Representation of Quivers

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Homological Algebra

2.1 Chain Complexes

Definition 2.1.1. A *chain complex* $\mathbf{C}_\bullet$ consists of a sequence of $\mathbb{R}$-modules C_i ($i \in \mathbb{Z}$) and morphisms of the form,

$$\mathbf{C}: \qquad \ldots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} C_{-2} \xrightarrow{\delta_{-2}} \ldots$$

such that $\delta_{n-1}\delta_n = 0$ for all n, i.e. the composition of any two consecutive maps is zero. The maps δ_n are called the *differentials* of C.

Remark 2.1.2. It is convention that the map δ_n starts at C_n.

Example 2.1.3. If we have a field K then we can create the following chain complex:

$$\mathbf{C}: \qquad \ldots \to 0 \to K^2 \xrightarrow{\begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 0 \end{pmatrix}} K^3 \xrightarrow{\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}} K \to 0 \to \ldots$$

We can clearly see that the maps uphold the $\delta^2 = 0$ condition as,

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

Example 2.1.4. If we consider the sequence,

$$\ldots \to 0 \to K \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} K^2 \xrightarrow{\begin{pmatrix} 1 & 0 \end{pmatrix}} K \to 0 \to \ldots$$

however,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \neq 0.$$

Hence, $\delta^2 \neq 0$ and so the sequence is not a chain complex. However, if we change the second map slightly we obtain the chain complex,

$$\mathbf{C}: \qquad \ldots \to 0 \to K \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} K^2 \xrightarrow{\begin{pmatrix} 0 & 1 \end{pmatrix}} K \to 0 \to \ldots$$

since,

$$\begin{pmatrix}1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = 1 \neq 0.$$

Definition 2.1.5. If $\mathbf{C}$ is a chain complex then its *homology* is defined to be,

$$H_n(\mathbf{C}) = \frac{Ker(\delta_n : C_n \to C_{n-1})}{Im(\delta_{n+1} : C_{n+1} \to C_n)} = \frac{Z_n(\mathbf{C})}{B_n(\mathbf{C})}.$$

This becomes an $\mathbb{R}$-module and, since δ^2, it follows that $B_n(\mathbf{C}) \subseteq Z_n(\mathbf{C})$.

The following Lemma is the solution to Exercise 6.1 in [3].

Lemma 2.1.6. *If $\mathbf{C}$ is a chain complex with $C_n = 0$ for some n then $H_n(\mathbf{C}) = 0$.*

Proof. Well suppose we have such a chain complex,

$$\mathbf{C}: \qquad \ldots \to C_{n+1} \xrightarrow{\delta_{n+1}} 0 \xrightarrow{\delta_n} C_{n-1} \to \ldots$$

the the homology is,

$$H_n(\mathbf{C}) = \frac{Ker(\delta_n : 0 \to C_{n-1}}{Im(\delta_{n+1} : C_{n+1} \to 0)},$$

as the only element in C_n is the zero element, and so, $H_n(\mathbf{C}) = 0$, as required. $\square$

Examples 2.1.7 and 2.1.10 are taken from [1] and are included here because they are felt to be the clearest at demonstrating a chain complex and homology, however, the more general statement of the second example is presented as Proposition 2.1.8 because it is an interesting result.

Example 2.1.7. If we take a module M then we can make a chain complex;

$$\mathbf{C}: \qquad \ldots \to 0 \to M \to 0 \to \ldots$$

where M is at degree n. Then the homology will be:

$$H_i(\mathbf{C}) = \begin{cases} \frac{Ker(M \to 0)}{Im(0 \to M)} = M & i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.1.8. *If we have a module homomorphism between R-modules, $f : M \to N$, the we get the chain complex,*

$$\mathbf{C}: \qquad \underset{deg}{\ldots} \to \underset{n+2}{0} \to \underset{n+1}{M} \xrightarrow{f} \underset{n}{N} \to \underset{n-1}{0} \to \ldots,$$

and the homology becomes,

$$H_i(\mathbf{C}) = \begin{cases} \frac{N}{Im(f)} = Coker(f) & i = n \\ Ker(f) & i = n+1 \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. Firstly, at degree n we have that,

$$H_n(\mathbf{C}) = \frac{Ker(N \to 0)}{Im(M \xrightarrow{f} N)} = \frac{N}{Im(f)} = Coker(f).$$

Then at degree $n+1$ we have that,

$$H_{n+1}(\mathbf{C}) = \frac{Ker(M \xrightarrow{f} N)}{Im(0 \to M)} = Ker(f).$$

Finally, it is clear that everywhere else there is no homology. $\square$

Notation 2.1.9. Here,

$$Coker(f) = \frac{\text{Codomain of } f}{\text{Image of } f},$$

is the *cokernel* of the map f.

Example 2.1.10. We can have a chain complex of $\mathbb{Z}$-modules,

$$\mathbf{C}: \qquad \underset{\text{deg}}{\ldots} \to \underset{2}{0} \to \underset{1}{\mathbb{Z}} \xrightarrow{a} \underset{0}{\mathbb{Z}} \to \underset{-1}{0} \to \ldots$$

where the map a is right multiplication by some $a \in \mathbb{Z}$. The homology is,

$$H_i(\mathbf{C}) = \begin{cases} \frac{Ker(\mathbb{Z} \to 0)}{Im(\mathbb{Z} \xrightarrow{a} \mathbb{Z})} = \frac{\mathbb{Z}}{a\mathbb{Z}} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that,

$$H_0(C) = \frac{\mathbb{Z}}{a\mathbb{Z}} = \frac{\text{Codomain of } f}{\text{Image of } f} = Coker(a).$$

Also,

$$H_1(C) = \frac{Ker(\mathbb{Z} \xrightarrow{a} \mathbb{Z})}{Im(0 \to \mathbb{Z})} = Ker(a) = 0,$$

because $Ker(a)$ is empty.

Definition 2.1.11.

- The elements of $B_n(\mathbf{C})$ are called $n-$*boundaries*.
- The elements of $Z_n(\mathbf{C})$ are called $n-$*cycles*.

Remark 2.1.12. If $x \in Z_n(\mathbf{C})$ then its image in $H_n(\mathbf{C})$ is usually written as $[x]$.

Definition 2.1.13. A chain complex $\mathbf{C}$ is said to be:

- *acyclic* if $H_n(\mathbf{C}) = 0$ for all n.
- *bounded above* if there exists some $n \in \mathbb{N}$, $C_k = 0$ for all $k > n$.
- *bounded below* if for some $n \in \mathbb{N}$, $C_k = 0$ for all $k < n$.
- *bounded* if it is bounded above and below.
- *non-negative* if $C_n = 0$ for $n < 0$.

Example 2.1.14. All the chain complexes in the previous examples are bounded both above and below, however, neither is acyclic as they both have instances where the homology is non-zero. The chain complex in Example 2.1.10 is non-negative because $C_n \neq 0$ only when $n = 0, 1$.

Example 2.1.15. If we take another look at the chain complex in Example 2.1.4,

$$\mathbf{C}: \qquad \underset{\text{deg}}{\dots} \to 0 \to \underset{1}{K} \xrightarrow{\binom{1}{0}} \underset{0}{K^2} \xrightarrow{(0\ 1)} \underset{-1}{K} \to 0 \to \dots$$

the homologies are,

$$H_1(\mathbf{C}) = \frac{Ker(K \xrightarrow{\binom{1}{0}} K^2)}{Im(0 \to K)} \cong \frac{K}{K} \cong 0,$$

$$H_0(\mathbf{C}) = \frac{Ker(K^2 \xrightarrow{(1\ 0)} K)}{Im(K \xrightarrow{\binom{1}{0}} K^2)} \cong \frac{K}{K} \cong 0,$$

$$H_{-1}(\mathbf{C}) = \frac{Ker(K \to 0)}{Im(K^2 \xrightarrow{(1\ 0)} K))} \cong \frac{K}{K} \cong 0.$$

Thus $H_n(\mathbf{C}) = 0$ for all n and so $\mathbf{C}$ is an acyclic chain complex. Later in the report, we will see that $\mathbf{C}$ is in fact a short exact sequence.

Example 2.1.16. The chain complex,

$$\mathbf{C}: \qquad \underset{\text{deg}}{\dots} \xrightarrow{3} \underset{1}{\frac{\mathbb{Z}}{9\mathbb{Z}}} \xrightarrow{3} \underset{0}{\frac{\mathbb{Z}}{9\mathbb{Z}}} \xrightarrow{3} \underset{-1}{\frac{\mathbb{Z}}{9\mathbb{Z}}} \to \dots$$

where the differentials are the maps,

$$\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \to \frac{\mathbb{Z}}{9\mathbb{Z}},\ z + 9\mathbb{Z} \mapsto 3z + 9\mathbb{Z},$$

is unbounded. It is also acyclic, since the homology is,

$$H_n(\mathbf{C}) = \frac{Ker(\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \to \frac{\mathbb{Z}}{9\mathbb{Z}})}{Im(\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \to \frac{\mathbb{Z}}{9\mathbb{Z}})} \cong \frac{\mathbb{Z}/3\mathbb{Z}}{\mathbb{Z}/3\mathbb{Z}} \cong 0,$$

for all n.

Definition 2.1.17. A *cochain complex* $\mathbf{C}^\bullet$ consists of a sequence of $\mathbb{R}$-modules C^i $(i \in \mathbb{Z})$ and morphisms of the form,

$$\mathbf{C}: \qquad \dots \xrightarrow{\delta^{-3}} C^{-2} \xrightarrow{\delta^{-2}} C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \dots$$

such that $\delta^{n-1}\delta^n = 0$ for all n, i.e. the composition of any two consecutive maps is zero.

Remark 2.1.18. Chain and cochain complexes can be thought of as almost identical constructs with the only difference being thenumbering of the chain. The degree of a chain complex *decreases* from left to right, whereas, the degree of a cochain complex *increses* from left to right. So, we can compute one from the other by setting $C^{-n} = C_n$, or equivalently $C^n = C_{-n}$; this is called *renumbering.*

Definition 2.1.19. If $\mathbf{C}$ is a cochain complex then its *cohomology* is defined to be,

$$H^n(\mathbf{C}) = \frac{Ker(\delta^n : C^n \to C^{n+1})}{Im(\delta^{n-1} : C^{n-1} \to C^n)} = \frac{Z^n(\mathbf{C})}{B^n(\mathbf{C})}.$$

- The elements of $B_n(\mathbf{C})$ are called $n-$*coboundaries.*
- The elements of $Z_n(\mathbf{C})$ are called $n-$*cocycles.*

Example 2.1.20. We can renumber the chain complex in Example 2.1.10 to get the cochain complex,

$$\mathbf{C}: \qquad \underset{\deg}{\ldots} \to \underset{-2}{0} \to \underset{-1}{\mathbb{Z}} \xrightarrow{a} \underset{0}{\mathbb{Z}} \to \underset{1}{0} \to \ldots$$

Its cohomology is,

$$H^i(\mathbf{C}) = \begin{cases} \frac{Ker(\mathbb{Z} \to 0)}{Im(\mathbb{Z} \xrightarrow{a} \mathbb{Z})} = \frac{\mathbb{Z}}{a\mathbb{Z}} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.1.21. Let $\mathbf{C}$ be a chain complex of left R-modules. If M is a left R-module then $Hom(\mathbf{C}, M)$ is the cochain complex where,

$$Hom(\mathbf{C}, M)^n = Hom(C_n, M),$$

and the differentials,

$$\delta^n : Hom(\mathbf{C}, M)^n \to Hom(C, M)^{n+1},$$

are induced by the differentials of $\mathbf{C}$, $\delta_n : C_{n+1} \to C_n$. The cohomology of this cochain complex is denoted $H^n(\mathbf{C}, M)$.

The following example is a generalised version of one found in [1].

Example 2.1.22. Consider the acyclic chain complex,

$$\mathbf{C}: \qquad \underset{\deg}{\ldots} \to 0 \to \underset{1}{\mathbb{Z}} \xrightarrow{n} \underset{0}{\mathbb{Z}} \xrightarrow{nat} \underset{-1}{\frac{\mathbb{Z}}{n\mathbb{Z}}} \to \ldots$$

So applying $Hom(-, \mathbb{Z})$ we gives the cochain complex,

$$\mathbf{C}': \qquad \underset{\deg}{\ldots} \to 0 \to \underset{1}{\mathbb{Z}} \xrightarrow{n} \underset{0}{\mathbb{Z}} \xrightarrow{nat} \underset{-1}{\frac{\mathbb{Z}}{n\mathbb{Z}}} \to \ldots$$

which has cohomology,

$$H^i(\mathbf{C}', \mathbb{Z}) = \begin{cases} \frac{Ker(\mathbb{Z} \to 0)}{Im(\mathbb{Z} \xrightarrow{n} \mathbb{Z})} \cong \frac{\mathbb{Z}}{n\mathbb{Z}} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that despite the chain complex being acyclic, its cohomology induced by $Hom(-, \mathbb{Z})$ is not zero everywhere.

Chapter 3

Representation of Quivers

3.1 Quivers and Path Algebras

Definition 3.1.1. A *quiver* is defined as the tuple of sets and functions, $\mathbf{Q} = (Q_0,\, Q_1,\, s,\, t : Q_1 \to Q_0)$ such that:

- Q_0 is the set of vertices, which we will set to be the finite set $\{1, 2, \ldots, n\}$.
- Q_1 is the set of arrows, which we will also set to be finite.
- Functions s, t such that an arrow $\rho \in Q_1$ *starts* at the vertex $s(\rho) \in Q_0$ and *terminates* at the vertex $t(\rho) \in Q_0$, i.e. $\rho : s(\rho) \to t(\rho)$.

Example 3.1.2. A quiver $Q = (Q_0, Q_1, s, t : Q_1 \to Q_0)$ where $Q_0 = \{1, 2, 3, 4\}$, $Q_1 = \{\alpha, \beta\}$, and s, t are defined such that;

$$\begin{aligned} s :& Q_1 \to Q_0, \quad \alpha \mapsto 1,\ \beta \mapsto 2,\ \gamma \mapsto 4 \\ t :& Q_1 \to Q_0, \quad \alpha \mapsto 2,\ \beta \mapsto 3,\ \gamma \mapsto 3, \end{aligned}$$

looks like,

$$\mathbf{Q} : \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"'] & 3 & 4 \arrow[l, "\gamma"] \end{tikzcd}$$

Definition 3.1.3. A *non-trivial path*, p, in a quiver is a sequence of arrows $\rho_1, \ldots, \rho_n$ which satisfies $t(\rho_{i+1}) = s(\rho_i)$ for all $1 \leq i < n$, i.e. the start of an arrow is where the previous arrow terminated. The starting and terminatinating vertex of a path p are denoted $s(p)$ and $t(p)$, respectively.

Notation 3.1.4. In this report the arrows in a path will be ordered the same way as the composition of functions, as in [2], however, be aware that other publications may order the arrows the opposite way.

Definition 3.1.5. The *trivial path* is the path which contains no arrows, i.e. it is a single vertex, and is denoted e_i where the vertex is i.

Example 3.1.6. The paths of the quiver in Example 3.1.2 are:

$$p_1 = e_1, \quad p_2 = e_2, \quad p_3 = e_3, \quad p_4 = e_4, \quad p_5 = \alpha, \quad p_6 = \beta, \quad p_7 = \gamma, \quad p_8 = \beta\alpha.$$

However, $\gamma\beta\alpha$ is not a path because $t(\gamma) = 3 \neq s(\beta) = 2$.

Definition 3.1.7. A *path algebra* kQ is the k-alegbra which has the basis all the paths in Q, and the product of two paths p, q is defined as,

$$pq = \begin{cases} \text{obvious composition} & \text{if } t(q) = s(p), \\ 0 & \text{otherwise.} \end{cases}$$

This multiplication is assosciative.

Example 3.1.8. If we once again use the quiver Q from Example 3.1.2, then, from Example 3.1.6, we know the basis of the path algebra kQ will be,

$$\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \beta\alpha\}.$$

The product of β and α is $\beta\alpha$, but the product of α and β is zero, and the product of e_2 and α is just α.

Example 3.1.9. If Q is the following quiver,

$$\mathbf{Q}: \quad \begin{tikzcd} 1 \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

forms the path algebra $kQ \cong k[X, Y]$, the free, assosciative algebra on two letters. In fact, if we have a quiver with a single vertex and n loops, then this can be associated with the free, assosciated algebra on n letters.

Example 3.1.10. If we have the quiver,

$$\mathbf{Q}: \quad \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & 3 \arrow[r, "\gamma"] & 4 \end{tikzcd}$$

the the path algebra, $kQ \cong UT_4(k)$ by the isomorphism,

$$\begin{aligned} &\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 \alpha + \\ &\qquad \lambda_6 \beta + \lambda_7 \gamma + \lambda_8 \beta\alpha + \lambda_9 \gamma\beta\alpha + \lambda_{10} \gamma\beta \mapsto \begin{pmatrix} \lambda_4 & \lambda_7 & \lambda_{10} & \lambda_9 \\ 0 & \lambda_3 & \lambda_6 & \lambda_8 \\ 0 & 0 & \lambda_2 & \lambda_5 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix} \end{aligned}$$

Generally, a quiver of the form,

$$\mathbf{Q'}: \quad \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & \cdots \arrow[r, "\gamma"] & n \end{tikzcd}$$

induces a path alegbra $kQ' \cong UT_n(k)$ for any n.

Example 3.1.11. In fact, we find that if Q is the same quiver as above in Example 3.1.10, then $kQ \cong LT_4(k)$ as well, through the isomorphism,

$$\begin{aligned} &\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 \alpha + \\ &\qquad \lambda_6 \beta + \lambda_7 \gamma + \lambda_8 \beta\alpha + \lambda_9 \gamma\beta\alpha + \lambda_{10} \gamma\beta \mapsto \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ \lambda_5 & \lambda_2 & 0 & 0 \\ \lambda_8 & \lambda_6 & \lambda_3 & 0 \\ \lambda_9 & \lambda_{10} & \lambda_7 & \lambda_4 \end{pmatrix}. \end{aligned}$$

Once again, this extends to the general case that, $kQ' \cong LT_n(k)$.

The following results about the idempotents of path alegbras are from [2], however, where possible we have provided or expanded on the given proof.

Bibliography

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