Lecture 1: 29 September 2025

Complex numbers: revision

1 Chapter 1: complex numbers

1.1 Basics

Complex number z with real part $x \in \mathbb{R}$ and imaginary part $y \in \mathbb{R}$ is a point on the plane $\mathbb{R}^2$ with coordinates x and y, i.e. $z = (x, y)$. Notation: $1 := (1, 0), i = (0, 1)$, so that $z = x + iy$. Notation: $\operatorname{Re} z = x,\ \operatorname{Im} z = y$.

Definition 1.1. Standard form: $z = x + iy$ with the following rules of addition

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2),$$

and multiplication

$$\begin{aligned} z_1 z_2 = z_2 z_1 &= (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 + iy_2 x_2 + ix_1 y_2 + i^2 y_1 y_2 \\ &= (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + y_1 x_2). \end{aligned}$$

Complex plane = Argand plane. Notation: $\mathbb{C}$.

Consequence: $i^2 = -1$.

Definition 1.2. Modulus: $|z| = \sqrt{x^2 + y^2} \geq 0$. Complex conjugate: $\bar{z} = x - iy$.

Observations:

1. $|z_1 - z_2|$ is the distance between z_1 and z_2,
2. $\bar{z}$ is the reflection of z in the real axis. $\bar{\bar{z}} = z$.

3.

$$\operatorname{Re} z = \frac{z + \bar{z}}{2}, \quad \operatorname{Im} z = \frac{z - \bar{z}}{2i}.$$

Proposition 1.3. *1.* $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$,

2. $\overline{z_1 \, z_2} = \overline{z_1} \, \overline{z_2}$, $\quad \overline{z_1 \, \overline{z_2}} = \overline{z_1} \, z_2$.

3. $\bar{z} \, z = |z|^2$,

4. $|z_1 \, z_2| = |z_1| \, |z_2|$.

How to write complex numbers in the standard form? Example:

$$\frac{1}{z} = \frac{\bar{z}}{z \, \bar{z}} = \frac{\bar{z}}{|z|^2} = \frac{x - iy}{x^2 + y^2} = \frac{x}{x^2 + y^2} - i \, \frac{y}{x^2 + y^2}.$$

1.2 Inequalities

Lemma 1.4. *1.* $|\operatorname{Re} z| \leq |z|$, $|\operatorname{Im} z| \leq |z|$,

2. Triangle inequality: $|z + w| \leq |z| + |w|$,

3. $\big| |z| - |w| \big| \leq |z - w|$.

Proof. Set $z = x + iy$, so

$$x^2 + y^2 = |z|^2,$$

and hence $|x| \leq |z|, |y| \leq |z|$.

Triangle inequality:

$$\begin{aligned}
|z + w|^2 &= (z + w)\overline{(z + w)} = (z + w)(\bar{z} + \overline{w}) \\
&= |z|^2 + |w|^2 + w\bar{z} + z\overline{w} = |z|^2 + |w|^2 + 2\operatorname{Re}(w\bar{z}) \\
&\leq |z|^2 + |w|^2 + 2|w\bar{z}| = |z|^2 + |w|^2 + 2|w| \, |z| \\
&= (|z| + |w|)^2,
\end{aligned}$$

so $|z + w| \leq |z| + |w|$, as claimed.

(3) is an exercise. □

1.3 Polar form

Assume $z \neq 0$. Write $r := |z|$, $x = r\cos\theta$, $y = r\sin\theta$, so

$$z = x + iy = r(\cos\theta + i\sin\theta).$$

This is the polar form of the complex number z. θ is called *the argument* of z. Notation: $\theta = \arg z$.

Note that $\theta + 2\pi k$ is also an argument for every $k \in \mathbb{Z}$!

Definition 1.5. Let $z \neq 0$. Then *the principal argument* of z is the unique argument satisfying $\theta \in (-\pi, \pi]$. Notation: $\theta = \operatorname{Arg} z$.

Back to polar form. Introduce the notation

$$\cos\theta + i\sin\theta =: e^{i\theta}, \quad \text{so} \quad z = re^{i\theta}.$$

Observe:

$$\overline{e^{i\theta}} = \cos\theta - i\sin\theta = \cos(-\theta) + i\sin(-\theta) = e^{-i\theta}.$$

Properties:

1. $|e^{i\theta}| = 1$,
2. $$\cos\theta = \operatorname{Re} e^{i\theta} = \frac{e^{i\theta} + e^{-i\theta}}{2}, \quad \sin\theta = \operatorname{Im} e^{i\theta} = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$$
3. If $e^{i\phi} = e^{i\theta}$, then $\phi = \theta + 2\pi k$ with some $k \in \mathbb{Z}$.

Lemma 1.6. *Let $z_1 = r_1 e^{i\theta_1}, z_2 = r_2 e^{i\theta_2}$. Then*

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

Proof.

$$\begin{aligned} z_1 z_2 &= r_1 r_2 \big(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\big) \\ &= r_1 r_2 \big(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\big) = r_1 r_2 e^{i(\theta_1 + \theta_2)}. \end{aligned}$$

□

Consequence:

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}.$$

This implies De Moivre's formula:

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta).$$

1.4 Complex roots

Objective: to find nth root of a number $z \neq 0$. This means that we want to solve the equation

$$w^n = z.$$

Represent $z = \rho e^{i\theta}$ and $w = re^{i\phi}$. Then

$$w^n = r^n e^{in\phi} = \rho e^{i\theta}.$$

Take the modulus: $r^n = \rho$, so $r = \rho^{1/n}$, the arithmetic root! Thus

$$e^{in\phi} = e^{i\theta}.$$

We have observed earlier that $n\phi = \theta + 2\pi k$, $k \in \mathbb{Z}$. Thus there are infinitely many values of ϕ:

$$\phi = \phi_k = \frac{\theta}{n} + \frac{2\pi k}{n}.$$

However, if $k_1 - k_2$is a multiple of 2π, then the arguments ϕ_{k_1}, ϕ_{k_2} correspond to the same complex number. Thus there are exactly n *distinct* values of w: for $k = 0, 1, \ldots, n-1$. They all are called roots of z. Each roots is called a branch of the root function.

If $\theta = \operatorname{Arg} z$ and $k = 0$, then the barnch is *principal*:

$$w = \rho^{1/n} e^{i\theta/n}.$$

Exanple: square root, i.e. $n = 2$:

$$\sqrt{z} = \sqrt{\rho} e^{i\theta/2} e^{i2\pi/2\cdot 0} = \sqrt{\rho}\, e^{i\theta/2},\ \theta \in (-\pi, \pi].$$

Observe: Re $\sqrt{z} \geq 0$. The other branch is

$$\sqrt{z} = \sqrt{\rho} e^{i\theta/2} e^{i2\pi/2\cdot 1} = \sqrt{\rho}\, e^{i\theta/2 + i\pi} = -\sqrt{\rho}\, e^{i\theta/2}.$$

Re $\sqrt{z} \leq 0$.

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Complex numbers: revision

1.5 Geometry and topology of the complex plane

1.5.1 Sequences and limits

Look at sequences of complex numbers: $z_n \in \mathbb{C}$, $n = 1, 2, \ldots$. All very similar to real sequences.

Definition 1.7. A sequence $\{z_n\}$ is said to converge to $w \in \mathbb{C}$ if the real sequence $|z_n - w| \to 0$ as $n \to \infty$. In more detail, z_n converges to w if for any $\varepsilon > 0$ there exists a number $N \in \mathbb{R}$ such that $|z_n - w| < \varepsilon$ for all $n > N$. This is the standar $\varepsilon - N$ definition of the limit., as in Real Analysis. Notation: $z_n \to w$ as $n \to \infty$, $\lim_{n\to\infty} z_n = w$.

A sequence z_n is said to be Cauchy if $|z_n - z_m| \to 0$ as $n, m \to \infty$. In other words, $\{z_n\}$ is Cauchy if for any $\varepsilon > 0$ there is a number $N \in \mathbb{R}$ such that $|z_n - z_m| < \varepsilon$ for all $n, m > N$.

Proposition 1.8. *$z_n \to w$ iff $\operatorname{Re} z_n \to \operatorname{Re} w$ and $\operatorname{Im} z_n \to \operatorname{Im} w$.*

If $z_n \to w$, then $\overline{z_n} \to \overline{w}$ and $|z_n| \to |w|$.

z_n converges iff z_n is Cauchy! This means that $\mathbb{C}$ is complete!

1.5.2 Subsets of the complex plane

Definition 1.9. A circle of radius $r > 0$ centred at $z_0 \in \mathbb{C}$:

$$S(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\},$$

An open disk:

$$D(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\}.$$

A closed disk:

$$\overline{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| \le r\}.$$

A punctured disk:

$$D'(z_0, r) = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}.$$

Upper/lower half-plane:

$$\Pi_\pm = \{z \in \mathbb{C} : \pm \operatorname{Im} z > 0\}.$$

Definition 1.10. A point $z \in \Omega \subset \mathbb{C}$ is said to be an interior point of Ω if there is a radius $r > 0$ such that $D(z, r) \in \Omega$.

Int Ω is the set of all interior points.

Ω is said to be open if Int $\Omega = \Omega$.

Ω is said to be closed if its complement $\mathbb{C} \setminus \Omega$ is open.

Example: Π_+ is open. Indeed, let $z = x + iy \in \Pi_+$, i.e. $y > 0$, Take $r = y$. Then for every $w \in D(z, r)$ we have

$$\operatorname{Im} w = \operatorname{Im} z + \operatorname{Im}(w - z) = y + \operatorname{Im}(w - z) \geq y - |w - z| > y - r = 0.$$

1.6 Functions

1.6.1 Definitions

Write $f : \Omega \mapsto \mathbb{C}$, where $\Omega \subset \mathbb{C}$.

Notation: $f(z) = u(x, y) + iv(x, y)$ with real-valued u and v.

Examples:

1. Polynomials. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0,$$

where $a_n \neq 0$. This is a polynomial of degree n. Defined everywhere.

For two polynomials $P(z)$ and $Q(z)$ the function

$$R(z) = \frac{P(z)}{Q(z)}$$

is called *rational*.

2.

$$g(z) = \frac{z}{|z|}.$$

Domain: $\mathbb{C} \setminus \{0\}$.

Mapping properties.

1.6.2 Continuity

Definition 1.11. Let $f : \Omega \mapsto \mathbb{C}$. Pick a point $z_0 \in \mathbb{C}$. We say that

$$\lim_{z \to z_0} f(z) = w,$$

if for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(z) - w| < \varepsilon, \quad \text{as long as} \quad z \in D'(z_0, \delta) \cup \Omega.$$

f is continuous at $z_0 \in \Omega$ if $\lim_{z \to z_0} f(z) = f(z_0)$.

f is said to be continuous on Ω if it is continuous at every point of Ω.

Proposition 1.12. *1. f is continuous at z_0 iff $\operatorname{Re} f$ and $\operatorname{Im} f$ are continuous at z_0;*

2. AOL is applicable, i.e. if f, g are continuous at z_0 then $f + g$, fg are continuous as well. Also $1/f$ is continuous as long as $f(z_0) \neq 0$.

Examples:

1. $f(z) = z$,
2. Polynomials.
3. Rational functions away from the zeroes of the denominator.
4. $g(z) = \operatorname{Arg} z$ is not continuous, it has a jump on the negative real half-line.

Lecture 3: 6 October 2025

Differentiation

2 Chapter 2: differentiation, holomorphic functions

2.1 The basics

Denote $f(z) = u(x, y) + iv(x, y)$, where u and v are real-valued.

Definition 2.1. Let $f : \Omega \mapsto \mathbb{C}$ where $\Omega \subset \mathbb{C}$ is open, and let $z_0 \in \Omega$. The derivative of f at z_0 is defined as the limit

$$f'(z_0) = \frac{d}{dz} f(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h},$$

if it exists. In this case we say that f is differentiable at z_0.

If f is differentiable at all points $z_0 \in \Omega$, then we say that f is *holomorphic* on Ω.

holomorphic = regular=analytic.

If f is holomorphic on $\mathbb{C}$, then f is said to be entire.

he notation for the set of all functions holomorphic on Ω is $H(\Omega)$.

As in Real Analysis,

Lemma 2.2. *If f is differentiable at z_0, it is continuous at z_0.*

No proof.

Proposition 2.3.

1. *If c is a constant, then*

$$\frac{d}{dz} c = 0, \quad \frac{d}{dz}(cf(z)) = c \frac{d}{dz} f(z),$$

2.

$$\frac{d}{dz} z^n = nz^{n-1}, \quad n = 1, 2, \ldots,$$

3. *The usual rules of differentiation apply, including the chain rule.*

Examples.

1. Polynomials are entire functions,
2. Rational functions are differentiable everywhere except for the zeroes of the denominator.
3. Let $f(z) = \operatorname{Re} z$. Denote $h = a + ib$ and find f':

$$\frac{f(z+h) - f(z)}{h} = \frac{\operatorname{Re}(z+h) - \operatorname{Re} z}{h} = \frac{x + a - x}{a + ib} = \frac{a}{a + ib}.$$

If $a = 0$ and $b \to 0$, then the right-hand side equals 0. On the other hand, if $b = 0$ and $a \to 0$, then the right-hand side equals 1. Therefore the limit does not exist, and hence the function is not differentiable.

2.2 The Cauchy-Riemann equations

Recall: $z = x + iy$, $f(z) = u(x, y) + iv(x, y)$.

Remind definitions of partial derivatives.

Theorem 2.4. *Assume that $f'(z)$ exists. Then the partial derivatives u_x, u_y, v_x, v_y exist at z and*

$$\begin{aligned} f'(z) &= u_x(x, y) + iv_x(x, y) \\ &= v_y(x, y) - iu_y(x, y), \end{aligned}$$

and hence the Cauchy-Riemann equations (CRE) hold:

$$u_x = v_y, \quad u_y = -v_x.$$

The "converse" is true

Theorem 2.5. *Assume that two real-valued functions on Ω, satisfy CRE, and their partial derivatives u_x, u_y, v_x, v_y are continuous on Ω. Then $f = u + iv$ is holomorphic on Ω.*

Proof of Theorem 2.4. Let $h \in \mathbb{R}$. Then

$$\begin{aligned} \lim_{t \to 0} \frac{f(z+h) - f(z)}{h} &= \lim_{t \to 0} \left[\frac{u(x+t, y) - u(x, y)}{t} + i \frac{v(x+t, y) - v(x, y)}{t} \right] \\ &= \lim_{t \to 0} \frac{u(x+t, y) - u(x, y)}{t} + i \lim_{t \to 0} \frac{v(x+t, y) - v(x, y)}{t} \\ &= u_x(x, y) + iv_x(x, y). \end{aligned}$$

Now let $h = it$, $t \in \mathbb{R}$. Then

$$\lim_{t\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{t\to 0} \left[\frac{u(x, y+t) - u(x,y)}{it} + i\frac{v(x, y+t) - v(x,y)}{it} \right]$$
$$= \lim_{t\to 0} \frac{v(x, y+t) - v(x,y)}{t} - i \lim_{t\to 0} \frac{u(x, y+t) - u(x,y)}{t}$$
$$= v_y(x,y) - iu_y(x,y).$$

□

Examples.

1. $g(z) = z^2 = x^2 - y^2 + 2ixy$, so $u = x^2 - y^2$, $v = 2xy$. Then

$$u_x = 2x,\ \ u_y = -2y,$$
$$v_x = 2y,\ \ v_y = 2x.$$

   Clearly, CRE hold!

2. Let

$$f(z) = e^x \left(\cos y + i \sin y \right) = e^x e^{iy} \approx e^{x+iy} = e^z.$$

   Check CRE for $u = e^x\ \cos y$, $v = e^x\ \sin y$:

$$u_x = e^x \cos y,\ \ u_y = -e^x \sin y,$$
$$v_x = e^x \sin y,\ \ v_y = e^x \cos y.$$

   Hence CRE hold for all x, y. All partial derivatives are continuous, and therefore, by Theorem 2.5, $f(z)$ is holomorphic on $\mathbb{C}$. Moreover,

$$f'(z) = u_x + iv_x = f(z).$$

2.3 Properties of differentiable functions

Definition 2.6. Let $z_1, z_2 \in \mathbb{C}$. Then the set

$$[z_1, z_2] = \{z = (1-\alpha)z_1 + \alpha z_2, \alpha \in [0,1]\}$$

is called the segment joining z_1 and z_2.

The set S is said to be convex if for any two points $z_1, z_2 \in S$ the segment $[z_1, z_2]$ also belongs to S.

Examples.

1. Disk $D(z, r)$ is convex. Indeed, let $z_1, z_2 \in D(z, r)$. Then

$$\begin{aligned}|(1-\alpha)z_1 + \alpha z_2 - z| &= |(1-\alpha)z_1 + \alpha z_2 - (1-\alpha)z - \alpha z| \\ &= |(1-\alpha)(z_1 - z) + \alpha(z_2 - z)| \\ &\leq (1-\alpha)|z_1 - z| + \alpha|z_2 - z| \leq (1-\alpha)r + \alpha r = r.\end{aligned}$$

2. Π_+ is convex, DIY.

Definition 2.7. Let $z_1, z_2, \ldots, z_n \in \mathbb{C}$. Then the set

$$\Gamma = [z_1, z_2] \cup [z_2, z_3] \cup \cdots \cup [z_{n-1}, z_n]$$

is called a polygonal path joining z_1 and z_n.

We say that $S \subset \mathbb{C}$ is connected (or polygonally connected) if for any two points $a, b \in S$ there is a polygonal path in S that joins them.

An open connected set is called domain (or region).

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Theorem 2.8. *Let Ω be a domain. Suppose $f \in H(\Omega)$. Then*

1. *If $f' = 0$ on Ω, then $f = const$ on Ω.*
2. *If $|f(z)| = const$ on Ω, then $f = const$ on Ω.*

Proof. Write

$$f' = u_x + iv_x = v_y - iu_y = 0,$$

so $u_x = u_y = v_x = v_y = 0$. Thus u and v are constant along the lines parallel to coordinate axes x and y.

Want: for any $a, b \in \Omega$ prove that $f(a) = f(b)$. Join a and b with a polygonal path whose seghments are parallel to the x or y axis. Then $f(a) = f(b)$, as required.

Let $|f| = c \geq 0$. If $c - 0$, then $f = 0$ and there is nothing left to prove. Assume that $c > 0$. Write: $|f|^2 = u^2 + v^2 = c^2$ and differentiate:

$$\begin{aligned}u_x u + v_x v &= 0, \\ u_y u + v_y v &= 0.\end{aligned}$$

Using $u_x = v_y$ and $u_y = -v_x$ rewrite:

$$u_x u - u_y v = 0,$$
$$u_y u + u_x v = 0.$$

Multiply the first one by u and the second one by v:

$$u_x u^2 - u_y uv = 0,$$
$$u_y uv + u_x v^2 = 0.$$

Add them up:

$$0 = u_x u^2 + u_x v^2 = u_x(u^2 + v^2) = c^2 u_x.$$

Thus $u_x = v_y = 0$. In a similar way, $u_y = -v_x = 0$, so $f'(z) = 0$, and hence, by Part 1, $f = const$ on Ω. □

Question: what if $f \in H(\Omega)$ and $\operatorname{Im} f = 0$? What can we say about f?

2.4 Harmonic functions

Assume that u, v have continuous second order derivatives:

$$u_{xx}, u_{xy} = u_{yx}, u_{yy}, \quad v_{xx}, v_{xy} = v_{yx}, v_{yy}.$$

Differentiate CRE:

$$u_{xx} = v_{yx},\ u_{yy} = -v_{xy} = -v_{yx}.$$

Add up:

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Laplace operator. In the same way $\Delta v = 0$!

Definition 2.9. If u has continuous partial derivatives of first and second order, and $\Delta u = 0$, then u is said to be a harmonic function.

Definition 2.10. An ordered pair (u, v) of harmonic functions u and v on Ω are called harmonic conjugates if $u + iv$ is holomorphic on Ω.

Function v is said to be a harmonic conjugate of u.

Example. Let $u = 2x - x^3 + 3xy^2$. Find a harmonic conjugate of u and find the function $f = u + iv$. Is u harmonic? Calculate:

$$u_{xx} = -6x, u_{yy} = 6x,$$

so $\Delta u = 0$. Now we use CRE: $v_x = -u_y, v_y = u_x$:

$$v_y = 2 - 3x^2 + 3y^2.$$

Therefore

$$v(x, y) = \int (2 - 3x^2 + 3y^2)dy = 2y - 3x^2 y + y^3 + \phi(x).$$

Consequently, using the other CRE, i.e. $v_x = -u_y$ we get

$$v_x = -6xy + \phi'(x) = -6xy,$$

so $\phi'(x) = 0$, i.e. $\phi = const$.

Put together:

$$\begin{aligned} f &= u + iv = 2x - x^3 + 3xy^2 + i(2y - 3x^2 y + y^3) + ic \\ &= 2(x + iy) - (x^3 - 3x^2 y + 3ix^2 y - iy^3) + ic \\ &= 2z - z^3 + ic. \end{aligned}$$

3 Power series

3.1 Basics

Recall some definitions from Analysis 1 and 2.

Let a_k, $k = 1, 2, \ldots$, be a complex sequence. Then the sum

$$\sum_{k=1}^{\infty} a_k$$

is called complex series. We say that the series converges if the sequence

$$S_n = \sum_{k=1}^{n} a_k$$

of partial sums converges and we write

$$S = \sum_{k=1}^{\infty} = \lim_{n\to\infty} S_n.$$

Convergence of S_n amounts to convergence of $\sum \operatorname{Re} a_k$ and $\sum \operatorname{Im} a_k$.

Facts:

1. If $\sum a_k$ converges, then $a_k \to 0$ as $k \to \infty$, and hence the sequence a_k is bounded.
2. If $\sum a_k$ and $\sum b_k$ converge then for any two complex constants A, B the series $\sum(Aa_k + Bb_k)$ also converge.
3. $\sum a_k$ is said to converge absolutely, if $\sum |a_k|$ converges. Absolute convergence implies convergence.
4. Tests: Comparison, Root, Ratio.

Example: geometric series. Consider $\sum z^k$ and apply the Root test:

$$\lim_{k\to\infty} |z^k|^{\frac{1}{k}} = |z|.$$

Thus for $|z| < 1$ we have absolute convergence and for $|z| > 1$ we have divergence. The known formula,

$$\sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \quad \text{if} \quad |z| < 1,$$

supports this conclusion.

3.2 Power series

Let $z_0 \in \mathbb{C}$ be fixed and let $\{a_k\}$ be a fixed sequence of complex numbers. Then the series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \tag{1}$$

is called a power series. It is a function of $z \in \mathbb{C}$.

Definition 3.1. Radius of convergence is defined to be

$$\begin{aligned} R = \ & \sup\{|z - z_0| : \sum_k |a_k||z - z_0|^k \quad \text{converges}\} \\ = \ & \sup\{r \geq 0 : \sum_k |a_k| r^k \quad \text{converges}\}. \end{aligned}$$

R can be ∞.

Theorem 3.2. *Let $R > 0$. Then the series converges absolutely for all $z \in D(z_0, R)$. If $R < \infty$, then the series diverges for all $|z| > R$.*

See Theorem 4.1, Analysis 2.
Example.

1. Find Radius of convergence for

$$\sum_k k^{2023} \pi^k (z - e)^{2k}.$$

Can drop e as the radius of convergence does not depend on z_0 in (1). Ratio test:

$$\frac{(k+1)^{2023} \pi^{k+1} |z|^{2k+2}}{k^{2023} \pi^k |z|^{2k}} = \left(1 + \frac{1}{k}\right)^{2023} \pi |z|^2 \to \pi |z|^2, \ k \to \infty.$$

Therefore, if $\pi|z|^2 < 1$ then we have abs convergence, and if $\pi|z|^2 > 1$ – divergence. Consequently, $R = \pi^{-1/2}$ is the radius of convergence.

2. Exponential function:

$$\exp(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k.$$

Ratio test:

$$\frac{|z|^{k+1}\, k!}{(k+1)!|z|^k} = \frac{|z|}{k+1} \to 0, \ k \to \infty.$$

Therefore $R = \infty$.

3.3 Differentiability of power series

Differentiate (1) term by term:

$$\sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}. \tag{2}$$

Does this series represent the derivative of f?

Lemma 3.3. *The radii of convergence of* (1) *and* (2) *coincide.*

See Lemma 4.1, Analysis 2.

Theorem 3.4. *Let $R > 0$ be the radius of convergence of* (1)*. Then $f \in H(D(z_0, R))$ and f' coincides with* (2).

See Theorem 4.2, Analysis 2.

Observe that f is in fact infinitely differentiable and each consequtive derivative is obtained by differentiating term by term.

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3.4 Exponential function

Definition as before.

Theorem 3.5. *1.*

$$\frac{d}{dz} \exp(z) = \exp(z), \quad \text{for all} \quad z \in \mathbb{C},$$

2. $\exp(0) = 1$,

3.

$$\exp(w + z) = \exp(w) \exp(z), \quad w, z \in \mathbb{C},$$

4. $\exp(z) \neq 0$, $z \in \mathbb{C}$.

Proof. 1. See definition.

2. See definition.

3. Fix a $p \in \mathbb{C}$, and let
$$f(z) = \exp(p - z)\, \exp(z).$$
Differentiate:
$$f'(z) = -\exp(p - z)\, \exp(z) + \exp(p - z)\, \exp(z) = 0,$$
and hence $f(z) = const$. To find the constant find $f(0)$:
$$f(0) = \exp(p).$$
Therefore
$$\exp(p - z) \exp(z) = \exp(p).$$
Take $p = w + z$.

4. Write $\exp(z) \exp(-z) = \exp(0) = 1$, so $\exp(z) \neq 0$ for any $z \in \mathbb{C}$.

□

Corollary 3.6. *If $f'(z) = f(z)$, $f(0) = 1$, then $f(z) = \exp(z)$.*

Proof. Let $g(z) = f(z) \exp(-z)$, so $g(0) = f(0) = 1$. Differentiate:
$$g'(z) = f'(z) \exp(-z) - f(z) \exp(-z) = 0,$$
and hence $g(z) = const = g(0) = 1$. Therefore $f(z) = \exp(z)$. □

Corollary 3.7. *Let $z = x + iy$. Then*
$$\exp(z) = e^x(\cos y + i \sin y) = e^z.$$

Proof. Recall that $de^z/dz = e^z$ and $e^0 = 1$. Thus, by Corollary 3.6, $e^z = \exp(z)$. □

Corollary 3.8.
$$\exp(z + 2\pi k i) = \exp(z), \quad k \in \mathbb{Z}, \tag{3}$$

Proof. Since $\cos(y + 2\pi k) = \cos y$ and $\sin(y + 2\pi k) = \sin y$, we have the required periodicity (3). □

3.5 Trigonometric and hyperbolic functions

$$\cos z = \frac{e^{iz} + e^{-iz}}{2}, \quad \sin z = \frac{e^{iz} - e^{-iz}}{2i},$$
$$\cosh z = \frac{e^{z} + e^{-z}}{2}, \quad \sinh z = \frac{e^{z} - e^{-z}}{2}.$$

The familiar formulae hold for their derivatives and the series expansions are the same as in Real Analysis.

3.6 Logarithm

Real Analysis: e^x is strictly increasing on $\mathbb{R}$, hence injectivity hence the inverse exists, i.e. the equation $t = e^x$ is uniquely solvable for each $t > 0$. We say in this case that $x = \ln t$.

Let us solve for w the equation $e^w = z$ with $z \neq 0$. Write $w = u + iv$, so

$$z = e^{u+iv} = e^u\, e^{iv},$$

i.e. $|z| = e^u$ and $v = \arg z$. Consequently,

$$w = \ln|z| + i \arg z.$$

We call w the logarithm of z. The principal logarithm:

$$\operatorname{Log} z = \ln\,|z| + i\operatorname{Arg} z,\ -\pi < \operatorname{Arg} z \leq \pi.$$

For the other ones:

$$\log_k z = \operatorname{Log} z + 2\pi k i,\ k \in \mathbb{Z}.$$

These are called branches of the logarithm.

The range of Log belongs to

$$S_0 = \{z = x + iy : -\pi < y \leq \pi\}.$$

Theorem 3.9.

$$\begin{aligned} \exp(\operatorname{Log} z) &= z, \quad \text{for all} \quad z \neq 0, \\ \operatorname{Log}(\exp(z)) &= z, \quad \text{for all} \quad z \in S_0. \end{aligned}$$

Hence exp *is injective on the strip* S_0*, its range coincides with* $\mathbb{C} \setminus \{0\}$ *and its inverse is* Log.

Also, Log *is injective on* $\mathbb{C} \setminus \{0\}$*, its range coincides with* S_0 *and its inverse is* exp.

Thus exp *on* S_0 *and* Log *are inverse to each other.*

Proof. Check the identities:

$$\begin{aligned} \exp(\operatorname{Log} z) &= e^{\ln|z| + i\operatorname{Arg} z} = e^{\ln|z|} e^{i\operatorname{Arg} z} = |z| e^{i\operatorname{Arg} z} = z,\ z \neq 0, \\ \operatorname{Log}(\exp(z)) &= \ln|\exp z| + i\operatorname{Arg}(e^z) = \ln e^x + i\operatorname{Arg}(e^x e^{iy}) = x + iy = z,\ z \in S_0. \end{aligned}$$

Injectivity: assume $\exp(z_1) = \exp(z_2)$. Then

$$z_1 = \operatorname{Log}(\exp(z_1)) = \operatorname{Log}(\exp(z_2)) = z_2.$$

Assume $\operatorname{Log} z_1 = \operatorname{Log} z_2$. Then

$$z_1 = \exp(\operatorname{Log} z_1) = \exp(\operatorname{Log} z_2) = z_2.$$

$\square$

The other branches are inverses of e^z defined on the strips

$$S_k = \{z = x + iy : -\pi + 2\pi k < y \leq \pi + 2\pi k\}, k \in \mathbb{Z}.$$

Where is $\operatorname{Log} z$ differentiable?

Theorem 3.10. *Let*

$$S_{00} = \{z = x + iy : -\pi < y < \pi\}.$$

Then the range of $\exp(z)$ *defined on* S_{00} *is the set*

$$U = \mathbb{C} \setminus \{z : \operatorname{Re} z \leq 0,\ \operatorname{Im} z = 0\}.$$

The inverse of $\exp(z)$*, i.e.* Log z *is holomorphic on* U *and*

$$\frac{d}{dz} \operatorname{Log} z = \frac{1}{z}.$$

The domain U is the complex plane with a cut along the negative real semi-axis. The cut is called the *branch cut*. The point $z = 0$ is the *branch point*.

Proof. Range: DIY.
Derivative: see Real Analysis. □

3.7 General exponent

Let $z \in \mathbb{C}, z \neq 0$, and $\alpha \in \mathbb{C}$. Then z^α is defined as

$$z^\alpha = e^{\alpha \log z}.$$

The principal branch is

$$z^\alpha = e^{\alpha \operatorname{Log} z}.$$

Example. Find the principal branch of i^i.

$$\begin{aligned} i^i &= e^{i \operatorname{Log} i} = e^{i(\ln|i| + i \operatorname{Arg} i)} \\ &= e^{i(i\frac{\pi}{2})} = e^{-\frac{\pi}{2}}. \end{aligned}$$

For the other branches:

$$i^i = e^{-\frac{\pi}{2} - 2\pi k}, \quad k \in \mathbb{Z}.$$

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4 Conformal mappings

4.1 Paths

Definition 4.1. Let $[a, b] \subset \mathbb{R}$. A continuous function $\gamma : [a, b] \mapsto \mathbb{C}$ is called a path.

$\gamma(a)$ is the initial point, $\gamma(b)$ is the terminal point.

The image = a curve. Notation γ^*.

If the same curve is produced by two different paths, then we say that these paths yield two different *parametrisations* of the curve.

Assume that γ is differentiable, i.e.

$$\gamma'(t) = (\operatorname{Re}\gamma)'(t) + i\,(\operatorname{Im}\gamma)'(t)$$

exists for $t \in (a, b)$. Then we can find the tangent line. If $\gamma'(t_0) \neq 0$, then the tangent line is $\gamma(t_0)+\gamma'(t_0)s$, $s \in \mathbb{R}$. The angle between the line and the real axis is $\arg\gamma'(t_0)$.

Examples:

1. $\gamma_1(t) = it, t \in [-1, 1]$.
2. $\gamma_2(t) = e^{it},\ t \in [-\pi, \pi]$.

Another example: $\gamma(t) = t + it^2$.

4.2 Conformal mappings

Let $\gamma_1, \gamma_2 : [-1, 1] \mapsto \mathbb{C}$ be such that $\gamma_1(0) = \gamma_2(0) = \zeta$, and $\gamma_1'(0) \neq 0$, $\gamma_2'(0) \neq 0$.

The angles between the tangent line and the positive difection of the x-axis are $\arg\gamma_1'(0)$ and $\arg\gamma_2'(0)$. Thus the angle between the lines is

$$\theta = \arg\gamma_1'(0) - \arg\gamma_2'(0).$$

Assume now that $\gamma_1, \gamma_2 \subset \Omega$ and that $f \in H(\Omega)$. Two new paths: $\beta_1(t) = f(\gamma_1(t))$, $\beta_2(t) = f(\gamma_2(t))$. They intersect at $f(\zeta)$. Let us find the angle between β_1 and β_2 at this point:

$$\phi = \arg\beta_1'(0) - \arg\beta_2'(0).$$

Calculate the derivatives using the Chain Rule:

$$\beta_1'(t) = f'(\gamma_1(t))\gamma_1'(t), \quad \beta_2'(t) = f'(\gamma_2(t))\gamma_2'(t).$$

Theorem 4.2. *Let $f \in H(\Omega)$, and let $\gamma_1, \gamma_2 : [-1, 1] \mapsto \Omega$. Then, if $f'(\zeta) \neq 0$, then $\phi = \theta \mod 2\pi$.*

Proof. Write:

$$\begin{aligned}\phi &= \arg\beta_1'(0) - \arg\beta_2'(0) \\ &= \arg\left(f'(\gamma_1(0))\gamma_1'(0)\right) - \arg\left(f'(\gamma_2(0))\gamma_2'(0)\right) \\ &= \arg\left(f'(\gamma_1(0))\right) + \arg\gamma_1'(0) - \arg\left(f'(\gamma_2(0))\right) - \arg\gamma_2'(0) \\ &= \arg\gamma_1'(0) - \arg\gamma_2'(0) \mod 2\pi = \theta \mod 2\pi.\end{aligned}$$

□

Definition 4.3. A complex valued function f is said to be conformal in the domain Ω if $f \in H(\Omega)$ and $f'(z) \neq 0$ for all $z \in \Omega$.

Examples.

1. e^z is conformal on $\mathbb{C}$.
2. z^2 is conformal on $\mathbb{C} \setminus \{0\}$.
3. $\operatorname{Log} z$ is conformal on U.

4.3 Linear fractional transformations (Möbius transformations)

$$f(z) = \frac{az + b}{cz + d},$$

where a, b, c, d are complex constants. Put them in the matrix:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Always assume that A is non-degenerate, i.e.

$$\det A = ad - bc \neq 0. \tag{4}$$

This ensures that the denominator is not always zero, since at least one of the coefficients c or d is non-zero. The function is always defined as long as $cz + d \neq 0$.

The derivative is

$$f'(z) = \frac{ad - bc}{(cz + d)^2},$$

which is distinct from zero for all z such that $cz + d \neq 0$. Thus f is conformal.

Theorem 4.4. *Non-degenerate LFT's form a group with the operation of composition:*

$$(f_1 \circ f_2)(z) = f_1(f_2(z)).$$

Proof. Need to prove:

1. Group operation,
2. Associative,
3. Identity element,
4. Existence of inverse.

1. First we prove that the composition is a group operation, i.e. composition of two LFT's is again an LFT. Indeed, let

$$f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}, \ f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2},$$

with the coefficient matrices

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}.$$

Then

$$f_1(f_2(z)) = \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} = \frac{(a_1 a_2 + c_2 b_1)z + a_1 b_2 + d_2 b_1}{(c_1 a_2 + c_2 d_1)z + c_1 b_2 + d_1 d_2}.$$

Observe that the coefficients can be obtained by multiplying two matrices:

$$A_1 A_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}.$$

The condition (4) is also satisfied for this new transformation, since the determinant of the product of two matrices equals the product of determinants:

$$\det(A_1 A_2) = \det A_1 \det A_2 \neq 0.$$

Thus the new transformation satisfies (4) again.

2. For any three functions f, g, h (not necessarily LFT), we always have

$$((f \circ g) \circ h)(z) = f(g(h(z))) = (f \circ (g \circ h))(z),$$

so the composition is an associative operation. DIY.

3. The identity. Let $e(z) = z$. Clearly, $e(z)$ is non-degenerate, and for any LFT f we have $f \circ e = e \circ f = f$.

4. Let us find the inverse. To this end, for a fixed w we need to solve the equation

$$w = \frac{az+b}{cz+d},$$

for z. Calculate:

$$w(cz+d) = az+b, \text{ i.e. } z(cw-a) = -wd+b, \text{ so } z = g(w) = \frac{-dw+b}{cw-a}, \ cw-a \neq 0.$$

It is immediate to check that g is non-degenerate: $(-a)(-d) - bc = ad - bc \neq 0$. Thus f is one-to-one as a mapping from the domain of f onto the domain of g.

Conclusion: non-degenerate LFT's form a group, as claimed. □

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Examples.

1. Let $a = d + is$, $s > 0$, and let

$$f(z) = \frac{z-a}{z-\overline{a}},$$

Since the root of the denominator is in the lower half-plane, we have $f \in H(\Pi_+)$. For every point $z \in \Pi_+$:

$$|z-a| < |z-\overline{a}|,$$

so that

$$|f(z)| = \frac{|z-a|}{|z-\overline{a}|} < 1, \quad z \in \Pi_+.$$

In other words, f maps Π_+ into the unit disk $D(0,1)$. Alternatively, we can calculate directly,

$$|f(z)|^2 = \frac{(x-d)^2 + (y-s)^2}{(x-d)^2 + (y+s)^2} < 1, \quad \forall x \in \mathbb{R}, y > 0,$$

since

$$\begin{aligned}(x-d)^2 + (y-s)^2 - [(x-d)^2 + (y+s)^2] \\ = (y-s)^2 - (y+s)^2 = -4sy < 0.\end{aligned}$$

Now we prove that f maps *onto* the disk $D(0,1)$, i.e. for every $w \in D(0,1)$ there exists a $z \in \Pi_+$, such that $f(z) = w$. The inverse is given by

$$z = d + is\frac{1+w}{1-w}.$$

The point 1 is not in the disk $D(0,1)$, and hence this function is holomorphic in $D(0,1)$. For any $w \in D(0,1)$ the right-hand side has a positive imaginary part. Indeed, rewrite:

$$\begin{aligned}z = d + is\frac{(1+w)(1-\overline{w})}{|1-w|^2} &= d + is\frac{1-|w|^2+w-\overline{w}}{|1-w|^2} \\ = d - \frac{2s\,\mathrm{Im}\,w}{|1-w|^2} + is\frac{1-|w|^2}{|1-w|^2},\end{aligned}$$

and the imaginary part is positive since $|w| < 1$ and $s > 0$. Therefore, the mapping f is onto. Observe also that the found z is uniquely defined, and hence the map f is bijective.

2. Let $\alpha \in D(0,1)$ be arbitrary. The function

$$\phi(z) = \frac{z-\alpha}{\overline{\alpha}z - 1}$$

maps the disk $D(0,1)$ onto itself. Note first that it is holomorphic in $D(0,1)$, since the denominator never turns into zero for $z \in D(0,1)$ due to the condition $|\alpha| < 1$.

To prove that $|\phi(z)| < 1$ for all $z \in D(0,1)$, calculate:

$$\begin{aligned}|z-\alpha|^2 &= |z|^2 + |\alpha|^2 - 2\,\mathrm{Re}(z\overline{\alpha}), \\ |\overline{\alpha}z - 1|^2 &= |\alpha|^2\,|z|^2 + 1 - 2\,\mathrm{Re}(z\overline{\alpha}),\end{aligned}$$

so

$$\begin{aligned}|\overline{\alpha}z-1|^2 - |z-\alpha|^2 &= |\alpha|^2\,|z|^2 + 1 - |z|^2 - |\alpha|^2 \\ &= (1-|\alpha|^2)\,(1-|z|^2) > 0.\end{aligned}$$

Hence $|\phi(z)| < 1$, $z \in D(0,1)$, i.e. ϕ maps $D(0,1)$ into $D(0,1)$.

Now it is easy to check that the inverse coincides with ϕ, i.e. $(\phi \circ \phi)(z) = z$. Thus our argument actually shows that ϕ is *onto*, as required!

3. $f(z) = e^z$. What is the image of f on the domain

$$S_{00} = \{z : 0 < \mathrm{Im}\,z < \pi\}?$$

DIY.

5 Integration and Cauchy's formula

5.1 Paths

Recall paths: continuous $\gamma : [a,b] \mapsto \mathbb{C}$.

The union of two paths: suppose that $\gamma_1 : [a_1, b_1] \mapsto \mathbb{C}$, $\gamma_2 : [a_2, b_2] \mapsto \mathbb{C}$ are such that $\gamma_1(b_1) = \gamma_2(a_2)$. Then we define the path $\gamma := \gamma_1 \cup \gamma_2$ as follows: $\gamma : [a_1, b_1 + b_2 - a_2]$ and

$$\gamma(t) = \begin{cases} \gamma_1(t), \ t \in [a_1, b_1], \\ \gamma_2(t + a_2 - b_1), \ t \in [b_1, b_1 + b_2 - a_2]. \end{cases}$$

A reverse path. Define

$$-\gamma(t) = \gamma(a + b - t), t \in [a.b].$$

Clearly, $\gamma(a) = -\gamma(b)$, $\gamma(b) = -\gamma(a)$.

Definition 5.1. 1. γ is closed if $\gamma(a) = \gamma(b)$.

2. γ is simple if if it does not intersect itself, i.e. for all $t, s \in [a,b]$ such that $|t - s| < |b - a|$ we have $\gamma(t) \neq \gamma(s)$.

3. γ is smooth if it is differentiable and the derivative is continuous on $[a,b]$.

4. γ is piece-wise smooth if γ is a union of finitely many smooth paths.

5. γ is a contour if it is simple, closed, piece-wise smooth.

Examples.

1. $\gamma_1(t) = t, t \in [-1, 1]$,

2. $g_2(t) = e^{it}, t \in [0, \pi]$ is simple, smooth.

3. $\gamma = \gamma_1 \cup \gamma_2$ is a contour.

4. $\gamma_3(t) = \sin t$, $t \in [0, \pi]$ is closed, not simple, smooth.

Theorem 5.2 (Jordan curve theorem)**.** *Let γ be a simple closed path. Then the complement of γ^* is a union of two disjoint domains:*

- a bounded domain, called the interior of γ, $\operatorname{Int}\gamma$, and

- an unbounded domain, called the exterior of γ, $\operatorname{Ext}\gamma$.

Thus $\mathbb{C} = \gamma^* \cup \operatorname{Int}\gamma \cup \operatorname{Ext}\gamma$.

Orientation. We say that the closed simple path is positively oriented if the point $\gamma(t)$ rotates about every point in the interior of γ counterclockwise.

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5.2 Integration

From now on all the paths are assumed to be piece-wise smooth.

Let $F(t) = A(t) + iB(t), t \in [a, b]$, with real-valued piece-wise continuous A, B. Then we define

$$\int_a^b F(t)\,dt = \int_a^b A(t)\,dt + i\int_a^b B(t)\,dt.$$

Definition 5.3. Let f be continuous on a domain Ω. Let $\gamma : [a, b] \mapsto \Omega$ be a path. Then the integral of f along γ is defined to be

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\,\gamma'(t)\,dt.$$

The length of the path is defined to be

$$L(\gamma) = \int_a^b |\gamma'(t)|\,dt.$$

We'll see later that the integral does not depend on the parametrisation. This reminds us of the change of variable formula in real analysis:

$$\int_a^b f(x)dx = \int_\alpha^\beta f(\phi(t))\,\phi'(t)\,dt, \quad x = \phi(t),\ dx = \phi'(t)dt,$$

and $\phi(\alpha) = a, \phi(\beta) = b$.

Example.

1. Let $f(z) = z^n, \quad n \in \mathbb{Z}, n \neq -1$, and let $\gamma(t) = re^{it}, t \in [0, 2\pi]$, with $r > 0$. Then

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_0^{2\pi} r^n e^{int}\, rie^{it} dt \\ &= ir^{n+1}\int_0^{2\pi} e^{i(n+1)t}\,dt = ir^{n+1}\left.\frac{e^{i(n+1)t}}{i(n+1)}\right|_0^{2\pi} = 0.\end{aligned}$$

2. Let $f(z) = z^{-1}$. Then

$$\int_\gamma f(z)\,dz = i\int_0^{2\pi} dt = 2\pi i.$$

$$\boxed{\int_\gamma \frac{1}{z}\,dz = 2\pi i.}$$

3. Segment $[a, a+h]$ with complex a, h. Let $\gamma(t) = a + th, t \in [0,1]$. Then

$$\int_{[a,a+h]} dz = \int_0^1 h dt = h.$$

4. Let $\gamma(t) = re^{it}, t \in [0, 2\pi]$. Then $\gamma'(t) = ire^{it}$ and

$$L(\gamma) = \int_0^{2\pi} |ire^{it}|\,dt = r\int_0^{2\pi} dt = 2\pi r.$$

5.3 Properties of integrals

Theorem 5.4. *Let $\gamma_j : [a_j, b_j] \mapsto \gamma^*, j = 1, 2$, be two parametrisations of γ^*. Assume that there is a function ψ mapping $[a_1, b_1]$ onto $[a_2, b_2]$, with a continuous derivative, such that $\psi(a_1) = a_2, \psi(b_1) = b_2$, and $\gamma_1 = \gamma_2 \circ \psi$, i.e. $\gamma_1(t) = \gamma_2(\psi(t))$ for all $t \in [a_1, b_1]$. Then*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

If $\psi'(s) \geq 0$ for all $s \in [a_1, b_1]$, then the length of the path does not depend on parametrisation.

Proof. Write:

$$\int_{\gamma_2} f(z)\,dz = \int_{a_2}^{b_2} f(\gamma_2(t))\gamma_2'(t)dt.$$

Change the variable $t = \psi(s)$:

$$\begin{aligned} &= \int_{a_1}^{b_1} f(\gamma_2(\psi(s)))\gamma_2'(\psi(s))\psi'(s)ds \\ &= \int_{a_1}^{b_1} f(\gamma_1(s))\gamma_1'(s)ds = \int_{\gamma_1} f(z)\,dz. \end{aligned}$$

For the length:

$$\begin{aligned} L(\gamma_2) &= \int_{a_2}^{b_2} |\gamma_2'(t)|dt = \int_{a_1}^{b_1} |\gamma_2'(\psi(s))|\psi'(s)ds \\ &= \int_{a_1}^{b_1} |\gamma_2'(\psi(s))\,\psi'(s)|ds = \int_{a_1}^{b_1} |\gamma_1'(s)|ds = L(\gamma_1). \end{aligned}$$

□

Theorem 5.5. *1.*

$$\int_\gamma f(z)dz = -\int_{-\gamma} f(z)dz.$$

2. If $\gamma = \gamma_1 \cup \gamma_2$, then

$$\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz.$$

3. $\int_\gamma cf(z)dz = c\int_\gamma f(z)dz$ for all complex constants c.

4.

$$\int_\gamma (f(z) + g(z))dz = \int_\gamma f(z)dz + \int_\gamma g(z)dz.$$

Proof. 2-4: DIY. The proof of 2 uses Theorem 5.4.

Proof of 1:

$$\begin{aligned} \int_{-\gamma} f(z)dz &= -\int_a^b f(\gamma(a+b-t))\gamma'(a+b-t)dt \\ &= \int_b^a f(\gamma(s))\gamma'(s)ds = -\int_a^b f(\gamma(s))\gamma'(s)ds = -\int_\gamma f(z)dz, \end{aligned}$$

as required. □

5.4 The Standard Integral Bound (SIB)

Recall the result from Real Analysis:

$$\left|\int_a^b f(x)\,dx\right| \le \max_{x\in[a,b]} |f(x)|\,(b-a).$$

Lemma 5.6. *For any continuous complex-valued function g we have the inequality*

$$\left|\int_a^b g(t)\,dt\right| \le \int_a^b |g(t)|\,dt.$$

Proof. Write:

$$J = \int_a^b g(t)\,dt, \quad \text{so} \quad J = |J|\,e^{i\theta},$$

and hence

$$\begin{aligned} |J| = J\,e^{-i\theta} &= \int_a^b e^{-i\theta}\,g(t)\,dt \\ &= \operatorname{Re}\int_a^b e^{-i\theta}\,g(t)\,dt = \int_a^b \operatorname{Re}\,(e^{-i\theta}\,g(t))\,dt \\ &\le \int_a^b |e^{-i\theta}\,g(t)|\,dt = \int_a^b |g(t)|\,dt, \end{aligned}$$

as required. □

Theorem 5.7 (SIB)**.** *Assume that f is continuous on γ. Then*

$$\left| \int_\gamma f(z)\,dz \right| \le \max_{z\in\gamma^*} |f(z)|\; L(\gamma).$$

Proof. Estimate using Lemma 5.6:

$$\begin{aligned} \left| \int_\gamma f(z)dz \right| &= \left| \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \right| \\ &\le \int_a^b |f(\gamma(t))|\,|\gamma'(t)|\,dt. \end{aligned}$$

Denote $M = \max_{z\in\gamma^*} |f(z)|$. Then the integral does not exceed

$$M\int_a^b |\gamma'(t)|\,dt = ML(\gamma),$$

as claimed. □

Example.

1. Let $\gamma_1(t) = e^{it}, t \in [0,\pi]$, $\gamma_2(t) = -t, t \in [-1,1]$. Find $\int_{\gamma_1} z^2 dz$ and $\int_{\gamma_2} z^2 dz$.

5.5 Antiderivatives (primitives)

Definition 5.8. Assume that f is continuous on Ω, and assume that there exists a function $F \in H(\Omega)$ such that $F'(z) = f(z)$. Then F is called an anti-derivative (or primitive) of f.

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Theorem 5.9. *("Fundamental Theorem of Calculus") Let f be continuous on Ω and let F be its primitive. Let $\gamma : [a, b] \to \Omega$ be a path such that $\gamma(a) = z_1, \gamma(b) = z_2$. Then*

$$\int_\gamma f(z)dz = F(z_2) - F(z_1). \tag{5}$$

Proof. Write:

$$\begin{aligned}\int_\gamma f(z)dz &= \int_a^b f(\gamma(t))\,\gamma'(t)\,dt \\ &= \int_a^b F'(\gamma(t))\,\gamma'(t)\,dt = \int_a^b \frac{d}{dt}F(\gamma(t))\,dt \\ &= F(\gamma(b)) - F(\gamma(a)) = F(z_2) - F(z_1).\end{aligned}$$

□

Corollary 5.10. *Let γ be a contour, and let f be as in Theorem 5.9. Then*

$$\int_\gamma f(z)\,dz = 0.$$

Proof. Since $z_1 = z_2$, the R.H.S. of (5) equals zero. □

Example.

1. $f(z) = z^n, n \neq -1, \gamma(t) = re^{it}, t \in [0, 2\pi]$, $r > 0$. f is holomorphic on $\mathbb{C} \setminus \{0\}$.
   A primitive: $F(z) = \frac{z^{n+1}}{n+1}$, so by Corollary 5.10,
   $$\int_\gamma f(z)\,dz = 0.$$
2. The same contour, and $f(z) = z^{-1}$. Is $\operatorname{Log} z$ a "good" primitive? No! Calculate directly:
   $$\int_\gamma \frac{1}{z}\,dz = \int_0^{2\pi} \frac{1}{re^{it}}\,ire^{it}\,dt = 2\pi i.$$

5.6 The Cauchy Theorem

5.6.1 Statement

Theorem 5.11. *Let $f \in H(\Omega)$, and let $\gamma \subset \Omega$ be a contour such that* $\operatorname{Int}\gamma \subset \Omega$. *Then*

$$\int_\gamma f(z)\, dz = 0. \tag{6}$$

Definition 5.12. We say that Ω is simply connected if for all contours $\gamma \subset \Omega$ we also have $\operatorname{Int}\gamma \subset \Omega$.

Theorem 5.11 restated:
Assume that $f \in H(\Omega)$ and that Ω is simply connected. Then for all contours $\gamma \subset \Omega$ we have (6).

Proof of Theorem 5.11 for triangular contours. Assume that γ is a triangular contour such that $\gamma \subset \Omega$ and $\operatorname{Int}\gamma \subset \Omega$. Denote $\Delta = \gamma^* \cup \operatorname{Int}(\gamma)$. Let us partition Δ into four triangles by joining the midpoints of its sides. Denote the obtained contours by $\sigma_1, \sigma_2, \sigma_3, \sigma_4$, assuming the standard orientation for each of them (counterclockwise). Thus

$$I = \int_\gamma f(z)dz = \sum_{k=1}^4 \int_{\sigma_k} f(z)dz.$$

Take σ_j for which the modulus of the integral is the largest and call it γ_1, so that

$$|I| \le 4\left|\int_{\gamma_1} f(z)dz\right|.$$

Note that $L(\gamma_1) = L(\gamma)/2$. Repeat the partition procedure with the triangle

$$\Delta_1 = \gamma_1^* \cup \operatorname{Int}(\gamma_1).$$

Thus we obtain a sequence of triangular contours $\gamma_0, \gamma_1, \ldots$ and triangles $\Delta_k = \gamma_k^* \cup \operatorname{Int}(\gamma_k)$ such that

1. $\gamma_0 = \gamma$;
2. $\Delta_{k+1} \subset \Delta_k$;
3. $L(\gamma_k) = 2^{-k} L(\gamma_0)$;
4. $|I| \le 4^k \left|\int_{\gamma_k} f(z)dz\right|$.

Claim: the set $\cap_{k=0}^\infty \Delta_k$ is not empty. Indeed, pick a sequence $z_k \in \Delta_k \subset \Delta$. Let $z_l \in \Delta_l$ If $l \ge k$, then $z_l, z_k \in \Delta_k$, so that $|z_k - z_l| < L(\gamma_k)$. If $k \ge l$, then $|z_k - z_l| < L(\gamma_l)$, so that in general,

$$|z_k - z_l| < L(\gamma_k) + L(\gamma_l) \to 0,$$

as $k, l \to \infty$. Therefore $\{z_k\}$ is a Cauchy sequence, and as such, it has a limit $\xi = \lim_{j\to\infty} z_k$. Since the sequence $z_k \in \Delta_n$ for all $k \geq n$, Δ_n is closed, and the point ξ is its limit point, we have $\xi \in \Delta_n$ for all n, and hence $\xi \in \cap_{k=0}^{\infty} \Delta_k$.

Now, as f is differentiable at ξ, we have

$$\frac{f(z) - f(\xi)}{z - \xi} \to f'(\xi), z \to \xi,$$

or, more precisely, for any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\left| \frac{f(z) - f(\xi)}{z - \xi} - f'(\xi) \right| < \epsilon$$

for all $z \in D(\xi, \delta)$. Rewrite this as

$$\left| f(z) - f(\xi) - f'(\xi)(z - \xi) \right| < \epsilon |z - \xi|, \qquad z \in D(\xi, \delta).$$

For all $z \in \Delta_k$ we have

$$|z - \xi| < L(\gamma_k) = \frac{1}{2^k} L(\gamma).$$

Therefore, for a large k we definitely have $\Delta_k \subset D(\xi; \delta)$. Consequently,

$$\max_{z \in \gamma_k} \left| f(z) - f(\xi) - f'(\xi)(z - \xi) \right| < \epsilon \max_{z \in \gamma_k} |z - \xi| < \epsilon 2^{-k} L(\gamma).$$

Recall that the functions $1, z$ have antiderivatives, so that

$$\int_{\gamma_k} (f(\xi) - (z - \xi) f'(\xi)) dz = 0,$$

by Corollary 5.10. Therefore

$$\int_{\gamma_k} f(z) dz = \int_{\gamma_k} (f(z) - f(\xi) - (z - \xi) f'(\xi)) dz,$$

and, as a consequence, by Theorem 5.7 (SIB),

$$\begin{aligned} \left| \int_{\gamma_k} f(z) dz \right| &= \left| \int_{\gamma_k} (f(z) - f(\xi) - (z - \xi) f'(\xi)) dz \right| \\ &\leq \epsilon 2^{-k} L(\gamma) L(\gamma_k) = \epsilon 2^{-2k} L(\gamma)^2. \end{aligned}$$

Consequently,

$$|I| \leq 4^k \left| \int_{\gamma_k} f dz \right| \leq 4^k \epsilon 2^{-2k} L^2(\gamma) = \epsilon L^2(\gamma).$$

Since ϵ is arbitrary, $I = 0$ as required. □

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Corollary 5.13. *Let $f \in H(\Omega)$, and let $\gamma \subset \Omega$ be a polygonal contour such that* $\mathrm{Int}\gamma \cup \gamma^* \subset \Omega$. *Then* $\int_\gamma f(z)dz = 0$.

Proof. Indeed, split the polygon enclosed within the contour γ into triangles and use Theorem 5.11 with triangular γ for each of them. ◻

Using the above objects we can now complete the proof of Theorem 5.11. We begin the proof with

Theorem 5.14 (The anti-derivative thm)**.** *Assume that Ω is convex, that f is continuous on Ω, and that*

$$\int_\gamma f(z)\,dz = 0$$

for all triangular contours in Ω. Then for any point $a \in \Omega$ the function

$$F(z) = \int_{[a,z]} f(w)\,dw$$

is an anti-derivative of f.

Proof. Want to prove that $F'(z) = f(z)$, i.e. for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| < \varepsilon,$$

as soon as $|h| < \delta$. We always assume that $D(z,\delta) \subset \Omega$. Observe:

$$F(z+h) - F(z) = \int_{[a,z+h]} f(w)\,dw - \int_{[a,z]} f(w)\,dw = \int_{[z,z+h]} f(w)\,dw.$$

Thus

$$\frac{F(z+h) - F(z)}{h} - f(z) = \frac{1}{h}\int_{[z,z+h]} f(w)\,dw - f(z).$$

Remind: $\int_{[z,z+h]} dw = h$, so the right-hand side equals

$$\frac{1}{h}\int_{[z,z+h]} f(w)\,dw - f(z)\frac{1}{h}\int_{[z,z+h]} dw = \frac{1}{h}\int_{[z,z+h]} [f(w) - f(z)]\,dw.$$

Since f is continuous, there is a $\delta > 0$ such that $|f(w) - f(z)| < \varepsilon$ if $|w - z| < \delta$. This inequality holds if $w \in [z, z+h]$ for $|h| < \delta$. By the SIB (see Thm 5.7),

$$\left|\frac{F(z+h) - F(z)}{h} - f(z)\right| \leq \frac{1}{|h|} \max_{w \in [z,z+h]} |f(w) - f(z)|\,|h| < \varepsilon,$$

as long as $|h| < \delta$, as required. □

Sketch of the proof of Theorem 5.11. We have the contour $\gamma = \gamma(t), t \in [a,b]$, $\gamma(a) = \gamma(b)$. Let $t_0 = a, t_1, \ldots, t_n = b$ be a sequence and let $\delta > 0$ be some constant such that the open disks $D_k = D_k(\gamma(t_k), \delta)$ for all $k = 0, 1, \ldots, n$ satisfy the following conditions:

1. $D_k \subset \Omega$,
2. $D_k \cap D_{k+1} \neq \varnothing$,
3. $\gamma([t_k, t_{k+1}]) \subset D_k$.

The last condition means that the disks centered at $z_k = \gamma(t_k)$ form a cover of the curve γ^*, in such a way that the part of the curve between z_k and z_{k+1} is contained in $D_k(z_k, \delta)$. We denote by $\gamma_k : [t_k, t_{k+1}] \mapsto \mathbb{C}$, the associated part of the contour, so that γ is a join:

$$\gamma = \bigcup_{k=0}^{n-1} \gamma_k.$$

Denote by σ_k the straight path, joining z_k and z_{k+1}, so $\sigma_k^* = [z_k, z_{k+1}]$. By taking a sufficiently small $\delta > 0$ we can ensure that

$$\sigma = \bigcup_{k=0}^{n-1} \sigma_k$$

is a contour. Note that σ is a polygonal contour.

Consider the disk D_k. Since it is convex, by Theorem 5.14 the function f has a primitive in D_k and consequently, by Theorem 5.9

$$\int_{\gamma_k} f(z)dz = \int_{\sigma_k} f(z)dz.$$

Summing them up over k, we get

$$\int_{\gamma} f(z)dz = \int_{\sigma} f(z)dz.$$

By Corollary (5.13), the last integral equals zero, which leads to the proclaimed result. □

Example. Let $S(z_0, R)$ be a circular positively oriented contour of radius $R > 0$ centred at z_0. For all $R < 3$ we have

$$\int_{S(0,R)} \frac{1}{z^2 + 9}\, dz = 0$$

5.7 The Keyhole Lemma

Lemma 5.15. *let $f \in H(\Omega)$. Let $\gamma_1, \gamma_2 \subset \Omega$ be such that $\gamma_1 \subset \operatorname{Int}\gamma_2$ and $\operatorname{Ext}\gamma_1 \cap \operatorname{Int}\gamma_2 \subset \Omega$. Then*

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz.$$

Proof. Connect γ_2 to γ_1 with a segment η. A new contour:

$$\gamma = \gamma_2 \cup \eta \cup (-\gamma_1) \cup (-\eta).$$

The domain enclosed within γ lies inside Ω, and hence by Cauchy's Theorem,

$$\int_\gamma f(z)\,dz = 0.$$

Expand:

$$\int_\gamma = \int_{\gamma_2} + \int_\eta - \int_{\gamma_1} - \int_\eta = \int_{\gamma_2} - \int_{\gamma_1} = 0.$$

The proof is complete. $\square$

Theorem 5.16 (The Keyhole Lemma for Multiple Contours)**.** *Let $f \in H(\Omega)$, and let $\gamma_1, \gamma_2, \ldots, \gamma_m, \gamma$ be some contours in Ω, such that $\gamma_j \subset \operatorname{Int}\gamma$, $\overline{\operatorname{Int}\gamma_j}$ are pairwise disjoint, and*

$$\left(\bigcap_{j=1}^m \operatorname{Ext}\gamma_j\right) \bigcap \operatorname{Int}\gamma \subset \Omega.$$

Then

$$\int_\gamma f(z)\,dz = \sum_{j=1}^m \int_{\gamma_j} f(z)\,dz.$$

Proved as Lemma 5.15.

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Corollary 5.17. *Let γ be a contour, and let $z_0 \in \operatorname{Int}\gamma$. Then*

$$\int_\gamma \frac{1}{z - z_0}\,dz = 2\pi i.$$

Proof. Assume that $z_0 = 0$. Let $S(0, r)$ be a contour s.t. $S(0, r) \subset \operatorname{Int}\gamma$. The function z^{-1} is holomorphic on $\mathbb{C} \setminus \{0\}$. By Lemma 5.15,

$$\int_\gamma \frac{1}{z}\, dz = \int_{S(0,r)} \frac{1}{z}\, dz = 2\pi i,$$

as required.

For general z_0, consider the contour $\tilde{\gamma}(t) = \gamma(t) - z_0$. Then

$$\int_\gamma \frac{1}{z - z_0}\, dz = \int \frac{1}{\gamma(t) - z_0}\, \gamma'(t)\, dt = \int \frac{1}{\tilde{\gamma}'(t)}\, \tilde{\gamma}'(t)\, dt = \int_{\tilde{\gamma}} \frac{1}{z}\, dz = 2\pi i.$$

□

5.8 The Cauchy Formula

Theorem 5.18. *Let Ω be simply connected, and let $\gamma \subset \Omega$ be a contour. Assume $f \in H(\Omega)$. Then for any $z_0 \in \operatorname{Int}\gamma$ we have*

$$f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z - z_0}\, dz.$$

Proof. Rewrite:

$$\begin{aligned} \int_\gamma \frac{f(z)}{z - z_0}\, dz &= \int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz + f(z_0) \int_\gamma \frac{1}{z - z_0}\, dz \\ &= \int_\gamma \frac{f(z) - f(z_0)}{z - z_0}\, dz + 2\pi i\, f(z_0). \end{aligned}$$

Denote

$$g(z) = \frac{f(z) - f(z_0)}{z - z_0}.$$

It remains to prove that $\int_\gamma g(z)\, dz = 0$. This function is holomorphic in $\Omega \setminus \{z_0\}$ and $g(z) \to f'(z_0)$ as $z \to z_0$, i.e. for any $\varepsilon > 0$ there is a $\delta > 0$ s.t. $|g(z) - f'(z_0)| < \varepsilon$ if $z \in D'(z_0, \delta)$. Assume that $D(z_0, \delta) \subset \Omega$. Use this fact with $\varepsilon = 1$:

$$|g(z) - f'(z_0)| < 1,\ z \in D'(z_0, \delta),$$

and therefore,

$$|g(z)| \leq 1 + |f'(z_0)| =: M,\ z \in D'(z_0, \delta).$$

By Lemma 5.15, for any $r \in (0, \delta)$ we have

$$\int_\gamma g(z)\,dz = \int_{S(z_0,r)} g(z)\,dz.$$

Now we use Theorem 5.7:

$$\left| \int_\gamma g(z)\,dz \right| = \left| \int_{S(z_0,r)} g(z)\,dz \right| \le M \cdot 2\pi r.$$

Since $r > 0$ is arbitrary, the l.h.s. equals zero, as required. □

Examples.

1.

$$\int_{S(0,1)} \frac{e^{iz}}{z}\,dz = 2\pi i$$

2. Let $R > 3$. Find

$$\int_{S(0,R)} \frac{1}{z^2+9}\,dz.$$

Write

$$\frac{1}{z^2+9} = \frac{1}{6i}\frac{1}{z-3i} - \frac{1}{6i}\frac{1}{z+3i}.$$

Therefore the integral equals

$$\frac{1}{6i}\int_{S(0,R)} \frac{1}{z-3i}\,dz - \frac{1}{6i}\int_{S(0,R)} \frac{1}{z+3i}\,dz = 2\pi i\,\frac{1}{6i}(1-1) = 0.$$

Alternative method, using Theorem 5.16.

3.

$$\int_{S(2i,2)} \frac{\cos z}{1+z^2}\,dz.$$

Write:

$$\frac{\cos z}{1+z^2} = \frac{f(z)}{z-i}, \quad \text{where} \quad f(z) = \frac{\cos z}{z+i}.$$

Therefore, the integral equals

$$\int_{S(2i,2)} \frac{f(z)}{z-i}\,dz = 2\pi i f(i) = 2\pi i\,\frac{\cos i}{2i} = \pi\cos i = \pi\cosh 1.$$

4.

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx = ?$$

Recall: $\cos x = \operatorname{Re} e^{ix}$, so

$$\int_{-\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx = \operatorname{Re} I, \qquad I = \int_{-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\, dx.$$

The definition:

$$I = \lim_{R\to\infty} I_R, \qquad I_R = \int_{-R}^{R} \frac{e^{ix}}{1+x^2}\, dx.$$

Two paths: $[-R,R]$ and $\gamma_R(t) = Re^{it}, t \in [0,\pi]$. Assume that $R \geq 2$, so that the singularities are inside the contour. Then by the Cauchy Integral Formula,

$$\int\limits_{[-R,R]\cup\gamma_R} \frac{e^{iz}}{1+z^2}\, dz = 2\pi i \left.\frac{e^{iz}}{z+i}\right|_{z=i} = \pi e^{-1}.$$

Want to show that $\int_{\gamma_R} \to 0$ as $R \to \infty$. Use the SIB. Estimate:

$$|e^{iz}| = |e^{i(x+iy)} = |e^{ix}|\, e^{-y} \leq 1,\ y \geq 0.$$

Also, $\max_{z\in\gamma_R} |1+z^2| \geq R^2 - 1$. Thus

$$\max_{z\in\gamma_R} \left|\frac{e^{iz}}{1+z^2}\right| \leq \frac{1}{R^2-1}.$$

By the SIB,

$$\left|\int\limits_{\gamma_R} \frac{e^{iz}}{1+z^2}\, dz\right| \leq \frac{1}{R^2-1}\pi R \to 0, \quad R \to \infty.$$

Therefore,

$$I = \lim_{R\to\infty} I_R = \lim_{R\to\infty} \Bigg[\int\limits_{[-R,R]\cup\gamma_R} \frac{e^{iz}}{1+z^2}\, dz - \int\limits_{\gamma_R} \frac{e^{iz}}{1+z^2}\, dz \Bigg] = \pi e^{-1}.$$

Conclusion:

$$\int_{\infty}^{\infty} \frac{\cos x}{1+x^2}\, dx = \pi e^{-1}.$$

Why don't we extend cos into the complex plane?

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5.9 The Cauchy formulas for derivatives

Recall: if $z \in \operatorname{Int}\gamma$, then

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta.$$

Differentiate formally:

$$f'(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta, \quad f''(z) = \frac{2}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^3}\, d\zeta.$$

Theorem 5.19. *Let $f \in H(\Omega)$. Then f is infinitely differentiable on Ω. Moreover, for each $z \in \Omega$ and any contour $\gamma \in \Omega$ such that* $\operatorname{Int}\gamma \subset \Omega$, *and $z \in \operatorname{Int}\gamma$, we have*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta, \quad n = 0, 1, \ldots.$$

Proof. Prove for $n = 1$. Fix a $z \in \operatorname{Int}\gamma$ and denote

$$I = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta,$$

and

$$\begin{aligned} g(h) = \frac{f(z+h) - f(z)}{h} &= \frac{1}{2\pi i}\frac{1}{h} \int_\gamma f(\zeta)\Big[\frac{1}{\zeta - z - h} - \frac{1}{\zeta - z}\Big] d\zeta \\ &= \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)}\, d\zeta. \end{aligned}$$

Let $\delta > 0$ be such that

$$\inf_{w \in D(z,\delta), \zeta \in \gamma} |w - \zeta| = s > 0.$$

Assume that $|h| < \delta$. Want: $g(h) \to I$ as $h \to 0$. Write

$$\begin{aligned} g(h) - I &= \frac{1}{2\pi i} \int_\gamma f(\zeta)\Big[\frac{1}{(\zeta - z - h)(\zeta - z)} - \frac{1}{(\zeta - z)^2}\Big] d\zeta \\ &= \frac{h}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z - h)(\zeta - z)^2}\, d\zeta. \end{aligned}$$

In order to use the SIB, estimate:

$$\max_{\zeta\in\gamma}\left|\frac{f(\zeta)}{(\zeta-z-h)(\zeta-z)^2}\right| \le \frac{\max_{\zeta\in\gamma}|f(\zeta)|}{s^3},$$

and hence

$$|g(h)-I| \le \frac{|h|}{2\pi\, s^3}\,\max_{\zeta\in\gamma}|f(\zeta)|\,L(\gamma)\to 0,\ h\to 0.$$

Therefore $g(h)\to I$ as $h\to 0$, as required.

The higher n are done by induction. □

Example. Evaluate

$$I=\int_\gamma \frac{\cos z}{z^2(z-1)}\,dz,\qquad \gamma = S(0,1/2).$$

By Theorem 5.19,

$$I = 2\pi i\,\frac{d}{dz}\frac{\cos z}{z-1}\bigg|_{z=0} = 2\pi i\left(-\frac{\sin z}{z-1}-\frac{\cos z}{(z-1)^2}\right)\bigg|_{z=0} = -2\pi i.$$

Theorem 5.20 (Morera's Theorem)**.** *Let f be continuous on Ω and assume that $\int_\gamma f(z)\,dz = 0$ for every triangular contour such that* $\operatorname{Int}\gamma\subset\Omega$. *Then f is holomorphic on Ω.*

Proof. Pick a $z\in\Omega$ and a radius $R>0$ such that $D(z,R)\subset\Omega$. Since $D(z,R)$ is convex, the function f has an antiderivative F on $D(z,R)$, by Theorem 5.14, i.e. $F\in H(D(z,R))$ and $F'(w)=f(w)$ for all $w\in D(z,R)$. By Theorem 5.19, f is holomorphic on $D(z,R)$, and hence on all of Ω. □

5.10 Cauchy's inequalities, Liouville's Theorem

Theorem 5.21 (Cauchy's inequalities)**.** *Let $f\in H(\Omega)$, and assume that $\overline{D}(z_0,R)\subset\Omega$. Denote*

$$M=\max_{|z-z_0|=R}|f(z)|.$$

Then

$$|f^{(k)}(z_0)|\le k!\,\frac{M}{R^k},\quad k=0,1,2,\dots.$$

Proof. Recall the Cauchy formula for the kth derivative:

$$f^{(k)}(z_0)=\frac{k!}{2\pi i}\int_{S(z_0,R)}\frac{f(\zeta)}{(\zeta-z)^{k+1}}\,d\zeta.$$

Use the SIB:

$$|f^{(k)}(z_0)|\le\frac{k!}{2\pi}M\frac{1}{R^{k+1}}2\pi R = k!\,\frac{M}{R^k},$$

as claimed. □

Theorem 5.22 (Liouville's Theorem)**.** *Suppose that f is entire, and that $|f(z)| \le M$ for all $z \in \mathbb{C}$, with some constant $M \ge 0$. Then $f(z) = const$.*

Proof. For any $z \in \mathbb{C}$ the function f is holomorphic on $D(z, R)$ for any R. Therefore, by Cauchy's inequality,

$$|f'(z)| \le \frac{M}{R}.$$

Since $R > 0$ is arbitrary, $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore $f(z) = const$, as required. $\square$

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Example. Suppose that f is entire and that $|f(z)| \le C\sqrt{|z|}$ for all $|z| \ge 1$. Prove that f is constant.
For each $R > 0$ and each $z \in \mathbb{C}$ the function f is holomorphic on $D(z, R)$. By Cauchy's inequality,

$$|f'(z)| \le \max_{|w-z|=R} |f(w)| \frac{1}{R} \le C \max_{|w-z|=R} \sqrt{|w|} \frac{1}{R}.$$

Estimate:

$$\sqrt{|w|} \le \sqrt{|w - z| + |z|} = \sqrt{R + |z|},$$

so

$$\begin{aligned} |f'(z)| &\le \max_{|w-z|=R} |f(w)| \frac{1}{R} \le C \max_{|w-z|=R} \left(\sqrt{|w - z| + |z|}\right) \frac{1}{R} \\ &= C \frac{\sqrt{R + |z|}}{R} \to 0, \quad R \to \infty. \end{aligned}$$

Therefore $f'(z) = 0$ for all $z \in \mathbb{C}$, and hence $f(z) = const$.

5.11 The Fundamental Theorem of Algebra

Lemma 5.23. *Let $p(z)$ be a non-constant polynomial. Then there exists a $w \in \mathbb{C}$ such that $p(w) = 0$.*

Proof. Assume that $p(z) \neq 0$ for all $z \in \mathbb{C}$. Thus the function $g(z) = p(z)^{-1}$ is entire. We shall show that g is bounded on $\mathbb{C}$, and hence, by Liouville, constant. To this end return to $p(z)$ and write:

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \; a_n \neq 0.$$

Then

$$p(z) = z^n(a_n + a_{n-1}z^{-1} + \cdots + a_0 z^{-n}).$$

The bracket tends to a_n as $|z| \to \infty$. Therefore, for sufficiently large z,

$$|p(z)| \geq |a_n| \frac{|z|^n}{2}.$$

Consequerntly, there exists a radius $R > 0$ such that $|p(z)| \geq 1$ for all $|z| \geq R$, and hence $|g(z)| \leq 1$ for $|z| \geq R$.

Inside the disk $\overline{D}(0, R)$ the function g is continuous, and hence bounded: $\max_{z \in \overline{D}(0,R)} |g(z)| \leq M$. Thus $|g(z)| \leq M+1$ for all $z \in \mathbb{C}$. By Liouville's Theorem, $g(z) = const$, and $p(z) = const$. Contradiction. Therefore p has at least one zero. □

Corollary 5.24. *Let p be a polynomial of degree $n \geq 1$. Then it has exactly n zeroes $z_1, z_2, \ldots, z_n$ on $\mathbb{C}$ (counting multiplicity), and*

$$p(z) = a_n(z - z_1)(z - z_2) \cdots (z - z_n).$$

Proof. By induction. If $n = 1$, then $p(z) = a_1 z + a_0 = a_1(z + a_0 a_1^{-1})$.

Assume that the result holds for $k = n$, and prove it for $k = n + 1$. Let

$$p(z) = a_{n+1}z^{n+1} + a_n z^n + \cdots + a_0, \quad a_{n+1} \neq 0.$$

By Lemma 5.23, $p(z)$ has a zero z_{n+1}, so $p(z_{n+1}) = 0$. Polynomial division:

$$p(z) = (z - z_{n+1})q(z) + C,$$

where q is a polynomial of degree n, and C is a constant. Observe: $C = p(z_{n+1}) = 0$. By the induction assumption,

$$q(z) = b_n(z - z_1)(z - z_2) \cdots (z - z_n), \quad \text{with some} \quad b_n \neq 0.$$

Therefore

$$p(z) = b_n(z - z_1)(z - z_2) \cdots (z - z_n)(z - z_{n+1}).$$

The coefficient by z^{n+1} equals b_n, so that $b_n = a_{n+1}$, as claimed. □

5.12 Uniform convergence

Definition 5.25. Let $f_n = f_n(z)$, $n = 1, 2, \ldots\ldots$, be a sequence of functions defined on the set $S \subset \mathbb{C}$. Then we say that f_n converges to the function f pointwise as $n \to \infty$, if for each $z \in S$ we have $f_n(z) \to f(z), n \to \infty$. In other words, for any $\varepsilon > 0$ there is an $N = N(\varepsilon, z)$ s.t.

$$|f_n(z) - f(z)| < \varepsilon, \quad \text{if} \quad n > N.$$

We say that f_n converges to f uniformly on S as $n \to \infty$, if

$$\sup_{z \in S} |f_n(z) - f(z)| \to 0, \quad \text{as} \quad n \to \infty.$$

In other words, for any $\varepsilon > 0$ there is an $N = N(\varepsilon)$ s.t.

$$\sup_{z \in S} |f_n(z) - f(z)| < \varepsilon, \quad \text{if} \quad n > N.$$

We say that the series $\sum_{k=1}^{\infty} u_k(z)$ convergence uniformly on S if the sequence of partial sums

$$f_n(z) = \sum_{k=1}^{n} u_k(z)$$

converges uniformly on S.

Remark.

1. Uniform convergence implies pointwise convergence.
2. Assume that f_n converges uniformly and its pointwise limit is f. Then f_n converges to f uniformly.

Examples.

1. $f_n(z) = z^n, S = D(0, 1)$. For each $z \in D(0, 1)$ we have $|f_n(z)| = |z|^n \to 0$ as $n \to \infty$.
   Is it uniform? It is not:

   $$\sup_{|z|<1} |f_n(z)| = \sup_{|z|<1} |z|^n = 1^n = 1 \not\to 0, n \to \infty.$$

   However, it converges uniformly in the disk $D(0, r)$ for each $r < 1$:

   $$\sup_{|z|<r} |f_n(z)| = \sup_{|r|<1} |z|^n = r^n \to 0, n \to \infty.$$

2. Let

$$f(z) = \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \quad z \in D(0,1).$$

Which type of convergence is this? Analyse the partial sums:

$$f_n(z) = \sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}.$$

The sequence z^{n+1} converges to zero pointwise, but not uniformly.

If, however, $S = D(0,r), r < 1$, then the convergence is uniform.

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3. Similarly for the series

$$\sum_{k=0}^{\infty} \frac{1}{w^k}, \quad |w| > 1.$$

Proposition 5.26 (Weierstrass M-test). *Let M_k be a sequence of real numbers such that $M_k \geq 0$ and $\sum_k M_k < \infty$. Let $u_k(z)$ be a sequence of complex functions on S such that $\sup_{z\in S} |u_k(z)| \leq M_k$. Then the series $\sum_k u_k(z)$ converges uniformly on S.*

Proof. Check that the sequence

$$f_n(z) = \sum_{k=1}^{n} u_k(z)$$

is Cauchy. Assuming that $n < m$, write:

$$|f_m(z) - f_n(z)| = \left| \sum_{k=n+1}^{m} u_k(z) \right| \leq \sum_{k=n+1}^{m} |u_k(z)| \leq \sum_{k=n+1}^{m} M_k \to 0, \ m, n \to \infty. \tag{7}$$

Therefore, for each $z \in S$ the sequence $f_n(z)$ is convergent. Denote $f(z) = \lim_{n\to\infty} f_n(z)$. Passing to the limit as $m \to \infty$ in (7):

$$|f_n(z) - f(z)| \leq \sum_{k=n+1}^{\infty} M_k,$$

This means that the sup of the left-hand side is also bounded by the sum $\sum_{k=n+1}^{\infty} M_k$ which tends to zero as $n \to \infty$. THis, by definition, implies the uniform convergence. □

Theorem 5.27. *Let $\gamma : [a,b] \mapsto \mathbb{C}$, be piece-wise smooth. Let $f_n(z)$ be a sequence of continuous functions defined on γ^*. Suppose that f_n converges uniformly on γ^* to some continuous function f. Then*

$$\lim_{n\to\infty} \int_\gamma f_n(z)\,dz = \int_\gamma f(z)\,dz. \tag{8}$$

Suppose that the u_k is a sequence of continuous functions, and the series $\sum_{k=1}^\infty u_k(z)$ converges to a continuous function uniformly on γ^*. Then

$$\sum_{k=1}^\infty \int_\gamma u_k(z)\,dz = \int_\gamma \sum_{k=1}^\infty u_k(z)\,dz. \tag{9}$$

Proof. Observe that (8) implies (9). Indeed, let

$$f_n(z) = \sum_{k=1}^n u_k(z), \quad f(z) = \lim_{n\to\infty} f_n(z) = \sum_{k=1}^\infty u_k(z),$$

and use (8).

Proof of (8). Estimate using the SIB:

$$\begin{aligned}\left|\int_\gamma f_n(z)\,dz - \int_\gamma f(z)\,dz\right| &= \left|\int_\gamma (f_n(z) - f(z))\,dz\right| \\ &\le \max_{z\in\gamma^*} |f_n(z) - f(z)|\, L(\gamma) \to 0, \quad n\to\infty,\end{aligned}$$

since f_n converges to f uniformly. □

5.13 Taylor series

Theorem 5.28 (Taylor's Theorem)**.** *Suppose that $f \in H(D(z_0, R))$ with some $z_0 \in \mathbb{C}$ and $R > 0$. Then for all $z \in D(z_0, R)$ the function f can be represented as the absolutely convergent series*

$$f(z) = \sum_{k=0}^\infty a_k (z - z_0)^k, \quad a_k = \frac{f^{(k)}(z_0)}{k!} = \frac{1}{2\pi i} \int_{S(0,r)} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\,d\zeta,$$

where $r \in (0, R)$ is arbitrary.

Proof. Suppose that $z_0 = 0$. Fix $z \in D(0, R)$ and pick a number $r : |z| < r < R$. By the Cauchy formula,

$$f(z) = \frac{1}{2\pi i} \int_{S(0,r)} \frac{f(\zeta)}{\zeta - z}\,d\zeta.$$

Expand:

$$\frac{1}{\zeta - z} = \frac{1}{\zeta(1 - z\zeta^{-1})} = \frac{1}{\zeta}\sum_{k=0}^{\infty}(z\zeta^{-1})^k. \tag{10}$$

Since $\delta := |z\zeta^{-1}| = |z|r^{-1} < 1$ and the series $\sum_k \delta^k$ converges, the M-test entails that (10) converges uniformly in $\zeta \in S(0, r)$. Therefore, by Theorem 5.27,

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{S(0,r)} f(\zeta)\frac{1}{\zeta}\sum_{k=0}^{\infty}(z\zeta^{-1})^k\, d\zeta \\
&= \frac{1}{2\pi i}\sum_{k=0}^{\infty} z^k \int_{S(0,r)} \frac{f(\zeta)}{\zeta^{k+1}}\, d\zeta \\
&= \sum_{k=0}^{\infty} a_k z^k, \quad a_k = \frac{1}{2\pi i}\int_{S(0,r)} \frac{f(\zeta)}{\zeta^{k+1}}\, d\zeta = \frac{f^{(k)}(0)}{k!}.
\end{aligned}$$

Theorem proved for $z_0 = 0$. For general z_0 apply the above to the function $g(z) = f(z + z_0)$. □

Remark.

1. If f is entire then its Taylor series about any point $z_0 \in \mathbb{C}$ converges absolutely for all $z \in \mathbb{C}$.
2. Cauchy's inequalities for the coefficients a_k, $k = 0, 1, \ldots$. It follows from Theorem 5.21 that for every $r \in (0, R)$,

$$|a_k| = \frac{|f^{(k)}(z_0)|}{k!} \le \frac{M(r)}{r^k}, \quad M(r) = \max_{|z - z_0| = r} |f(z)|.$$

Example. Let

$$f(z) = \frac{1}{z^2 + 9}.$$

The function is holomorphic on the disk $D(0, 3)$, and hence we can expand it in the Taylor series at $z_0 = 0$ that converges absolutely for all $z \in D(0, 3)$. Radius of convergence = 3.

Corollary 5.29. *Suppose that f is entire and that $|f(z)| \le C|z|^{n+\alpha}$ for all $|z| \ge 1$, with some $n = 0, 1, \ldots$ and some $\alpha \in [0, 1)$. Then f is a polynomial of degree at most n.*

Proof. By Theorem 5.28,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k,$$

for all $z \in \mathbb{C}$, where $a_k = f^{(k)}(0)/k!$. Rewrite:

$$f(z) = \sum_{k=0}^{n} a_k z^k + \sum_{k=n+1}^{\infty} a_k z^k,$$

and prove that $a_k = 0$ for all $k \geq n+1$. By Cauchy's inequalities,

$$\begin{aligned} |a_k| &= \frac{1}{k!}\,|f^{(k)}(0)| \leq \frac{\max_{|z|=R}\ |f(z)|}{R^k} \\ &\leq C\frac{R^{n+\alpha}}{R^k} = CR^{n+\alpha-k}. \end{aligned}$$

If $k \geq n+1$, we have $n+\alpha-k \leq \alpha-1 < 0$. Since R is arbitrary, and $R^{n+\alpha-k} \to 0$ as $R \to \infty$, we have $a_k = 0$ for all $k \geq n+1$. □

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6 Singularities and zeroes

6.1 Laurent series

Consider the series

$$g(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k.$$

Rewrite: $g(z) = g_1(z) + g_2(z)$, where

$$g_1(z) = \sum_{k=-\infty}^{-1} a_k (z - z_0)^k = \sum_{k=1}^{\infty} a_{-k} (z - z_0)^{-k},$$
$$g_2(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k.$$

Let R_2 be the radius of convergence for g_2. Thus for g_2 we have absolute convergence for all $|z-z_0| < R_2$.

For g_1: denote $w = (z - z_0)^{-1}$, so $g_2 = \sum_{k=1}^{\infty} a_{-k} w^k$. If $r_1 > 0$ is the radius of convergence for this series, then it converges absolutely for all $|w| = |z - z_0|^{-1} < r_1$, or, which is the same, for all $|z - z_0| > r_1^{-1} = R_1$. Thus $g_1 + g_2$ converges in the ring (or annulus)

$$D_{R_1,R_2}(z_0) = \{z : R_1 < |z - z_0| < R_2\}.$$

Theorem 6.1 (Laurent's Theorem)**.** *Assume that $f \in H(D_{R_1,R_2}(z_0))$, where $0 \le R_1 < R_2 \le \infty$. Then f can be represented by a convergent series*

$$f(z) = \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k,$$

which converges absolutely for all $z \in D_{R_1,R_2}(z_0)$, and

$$a_k = \frac{1}{2\pi i} \int_{S(z_0,r)} \frac{f(\zeta)}{(\zeta - z_0)^{k+1}}\, d\zeta, \quad k \in \mathbb{Z},$$

for any $r \in (R_1, R_2)$.

This is the Laurent series of the function f. The proof is similar to that of Theorem 5.28.

Example. Let

$$g(z) = \frac{1}{z(1-z)}.$$

Find all possible Laurent expansions.

There are four options: $0 < |z| < 1$, $|z| > 1$, $0 < |z-1| < 1$, $|z-1| > 1$. Do the last one. Looking for expansion around $z_0 = 1$:

$$g(z) = \sum_{k=-\infty}^{\infty} a_k (z-1)^k$$

Rewrite:

$$g(z) = -\frac{1}{z} + \frac{1}{z-1}.$$

Expand:

$$\begin{aligned} -\frac{1}{z} &= -\frac{1}{1+(z-1)} = -\frac{1}{z-1}\frac{1}{1+\frac{1}{z-1}} \\ &= -\frac{1}{z-1}\sum_{k=0}^{\infty}(-1)^k (z-1)^{-k} = \sum_{k=-\infty}^{-1}(-1)^k (z-1)^k. \end{aligned}$$

Therefore,

$$g(z) = (z-1)^{-1} + \sum_{k=-\infty}^{-1}(-1)^k (z-1)^k = \sum_{k=-\infty}^{-2}(-1)^k (z-1)^k.$$

Theorem 6.2. *[Cauchy's inequalities] Assume that $f \in H(D_{R_1,R_2}(z_0))$. Let $r \in (R_1, R_2)$. Denote*

$$M(r) = \max_{|z-z_0|=r} |f(z)|.$$

Then the coefficients a_k in the Laurent expansion satisfy the bounds

$$|a_k| \leq \frac{M(r)}{r^k}, \quad k \in \mathbb{Z}.$$

Proof. DIY. □

Corollary 6.3. *Let $f \in H(D'(z_0, R))$. Assume that $|f(z)| \leq M$ for all z. Then $a_k = 0$ for all $k = -1, -2, \ldots$.*

Proof. Let $0 < r < R$. By Cauchy's inequalitites (Theorem 6.2),

$$|a_k| \leq \frac{M}{r^k}, k \in \mathbb{Z}.$$

If $k \leq -1$, the right-hand side tends to zero as $r \to 0$, which implies that $a_k = 0$ for all $k = -1, -2, \ldots$, as claimed. □

6.2 Isolated singularities

Definition 6.4. We say that f has an isolated singularity at z_0 if f is holomorphic on $D'(z_0, R)$ with some $R > 0$.

Examples.

1. $1/z$ - isolated,
2. $\operatorname{Log} z$, the point $z = 0$ is not isolated.
3. $$\frac{1}{\sin \frac{1}{z}}.$$

   Singularities: Singularities: $z = 0$ and the points z_k such that $z_k^{-1} = \pi k, k \neq 0$, i.e. $z_k = (\pi k)^{-1}$. $\{z_k\}$ are isolated. But $z = 0$ is not!

Since $D'(z_0, R)$ is a ring, we have an expansion

$$\begin{aligned} f(z) &= \sum_{k=-\infty}^{\infty} a_k (z - z_0)^k \\ &= \sum_{k=-\infty}^{-1} a_k (z - z_0)^k + \sum_{k=0}^{\infty} a_k (z - z_0)^k. \end{aligned}$$

Definition 6.5. The series containing all negative powers of $z - z_0$ is called the principal part of the function f.

The coefficient a_{-1} is called the residue of f at the point z_0. Notation: $\operatorname{Res}(f, z_0) = a_{-1}$.

From Theorem 6.1:

$$\operatorname{Res}(f, z_0) = \frac{1}{2\pi i} \int_\gamma f(\zeta) d\zeta,$$

where γ is a contour such that neither γ not $\operatorname{Int}\gamma$ contain any other singularities of f.

Theorem 6.6 (Cauchy's residue Theorem)**.** *Let $f \in H(\Omega)$ apart from some isolated singularities. Let $\gamma \subset \Omega$ be a contour s.t.* $\operatorname{Int}\gamma \subset \Omega$ *and* $\operatorname{Int}\gamma$ *contains a finite number of isolated singularities at $z_1, z_2, \dots, z_n$, and there are no singularities on γ. Then*

$$\int_\gamma f(z) dz = 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f, z_k).$$

Proof. Let $\gamma_1, \gamma_2, \ldots, \gamma_n$ be contours such that $z_j \in \operatorname{Int}\gamma_j$, $j = 1, 2, \ldots, n$ and $\overline{\operatorname{Int}\gamma_j} \cap \overline{\operatorname{Int}\gamma_k} = \varnothing$, $j \neq k$, and $\operatorname{Int}\gamma_j \subset \operatorname{Int}\gamma$. Then, by Theorem 5.16(Keyhole Lemma for multiple contours),

$$\int_\gamma g(z)\, dz = \sum_{j=1}^{n} \int_{\gamma_j} f(z)\, dz = 2\pi i \sum_{j=1}^{n} \operatorname{Res}(f, z_j),$$

as claimed. □

Example, revisited.

$$\int_{S(0,2)} \frac{\cos z}{1+z^2}\, dz = 2\pi i(\operatorname{Res}(f, i) + \operatorname{Res}(f, -i)).$$

Find the residues:

$$\operatorname{Res}(f, i) = \frac{1}{2\pi i} \int_{S(i,1/2)} \frac{\cos z}{z+i} \frac{1}{z-i}\, dz = \frac{1}{2\pi i} \frac{\cos i}{2i} = -\frac{\cosh 1}{4\pi},$$

and

$$\operatorname{Res}(f, -i) = \frac{1}{2\pi i} \int_{S(-i,1/2)} \frac{\cos z}{z-i} \frac{1}{z+i}\, dz = \frac{1}{2\pi i} \frac{\cos(-i)}{-2i} = \frac{\cosh 1}{4\pi}.$$

Thus, the integral = 0.

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6.3 Classification of isolated singularities

Assume that $f \in H(D'(z_0, R))$. Three cases.

1. Pole. Principal Part (PP) contains finitely terms, i.e. for some number M we have $a_{-n} = 0$ for all $n \geq M + 1$ and $a_{-M} \neq 0$:

$$\sum_{k=-\infty}^{-1} a_k (z - z_0)^k = \frac{a_{-1}}{z - z_0} + \frac{a_{-2}}{(z - z_0)^2} + \cdots + \frac{a_{-M}}{(z - z_0)^M}.$$

   In this case we say that the function f has a pole of order M at z_0. If $M = 1$ then we say that the pole is simple.

   Example: $(z + 1)^{-1}$ has a simple pole at the point $z_0 = -1$.

   The function

$$\frac{1}{z^2 + 9} = \frac{1}{6i} \frac{1}{z + 3i} - \frac{1}{6i} \frac{1}{z - 3i}$$

   has two simples poles, st $z_0 = 3i$ and $z_0 = -3i$.

2. Essential singularity. We say that f has an essential singularity at z_0 if its principal part contains infinitely many terms, i.e there is no number $n \in \mathbb{N}$ such that $a_{-k} = 0$ for all $k > n$.

   Example: $\sin \frac{1}{z}$, $z \neq 0$. Indeed, use Taylor's series for sin:

$$f(z) = \sin \frac{1}{z} = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k + 1)!} \frac{1}{z^{2k+1}}.$$

   $\operatorname{Res}(f, 0) = 1$.

3. Removable singularity We say that f has a removable singularity at z_0 if the principal part equals zero, i.e.

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k, \quad z \in D'(z_0, R).$$

   If we set $f(z_0) = a_0$, then f becomes holomorphic on $D(z_0, R)$.

Example. Define on $\mathbb{C} \setminus 0$:

$$g(z) = \frac{\sin z}{z} = \frac{1}{z}\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k+1)!}.$$

Thus we have a removable singularity. For this function we define $g(0) = 1$, which makes it entire.

Theorem 6.7. *Suppose that $f \in H(D'(z_0, R))$ and that $|f(z)| \leq M$ with some positive number M for all $z \in D'(z_0, R)$. Then z_0 is a removable singularity.*

Proof. See Corollary 6.3. □

More examples.

1.

$$\begin{aligned} g(z) = z^{-4}\sin z &= z^{-4}\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)!} \\ &= z^{-3} - \frac{1}{6}z^{-1} + \frac{1}{5!}z + \dots . \end{aligned}$$

Principal part = $z^{-3} - 1/6\, z^{-1}$. Pole of order 3. The residue is $-1/6$.

2.

$$\begin{aligned} \frac{1-\cos z}{z^2} &= \frac{1}{z^2}\Big[1 - \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k)!}\Big] \\ &= \frac{1}{z^2}\Big[1 - 1 + \frac{z^2}{2} - \frac{z^4}{4!} + \dots\Big] \\ &= \frac{1}{2} - \frac{z^2}{4} + \dots . \end{aligned}$$

This is a removable singularity, residue is 0.

3.

$$\begin{aligned} \frac{1}{e^z - 1} &= \frac{1}{1 + z + \frac{z^2}{2} + \dots - 1} = \frac{1}{z + \frac{z^2}{2} + \dots} \\ &= \frac{1}{z(1 + \frac{z}{2} + \dots)} = \frac{1}{z}\,\frac{1}{1 + \frac{z}{2} + \dots} \\ &= \frac{1}{z}\Big(1 - \frac{z}{2} + \dots\Big) = \frac{1}{z} - \frac{1}{2} + \dots . \end{aligned}$$

Thus $z_0 = 0$ is a simple pole with residue 1.

Proposition 6.8. *Assume that $f \in H(D'(z_0, R))$. Then f has a pole of order m at z_0 if and only if f can be represented as*

$$f(z) = \frac{h(z)}{(z - z_0)^m},$$

where h is holomorphic in $D(z_0, R)$ and $h(z_0) \neq 0$.

If $m = 1$, then $\operatorname{Res}(f, z_0) = h(z_0)$. *If $m \geq 2$, then*

$$\operatorname{Res}(f, z_0) = \frac{1}{(m-1)!} h^{(m-1)}(z_0).$$

Proof. Let f have a pole of order m:

$$\begin{aligned} f(z) &= \sum_{k=-m}^{\infty} a_k (z - z_0)^k \quad (a_{-m} \neq 0) \\ &= (z - z_0)^{-m} \sum_{k=-m}^{\infty} a_k (z - z_0)^{k+m} = (z - z_0)^{-m} \sum_{k=0}^{\infty} a_{k-m} (z - z_0)^k \\ &= (z - z_0)^{-m} h(z), \quad h(z) = \sum_{k=0}^{\infty} a_{k-m} (z - z_0)^k. \end{aligned}$$

so $h \in H(D(z_0, R)$ and $h(z_0) = a_{-m} \neq 0$.

The converse. Suppose that $f(z) = h(z)(z - z_0)^{-m}$ with $h(z_0) \neq 0$ and $h \in H(D(z_0, R))$. Need to show that f has a pole of order m. Write:

$$h(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k, \; b_0 \neq 0.$$

Then

$$\begin{aligned} f(z) &= (z - z_0)^{-m} \sum_{k=0}^{\infty} b_k (z - z_0)^k \\ &= \sum_{k=0}^{\infty} b_k (z - z_0)^{k-m} = \sum_{k=-m}^{\infty} b_{k+m} (z - z_0)^k, \end{aligned}$$

as required.

Residues. Let $m = 1$. Then

$$f(z) = \frac{b_0}{z - z_0} + b_1 + \dots.$$

Thus $\operatorname{Res}(f, z_0) = b_0 = h(z_0)$.

Let $m \geq 2$. Then

$$f(z) = \frac{b_0}{(z-z_0)^m} + \frac{b_1}{(z-z_0)^{m-1}} + \cdots + \frac{b_{m-1}}{z-z_0} + \ldots.$$

Consequently,

$$\operatorname{Res}(f, z_0) = b_{m-1} = \frac{h^{(m-1)}(z_0)}{(m-1)!},$$

as claimed. $\square$

Example.

1. Let

$$f(z) = \frac{z+1}{(z-1)^3(z+3)}.$$

   Consider $z_0 = -3$ and define

$$h(z) = (z+3)f(z) = \frac{z+1}{(z-1)^3}.$$

   Observe: $h(-3) = 1/32 \neq 0$. Thus it is a simple pole of f and $\operatorname{Res}(f, -3) = 1/32$. Let $z_0 = 1$. Define

$$h(z) = (z-1)^3 f(z) = \frac{z+1}{z+3}, \quad h(1) = 1/2,$$

   so a pole of order 3 and $\operatorname{Res}(f, 1) = h''(1)/2 = -1/32$.

2.

$$f(z) = \frac{z^2}{z^4 + 5z^2 + 4}.$$

   Factorize: $z^4 + 5z^2 + 4 = (z^2+4)(z^2+1)$. Four singularities:

$$f(z) = \frac{z^2}{(z+2i)(z+i)(z-2i)(z-i)}.$$

   Let $z_0 = i$ and define

$$h(z) = (z-i)f(z) = \frac{z^2}{(z^2+4)(z+i)}.$$

h is holomorphic around z_0, so the pole of order 1 and

$$\operatorname{Res}(f, i) = h(i) = \frac{i^2}{2i(i^2 + 4)} = -\frac{1}{6i}.$$

Let $z_0 = 2i$ and define

$$h(z) = (z - 2i)f(z) = \frac{z^2}{(z + 2i)(z^2 + 1)}.$$

h is holomorphic around z_0, so the pole of order 1 and

$$\operatorname{Res}(f, 2i) = h(2i) = \frac{(2i)^2}{4i(2i)^2 + 1)} = \frac{1}{3i}.$$

Lecture 17: 1 December 2025

6.4 Evaluating integrals using residues

6.4.1 Trigonometric integrals

Integrals of the form

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)\, d\theta.$$

Define $z = e^{i\theta}$, so

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{1}{2}\left(z + \frac{1}{z}\right),$$
$$\sin\theta = \frac{e^{i\theta} + e^{-i\theta}}{2i} = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

Note also that $d\theta = \frac{dz}{iz}$. Thus the integral coincides with

$$\int_{S(0,1)} f\left(\frac{1}{2}(z + \frac{1}{z}), \frac{1}{2i}(z - \frac{1}{z})\right) \frac{dz}{iz}.$$

For example, consider

$$\begin{aligned}
I &= \int_0^{2\pi} \frac{1}{5-4\cos\theta}\,d\theta = \int_{S(0,1)} \frac{1}{5-2(z+z^{-1})}\,\frac{dz}{iz} \\
&= -i\int_{S(0,1)} \frac{1}{5z-2z^2-2}\,dz = i\int_{S(0,1)} \frac{1}{(2z-1)(z-2)}\,dz \\
&= i\cdot 2\pi i\,\mathrm{Res}\,(f,1/2) = \frac{2\pi}{3}.
\end{aligned}$$

Thus $I = 2\pi/3$.

6.4.2 Improper real integrals

Find

$$\int_{-\infty}^{\infty} \frac{\cos x}{x^4+5x^2+4}\,dx = \mathrm{Re}\,I, \quad I = \int_{-\infty}^{\infty} f(x)\,dx.$$

Denote

$$f(z) = \frac{e^{iz}}{z^4+5z^2+4}, \quad z\in\mathbb{C}.$$

Rewrite

$$I = \lim_{R\to\infty} I_R, \quad I_R = \int_{-R}^{R} f(x)\,dx.$$

Factorise:From the factorisation

$$z^4+5z^2+4 = (z^2+4)(z^2+1)$$

we see that f has singularities at $\pm i, \pm 2i$, and they are simple poles. Close the contour using the semi-circular path $\gamma_R(t) = Re^{it}$, $t\in[0,\pi]$. Consequently,if $R>2$, then

$$\int_{[-R,R]\cup\gamma_R} f(z)dz = 2\pi i(\mathrm{Res}\,(f,i) + \mathrm{Res}\,(f,2i)).$$

Find the residues. For $z_0 = i$ represent:

$$f(z) = \frac{1}{z-i}\,\frac{e^{iz}}{(z+i)(z^2+4)},$$

and for $z_0 = 2i$:

$$f(z) = \frac{1}{z-2i}\,\frac{e^{iz}}{(z+2i)(z^2+1)},$$

so

$$\operatorname{Res}(f,i) = \frac{e^{-1}}{2i\cdot 3} = \frac{1}{6i}e^{-1}, \quad \operatorname{Res}(f,2i) = \frac{e^{-2}}{4i\cdot(-3)} = -\frac{1}{12i}e^{-2}.$$

Consequently,

$$\int_{[-R,R]\cup\gamma_R} f(z)dz = 2\pi i\Big(\frac{1}{6i}e^{-1} - \frac{1}{12i}e^{-2}\Big) = \frac{\pi}{6}\Big(2e^{-1} - e^{-2}\Big).$$

It remains to prove that

$$\int_{\gamma_R} f(z)\,dz \to 0, \quad R\to\infty.$$

Use the SIB:

$$\Big|\int_{\gamma_R} f(z)\,dz\Big| \le \max_{|z|=R} |f(z)| \cdot \pi R.$$

Estimate the maximum:

$$\max_{|z|=R} |f(z)| \le \max_{|z|=R, \operatorname{Im} z\ge 0} \frac{|e^{iz}|}{(|z|^2-4)(|z|^2-1)} = \frac{1}{(R^2-4)(R^2-1)} \le \frac{C}{R^4}, R\to\infty.$$

Therefore,

$$I = \lim_{R\to\infty}\Big(I_R + \int_{\gamma_R} f(z)dz\Big) = \frac{\pi}{6}\Big(2e^{-1} - e^{-2}\Big).$$

6.5 Jordan's Lemma

Want to study integrals of the form

$$I_R = \int_{\gamma_R} e^{i\alpha z} f(z)\,dz,$$

where

$$\gamma_R(t) = Re^{it}, \quad t\in[0,\pi].$$

Lemma 6.9. *Suppose that f is continuous on $\{z : \operatorname{Im} z \ge 0, |z| > r\}$, with some $r > 0$, and that*

$$M(R) := \max_{|z|=R, \operatorname{Im} z\ge 0} |f(z)| \to 0, \quad R\to\infty.$$

Then for any $\alpha > 0$, we have $I_R \to 0, R\to\infty$.

If $\alpha < 0$, then instead of γ_R we take the path in the lower half-plane, i.e. $Re^{it}, t \in [-\pi, 0]$.

Proof. Write:

$$I_R = iR \int_0^\pi e^{i\alpha R(\cos t + i \sin t)} e^{it} f(Re^{it})\, dt.$$

Therefore,

$$\begin{aligned} |I_R| &\le R \int_0^\pi e^{-\alpha R \sin t} |f(Re^{it})|\, dt \le M(R) R \int_0^\pi e^{-\alpha R \sin t}\, dt \\ &\le 2M(R) R \int_0^{\pi/2} e^{-\alpha R \sin t}\, dt. \end{aligned}$$

Observe:

$$\frac{2t}{\pi} \le \sin t \le t, \quad t \in [0, \pi/2].$$

Thus

$$\begin{aligned} |I_R| &\le 2M(R) R \int_0^{\pi/2} e^{-\alpha R \frac{2t}{\pi}}\, dt \le 2M(R) R \int_0^\infty e^{-\alpha R \frac{2t}{\pi}}\, dt \\ &= 2M(R) R \cdot \frac{\pi}{2\alpha R} = \frac{\pi}{\alpha} M(R) \to 0, \quad R \to \infty, \end{aligned}$$

as claimed. □

Example. Find

$$\int_{-\infty}^\infty \frac{x \sin x}{x^2 + a^2}\, dx, \quad a > 0.$$

Rewrite as:

$$\operatorname{Im} I, \quad I = \int_{-\infty}^\infty f(x)\, dx, \quad f(z) = \frac{z e^{iz}}{z^2 + a^2}.$$

As before,

$$I = \lim_{R \to \infty} I_R, \quad I_R = \int_{-R}^R f(x)\, dx.$$

Let $\gamma_R(t) = Re^{it},\ t \in [0, \pi]$, and find the integral

$$\int_{[-R,R] \cup \gamma_R} f(z) dz = 2\pi i \operatorname{Res}(f, ia).$$

Find the residue by rewriting

$$f(z) = \frac{1}{z - ia} \cdot \frac{ze^{iz}}{z + ia},$$

so

$$\operatorname{Res}(f, ia) = \frac{iae^{-a}}{2ia} = \frac{1}{2}\, e^{-a},$$

and hence

$$\int_{[-R,R]\cup\gamma_R} f(z)dz = \pi i\, e^{-a}.$$

It remains to show that

$$\int_{\gamma_R} f(z)dz \to 0, \quad R \to \infty.$$

In order to apply Jordan's Lemma estimate:

$$M(R) = \max_{|z|=R, \operatorname{Im} z \geq 0} \left| \frac{z}{z^2 + a^2} \right| \leq \frac{R}{R^2 - a^2} \to 0, \quad R \to \infty.$$

Therefore,

$$I = \lim_{R\to\infty} \left[\int_{[-R,R]\cup\gamma_R} f(z)\, dz - \int_{\gamma_R} f(z)\, dz \right] = \pi i e^{-a},$$

and hence

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + a^2}\, dx = \operatorname{Im} I = \pi e^{-a}.$$

6.6 Zeroes

6.6.1 Unique continuation

Definition 6.10. If $H \in H(\Omega)$, then a point z_0 is called a zero of f if $f(z_0) = 0$.

A zero is said to have order $m = 1, 2, \dots,$ if the Taylor expansion of f at z_0 has the form

$$\begin{aligned} f(z) &= \sum_{k=m}^{\infty} a_k (z - z_0)^k, \\ &= a_m (z - z_0)^m + a_{m+1} (z - z_0)^{m+1} + \dots, \qquad \text{where} \quad a_m \neq 0. \end{aligned}$$

In other words, $a_0 = a_1 = \cdots = a_{m-1} = 0$.

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Set of zeroes: $Z(f)$.

Proposition 6.11. *Let $f \in H(\Omega)$. Then $z_0 \in \Omega$ is a zero of order m iff there exists a radius $r > 0$ such that for all $z \in D(z_0, r)$ the function f can be represented as*

$$f(z) = (z - z_0)^m g(z),$$

where $g \in H(D(z_0, r))$ and $g(z) \neq 0$ for all $z \in D(z_0, r)$.

Proof. Assume z_0 is a zero of order m, so that in some disk around z_0 we have

$$\begin{aligned} f(z) &= \sum_{k=m}^{\infty} a_k (z - z_0)^k = (z - z_0)^m \sum_{k=m}^{\infty} a_k (z - z_0)^{k-m} \\ &= (z - z_0)^m \sum_{k=0}^{\infty} a_{k+m} (z - z_0)^k = (z - z_0)^m g(z), \end{aligned}$$

where

$$g(z) = \sum_{k=0}^{\infty} a_{k+m} (z - z_0)^k.$$

As $a_m \neq 0$, we have $g(z_0) = a_m \neq 0$. Since g is continuous, and $g(z_0) \neq 0$, there is a radius $r > 0$ such that $g(z) \neq 0$ for all $z \in D(z_0, r)$.

The converse is proved by reversing the argument. □

Corollary 6.12. *A function f has a zero of order m at z_0 iff $1/f$ has a pole of order m at z_0.*

Proof. Let z_0 be a zero, so

$$f(z) = (z - z_0)^m g(z), \quad \text{with} \quad g(z) \neq 0, \quad z \in D(z_0, r).$$

Therefore

$$\frac{1}{f(z)} = \frac{1}{(z - z_0)^m} \frac{1}{g(z)} = \frac{h(z)}{(z - z_0)^m}, \qquad h(z) = \frac{1}{g(z)},$$

where $h(z_0) \neq 0$. Use Proposition 6.8.

The converse is proved by reversing the argument. □

For the next theorem recall the definition of an accumulation point. Let $S \subset \mathbb{C}$ be a set. We say that $a \in \mathbb{C}$ is its accumulation point if there is a sequence $a_n \in S$, $n = 1, 2, \ldots$, such that $a_n \neq a$ and $a_n \to a$ as $n \to \infty$.

Examples: $S = \{n^{-1}, n \in \mathbb{N}\}$, segment $[0, 2]$, disk $D(0, 1)$.

Theorem 6.13. *Let $H(\Omega)$. Suppose that the set of zeroes $Z(f)$ has an accumulation point in Ω. Then $f = 0$ in Ω.*

Proof. Let $z_0 \in \Omega$ an accumulation point of zeroes, i.e. there is a sequence $z_j \in Z(f)$ such that $z_j \to z_0$, $j \to \infty$. By continuity of f, we have $f(z_0) = 0$. Let $\delta > 0$ be such that $D(z_0, \delta) \subset \Omega$. Assume that $f \not\equiv 0$ in $D(z_0, \delta)$. Therefore its Taylor series

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

has some non-zero coeffricients a_k. Let $a_m, m \geq 1$ be the first non-zero coefficient, so that z_0 is a zero of order $m \geq 1$. By Proposition 6.11,

$$f(z) = (z - z_0)^m g(z), \quad g(z) \neq 0, z \in D(z_0, r),$$

and hence $f(z) \neq 0$ for all $z \in D'(z_0, r)$. This contradicts the fact that there is a sequence of zeroes converging to z_0. Thus $f \equiv 0$ on $D(z_0, \delta)$.

Extend this result to all of Ω. We want to prove that $f(w) = 0$ for all $w \in \Omega$. Pick a w and connect z_0 with w by a polygonal path Γ. Cover this path with disks $D(w_n, \delta)$, $n = 1, 2, \ldots, N$, of appropriately small radius $\delta > 0$ in the following way:

$$z_0 = w_1, w = w_N,$$
$$D(w_n, \delta) \subset \Omega, \ \Gamma \subset \cup_n D(w_n, \delta), \ w_{j+1} \in D(w_j, \delta).$$

Induction base: we already know that $f(z) = 0$ for all $z \in D(z_0, \delta)$.

Assume that $f(z) = 0$ for $z \in D(w_k, \delta)$ and prove that $f(z) = 0$ for $z \in D(w_{k+1}, \delta)$. Indeed, since $w_{k+1} \in D(w_k, \delta)$, the point w_{k+1} is an accumulation point of zeroes, and hence by the first part of the proof, $f(z) = 0$ in $D(w_{k+1}, \delta)$.

By induction, f is zero in every disk, and hence $f(w) = f(w_N) = 0$, as required. □

Theorem 6.14 (Unique continuation property)**.** *Suppose that $f, g \in H(\Omega)$ and that $f(z) = g(z)$ on a set $S \subset \Omega$ that has an accumulation point in Ω. Then $f(z) = g(z)$ for all $z \in \Omega$.*

Proof. Let $h = f - g$, so $S \subset Z(h)$. By Theorem 6.13, $h = 0$ for all $z \in \Omega$. □

Example. Does there exist a function $f \in H(D(0, 1))$ such that

$$f(1/n) = 1/n^2, \tag{11}$$

for all $n = 2, 3, \ldots$. If Yes, then how many such functions are there?

Answer: $f(z) = z^2$. Indeed, f satisfies (11) and the sequence $1/n$ converges to zero. Thus there is only one such functions.

6.6.2 Counting zeroes and poles

Lemma 6.15. *Suppose $f \in H(D(0,R))$ and that z_0 is a zero of order m. Then*

$$\mathrm{Res}\,(f' f^{-1}, z_0) = m.$$

Proof. By Proposition 6.11,

$$f(z) = (z - z_0)^m g(z),\ \ g(z) \neq 0,\ \ z \in D(z_0, r), r < R.$$

Thus

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{m(z - z_0)^{m-1} g(z) + (z - z_0)^m g'(z)}{(z - z_0)^m g(z)} \\ &= \frac{m}{z - z_0} + \frac{g'(z)}{g(z)}.\end{aligned}$$

This means that the function f'/f has a simple pole at z_0 with residue m, and hence

$$\mathrm{Res}\,(f' f^{-1}, z_0) = m.$$

□

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Lemma 6.16. *Suppose $f \in H(D'(0,R))$ and that z_0 is a pole of order p. Then*

$$\mathrm{Res}\,(f' f^{-1}, z_0) = -p.$$

Proof. By Proposition 6.8,

$$f(z) = (z - z_0)^{-p} h(z), \quad h(z_0) \neq 0.$$

Let $r < R$ be a radius such that $h(z) \neq 0$ for all $z \in D(z_0, r)$. Thus

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{-p(z - w_z)^{-p-1} h(z) + (z - z_0)^{-p} h'(z)}{(z - z_0)^{-p} h(z)} \\ &= \frac{-p}{z - z_0} + \frac{h'(z)}{h(z)},\ z \in D(z_0, r).\end{aligned}$$

This means that the function f'/f has a simple pole at z_0 with residue $-p$, and hence

$$\mathrm{Res}\,(f' f^{-1}, z_0) = -p.$$

□

Theorem 6.17 (The argument principle)**.** *Let f be holomorphic on Ω except for some poles. Suppose that $\gamma \subset \Omega$ is a contour such that* $\operatorname{Int}\gamma \subset \Omega$*, and that there are no zeroes or poles on γ. Denote by N and P the number of zeroes and poles respectively in* $\operatorname{Int}\gamma$*, counting their multiplicity. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = N - P.$$

Proof. Since f is not a zero function, it can have only isolated zeroes in Ω. Let $z_1, z_2, \dots, z_k$ and $w_1, w_2, \dots, w_l$ be the zeroes and poles inside γ of orders

$$m_1, m_2, \dots, m_k \quad \text{and} \quad p_1, p_2, \dots, p_l$$

respectively, so that

$$\sum_{j=1}^k m_j = N, \quad \sum_{j=1}^l p_j = P.$$

Due to the Cauchy Residue Theorem,

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{j=1}^k \operatorname{Res}(f'f^{-1}, z_j) + \sum_{j=1}^l \operatorname{Res}(f'f^{-1}, w_j).$$

By Lemmata 6.15 and 6.16,

$$\operatorname{Res}(f'f^{-1}, z_j) = m_j, \quad \operatorname{Res}(f'f^{-1}, w_j) = -p_j.$$

Collecting contributions from all zeroes and poles we arrive at the claimed result. □

Remark. Rewrite the integral in the theorem:

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi i}\int_a^b \frac{f'(\gamma(t))}{f(\gamma(t))}\gamma'(t)\,dt.$$

Let $\beta(t) = f(\gamma(t))$ be a new path, so that the last integral coincides with

$$\frac{1}{2\pi i}\int_a^b \frac{1}{\beta(t)} f'(\gamma(t))\gamma'(t)\,dt = \frac{1}{2\pi i}\int_a^b \frac{1}{\beta(t)}\beta'(t)\,dt = \frac{1}{2\pi i}\int_{f(\gamma)} \frac{1}{w}\,dw.$$

Thus the integral counts the times that the path $f(\gamma)$ loops around 0.

Theorem 6.18 (Rouché's Theorem)**.** *Let $f, g \in H(\Omega)$, and let γ be a positively oriented contour such that $\overline{\operatorname{Int}\gamma} \subset \Omega$ and such that f has no zeroes on γ and $|g(z)| < |f(z)|$ for all $z \in \gamma$. Then f and $f + g$ have the same number of zeroes inside* $\operatorname{Int}\gamma$*.*

Note that the conditions are imposed on the contour, but the conclusions concern the interior of the contour!!!

Proof. Let $t \in [0, 1]$. Denote by $N(t)$ the number of zeroes of $f(z) + tg(z)$ inside γ. Since $|f(z)| > |g(z)|$ on γ, we have

$$|f(z) + tg(z)| \geq |f(z)| - t|g(z)| \geq |f(z)| - |g(z)| \geq d > 0, \quad \forall z \in \gamma,\ \forall t \in [0, 1]. \tag{12}$$

By Theorem 6.17, $N(t)$ is given by

$$N(t) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z) + tg'(z)}{f(z) + tg(z)} dz.$$

This is an integer number, depending on the parameter t. Therefore, if we show that this function is continuous, this would mean that $N(t) = const$ for $t \in [0, 1]$, thus proving the required result. To this end, write:

$$N(t) - N(s) = \frac{t - s}{2\pi i} \int_\gamma \frac{(g'f - f'g)(z)}{(f + tg)(z)(f + sg)(z)} dz.$$

Let us estimate the integral using again Theorem 5.7(SIB). Taking

$$M = \max_{z \in \gamma^*} |(g'f - f'g)(z)|,$$

and applying (12), we obtain:

$$|N(t) - N(s)| \leq \frac{|t - s|}{2\pi} \frac{M}{d^2} L(\gamma) \to 0, \text{ as } |t - s| \to 0.$$

Therefore N is indeed continuous on $[0, 1]$. This completes the proof. □

Example 6.19. Prove that the polynomial $z^4 + 100z + 13$ has exactly three zeroes inside the annulus $1 < |z| < 10$.

Clearly, the total number of roots is four. For $|z| = 1$ write $f_1(z) = 100z$, $g_1(z) = z^4 + 13$. Observe that $|f_1(z)| = 100 > 14 \geq |g_1(z)|$. Therefore the functions $f = f_1 + g_1$ and f_1 have the same number of roots inside $S(0, 1)$, that is one.

For $|z| = 10$ write $f_2(z) = z^4$, $g_2(z) = 100z + 13$. Then $|f_2(z)| = 10^4 > 1013 \geq |g_2(z)|$ for $|z| = 10$. Consequently, the number of roots of f and f_2 inside $S(0, 10)$ is the same, that is four. Thus the annular region $1 < |z| < 10$ contains three roots.