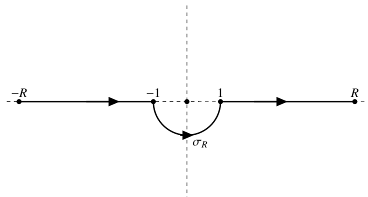

Figure 1: Path σ_R

1 More real integrals

1.1 Indenting

Consider the case where the integrand may have **poles** on the real axis. The plan is to skip around them and then to proceed using Jordan's Lemma or the Standard Integral Bound.

Evaluate

$$I = \int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx = \lim_{R\to\infty} I_R, \quad I_R = \int_{-R}^{R} \frac{\sin x}{x}\,dx,$$

where $R > 0$.

Singularity of the function

$$f(z) = \frac{\sin z}{z}$$

at $z = 0$ is removable, so I_R is well-defined.

However we cannot replace I_R by

$$\operatorname{Im} \int_{-R}^{R} \frac{e^{ix}}{x}\,dx,$$

since the integrand has a singularity at zero.

To circumvent this obstacle observe that the integral along $[-R, R]$ coincides with the integral along the new path σ_R, see Fig. 1.

Now we rewrite

$$\frac{\sin z}{z} = \frac{1}{2i}\frac{e^{iz}}{z} - \frac{1}{2i}\frac{e^{-iz}}{z} = \frac{1}{2i}(f_1(z) - f_2(z)),$$

and consider separately the integrals

$$\int_{\sigma_R} f_1(z)\,dz \quad \text{and} \quad \int_{\sigma_R} f_2(z)\,dz.$$

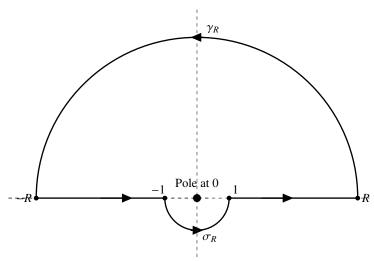

Figure 2: Contour Γ_R

Begin with f_1. As usual, close the contour with a semi-circular part in the upper half-plane:

$$\Gamma_R = \gamma_R \cup \sigma_R,$$

see Fig. 2.

By the Cauchy Residue Theorem,

$$\int_{\Gamma_R} \frac{e^{iz}}{z}\,dz = 2\pi i \operatorname{Res}(f_1, 0) = 2\pi i \lim_{z\to 0} z\,\frac{e^{iz}}{z} = 2\pi i.$$

Estimate the integral along γ_R using Jordan's Lemma. Since

$$\max_{z\in\gamma_R} \frac{1}{|z|} = \frac{1}{R} \to 0,\ R \to \infty,$$

we conclude that

$$\int_{\gamma_R} \frac{e^{iz}}{z}\,dz \to 0, \quad R \to \infty.$$

Conclusion:

$$\lim_{R\to\infty} \int_{\sigma_R} \frac{e^{iz}}{z}\,dz = \lim_{R\to\infty} \int_{\Gamma_R} \frac{e^{iz}}{z}\,dz = 2\pi i.$$

Note that the SIB would not be applicable. Indeed, the SIB gives

$$\left|\int_{\gamma_R} \frac{e^{iz}}{z}\,dz\right| \le \pi R \max_{z\in\gamma_R} \frac{|e^{iz}|}{|z|} \le \pi,$$

which is not enough.

Let us find

$$\int_{\sigma_R} \frac{e^{-iz}}{z}\,dz.$$

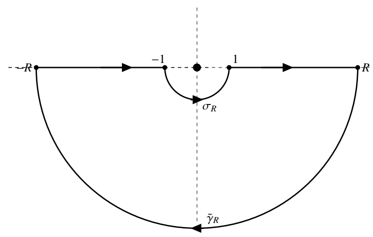

Figure 3: Contour $\tilde{\Gamma}_R$

Now we close the contour in the lower half-plane:

$$\tilde{\Gamma}_R = \sigma_R \cup \tilde{\gamma}_R,$$

see Fig. 3. We take $\tilde{\gamma}_R$ in the lower half-plane, since the exponential e^{-iz} contains $-i$.

Using Jordan's Lemma, as before, we get

$$\int_{\tilde{\gamma}_R} \frac{e^{-iz}}{z}\, dz \to 0, \quad R \to \infty,$$

and hence

$$\lim_{R\to\infty} \int_{\sigma_R} \frac{e^{-iz}}{z}\, dz = \lim_{R\to\infty} \int_{\sigma_R \cup \tilde{\gamma}_R} \frac{e^{-iz}}{z}\, dz = 0.$$

Put all formulae together:

$$\int_{-\infty}^{\infty} \frac{\sin x}{x}\, dx = \frac{1}{2i} \lim_{R\to\infty} \Big[\int_{\sigma_R} \frac{e^{iz}}{z}\, dz + \int_{\sigma_R} \frac{e^{-iz}}{z}\, dz \Big] = \frac{2\pi i}{2i} = \pi.$$

1.2 Branch Cuts 1

Some functions such as $\log(z)$ and z^α are not defined on the whole plane. So when constructing a contour for such functions we must stay within the domain of definition. That is we must avoid at all costs the cut where the function is not defined.

Assume that $\alpha \in (0, 2)$ and evaluate

$$I = \int_0^\infty \frac{x^{\alpha-1}}{1+x^2}\, dx = \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} I_{\varepsilon,R}, \quad I_{\varepsilon,R} = \int_\varepsilon^R \frac{x^{\alpha-1}}{1+x^2}\, dx.$$

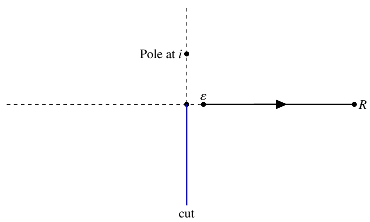

Figure 4: Path for $I_{\varepsilon,R}$

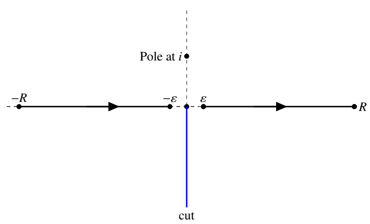

Figure 5: Two symmetric intervals

where $R > \varepsilon > 0$. Let

$$f(z) = \frac{z^{\alpha-1}}{1+z^2},$$

where $z^{\alpha-1} = \exp((\alpha-1)\log z)$, $\arg z \in [-\pi/2, 3\pi/2)$, i.e. the branch cut runs along the negative imaginary semi-axis, see Fig. 4 for the path and the cut.

Compare with the integral along $[-R, -\varepsilon]$, see Fig 5. Remembering that $\log(-t) = \ln t + i\pi$ for all $t > 0$, we get

$$(-t)^{\alpha-1} = e^{(\alpha-1)\log(-t)} = e^{(\alpha-1)\ln t + i(\alpha-1)\pi} = t^{\alpha-1}e^{i(\alpha-1)\pi}.$$

Therefore

$$\int_{-R}^{-\varepsilon} f(z)dz = \int_{\varepsilon}^{R} \frac{(-t)^{\alpha-1}}{1+t^2}dt = e^{i(\alpha-1)\pi}\int_{\varepsilon}^{R} \frac{t^{\alpha-1}}{1+t^2}dt = e^{i(\alpha-1)\pi}I_{\varepsilon,R}.$$

Build a new contour:

$$\Gamma_{\varepsilon,R} = [\varepsilon, R] \cup \varkappa_R \cup [-R, -\varepsilon] \cup \gamma_\varepsilon,$$

see Fig. 6.

Denote

$$J_{\varepsilon,R} = \int\limits_{\Gamma_{\varepsilon,R}} f(z)dz.$$

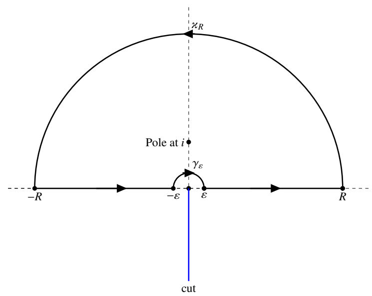

Figure 6: Contour $\Gamma_{\varepsilon,R}$

By the Cauchy Residue Theorem,

$$J_{\varepsilon,R} = 2\pi i \operatorname{Res}(f,i) = 2\pi i \lim_{z\to i}(z-i)\frac{z^{\alpha-1}}{(z+i)(z-i)} = 2\pi i \frac{i^{\alpha-1}}{2i} = \pi\, e^{i(\alpha-1)\frac{\pi}{2}}.$$

Furthermore,

$$\int_{-R}^{-\varepsilon} + \int_{\varepsilon}^{R} = (1+e^{i(\alpha-1)\pi}) I_{\varepsilon,R} \to (1+e^{i(\alpha-1)\pi}) \int_0^\infty f(t)dt,$$

as $\varepsilon \to 0, R \to \infty$.

Estimate the integrals along $\varkappa_R$ and γ_ε, using the SIB:

$$\left|\int\limits_{\varkappa_R} \frac{z^{\alpha-1}}{1+z^2}\,dz\right| \le \pi R\,\frac{R^{\alpha-1}}{R^2-1} = \pi\,\frac{R^\alpha}{R^2-1} \to 0,$$

as $R \to \infty$, since $\alpha < 2$.

Now,

$$\left|\int\limits_{\gamma_\varepsilon} \frac{z^{\alpha-1}}{1+z^2}\,dz\right| \le \pi\varepsilon\,\frac{\varepsilon^{\alpha-1}}{1-\varepsilon^2} = \pi\frac{\varepsilon^\alpha}{1-\varepsilon^2} \to 0,$$

as $\varepsilon \to 0$, since $\alpha > 0$.

Put all components together:

$$\lim_{\substack{\varepsilon\to 0\\ R\to\infty}} J_{\varepsilon,R} = (1+e^{i(\alpha-1)\pi}) \int_0^\infty f(t)dt = \pi e^{i(\alpha-1)\frac{\pi}{2}}.$$

Consequently,

$$\begin{aligned} I &= \frac{\pi e^{i(\alpha-1)\frac{\pi}{2}}}{1+e^{i(\alpha-1)\pi}} = \frac{\pi}{2}\,\frac{2}{e^{-i(\alpha-1)\frac{\pi}{2}}+e^{i(\alpha-1)\frac{\pi}{2}}} \\ &= \frac{\pi}{2}\,\frac{1}{\cos\left((\alpha-1)\frac{\pi}{2}\right)} = \frac{\pi}{2\sin(\alpha\frac{\pi}{2})}. \end{aligned}$$

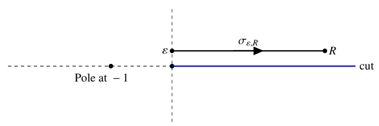

Figure 7: Path $\sigma_{\varepsilon,R}$ and the branch cut

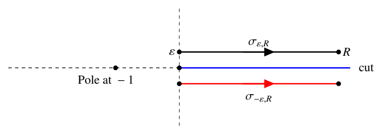

Figure 8: Paths σ_R and $\sigma_{-\varepsilon,R}$

1.3 Branch cuts 2

Assume that $\alpha \in (0,1)$ and evaluate

$$I = \int_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx = \lim_{R\to\infty} I_R, \quad I_R = \int_0^R \frac{x^{\alpha-1}}{1+x}\,dx.$$

Let $f(z) = \frac{z^{\alpha-1}}{1+z}$, where $z^{\alpha-1} = \exp((\alpha-1)\log z)$, $\arg z \in [0, 2\pi)$, i.e. the branch cut runs along the positive real semi-axis, see Fig. 7.

Compare I_R with the integral along the new path σ_R, see Fig 7:

$$I_{\varepsilon,R} = \int_{\sigma_{\varepsilon,R}} f(z)\,dz \to \int_0^R f(z)\,dz, \quad \varepsilon \to 0.$$

Indeed, parametrise: $\sigma_{\varepsilon,R}(t) = t + i\varepsilon, t \in [0,R]$, and write:

$$(t+i\varepsilon)^{\alpha-1} = e^{(\alpha-1)\ln|t+i\varepsilon|+i(\alpha-1)\arg(t+i\varepsilon)} \to e^{(\alpha-1)\ln t} = t^{\alpha-1}, \varepsilon \to 0,$$

and $1 + (t+i\varepsilon) \to 1+t$, $\varepsilon \to 0$.

Now compare with the integral along $\sigma_{-\varepsilon,R}$, see Fig 8. To this end parametrise: $\sigma_{-\varepsilon,R}(t) = t - i\varepsilon, t \in [0,R]$, and remember that $\arg(t-i\varepsilon) = 2\pi - \arg(t+i\varepsilon)$. Therefore,

$$\begin{aligned}\log(t-i\varepsilon) &= \ln|t-i\varepsilon| + i\arg(t-i\varepsilon)\\ &= \ln|t+i\varepsilon| + 2\pi i - i\arg(t+i\varepsilon).\end{aligned}$$

Thus

$$\begin{aligned}(t-i\varepsilon)^{\alpha-1} &= e^{(\alpha-1)\log(t-i\varepsilon)}\\ &= e^{(\alpha-1)2\pi i}|t+i\varepsilon|^{\alpha-1}e^{-i(\alpha-1)\arg(t+i\varepsilon)} \to e^{(\alpha-1)2\pi i}t^{\alpha-1}.\end{aligned}$$

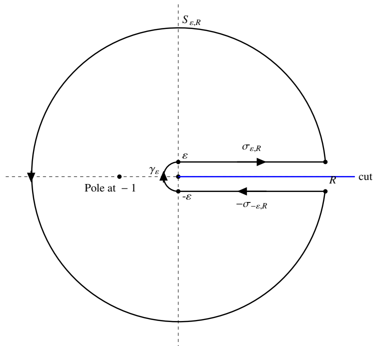

Figure 9: Countour $\Gamma_{\varepsilon,R}$

Thus

$$\int_{\sigma_{-\varepsilon,R}} f(z)\,dz \to e^{(\alpha-1)2\pi i}\int_0^R f(t)dt,\ \ \varepsilon \to 0.$$

Complete these two paths to a contour:

$$\Gamma_{\varepsilon,R} = \sigma_{\varepsilon,R} \cup S_{\varepsilon,R} \cup (-\sigma_{-\varepsilon,R}) \cup \gamma_\varepsilon,$$

see Fig.9.

Denote

$$J_{\varepsilon,R} = \int\limits_{\Gamma_{\varepsilon,R}} f(z)dz.$$

By the Cauchy Residue Theorem,

$$J_{\varepsilon,R} = 2\pi i\,\mathrm{Res}(f,-1) = 2\pi i \lim_{z\to -1}(z+1)\frac{z^{\alpha-1}}{z+1}$$
$$= 2\pi i\,(-1)^{\alpha-1} = 2\pi i\, e^{i(\alpha-1)\pi}.$$

Furthermore,

$$\int\limits_{\sigma_{\varepsilon,R}} + \int\limits_{-\sigma_{-\varepsilon,R}} = \int\limits_{\sigma_{\varepsilon,R}} - \int\limits_{\sigma_{-\varepsilon,R}} \to (1-e^{(\alpha-1)2\pi i})\int_0^R f(t)dt,\ \varepsilon \to 0.$$

Estimate the integrals along $S_{\varepsilon,R}$ and γ_ε, using the SIB:

$$\Bigg|\int\limits_{S_{\varepsilon,R}} \frac{z^{\alpha-1}}{1+z}\,dz\Bigg| \le 2\pi R\,\frac{R^{\alpha-1}}{R-1} = 2\pi\,\frac{R^\alpha}{R-1} \to 0,$$

as $R \to \infty$, since $\alpha < 1$.

Now,

$$\left| \int\limits_{\gamma_\varepsilon} \frac{z^{\alpha-1}}{1+z}\, dz \right| \le \pi\varepsilon \frac{\varepsilon^{\alpha-1}}{1-\varepsilon} = \pi \frac{\varepsilon^{\alpha}}{1-\varepsilon} \to 0,$$

as $\varepsilon \to 0$, since $\alpha > 0$.

Put all components together:

$$\lim_{R\to\infty} \lim_{\varepsilon\to 0} J_{\varepsilon,R} = (1 - e^{2i(\alpha-1)\pi}) \lim_{R\to\infty} \int_0^R f(t)dt = 2\pi i e^{i(\alpha-1)\pi}.$$

Consequently,

$$\begin{aligned} I &= \frac{2\pi i e^{i(\alpha-1)\pi}}{1 - e^{2i(\alpha-1)\pi}} = \pi \frac{2i}{e^{-i(\alpha-1)\pi} - e^{i(\alpha-1)\pi}} \\ &= \frac{\pi}{\sin\big(-(\alpha-1)\pi\big)} = \frac{\pi}{\sin(\pi - \alpha\pi)} = \frac{\pi}{\sin(\alpha\pi)}. \end{aligned}$$