

MATH0013 Complex Analysis 2022/23
Main exam: questions and solutions

Duration: 2 hours

1. (a) What does it mean for a function f defined on a domain $\Omega \subset \mathbb{C}$ to be conformal on Ω ?
- (b) Consider the linear fractional transformation

$$f(z) = \frac{az + b}{cz + d}, \quad z \in \mathbb{C}, cz + d \neq 0,$$

with some complex coefficients a, b, c, d . We say that this transformation is non-degenerate if the coefficients satisfy the condition

$$ad - bc \neq 0. \tag{1}$$

- i. Show that a non-degenerate f is conformal for all z such that $cz + d \neq 0$.
- ii. Prove that the composition of two non-degenerate linear fractional transformations is again a non-degenerate linear fractional transformation.

Solution.

- (a) f is conformal on Ω if f is holomorphic on Ω and $f'(z) \neq 0$ for all $z \in \Omega$.
- (b) i. Since f is a rational function it is holomorphic away from the roots of the denominator, i.e. for $cz + d \neq 0$. Furthermore, a direct calculation shows that

$$f'(z) = \frac{a(cz + d) - c(az + b)}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2},$$

and hence, in view of (1), the right hand side is non-zero for all $cz + d \neq 0$. This implies the required conformality.

- ii. Let

$$f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}, \quad f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$$

with $a_1d_1 - b_1c_1 \neq 0, a_2d_2 - b_2c_2 \neq 0$. Calculate the composition

$$f_1(f_2(z)) = \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} = \frac{(a_1 a_2 + c_2 b_1)z + a_1 b_2 + d_2 b_1}{(c_1 a_2 + c_2 d_1)z + c_1 b_2 + d_1 d_2}.$$

Observe that the coefficients can be obtained by multiplying two matrices:

$$A_1 A_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}.$$

The condition (1) is also satisfied for this new transformation, since the determinant of the product of two matrices equals the product of determinants:

$$\det(A_1 A_2) = \det A_1 \det A_2 \neq 0,$$

as required.

2. (a) Let f be holomorphic in the punctured disk $D'(z_0, r) = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$. Classify the possible singularity types of the function f at the point z_0 .
- (b) Locate and classify the singularities of the following functions:

$$\frac{1 - \cos z}{z^2}, \quad z^{2022} \sin \frac{1}{z}.$$

Find the residues at every isolated singularity. Explain your answer.

Solution.

- (a) The type of singularity depends on the Laurent expansion in the neighbourhood $D'(z_0, r)$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n.$$

The part of this series containing only negative powers of $(z - z_0)$ is called the principle part:

$$\sum_{n=-\infty}^{-1} a_n(z - z_0)^n.$$

If the principle part equals zero, i.e. $a_n = 0$ for all $n < 0$, then the singularity is said to be removable.

If the principle part contains finitely many terms, then the function is said to have a pole at z_0 . The number k , such that $a_{-k} \neq 0$ and $a_{-n} = 0$, for all $n > k$, is said to be the order of the pole.

If the principal part contains infinitely many terms then the singularity is said to be essential.

- (b) The first function is holomorphic except at $z = 0$. Expand around zero:

$$\begin{aligned} \frac{1 - \cos z}{z^2} &= \frac{1}{z^2} \left(1 - 1 + \frac{z^2}{2} - \frac{z^4}{4!} + \dots \right) \\ &= \frac{1}{2} - \frac{z^2}{4!} + \dots \end{aligned}$$

Therefore the principal part of the expansion vanishes, and hence the singularity is removable. As a result, the residue equals zero.

The second function is holomorphic except at $z = 0$. Expand around zero:

$$\begin{aligned} z^{2022} \sin \frac{1}{z} &= z^{2022} \sum_{k=0}^{\infty} (-1)^k \frac{z^{-(2k+1)}}{(2k+1)!} \\ &= \sum_{k=0}^{1010} (-1)^k \frac{z^{-(2k+1)+2022}}{(2k+1)!} - \frac{1}{2023!} \frac{1}{z} + \sum_{k=1012}^{\infty} (-1)^k \frac{z^{-(2k+1)+2022}}{(2k+1)!}. \end{aligned}$$

The last sum is the principal part of the function and it contains infinitely many terms. Thus the singularity at zero is essential. By definition, the residue equals $-\frac{1}{2023!}$.

3. (a) Let f be holomorphic on the disk $D(a, r)$, and let $Z \subset D(a, r)$ be the set of its zeroes. Suppose that the point a is an accumulation point of Z , i.e. for every $\varepsilon > 0$ the punctured disk $D'(a, \varepsilon)$ contains at least one zero of f . Prove that $f(z) = 0$ for all $z \in D(a, r)$.
- (b) Suppose that the radius of convergence of the series $\sum_{n=0}^{\infty} a_n z^n$ equals 1.
- Show that for any $N \in \mathbb{N}$ the series $\sum_{n=0}^{\infty} a_n N^{-n} z^n$ converges absolutely on the disk $D(0, N)$,
 - Prove that the function $\sum_{n=0}^{\infty} \frac{a_n}{n^n} z^n$ is entire.

Solution.

- (a) By Taylor Theorem, the function f can be represented as a series

$$f(z) = \sum_{n \geq 0} a_n (z - a)^n$$

which converges absolutely for all $z \in D(a, r)$. Assume that $f(z)$ is not identically zero in $D(a, r)$. Then there is at least one coefficient $a_m \neq 0$. Let $a_m, m \geq 1$ be the first non-zero coefficient. Thus

$$\begin{aligned} f(z) &= (z - a)^m \sum_{n \geq m} a_n (z - a)^{n-m} = (z - a)^m g(z), \\ g(z) &= a_m + a_{m+1} z + \dots \end{aligned}$$

The series defining $g(z)$ has the same radius of convergence as the series for f , and hence it defines a holomorphic function on $D(a, r)$, such that $g(a) = a_m \neq 0$.

Alternatively, one can invoke the theorem proved in the lectures saying that $f(z) = (z - a)^m g(z)$ with a holomorphic g , such that $g(a) \neq 0$.

Since g is continuous, there is a disk $D(a, \epsilon)$ such that $g(z) \neq 0$ for all $z \in D(a, \epsilon)$.

Thus the function f has no zeroes in $D(a, \epsilon)$ except for a . This means that Z cannot have a as an accumulation point, which is a contradiction. Thus $f(z) = 0$ for all $z \in D(a, r)$, as claimed.

- (b) Since the radius of convergence of the initial series equals 1, the new series

$$g_N(z) = \sum_{n=0}^{\infty} a_n \left(\frac{z}{N}\right)^n = \sum_{n=0}^{\infty} \frac{a_n}{N^n} z^n \tag{2}$$

converges absolutely for all $|zN^{-1}| < 1$, i.e. on the disk $D(0, N)$.

In order to prove that the function

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n^n} z^n, \quad (3)$$

is entire, one needs to show that the radius of convergence is infinite. In other words, it suffices to show that (3) converges in the disk $D(0, N)$ for all $N > 0$. Now we use the Comparison Principle observing that

$$\left| \frac{a_n}{n^n} \right| \leq \frac{|a_n|}{N^n}$$

for all $n \geq N$. Since g_N converges in the disk $D(0, N)$, so does the series (3), as required.

4. (a) As usual, we denote $z = x + iy$. Let $f(z) = u(x, y) + iv(x, y)$ be an entire function. Suppose that $u(x, y) \leq x$ for all $z \in \mathbb{C}$. Show that $f(z) = z + c$ with a complex constant c . Justify your answer.
- (b) Using the Cauchy Residue Theorem prove that

$$\int_0^{2\pi} \frac{\sin^2 \theta}{5 + 4 \cos \theta} d\theta = \frac{\pi}{4}.$$

Solution.

- (a) Let $g(z) = e^{f(z)-z}$. Then $|g(z)| = e^{u(x,y)-x}$. Since $u(x, y) \leq x$, we have $|g(z)| \leq 1$ for all z . And since $g(z)$ is entire, $g(z)$ must be constant by Louville's theorem.

Therefore, $g'(z) = 0$. That is, $(f'(z) - 1)e^{f(z)-z} = 0$ and hence $f'(z) = 1$ for all z . So $f(z) = z + c$ for some complex constant c .

- (b) Use the substitution $z = e^{i\theta}$, so that

$$\sin \theta = \frac{1}{2i}(z - z^{-1}), \quad \cos \theta = \frac{1}{2}(z + z^{-1}), \quad d\theta = -iz^{-1}dz.$$

Thus the integral equals

$$\begin{aligned} I &= \frac{i}{4} \int_{|z|=1} \frac{(z - z^{-1})^2}{5 + 2(z + z^{-1})} \frac{dz}{z} \\ &= \frac{i}{4} \int_{|z|=1} \frac{(z^2 - 1)^2}{z^2(2z^2 + 5z + 2)} dz = \frac{i}{4} \int_{|z|=1} \frac{(z^2 - 1)^2}{z^2(2z + 1)(z + 2)} dz. \end{aligned}$$

The denominator has three roots: $0, -1/2, -2$. Only two of them, $-1/2$ and 0 are inside the contour. The root $-1/2$ is a simple pole, and the root 0 is a double pole of the function

$$f(z) = \frac{i}{4} \frac{(z^2 - 1)^2}{z^2(2z + 1)(z + 2)}.$$

Find the residues:

$$\begin{aligned} \text{Res}(f, -1/2) &= \lim_{z \rightarrow -1/2} (z + 1/2)f(z) \\ &= \frac{i}{4} \lim_{z \rightarrow -1/2} \frac{(z^2 - 1)^2}{2z^2(z + 2)} = \frac{i}{4} \frac{(1/4 - 1)^2}{2/4 \cdot 3/2} = \frac{i}{4} \cdot \frac{3}{4}, \end{aligned}$$

and

$$\begin{aligned}\text{Res}(f, 0) &= \frac{i}{4} \lim_{z \rightarrow -1/2} \frac{d}{dz} (z^2 f(z)) \\ &= \frac{i}{4} \lim_{z \rightarrow 0} \frac{d}{dz} \frac{(z^2 - 1)^2}{2z^2 + 5z + 2} \\ &= \frac{i}{4} \lim_{z \rightarrow 0} \frac{2(z^2 - 1) \cdot 2z(2z^2 + 5z + 2) - (z^2 - 1)^2(4z + 5)}{(2z^2 + 5z + 2)^2} = -\frac{i}{4} \cdot \frac{5}{4}.\end{aligned}$$

By the Cauchy Residue Theorem,

$$\begin{aligned}I &= 2\pi i \text{Res}(f, -1/2) + 2\pi i \text{Res}(f, 0) \\ &= 2\pi i \frac{i}{4} \left(\frac{3}{4} - \frac{5}{4} \right) = -\frac{\pi}{2} \cdot \left(-\frac{1}{2} \right) = \frac{\pi}{4},\end{aligned}$$

as required.

5. Let f be a function holomorphic on the domain $\Lambda = \{z : |z| > 1\}$. Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad z \in \Lambda,$$

be its Laurent expansion. Assuming that f is a bounded function, prove that $a_n = 0$ for all $n > 0$.

Solution. Assume that $|f(z)| \leq M$, for all $|z| > 1$. Recall the formula for the Laurent coefficients:

$$a_n = \frac{1}{2\pi i} \int_{S(0,R)} \frac{f(z)}{z^{n+1}} dz, \quad n \in \mathbb{Z},$$

where $R > 1$ is arbitrary.

Estimate a_n using the Standard Integral Bound:

$$|a_n| \leq \frac{1}{2\pi} 2\pi R \cdot \frac{\max_{|z|=R} |f(z)|}{R^{n+1}} \leq MR^{-n}.$$

If $n > 0$, then the right-hand side tends to zero as $R \rightarrow \infty$. Therefore $a_n = 0$, as claimed.