

MATH0013 Complex Analysis 2022/23  
Main exam: questions and solutions

Duration: 2 hours

1. (a) What does it mean for a function  $f$  defined on a domain  $\Omega \subset \mathbb{C}$  to be conformal on  $\Omega$ ?
- (b) Consider the linear fractional transformation

$$f(z) = \frac{az + b}{cz + d}, \quad z \in \mathbb{C}, cz + d \neq 0,$$

with some complex coefficients  $a, b, c, d$ . We say that this transformation is non-degenerate if the coefficients satisfy the condition

$$ad - bc \neq 0. \tag{1}$$

- i. Show that a non-degenerate  $f$  is conformal for all  $z$  such that  $cz + d \neq 0$ .
- ii. Prove that the composition of two non-degenerate linear fractional transformations is again a non-degenerate linear fractional transformation.

*Solution.*

- (a)  $f$  is conformal on  $\Omega$  if  $f$  is holomorphic on  $\Omega$  and  $f'(z) \neq 0$  for all  $z \in \Omega$ .
- (b) i. Since  $f$  is a rational function it is holomorphic away from the roots of the denominator, i.e. for  $cz + d \neq 0$ . Furthermore, a direct calculation shows that

$$f'(z) = \frac{a(cz + d) - c(az + b)}{(cz + d)^2} = \frac{ad - bc}{(cz + d)^2},$$

and hence, in view of (1), the right hand side is non-zero for all  $cz + d \neq 0$ . This implies the required conformality.

- ii. Let

$$f_1(z) = \frac{a_1z + b_1}{c_1z + d_1}, \quad f_2(z) = \frac{a_2z + b_2}{c_2z + d_2}$$

with  $a_1d_1 - b_1c_1 \neq 0, a_2d_2 - b_2c_2 \neq 0$ . Calculate the composition

$$f_1(f_2(z)) = \frac{a_1 \frac{a_2z+b_2}{c_2z+d_2} + b_1}{c_1 \frac{a_2z+b_2}{c_2z+d_2} + d_1} = \frac{(a_1a_2 + c_2b_1)z + a_1b_2 + d_2b_1}{(c_1a_2 + c_2d_1)z + c_1b_2 + d_1d_2}.$$

Observe that the coefficients can be obtained by multiplying two matrices:

$$A_1A_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{pmatrix}.$$

The condition (1) is also satisfied for this new transformation, since the determinant of the product of two matrices equals the product of determinants:

$$\det(A_1A_2) = \det A_1 \det A_2 \neq 0,$$

as required.

2. (a) Let  $f$  be holomorphic in the punctured disk  $D'(z_0, r) = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$ . Classify the possible singularity types of the function  $f$  at the point  $z_0$ .
- (b) Locate and classify the singularities of the following functions:

$$\frac{1 - \cos z}{z^2}, \quad z^{2022} \sin \frac{1}{z}.$$

Find the residues at every isolated singularity. Explain your answer.

*Solution.*

- (a) The type of singularity depends on the Laurent expansion in the neighbourhood  $D'(z_0, r)$ :

$$f(z) = \sum_{n=-\infty}^{\infty} a_n(z - z_0)^n.$$

The part of this series containing only negative powers of  $(z - z_0)$  is called the principle part:

$$\sum_{n=-\infty}^{-1} a_n(z - z_0)^n.$$

If the principle part equals zero, i.e.  $a_n = 0$  for all  $n < 0$ , then the singularity is said to be removable.

If the principle part contains finitely many terms, then the function is said to have a pole at  $z_0$ . The number  $k$ , such that  $a_{-k} \neq 0$  and  $a_{-n} = 0$ , for all  $n > k$ , is said to be the order of the pole.

If the principal part contains infinitely many terms then the singularity is said to be essential.

- (b) The first function is holomorphic except at  $z = 0$ . Expand around zero:

$$\begin{aligned} \frac{1 - \cos z}{z^2} &= \frac{1}{z^2} \left( 1 - 1 + \frac{z^2}{2} - \frac{z^4}{4!} + \dots \right) \\ &= \frac{1}{2} - \frac{z^2}{4!} + \dots \end{aligned}$$

Therefore the principal part of the expansion vanishes, and hence the singularity is removable. As a result, the residue equals zero.

The second function is holomorphic except at  $z = 0$ . Expand around zero:

$$\begin{aligned} z^{2022} \sin \frac{1}{z} &= z^{2022} \sum_{k=0}^{\infty} (-1)^k \frac{z^{-(2k+1)}}{(2k+1)!} \\ &= \sum_{k=0}^{1010} (-1)^k \frac{z^{-(2k+1)+2022}}{(2k+1)!} - \frac{1}{2023!} \frac{1}{z} + \sum_{k=1012}^{\infty} (-1)^k \frac{z^{-(2k+1)+2022}}{(2k+1)!}. \end{aligned}$$

The last sum is the principal part of the functions and it contains infinitely many terms. Thus the singularity at zero is essential. By definition, the residue equals  $-\frac{1}{2023!}$ .

3. (a) Let  $f$  be holomorphic on the disk  $D(a, r)$ , and let  $Z \subset D(a, r)$  be the set of its zeroes. Suppose that the point  $a$  is an accumulation point of  $Z$ , i.e. for every  $\varepsilon > 0$  the punctured disk  $D'(a, \varepsilon)$  contains at least one zero of  $f$ . Prove that  $f(z) = 0$  for all  $z \in D(a, r)$ .
- (b) Suppose that the radius of convergence of the series  $\sum_{n=0}^{\infty} a_n z^n$  equals 1.
- Show that for any  $N \in \mathbb{N}$  the series  $\sum_{n=0}^{\infty} a_n N^{-n} z^n$  converges absolutely on the disk  $D(0, N)$ ,
  - Prove that the function  $\sum_{n=0}^{\infty} \frac{a_n}{n^n} z^n$  is entire.

*Solution.*

- (a) By Taylor Theorem, the function  $f$  can be represented as a series

$$f(z) = \sum_{n \geq 0} a_n (z - a)^n$$

which converges absolutely for all  $z \in D(a, r)$ . Assume that  $f(z)$  is not identically zero in  $D(a, r)$ . Then there is at least one coefficient  $a_m \neq 0$ . Let  $a_m, m \geq 1$  be the first non-zero coefficient. Thus

$$f(z) = (z - a)^m \sum_{n \geq m} a_n (z - a)^{n-m} = (z - a)^m g(z),$$

$$g(z) = a_m + a_{m+1}z + \dots$$

The series defining  $g(z)$  has the same radius of convergence as the series for  $f$ , and hence it defines a holomorphic function on  $D(a, r)$ , such that  $g(a) = a_m \neq 0$ .

Alternatively, one can invoke the theorem proved in the lectures saying that  $f(z) = (z - a)^m g(z)$  with a holomorphic  $g$ , such that  $g(a) \neq 0$ .

Since  $g$  is continuous, there is a disk  $D(a, \epsilon)$  such that  $g(z) \neq 0$  for all  $z \in D(a, \epsilon)$ .

Thus the function  $f$  has no zeroes in  $D(a, \epsilon)$  except for  $a$ . This means that  $Z$  cannot have  $a$  as an accumulation point, which is a contradiction. Thus  $f(z) = 0$  for all  $z \in D(a, r)$ , as claimed.

- (b) Since the radius of convergence of the initial series equals 1, the new series

$$g_N(z) = \sum_{n=0}^{\infty} a_n \left( \frac{z}{N} \right)^n = \sum_{n=0}^{\infty} \frac{a_n}{N^n} z^n \quad (2)$$

converges absolutely for all  $|zN^{-1}| < 1$ , i.e. on the disk  $D(0, N)$ .

In order to prove that the function

$$f(z) = \sum_{n=0}^{\infty} \frac{a_n}{n^n} z^n, \quad (3)$$

is entire, one needs to show that the radius of convergence is infinite. In other words, it suffices to show that (3) converges in the disk  $D(0, N)$  for all  $N > 0$ . Now we use the Comparison Principle observing that

$$\left| \frac{a_n}{n^n} \right| \leq \frac{|a_n|}{N^n}$$

for all  $n \geq N$ . Since  $g_N$  converges in the disk  $D(0, N)$ , so does the series (3), as required.

4. (a) As usual, we denote  $z = x + iy$ . Let  $f(z) = u(x, y) + iv(x, y)$  be an entire function. Suppose that  $u(x, y) \leq x$  for all  $z \in \mathbb{C}$ . Show that  $f(z) = z + c$  with a complex constant  $c$ . Justify your answer.
- (b) Using the Cauchy Residue Theorem prove that

$$\int_0^{2\pi} \frac{\sin^2 \theta}{5 + 4 \cos \theta} d\theta = \frac{\pi}{4}.$$

*Solution.*

- (a) Let  $g(z) = e^{f(z)-z}$ . Then  $|g(z)| = e^{u(x,y)-x}$ . Since  $u(x, y) \leq x$ , we have  $|g(z)| \leq 1$  for all  $z$ . And since  $g(z)$  is entire,  $g(z)$  must be constant by Liouville's theorem.

Therefore,  $g'(z) = 0$ . That is,  $(f'(z) - 1)e^{f(z)-z} = 0$  and hence  $f'(z) = 1$  for all  $z$ . So  $f(z) = z + c$  for some complex constant  $c$ .

- (b) Use the substitution  $z = e^{i\theta}$ , so that

$$\sin \theta = \frac{1}{2i}(z - z^{-1}), \quad \cos \theta = \frac{1}{2}(z + z^{-1}), \quad d\theta = -iz^{-1}dz.$$

Thus the integral equals

$$\begin{aligned} I &= \frac{i}{4} \int_{|z|=1} \frac{(z - z^{-1})^2}{5 + 2(z + z^{-1})} \frac{dz}{z} \\ &= \frac{i}{4} \int_{|z|=1} \frac{(z^2 - 1)^2}{z^2(2z^2 + 5z + 2)} dz = \frac{i}{4} \int_{|z|=1} \frac{(z^2 - 1)^2}{z^2(2z + 1)(z + 2)} dz. \end{aligned}$$

The denominator has three roots: 0,  $-1/2$ ,  $-2$ . Only two of them,  $-1/2$  and 0 are inside the contour. The root  $-1/2$  is a simple pole, and the root 0 is a double pole of the function

$$f(z) = \frac{i}{4} \frac{(z^2 - 1)^2}{z^2(2z + 1)(z + 2)}.$$

Find the residues:

$$\begin{aligned} \text{Res}(f, -1/2) &= \lim_{z \rightarrow -1/2} (z + 1/2)f(z) \\ &= \frac{i}{4} \lim_{z \rightarrow -1/2} \frac{(z^2 - 1)^2}{2z^2(z + 2)} = \frac{i}{4} \frac{(1/4 - 1)^2}{2/4 \cdot 3/2} = \frac{i}{4} \cdot \frac{3}{4}, \end{aligned}$$

and

$$\begin{aligned}\operatorname{Res}(f, 0) &= \frac{i}{4} \lim_{z \rightarrow -1/2} \frac{d}{dz}(z^2 f(z)) \\ &= \frac{i}{4} \lim_{z \rightarrow 0} \frac{d}{dz} \frac{(z^2 - 1)^2}{2z^2 + 5z + 2} \\ &= \frac{i}{4} \lim_{z \rightarrow 0} \frac{2(z^2 - 1) \cdot 2z(2z^2 + 5z + 2) - (z^2 - 1)^2(4z + 5)}{(2z^2 + 5z + 2)^2} = -\frac{i}{4} \cdot \frac{5}{4}.\end{aligned}$$

By the Cauchy Residue Theorem,

$$\begin{aligned}I &= 2\pi i \operatorname{Res}(f, -1/2) + 2\pi i \operatorname{Res}(f, 0) \\ &= 2\pi i \frac{i}{4} \left( \frac{3}{4} - \frac{5}{4} \right) = -\frac{\pi}{2} \cdot \left( -\frac{1}{2} \right) = \frac{\pi}{4},\end{aligned}$$

as required.

5. Let  $f$  be a function holomorphic on the domain  $\Lambda = \{z : |z| > 1\}$ . Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad z \in \Lambda,$$

be its Laurent expansion. Assuming that  $f$  is a bounded function, prove that  $a_n = 0$  for all  $n > 0$ .

*Solution.* Assume that  $|f(z)| \leq M$ , for all  $|z| > 1$ . Recall the formula for the Laurent coefficients:

$$a_n = \frac{1}{2\pi i} \int_{S(0,R)} \frac{f(z)}{z^{n+1}} dz, \quad n \in \mathbb{Z},$$

where  $R > 1$  is arbitrary.

Estimate  $a_n$  using the Standard Integral Bound:

$$|a_n| \leq \frac{1}{2\pi} 2\pi R \cdot \frac{\max_{|z|=R} |f(z)|}{R^{n+1}} \leq M R^{-n}.$$

If  $n > 0$ , then the right-hand side tends to zero as  $R \rightarrow \infty$ . Therefore  $a_n = 0$ , as claimed.