Representation of Quivers

Chanelle Lee
Student ID: 200646370
Supervisor: William Crawley-Boevey

May 8, 2015

Contents

Chapter 1

Introduction

This project aims to cover interesting topics from the area of quiver representations. The first chapter will give a brief introduction to the topics in Homological Algebra needed for bulk of the report. It will cover chain complexes, exact sequences, short exact sequences, and extensions, most importantly Ext^1. The the second chapter will introduce quivers and look at some interesting results about path algebras before moving onto representations of quivers and their relationship with the modules of the corresponding path algebra. Next we will cover Dynkin and Euclidean diagrams before finishing with a proof of Gabriel's Theorem. The third chapter will cover Auslander-Reiten quivers.

Chapter 2

Homological Algebra

2.1 Chain Complexes

Definition 2.1.1. A *chain complex* $\mathbf{C}_\bullet$ consists of a sequence of $\mathbb{R}$-modules C_i $(i \in \mathbb{Z})$ and morphisms of the form,

$$\mathbf{C}: \qquad \dots \xrightarrow{\delta_3} C_2 \xrightarrow{\delta_2} C_1 \xrightarrow{\delta_1} C_0 \xrightarrow{\delta_0} C_{-1} \xrightarrow{\delta_{-1}} C_{-2} \xrightarrow{\delta_{-2}} \dots$$

such that $\delta_{n-1}\delta_n = 0$ for all n, i.e. the composition of any two consecutive maps is zero. The maps δ_n are called the *differentials* of C.

Remark 2.1.2. It is convention that the map δ_n starts at C_n.

Example 2.1.3. If we have a field K then we can create the following chain complex:

$$\mathbf{C}: \qquad \dots \to 0 \to K^2 \xrightarrow{\begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 0 \end{pmatrix}} K^3 \xrightarrow{(0\ 0\ 1)} K \to 0 \to \dots$$

We can clearly see that the maps uphold the $\delta^2 = 0$ condition as,

$$\begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

Example 2.1.4. If we consider the sequence,

$$\dots \to 0 \to K \xrightarrow{\delta_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}} K^2 \xrightarrow{\delta_1 = (1\ 0)} K \to 0 \to \dots$$

however,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \neq 0.$$

Hence, $\delta_1\delta_2 \neq 0$, and so the sequence is not a chain complex. However, if we change δ_2 slightly we obtain the chain complex,

$$\mathbf{C}: \qquad \dots \to 0 \to K \xrightarrow{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} K^2 \xrightarrow{(0\ 1)} K \to 0 \to \dots$$

since,

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \neq 0.$$

Definition 2.1.5. For a chain complex,

$$\mathbf{C}: \qquad \ldots \to C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \to \ldots$$

we define:

$$Z_n(\mathbf{C}) = ker(\delta_n) \quad \& \quad B_n(\mathbf{C}) = im(\delta_{n+1}).$$

To explore the relationship between $Z_n(\mathbf{C})$ and $B_n(\mathbf{C})$, the following is a solution to Exercise 2.14 in [3].(Note the example will not be provided here as exactness has not yet been covered.)

Lemma 2.1.6. *Let,*

$$A \xrightarrow{f} B \xrightarrow{g} C,$$

be a sequence of module maps. Then $gf = 0$ if and only if $im(f) \subseteq im(g)$.

Proof. Suppose $gf = 0$, i.e., for all $a \in A$, $g(f(a)) = 0$ which implies that $f(a) \in ker(g)$ for all $a \in A$. Hence, $im(f) \subseteq ker(g)$.
Conversely, suppose that $im(f) \subseteq ker(g)$. Every element $b \in im(f)$ is also an element of $ker(g)$, thus $gf = 0$. $\square$

Definition 2.1.7. If $\mathbf{C}$ is a chain complex then its *homology* is defined to be,

$$H_n(\mathbf{C}) = \frac{ker(\delta_n : C_n \to C_{n-1})}{im(\delta_{n+1} : C_{n+1} \to C_n)} = \frac{Z_n(\mathbf{C})}{B_n(\mathbf{C})},$$

which is an $\mathbb{R}$-module.

The subsequent proposition is the solution to Exercise 6.1 in [3].

Proposition 2.1.8. *If $\mathbf{C}$ is a chain complex with $C_n = 0$ for some n then $H_n(\mathbf{C}) = 0$.*

Proof. Well suppose we have such a chain complex,

$$\mathbf{C}: \qquad \ldots \to C_{n+1} \xrightarrow{\delta_{n+1}} 0 \xrightarrow{\delta_n} C_{n-1} \to \ldots$$

the the homology is,

$$H_n(\mathbf{C}) = \frac{ker(\delta_n : 0 \to C_{n-1}}{im(\delta_{n+1} : C_{n+1} \to 0)},$$

as the only element in C_n is the zero element, and so, $H_n(\mathbf{C}) = 0$, as required. $\square$

Examples 2.1.9 and 2.1.12 are taken from [1] and are included here because they are felt to be the clearest at demonstrating a chain complex and homology, however, the more general statement of the second example is presented as Proposition 2.1.10 because it is an interesting result.

Example 2.1.9. If we take a module M then we can make a chain complex;

$$\mathbf{C}: \qquad \ldots \to 0 \to M \to 0 \to \ldots$$

where M is at degree n. Then the homology will be:

$$H_i(\mathbf{C}) = \begin{cases} \frac{ker(M \to 0)}{im(0 \to M)} = M & i = n, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.1.10. *If we have a module homomorphism between R-modules, $f : M \to N$, the we get the chain complex,*

$$\mathbf{C}: \qquad \underset{deg}{\ldots} \to \underset{n+2}{0} \to \underset{n+1}{M} \xrightarrow{f} \underset{n}{N} \to \underset{n-1}{0} \to \ldots,$$

and the homology becomes,

$$H_i(\mathbf{C}) = \begin{cases} \frac{N}{im(f)} = coker(f) & i = n \\ ker(f) & i = n+1 \\ 0 & otherwise. \end{cases}$$

Proof. Firstly, at degree n we have that,

$$H_n(\mathbf{C}) = \frac{ker(N \to 0)}{im(M \xrightarrow{f} N)} = \frac{N}{im(f)} = coker(f).$$

Then at degree $n+1$ we have that,

$$H_{n+1}(\mathbf{C}) = \frac{ker(M \xrightarrow{f} N)}{im(0 \to M)} = ker(f).$$

Finally, it is clear that everywhere else there is no homology. $\square$

Notation 2.1.11. Here,

$$Coker(f) = \frac{\text{Codomain of } f}{\text{Image of } f},$$

is the *cokernel* of the map f.

Example 2.1.12. We can have a chain complex of $\mathbb{Z}$-modules,

$$\mathbf{C}: \qquad \underset{\text{deg}}{\ldots} \to \underset{2}{0} \to \underset{1}{\mathbb{Z}} \xrightarrow{a} \underset{0}{\mathbb{Z}} \to \underset{-1}{0} \to \ldots$$

where the map a is right multiplication by some $a \in \mathbb{Z}$. The homology is,

$$H_i(\mathbf{C}) = \begin{cases} \frac{ker(\mathbb{Z} \to 0)}{im(\mathbb{Z} \xrightarrow{a} \mathbb{Z})} = \frac{\mathbb{Z}}{a\mathbb{Z}} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Note that,

$$H_0(C) = \frac{\mathbb{Z}}{a\mathbb{Z}} = \frac{\text{Codomain of } f}{\text{Image of } f} = coker(a).$$

Also,

$$H_1(C) = \frac{ker(\mathbb{Z} \xrightarrow{a} \mathbb{Z})}{im(0 \to \mathbb{Z})} = ker(a) = 0,$$

because $ker(a)$ is empty.

Definition 2.1.13.

- The elements of $B_n(\mathbf{C})$ are called $n-$*boundaries*.
- The elements of $Z_n(\mathbf{C})$ are called $n-$*cycles*.

Remark 2.1.14. If $x \in Z_n(\mathbf{C})$ then its image in $H_n(\mathbf{C})$ is usually written as $[x]$.

Definition 2.1.15. A chain complex $\mathbf{C}$ is said to be:

- *acyclic* if $H_n(\mathbf{C}) = 0$ for all n.
- *bounded above* if there exists some $n \in \mathbb{N}$, $C_k = 0$ for all $k > n$.
- *bounded below* if for some $n \in \mathbb{N}$, $C_k = 0$ for all $k < n$.
- *bounded* if it is bounded above and below.
- *non-negative* if $C_n = 0$ for $n < 0$.

Example 2.1.16. All the chain complexes in the previous examples are bounded both above and below, however, neither is acyclic as they both have instances where the homology is non-zero. The chain complex in Example 2.1.12 is non-negative because $C_n \neq 0$ only when $n = 0, 1$.

Example 2.1.17. If we take another look at the chain complex in Example 2.1.4,

$$\mathbf{C}: \qquad \underset{\text{deg}}{\ldots} \to 0 \to \underset{1}{K} \xrightarrow{\binom{1}{0}} \underset{0}{K^2} \xrightarrow{(0\ 1)} \underset{-1}{K} \to 0 \to \ldots$$

the homologies are,

$$\begin{aligned}
H_1(\mathbf{C}) &= \frac{ker(K \xrightarrow{\binom{1}{0}} K^2)}{im(0 \to K)} \cong \frac{K}{K} \cong 0,\\
H_0(\mathbf{C}) &= \frac{ker(K^2 \xrightarrow{(1\ 0)} K)}{im(K \xrightarrow{\binom{1}{0}} K^2)} \cong \frac{K}{K} \cong 0,\\
H_{-1}(\mathbf{C}) &= \frac{ker(K \to 0)}{im(K^2 \xrightarrow{(1\ 0)} K))} \cong \frac{K}{K} \cong 0.
\end{aligned}$$

Thus $H_n(\mathbf{C}) = 0$ for all n and so $\mathbf{C}$ is an acyclic chain complex. Later in the report, we will see that $\mathbf{C}$ is in fact a short exact sequence.

Example 2.1.18. The chain complex,

$$\mathbf{C}: \qquad \underset{\text{deg}}{\ldots} \xrightarrow{3} \underset{1}{\frac{\mathbb{Z}}{9\mathbb{Z}}} \xrightarrow{3} \underset{0}{\frac{\mathbb{Z}}{9\mathbb{Z}}} \xrightarrow{3} \underset{-1}{\frac{\mathbb{Z}}{9\mathbb{Z}}} \to \ldots$$

where the differentials are the maps,

$$\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \to \frac{\mathbb{Z}}{9\mathbb{Z}},\ z + 9\mathbb{Z} \mapsto 3z + 9\mathbb{Z},$$

is unbounded. It is also acyclic, since the homology is,

$$H_n(\mathbf{C}) = \frac{ker(\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \to \frac{\mathbb{Z}}{9\mathbb{Z}})}{im(\delta_n : \frac{\mathbb{Z}}{9\mathbb{Z}} \to \frac{\mathbb{Z}}{9\mathbb{Z}})} \cong \frac{\mathbb{Z}/3\mathbb{Z}}{\mathbb{Z}/3\mathbb{Z}} \cong 0,$$

for all n.

Definition 2.1.19. A *cochain complex* $\mathbf{C}^{\bullet}$ consists of a sequence of $\mathbb{R}$-modules C^i $(i \in \mathbb{Z})$ and morphisms of the form,

$$\mathbf{C} : \qquad \dots \xrightarrow{\delta^{-3}} C^{-2} \xrightarrow{\delta^{-2}} C^{-1} \xrightarrow{\delta^{-1}} C^0 \xrightarrow{\delta^0} C^1 \xrightarrow{\delta^1} C^2 \xrightarrow{\delta^2} \dots$$

such that $\delta^{n-1}\delta^n = 0$ for all n, i.e. the composition of any two consecutive maps is zero.

Remark 2.1.20. Chain and cochain complexes can be thought of as almost identical constructs with the only difference being thenumbering of the chain. The degree of a chain complex *decreases* from left to right, whereas, the degree of a cochain complex *increses* from left to right. So, we can compute one from the other by setting $C^{-n} = C_n$, or equivalently $C^n = C_{-n}$; this is called *renumbering.*

Definition 2.1.21. If $\mathbf{C}$ is a cochain complex then its *cohomology* is defined to be,

$$H^n(\mathbf{C}) = \frac{ker(\delta^n : C^n \to C^{n+1})}{im(\delta^{n-1} : C^{n-1} \to C^n)} = \frac{Z^n(\mathbf{C})}{B^n(\mathbf{C})}.$$

- The elements of $B_n(\mathbf{C})$ are called $n-$*coboundaries*.
- The elements of $Z_n(\mathbf{C})$ are called $n-$*cocycles*.

Example 2.1.22. We can renumber the chain complex in Example 2.1.12 to get the cochain complex,

$$\mathbf{C} : \qquad \underset{\deg}{\dots} \to \underset{-2}{0} \to \underset{-1}{\mathbb{Z}} \xrightarrow{a} \underset{0}{\mathbb{Z}} \to \underset{1}{0} \to \dots$$

Its cohomology is,

$$H^i(\mathbf{C}) = \begin{cases} \frac{ker(\mathbb{Z} \longrightarrow 0)}{im(\mathbb{Z} \xrightarrow{a} \mathbb{Z})} = \frac{\mathbb{Z}}{a\mathbb{Z}} & i = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.1.23. Let $\mathbf{C}$ be a chain complex of left R-modules. If M is a left R-module then $Hom(\mathbf{C}, M)$ is the cochain complex where,

$$Hom(\mathbf{C}, M)^n = Hom(C_n, M),$$

and the differentials,

$$\delta^n : Hom(\mathbf{C}, M)^n \to Hom(C, M)^{n+1},$$

are induced by the differentials of $\mathbf{C}$, $\delta_n : C_{n+1} \to C_n$. The cohomology of this cochain complex is denoted $H^n(\mathbf{C}, M)$.

The following example is a generalised version of one found in [1].

Example 2.1.24. Consider the acyclic chain complex,

$$\mathbf{C}: \qquad \underset{\text{deg}}{\ldots} \to 0 \to \underset{1}{\mathbb{Z}} \xrightarrow{n} \underset{0}{\mathbb{Z}} \xrightarrow{nat} \underset{-1}{\frac{\mathbb{Z}}{n\mathbb{Z}}} \to \ldots$$

So applying $Hom(-,\mathbb{Z})$ we gives the cochain complex,

$$\mathbf{C}': \qquad \underset{\text{deg}}{\ldots} \to 0 \to \underset{1}{\mathbb{Z}} \xrightarrow{n} \underset{0}{\mathbb{Z}} \xrightarrow{nat} \underset{-1}{\frac{\mathbb{Z}}{n\mathbb{Z}}} \to \ldots$$

which has cohomology,

$$H^i(\mathbf{C}',\mathbb{Z}) = \begin{cases} \frac{ker(\mathbb{Z}\to 0)}{im(\mathbb{Z}\xrightarrow{n}\mathbb{Z})} \cong \frac{\mathbb{Z}}{n\mathbb{Z}} & i=0, \\ 0 & \text{otherwise.} \end{cases}$$

Notice that despite the chain complex being acyclic, its cohomology induced by $Hom(-,\mathbb{Z})$ is not zero everywhere.

Definition 2.1.25. A *chain map* $f : \mathbf{C} \to \mathbf{D}$, with $\mathbf{C}$ and $\mathbf{D}$ chain complexes, is given by a homomorphism $f_n : \mathbf{C}_n \to \mathbf{D}_n$ for each n, such that each square in the following diagram commutes,

$$\begin{tikzcd}
\cdots \arrow[r, "\gamma_{n+2}"] & \mathbf{C}_{n+1} \arrow[r, "\gamma_{n+1}"] \arrow[d, "f_{n+1}"'] & \mathbf{C}_n \arrow[r, "\gamma_n"] \arrow[d, "f_n"'] & \mathbf{C}_{n-1} \arrow[r, "\gamma_{n-1}"] \arrow[d, "f_{n-1}"'] & \cdots \\
\cdots \arrow[r, "\delta_{n+2}"] & \mathbf{D}_{n+1} \arrow[r, "\delta_{n+1}"] & \mathbf{D}_n \arrow[r, "\delta_n"] & \mathbf{D}_{n-1} \arrow[r, "\delta_{n-1}"] & \cdots
\end{tikzcd}$$

Note that there is an equivalent notion of a *cochain map* of cochain complexes.

Example 2.1.26. If we have the following diagram,

$$\begin{tikzcd}
\mathbf{C}: & \cdots \arrow[r] & 0 \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r, "n"] \arrow[d, "f_1"', "nat"] & \mathbb{Z} \arrow[r, "nat"] \arrow[d, "f_0"', "nat"] & \frac{\mathbb{Z}}{n\mathbb{Z}} \arrow[r] \arrow[d, equal, "f_{-1}"'] & 0 \arrow[r] \arrow[d, equal] & \cdots \\
\mathbf{D}: & \cdots \arrow[r] & 0 \arrow[r] & \frac{\mathbb{Z}}{n\mathbb{Z}} \arrow[r, "n"] & \frac{\mathbb{Z}}{n^2\mathbb{Z}} \arrow[r, "nat"] & \frac{\mathbb{Z}}{n\mathbb{Z}} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

then we can see that our non trivial chain maps are,

$$f_1 : \mathbb{Z} \to \frac{\mathbb{Z}}{n\mathbb{Z}}, \quad z \mapsto z + n\mathbb{Z},$$
$$f_0 : \mathbb{Z} \to \frac{\mathbb{Z}}{n^2\mathbb{Z}}, \quad z \mapsto z + n^2\mathbb{Z},$$
$$f_1 : \frac{\mathbb{Z}}{n\mathbb{Z}} \to \frac{\mathbb{Z}}{n\mathbb{Z}}, \quad z + n\mathbb{Z} \mapsto z + n\mathbb{Z},$$

and they satisfy,

$$f_0(\gamma_1(z)) = f_0(nz) = nz + n\mathbb{Z} = 0 \quad \& \quad \delta_1(f_1(z)) = \delta_1(z + n\mathbb{Z}) = nz + n\mathbb{Z} = 0.$$

Simmilarly, we can see that the other chain maps make the diagram commute.

2.2 Exact Sequences

Definition 2.2.1. A finite or infinite sequence of R-morphisms and left R-modules,

$$\ldots \to L \xrightarrow{f} M \xrightarrow{g} N \to \ldots$$

is said to be *exact* at M if $imf = kerg$. A finite or infinite sequence of R-morphisms and left R-modules,

$$\ldots \xrightarrow{f_{n+2}} M_{n+1} \xrightarrow{f_{n+1}} M_n \xrightarrow{f_n} M_{n-1} \xrightarrow{f_{n-1}} \ldots$$

is said to be an *exact sequence* if it is exact at every M_i, i.e. $imf_i = kerf_{i+1}$.

Example 2.2.2. The finite sequence,

$$0 \to K \xrightarrow{f} K^2 \xrightarrow{g} K^2 \xrightarrow{h} K^2 \to 0,$$

where $f(x) = (x, 0)$, $g(x, y) = (0, y)$, and $h(x, y) = (x, 0)$ is an exact sequence since $im(f) = \{(x, 0) : x \in K\} = ker(g)$ and $im(g) = \{(0, y) : u \in K\} = ker(h)$. However, if we instead consider the finite sequence,

$$0 \to K \xrightarrow{f} K^2 \xrightarrow{g} K^2 \xrightarrow{h'} K^2 \to 0,$$

where f and g are the same as above, but $h'(x, y) = (x, y)$. Obviously, once again we have that $im(f) = ker(g)$ and $im(g) = \{(0, y) : y \in K\}$, however, $ker(h') = \{(0, 0)\}$ and so $im(g) \neq ker(h')$. Hence this is not an exact sequence, because it is not exact everywhere.

Definition 2.2.3. A *short exact sequence* is an exact sequence of the form,

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0.$$

This is also referred to as and *extension of* N *by* L.

Example 2.2.4. The sequence,

$$0 \to K \xrightarrow{f} K^2 \xrightarrow{g} K \to 0,$$

where $f(x) = (x, 0)$ and $g(x, y) = y$ is a short exact sequence, since $im(f) = \{(x, 0) : x \in K\} = ker(g)$.

Example 2.2.5. Notice that the sequence,

$$0 \to \mathbb{Z} \xrightarrow[6]{f} \mathbb{Z} \xrightarrow{\eta} \frac{\mathbb{Z}}{3\mathbb{Z}} \to 0,$$

has $ker(\eta) = \{z : z + s\mathbb{Z} = 0 \cong 3\mathbb{Z}$ and $im(f) = \{6z : z \in \mathbb{Z}\} \cong 6\mathbb{Z}$. Hence, as $6\mathbb{Z} \subsetneq 3\mathbb{Z}$, $im(f) \neq ker(\eta)$ and the sequence is not exact.

Notice that you can obtain a chain complex from an exact sequence simply by deciding the degree of any term of the sequence. To consider the result when a chain complex is an exact sequence, the following is in response to Exercise 1.1.5 from [4]. (Note that the third condition has been omitted as we are not consider quasi-isomerism in this report.)

Theorem 2.2.6. *The following are equivalent for every chain complex* $\mathbf{C}$*:*

1. $\mathbf{C}$ *is an exact sequence, that is, exact at every* C_n*.*
2. $\mathbf{C}$ *is acyclic, that is,* $H_n(\mathbf{C}) = 0$ *for all* n*.*

Proof. Suppose the chain complex,

$$\mathbf{C}: \qquad \ldots \to C_{n+1} \xrightarrow{\delta_{n+1}} C_n \xrightarrow{\delta_n} C_{n-1} \xrightarrow{\delta_{n-1}} \ldots$$

is an exact sequence, so we have that $im(\delta_{n+1}) = ker(\delta_n)$ for all n, i.e. $B_n(\mathbf{C}) = Z_n(\mathbf{C})$ for all n. Hence, when we compute the homology of $\mathbf{C}$, we find that,

$$H_n(\mathbf{C}) = \frac{Zn(\mathbf{C})}{B_n(\mathbf{C})} = 0 \quad \forall n.$$

Thus, $\mathbf{C}$ is acyclic.
Conversely, suppose $\mathbf{C}$ is acyclic, then we have that $H_n(\mathbf{C}) = 0$ for all n. Therefore, $Z_n(\mathbf{C})/B_n(\mathbf{C}) = 0$ for all n, by definition, but this is only possibly if either $Z_n(\mathbf{C}) \subset B_n(\mathbf{C})$, which is impossible from Definition 2.1.6, or $Z_n(\mathbf{C}) = B_n(\mathbf{C})$. Thus, $ker(\delta_n) = im(\delta_{n+1})$ and so $\mathbf{C}$ is exact. □

Remark 2.2.7. The above theorem makes it clear that exact sequences can be thought of as acyclic chain complexes. By the same token, homology can be thought of as a chain complex's deviation from exactness.

The folllowing is provided as an example in [6], however, it was felt too useful as a method of constructing exact sequences to be simply an example and so is presented on this report as a lemma.

Proposition 2.2.8. *The sequence,*

$$0 \to ker(f) \xrightarrow{\iota} M \xrightarrow{f} N \xrightarrow{\eta} coker(f) \to 0,$$

where ι, η *are the inclusion, and natural maps respectively, is exact. In addition the sequence,*

$$0 \to ker(f) \xrightarrow{\iota} M \xrightarrow{\eta} \frac{M}{im(f)} \to 0,$$

is a short exact sequence.

Proof. It is clear here that we have $im(\iota) = ker(f)$ as ι is the inclusion map and since $coker(f) = N/im(f)$ we also have that $im(f) = ker(\eta)$. Hence the sequence is exact. The second result follows a similar argument. □

Example 2.2.9. Consider the map $f : \mathbb{Z}^2 \to \mathbb{Z}$, $(x, y) \mapsto 2x$. Then $ker(f) = \{(0, y) : y \in \mathbb{Z}\} \cong \mathbb{Z}$, and $im(f) = 2\mathbb{Z}$ giving that $coker(f) = \frac{\mathbb{Z}}{2\mathbb{Z}}$. It follows from Lemma 2.2.8 that the sequence,

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{Z}^2 \xrightarrow{f} \mathbb{Z} \xrightarrow{\eta} \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0,$$

is exact. In addition, the following is a short exact equation,

$$0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{Z}^2 \xrightarrow{\eta} \frac{\mathbb{Z}}{2\mathbb{Z}} \to 0.$$

Here are some simple properties of exact sequences as detailed by Rotman in Proposition 2.18 of [3]. Also included is the solution to Exercise 2.16 (ii) from [3]; it caught our attention as an interesting property, with such a simple proof, which shows the rigidity of the exactness condition.

Proposition 2.2.10. **(i)** *A sequence* $0 \to L \xrightarrow{f} M$ *is exact if and only if* f *is injective.*

(ii) *A sequence* $M \xrightarrow{g} N \to 0$ *is exact if and only if* g *is surjective.*

(iii) *A sequence* $0 \to L \xrightarrow{h} M \to 0$ *is exact if and only if* h *is an isomorphism.*

(iv) *If* : $L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} L'$ *then* f *is surjective if and only if* h *is injective.*

Proof. **(i)** As the image of $0 \to L$ is $\{0\}$, exactness gives $ker(f) = \{0\}$, and so f is injective. Conversely, given some map $f : L \to M$, there is an exact sequence $ker(f) \xrightarrow{\iota} L \to fM$, where ι is the inclusion map, (see Proposition 2.2.8). If f is injective, then $ker(f) = \{0\}$.

(ii) The kernel of $N \to 0$ is N, so the exactness gives $im(g) = N$, and so g is surjective. Conversely, given $g : M \to N$, there is an exact sequence $M \xrightarrow{g} N \xrightarrow{\pi} coker(g)$, where π is the natural map, (see Proposition 2.2.8). If g is surjective, then $N = im(g)$ and $coker(g) = \{0\}$.

(iii) Part (i) shows that the sequence is exact at L if and only if h is injective, and part (ii) shows that the sequence is exact at M if and only if h is surjective. Therefore, the sequence is exact everywhere if and only if h is an isomorphism.

(iv) Suppose f is surjective, them $im(f) = M$, and exactness at M gives $ker(g) = M$. Hence, $im(g) = \{0\}$, and exactness at N gives $ker(h) = \{0\}$ and so h is injective. The converse is similar.

$\square$

Definition 2.2.11. Two short exact sequences,

$$\mathbf{E}: \quad 0 \to L \to M \to N \to 0 \qquad \& \qquad \mathbf{E}': \quad 0 \to L \to M' \to N \to 0,$$

are *equivalent* if there is a map $\xi : M \to M'$ such that the diagram,

$$\begin{tikzcd}
\mathbf{E}: & 0 \arrow[r] & L \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d, "\xi"'] & N \arrow[r] \arrow[d, equal] & 0 \\
\mathbf{E}': & 0 \arrow[r] & L \arrow[r] & M' \arrow[r] & N \arrow[r] & 0
\end{tikzcd}$$

commutes.

Example 2.2.12. Consider the two short exact sequences,

$$\mathbf{E}: \quad 0 \to K \xrightarrow{f} K^2 \xrightarrow{g} K \to 0 \quad \& \quad \mathbf{E}': \quad 0 \to K \xrightarrow{f'} K^2 \xrightarrow{g'} K \to 0,$$

where $f : x \mapsto (x, 0)$, $g : (x, y) \mapsto y$, $f' : x \mapsto (0, x)$ and $g' : (x, y) \mapsto x$. Then we can construct the diagram:

$$\begin{tikzcd}
\mathbf{E}: & 0 \arrow[r] & K \arrow[r] \arrow[d, equal] & K^2 \arrow[r] \arrow[d, "\xi"'] & K \arrow[r] \arrow[d, equal] & 0 \\
\mathbf{E}': & 0 \arrow[r] & K \arrow[r] & K^2 \arrow[r] & K \arrow[r] & 0
\end{tikzcd}$$

This diagram commutes when ξ is the map $\xi : K^2 \to K^2$, $(x, y) \mapsto (y, x)$, hence these two short exact equations are equivalent.

For a nonexample we turn to Exercise III 1.1 in [5].

Example 2.2.13. Consider the two short exact sequences,

$$\mathbf{E}: \quad 0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z} \xrightarrow{g} \frac{\mathbb{Z}}{3\mathbb{Z}} \to 0 \quad \& \quad \mathbf{E}': \quad 0 \to \mathbb{Z} \xrightarrow{f'} \mathbb{Z} \xrightarrow{g'} \frac{\mathbb{Z}}{3\mathbb{Z}} \to 0,$$

where $f, f' : x \mapsto 3x$, $g : x \mapsto x(\text{mod } 3)$ and $g' : x \mapsto 2x(\text{mod } 3)$. Then we can construct the diagram:

$$\begin{tikzcd}
\mathbf{E}: & 0 \arrow[r] & \mathbb{Z} \arrow[r] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d, "\xi"'] & \frac{\mathbb{Z}}{3\mathbb{Z}} \arrow[r] \arrow[d, equal] & 0 \\
\mathbf{E}': & 0 \arrow[r] & \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[r] & \frac{\mathbb{Z}}{3\mathbb{Z}} \arrow[r] & 0
\end{tikzcd}$$

Assume $\mathbf{E}$ and $\mathbf{E}'$ are equivalent, so we there must exist some ξ such that the diagram commutes. The second square in the diagram will commute if and only if $\xi = id_{\mathbb{Z}}$, since $f = f'$, however, this then means the third square will never commute, as $g \neq g'$. Thus no such ξ exists and the two short exact sequences are not equivalent.

For the discussion of split exact sequences, this report will first follow [6] in defining a section and refraction and then use Schiffler's Proposition 1.8 to prove that the conditions in Definition 1.23 in [1] are equivalent. This will obtain us the Splitting Lemma. However, the proof of Proposition 1.8 will be ommitted as it is

Definition 2.2.14.

- A map $f : L \to M$ is called a *section* if there exists a map $h : M \to L$ such that $hf = id_L$.
- A map $g : M \to N$ is called a *retraction* if there exists a map $h : N \to M$ such that $gh = id_N$.

Definition 2.2.15. A short exact sequence,

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0,$$

is said to be *split* if f is a section.

Lemma 2.2.16. *(Splitting Lemma) If the short exact sequence,*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0,$$

is split then the following equivalent conditions hold:

(i) *f is a section.*

(ii) *g is a retraction.*

(iii) *$im(f)$ is a direct summand of M.*

Proof. Proof omitted. See Proposition 1.8 in [6] and the proof given on [7]. □

Example 2.2.17. The short exact sequence,

$$0 \to \mathbb{Z} \xrightarrow{f} \mathbb{Z}^2 \xrightarrow{g} \mathbb{Z} \to 0,$$

where $f : x \mapsto (x, 0)$ and $g : (x, y) \mapsto y$, splits. Then

(i) f is a section as the map $h : \mathbb{Z}^2 \to \mathbb{Z}$, $(x, y) \mapsto x$ satifies $hf = id_{\mathbb{Z}}$.

(ii) g is a retraction as the map $h' : \mathbb{Z} \to \mathbb{Z}^2$, $y \mapsto (x, y)$ satifies $gh = id_{\mathbb{Z}}$.

(iii) $im(f) \cong \mathbb{Z}$ is a direct summand of $\mathbb{Z}^2 \cong \mathbb{Z} \oplus \mathbb{Z}$.

Rather than provide the following result as a proposition and prove it directly as Rotman does in [3], it will be given as a corollary of the splitting lemma as in [6].

Corollary 2.2.18. *If the sequence,*

$$0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0,$$

is split exact, then

$$M \cong L \oplus N.$$

Proof. Since f is injective as the seqence is short exact, we have $L \cong im(f) \cong ker(g)$. The since g is surjective as the sequence is short exact, the first isomorphism theorem implies $N \cong {}^{M}/_{ker(g)}$. Then Lemma 2.2.16 gives M is a direct summand of $im(f)$, hence, $M \cong L \oplus N$. □

Example 2.2.19. Notice that in the previous example $\mathbb{Z} \cong im(f) \cong ker(g)$ and $\mathbb{Z} \cong {}^{\mathbb{Z}^2}/_{ker(g)} \cong \mathbb{Z}$. Hence, $\mathbb{Z}^2 \cong \mathbb{Z} \oplus \mathbb{Z}$, as expected.

Definition 2.2.20. Given a short exact seqence,

$$\mathbf{E} : \qquad 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0,$$

and a map $\theta : L \to L'$ then the short exact sequence,

$$\mathbf{E}' : \qquad 0 \to L' \xrightarrow{f'} M' \xrightarrow{g'} N \to 0,$$

fitting into the commutative diagram

$$\begin{tikzcd}
\mathbf{E}\colon & 0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\theta"] & M \arrow[r, "g"] \arrow[d, "\phi"] & N \arrow[r] \arrow[d, equal] & 0 \\
\mathbf{E}'\colon & 0 \arrow[r] & L' \arrow[r, "f'"'] & M' \arrow[r, "g'"'] & N \arrow[r] & 0
\end{tikzcd}$$

is defined to be the *pushout of* $\mathbf{E}$ *along* θ.

Proposition 2.2.21. *Let* $\mathbf{E}$, $\mathbf{E}'$ *and* $\theta : L \to L'$ *be defined as above in Definition 2.2.20. Then the pushout of* $\mathbf{E}$ *along* θ *exists and is unique up to equivalence.*

Proof. Existence: Firstly, set

$$M' = \frac{(L' \oplus M)}{\{(\theta(l), -f(l)) : l \in L\}},$$

and let the maps be $f' : l' \mapsto \overline{(l', 0)}$, $g' : \overline{(l', m)} \mapsto g(m)$ and $\phi : m \mapsto \overline{(0, m)}$, where $\overline{(l', m)}$ denotes the class of $(l', m) \in L' \oplus M$ in M'. The diagram is then commutative as $\phi(f(l)) = \overline{(0, f(l))} = \overline{(0, 0)} = \overline{(\theta(l), 0)} = f'(\theta(l))$ for an $l \in L$ and $id_N(g(m)) = g(m) = g'(\theta(m))$ for any $m \in M$. The pushout is exact as $im(f') = \{\overline{(l', 0)} : l' \in L'\} = ker(g')$.
Uniqueness: The sequence

$$\xi' \qquad 0 \to L \xrightarrow{\left(\begin{smallmatrix} \theta \\ -f \end{smallmatrix}\right)} L' \oplus M \xrightarrow{\left(\begin{smallmatrix} f' & \phi \end{smallmatrix}\right)} M' \to 0,$$

is exact as $im(\left(\begin{smallmatrix} \theta \\ -f \end{smallmatrix}\right)) = \{\left(\begin{smallmatrix} \theta(l) \\ -f(l) \end{smallmatrix}\right) : l \in L\} = ker(\left(\begin{smallmatrix} f' & \phi \end{smallmatrix}\right))$. Exactness then implies that $\left(\begin{smallmatrix} f' & \phi \end{smallmatrix}\right)$ is surjective and so by the first isomorphism theorem (see proof of Corollary 2.2.18) we have,

$$M' \cong \frac{L' \oplus M}{ker(\left(\begin{smallmatrix} f' & \phi \end{smallmatrix}\right))} \cong \frac{L' \oplus M}{\{(\theta(l), -f(l)) : l \in L\}}.$$

This gives equivalence between $\mathbf{E}'$ and $\mathbf{E}''$. ☐

Example 2.2.22. Consider the short exact sequence,

$$\mathbf{E} : \qquad 0 \to \mathbb{Z} \xrightarrow[f]{n} \mathbb{Z} \xrightarrow[g]{nat} \frac{\mathbb{Z}}{n\mathbb{Z}} \to 0,$$

and the $\mathbb{Z}$-module map $\theta : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is the natural one. Now we can use Proposition 2.2.21 to construct the pushout of $\mathbf{E}$ along θ.
Firstly, construct the commutative diagram,

$$\begin{tikzcd}
\mathbf{E}\colon & 0 \arrow[r] & \mathbb{Z} \arrow[r, "n", "f"'] \arrow[d, "nat"', "\theta"] & \mathbb{Z} \arrow[r, "nat", "g"'] \arrow[d, "\phi"] & \frac{\mathbb{Z}}{n\mathbb{Z}} \arrow[r] \arrow[d, equal] & 0 \\
\mathbf{E}'\colon & 0 \arrow[r] & \frac{\mathbb{Z}}{n\mathbb{Z}} \arrow[r, "f'"'] & M' \arrow[r, "g'"'] & \frac{\mathbb{Z}}{n\mathbb{Z}} \arrow[r] & 0
\end{tikzcd}$$

and now we want to find M'. As $\theta : x \mapsto x + n\mathbb{Z}$ and $f : x \mapsto nx$, we have,

$$M' = \frac{\frac{\mathbb{Z}}{n\mathbb{Z}} \bigoplus \mathbb{Z}}{\{(x + n\mathbb{Z}, -nx) : x \in \mathbb{Z}\}},$$

and we can set $f' : \mathbb{Z}/n\mathbb{Z} \to M'$, $x + n\mathbb{Z} \mapsto \overline{(x + n\mathbb{Z}, 0)}$, $g' : M' \to \mathbb{Z}/n\mathbb{Z}$, $\overline{(x + n\mathbb{Z}, y)} \mapsto y + n\mathbb{Z}$, and $\phi : \mathbb{Z} \to M'$, $x \mapsto \overline{(0, x)}$. It is clear that this makes the above diagram commutative and gives $\mathbf{E}'$ exact. Furthermore, the map ϕ is onto as,

$$\begin{aligned} M' \ni \overline{(x + n\mathbb{Z}, y)} &= \overline{(x + n\mathbb{Z}, -nx)} + \overline{(o, y + na)}, \\ &= \overline{(0, y + nx)} \in im(\phi). \end{aligned}$$

We can also see that,

$$\begin{aligned} ker(\phi) &= \{y \in \mathbb{Z} : \overline{(0, y)} = 0\}, \\ &= \{y \in \mathbb{Z} : \overline{(0, y)} \text{ is of the form } \overline{(x + n\mathbb{Z}, -nx)}\}, \\ &= \{y \in \mathbb{Z} : n|y,\ x = \frac{-b}{n} \text{ and } x + n\mathbb{Z} = 0\}, \\ &= \{y \in \mathbb{Z} : n^2|y\} \cong n^2\mathbb{Z}, \end{aligned}$$

and then the First Isomorphism Theorem gives,

$$M' \cong \frac{\mathbb{Z}}{ker(\phi)} \cong \frac{\mathbb{Z}}{n^2\mathbb{Z}}.$$

Hence, the pushout of $\mathbf{E}$ along θ is,

$$\mathbf{E}' : \qquad 0 \to \frac{\mathbb{Z}}{n\mathbb{Z}} \xrightarrow{f'} \frac{\mathbb{Z}}{n^2\mathbb{Z}} \xrightarrow{g'} \frac{\mathbb{Z}}{n\mathbb{Z}} \to 0,$$

with the maps f', g' defined as above.

Definition 2.2.23. Given a short exact seqence,

$$\mathbf{E} : \qquad 0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0,$$

and a map $\psi : N'' \to N$ then the short exact sequence,

$$\mathbf{E}'' : \qquad 0 \to L \xrightarrow{f''} M'' \xrightarrow{g''} N'' \to 0,$$

fitting into the commutative diagram

$$\begin{tikzcd} \mathbf{E}'': & 0 \arrow[r] & L \arrow[r, "f''"] \arrow[d, equal] & M'' \arrow[r, "g''"] \arrow[d, "\phi'"] & N'' \arrow[r] \arrow[d, "\psi"] & 0 \\ \mathbf{E}: & 0 \arrow[r] & L \arrow[r, "f"'] & M \arrow[r, "g"'] & N \arrow[r] & 0 \end{tikzcd}$$

is defined to be the *pullback* of $\mathbf{E}$ along ψ.

Proposition 2.2.24. *Let* $\mathbf{E}$, $\mathbf{E}''$ *and* $\psi : N'' \to N$ *be defined as above in Definition 2.2.23. Then the pullback of* $\mathbf{E}$ *along* ψ *exists and is unique up to equivalence.*

Proof. Existence: Firstly, set

$$M'' = \{(m, n'') \in M \oplus N'' : g(m) = \psi(n'')\},$$

and let the maps be $f'' : L \to M''$, $l \mapsto (f(l), 0)$ (if $g(f(l)) = \psi(0)$, 0 otherwise), $g'' : M'' \to N''$, $(m, n'') \mapsto n''$, and $\phi' : M'' \to M$, $(m, n'') \mapsto m$. The diagram is then commutative by diagram chasing and the pullback is exact as $im(f) = \{(f(l), 0) : l \in L \text{ and } g(f(l)) = \psi(0)\} = \{(m, 0) : m \in M \text{ and } g(m) = \psi(0)\} = ker(g'')$. Uniqueness: Consider the commutative diagram of short exact sequences,

$$\begin{tikzcd}
\xi'': & 0 \arrow[r] & M'' \arrow[r, "\begin{pmatrix}\phi' & g''\end{pmatrix}"] \arrow[d] & M \oplus N'' \arrow[r, "\begin{pmatrix} g \\ -\psi \end{pmatrix}"] \arrow[d, equal] & N \arrow[r] \arrow[d, equal] & 0 \\
\widetilde{\xi''}: & 0 \arrow[r] & \widetilde{M''} \arrow[r] & M \oplus N'' \arrow[r, "\begin{pmatrix} g \\ -\psi \end{pmatrix}"'] & N \arrow[r] & 0
\end{tikzcd}$$

where M'' and $\widetilde{M''}$ are from two pullbacks, $\mathbf{E}''$ and $\widetilde{\mathbf{E}''}$, of $\mathbf{E}$ along ψ. By diagram chasing we can see that there is in fact an isomorphism between M'' and $\widetilde{M''}$ and so $\mathbf{E}''$ and $\widetilde{\mathbf{E}''}$ are equivalent as short exact sequences. □

Example 2.2.25. Consider the short exact sequence,

$$\mathbf{E}: \qquad 0 \to \mathbb{Z} \xrightarrow[f]{n} \mathbb{Z} \xrightarrow[g]{nat} \frac{\mathbb{Z}}{n\mathbb{Z}} \to 0,$$

and the $\mathbb{Z}$-module map $\psi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ is the natural one. Now we can use Proposition 2.2.24 to construct the pullback of $\mathbf{E}$ along ψ.
Firstly, construct the commutative diagram,

$$\begin{tikzcd}
\mathbf{E}'': & 0 \arrow[r] & \mathbb{Z} \arrow[r, "f''"] \arrow[d, equal] & M'' \arrow[r, "g''"] \arrow[d, "\phi'"] & \mathbb{Z} \arrow[r] \arrow[d, "nat"', "\psi"] & 0 \\
\mathbf{E}: & 0 \arrow[r] & \mathbb{Z} \arrow[r, "n", "f"'] & \mathbb{Z} \arrow[r, "nat", "g"'] & \frac{\mathbb{Z}}{n\mathbb{Z}} \arrow[r] & 0
\end{tikzcd}$$

and now to find M''. As both $\psi, g : x \mapsto x + n\mathbb{Z}$, we have,

$$\begin{aligned}
M'' &= \{(x, y) \in \mathbb{Z} \oplus \mathbb{Z} : x + n\mathbb{Z} = y + n\mathbb{Z}\}, \\
&= \{(x, y) \in \mathbb{Z} \oplus \mathbb{Z} : n \mid x - y\}, \\
&\cong \mathbb{Z} \oplus \mathbb{Z}, \text{ by considering the map } (a, b) \mapsto (a, a + nb), \\
&\cong \mathbb{Z}^2,
\end{aligned}$$

and we can set $f'' : \mathbb{Z} \to M''$, $x \mapsto \overline{(nx, 0)}$, $g'' : M'' \to \mathbb{Z}$, $\overline{(x, y)} \mapsto y$ and $\phi' : M'' \to \mathbb{Z}$, $\bar{(x, y)} \mapsto m$. It is clear that this makes the diagram commutative and gives $\mathbf{E}''$ exact. Hence, the pullback of $\mathbf{E}$ along ψ is,

$$\mathbf{E}'' : \quad 0 \to \mathbb{Z} \xrightarrow{f''} \mathbb{Z}^2 \xrightarrow{g''} \mathbb{Z} \to 0,$$

with the maps f'', g'' defined as above.

Now this report will start to consider the results possible by considering exact sequences of ....
The following theorem is from [1] and the proof that the connecting map exists and is well defined taken from [1] and the proof that the sequence induced by this map is exact from [4]. Both proofs were adapted to use the notation used in this report and some more explanation given when we felt it was necessary.

Theorem 2.2.26. *(Long Exact Sequence)*
Let $0 \to \mathbf{C} \xrightarrow{f} \mathbf{D} \xrightarrow{g} \mathbf{E} \to 0$ *be a short exact sequence of chain complexes, meaning that* f *and* g *are chain maps and for each* n *the maps*

$$0 \to C_n \xrightarrow{f_n} D_n \xrightarrow{g_n} E_n \to 0,$$

form a short exact sequence. Then there are connecting maps $c_n : H_n(\mathbf{E}) \to H_{n-1}(\mathbf{C})$ *giving a long exact sequence,*

$$\ldots \to H_{n+1}(\mathbf{E}) \xrightarrow{c_{n+1}} H_n(\mathbf{C}) \xrightarrow{f*} H_n(\mathbf{D}) \xrightarrow{g*} H_n(\mathbf{E}) \xrightarrow{c_n} H_{n-1}(\mathbf{C}) \to \ldots$$

Proof. Have diagram,

$$\begin{tikzcd}
0 \arrow[r] & C_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\gamma_{n+1}"] & D_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\delta_{n+1}"] & E_{n+1} \arrow[r] \arrow[d, "\epsilon_{n+1}"] & 0 \\
0 \arrow[r] & C_n \arrow[r, "f_n"'] \arrow[d, "\gamma_n"] & D_n \arrow[r, "g_n"'] \arrow[d, "\delta_n"] & E_n \arrow[r] \arrow[d, "\epsilon_n"] & 0 \\
0 \arrow[r] & C_{n-1} \arrow[r, "f_{n-1}"'] & D_{n-1} \arrow[r, "g_{n_1}"'] & E_{n-1} \arrow[r] & 0
\end{tikzcd}$$

- Define the connecting map $c_n : H_n(\mathbf{E}) \to H_{n-1}(\mathbf{C})$ as follows:
  Let $\overline{x}$ be a typical element of $H_n(\mathbf{E})$, with $x \in ker(\epsilon_n) = Z_n(\mathbf{E})$. Then as g_n is surjective we can lift $x \in E_n$ to some element $y \in D_n$ such that $g_n(y) = x$. From the commutativity of the diagram,

  $$g_{n-1}(\delta_n(y)) = \epsilon_n(g_n(y)) = \epsilon_n(x) = 0,$$

  since $x \in ker(\epsilon_n)$. Hence, $\delta_n(y) \in ker(g_{n-1}) \cong im(f_{n-1})$. Then as f_{n-1} is injective there exists some $z \in C_{n-1}$ such that $f_{n-1}(z) = \delta_n(y)$. Thus define $c_n(\overline{x}) = \overline{z}$.(Note $z \in ker(\gamma_{n-1})$ as $\delta_{n-1}(\delta_n(y)) = 0$, then by commutivity $f_{n-2}(\gamma_{n-1}(z)) = 0$ and as f_{n-2} is injective.)

- Not dependent on choice of x or y: Say $y, y' \in D_n$ have images $g_n(y) = x$, $g_n(y') = x'$ where $x, x' \in ker(\epsilon_n) = Z_n(\mathbf{E})$ with $\overline{x} = \overline{x'}$. Thus $g_n(y) - g_n(y') \in im(\epsilon_{n+1}) = B_n(\mathbf{E})$. Therefore,

$$\begin{aligned} g_n(y - y') &= \epsilon_{n+1}(g_{n+1}(u)), \text{ for some } u \in D_{n+1}, \\ &= g_n(\delta_{n+1}(u)). \end{aligned}$$

Hence, $y - y' - \delta_{n+1}(u) = f_n(v)$ for some $v \in C_n$ by the injectivity of f_n. If $f_{n-1}(z) = \delta_n(y)$ and $f_{n-1}(z') = \delta_n(y')$ then,

$$\begin{aligned} f_{n-1}(z - z') &= \delta_n(y - y'), \\ &= \delta_n(y - y' - \delta_(u)), \text{ as } \delta_n(\delta_{n+1}(u)) = 0 \text{ as chain complex}, \\ &= \delta_n(f_n(v)), \\ &= f_{n-1}(\gamma_n(v)), \end{aligned}$$

implying $z - z' = \gamma_n(v) \in im(\gamma_n) = B_{n-1}(\mathbf{C})$. Hence, $\overline{z} = \overline{z'}$ in $H_{n-1}(\mathbf{C})$.

- Exactness at $H_n(\mathbf{C})$: $\quad H_{n+1}(\mathbf{E}) \xrightarrow{c_{n+1}} H_n(\mathbf{C}) \xrightarrow{f_*} H_n(\mathbf{D})$
$im(c_{n+1}) \subseteq ker(f_*)$: We have $f_*(c_{n+1}(\overline{x})) = \overline{f_n(z)}$ for some $z \in C_n$. But,

$$f_n(z) = \delta_{n+1} g_{n+1}^{-1}(x) \in B_{n+1}(\mathbf{D});$$

that is, $f_*(c_{n+1}(\overline{x}) = 0$.
$ker(f_*) \subseteq im(c_{n+1})$: If $f_*(\overline{z}) = \overline{f_n(z)} = \overline{0}$ then $f_n(z) = \delta_{n+1}(y')$ for some $y' \in D_{n+1}$. Since g is a chain map,

$$\begin{aligned} \epsilon_{n+1}(g_{n+1}(y')) &= g_n(\delta_{n+1}(y')), \\ &= g_n(f_n(z)), \\ &= 0, \text{ by exactness of the original sequence}, \end{aligned}$$

and so $g_{n+1}(y') \in Z_n(\mathbf{E})$. However,

$$\begin{aligned} c_{n+1}(\overline{g_{n+1}(y')}) &= \overline{f_n^{-1}(\delta_{n+1}(g_{n+1}^{-1}(g_{n+1}(y'))))}, \\ &= \overline{f_n^{-1}\delta_{n+1}(y')}, \\ &= \overline{f_n^{-1}(f_n(z))}, \\ &= \overline{z}. \end{aligned}$$

- Exactness at $H_n(\mathbf{D})$: $\quad H_n(\mathbf{C}) \xrightarrow{f_*} H_n(\mathbf{D}) \xrightarrow{g_*} H_n(\mathbf{E})$
$im(f_*) \subseteq ker(g_*)$: True as,

$$g_* f_* = (gf)_* = 0_* = 0.$$

$ker(g_*) \subseteq im(f_*)$: If $g_*(\overline{y}) = \overline{g_n(y)} = \overline{0}$, then $g_n(y) = \epsilon_{n+1}$ for some $x' \in E_{n+1}$. However, g is surjective meaning $x' = g_{n+1}$ for some $y' \in C_{n+1}$, hence,

$$\begin{aligned} g_n(y) &= \epsilon_{n+1}(g_{n+1}(y')), \\ &= g_n(\delta_{n+1}(y')), \end{aligned}$$

since g is a chain map and so $g_n(y-\delta)n+1(y')=0$. By exactness, there is $z \in C_n$ with $f_n(z) = y - \delta_{n+1}(y')$. Now $z \in Z_n(\mathbf{C})$ because,

$$\begin{aligned} f_{n-1}(\gamma_n(z)) &= \delta_n(f_n(z)), \\ &= \delta_n(y - \delta_{n+1}(y')), \\ &= \delta_n(y) - \delta_n(\delta_{n+1}(y')), \\ &= 0, \text{ as } y \in Z_n(\mathbf{C}) \text{ and } \delta_n\delta_{n+1} = 0, \end{aligned}$$

then because f_{n-1} is injective, $\gamma_n(z) = 0$. Therefore,

$$\begin{aligned} f_*(\overline{z}) &= \overline{f_n(z)}, \\ &= \overline{y - \delta_{n+1}(y')}, \\ &= \overline{y}. \end{aligned}$$

- Exactness at $H_n(\mathbf{E})$: $\quad H_n(\mathbf{D}) \xrightarrow{g_*} H_n(\mathbf{E}) \xrightarrow{c_n} H_{n-1}(\mathbf{C})$
  $im(g_*) \subseteq ker(c_n)$: If $g_*(\overline{y}) = \overline{g_n(y)} \in im(g_*)$, then $c_n(\overline{g_n(y')}) = \overline{z}$ where,
  $$f_{n-1}(z) = \delta_n(g_n^{-1}(g_n(y'))).$$
  Since this formula is independent of the choice of lifting of $g_n(y')$, let us choose $g_n^{-1}(g_n(y')) = y'$. Now $\delta_n(g_n^{-1}(g_n(y')) = \delta_n(y') = 0$ because $y' \in Z_n(\mathbf{D})$. Thus, $f_{n-1}(z) = 0$ and hence $z = 0$ because f_{n-1} is injective.
  $ker(c_n) \subseteq im(g_*)$: If $c_n(x) = \overline{0}$, then $z' = f_{n-1}^{-1} = f_{n-1}^{-1}(\delta_n(g_n^{-1}(x))) \in B_n(\mathbf{C})$, that is $z' = \gamma_n(z)$ for some $z \in C_n$. However,
  $$\begin{aligned} f_{n-1}(z') &= f_{n-1}(\gamma_n(z)), \\ &= \delta_n(f_n(z)), \\ &= \delta_n(g_n^{-1}(x)), \end{aligned}$$
  so that $\delta_n(g_n^{-1} - f_n(z)) = 0$, that is $g_n^{-1} - f_n(z) \in ker(\delta_n)$. Exactness of the original sequence gives $g_n f_n = 0$, so that,
  $$\begin{aligned} g_*(\overline{g_n^{-1}(x) - f_n(z)}) &= \overline{g_n(g_n^{-1}(x) - gn(f_n(z),} \\ &= \overline{x}. \end{aligned}$$

□

Note that this report is following Crawley-Boevey's example in [1] by proving the Long Exact Sequence theorem directly and taking the Snake Lemma as a corollary, unlike in other reading material, like [4] where the Long Exact Sequence is given as a corollary of the Snake Lemma. If the reader would like a direct proof of the Snake Lemma, we direct them, as Weibel does in [4], to *It's My Turn* (Rastar-Martin Elfand Studios, 1980), where a proof is given in the beginning of the movie.

Corollary 2.2.27. *(Snake Lemma)*
If you have a commutative diagram with exact rows,

$$\begin{tikzcd} 0 \arrow[r] & L \arrow[r] \arrow[d, "\theta"'] & M \arrow[r] \arrow[d, "\phi"'] & N \arrow[r] \arrow[d, "\psi"'] & 0 \\ 0 \arrow[r] & L' \arrow[r] & M' \arrow[r] & N' \arrow[r] & 0 \end{tikzcd}$$

you get an exact sequence,

$$0 \to ker(\theta) \to ker(\phi) \to ker(\psi) \to coker(\theta) \to coker(\phi) \to coker(\psi) \to 0$$

Proof. Consider $L \to L'$, $M \to M'$ and $N \to N'$ as chain complexes and use Theorem 2.2.26. □

Remark 2.2.28. A short exact sequence of cochain complexes $0 \to \mathbf{C} \to \mathbf{D} \to \mathbf{E} \to 0$ gives a long exact sequence,

$$\ldots \to H^{n-1}(\mathbf{E}) \to H^n(\mathbf{C}) \to H^n(\mathbf{D}) \to H^n(\mathbf{E}) \to H^{n+1}(\mathbf{C}) \to H^{n+1}(\mathbf{D}) \to \ldots$$

by renumbering in Theorem 2.2.26.

2.3 Homotopy and Quasi-Isomorphism

For this section the report relies heavily on [1], with some added details, such as the definition for null homotopic and the diagram, come from [4] and [3]. However, all examples are original work.

Definition 2.3.1. If $f, g : \mathbf{C} \to \mathbf{D}$ are chain maps, then f and g are *homotopic*, denoted $f \simeq g$, if for each n there are maps $h_n : C_n \to D_{n+1}$ such that,

$$f_n - g_n = h_{n-1}\gamma_n + \delta_{n+1}h_n,$$

where the maps are defined ,

$$\begin{tikzcd}
\cdots \arrow[r] & C_{n+1} \arrow[r, "\gamma_{n+1}"] \arrow[d, "k_{n+1}"'] & C_n \arrow[r, "\gamma_n"] \arrow[dl, "h_n"] \arrow[d, "k_n"] & C_{n-1} \arrow[r, "\gamma_{n-1}"] \arrow[dl, "h_{n-1}"] \arrow[d, "k_{n-1}"] & \cdots \\
\cdots \arrow[r] & D_{n+1} \arrow[r, "\delta_{n+1}"'] & D_n \arrow[r, "\delta_n"'] & D_{n-1} \arrow[r, "\delta_{n-1}"'] & \cdots
\end{tikzcd}$$

with k_n representing either f_n or g_n.
A chain map $f : \mathbf{C} \to \mathbf{D}$ is considered *null-homotopic* if $f \simeq 0$, where 0 is the zero chain map.

Example 2.3.2. Consider the two chain complexes,

$$\mathbf{C}: \quad \ldots \to 0 \to \mathbb{Z} \xrightarrow{3} \mathbb{Z} \to 0 \to \ldots \quad \& \quad \mathbf{D}: \quad \ldots \to \frac{\mathbb{Z}}{9\mathbb{Z}} \xrightarrow{3} \frac{\mathbb{Z}}{9\mathbb{Z}} \xrightarrow{3} \frac{\mathbb{Z}}{9\mathbb{Z}} \to \ldots,$$

such that $C_1, C_0 = \mathbb{Z}$ and $D_n = \mathbb{Z}/9\mathbb{Z}$. Then the two chain maps $f, g : \mathbf{C} \to \mathbf{D}$, where $f_0, f_1 : x \mapsto 4x + 9\mathbb{Z}$ and $g_0, g_1 : x \to x + 9\mathbb{Z}$ are homotopic where $h_0 : \mathbb{Z} \to \mathbb{Z}/9\mathbb{Z}, x \to x + 9\mathbb{Z}$ and $h_1 : \mathbb{Z} \to \mathbb{Z}/9\mathbb{Z}, x \mapsto 3x + 9\mathbb{Z}$.

Proposition 2.3.3. *If $f, g : \mathbf{C} \to \mathbf{D}$ are homotopic then for each n they induce exactly the same map $H_n(\mathbf{C}) \to H_n(\mathbf{D})$.*

Proof. Let $\overline{x} \in H_n(\mathbf{C})$ such that $x \in Z_n(\mathbf{C})$. Then,

$$\begin{aligned} H_n(f)(\overline{x}) - H_n(g)(\overline{x}) &= \overline{f_n(x)} - \overline{g_n(x)}, \\ &= \overline{h_{n-1}\gamma_n(x) - \delta_{n+1}h_n(x)}, \\ &= \overline{\delta_{n+1}h_n(x)}, \text{ as } x \in Z_n(\mathbf{C}) = ker(\gamma_n), \\ &= 0, \text{ as } \delta_{n+1}h_n(x) \in B_n(\mathbf{D}) = im(\delta_{n+1}). \end{aligned}$$

□

Proposition 2.3.4. *If $f, g : \mathbf{C} \to \mathbf{D}$ are homotopic and N is an R-module, then the induced cochain maps $Hom(\mathbf{D}, N) \to Hom(\mathbf{C}, N)$ are homotopic.*

Proof. A homotopy is given by maps $h_n : C_n \to D_{n+1}$ such that,

$$f_n - g_n = h_{n-1}\gamma_n + \delta_{n+1}h_n.$$

Let $h^n : Hom(\mathbf{D}, N)^n \to Hom(\mathbf{C}, N)^{n-1}$ be $Hom(h_{n-1}, N)$, as $Hom(D, N)^n \cong Hom(D_n, N)$ and $Hom(C, N)^{n-1} \cong Hom(C_{n-1}, N)$. □

Definition 2.3.5. A chain map $f : \mathbf{C} \to \mathbf{D}$ is a *homotopy equivalence* if there is a chain map $g : \mathbf{D} \to \mathbf{C}$ such that gf and fg are homotopic to the respective identity maps of $\mathbf{C}$ and $\mathbf{D}$.
If there exists a homotopy equivalence, $f : \mathbf{C} \to \mathbf{D}$, $\mathbf{C}$ and $\mathbf{D}$ are said to be *homotopy equivalent.*

Remark 2.3.6. The reader is advised not to confuse these definitions: Note that the two chain maps $f, g : \mathbf{C} \to \mathbf{D}$ can be homotopic, whilst two chain complexes $\mathbf{C}, \mathbf{D}$ can be homotopy equivalent.

Proposition 2.3.7. *A homotopy equivalence $f : \mathbf{C} \to \mathbf{D}$ of chain complexes induces a homotopy equivalence of cochain complexes $Hom(\mathbf{D}, N) \to Hom(\mathbf{C}, N)$. In particular, $H^n(\mathbf{D}, N) \cong H^n(\mathbf{C}, N)$.*

Proof. Omitted. □

Definition 2.3.8. A chain map $f : \mathbf{C} \to \mathbf{D}$ is a *quasi-isomorphism* if for each n the map $H_n(\mathbf{C}) \to H_n(\mathbf{D})$ is an isomorphism.

Proposition 2.3.9. *If $f : \mathbf{C} \to \mathbf{D}$ is a homotopy equivalence then it is a quasi-isomorphism.*

Proof. Assume that $f : \mathbf{C} \to \mathbf{D}$ is a homotopy equivalence, then there exists some chain map $g : \mathbf{D} \to \mathbf{C}$ such that $fg \simeq id_{\mathbf{D}}$ and $gf \simeq id_{\mathbf{C}}$. By Proposition 2.3.3, since $fg \simeq id_{\mathbf{D}}$ then they both induce the same map $H_n(\mathbf{D}) \to H_n(\mathbf{D})$ and similarly gf and $id_{\mathbf{C}}$ induce the same map $H_n(\mathbf{C}) \to H_n(\mathbf{C})$. Hence, f induces an isomorphism $H_n(\mathbf{C}) \to H_n(\mathbf{D})$ as it has an inverse and so is a quasi-isomorphism. □

Example 2.3.10. This example shows that the converse to Proposition 2.3.9 is not true. Consider the chain complexes,

$$\begin{tikzcd}
 & & & H_1 = 0 & H_0 = \frac{\mathbb{Z}}{2\mathbb{Z}} & & \\
\mathbf{C} & \cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "2"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, "nat"] & 0 \arrow[r] \arrow[d] & \cdots \\
\mathbf{D} & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \frac{\mathbb{Z}}{2\mathbb{Z}} \arrow[r] & 0 \arrow[r] & \cdots \\
 & & & & H_0 = \frac{\mathbb{Z}}{2\mathbb{Z}} & &
\end{tikzcd}$$

and so $H_0(\mathbf{C}) = \frac{\mathbb{Z}}{2\mathbb{Z}} = H_0(\mathbf{D})$ and so f is a quasi-isomorphism. However, it is not a homotopy equivalence.

Definition 2.3.11. A chain complex $\mathbf{C}$ is *contractible* if it is homotopy equivalent to the zero complex.
Equivalent condition: $id_{\mathbf{C}}$ is homotopic to $0_{\mathbf{C}}$.
Equivalent condition: There are maps $h_n : C_n \to C_{n+1}$ with,

$$id_{C_n} = h_{n-1}\gamma_n + \delta_{n+1}h_n, \text{ for all } n.$$

This is called a *contracting homotopy.*

Theorem 2.3.12. *A chain complex* $\mathbf{C}$ *is contractible if and only if it is acyclic and all of the short exact sequences,*

$$0 \to Z_(\mathbf{C} \xrightarrow{\iota_n} C_n \xrightarrow{\gamma_n} B_{n-1}(\mathbf{C}) \to 0,$$

are split, where ι_n *is the inclusion.*

Proof. $\Rightarrow$: If $\mathbf{C}$ is contractible then it is quasi-isomorphic to the zero complex, so acyclic by Proposition 2.3.9. Let h be the contracting homotopy. Let $s : B_{n-1}(\mathbf{C}) \to C_n$ be the restriction of $h_{n-2} : C_{n-1} \to C_n$. If $x \in B_{n-1}(\mathbf{C}) = im(\gamma_n)$,

$$\begin{aligned}
B_{n-1}(\mathbf{C} \ni x &= id_{C_{n-1}}(x), \\
&= (h_{n-1}\gamma_{n-1} + \gamma_n h_{n-1})(x), \\
&= h_{n-2}(\gamma_{n-1}(x)) + \gamma_n(h_{n-1}(x)), \\
&= \gamma_n(h_{n-1}(x)), \\
&= \gamma_n(s(x)).
\end{aligned}$$

Thus s makes γ_n is a retraction in the short exact sequence. $\Leftarrow$: Now suppose that $\mathbf{C}$ is acyclic and all the exact sequences are split. Then for all n there are sections,

$$s_{n-1} : B_{n-1}(\mathbf{C}) \to C_n.$$

If $x \in C_n$ then $x - s_{n-1}\gamma_n x \in Z_n(\mathbf{C}) = B_n(\mathbf{C})$ so we can define a function $h_n : C_n \to C_{n-1}$ by,

$$h_n(x) = s_n(x - s_{n-1}\gamma_n x).$$

Then,

$$\begin{aligned}(h_{n-1}\gamma_n + \gamma_{n+1}h_n)(x) &= s_{n-1}(\gamma_n x - s_{n-2}\gamma_{n-1}\gamma_n x) + \gamma_{n+1}s_n(x - s_{n-1}\gamma_n x),\\ &= s_{n-1}\gamma_n x + \gamma_{n+1}s_n x - \gamma_{n+1}s_n s_{n-1}\gamma_n x,\\ &= s_{n-1}\gamma_n x + \gamma_{n+1}s_n x + (x - s_{n-1}\gamma_n x),\\ &= x.\end{aligned}$$

□

2.4 Projective and Injective Resolutions

In this section we look at projective and injective modules and resolutions, for this the reader will need some prior knowledge of free modules, which will not be covered in this report. The reader is advised to look at [5], [4] and [3] for more information regarding free modules.
The following properties for projective modules come from [1] with added details from either my own work or [3]. In order to have a clear definition of a projective modules, rather than as a culmination of properties, we have promoted the first property from [1] to a definition and to aid this we give a separate defintion for a lifting as in [3].

Definition 2.4.1. Suppose we have a map $g : M \to N$ then a *lifting* of a map $p : A \to N$ is a map $q : A \to M$ with $gq = p$, i.e. the following diagram commutes.

$$\begin{tikzcd} & A \arrow[dl, "q"'] \arrow[d, "p"] \\ M \arrow[r, "g"] & N \end{tikzcd}$$

Definition 2.4.2. A module P is *projective* if, whenever $g : M \to N$ is surjective and $p : P \to N$ is any map, there exists a lifting $q : P \to M$ making the following diagram commutes:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists q"'] \arrow[d, "p"] \\ M \arrow[r, two heads, "g"] & N \end{tikzcd}$$

Proposition 2.4.3. *The following for a module P are equivalent:*

(i) *P is projective.*

(ii) *The sequence $0 \to Hom(P, L) \to Hom(P, M) \to Hom(P, N) \to 0$ is exact for any short exact sequence $0 \to L \to M \to N \to 0$.*

(iii) *Any short exact sequence $0 \to L \to M \to P \to 0$ splits.*

(iv) *P is isomorphic to a direct summand of free modules.*

Proof. (i) $\Rightarrow$ (ii): As P is projective we have that,

$$\begin{tikzcd} & & & P \arrow[dl, dashed, "\exists q"'] \arrow[d, "p"] & \\ 0 \arrow[r] & L \arrow[r, "f"] & M \arrow[r, two heads, "g"] & N \arrow[r] & 0 \end{tikzcd}$$

Hence, we have $q \in Hom(P, M)$ such that $gq = p \in Hom(P, N)$ which can also we written $p = gq = g_*(q) \in im(g_*)$, defining a surjective map, $g_* : Hom(P, M) \to Hom(P, N)$. Thus,

$$0 \to Hom(P, L) \to Hom(P, M) \xrightarrow{g_*} Hom(P, N) \to 0$$

is exact by Proposition 2.2.10.
(ii) $\Rightarrow$ (iii): By assumption, if we have the short exact sequence $0 \to L \to M \to P \to 0$ then we have an exact sequence,

$$0 \to Hom(P, L) \to Hom(P, M) \to Hom(P, P) \to 0.$$

Then we can lift the identity map $id_P \in Hom(P, P)$ to some $q \in Hom(P, M)$, such that:

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists q"'] \arrow[d, "p"] \\ M \arrow[r, two heads, "g"] & P \end{tikzcd}$$

Thus $g : M \to P$ is a retraction as we have $q : P \to M$ such that $gq = id_P$. Hence, $0 \to L \to M \to P \to 0$ splits by Lemma 2.2.16.
(iii)$\Rightarrow$ (iv): By choosing a generating set of P we get a surjection from a free module F onto P, $g : F \to P$, and this gives the short exact sequence,

$$0 \to ker(g) \to F \xrightarrow{g} P \to 0,$$

and this splits by assumption. Hence, $F \cong ker(g) \oplus P$ and so P is a direct summand of F by Corollary 2.2.18.
(iv) $\Rightarrow$ (i): By assumption, P is a direct summand of F and so there are maps $r : F \to P$ and $s : P \to F$ with $rs = id_P$. Now consider the diagram,

$$\begin{tikzcd} F \arrow[r, shift left, "r"] \arrow[d, dashed, "\exists t"'] & P \arrow[l, shift left, "s"] \arrow[d, "p"] \\ M \arrow[r, two heads, "g"] & N \end{tikzcd}$$

where g is surjective. The composite map $pr : F \to N$; since F is free, it is surjective, and so there is a map $t : F \to M$ with $gt = pr$. Define $q : P \to M$ by $q = ts$ and then we can see that $gq = p$ as,

$$gq = gts = prs = pid_P = p.$$

Hence P is projective. $\square$

Now we have established some properties of projective modules we can prove a corollary of the Long Exact Sequence Theorem in [1].

Corollary 2.4.4. *(Corollary to Theorem 2.2.26)*
If $\mathbf{P}$ is a chain complex of projective R-modules and $0 \to L \to M \to N \to 0$ is a short exact sequence of R-modules then you get a long exact sequence in cohomology.

$$\ldots \to H^{n-1}(\mathbf{P}, N) \to H^n(\mathbf{P}, L) \to H^n(\mathbf{P}, M) \to H^n(\mathbf{P}, N) \to H^{n+1}(\mathbf{P}, L) \to \ldots$$

Proof. Since P_n is projective for every n, Proposition 2.4.3, tells us we get an exact sequence,

$$0 \to Hom(P_n, L) \to Hom(P_n, M) \to Hom(P_n, N) \to 0.$$

Then use Remark **??**. □

As with projective modules we will use the first property given in [1] to define injective modules, and then provide the properties as in [1] with some details from [3].

Definition 2.4.5. A module I is *injective* if, whenever $f : L \to M$ is an injection, there exists a map $j : M \to I$ which extends any map $i : M \to I$, making the following diagram commute.

$$\begin{tikzcd}
I & \\
L \arrow[u, "i"] \arrow[r, "g"] & M \arrow[ul, dashed, "j"']
\end{tikzcd}$$

Proposition 2.4.6. *The following properties of a module I are equivalent:*

(i) *I is injective.*

(ii) *The sequence $0 \to Hom(N, I) \to Hom(M, I) \to Hom(L, I) \to 0$ is exact for any short exact sequence $0 \to L \to M \to N \to 0$.*

(iii) *Any short exact sequence $0 \to I \to M \to N \to 0$ splits.*

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii): Dual to Proposition 2.4.3 for projectives. (iii) $\Rightarrow$ (i): Consider the $0 \to L \to M \to {}^{M}/_{L} \to 0$ and construct it's pushout along $\theta : L \to I$:

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r, "f"] \arrow[d, "\theta"] & M \arrow[r, "g"] \arrow[d, "\phi"] & \frac{M}{L} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & I \arrow[r, "h"] & M' \arrow[r] \arrow[l, dashed, bend left, "\exists r"] & \frac{M}{L} \arrow[r] & 0
\end{tikzcd}$$

By assumption $0 \to I \to M' \to {}^{M}\!/_{L} \to 0$ splits and so $h : I \to M'$ has a retraction $r : M' \to I$. Then,

$$r\phi f = rh\theta = \theta,$$

so $r\phi : M \to I$ extends $\theta : L \to I$. Hence I is injective. $\square$

Definition 2.4.7. If M is an R-module, then a *projective resolution* of M is an exact sequence,

$$\mathbf{P} : \quad \ldots \to P_2 \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0$$

where each P_i is a projective module.

Proposition 2.4.8. *A projective resolution* $\mathbf{P}$ *of* M *is equivalent to giving a non-negative chain complex* $\mathbf{P}_\bullet$ *of projective modules and a quasi-isomorphism* $f : \mathbf{P}_\bullet \to \mathbf{M}$*, where all the* f_n *are trivial apart from* $f_0 : P_0 \twoheadrightarrow M$*.*

Proof. Consider the diagram,

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r] \arrow[d] & P_1 \arrow[r] \arrow[d] & P_0 \arrow[r] \arrow[d, two heads, "f_0"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

letting $f_0 : P_0 \twoheadrightarrow M$ be is any surjection and setting $\theta : P_1 \twoheadrightarrow ker(f_0)$, $\phi : P_2 \twoheadrightarrow ker(\phi)$, etc, as surjections . Then we have $H_n(\mathbf{P}_\bullet) = 0 = H_n(\mathbf{M})$ for all $n \geqslant 1$. At $n = 0$ we have $H_0(\mathbf{P}_\bullet) = {}^{P_0}\!/_{ker(f_0)}$ and $H_0(\mathbf{M}) = M$, but as f_0 is a surjection, the first isomorphism theorem gives $M \cong {}^{P_0}\!/_{ker(f_0)}$ and hence $H_n(\mathbf{P}_\bullet \cong H_n(\mathbf{M})$ for all n and so $f : \mathbf{P}_\bullet \to \mathbf{M}$ is a quasi-isomorphism. $\square$

Example 2.4.9. Let $M = {}^{\mathbb{Z}}\!/_{n\mathbb{Z}}$ be considered as a $\mathbb{Z}$-module. Then it has a projective resolution,

$$\ldots \to 0 \to \mathbb{Z} \xrightarrow{n} \mathbb{Z} \to \frac{\mathbb{Z}}{n\mathbb{Z}} \to 0.$$

Also, we can see that there is a quasi-isomorphism $f : \mathbf{P}_\bullet \to \mathbf{M}$,

$$\begin{tikzcd}
\cdots \arrow[r] & 0 \arrow[r] \arrow[d] & \mathbb{Z} \arrow[r, "n"] \arrow[d] & \mathbb{Z} \arrow[r] \arrow[d, two heads, "nat"] & 0 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & \frac{\mathbb{Z}}{n\mathbb{Z}} \arrow[r] & 0 \arrow[r] & \cdots
\end{tikzcd}$$

since the homologies are isomorphic at every n.

Definition 2.4.10. Given a projective resolution $\mathbf{P}$ of M,

$$\mathbf{P} : \quad \ldots \to P_n \xrightarrow{d_n} P_{n-1} \to \ldots \to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\epsilon} M \to 0,$$

the *syzygies* of M are the modules $\Omega^n M = im(d_n : P_n \to P_{n-1}$ and $\Omega^0 M = M$.

Note that here we used Crawley-Boevey's notation for the syzygies from [1] rather than Rotman's in [3]. However, they are clearly the same definition as exactness of the projective resolution gives,

$$K_n = ker(d_n : P_n \to P_{n-1}) = im(d_{n+1} : P_{n+1} \to P_n) = \Omega^{n+1}M,$$

and as the following remark demonstrates it is useful to recall this result:

Lemma 2.4.11. *Defintion 2.4.9 implies that there are exact sequences,*

$$0 \to \Omega^{n+1}M \to P_n \to \Omega^n M \to 0.$$

Proof. Note that $\Omega^{n+1}M = im(d_{n+1} : P_{n+1} \to P_n) = ker(d_n : P_n \to P_{n-1})$ and $\Omega^n M = im(d_n : P_n \to P_{n-1})$. $\square$

Definition 2.4.12. If M is an R-module, then an *injective resolution* of M is an exact sequence,

$$\mathbf{I} : \quad 0 \to M \xrightarrow{\eta} I^0 \xrightarrow{d^0} I^1 \xrightarrow{d^1} I^2 \to \dots,$$

where each I^i is an injective module.

Since the defintion for a cosyzygy is not given in [1] we turn to [3] for the following definition.

Definition 2.4.13. Given an injective resolution,

$$\mathbf{I} : \quad 0 \to M \xrightarrow{\eta} I^0 \xrightarrow{d^0} I^1 \to \dots \to I^n \xrightarrow{d^n} I^{n+1} \to \dots$$

the *cosyzygies* of M are the modules $\mho^n = coker(d^{n-1})$, for $n \geqslant 1$ and with $\mho^0 M = coker(\eta)$.

Theorem 2.4.14. *(Comparison Theorem)*
Any map of modules $f : M \to M'$ can be lifted to a map of projective resolutions.

$$\begin{tikzcd}
\cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, dashed, "\hat{f}_2"] & P_1 \arrow[r, "d_1"] \arrow[d, dashed, "\hat{f}_1"] & P_0 \arrow[r, "\epsilon"] \arrow[d, dashed, "\hat{f}_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & P_2' \arrow[r, "d_2'"'] & P_1' \arrow[r, "d_1'"'] & P_0' \arrow[r, "\epsilon'"'] & M' \arrow[r] & 0
\end{tikzcd}$$

Moreover, any two such lifts are homotopic as chain maps $\mathbf{P} \to \mathbf{P}'$.

Proof. Firstly, we show the existence of the lift. Consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Omega^1 M \arrow[r] \arrow[d, dashed, "\Omega^1 f"] & P_0 \arrow[r] \arrow[d, dashed, "\hat{f}_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \Omega^1 M' \arrow[r] & P_0' \arrow[r] & M' \arrow[r] & 0
\end{tikzcd} \tag{2.1}$$

Now as these are exact sequences by Lemma 2.4.10, the map $\epsilon' : P_0' \to M'$ is surjective. Then since P_0 is projective we can lift the composite map $f\epsilon : P_0 \to M'$,

$$\begin{tikzcd} & P_0 \arrow[dl, dashed, "\hat{f}_0"'] \arrow[d, "f\epsilon"] \\ P_0' \arrow[r, two heads, "\epsilon'"'] & M' \end{tikzcd}$$

and so have a map $\hat{f}_0$ making the right hand square of 2.1 commute. Then by diagram chasing there is a map $\Omega^1 f$ making the left hand square of 2.1 commute. The same argument gets $\hat{f}_1$ and $\Omega^2 f$.

$$\begin{tikzcd} 0 \arrow[r] & \Omega^2 M \arrow[r] \arrow[d, "\Omega^2 f"] & P_1 \arrow[r] \arrow[d, "\hat{f}_1"] & \Omega^1 M \arrow[r] \arrow[d, "\Omega^1 f"] & 0 \\ 0 \arrow[r] & \Omega^2 M' \arrow[r] & P_1' \arrow[r] & \Omega^1 M' \arrow[r] & 0 \end{tikzcd}$$

etc.

Finally, to show that any two lifts are homotopic it is equivalent to show that any lift of the zero map $M \to M'$ is homotopic to zero. Now,

$$\begin{tikzcd} \cdots \arrow[r] & P_2 \arrow[r, "d_2"] \arrow[d, "\hat{f}_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "\hat{f}_1"] & P_0 \arrow[r, "\epsilon"] \arrow[d, "\hat{f}_0"] & M \arrow[r] \arrow[d, "0"] & 0 \\ \cdots \arrow[r] & P_2' \arrow[r, "d_2'"'] & P_1' \arrow[r, "d_1'"'] & P_0' \arrow[r, "\epsilon'"'] & M' \arrow[r] & 0 \end{tikzcd}$$

Then $\epsilon\hat{f}_0 = 0$ by commutativity, and so $\hat{f}_0$ has an image contained in $\Omega^1 M'$. Now $d_1' : P_1' \twoheadrightarrow \Omega^1 M'$ is surjective and P_0 is projective so $\hat{f}_0$ lifts to a map $h_0 : P_0 \to P_1'$.

$$\begin{tikzcd} & P_0 \arrow[dl, dashed, "h_0"'] \arrow[d, "\hat{f}_0"] \\ P_1 \arrow[r, two heads] & \Omega^1 M' \end{tikzcd}$$

Thus $\hat{f}_0 = d_1' h_0$.

Now suppose we have constructed $h_0, h_1, \ldots, h_{n-1}$ with,

$$\hat{f}_i = d_{i+1} h_i + h_{i-1} d_i, \text{ for } 0 < i < n.$$

Then,

$$\begin{aligned} d_n'(\hat{f}_n - h_{n-1} d_n) &= d_n' \hat{f}_n - d_n' h_{n-1} d_n, \\ &= \hat{f_{n-1}} d_n - d_n' h_{n-1} d_n, \ (d_n' \hat{f}_n = \hat{f_{n-1}} d_n) \text{ by commutativity}, \\ &= (\hat{f_{n-1}} - d_n' h_{n-1}) d_n. \end{aligned}$$

If $n = 1$ then $(\hat{f}_0 - d'_1 h_0)d_1 = 0$ and if $n > 1$ $(f_{n-1} - d'_n h_{n-1})d_n = (f_{n-1} - h_{n-2}d_{n-1})d_n$, so also zero. Thus $f_n - h_{n-1}d_n$ has image contained in $\Omega^{n+1}M'$. Thus it lifts to a map $h_n : P_n \to P'_{n+1}$. □

Corollary 2.4.15. *If $\mathbf{P}$ and $\mathbf{P}'$ are projective resolutions of M then there is a homotopy equivalence $f : \mathbf{P} \to \mathbf{P}'$ such that the triangle,*

$$\begin{tikzcd} \mathbf{P} \arrow[d] \arrow[dr, "f"] & \\ \mathbf{P}' \arrow[r] & M(\mathit{in\ deg}\ 0) \end{tikzcd}$$

commutes. Moreover, f is unique up to homotopy.

Proof. The identity map $id_M : M \to M$ lifts to chain maps $f : \mathbf{P} \to \mathbf{P}'$ and $g : \mathbf{P}' \to \mathbf{P}$. Now $gf : \mathbf{P} \to \mathbf{P}$ is a lift of the identity map id_M, so it is homotopic to $id_{\mathbf{P}}$. Similarly, $fg : \mathbf{P}' \to \mathbf{P}'$ is homotopic to $id_{\mathbf{P}'}$.
The uniqueness of f up to homotopy is part of the Comparison Theorem (Thm 2.4.12). □

2.5 Ext

Definition 2.5.1. If M and N are R-modules then $Ext^n(M, N)$, or more precisely $Ext^n_R(M, N)$, is defined as follows:

- Choose a projective resolution $\mathbf{P}$ of M.
- Set $Ext^n(M, N) = H^n(\mathbf{P}, N)$.

Thus if,

$$\ldots \to P_2 \to P_1 \to P_0 \to M \to 0,$$

is the projective resolution, then $Ext^n(M, N)$ is the cohomology in degree n of the cochain complex,

$$0 \to Hom(P_0, N) \to Hom(P_1, N) \to Hom(P_2, N) \to \ldots$$

Remark 2.5.2. Note that this definition does not depend on the choice of projective resolution. If $\mathbf{P}'$ is another projective resolution there there is a homotopy equivalence $f : \mathbf{P} \to \mathbf{P}'$ by Corollary **??**. This gives a homotopy equivalence of cochain complexes $f^* : Hom(\mathbf{P}', N) \to Hom(P, N)$. Proposition 2.3.9 says this is a quasi-isomorphism and so induces isomorphisms on the cohomology $H^n(\mathbf{P}', N) \to H^n(\mathbf{P}, N)$. Moreover, the homotopy equivalence f is unique up to homotopy, so the cochain map is unique up to homotopy. Thus the map $H^n(\mathbf{P}', N) \to H^n(\mathbf{P}, N)$ is uniquely determined.

Proposition 2.5.3. (i) *If $\theta : N \to N'$ is a map there is a natural map $Ext^n(M, N) \to Ext^n(M, N')$.*

(ii) *If $\phi : M'' \to M$ is a map there is a natural map $Ext^n(M, N) \to Ext^n(M'', N)$.*

Proof. **(i)** As $\mathbf{P}$ is a projective resolution of M, $\theta : N \to N'$ induces a chain map $Hom(\mathbf{P}, N) \to Hom(\mathbf{P}', N)$.

(ii) $\phi : M'' \to M$ lifts to a chain map $\mathbf{P}'' \to \mathbf{P}$ of projective resolutions unique up to homotopy (Thm 2.4.12) and so induces a cochain map $Hom(\mathbf{P}, N) \to Hom(\mathbf{P}'', N)$ unique up to homotopy, giving unique maps on Ext^n.

□

Proposition 2.5.4. **(i)** *$Ext^n(M, N \oplus N') \cong Ext^n(M, N) \oplus Ext^n(M, N')$.*

(ii) *$Ext^n(M \oplus M', N) \cong Ext^n(M, N) \oplus Ext^n(M', N)$.*

Proof. **(i)** If $\mathbf{P}$ is a projective resolution of M then $Hom(\mathbf{P}, N \oplus N') \cong Hom(\mathbf{P}, N) \oplus Hom(\mathbf{P}, N')$.

(ii) If $\mathbf{P}'$ is a projective resolution of M' then $\mathbf{P} \oplus \mathbf{P}'$ is a projective resolution of $M \oplus M'$ and $Hom(\mathbf{P} \oplus \mathbf{P}', N) \cong Hom(\mathbf{P}, N) \oplus Hom(\mathbf{P}', N)$.

□

Lemma 2.5.5.

$$Ext^n(M, N) \cong \begin{cases} 0 & n < 0, \\ Hom(M, N) & n = 0, \\ coker(Hom(P_{n-1}, N) \xrightarrow{\iota^*_{n-1}} Hom(\Omega^n M, N) & n > 0, \end{cases}$$

where $0 \to \Omega^n M \xrightarrow{\iota_n} P_{n-1} \to \Omega^{n-1} M \to 0$.

Proof. $n < 0$: The claim is clear as $Hom(\mathbf{P}, N)$ is a non-negative cochain complex.
$n \geqslant 0$: By definition,

$$Ext^n(M, N) = H^n(\mathbf{P}, N) = \frac{ker(d^*_{n+1} : Hom(P_n, N) \to Hom(P_{n+1}, N))}{im(d^*_n : Hom(P_{n-1}, N) \to Hom(P_n, N))}.$$

Thus,

$$\begin{aligned} Ext^n(M, N) &= \frac{ker(d^*_{n+1} : Hom(P_n, N) \to Hom(P_{n+1}, N))}{im(d^*_n : Hom(P_{n-1}, N) \to Hom(P_n, N))}, \\ &\cong \frac{codomain(\gamma : Hom(P_{n-1}, N) \to Ker(d^*_{n+1}))}{im(\gamma : Hom(P_{n-1}, N) \to Ker(d^*_{n+1}))}, \\ &\cong coker(\gamma : Hom(Hom(P_{n-1}, N) \to Ker(d^*_{n+1}))). \end{aligned}$$

Now there is an exact sequence,

$$P_{n+1} \xrightarrow{d_{n+1}} P_n \xrightarrow{d_n} \Omega^n M \to 0,$$

which then induces the exact sequence,

$$0 \to Hom(\Omega^n M, N) \twoheadrightarrow Hom(P_n, N) \xrightarrow{d^*_{n+1}} Hom(P_{n+1}, N),$$

and so, $ker(d^*_{n+1}) \cong im(Hom(\Omega^n M, N)) \twoheadrightarrow Hom(P_n, N)) \cong Hom(\Omega^n M, N)$.
Note that if we let $P_{-1} = 0$ and $\iota_0 : M \to 0$ then at $n = 0$ we have,

$$\begin{aligned} coker(Hom(P_{-1}, N) \xrightarrow{\iota^*_0} Hom(\Omega^0 M, N)) &= coker(Hom(0, N) \xrightarrow{\iota^*_0} Hom(M, N), \\ &\cong Hom(M, N). \end{aligned}$$

□

Proposition 2.5.6. *If X is a module then for any short exact sequence $0 \to L \to M \to N \to 0$ you get a long exact sequence,*

$$\begin{aligned} 0 \to & Hom(X, L) \to Hom(X, M) \to Hom(X, N) \to \\ \to & Ext^1(X, L) \to Ext^1(X, M) \to Ext^1(X, N) \to \\ \to & Ext^2(X, L) \to \dots \end{aligned}$$

It is called the long exact sequence for $Hom(X, -)$.

Proof. This is the long exact sequence in cohomology for the chain complex $\mathbf{P}$, a projective resolution of X. (Corollary **??**). $\square$

Lemma 2.5.7. *$Ext^n(M, N) = 0$ for $n > 0$ if either:*

(i) *M is projective, or,*

(ii) *N is injective.*

Proof. **(i)** If M is projective you can use the projective resolution,

$$\dots \to 0 \to 0 \to M \to M \to 0.$$

(ii) If N is injective then the exact sequence,

$$0 \to \Omega^n M \xrightarrow{\iota_n} P_{n-1} \to \Omega^{n-1} M \to 0,$$

gives an exact sequence,

$$0 \to Hom(\Omega^{n-1} M, N) \to Hom(P_{n-1}, N) \to Hom(\Omega^n M, N) \to 0,$$

so $coker(\iota_n^*) = 0$.

$\square$

Proposition 2.5.8. *If $0 \to N \to I^0 \to I^1 \to \dots$ is an injective resolution of N then you can complute $Ext^n(M, N)$ as the cohomology of the cochain complex $Hom(M, \mathbf{I})$,*

$$0 \to Hom(M, I^0) \to Hom(M, I^1) \to Hom(M, I^2) \to \dots$$

Proof. Firstly, we break the injective resolution into short exact sequences,

$$\begin{array}{ccccccccc} & & N & & & & & & \\ & & \| & & & & & & \\ 0 & \longrightarrow & \mho^0 N & \longrightarrow & I^0 & \longrightarrow & \mho^1 N & \longrightarrow & 0 \\ 0 & \longrightarrow & \mho^1 N & \longrightarrow & I^1 & \longrightarrow & \mho^2 N & \longrightarrow & 0 \end{array}$$

and so on. You then get a long exact sequence,

$$\begin{aligned} 0 \to & Hom(M, \mho^i N) \to Hom(M, I^i) \to Hom(M, \mho^{i+1} N) \to \\ \to & Ext^1(M, \mho^i N) \twoheadrightarrow 0 \hookrightarrow Ext^1(M, \mho^{i+1} N) \to \\ \to & Ext^2(M, \mho^i N) \twoheadrightarrow 0 \hookrightarrow \dots \end{aligned}$$

Thus,

$$\begin{aligned} Ext^n(M,N) &\cong Ext^{n-1}(M,\mho^i N) \cong \cdots \cong Ext^1(M,\mho^{n-1}N), \text{ by dimension shifting,} \\ &\cong coker(Hom(M,I^{n-1}) \to Hom(M,\mho^n N)). \end{aligned}$$

Now, $0 \to \mho^n N \to I^n \to I^{n+1}$ is exact, so,

$$0 \to Hom(M,\mho^n N) \to Hom(M,I^n) \to Hom(M,I^{n+1}),$$

is exact. The claim follows. $\square$

Chapter 3

Representation of Quivers

3.1 Quivers and Path Algebras

Definition 3.1.1. A *quiver* is defined as the tuple of sets and functions, $\mathbf{Q} = (Q_0,\, Q_1,\, s,\, t : Q_1 \to Q_0)$ such that:

- Q_0 is the set of vertices, which we will set to be the finite set $\{1, 2, \ldots, n\}$.
- Q_1 is the set of arrows, which we will also set to be finite.
- Functions s, t such that an arrow $\rho \in Q_1$ *starts* at the vertex $s(\rho) \in Q_0$ and *terminates* at the vertex $t(\rho) \in Q_0$, i.e. $\rho : s(\rho) \to t(\rho)$.

Example 3.1.2. A quiver $Q = (Q_0, Q_1, s, t : Q_1 \to Q_0)$ where $Q_0 = \{1, 2, 3, 4\}$, $Q_1 = \{\alpha, \beta\}$, and s, t are defined such that;

$$\begin{aligned} s :& Q_1 \to Q_0, \quad \alpha \mapsto 1,\ \beta \mapsto 2,\ \gamma \mapsto 4 \\ t :& Q_1 \to Q_0, \quad \alpha \mapsto 2,\ \beta \mapsto 3,\ \gamma \mapsto 3, \end{aligned}$$

looks like,

$$\mathbf{Q} : \quad \begin{tikzcd} 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"'] & 3 & 4 \arrow[l, "\gamma"] \end{tikzcd}$$

Definition 3.1.3. A *non-trivial path*, p, in a quiver is a sequence of arrows $\rho_1, \ldots, \rho_n$ which satisfies $t(\rho_{i+1}) = s(\rho_i)$ for all $1 \leqslant i < n$, i.e. the start of an arrow is where the previous arrow terminated. The starting and terminatinating vertex of a path p are denoted $s(p)$ and $t(p)$, respectively.

Notation 3.1.4. In this report the arrows in a path will be ordered the same way as the composition of functions, as in [2], however, be aware that other publications may order the arrows the opposite way.

Definition 3.1.5. The *trivial path* is the path which contains no arrows, i.e. it is a single vertex, and is denoted e_i where the vertex is i.

Example 3.1.6. The paths of the quiver in Example 3.1.2 are:

$$p_1 = e_1, \quad p_2 = e_2, \quad p_3 = e_3, \quad p_4 = e_4, \quad p_5 = \alpha, \quad p_6 = \beta, \quad p_7 = \gamma, \quad p_8 = \beta\alpha.$$

However, $\gamma\beta\alpha$ is not a path because $t(\gamma) = 3 \neq s(\beta) = 2$.

Definition 3.1.7. A *path algebra* kQ is the k-alegbra which has the basis all the paths in Q, and the product of two paths p, q is defined as,

$$pq = \begin{cases} \text{obvious composition} & \text{if } t(q) = s(p), \\ 0 & \text{otherwise.} \end{cases}$$

This multiplication is assosciative.

Example 3.1.8. If we once again use the quiver Q from Example 3.1.2, then, from Example 3.1.6, we know the basis of the path algebra kQ will be,

$$\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \beta\alpha\}.$$

The product of β and α is $\beta\alpha$, but the product of α and β is zero, and the product of e_2 and α is just α.

Example 3.1.9. If Q is the following quiver,

$$\mathbf{Q}: \begin{tikzcd} \textcircled{1} \arrow[loop left, "\alpha"] \arrow[loop right, "\beta"] \end{tikzcd}$$

forms the path algebra $kQ \cong k[X, Y]$, the free, assosciative algebra on two letters. In fact, if we have a quiver with a single vertex and n loops, then this can be associated with the free, assosciated algebra on n letters.

Example 3.1.10. If we have the quiver,

$$\mathbf{Q}: \begin{tikzcd} \textcircled{1} \arrow[r, "\alpha"] & \textcircled{2} \arrow[r, "\beta"] & \textcircled{3} \arrow[r, "\gamma"] & \textcircled{4} \end{tikzcd}$$

the the path algebra, $kQ \cong UT_4(k)$ by the isomorphism,

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 \alpha +$$
$$\lambda_6 \beta + \lambda_7 \gamma + \lambda_8 \beta\alpha + \lambda_9 \gamma\beta\alpha + \lambda_{10} \gamma\beta \mapsto \begin{pmatrix} \lambda_4 & \lambda_7 & \lambda_{10} & \lambda_9 \\ 0 & \lambda_3 & \lambda_6 & \lambda_8 \\ 0 & 0 & \lambda_2 & \lambda_5 \\ 0 & 0 & 0 & \lambda_1 \end{pmatrix}$$

Generally, a quiver of the form,

$$\mathbf{Q}': \begin{tikzcd} \textcircled{1} \arrow[r, "\alpha"] & \textcircled{2} \arrow[r, "\beta"] & \cdots \arrow[r, "\gamma"] & \textcircled{n} \end{tikzcd}$$

induces a path alegbra $kQ' \cong UT_n(k)$ for any n.

Example 3.1.11. In fact, we find that if Q is the same quiver as above in Example 3.1.10, then $kQ \cong LT_4(k)$ as well, through the isomorphism,

$$\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3 + \lambda_4 e_4 + \lambda_5 \alpha +$$
$$\lambda_6 \beta + \lambda_7 \gamma + \lambda_8 \beta\alpha + \lambda_9 \gamma\beta\alpha + \lambda_{10} \gamma\beta \mapsto \begin{pmatrix} \lambda_1 & 0 & 0 & 0 \\ \lambda_5 & \lambda_2 & 0 & 0 \\ \lambda_8 & \lambda_6 & \lambda_3 & 0 \\ \lambda_9 & \lambda_{10} & \lambda_7 & \lambda_4 \end{pmatrix}.$$

Once again, this extends to the general case that, $kQ' \cong LT_n(k)$.

Example 3.1.12. In a more general case, as long as there is only one path between any two vertices, we can identify kQ with the following subalgebra of $M_n(k)$,

$$\{M \in M_n(k) : M_ij = 0 \text{ if there is no path from } j \text{ to } i\}.$$

For instance, if we have the quiver,

$$\mathbf{Q}: \quad \begin{tikzcd} & & & 5 \arrow[dl, "\delta"'] \arrow[r, "\epsilon"] & 6 \\ 1 \arrow[r, "\alpha"] & 2 \arrow[r, "\beta"] & 3 \arrow[dr, "\gamma"] & & \\ & & & 4 & \end{tikzcd}$$

the we can see that the path algebra kQ is isomorphic to the subalgebra with matrices of the form,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

Remark 3.1.13. Note that the idea here has some parallels to similar results for directed graphs in graph theory.

The following results about the idempotents of path alegebras are from [2], however, we have either given a proof or expanded upon the one given. For the following results we set $A = kQ$ and the e_i are the trivial paths of Q.

Lemma 3.1.14. *The e_i are orthogonal, idempotents in A, i.e. $e_ie_i = e_i$ and $e_ie_j = 0$ where $i \neq j$. Thus $\sum_{i=1}^{n} e_i = 1_A$.*

Proof. Well, obviously, $e_ie_j = 0$ when$i \neq j$ because $t(e_j) \neq s(e_i)$ because they are the trivial paths at different vertices, i and j. Similarly, if we have the product e_ie_i then the composition makes sense here because $t(e_i) = s(e_i)$, but the composition is just e_i becuase if we travel along the trivial path e_i twice, then this is just the same as travelling the trivial path.
If we have some path p of the quiver, then bear in mind that,

$$e_ip = \begin{cases} e_ip = p & \text{if } i = t(p), \\ 0 & \text{otherwise.} \end{cases}$$

Hence, the sum of the trivial paths acting on our path p becomes,

$$\big(\sum_{i=1}^{n} e_i\big)p = (e_1 + e_2 + \cdots + e_n)p = e_ip = p = 1_Ap,$$

because only one of the e_i in the sum will satisfy $i = t(p)$ and all the rest will give zero. Similary, we can show that $p\big(\sum_{i=1}^{n} e_i\big) = p1_A$. $\square$

Lemma 3.1.15. *The bases of the spaces Ae_i and e_jA are all the paths starting at i and all the paths terminating at j, respectively. It then follows that the space e_jAe_i has as a basis the paths that start at i and terminate at j.*

Proof. For some $i \in Q_0$, and A has the basis $\{p_1, p_2, \dots, p_n\}$, then as Ae_i is a subspace of A the basis of Ae_i must be a subset of the basis of A, so,

$$Ae_i = \{ae_i : a \in A\} = span\{p_1e_i + p_2e_i + \cdots + p_ne_i\},$$

and we know that,

$$p_re_i = \begin{cases} p_r & \text{if } s(p_r) = i, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we can see that the basis of Ae_i must be all the paths starting at i. Similarly, as e_jA is a subspace of A snd so following a similar argument we can see that its basis must be all the paths terminating at j. The result for e_jAe_i follows from these as it simply the intersection of the two spaces. □

Lemma 3.1.16. *$A = \oplus_{i=1}^n Ae_i$, so each Ae_i is a projective left A-module.*

Proof. From Lemma 3.1.15 we know that for each i, the basis of Ae_i are all the paths starting at i and so $\oplus_{i=1}^n Ae_i$ must have as a basis all the paths starting at every vertex in Q, hence all the paths in Q. Thus $A = \oplus_{i=1}^n Ae_i$. Also, each Ae_i is obviously a left A-module with the action defined as multiplication by e_i. Hence, each Ae_i is a projective, left A-module. □

Remark 3.1.17. Similarly, $A = \oplus_{i=1}^n e_jA$, so each e_jA is a projective, right A-module.

Lemma 3.1.18. *If X is a left A-module, then $Hom_A(Ae_i, X) \cong e_iX$.*

Proof. Well, we want to show that there are some maps f and g such that they satisfy,

$$Hom_A(Ae_i, X) \underset{g}{\overset{f}{\longleftrightarrow}} e_iX$$

Well, first consider the map $\theta : Ae_i \to X$ and so $\theta \in Hom_A(Ae_i, X)$, then we can have f such that $f(\theta) = \theta(e_i) = \theta(e_i^2) = e_i\theta(e_i) \in e_iX$. Hence, f maps a homomorphism from $Hom_A(Ae_i, X)$ to an element in e_iX. Now consider the map $g(x) : Ae_i \to X$, $g(x)(a) = ax$. We can check this is an A-module homomorphism as,

$$\begin{aligned} g(x)(a+b) = (a+b)x = ax + bx = g(x)(a) + g(x)(b) \qquad & a, b \in Ae_i, \\ g(x)(\lambda a) = \lambda ax = \lambda g(x)(a) \qquad & \lambda \in A, a \in Ae_i. \end{aligned}$$

Hence, $g \in Hom_A(Ae_i, X)$. However, we can also have that $g : eX \to Hom_A(Ae_i, X)$ through $x \mapsto g(x)(r)$.
So now we need to show that f and g are inverse contructions of one another. We have for some $\theta \in Hom_A(Ae_i, X)$,

$$\begin{aligned} \theta &\xrightarrow{f} \theta(e_i) \\ g(\theta(e_i))(a) = a\theta(e_i) = \theta(ae_i) &\xleftarrow[g]{} \theta(e_i) \end{aligned}$$

but $a \in Ae_i$ and so $a = \lambda e_i$ for some $\lambda \in A$, hence $ae_i = \lambda e_i^2 = \lambda e_i = a$. Thus, $g(\theta(e_i))(a) = \theta(ae_i) = \theta(a)$ so $g(\theta(e_i)) = \theta$ and f and g are inverses. □

Lemma 3.1.19. *If $0 \neq a \in Ae_i$ and $0 \neq b \in e_iA$ then $ab \neq 0$.*

Proof. We know that a and b must have the forms, $a = \lambda_1 p_1 + \dots \lambda_n p_n$ and $b = \mu_1 q_1 + \dots \mu_m q_m$ where the p are paths starting at vertex i, the q are paths termination at vertex i and p_r, q_r are paths of length r. Then the longest path in the product ab must be $\lambda_n \mu_m p_n q_m$ and so $\lambda_n \mu_m \neq 0$ and $p_n q_m \neq 0$, so the product $ab \neq 0$. $\square$

Lemma 3.1.20. *The e_i are primitive idempotents, i.e. Ae_i is an indecomposable module.*

Proof. From Lemma 3.1.18 we know that $End_A(Ae_i) \cong e_iAe_i$ and if this contains an idempotent ϵ, then $\epsilon^2 = \epsilon = \epsilon e_i$, so $\epsilon(e_i - \epsilon) = 0$, but from Lemma 3.1.19 we know that this can not happen if $\epsilon, (e_i - \epsilon) \neq 0$, thus we must have that either $\epsilon = 0$ or $\epsilon = e_i$ and the result follows. $\square$

Lemma 3.1.21. *If $e_i \in Ae_jA$ then $i = j$.*

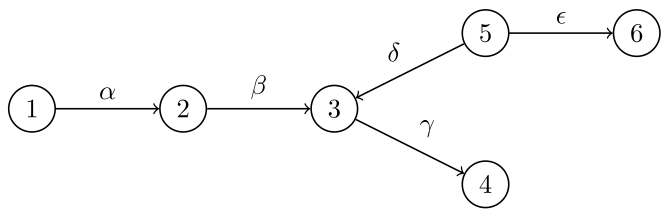
Proof. For similar reasoning as in Lemma **??**, we can see that Ae_jA has as a basis all the paths in A which pass through the vertex j and, by the definition of the trivial path, e_i cannot pass through the vertex j unless $i = j$ and so if $e_i \in Ae_jA$ we must have that $i = j$. $\square$

Lemma 3.1.22. *The e_i are inequivalent, i.e. $Ae_i \ncong Ae_j$ for $i \neq j$.*

Proof. Two idempotent elements e_i, e_j are said to be equivalent iff there exists some $u \in e_iAe_j$ and $v \in e_jAe_i$ such that $uv = e_i$ and $vu = e_j$. From Lemma 3.1.18 we can see that $Hom(Ae_i, Ae_j) \cong e_iAe_j$ and $Hom(Ae_j, Ae_i) \cong e_jAe_i$, and so inverse isomorphisms give elements u and v as described above with $uv = e_i$ and $vu = e_j$. However, this means that the path e_i must pass through the vertex j at some point, and vice versa, which contradicts Lemma . $\square$

Example 3.1.23. Consider the quiver from Example 3.1.12,

Q :



then we have have the following examples displaying the previous results for the idempotents of kQ. Let $A = kQ$.

1. We can see that $e_1e_1 = e_1$ and $e_1e_2 = 0$. Also, $1_A = e_1+e_2+e_3+e_4+e_5+e_6$ and so

$$\begin{aligned} 1_A\gamma\beta &= (e_1 + e_2 + e_3 + e_4 + e_5 + e_6)\gamma\beta, \\ &= e_1\gamma\beta + e_2\gamma\beta + e_3\gamma\beta + e_4\gamma\beta + e_5\gamma\beta + e_6\gamma\beta, \\ &= e_4\gamma\beta, \\ &= \gamma\beta. \end{aligned}$$

2. Now we can see that Ae_1 is spanned by all the paths starting at vertex 1, as,

$$\begin{aligned} Ae_1 &= \{ae_1 : a \in A\}, \\ &= span_k\{\sum pe_1 : p \text{ is in the basis of } A\}, \\ &= span_k\{e_1e_1 + e_2e_1 + e_3e_1 + e_4e_1 + e_5e_1 + e_6e_1 + \alpha e_1 + \\ &\qquad \beta e_1 + \gamma e_1 + \delta e_1 + \epsilon e_1 + \beta\alpha e_1 + \gamma\beta e_1 + \gamma\delta e_1 + \gamma\beta\alpha e_1\}, \\ &= span_k\{e_1 + \alpha + \beta\alpha + \gamma\beta\alpha\}, \\ &= span_k\{\text{all paths starting at vertex 1}\}. \end{aligned}$$

Similarly, we can see that,

$$\begin{aligned} e_3A &= \{e_3a : a \in A\}, \\ &= span_k\{e_3 + \beta + \delta + \beta\alpha\}, \\ &= span_k\{\text{all the paths terminating at vertex 3}\}. \end{aligned}$$

Then we can see that,

$$\begin{aligned} e_3Ae_1 &= \{e_3ae_1 : a \in A\}, \\ &= span_k\{\beta\alpha\}, \\ &= span_k\{\text{all paths starting at vertex 1 and terminating at vertex 3}\}. \end{aligned}$$

3. We can see that,

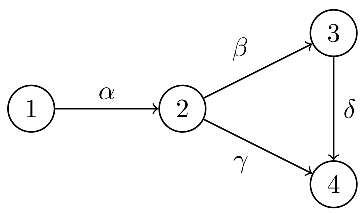
$$\begin{aligned} \oplus_{i=1}^{6} Ae_i &= Ae_1 \oplus Ae_2 \oplus Ae_3 \oplus Ae_4 \oplus e_5 \oplus e_6, \\ &= span_k\{e_1 + \alpha + \beta\alpha + \gamma\beta\alpha\} \oplus span_k\{e_2 + \beta + \gamma\beta\} \oplus span_k\{e_3 + \gamma\} \oplus span_k\{e_4\} \oplus span_k\{e_5 + \delta + \\ &= A. \end{aligned}$$

The following properties of path algebras are from [2], where they are given as exercises. In this report we will present them with proofs. Once again $A = kQ$.

Lemma 3.1.24. *A is finite dimensional if and only if Q has no oriented cycles.*

Proof. $\Rightarrow$: If Q has an oriented cycle then it will have a infinite number of paths as you can keep going round the cycle. This means that the basis of A, which is all the paths in Q, will be infinite, and so A will not be finite dimensional.
$\Leftarrow$: If Q has no oriented cycles then it must have a finite number of paths and so the basis of A will be finite and hence A will be finite dimensional. $\square$

Example 3.1.25. If we consider the quiver Q,

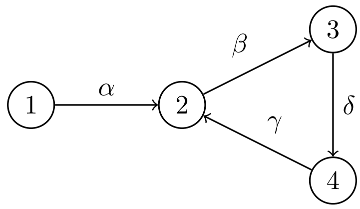


Q :

and we can see that Q has the paths $e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \beta\alpha, \gamma\alpha, \beta\delta, \delta\beta\alpha$, and so the basis of A is $\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \beta\alpha, \gamma\alpha, \beta\delta, \delta\beta\alpha\}$, which is finite, and

so A is finite dimensional.
However, if we have the quiver Q',

$\mathbf{Q'}$:



it has paths $e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \delta, \beta\alpha, \delta\beta, \gamma\delta, \delta\beta\alpha, \gamma\delta\beta\alpha, \beta\gamma\delta\beta\alpha, \delta\beta\gamma\delta\beta\alpha, \ldots$, and so on, an infinte number of paths, meaning that the basis of A is infinte and hence A is not finite dimensional.

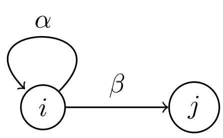
Lemma 3.1.26. *A is prime, i.e. $IJ \neq 0$ for two sided ideals $I, J \neq 0$ if and only if for all $i, j \in Q_0$ there exists a path i to j.*

Proof. Need to include proof here □

Lemma 3.1.27. *A is left noetherian if and only if, if there is an oriented cycle through the vertex i, then only one arrow starts at the vertex i.*

Proof. Let Ap represent the left-ideal whose basis are all the paths starting with p for any path p.
Suppose we have a quiver Q of the form,

$\mathbf{Q}$:



assuming $i \neq j$ and where α represents an orientated cycle, then we can see that,

$$A\beta + A\beta\alpha + \cdots + A\beta\alpha^n \subseteq A\beta + A\beta\alpha + \cdots + A\beta\alpha^{n+1}.$$

However, since $e_j\beta\alpha^{n+1} \notin A\beta + A\beta\alpha + \cdots + A\beta\alpha^n$ the inclusion is strict. Hence, if Q is of the form above, there is an ascending chain of left ideals,

$$A\beta \subset A\beta + A\beta\alpha \subset A\beta + A\beta\alpha + A\beta\alpha^2 \subset \ldots$$

which does not terminate and so A is not noetherian.
Suppose that for all $i \in Q_0$, if α an oriented cycle where $s(\alpha) = i$, then for all $\rho \in Q_1$, $\rho = \alpha_1$ or $s(\rho) \neq i$; where $\alpha = \alpha_1\alpha_2 \ldots \alpha_m$ for some $\alpha_1, \ldots, \alpha_m \in Q_1$. This means there are some $\rho_1, \ldots, \rho_n$ paths such that given any path $\beta \in A$ we have that $\beta = q\rho_j$ for some $j \in \{1, \ldots, n\}$ and some q a path in an oriented cycle. Now if we let Q' be the quiver, and A' the corresponding path algebra, where,

$$Q'_0 = Q_0 \ \& \ Q_1 = Q'_1 \cup \{a \in Q_1 : a \text{ is not in an oriented cycle of } Q\}.$$

So, for any basis element β of A, $\beta = qp_j$ where q is a path in A'. Following the previous two results we can see that,

$$A = A'\rho_1 + A'\rho_2 + \cdots + A'p_n \tag{3.1}$$

Claim. *If S is a subring of a ring R and R is finitely generated as a left S-module, then, if S is a noetherian ring, so is R.*

Proof. From the result that a finitely generated module over a noetherian ring is noetherian, gives us that R is noetherian as a left S-module. Let $I_1 \subseteq I_2 \subseteq \ldots$ be an ascending chain of left ideals in R, and each I_k is a left S-submodule of R; so by the above, this chain terminates. □

Using the above Claim and the earlier result that A is finitely generated as a left A'-module, we can see that our problem of proving that A is left noetherian reduces to proving that A' is left noetherian.
Now let $\alpha_j^{(l)}$ be the j^{th} arrow of the l^{th} oriented cycle in Q' and then,

$$Q_1' = \{\alpha_j^{(l)} : j \in \{1, \ldots, q_l\}, l \in \{1, \ldots, r\}\}.$$

Consider the endomorphism,

$$\begin{aligned} X_l :&A' \to A', \\ &e_t(p) \mapsto \alpha_{k-1}^{(l)}\alpha_{k-2}^{(l)} \ldots \alpha_1^{(l)}\alpha_{q_l}^{(l)} \ldots \alpha_k^{(l)}p, \text{ where } t(p) = t = s(\alpha_k^{(l)}), \\ &\text{otherwise } \mapsto 0. \end{aligned}$$

We can adapt Hilbert's Basis Theorem to show that $k[X_1, \ldots, X_r]$ is a noetherian ring and we can see that A' is a finitely generated $k[X_1, \ldots, X_r]$-module. Hence, A' is left noetherian, and so A is left noetherian. □

Definition 3.1.28. We can define the *length* of any path p by the following: Let,

$$length(p) = length(\sum_{\text{arrows } \rho} \lambda_\rho \rho) := max\{length(\rho) : \lambda_\rho \neq 0\},$$

where $\lambda_\rho \neq 0$ for some ρ.

Lemma 3.1.29. *The basis of $J(A)$ is $\{$path i to j : there is no path from j to $i\}$.*

Proof. Firstly, lets prove that,

$$J'(A) := span_k\{\text{paths } p : s(p) = i, t(p) = j \text{with no paths from } j \text{ to } i\} \subseteq J(A).$$

Given any $w \in J'(A)$ we have that,

1. $(wp)^2 = 0$ for any path p.

   Proof. There are no paths p such that $t(p) = s(w)$ and $s(p) = t(w)$, so $wpw = 0$, so $wpwp = (wp)^2 = 0$. □

2. $(w(p + p'))^2 = 0$ for any path p and $\lambda \in k$.

   Proof.

   $$(w(p + p'))^2 = (wp + wp')^2 = (wp)^2 + wpwp' + wp'wp + (wp')^2$$

   As wpw and $wp'w$ are both equal to zero by the above, we have that, $(w(p + p'))^2 = 0$. □

3. $(w(\lambda p))^2 = \lambda^2 (wp)^2 = 0$

Proof. By item 1. $\square$

Now let $w \in J'(A)$ and $z \in A$, then,

$$wz = u\Big(\sum_{\text{paths } p} \lambda_p p\Big) = w\Big(\sum_{\substack{\text{paths } p:\\ t(p)=s(w)}} \lambda_p p\Big) := \tau$$

and from above we can see that $\tau^2 = 0$, meaning $(wz)^2 = 0$, so, $(1 + wz)(1 - wz) = 1$, hence, $1 + wz \in U(A)$ for all $z \in A$. Thus $w \in J(A)$.
Now we want to prove that $J(A) \subseteq J'(A)$. Let w be a basis element of $J(A)$ coming from the basis of A. Suppose also that $length(w) = 0$, so $w = e_i$ for some $i \in Q_0$. Let,

$$M_i := \sum_{\substack{\text{paths } p:\\ p \neq e_i}} Ap,$$

then $M_i \trianglelefteq A$ and if $M_i \subsetneq N \trianglelefteq A$, then, $e_i \in N$, so, $N = A$; hence, M_i is a maximal left ideal, but $w \notin M_i$ contradicting that $w \in J(A)$.Hence, $length(w) > 0$.
Now suppose $w \notin J'(A)$, so there's a path p such that $s(w) = t(p)$ and $s(p) = t(w)$. Since $w \in J(A)$ and $p \in A$, $1 + wp \in U(A)$, so there exists some $z \in A$ such that $(1 + wp)z = 1$, with,

$$z = \sum_{l=0}^{m} \sum_{\substack{\text{paths } p:\\ length(p_l)=l}} \lambda_{p_l} p_l.$$

Since, $length(1) = length(l_1 + \cdots + l_n) = 0$, we have that,

$$\begin{aligned} 0 &= length((1 + wp)z),\\ &= length(z + wpz),\\ &= max(length(z), length(wpz)), \end{aligned}$$

so $wpz = 0$ as otherwise $length(wpz) = 0$, contradicting that $length(w) > 0$. So as $(1 + wp)z = z + wpz = 1$, $z = 1$, so $wp = 0$, a contradiction. $\square$

Lemma 3.1.30. *The centre of A is $k \times k \times \dots k[T] \times k[T] \times \dots$, with one factor for each connected component C of Q, and that the factor is $k[T]$ if and only if C is an oriented cycle.*

Proof. Firstly, we know that if our quiver Q is composed of n connected components $C_1, \dots, C_n$ then we our path algebra looks like $kQ = kC_1 \times kC_2 \times \cdots \times kC_n$.

Claim. *Where $Z(A)$ represents the centre of A, we have that,*

$$Z(A \times B) = Z(A) \times Z(B).$$

Proof. Suppose we have $(a, b) \in Z(A \times B)$, then we have that,

$$\begin{aligned} &(a, b)(a', b') = (a', b')(a, b) \forall (a', b') \in A \times B,\\ \Leftrightarrow &(aa', bb') = (a'a, b'b) \forall a' \in A, b' \in B,\\ \Leftrightarrow &aa' = a'a \;\&\; bb' = b'b \forall a' \in A, b' \in B,\\ \Leftrightarrow &a \in Z(A) \;\&\; b \in Z(B). \end{aligned}$$

Hence, $Z(A \times B) \cong Z(A) \times Z(B)$. $\square$

Now, using the claim, we can see that,

$$Z(kQ) \cong Z(kC_1) \times Z(kC_2) \times \cdots \times Z(kC_n),$$

Now, assume the connected component C_i is not an oriented cycle, and consider $a \in Z(kC_i)$, which is a linear combination of paths. Let us choose a path of maximal length $\rho_1 \dots \rho_m$, and say there exists some σ such that we can have $\sigma\rho_1 \dots \rho_m$ but with $t(\sigma) \neq s(\rho_m)$, but since $\rho_1 \dots \rho_m$ was maximal this path is involved $\sigma a = a\sigma$ as $a \in Z(A)$. So, $a\sigma$ is also a linear combination of paths, i.e. $\tau_1 \dots \tau_m \sigma$, wheere $\tau_1 \dots \tau_m$ is a path in a.This causes,

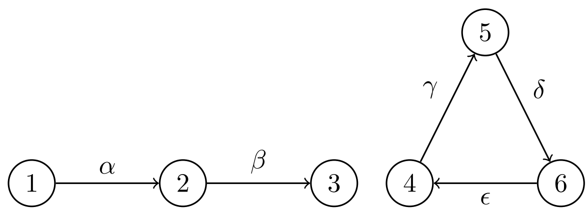
$$\sigma\rho_1 \dots \rho_m = \tau_1 \dots \tau_m \sigma,$$

which implies that $\sigma = \rho_m$, which causes an oriented cycle, hence a contradiction. Thus, $Z(kC_i) \cong k$.
Need to complete the case for oriented cycle connected component.

□

Example 3.1.31. Consider the quiver,

Q :



and so, $Z(kQ) = Z(kC_1) \times Z(kC_2)$, where C_2 is the oriented cycle component. Then,

$$\begin{aligned} Z(kC_1) =& \{a \in kC_1 : ac = ca \forall c \in kC_1\}, \\ &= \{\lambda(e_1 + e_2 + e_3) : \lambda \in k\}, \\ &\cong k, \end{aligned}$$

since none of the elements of kC_1 are commutative, apart from the identity. Also,

$$\begin{aligned} Z(kC_2) &= \{a \in kC_2 : ac = ca \forall c \in kC_2\}, \\ &= \{\lambda(\gamma\delta\epsilon + \delta\epsilon\gamma + \epsilon\gamma\delta) : \lambda \in k\}, \\ &\cong k[T], \text{ where } T = \gamma\delta\epsilon + \delta\epsilon\gamma + \epsilon\gamma\delta, \end{aligned}$$

because none of the elements of kC_2 are commutative other than the identity element. Thus, $Z(kQ) \cong k \times k[T]$.

3.2 Representations of Quivers

Definition 3.2.1. A *representation* X of a quiver Q is given by considering each vertex $i \in Q_0$ as a vector space X_i, and each arrow $\rho \in Q_1$ as a linear map $X_\rho : X_{s(\rho)} \to X_{t(\rho)}$.

Example 3.2.2. Let Q be the quiver,

$$\mathbf{Q}: \quad \textcircled{1} \longleftarrow \textcircled{2} \longrightarrow \textcircled{3}$$

then we can have representations X and Y,

$$\mathbf{X}: \quad k \xleftarrow[1]{\alpha} k \xrightarrow[1]{\beta} k \qquad \& \qquad \mathbf{Y}: \quad k \xleftarrow[1]{\gamma} k \xrightarrow{\delta} 0$$

Definition 3.2.3. A *morphism* $\theta : X \to X'$ between representationsis given by linear maps $\theta_i : X_i \to X'_i$ for each $i \in Q_0$ satisfy $X'_\rho \theta_{s(\rho)} = \theta_{t(\rho)} X_\rho$ for each $\rho \in Q_1$. The *composition* of morphisms, $\theta : X \to X'$ with $\phi : X' \to X''$ is given by $(\phi\theta)_i = \phi_i \theta_i$.

Example 3.2.4.

3.3 Standard Resolution

3.4 Bricks

3.5 Dynkin and Euclidean diagrams

Chapter 4

Auslander-Reiten Quivers

Bibliography

[1] Crawley-Boevey, W. *Cohomology and Central Simple Algebras.* [Online-PDF file]. [Accessed October 2014]. Available from: http://www1.maths.leeds.ac.uk/ pmtwc/cohom.pdf.

[2] Crawley-Boevey, W. *Representation of Quivers.* [Online-PDF file]. [Accessed October 2014]. Available from: http://www1.maths.leeds.ac.uk/ pmtwc/quivlecs.pdf.

[3] Rotman, J. J. 2009. *An Introduction to Homological Algebra* 2nd ed. New York: Springer Science+Business Media.

[4] Weibel, Charles A. 1995. *An Introduction to Homological Algebra* Cambridge University Press

[5] Hilton, P.J. & Stammbach, U. 1972. *A Course in Homological Algebra* 1st ed. Springer.

[6] Schiffler, Ralf. 2014. *Quiver Representations* Springer.

[7] *Splitting lemma - Wikipedia, the free encyclopedia.* [Online] [Accessed 23 April 2015]. Available at: http://en.wikipedia.org/wiki/Splitting_lemma.

[8] *Ext functor - Wikipedia, the free encyclopedia.* [ONLINE] [Accessed 29 April 2015]. Available at: http://en.wikipedia.org/wiki/Ext_functor#Ext_and_extensions.