The existence of a strong solution to the Navier-Stokes equations

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Abstract

In this work, a solution to the sixth Millennium Prize Problem is provided: the existence and uniqueness of a strong solution to the three-dimensional Navier-Stokes problem with periodic spatial boundary conditions.

Key words: sixth Millennium Prize problem, Navier-Stokes equation, strong solution.

1 Introduction

1.1 A short history of the problem

The problem of describing the dynamics of an incompressible fluid, due to its significance for both theory and application, has attracted the attention of many researchers. During the middle part of 2000, this problem was formulated as the *sixth Millennium Prize problem*, about the existence and smoothness of a solution to the Navier-Stokes equations for an incompressible viscous fluid [1].

The solution to this problem was the subject of many works even before its announcement as a Millennium Prize problem. As the number of these works is too large, I will not include a list. Deep results, in my opinion, were arrived at in the works of O. A. Ladyzhenskaya [2–5] and R. Temam [6, 7]. This problem has been of interest to many first-class mathematicians, who have been able to solve important mathematical problems, including problems in hydrodynamics. Substantial results have been arrived at in the works of such great mathematicians a A. N. Kolmogorov [8], J. Leray [9, 10], E. Hopf [11], J.-L. Lions [12, 13], M. I. Vishik [14], V. A. Solonnikov [15], and many others. Of course, this list is far from complete. It is not my goal to provide a full

*Ministry of Education and Science, Republic of Kazakhstan overview of existing works. With the exception of those publications, whose results I have directly used in the writing of this work (cf. [16–46]). My own works are included in this list, sometimes with coauthors, and so are the works of many other Kazakh mathematicians, whose positive influence I have constantly benefited from.

A full solution to the two-dimensional problem is provided by O. A. Ladyzhenskaya in [2]. In [5], she provides a sufficiently full analysis of the current state of the problem and an overview of existing literature and proposed solution methods. In part, the main problem of a global unique solution to the three-dimensional Navier-Stokes is cast as the problem of finding a special *a priori* bound for all possible solutions.

One must also keep in mind the existence of a large number of works containing mistakes or lacking proofs that have been published in little-known journals or merited electronic publication. Despite their mistakes, these works deserve respect.

I have been studying this problem since 1980. All of my works, written either by myself or with coauthors, published after 1982 and devoted to nonlinear equations, approximation techniques for solving equations and the final inverse problem, used ideas and technical approaches developed during my (and also some of my students' and coauthors') unsuccessful attempts to solve the problem of the strong solvability of the Navier-Stokes equation (cf. for example, [16–37]).

I would like to take this opportunity to express my gratitude to Professor M. Sadybekov for his careful reading of the work. He also reworked Section 4, which I had briefly outlined, and completely wrote Section 8. His comments proved very useful in the formulation of the final state of this article.

I dedicate this work to the family of my dear teachers: the mathematics professors T. I. Amanov, M. G. Gasymov, A. G. Kostuchenko, B. M. Levitan, P. I. Lizorkin, and also my school teachers I. Adykeev and A. M. Panivanov.

1.2 Statement of the problem

Let $Q \subset R^3$ be a domain in three dimensions, $\Omega = (0,a) \times Q, a > 0$. In this work we examine the case when Q is a three-dimensional cube centered at the origin and edges of length 2π, parallel to the coordinate axes.

The *Navier-Stokes problem* consists of finding the unknowns:

the velocity vector $u(t,x) = (u_1(t,x), u_2(t,x), u_3(t,x))$,
and the pressure—a scalar function $p(t,x)$

for points $x \in Q$ at time $t \in (0,a)$ that satisfy the system of equations

$$\begin{cases} \dfrac{\partial u_j}{\partial t} + \displaystyle\sum_{k=1}^{3} u_k \dfrac{\partial u_j}{\partial x_k} = \Delta u_j - \dfrac{\partial p}{\partial x_j} + f_j, \quad (t,x) \in \Omega, \qquad j = 1,2,3; \\ \operatorname{div} u \equiv \displaystyle\sum_{k=1}^{3} = 0, \qquad (t,x) \in \Omega. \end{cases} \tag{1.1}$$

Here, $f(t,x) = (f_1(t,x), f_2(t,x), f_3(t,x))$ is an external force, and $\Delta = \sum_{k=1}^{3} \frac{\partial^2}{\partial x^2}$ is the Laplacian for spatial dimensions, and the coefficient of viscosity, ν, is taken to be 1, without loss of generality.

To the system of equations in (1.1), we add initial and final time boundary conditions (along the spatial dimensions, we invoke periodic boundary conditions). Without loss of generality, we can take the initial conditions to be zero:

$$u(t,x)|_{t=0}, \quad x \in \bar{Q}; \tag{1.2}$$

$$\begin{aligned} u(t,x)|_{x_k=-\pi} &= u(t,x)|_{x_k=\pi}, \\ p(t,x)|_{x_k=-\pi} &= p(t,x)|_{x_k=\pi}, \qquad\qquad k = 1,2,3; \quad 0 \le t \le a. \\ \left.\frac{\partial u}{\partial x_k}\right|_{x_k=-\pi} &= \left.\frac{\partial u}{\partial x_k}\right|_{x_k=\pi}, \end{aligned} \tag{1.3}$$

The system of equations (1.1) and initial/boundary constraints (1.2), (1.3) do not allow a unique solution to the pressure $p(t,x)$. For this reason, we add the constraint

$$\int_Q p(t,x)\,dx = p_0, \;\; p_0 = \text{const} > 0. \tag{1.4}$$

In the problem, the sought-after quantities are the vector velocity function $u = (u_1, u_2, u_3)$ and the scalar pressure function p. We will denote *solution to the problem* as the pair $(u;p) = (u_1, u_2, u_3; p)$.

We could have examined the case where, instead of Q, a more general domain is used, and instead the period boundary conditions in (1.3), other conditions are used (for example, "sticky" boundary conditions). We do not wish to complicate this work with such extensions only for technical reasons. All of our basic innovations and analytical technique will be demonstrated in this work for the case of periodic boundary conditions.

Furthermore, I intend to write at least one other work devoted to the case of general initial/boundary problems for a system of hydrodynamic equations.

1.3 Essential symbols and definitions

With $L_2(\Omega)$, as usual, we denote the Hilbert space of Lebesgue vector functions $f(t,x) = (f_1(t,x), f_2(t,x), f_3(t,x)) \in R^3$ with the scalar product

$$(f,g) = \int_\Omega \langle f(t,x), g(t,x)\rangle\, dxdt \equiv \int_0^a \left(\int_Q \langle f(t,x), g(t,x)\rangle\, dx \right) dt$$

and the norm $\|f\| = \sqrt{(f,f)}$. Here and throughout, $\langle f, g\rangle$ is the scalar product of the vectors f and g in the Euclidian space R^3.

For the sake of the conciseness of our notation, we use the standard symbols:

$$\nabla = \left(\frac{\partial}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3} \right); \ \langle u, \nabla u \rangle = \left(\sum_{j=1}^{3} u_j \frac{\partial u_1}{\partial x_j}, \sum_{j=1}^{3} u_j \frac{\partial u_2}{\partial x_j}, \sum_{j=1}^{3} u_j \frac{\partial u_3}{\partial x_j}, \right);$$

$$\operatorname{grad} p \equiv \nabla p = \left(\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}, \frac{\partial p}{\partial x_3} \right).$$

Definition 1 *Given $f \in L_2(\Omega)$, we will call the solution $(u; p) = (u_1, u_2, u_3; p)$ of the problem* (1.1) – (1.4) ***strong*** *if*

$$\frac{\partial u}{\partial t}, \Delta u, \langle u, \nabla \rangle u, \operatorname{grad} p \in L_2(\Omega).$$

Definition 2 *We will say that the Navier-Stokes problem is* ***strongly solvable*** *in $L_2(\Omega)$, if for any $f \in L_2(\Omega)$, the problem* (1.1) – (1.4) *has a single strong solution.*

2 Statement of the main result

E. Hopf's classic result [11], which proved that the problem (1.1) – (1.4) has a generalized solution satisfying the bound

$$\sum_{k=1}^{3} \left(\|u_k(t, \cdot)\|^2_{L_2(Q)} + \int_0^t \|\operatorname{grad} u_k(\eta, \cdot)\|^2_{L_2(Q)} \, d\eta \right) \leq \sum_{k=1}^{3} \int_0^t \|f_k(\eta, \cdot)\|^2_{L_2(Q)} \, d\eta \tag{2.1}$$

is well known.

This estimate is arrived at easily if the k-th equation of the system is multiplied by u_k, and, nothing the equality $\operatorname{div} u = 0$, as well as the initial (1.2) and boundary (1.3) conditions, the equations are all integrated and summed for all $k = 1, 2, 3$.

In the case when the number of spatial variables is not less than three, the bound (2.1) is insufficient for perturbation theory. In my option, this fact in particular is one of the most important reasons why the problem of the strong solvability of the Navier-Stokes equations is a Millenium Prize problem.

The main result of this paper is

Theorem 1 *Given any $f \in L_2(\Omega)$, the problem* (1.1) – (1.4) *has a unique strong solution $(u; p)$ and this solution satisfies the bound*

$$\left\| \frac{\partial u}{\partial t} \right\| + \|\Delta u\| + \|(u, \nabla)u\| + \|\operatorname{grad} p\| \leq C \left(1 + \|f\| + \|f\|^l \right) \tag{2.2}$$

where $\|\cdot\|$ is the norm on $L_2(\Omega)$, and the constants $C > 0$ and $l \geq 1$ do not depend on $f \in L_2(\Omega)$.

This theorem presents a complete solution to the *sixth Millenium Prize problem* about the existence and smoothness of solutions to the Navier-Stokes equations for an incompressible viscous fluid [1]. This result also enables the use of perturbation theory, and improving the smoothness of a solution with increasing smoothness in the problem data.

We note that if the spatial dimension is greater than five, the strong solvability (in the sense of our definitions) would prevent the use of perturbation theory.

In this work I have concerned myself only with the three-dimensional case. Therefore, I have chosen definitions convenient for me. The case when the spatial dimension is greater than 3 will be examined in another work, which will also include certain cases of general boundary conditions.

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