Sheet 2, solutions

1. Verify the Cauchy-Riemann equations for

$$f(z) = e^{-y}(x\cos x - y\sin x) + ie^{-y}(y\cos x + x\sin x).$$

Find $f(z)$ as a function of z, calculate $f'(z)$.

Solution. For the function $f(z)$ we have $u = e^{-y}(x\cos x - y\sin x)$ and $v = e^{-y}(y\cos x + x\sin x)$. Therefore,

$$\frac{\partial u}{\partial x} = e^{-y}(\cos x - x\sin x - y\cos x) = \frac{\partial v}{\partial y},$$

$$\frac{\partial u}{\partial y} = -e^{-y}(x\cos x - y\sin x + \sin x), \quad \text{while} \quad \frac{\partial v}{\partial x} = e^{-y}(-y\sin x + \sin x + x\cos x)$$

One checks directly that $f(z) = ze^{iz}$.

Since $f'(z) = u_x + iv_x$, we get that

$$f'(z) = e^{-y}(\cos x - x\sin x - y\cos x) + ie^{-y}(-y\sin x + \sin x + x\cos x).$$

Alternatively, $f'(z) = e^{iz} + ize^{iz}$.

2. Apply the definition of derivatives to show that $f(z) = z^2 + \bar{z}$ is not differentiable at z_0 for any $z_0 \in \mathbb{C}$.

Solution.

We have

$$\lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z\to z_0} \frac{z^2 + \bar{z} - (z_0^2 + \bar{z}_0)}{z - z_0} = \lim_{z\to z_0} \frac{(z - z_0)(z + z_0) + \bar{z} - \bar{z}_0}{z - z_0}$$

$$= \lim_{z\to z_0} \left(z + z_0 + \frac{\overline{z - z_0}}{z - z_0}\right).$$

To show that the limit does not exist, we approch z_0 first horizontally and then vertically, i.e.
(i) $z - z_0 \in \mathbb{R}$. Then $\overline{z - z_0} = z - z_0$ so that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \left(z + z_0 + \frac{z - z_0}{z - z_0} \right) = 2z_0 + 1.$$

(ii) $z - z_0 \in i \cdot \mathbb{R}$, i.e. $z - z_0$ is purely imaginary. Then $\overline{z - z_0} = -(z - z_0)$ so that

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{z \to z_0} \left(z + z_0 - \frac{z - z_0}{z - z_0} \right) = 2z_0 - 1.$$

The two answers are different as $-1 \neq 1$. Therefore, the function f is not holomorphic at z_0.

Second solution: We use the Cauchy–Riemann equations. We have

$$f(z) = z^2 + \bar{z} = (x + iy)^2 + x - iy = (x^2 - y^2) + 2ixy + x - iy = u(x, y) + iv(x, y)$$

so that

$$u(x, y) = x^2 - y^2 + x, \quad v(x, y) = +2xy - y.$$

We compute

$$\frac{\partial u}{\partial x} = 2x + 1, \quad \frac{\partial v}{\partial y} = 2x - 1, \quad \frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y.$$

The Cauchy–Riemann equations are satisfied if

$$2x + 1 = 2x - 1 \quad \text{and} \quad -2y = 2y.$$

The first equation is never satisfied for any $x \in \mathbb{R}$. We conclude that the Cauchy–Riemann equations do not hold for any $z_0 \in \mathbb{C}$. Since a holomorphic function at a point satisfies the Cauchy–Riemann equations at that point, we conclude that f is not holomorphic at z_0 for $z_0 \in \mathbb{C}$.

3. Suppose that f is holomorphic in a region Ω. Prove that in any of the following cases

   (a) $\mathrm{Re}(f)$ is constant;

   (b) $\mathrm{Im}(f)$ is constant;

   (c) $\arg(f)$ is constant;

   (d) $\mathrm{Re}(f) = 5\,\mathrm{Im}(f)$;

   we can conclude that f is a constant. Do not use integration.

   Hint for (c): Can you find a complex number a such that the function $g(z) = af(z)$ is real-valued?

   Solution.

(a) Note that if u is constant, then $u_x = 0$ and $u_y = 0$. But $f'(z) = u_x + iv_x = u_x - iu_y = 0$. So the theorem proved in the lectures gives that f is constant.

(b) Note that if v is constant, then $v_x = v_y = 0$. But $f'(z) = v_y + iv_x = 0$. So again f is constant.

(c) Write

$$f(z) = F(z)e^{i\theta}$$

with a non-negative function F and a constant $\theta \in \mathbb{R}$. Therefore, the function $g(z) = e^{-i\theta} f(z)$ is real-valued and holomorphic. Since $\operatorname{Im} g = 0$, it is constant, as required.

(d) Notice that the equation $\operatorname{Re}(f) = 5 \operatorname{Im}(f)$ implies that the argument of f is constant, as $\tan \arg(f) = \operatorname{Im}(f)/\operatorname{Re}(f) = 1/5$. Now apply (c).

4. Show that the function

$$u(x, y) = \frac{1}{2} \ln(x^2 + y^2), \quad (x, y) \neq (0, 0),$$

is harmonic.

Let $\Omega = \{z = x + iy \in \mathbb{C} : x > 0, y > 0\}$ be the first quadrant of the complex plane. Find all functions $f(z)$, holomorphic in Ω, such that $\operatorname{Re} f(z) = u(x, y)$.

Solution. We have

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2},$$

and

$$\frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}, \quad \frac{\partial^2 u}{\partial x^2} = \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}.$$

It easily follows that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

i.e. u is harmonic. We find the harmonic conjugate v of u. It satisfies the Cauchy–Riemann equations. We get (with the substitution $y = xt \Longrightarrow dy = xdt$.):

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2} \Longrightarrow$$
$$v(x, y) = \int \frac{x}{x^2 + y^2}\, dy = \int \frac{x^2}{x^2(1 + t^2)} dt = \arctan(t) + c(x) = \arctan(y/x) + c(x).$$

We use the second equation to determine $c(x)$:

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = -\frac{y}{x^2 + y^2},$$

while

$$v(x, y) = \arctan(y/x) + c(x) \Longrightarrow \frac{\partial v}{\partial x} = \frac{-y/x^2}{1 + (y/x)^2} + c'(x) = -\frac{y}{x^2 + y^2} + c'(x).$$

This gives $c'(x) = 0$, i.e. $c(x) = k$ for a constant independent of x and y. The result is that

$$f(z) = \frac{1}{2}\ln(x^2 + y^2) + i\arctan(y/x) + ik.$$

Observing that

$$\arctan\frac{y}{x} = \operatorname{Arg} z,$$

we obtain

$$f(z) = \operatorname{Log} z + ik,$$

with an arbitrary real constant k.

5. Using the definition of derivative, prove rigorously that, if the function $f(z)$ is holomorphic on $D(0, R)$, then $g(z) = \overline{f(\bar{z})}$ is also holomorphic on $D(0, R)$.

Solution. We note that $z \to z_0 \Leftrightarrow \bar{z} \to \bar{z}_0$, as these are both equivalent to $x \to x_0$ and $y \to y_0$. We use the definition of the derivative of g at $z_0 \in D(0, R)$. We notice that this disc is symmetric with respect to the real axis, i.e. $z \in D(0, R) \Leftrightarrow \bar{z} \in D(0, R)$. We have

$$\lim_{z\to z_0} \frac{g(z) - g(z_0)}{z - z_0} = \lim_{z\to z_0} \frac{\overline{f(\bar{z})} - \overline{f(\bar{z}_0)}}{z - z_0} = \lim_{z\to z_0} \frac{\overline{f(\bar{z}) - f(\bar{z}_0)}}{\bar{z} - \bar{z}_0} = \lim_{z\to z_0} \overline{\frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0}}$$

$$= \overline{\lim_{\bar{z}\to \bar{z}_0} \frac{f(\bar{z}) - f(\bar{z}_0)}{\bar{z} - \bar{z}_0}} = \overline{\lim_{w\to \bar{z}_0} \frac{f(w) - f(\bar{z}_0)}{w - \bar{z}_0}} = \overline{f'(\bar{z}_0)},$$

with the substitution $w = \bar{z}$ and using the definition of the derivative of f at $\bar{z}_0$.

Remark: It is not enough to check the Cauchy–Riemann equations, as these are not equivalent to being holomorphic. In fact, the calculation below, showing that g satisfies the Cauchy–Riemann equations iff f does is not even easier and uses the chain rule from analysis.

If $f(z) = u(x, y) + iv(x, y)$, then $\overline{f(\overline{z})} = u(x, -y) - iv(x, -y)$. We calculate the partial derivatives with respect to x and y for $\overline{f(\overline{z})}$ using the chain rule from multivariable calculus:

$$\begin{aligned}
\frac{\partial u(x,-y)}{\partial x} &= \frac{\partial u}{\partial x}(x,-y)\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}(x,-y)\frac{\partial(-y)}{\partial x} = \frac{\partial u}{\partial x}(x,-y),\\
\frac{\partial u(x,-y)}{\partial y} &= \frac{\partial u}{\partial x}(x,-y)\frac{\partial x}{\partial y} + \frac{\partial u}{\partial y}(x,-y)\frac{\partial(-y)}{\partial y} = -\frac{\partial u}{\partial y}(x,-y),\\
\frac{\partial(-v(x,-y))}{\partial x} &= \frac{\partial(-v)}{\partial x}(x,-y)\frac{\partial x}{\partial x} + \frac{\partial(-v)}{\partial y}(x,-y)\frac{\partial(-y)}{\partial x} = -\frac{\partial v}{\partial x}(x,-y),\\
\frac{\partial(-v(x,-y))}{\partial y} &= \frac{\partial(-v)}{\partial x}(x,-y)\frac{\partial x}{\partial y} + \frac{\partial(-v)}{\partial y}(x,-y)\frac{\partial(-y)}{\partial y}\\
&= -\frac{\partial(-v)}{\partial y}(x,-y) = \frac{\partial v}{\partial y}(x,-y).
\end{aligned}$$

This gives:

$$\frac{\partial u(x,y)}{\partial x} = \frac{\partial v(x,y)}{\partial y} \Leftrightarrow \frac{\partial u(x,-y)}{\partial x} = \frac{\partial(-v(x,-y))}{\partial y},$$

and

$$\frac{\partial u(x,y)}{\partial y} = -\frac{\partial v(x,y)}{\partial x} \Leftrightarrow \frac{\partial u(x,-y)}{\partial y} = -\frac{\partial(-v(x,-y))}{\partial x}.$$

This proves the equivalence of the Cauchy–Riemann equations.

6. Let $g : \mathbb{R}^2 \to \mathbb{R}$ be harmonic and assume that all its second partial derivatives exist and are continuous. Show that the function f defined by

$$f = \frac{\partial g}{\partial x} - i\frac{\partial g}{\partial y}$$

is holomorphic.

Solution. From a theorem in class a set of conditions on u, v, and their derivatives under which f is holomorphic at z_0 are the following:

- u, v are continuous on D,
- $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ exist on D and are continuous at $z_0 = (x_0, y_0)$,
- The Cauchy–Riemann equations hold:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

We check the conditions for f. We have $f = u + iv$ with

$$u = \frac{\partial g}{\partial x}, \quad v = -\frac{\partial g}{\partial y}.$$

Since the partial derivatives of g are continuous, u and v are continuous. The partial derivatives of u and v exist and are

$$\frac{\partial u}{\partial x} = \frac{\partial^2 g}{\partial x^2}, \quad \frac{\partial u}{\partial y} = \frac{\partial^2 g}{\partial x \partial y}, \quad \frac{\partial v}{\partial x} = -\frac{\partial^2 g}{\partial y \partial x}, \quad \frac{\partial v}{\partial y} = -\frac{\partial^2 g}{\partial y^2}.$$

As the second partials of g are continuous at (x_0, y_0), the first partial derivatives of u and v are continuous at (x_0, y_0). Now we check the Cauchy–Riemann equations for u and v:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Leftrightarrow \frac{\partial^2 g}{\partial x^2} = -\frac{\partial^2 g}{\partial y^2} \Leftrightarrow \frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial y^2} = \Delta g = 0,$$

which holds, as g is harmonic.

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Leftrightarrow \frac{\partial^2 g}{\partial x \partial y} = -\left(-\frac{\partial^2 g}{\partial y \partial x}\right) = \frac{\partial^2 g}{\partial y \partial x},$$

which holds, as the mixed partial derivatives of g are equal.

7. Let f be holomorphic on the domain D in $\mathbb{C}$. Let u and v be the real and imaginary parts of f so that $f(z) = u(x, y) + iv(x, y)$. Suppose that

$$u(x, y) - 3v(x, y) = 5$$

for all $(x, y) \in D$. Show that f is a constant function.

Solution. Here $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$, so that $u(x, y), v(x, y) \in \mathbb{R}$. Assume $u - 3v = 5$.

We differentiate $u - 3v = 5$ in x and y to get

$$\begin{aligned} u_x - 3v_x &= 0 \\ u_y - 3v_y &= 0 \end{aligned}$$

Using the Cauchy–Riemann equations in the second equation above, we get the system

$$\left.\begin{array}{rl} u_x - 3v_x & = 0 \\ -v_x - 3u_x & = 0 \end{array}\right\} \Leftrightarrow \left\{\begin{array}{rrl} u_x & -3v_x & = 0 \\ -3u_x & -v_x & = 0 \end{array}\right. .$$

The system has a unique solution $u_x = v_x = 0$, since the determinant of the coefficients is non-zero:

$$\begin{vmatrix} 1 & -3 \\ -3 & -1 \end{vmatrix} = -1 - 9 = -10 \neq 0.$$

Therefore, we conclude that $f'(z) = u_x + iv_x = 0$ and f is constant.

8. Let the function $f = u + iv$ be holomorphic on the domain D. Suppose that u_x, u_y, v_x, v_y are continuous on D. Prove that $g = (u - v) + i(u + v)$ is holomorphic on D.

Solution. Denote $w = u - v$, $t = u + v$. Due to the CRE for the functions u, v, we have

$$\begin{aligned} w_x &= u_x - v_x = u_x + u_y, \\ w_y &= u_y - v_y = u_y - u_x, \\ t_x &= u_x + v_x = u_x - u_y = -w_y, \\ t_y &= u_y + v_y = u_y + u_x = w_x. \end{aligned}$$

Thus w, t satisfy CRE. Moreover, w_x, w_y, t_x, t_y are continuous on D. Therefore g is holomorphic on D.

9. Determine the radii of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}(z+1)^n, \quad \sum_{k=1}^{\infty}(3+(-1)^k)^k z^k, \quad \sum_{s=1}^{\infty} z^s \cosh s,$$

$$\sum_{q=1}^{\infty}(q + a^q)(z-i)^q, \quad \sum_{n=1}^{\infty}(z-e)^{n!}.$$

Here a is a positive number.

Solution.

(a) Use the ratio test:

$$\frac{n^n (n+1)!|z|^{n+1}}{(n+1)^{n+1} n! |z|^n} = \frac{n^n}{(n+1)^n} \frac{(n+1)!}{(n+1)n!}|z| = \frac{1}{\left(1+\frac{1}{n}\right)^n}|z| \to e^{-1}|z|.$$

Thus the radius of convergence is e.

(b) Split the series into two and study their radii of convergence separately:

$$\sum_{s=1} 2^{2s-1} z^{2s-1}, \quad \text{and} \quad \sum_{s=1} 4^{2s} z^{2s}.$$

Ratio test:

$$\frac{2^{2s+1}|z|^{2s+1}}{2^{2s-1}|z|^{2s-1}} = 4|z|^2 = (2|z|)^2.$$

Thus the radius of convergence of the first series is $1/2$. For the second series:

$$\frac{4^{2s+2}|z|^{2s+2}}{4^{2s}|z|^{2s}} = 16|z|^2 = (4|z|)^2.$$

Thus the radius of convergence of the second series equals $1/4$. The answer for the sum of the two series: $1/4$.

(c) Ratio test:

$$\frac{|z|^{s+1}\cosh(s+1)}{|z|^s\cosh s} = \frac{e^{s+1}+e^{-s-1}}{e^s+e^{-s}}|z| = \frac{e^{s+1}}{e^s}\frac{1+e^{-2(s+1)}}{1+e^{-2s}}|z| \to e|z|.$$

Thus the radius of convergence equals e^{-1}.

(d) Ratio test:

$$\frac{(q+1+a^{q+1})|z|^{q+1}}{(q+a^q)|z|^q} = \frac{(q+1+a^{q+1})}{(q+a^q)}|z|.$$

If $0 < a \le 1$, then

$$\frac{q+1+a^{q+1}}{q+a^q} \to 1, q \to \infty,$$

and hence the radius of convergence equals 1.

If $a > 1$, then

$$\frac{q+1+a^{q+1}}{q+a^q} \to a, q \to \infty,$$

and hence the radius of convergence equals a^{-1}.

(e) Ratio test:

$$\frac{|z|^{(n+1)!}}{|z|^{n!}} = |z|^{(n+1)!-n!} = |z|^{n!n}.$$

If $|z| > 1$, then the limit of the r.h.s. is infinity, and hence the series diverges. If $|z| < 1$, then the limit is zero, and hence the series converges. Answer: radius of convergence equals 1.

10. Find the domains of convergence of the following series and analyse their convergence on the boundary of these domains:

$$\sum_{n=2}^{\infty}(-1)^n\frac{z^{3n-1}}{\ln n}, \quad \sum_{n=1}^{\infty}\frac{4^n}{n}z^{kn}.$$

Here $k > 0$.

Solution.

(a) Ratio test:

$$\frac{|z|^{3n+2}\ln(n+1)}{|z|^{3n-1}\ln n} = \frac{\ln(n+1)}{\ln n}|z|^3 \to |z|^3,\ n \to \infty.$$

Thus radius of convergence equals 1.

On the boundary $z = e^{i\theta}$ and our series becomes:

$$\sum \frac{(-1)^n}{\ln n} e^{i(3n-1)\theta} = e^{-i\theta} \sum \frac{e^{in(3\theta-\pi)}}{\ln n}.$$

If $3\theta - \pi = 2\pi m \ \ (m \in \mathbb{Z})$, then the series is

$$\sum \frac{1}{\ln n},$$

and hence it diverges. Otherwise it converges according to the Dirichlet's Convergence Test.

(b) Ratio test:

$$\frac{4^{n+1}|z|^{k(n+1)}n}{4^n|z|^{kn}(n+1)} = 4|z|^k \frac{n}{n+1} \to 4|z|^k,\ n \to \infty.$$

Thus radius of convergence equals $4^{-\frac{1}{k}}$.

On the boundary, i.e. when

$$z = \frac{e^{i\theta}}{\sqrt[k]{4}},$$

we have series

$$\sum \frac{e^{ikn\theta}}{n}.$$

If $k\theta = 2\pi l (l \in \mathbb{Z})$ then it diverges. Otherwise it converges according to the Dirichlet's Convergence Test.

11. Find the radii of convergence R for the following power series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^n, \quad \sum_{k=1}^{\infty} \frac{(z+4)^{k!}}{k^2}, \quad \sum_{n=1}^{\infty} n! e^{-n^2} z^n.$$

Solution.

(a) Ratio test:

$$\frac{n|z|^{n+1}}{(n+1)|z^n|} \to |z|, n \to \infty.$$

Thus radius of convergence equals 1.

(b) Ratio test:

$$\frac{k^2|z|^{(k+1)!}}{(k+1)^2|z|^{k!}} = \frac{k^2}{(k+1)^2}|z|^{k!k}.$$

If $|z| > 1$, then the limit of the r.h.s. is infinity, and hence the series diverges. If $|z| < 1$, then the limit is zero, and hence the series converges. Answer: radius of convergence equals 1.

(c) Ratio test:

$$\frac{(n+1)!e^{-(n+1)^2}|z|^{n+1}}{n!e^{-n^2}|z|^n} = (n+1)e^{-(n+1)^2+n^2}|z| = (n+1)e^{-2n-1}|z| \to 0, n \to \infty.$$

So, the radius of convergence is infinity.

12. Let R be the radius of convergence of the series $f(z) = \sum a_k z^k$.

(a) Prove that the series $\sum \overline{a_k} z^k$ has the same radius of convergence.

(b) Prove that $\overline{f(\overline{z})} = \sum \overline{a_k} z^k$.

Solution.

(a) The series $\sum |a_k z^k|$ and $\sum |\overline{a_k} z^k|$ coincide, and hence, by the definition, the radii of convergence for them coincide.

(b) Let

$$f_n(z) = \sum_{k=0}^{n} a_k z^k,$$

so that $f_n(z) \to f(z)$ for all $|z| < R$. From the properties of convergent sequences it follows that $\overline{f_n(z)} \to \overline{f(z)}$ as $n \to \infty$, i.e.

$$\overline{f(z)} = \sum_{k=0}^{\infty} \overline{a_k}\, \overline{z}^k.$$

This formula holds for all $|z| < R$, and in particular, for $\overline{z}$, so that

$$\overline{f(\overline{z})} = \sum_{k=0}^{\infty} \overline{a_k}\, z^k,$$

as required.

13. Suppose that the series $\sum a_k z^k$ has radius of convergence R. Prove that

$$R = \sup\{|z| : a_k z^k \to 0, k \to \infty\}.$$

Hint: see proof of Theorem 4.1 from Analysis 2.

Solution. Recall that

$$R = \sup\{|z| : \sum |a_k z|^k \text{ converges}\}.$$

Denote

$$R_1 = \sup\{|z| : a_k z^k \to 0, k \to \infty\}.$$

Since convergence of the series $\sum |a_k z^k|$ implies that $a_k z^k \to 0, k \to \infty$, we conclude that $R \le R_1$.

In order to prove that $R_1 \le R$ assume the contrary, i.e. that $R < R_1$. Let $w \in \mathbb{C}$ be a number such that $R < |w| \le R_1$ and $a_k w^k \to 0$. Therefore the sequence $|a_k w^k|$ is bounded by some number C. Arguing as in the proof of Theorem 4.1. (Analysis 2), pick a point z such that $R < |z| < |w|$, so that

$$|a_k z^k| = |a_k w^k| \left|\frac{z}{w}\right|^k \le C \left|\frac{z}{w}\right|^k.$$

As $|z/w| < 1$, the geometric series $\sum \left|\frac{z}{w}\right|^k$ converges, and hence, by the Comparison principle, the series $\sum |a_k z^k|$ converges too. This contradicts Theorem 3.2, which says that for all $|z| > R$ this series diverges. Therefore $R_1 \le R$, as required.

14. Expand each of the following founctions in a power series in a disk centred at $z_0 = 0$:

$$\frac{1}{1+z^2}, \quad \frac{1}{(1-z)^2}.$$

In each case determine the radius of convergence.

Solution. In the first case use the geometric series:

$$\frac{1}{1+z^2} = \sum_{k=0}^{\infty} (-1)^k (z^2)^k = \sum_{k=0}^{\infty} (-1)^k z^{2k}.$$

The radius of convergence equals 1.

In the second case observe that

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\frac{1}{1-z},$$

and hence, by Theorem 3.5 from the lectures,

$$\frac{1}{(1-z)^2} = \frac{d}{dz}\sum_{k=0}^{\infty} z^k = \sum_{k=0}^{\infty} \frac{d}{dz} z^k = \sum_{k=1}^{\infty} k z^{k-1} = \sum_{k=0}^{\infty} (k+1) z^k.$$

The radius of convergence equals 1.

15. Determine for which values of z the following series converge absolutely:

$$\sum_{k=0}^{\infty}\left(\frac{z-1}{z+1}\right)^k, \qquad \sum_{k=1}^{\infty}\frac{1}{k^2}(z^k + z^{-k}), \qquad \sum_{k=0}^{\infty}\frac{z^k}{(1-z)^k}.$$

Solution.

(a) This is the series of the form $\sum w^k$ with $w = \frac{z-1}{z+1}$. Thus it converges absolutely for $|w| < 1$ and diverges for $|w| > 1$. The condition $|\frac{z-1}{z+1}| < 1$ is equivalent to $|z-1| < |z+1|$, i.e. it defines the right half-plane $\{z : \operatorname{Re} z > 0\}$.

(b) Both series should converge. The series $\sum k^{-2} z^k$ converges absolutely for all $|z| \le 1$ and diverges for $|z| > 1$. Thus the series $\sum k^{-2} z^{-k}$ converges absolutely for all $|z| \ge 1$ and diverges for $|z| < 1$. The intersection of these two sets gives the set of convergence: $\{z : |z| = 1\}$.

(c) The series has the form $\sum w^k$ with $w = z(1-z)^{-1}$. Thus it converges absolutely if $|w| < 1$ and diverges if $|w| > 1$. The condition $|\frac{z}{1-z}| < 1$ is equivalent to $|z| < |1-z|$, i.e. it defines the half-plane $\{z : \operatorname{Re} z < 1/2\}$.

16. Prove the formulae

$$\cos z_1 + \cos z_2 = 2\cos\frac{z_1+z_2}{2}\cos\frac{z_1-z_2}{2},$$

$$\sinh z_1 + \sinh z_2 = 2\sinh\frac{z_1+z_2}{2}\cosh\frac{z_1-z_2}{2},$$

for any $z_1, z_2 \in \mathbb{C}$.

Solution. Straightforward application of the definitions

$$\cos z = \frac{e^{iz}+e^{-iz}}{2}, \ \sinh z = \frac{e^z - e^{-z}}{2}.$$

17. Suppose that $f(z) = i^z$, where the power is defined with the principal branch of logarithm. Determine $f'(z)$ and $f'(i)$.

Solution. By definition

$$f(z) = i^z = e^{z\text{Log } i} = e^{iz\frac{\pi}{2}},$$

as $\text{Arg } i = \frac{\pi}{2}$. So by the chain rule,

$$f'(z) = i\frac{\pi}{2}e^{iz\frac{\pi}{2}}, \;\; f'(i) = i\frac{\pi}{2}e^{-\frac{\pi}{2}}.$$