

第3章 行列式

3.1 行列式的概念

行列式是 n 阶矩阵的一个特征量, 在求解线性方程组和确定矩阵的秩中起着重要的作用.

1. 二阶、三阶行列式

1) 二元线性方程组与二阶行列式

用消元法解二元线性方程组
$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2 \end{cases}.$$

(1) 第1个方程 $\times a_{22}$ - 第2个方程 $\times a_{12}$, 得

$$(a_{11}a_{22} - a_{12}a_{21})x_1 = b_1a_{22} - a_{12}b_2,$$

即

$$x_1 = \frac{b_1a_{22} - a_{12}b_2}{a_{11}a_{22} - a_{12}a_{21}}.$$

(2) 第1个方程 $\times a_{21}$ - 第2个方程 $\times a_{11}$, 得

$$(a_{11}a_{22} - a_{12}a_{21})x_2 = b_2a_{11} - a_{21}b_1,$$

即

$$x_2 = \frac{b_2a_{11} - a_{21}b_1}{a_{11}a_{22} - a_{12}a_{21}}.$$

若定义二阶行列式:

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21},$$

并记

$$D = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad D_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix},$$

则线性方程组的解可表为

$$x_1 = \frac{D_1}{D} = \frac{\begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}, \quad x_2 = \frac{D_2}{D} = \frac{\begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}}{\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}}.$$

例 1.1 求解二元线性方程组 $\begin{cases} 3x_1 - 2x_2 = 12 \\ 2x_1 + x_2 = 1 \end{cases}$.

解 由于

$$D = \begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3 - (-4) = 7 \neq 0,$$

$$D_1 = \begin{vmatrix} 12 & -2 \\ 1 & 1 \end{vmatrix} = 12 - (-2) = 14, \quad D_2 = \begin{vmatrix} 3 & 12 \\ 2 & 1 \end{vmatrix} = 3 - 24 = -21,$$

因此, 得

$$x_1 = \frac{D_1}{D} = \frac{14}{7} = 2, \quad x_2 = \frac{D_2}{D} = \frac{-21}{7} = -3.$$

2) 三元线性方程组与三阶行列式

用消元法解三元线性方程组 $\begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{cases}$.

解得

$$x_1 = \frac{b_1 a_{22} a_{33} + a_{12} a_{23} b_3 + a_{13} b_2 a_{32} - b_1 a_{23} a_{32} - a_{12} b_2 a_{33} - a_{13} a_{22} b_3}{a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31}},$$

$$x_2 = \frac{a_{11} b_2 a_{33} + b_1 a_{23} a_{31} + a_{13} a_{21} b_3 - a_{11} a_{23} b_3 - b_1 a_{21} a_{33} - a_{13} b_2 a_{31}}{a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31}},$$

$$x_3 = \frac{a_{11} a_{22} b_3 + a_{12} b_2 a_{31} + b_1 a_{21} a_{32} - a_{11} b_2 a_{32} - a_{12} a_{21} b_3 - b_1 a_{22} a_{31}}{a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31}}.$$

若定义三阶行列式:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31},$$

并记

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} a_{22} a_{33} + a_{12} a_{23} a_{31} + a_{13} a_{21} a_{32} - a_{11} a_{23} a_{32} - a_{12} a_{21} a_{33} - a_{13} a_{22} a_{31},$$

$$D_1 = \begin{vmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{vmatrix} = b_1 a_{22} a_{33} + a_{12} a_{23} b_3 + a_{13} b_2 a_{32} - b_1 a_{23} a_{32} - a_{12} b_2 a_{33} - a_{13} a_{22} b_3,$$

$$D_2 = \begin{vmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{vmatrix} = a_{11} b_2 a_{33} + b_1 a_{23} a_{31} + a_{13} a_{21} b_3 - a_{11} a_{23} b_3 - b_1 a_{21} a_{33} - a_{13} b_2 a_{31},$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{vmatrix} = a_{11} a_{22} b_3 + a_{12} b_2 a_{31} + b_1 a_{21} a_{32} - a_{11} b_2 a_{32} - a_{12} a_{21} b_3 - b_1 a_{22} a_{31},$$

则线性方程组的解可表为

$$x_1 = \frac{D_1}{D}, \quad x_2 = \frac{D_2}{D}, \quad x_3 = \frac{D_3}{D}.$$

2. n 阶行列式

定义 3. 1 把 n 阶矩阵 A 删去第 i 行第 j 列后得到的 $(n-1)$ 阶子矩阵称为对应于元

素 a_{ij} 的余子矩阵, 记为 S_{ij} .

例如, 矩阵

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{bmatrix}$$

中元素 a_{23} 的余子矩阵为

$$S_{23} = \begin{bmatrix} a_{11} & a_{12} & a_{14} & a_{15} \\ a_{31} & a_{32} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{54} & a_{55} \end{bmatrix}.$$

定义 3. 2 一阶矩阵 $[a_{11}]$ 的行列式定义为数 a_{11} , 并记作

$$\det[a_{11}] = |a_{11}| = a_{11}.$$

对任意一个 n 阶矩阵

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix},$$

其行列式用下列公式定义为

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j=1}^n a_{1j}(-1)^{1+j} \det(S_{1j}),$$

其中 S_{1j} 为 a_{1j} 的余子矩阵. n 阶矩阵的行列式叫做 **n 阶行列式**.

这个定义常被说成“行列式按第一行展开”.

由定义 3. 2, 二阶行列式

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}(-1)^{1+1}a_{22} + a_{12}(-1)^{1+2}a_{21} = a_{11}a_{22} - a_{12}a_{21}.$$

三阶行列式

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}(-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} + a_{12}(-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}(-1)^{1+3} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\ = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

定义 3. 3 表达式 $M_{ij} = \det S_{ij}$ 称为 n 阶矩阵 A 的元素 a_{ij} 的**余子式**; 表达式

$$A_{ij} = (-1)^{i+j} M_{ij} = (-1)^{i+j} \det S_{ij}$$

称为 n 阶矩阵 A 的元素 a_{ij} 的**代数余子式**. 则 n 阶行列式可写为

$$\det(A) = \sum_{j=1}^n a_{1j}(-1)^{1+j} M_{1j} = \sum_{j=1}^n a_{1j} A_{1j}.$$

例 1. 2 计算对角行列式

$$\begin{vmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{vmatrix}.$$

$$\begin{aligned} \text{解} \quad & \begin{vmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{vmatrix} = a_1 \begin{vmatrix} a_2 & & & \\ & a_3 & & \\ & & \ddots & \\ & & & a_n \end{vmatrix} = a_1 a_2 \begin{vmatrix} a_3 & & & \\ & a_4 & & \\ & & \ddots & \\ & & & a_n \end{vmatrix} \\ & = \cdots = a_1 a_2 \cdots a_n. \end{aligned}$$

例 1.3 计算下三角形行列式 $\begin{vmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$.

$$\begin{aligned} \text{解} \quad & \begin{vmatrix} a_{11} & & & \\ a_{21} & a_{22} & & \\ \vdots & \vdots & \ddots & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & & & \\ a_{32} & a_{33} & & \\ \vdots & \vdots & \ddots & \\ a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = a_{11} a_{22} \begin{vmatrix} a_{33} & & & \\ a_{43} & a_{44} & & \\ \vdots & \vdots & \ddots & \\ a_{n3} & a_{n4} & \cdots & a_{nn} \end{vmatrix} \\ & = \cdots = a_{11} a_{22} \cdots a_{nn}. \end{aligned}$$

n 阶行列式的定义, 给出了用递推公式计算 n 阶行列式的方法. 但在实际中, 用这种方法计算三阶以上的行列式, 计算量是非常大的. 因此本章将要探讨行列式的性质, 以得到化简行列式的方法.

定义 3.4 记

$$D = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \quad D^T = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix},$$

行列式 D^T 称为行列式 D 的**转置行列式**.