given a proof or expanded upon the one given. For the following results we set  $A = kQ$  and the  $e_i$  are the trivial paths of  $Q$ .

**Lemma 3.1.14.** *The  $e_i$  are orthogonal, idempotents in  $A$ , i.e.  $e_i e_i = e_i$  and  $e_i e_j = 0$  where  $i \neq j$ . Thus  $\sum_{i=1}^n e_i = 1_A$ .*

*Proof.* Well, obviously,  $e_i e_j = 0$  when  $i \neq j$  because  $t(e_j) \neq s(e_i)$  because they are the trivial paths at different vertices,  $i$  and  $j$ . Similarly, if we have the product  $e_i e_i$  then the composition makes sense here because  $t(e_i) = s(e_i)$ , but the composition is just  $e_i$  because if we travel along the trivial path  $e_i$  twice, then this is just the same as travelling the trivial path.

If we have some path  $p$  of the quiver, then bear in mind that,

$$e_i p = \begin{cases} e_i p = p & \text{if } i = t(p), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the sum of the trivial paths acting on our path  $p$  becomes,

$$\left( \sum_{i=1}^n e_i \right) p = (e_1 + e_2 + \cdots + e_n) p = e_i p = p = 1_A p,$$

because only one of the  $e_i$  in the sum will satisfy  $i = t(p)$  and all the rest will give zero. Similary, we can show that  $p \left( \sum_{i=1}^n e_i \right) = p 1_A$ .  $\square$



**Lemma 3.1.15.** *The bases of the spaces  $Ae_i$  and  $e_jA$  are all the paths starting at  $i$  and all the paths terminating at  $j$ , respectively. It then follows that the space  $e_jAe_i$  has as a basis the paths that start at  $i$  and terminate at  $j$ .*

*Proof.* For some  $i \in Q_0$ , and  $A$  has the basis  $\{p_1, p_2, \dots, p_n\}$ , then as  $Ae_i$  is a subspace of  $A$  the basis of  $Ae_i$  must be a subset of the basis of  $A$ , so,

$$Ae_i = \{ae_i : a \in A\} = \text{span}\{p_1e_i + p_2e_i + \dots + p_ne_i\},$$

and we know that,

$$p_re_i = \begin{cases} p_r & \text{if } s(p_r) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we can see that the basis of  $Ae_i$  must be all the paths starting at  $i$ . Similarly, as  $e_jA$  is a subspace of  $A$  and so following a similar argument we can see that its basis must be all the paths terminating at  $j$ . The result for  $e_jAe_i$  follows from these as it simply the intersection of the two spaces.  $\square$

**Lemma 3.1.16.**  *$A = \oplus_{i=1}^n Ae_i$ , so each  $Ae_i$  is a projective left  $A$ -module.*

*Proof.* From Lemma 3.1.15 we know that for each  $i$ , the basis of  $Ae_i$  are all the paths starting at  $i$  and so  $\oplus_{i=1}^n Ae_i$  must have as a basis all the paths starting at every vertex in  $Q$ , hence all the paths in  $Q$ . Thus  $A = \oplus_{i=1}^n Ae_i$ . Also, each  $Ae_i$  is obviously a left  $A$ -module with the action defined as multiplication by  $e_i$ . Hence, each  $Ae_i$  is a projective, left  $A$ -module.  $\square$

**Remark 3.1.17.** Similarly,  $A = \oplus_{i=1}^n e_jA$ , so each  $e_jA$  is a projective, right  $A$ -module.

**Lemma 3.1.18.** *If  $X$  is a left  $A$ -module, then  $\text{Hom}_A(Ae_i, X) \cong e_iX$ .*

*Proof.* Well, we want to show that there are some maps  $f$  and  $g$  such that they satisfy,

$$\text{Hom}_A(Ae_i, X) \xrightleftharpoons[g]{f} e_iX$$

Well, first consider the map  $\theta : Ae_i \rightarrow X$  and so  $\theta \in \text{Hom}_A(Ae_i, X)$ , then we can have  $f$  such that  $f(\theta) = \theta(e_i) = \theta(e_i^2) = e_i\theta(e_i) \in e_iX$ . Hence,  $f$  maps a homomorphism from  $\text{Hom}_A(Ae_i, X)$  to an element in  $e_iX$ . Now consider the map  $g(x) : Ae_i \rightarrow X$ ,  $g(x)(a) = ax$ . We can check this is an  $A$ -module homomorphism as,

$$\begin{aligned} g(x)(a+b) &= (a+b)x = ax + bx = g(x)(a) + g(x)(b) & a, b \in Ae_i, \\ g(x)(\lambda a) &= \lambda ax = \lambda g(x)(a) & \lambda \in A, a \in Ae_i. \end{aligned}$$

Hence,  $g \in \text{Hom}_A(Ae_i, X)$ . However, we can also have that  $g : e_iX \rightarrow \text{Hom}_A(Ae_i, X)$  through  $x \mapsto g(x)(r)$ .

So now we need to show that  $f$  and  $g$  are inverse constructions of one another. We have for some  $\theta \in \text{Hom}_A(Ae_i, X)$ ,

$$\begin{aligned} \theta &\xrightarrow{f} \theta(e_i) \\ g(\theta(e_i))(a) &= a\theta(e_i) = \theta(ae_i) \xleftarrow{g} \theta(e_i) \end{aligned}$$

but  $a \in Ae_i$  and so  $a = \lambda e_i$  for some  $\lambda \in A$ , hence  $ae_i = \lambda e_i^2 = \lambda e_i = a$ . Thus,  $g(\theta(e_i))(a) = \theta(ae_i) = \theta(a)$  so  $g(\theta(e_i)) = \theta$  and  $f$  and  $g$  are inverses.  $\square$

**Lemma 3.1.19.** *If  $0 \neq a \in Ae_i$  and  $0 \neq b \in e_iA$  then  $ab \neq 0$ .*

*Proof.* We know that  $a$  and  $b$  must have the forms,  $a = \lambda_1 p_1 + \dots \lambda_n p_n$  and  $b = \mu_1 q_1 + \dots \mu_m q_m$  where the  $p$  are paths starting at vertex  $i$ , the  $q$  are paths termination at vertex  $i$  and  $p_r, q_r$  are paths of length  $r$ . Then the longest path in the product  $ab$  must be  $\lambda_n \mu_m p_n q_m$  and so  $\lambda_n \mu_m \neq 0$  and  $p_n q_m \neq 0$ , so the product  $ab \neq 0$ .  $\square$

**Lemma 3.1.20.** *The  $e_i$  are primitive idempotents, i.e.  $Ae_i$  is an indecomposable module.*

*Proof.* From Lemma 3.1.18 we know that  $\text{End}_A(Ae_i) \cong e_i Ae_i$  and if this contains an idempotent  $\epsilon$ , then  $\epsilon^2 = \epsilon = \epsilon e_i$ , so  $\epsilon(e_i - \epsilon) = 0$ , but from Lemma 3.1.19 we know that this can not happen if  $\epsilon, (e_i - \epsilon) \neq 0$ , thus we must have that either  $\epsilon = 0$  or  $\epsilon = e_i$  and the result follows.  $\square$

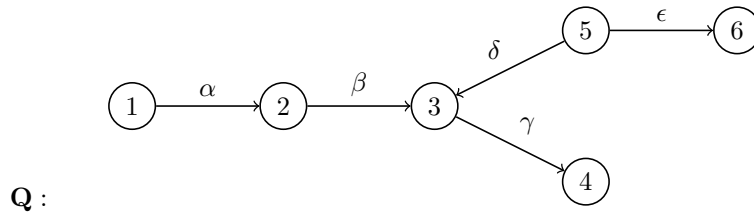
**Lemma 3.1.21.** *If  $e_i \in Ae_jA$  then  $i = j$ .*

*Proof.* For similar reasoning as in Lemma ??, we can see that  $Ae_jA$  has as a basis all the paths in  $A$  which pass through the vertex  $j$  and, by the definition of the trivial path,  $e_i$  cannot pass through the vertex  $j$  unless  $i = j$  and so if  $e_i \in Ae_jA$  we must have that  $i = j$ .  $\square$

**Lemma 3.1.22.** *The  $e_i$  are inequivalent, i.e.  $Ae_i \not\cong Ae_j$  for  $i \neq j$ .*

*Proof.* Two idempotent elements  $e_i, e_j$  are said to be equivalent iff there exists some  $u \in e_i Ae_j$  and  $v \in e_j Ae_i$  such that  $uv = e_i$  and  $vu = e_j$ . From Lemma 3.1.18 we can see that  $\text{Hom}(Ae_i, Ae_j) \cong e_i Ae_j$  and  $\text{Hom}(Ae_j, Ae_i) \cong e_j Ae_i$ , and so inverse isomorphisms give elements  $u$  and  $v$  as described above with  $uv = e_i$  and  $vu = e_j$ . However, this means that the path  $e_i$  must pass through the vertex  $j$  at some point, and vice versa, which contradicts Lemma .  $\square$

**Example 3.1.23.** Consider the quiver from Example 3.1.12,



then we have the following examples displaying the previous results for the idempotents of  $kQ$ . Let  $A = kQ$ .

1. We can see that  $e_1 e_1 = e_1$  and  $e_1 e_2 = 0$ . Also,  $1_A = e_1 + e_2 + e_3 + e_4 + e_5 + e_6$  and so

$$\begin{aligned}
1_A \gamma \beta &= (e_1 + e_2 + e_3 + e_4 + e_5 + e_6) \gamma \beta, \\
&= e_1 \gamma \beta + e_2 \gamma \beta + e_3 \gamma \beta + e_4 \gamma \beta + e_5 \gamma \beta + e_6 \gamma \beta, \\
&= e_4 \gamma \beta, \\
&= \gamma \beta.
\end{aligned}$$

2. Now we can see that  $Ae_1$  is spanned by all the paths starting at vertex 1, as,

$$\begin{aligned}
Ae_1 &= \{ae_1 : a \in A\}, \\
&= \text{span}_k\left\{\sum pe_1 : p \text{ is in the basis of } A\right\}, \\
&= \text{span}_k\{e_1e_1 + e_2e_1 + e_3e_1 + e_4e_1 + e_5e_1 + e_6e_1 + \alpha e_1 + \\
&\quad \beta e_1 + \gamma e_1 + \delta e_1 + \epsilon e_1 + \beta\alpha e_1 + \gamma\beta e_1 + \gamma\delta e_1 + \gamma\beta\alpha e_1\}, \\
&= \text{span}_k\{e_1 + \alpha + \beta\alpha + \gamma\beta\alpha\}, \\
&= \text{span}_k\{\text{all paths starting at vertex 1}\}.
\end{aligned}$$

Similarly, we can see that,

$$\begin{aligned}
e_3A &= \{e_3a : a \in A\}, \\
&= \text{span}_k\{e_3 + \beta + \delta + \beta\alpha\}, \\
&= \text{span}_k\{\text{all the paths terminating at vertex 3}\}.
\end{aligned}$$

Then we can see that,

$$\begin{aligned}
e_3Ae_1 &= \{e_3ae_1 : a \in A\}, \\
&= \text{span}_k\{\beta\alpha\}, \\
&= \text{span}_k\{\text{all paths starting at vertex 1 and terminating at vertex 3}\}.
\end{aligned}$$

3. We can see that,

$$\begin{aligned}
\oplus_{i=1}^6 Ae_i &= Ae_1 \oplus Ae_2 \oplus Ae_3 \oplus Ae_4 \oplus Ae_5 \oplus Ae_6, \\
&= \text{span}_k\{e_1 + \alpha + \beta\alpha + \gamma\beta\alpha\} \oplus \text{span}_k\{e_2 + \beta + \gamma\beta\} \oplus \text{span}_k\{e_3 + \gamma\} \oplus \text{span}_k\{e_4\} \oplus \text{span}_k\{e_5 + \delta\} \\
&= A.
\end{aligned}$$

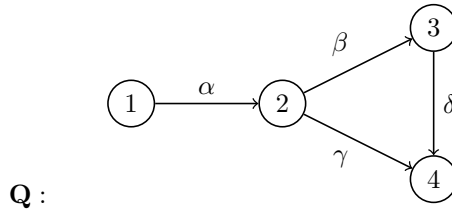
The following properties of path algebras are from [2], where they are given as exercises. In this report we will present them with proofs. Once again  $A = kQ$ .

**Lemma 3.1.24.** *A is finite dimensional if and only if Q has no oriented cycles.*

*Proof.*  $\Rightarrow$ : If  $Q$  has an oriented cycle then it will have a infinite number of paths as you can keep going round the cycle. This means that the basis of  $A$ , which is all the paths in  $Q$ , will be infinite, and so  $A$  will not be finite dimensional.

$\Leftarrow$ : If  $Q$  has no oriented cycles then it must have a finite number of paths and so the basis of  $A$  will be finite and hence  $A$  will be finite dimensional.  $\square$

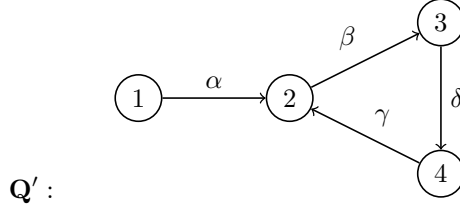
**Example 3.1.25.** If we consider the quiver  $Q$ ,



and we can see that  $Q$  has the paths  $e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \beta\alpha, \gamma\alpha, \beta\delta, \delta\beta\alpha$ , and so the basis of  $A$  is  $\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \beta\alpha, \gamma\alpha, \beta\delta, \delta\beta\alpha\}$ , which is finite, and

so  $A$  is finite dimensional.

However, if we have the quiver  $Q'$ ,



it has paths  $e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \beta\alpha, \delta\beta, \gamma\delta, \delta\beta\alpha, \gamma\delta\beta\alpha, \beta\gamma\delta\beta\alpha, \delta\beta\gamma\delta\beta\alpha, \dots$ , and so on, an infinite number of paths, meaning that the basis of  $A$  is infinite and hence  $A$  is not finite dimensional.

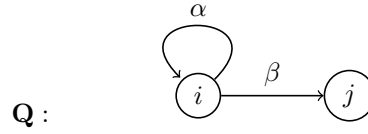
**Lemma 3.1.26.**  $A$  is prime, i.e.  $IJ \neq 0$  for two sided ideals  $I, J \neq 0$  if and only if for all  $i, j \in Q_0$  there exists a path  $i$  to  $j$ .

*Proof.* Need to include proof here □

**Lemma 3.1.27.**  $A$  is left noetherian if and only if, if there is an oriented cycle through the vertex  $i$ , then only one arrow starts at the vertex  $i$ .

*Proof.* Let  $Ap$  represent the left-ideal whose basis are all the paths starting with  $p$  for any path  $p$ .

Suppose we have a quiver  $Q$  of the form,



assuming  $i \neq j$  and where  $\alpha$  represents an orientated cycle, then we can see that,

$$A\beta + A\beta\alpha + \dots + A\beta\alpha^n \subseteq A\beta + A\beta\alpha + \dots + A\beta\alpha^{n+1}.$$

However, since  $e_j\beta\alpha^{n+1} \notin A\beta + A\beta\alpha + \dots + A\beta\alpha^n$  the inclusion is strict. Hence, if  $Q$  is of the form above, there is an ascending chain of left ideals,

$$A\beta \subset A\beta + A\beta\alpha \subset A\beta + A\beta\alpha + A\beta\alpha^2 \subset \dots$$

which does not terminate and so  $A$  is not noetherian.

Suppose that for all  $i \in Q_0$ , if  $\alpha$  an oriented cycle where  $s(\alpha) = i$ , then for all  $\rho \in Q_1$ ,  $\rho = \alpha_1$  or  $s(\rho) \neq i$ ; where  $\alpha = \alpha_1\alpha_2 \dots \alpha_m$  for some  $\alpha_1, \dots, \alpha_m \in Q_1$ . This means there are some  $\rho_1, \dots, \rho_n$  paths such that given any path  $\beta \in A$  we have that  $\beta = q\rho_j$  for some  $j \in \{1, \dots, n\}$  and some  $q$  a path in an oriented cycle. Now if we let  $Q'$  be the quiver, and  $A'$  the corresponding path algebra, where,

$$Q'_0 = Q_0 \text{ \& } Q'_1 = Q_1 \cup \{a \in Q_1 : a \text{ is not in an oriented cycle of } Q\}.$$

So, for any basis element  $\beta$  of  $A$ ,  $\beta = q\rho_j$  where  $q$  is a path in  $A'$ . Following the previous two results we can see that,

$$A = A'\rho_1 + A'\rho_2 + \dots + A'\rho_n \tag{3.1}$$

**Claim.** *If  $S$  is a subring of a ring  $R$  and  $R$  is finitely generated as a left  $S$ -module, then, if  $S$  is a noetherian ring, so is  $R$ .*

*Proof.* From the result that a finitely generated module over a noetherian ring is noetherian, gives us that  $R$  is noetherian as a left  $S$ -module. Let  $I_1 \subseteq I_2 \subseteq \dots$  be an ascending chain of left ideals in  $R$ , and each  $I_k$  is a left  $S$ -submodule of  $R$ ; so by the above, this chain terminates.  $\square$

Using the above Claim and the earlier result that  $A$  is finitely generated as a left  $A'$ -module, we can see that our problem of proving that  $A$  is left noetherian reduces to proving that  $A'$  is left noetherian.

Now let  $\alpha_j^{(l)}$  be the  $j^{\text{th}}$  arrow of the  $l^{\text{th}}$  oriented cycle in  $Q'$  and then,

$$Q'_1 = \{\alpha_j^{(l)} : j \in \{1, \dots, q_l\}, l \in \{1, \dots, r\}\}.$$

Consider the endomorphism,

$$\begin{aligned} X_l : A' &\rightarrow A', \\ e_t(p) &\mapsto \alpha_{k-1}^{(l)} \alpha_{k-2}^{(l)} \dots \alpha_1^{(l)} \alpha_{q_l}^{(l)} \dots \alpha_k^{(l)} p, \text{ where } t(p) = t = s(\alpha_k^{(l)}), \\ &\text{otherwise } \mapsto 0. \end{aligned}$$

We can adapt Hilbert's Basis Theorem to show that  $k[X_1, \dots, X_r]$  is a noetherian ring and we can see that  $A'$  is a finitely generated  $k[X_1, \dots, X_r]$ -module. Hence,  $A'$  is left noetherian, and so  $A$  is left noetherian.  $\square$

**Definition 3.1.28.** We can define the *length* of any path  $p$  by the following: Let,

$$\text{length}(p) = \text{length}\left(\sum_{\text{arrows } \rho} \lambda_\rho \rho\right) := \max\{\text{length}(\rho) : \lambda_\rho \neq 0\},$$

where  $\lambda_\rho \neq 0$  for some  $\rho$ .

**Lemma 3.1.29.** *The basis of  $J(A)$  is  $\{\text{path } i \text{ to } j : \text{there is no path from } j \text{ to } i\}$ .*

*Proof.* Firstly, let's prove that,

$$J'(A) := \text{span}_k\{\text{paths } p : s(p) = i, t(p) = j \text{ with no paths from } j \text{ to } i\} \subseteq J(A).$$

Given any  $w \in J'(A)$  we have that,

1.  $(wp)^2 = 0$  for any path  $p$ .

*Proof.* There are no paths  $p$  such that  $t(p) = s(w)$  and  $s(p) = t(w)$ , so  $wpw = 0$ , so  $wpw = (wp)^2 = 0$ .  $\square$

2.  $(w(p + p'))^2 = 0$  for any path  $p$  and  $\lambda \in k$ .

*Proof.*

$$(w(p + p'))^2 = (wp + wp')^2 = (wp)^2 + wpwp' + wp'wp + (wp')^2$$

As  $wpw$  and  $wp'w$  are both equal to zero by the above, we have that,  $(w(p + p'))^2 = 0$ .  $\square$

$$3. (w(\lambda p))^2 = \lambda^2(wp)^2 = 0$$

*Proof.* By item 1. □

Now let  $w \in J'(A)$  and  $z \in A$ , then,

$$wz = u\left(\sum_{\text{paths } p} \lambda_p p\right) = w\left(\sum_{\substack{\text{paths } p: \\ t(p)=s(w)}} \lambda_p p\right) := \tau$$

and from above we can see that  $\tau^2 = 0$ , meaning  $(wz)^2 = 0$ , so,  $(1 + wz)(1 - wz) = 1$ , hence,  $1 + wz \in U(A)$  for all  $z \in A$ . Thus  $w \in J(A)$ .

Now we want to prove that  $J(A) \subseteq J'(A)$ . Let  $w$  be a basis element of  $J(A)$  coming from the basis of  $A$ . Suppose also that  $\text{length}(w) = 0$ , so  $w = e_i$  for some  $i \in Q_0$ . Let,

$$M_i := \sum_{\substack{\text{paths } p: \\ p \neq e_i}} \lambda_p p,$$

then  $M_i \trianglelefteq A$  and if  $M_i \subsetneq N \trianglelefteq A$ , then,  $e_i \in N$ , so,  $N = A$ ; hence,  $M_i$  is a maximal left ideal, but  $w \notin M_i$  contradicting that  $w \in J(A)$ . Hence,  $\text{length}(w) > 0$ .

Now suppose  $w \notin J'(A)$ , so there's a path  $p$  such that  $s(w) = t(p)$  and  $s(p) = t(w)$ . Since  $w \in J(A)$  and  $p \in A$ ,  $1 + wp \in U(A)$ , so there exists some  $z \in A$  such that  $(1 + wp)z = 1$ , with,

$$z = \sum_{l=0}^m \sum_{\substack{\text{paths } p: \\ \text{length}(p_l)=l}} \lambda_{p_l} p_l.$$

Since,  $\text{length}(1) = \text{length}(l_1 + \dots + l_n) = 0$ , we have that,

$$\begin{aligned} 0 &= \text{length}((1 + wp)z), \\ &= \text{length}(z + wpz), \\ &= \max(\text{length}(z), \text{length}(wpz)), \end{aligned}$$

so  $wpz = 0$  as otherwise  $\text{length}(wpz) = 0$ , contradicting that  $\text{length}(w) > 0$ . So as  $(1 + wp)z = z + wpz = 1$ ,  $z = 1$ , so  $wp = 0$ , a contradiction. □

**Lemma 3.1.30.** *The centre of  $A$  is  $k \times k \times \dots \times k[T] \times k[T] \times \dots$ , with one factor for each connected component  $C$  of  $Q$ , and that the factor is  $k[T]$  if and only if  $C$  is an oriented cycle.*

*Proof.* Firstly, we know that if our quiver  $Q$  is composed of  $n$  connected components  $C_1, \dots, C_n$  then we our path algebra looks like  $kQ = kC_1 \times kC_2 \times \dots \times kC_n$ .

**Claim.** *Where  $Z(A)$  represents the centre of  $A$ , we have that,*

$$Z(A \times B) = Z(A) \times Z(B).$$

*Proof.* Suppose we have  $(a, b) \in Z(A \times B)$ , then we have that,

$$\begin{aligned} (a, b)(a', b') &= (a', b')(a, b) \forall (a', b') \in A \times B, \\ \Leftrightarrow (aa', bb') &= (a'a, b'b) \forall a' \in A, b' \in B, \\ \Leftrightarrow aa' &= a'a \ \& \ bb' = b'b \forall a' \in A, b' \in B, \\ \Leftrightarrow a &\in Z(A) \ \& \ b \in Z(B). \end{aligned}$$

Hence,  $Z(A \times B) \cong Z(A) \times Z(B)$ . □

Now, using the claim, we can see that,

$$Z(kQ) \cong Z(kC_1) \times Z(kC_2) \times \cdots \times Z(kC_n),$$

Now, assume the connected component  $C_i$  is not an oriented cycle, and consider  $a \in Z(kC_i)$ , which is a linear combination of paths. Let us choose a path of maximal length  $\rho_1 \dots \rho_m$ , and say there exists some  $\sigma$  such that we can have  $\sigma\rho_1 \dots \rho_m$  but with  $t(\sigma) \neq s(\rho_m)$ , but since  $\rho_1 \dots \rho_m$  was maximal this path is involved  $\sigma a = a\sigma$  as  $a \in Z(A)$ . So,  $a\sigma$  is also a linear combination of paths, i.e.  $\tau_1 \dots \tau_m \sigma$ , where  $\tau_1 \dots \tau_m$  is a path in  $a$ . This causes,

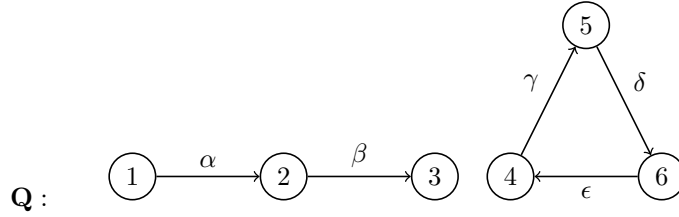
$$\sigma\rho_1 \dots \rho_m = \tau_1 \dots \tau_m \sigma,$$

which implies that  $\sigma = \rho_m$ , which causes an oriented cycle, hence a contradiction. Thus,  $Z(kC_i) \cong k$ .

*Need to complete the case for oriented cycle connected component.*

□

**Example 3.1.31.** Consider the quiver,



and so,  $Z(kQ) = Z(kC_1) \times Z(kC_2)$ , where  $C_2$  is the oriented cycle component. Then,

$$\begin{aligned} Z(kC_1) &= \{a \in kC_1 : ac = ca \forall c \in kC_1\}, \\ &= \{\lambda(e_1 + e_2 + e_3) : \lambda \in k\}, \\ &\cong k, \end{aligned}$$

since none of the elements of  $kC_1$  are commutative, apart from the identity. Also,

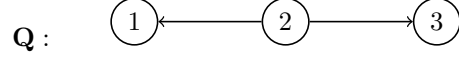
$$\begin{aligned} Z(kC_2) &= \{a \in kC_2 : ac = ca \forall c \in kC_2\}, \\ &= \{\lambda(\gamma\delta\epsilon + \delta\epsilon\gamma + \epsilon\gamma\delta) : \lambda \in k\}, \\ &\cong k[T], \text{ where } T = \gamma\delta\epsilon + \delta\epsilon\gamma + \epsilon\gamma\delta, \end{aligned}$$

because none of the elements of  $kC_2$  are commutative other than the identity element. Thus,  $Z(kQ) \cong k \times k[T]$ .

## 3.2 Representations of Quivers

**Definition 3.2.1.** A *representation*  $X$  of a quiver  $Q$  is given by considering each vertex  $i \in Q_0$  as a vector space  $X_i$ , and each arrow  $\rho \in Q_1$  as a linear map  $X_\rho : X_{s(\rho)} \rightarrow X_{t(\rho)}$ .

**Example 3.2.2.** Let  $Q$  be the quiver,



then we can have representations  $X$  and  $Y$ ,

$$\mathbf{X}: \quad k \xleftarrow[1]{\alpha} k \xrightarrow[1]{\beta} k \quad \& \quad \mathbf{Y}: \quad k \xleftarrow[1]{\gamma} k \xrightarrow{\delta} 0$$

**Definition 3.2.3.** A *morphism*  $\theta : X \rightarrow X'$  between representations is given by linear maps  $\theta_i : X_i \rightarrow X'_i$  for each  $i \in Q_0$  satisfy  $X'_\rho \theta_{s(\rho)} = \theta_{t(\rho)} X_\rho$  for each  $\rho \in Q_1$ . The *composition* of morphisms,  $\theta : X \rightarrow X'$  with  $\phi : X' \rightarrow X''$  is given by  $(\phi\theta)_i = \phi_i \theta_i$ .

**Example 3.2.4.**

### 3.3 Standard Resolution

### 3.4 Bricks

### 3.5 Dynkin and Euclidean diagrams

### 3.6 Gabriel's Theorem



## Chapter 4

# Auslander-Reiten Quivers

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