Dirichlet's test

There is a useful test that allows to study the convergence of series that are not absolutely convergent. It is called Dirichlet's test. It can be found, for example, in the book by T. Apostol, Mathematical Analysis.

Theorem 0.1 (Dirichlet's test)**.** *Let $\sum a_n$ be a series of complex terms whose partial sums form a bounded sequence, i.e.*

$$\sup_N \left| \sum_{n=0}^{N} a_n \right| < \infty.$$

Assume that b_n is a decreasing non-negative sequence converging to zero. Then the series $\sum a_n b_n$ converges.

Consider for example the series

$$\sum_{k=2}^{\infty} \frac{z^k}{\ln k} \tag{1}$$

where $|z| = 1$. It is clear that it does not converge absolutely, since

$$\sum_{k=2}^{\infty} \frac{1}{\ln k}$$

diverges. To ivestigate the convergence we use Dirichlet's test. To this end represent it is $\sum a_k b_k$ with

$$a_k = z^k, \quad b_k = \frac{1}{\ln k}.$$

The sequence b_k is decreasing and it converges to zero as $k \to \infty$. Let us estimate the partial sums

$$\sum_{k=0}^{N} a_k = \sum_{k=0}^{N} z^k, \quad |z| = 1.$$

It is clear that for $z = 1$ each partial sum equals N, and hence their sequence is unbounded. Assume that $z \neq 1$. Then

$$\sum_{k=0}^{N} z^k = \frac{z^{N+1} - 1}{z - 1},$$

and hence it remains bounded:

$$\left| \sum_{k=0}^{N} z^k \right| \leq \frac{2}{|z-1|} < \infty.$$

By Dirichlet's test, the series (1) converges for all z such that $|z| = 1$ and $z \neq 1$.