Sheet 1, solutions

1. Express each of the following in the form $x + iy$, $x, y \in \mathbb{R}$:

$$e^{4\pi i/3}; \quad e^{5\pi i/6}; \quad \frac{1}{(1+i)^3}; \quad \frac{1+2i}{2+i}.$$

Solution. Elementary.

2. Is it true that

$$\big||z| - |w|\big| \le |z + w|$$

for all $z, w \in \mathbb{C}$? Prove your answer.

Solution. An immediate consequence of the triangle inequality.

3. Describe geometrically the sets of points $z \in \mathbb{C}$ defined by the following relations:

(a) $|z - z_1| = |z - z_2|$ where $z_1, z_2 \in \mathbb{C}$;
(b) $1/z = \overline{z}$;
(c) $\mathrm{Re}(z) = 3$;
(d) $\mathrm{Re}(z) > c$, (respestively $\ge c$) where $c \in \mathbb{R}$;
(e) $\mathrm{Re}(az + b) > 0$, where $a, b \in \mathbb{C}$;
(f) $\mathrm{Im}(z) = c$ with $c \in \mathbb{R}$.

Solution.

(a) Write

$$z = \frac{z_1 + z_2}{2} + w,$$

so that

$$|w + a| = |w - a|, \quad \text{where} \quad a = \frac{z_2 - z_1}{2}.$$

This equation is equivalent to $\mathrm{Re}(wa) = 0$, or, which is the same, to $\cos(\theta + \phi) = 0$, where $\theta = \arg w$ and $\phi = \arg a$. This means that

$$\theta = -\phi \pm \frac{\pi}{2}.$$

Geometrically speaking, the vector w is orthogonal to a. This means that z lies on the straigh line orthogonal to $z_1 - z_2$ and passing through $\frac{z_1+z_2}{2}$.

(e) The inequality $\mathrm{Re}(az + b) > 0$ is equivalent to $\mathrm{Re}(e^{i\phi}z + b) > 0$ where $\phi = \arg a$. The inequality $\mathrm{Re}\, w > -\mathrm{Re}\, b$ defines a half-plane Π on the right of the vertical line $\mathrm{Re}\, w = -\mathrm{Re}\, b$. Thus to satisfy the original condition one rotates the plane Π by the angle $-\phi$.

4. (a) Find the square roots of $4 + 4\sqrt{3}i$.
   (b) Solve the quadratic equation $z^2 - (3 + 3i)z + 5i = 0$.
   (c) Find the value of $1 + i + i^2 + \cdots + i^n$. The answer depends on $n \mod 4$.

Solution.

(a) We find the modulus and argument of $4 + 4\sqrt{3}i$.

$$|4 + 4\sqrt{3}i| = \sqrt{4^2 + 4^2 \cdot 3} = \sqrt{16 + 16 \cdot 3} = \sqrt{64} = 8,$$

$$\tan t = y/x = \sqrt{3} \implies t = \pi/3.$$

So the polar form of $4 + 4\sqrt{3}i$ is

$$4 + 4\sqrt{3}i = 8(\cos(\pi/3) + i\sin(\pi/3)).$$

By De Moivre's formula the two square roots are

$$z_1 = \sqrt{8}\left(\cos\frac{\pi/3}{2} + i\sin\frac{\pi/3}{2}\right) = 2\sqrt{2}(\cos(\pi/6) + i\sin(\pi/6)) = \sqrt{6} + i\sqrt{2},$$

$$z_2 = \sqrt{8}\left(\cos\frac{\pi/3 + 2\pi}{2} + i\sin\frac{\pi/3 + 2\pi}{2}\right) = 2\sqrt{2}(\cos(7\pi/6) + i\sin(7\pi/6)) = -\sqrt{6} - i\sqrt{2}.$$

(b) We use the quadratic formula

$$z_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{3 + 3i \pm \sqrt{(3 + 3i)^2 - 20i}}{2}.$$

Since $(3 + 3i)^2 - 20i = 9 - 9 + 18i - 20i = -2i$, we need to find the square roots of $-2i = 2(\cos(3\pi/2) + i\sin(3\pi/2))$. We use De Moivre's formula again. The roots are:

$$\sqrt{2}(\cos(3\pi/4) + i\sin(3\pi/4)) = -1 + i, \quad \sqrt{2}(\cos(3\pi/4 + \pi) + i\sin(3\pi/4 + \pi)) = 1 - i.$$

Finally we get

$$z_{1,2} = \frac{3 + 3i \pm (-1 + i)}{2} = 1 + 2i \text{ or } 2 + i.$$

(c) We know that

$$i^k = \begin{cases} 1, & k \equiv 0 \pmod 4, \\ i, & k \equiv 1 \pmod 4, \\ -1, & k \equiv 2 \pmod 4, \\ -i, & k \equiv 3 \pmod 4. \end{cases}$$

We use the sum of the geometric progression to see that

$$1 + i + i^2 + \cdots + i^n = \frac{1 - i^{n+1}}{1 - i} = \frac{1 - i \cdot i^n}{1 - i} = \begin{cases} \frac{1-i}{1-i} = 1, & n \equiv 0 \pmod 4, \\ \frac{1-i \cdot i}{1-i} = \frac{2}{1-i} = 1 + i, & n \equiv 1 \pmod 4, \\ \frac{1+i}{1-i} = i, & n \equiv 2 \pmod 4, \\ \frac{1-i\cdot(-i)}{1-i} = 0, & n \equiv 3 \pmod 4. \end{cases}$$

5. Let $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$ with n a natural number. Show that

$$(1 - \omega)(1 - \omega^2)(1 - \omega^3) \cdots (1 - \omega^{n-1}) = n.$$

Solution. By De Moivre's theorem, $\omega^n = \cos(2\pi) + i\sin(2\pi) = 1$, i.e. ω is an n-th root of 1. By the same theorem, the other n-th roots of 1 are its powers $\omega^2, \omega^3, \ldots \omega^{n-1}, \omega^n = 1$. They all satisfy the n-th degree polynomial equation $z^n - 1 = 0$. We get the factorisation

$$z^n - 1 = (z - 1)(z - \omega)(z - \omega^2) \cdots (z - \omega^{n-1}).$$

This gives

$$\frac{z^n - 1}{z - 1} = (z - \omega)(z - \omega^2) \cdots (z - \omega^{n-1}),$$

and, since

$$z^n - 1 = (z-1)(z^{n-1} + z^{n-2} + \cdots + z + 1),$$

we get

$$z^{n-1} + z^{n-2} + \cdots + z + 1 = (z-\omega)(z-\omega^2)\cdots(z-\omega^{n-1}).$$

Substitute $z = 1$ to get the required result. There are n terms on the left-hand side.

6. Consider the equation for complex number z:

$$|z-4| = 2|z-1|.$$

(a) Show that such z describe the circle centered at $(0,0)$ with radius 2.

(b) Let

$$w = \frac{2z_1}{z_2} + \frac{2z_2}{z_1},$$

where z_1 and z_2 on the circle of (a). Show that w is real and $-4 \le w \le 4$.

(c) Assume that for the same number w we have $w = -4$. What relation exists between z_1 and z_2? Show that the triangle with vertices z_1, z_2 and $z_3 = 2iz_1$ is an isosceles triangle.

Solution.

(a) We have

$$|z-4| = 2|z-1| \Leftrightarrow |z-4|^2 = 4|z-1|^2 \Leftrightarrow (z-4)(\bar{z}-4) = 4(z-1)(\bar{z}-1)$$
$$\Leftrightarrow z\bar{z} - 4\bar{z} - 4z + 16 = 4(z\bar{z} - z - \bar{z} + 1) \Leftrightarrow |z|^2 - 4(z+\bar{z}) + 16 = 4|z|^2 - 4(z+\bar{z}) + 4$$
$$\Leftrightarrow 16 - 4 = 4|z|^2 - |z|^2 = 3|z|^2 \Leftrightarrow 4 = |z|^2 \Leftrightarrow |z| = 2.$$

(b) We have

$$\bar{w} = \frac{2\bar{z}_1}{\bar{z}_2} + \frac{2\bar{z}_2}{\bar{z}_1}.$$

We use that $|z_1|^2 = z_1\bar{z}_1 = 4 = z_2\bar{z}_2 = |z_2|^2$. This gives

$$\bar{w} = \frac{2\cdot 4/z_1}{4/z_2} + \frac{2\cdot 4/z_2}{4/z_1} = \frac{2z_2}{z_1} + \frac{2z_1}{z_2} = w.$$

This means that $\bar{w} = w$, which means that w is real. Moreover, using the triangle inequality

$$|w| = \left|\frac{2z_1}{z_2} + \frac{2z_2}{z_1}\right| \le \frac{2|z_1|}{|z_2|} + \frac{2|z_2|}{|z_1|} = 2 + 2 = 4, \tag{1}$$

as $|z_1| = |z_2| = 2$. This implies for the real number w that $-4 \le w \le 4$.

(c) To get $w = -4$, we must have $|w| = 4$, which gives in (1) equality of the left hand-side and the right-hand side. This means that we get = in the application of the triangle inequality. We investigate the use of the triangle inequality in (b). A careful reading of the proof of the triangle inequality $|z + w| \le |z| + |w|$ show that equality holds if $\mathrm{Re}(z\bar{w}) \ge 0$. If w is nonzero we get

$$z\bar{w} = \frac{z}{w}|w|^2 \ge 0,$$

which means that z/w is a nonnegative real number. In our case $z = 2z_1/z_2$ and $w = 2z_2/z_1$. This means

$$\frac{z_1/z_2}{z_2/z_1} \ge 0 \Leftrightarrow \frac{z_1^2}{z_2^2} \ge 0.$$

Setting $z_1 = (\cos t + i \sin t)z_2$, which is possible as they both lie of the circle of radius 2, we get $z_1^2 = (\cos(2t) + i\sin(2t))z_2^2$, which implies that $\cos(2t) + i\sin(2t) \ge 0$. There are only two such t: $t = 0$ and $t = \pi$. But $t = 0$ means $z_1 = z_2$, which gives $w = 4$, which is impossible. Therefore, $z_1 = (\cos\pi + i\sin\pi)z_2 = -z_2$.

The sides of the triangle have lengths

$$|z_1 - z_2| = |2z_1| = 4, |z_1 - 2iz_1| = |1 - 2i||z_1| = \sqrt{5} \cdot 2,$$
$$|z_2 - 2iz_1| = |- z_1 - 2iz_1| = |- 1 - 2i||z_1| = \sqrt{5} \cdot 2.$$

It follows that the triangle has two sides of equal length and it is isosceles.

7. Prove the Lagrange identity

$$\left|\sum_{i=1}^n a_i b_i\right|^2 = \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2 - \sum_{1\le i<j\le n} |a_i\bar{b}_j - a_j\bar{b}_i|^2.$$

Deduce the Cauchy–Schwarz inequality for the complex numbers a_i and b_i:

$$\left|\sum_{i=1}^n a_i b_i\right|^2 \le \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2.$$

Solution. We need to show the equivalent formula

$$\left|\sum_{i=1}^n a_i b_i\right|^2 + \sum_{1\le i<j\le n} |a_i\bar{b}_j - a_j\bar{b}_i|^2 = \sum_{i=1}^n |a_i|^2 \sum_{i=1}^n |b_i|^2.$$

We expand out using $|z|^2 = z\bar{z}$ to get

$$\left(\sum_{i=1}^{n} a_i b_i\right)\left(\sum_{j=1}^{n} \bar{a}_j \bar{b}_j\right) + \sum_{1\le i<j\le n} (a_i\bar{b}_j - a_j\bar{b}_i)(\bar{a}_i b_j - \bar{a}_j b_i)$$

$$= \sum_{i,j=1}^{n} a_i b_i \bar{a}_j \bar{b}_j + \sum_{i<j}(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2 - a_i\bar{a}_j b_i\bar{b}_j - a_j\bar{a}_i b_j\bar{b}_i)$$

$$= \sum_{i\neq j} a_i b_i \bar{a}_j \bar{b}_j + \sum_{i=j=1}^{n} |a_i|^2|b_i|^2 + \sum_{i<j}(|a_i|^2|b_j|^2 + |a_j|^2|b_i|^2) - \sum_{i<j}(a_i\bar{a}_j b_i\bar{b}_j + a_j\bar{a}_i b_j\bar{b}_i).$$

The first and the last sum are equal and, therefore, cancel. This can be seen as follows: In the last sum we have the condition $i < j$ while in the first only $i \neq j$. A pair (i, j) of unequal integers has either $i < j$ or $j < i$. In the second case $a_i b_i \bar{a}_j \bar{b}_j = a_{j'} b_{j'} \bar{a}_{i'} \bar{b}_{i'}$ with $j = i' < j' = i$, so we get the term $a_j b_j \bar{a}_i \bar{b}_i$ with $i < j$. The two summands in the middle give exactly

$$\sum_{i=1}^{n} |a_i|^2 \sum_{j=1}^{n} |b_j|^2$$

by the distributive law and the same thinking about $i \neq j$ vs $i < j$ and $j < i$.

The Cauchy–Schwarz inequality follows by noticing that

$$\sum_{1\le i<j\le n} |a_i\bar{b}_j - a_j\bar{b}_i|^2 \geq 0.$$

Therefore,

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 \leq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2.$$

8. Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1,$$

if $|a| < 1$ and $|b| < 1$.

Solution. We have

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1 \iff |a-b| < |1-\bar{a}b| \iff |a-b|^2 < |1-\bar{a}b|^2$$
$$\iff |a|^2 + |b|^2 - 2\,\mathrm{Re}(a\bar{b}) < 1 + |\bar{a}b|^2 - 2\,\mathrm{Re}(\bar{a}b)$$

(here we notice that $\mathrm{Re}(\bar{a}b) = \mathrm{Re}(a\bar{b})$ as they are mutually conjugate numbers)

$$\Leftrightarrow |a|^2 + |b|^2 < 1 + |a|^2|b|^2 \Leftrightarrow 0 < 1 + |a|^2|b|^2 - |a|^2 - |b|^2 = (1 - |a|^2)(1 - |b|^2).$$

The assumption $|a| < 1$ and $|b| < 1$ gives the result.

9. (a) Express $\cos(5t)$ in terms of $\cos(t)$. Express $\sin(5t)$ in terms of $\sin(t)$.
   (b) Write an expression in terms of square roots for $\sin(\pi/5)$.

Solution.

(a) By de Moivre's formula

$$(\cos t + i \sin t)^n = \cos(nt) + i \sin(nt)$$

for $n = 5$ we get by the binomial theorem

$$(\cos t + i \sin t)^5 = \cos^5 t + i5\cos^4 \sin t - 10\cos^3 t \sin^2 t - i10\cos^2 t \sin^3 t + 5\cos t \sin^4 t + i \sin^5 t.$$

$$\cos(5t) = \mathrm{Re}((\cos t + i \sin t)^5) = \cos^5 t - 10\cos^3 t \sin^2 t + 5\cos t \sin^4 t$$

$$= \cos^5 t - 10\cos^3 t(1 - \cos^2 t) + 5\cos t(1 - \cos^2 t)^2 = 16\cos^5 t - 20\cos^3 t + 5\cos t$$

$$\sin(5t) = ((\cos t + i \sin t)^5) = 5\cos^4 t \sin t - 10\cos^2 t \sin^3 t + \sin^5 t$$

$$= 5\sin t(1 - \sin^2 t)^2 - 10(1 - \sin^2 t)\sin^3 t + \sin^5 t = 16\sin^5 t - 20\sin^3 t + 5\sin t.$$

(b) We use the result for $\sin(5t)$ with $t = \pi/5$. This gives

$$0 = \sin(\pi) = 16\sin^5 t - 20\sin^3 t + 5\sin t = \sin t(16\sin^4 t - 20\sin^2 t + 5).$$

Since $\sin t \neq 0$ for $t = \pi/5$ we get the equation

$$16\sin^4 t - 20\sin^2 t + 5 = 0.$$

This is an equation in $\sin t$ of degree 4, but with a bit of care we see that since only even exponents appear, it is a quadratic equation in $x = \sin^2 t$. We get

$$16x^2 - 20x + 5 = 0 \Leftrightarrow x = \frac{20 \pm \sqrt{400 - 4 \cdot 16 \cdot 5}}{32} = \frac{20 \pm \sqrt{80}}{32} = \frac{5 \pm \sqrt{5}}{8}.$$

There are two square roots for each of these solutions giving a total of 4 values for x. However, since $x = \sin^2(t) > 0$ we reject the negative solutions. There are two options:

$$\sin(\pi/5) = \sqrt{\frac{5 \pm \sqrt{5}}{8}}.$$

However, since $(5+\sqrt{5})/8 > 7/8 > 1/2$, the first would give $\sin(\pi/5) > 1/\sqrt{2} = \sin(\pi/4)$. This is false as $\sin u$ is increasing on $[0, \pi/2]$. Therefore,

$$\sin(\pi/5) = \sqrt{\frac{5-\sqrt{5}}{8}}.$$

10. Let z_1, z_2 and z_3 be complex numbers satisfying $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 1$ and $z_1 z_2 z_3 = 1$. Show that

$$\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = 1$$

and compute z_1, z_2, z_3.

Solution. We have $|z_1|^2 = z_1\bar{z}_1 = 1$, $|z_2|^2 = z_2\bar{z}_2 = 1$, $|z_3|^2 = z_3\bar{z}_3 = 1$. These imply that $1/z_1 = \bar{z}_1$, $1/z_2 = \bar{z}_2$, $1/z_3 = \bar{z}_3$. So we get

$$\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = \bar{z}_1 + \bar{z}_2 + \bar{z}_3 = \overline{z_1 + z_2 + z_3} = \bar{1} = 1.$$

Moreover,

$$1 = \frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = \frac{z_2z_3 + z_1z_3 + z_1z_2}{z_1z_2z_3} = z_2z_3 + z_1z_3 + z_1z_2.$$

We now have found expressions for all the elementary symmetric polynomials for z_1, z_2 and z_3. Viète's formulas give us the relation with the polynomial with these three roots. This can be seen in a more elementary way as follows:

$$(z - z_1)(z - z_2)(z - z_3) = z^3 - (z_1 + z_2 + z_3)z^2 + (z_1z_2 + z_1z_3 + z_2z_3)z - z_1z_2z_3.$$

In our case we get the equation

$$z^3 - z^2 + z - 1 = 0 \Leftrightarrow z^2(z-1) + (z-1) = 0 \Leftrightarrow (z^2+1)(z-1) = 0 \Leftrightarrow z = 1, i, -i.$$

There are 6 solutions in terms of z_1, z_2 and z_3, which we get by permuting the three roots $1, i, -i$.

11. Let $ABCD$ be a trapezium with parallel sides AB and CD. Show that the midpoints of the diagonals AC and BD are endpoints of a segment parallel to the sides AB and CD and of length half the difference of AB and CD.

You must use complex numbers, not geometric arguments.

Solution. We recall that if A and B correspond to the complex numbers z and w then the vector AB corresponds to $w - z$. In our case we translate $ABCD$ so that D is at the origin and let C correspond to ζ. The diagonal DB as a vector corresponds to w and its midpoint corresponds to $w/2$. The midpoint of the diagonal CA corresponds to $(z + \zeta)/2$. The segment of the midpoints corresponds to the complex number

$$\frac{z+\zeta}{2} - \frac{w}{2}.$$

Since AB is parallel to DC the corresponding vector are multiple of each other $\lambda AB = DC$, $\lambda > 0$, which means the same for the complex numbers:

$$\lambda(w - z) = \zeta.$$

This gives for the midpoint segment

$$\frac{z - w + \lambda(w - z)}{2} = \frac{-1 + \lambda}{2}(w - z).$$

This is a real multiple of $w - z$, which means that the corresponding vectors are parallel, i.e. the midpoint segment is parallel to AB, as claimed.

Furthermore, the length of the midpoint segment is

$$\frac{1}{2}|\lambda - 1|\,|w - z|. \tag{2}$$

The difference of the lengths of the parallel sides is $|w - z| - |\zeta|$ or $|\zeta| - |w - z|$, i.e.

$$||w - z| - |\zeta|| = ||w - z| - \lambda|w - z|| = |1 - \lambda||w - z|,$$

which exactly twice the number in (2), as required.

12. Prove that the following sets are open:

    (a) The left half-plane $\{z : \operatorname{Re} z < 0\}$;

    (b) The open disk $D(z_0, r)$ for any $z_0 \in \mathbb{C}$ and $r > 0$;

    (c) The set $\{z \in \mathbb{C} : \operatorname{Re} z \cdot \operatorname{Im} z > 0\}$.

Solution. The first two sets are handled as in the lectures.

Let us prove (c). Denote $\mathcal{D} = \{z \in \mathbb{C} : \operatorname{Re} z \cdot \operatorname{Im} z > 0\}$. Let $z = x + iy \in \mathcal{D}$. We need to prove that there exists an $\varepsilon > 0$ such that $D(z, \varepsilon) \subset \mathcal{D}$. Let

$$\varepsilon = \min\{|x|, |y|\}.$$

Then for each $w = t + is \in D(z, \varepsilon)$ we have

$$|t - x|^2 + |s - y|^2 < \varepsilon^2,$$

so that $|t - x| < \varepsilon \leq |x|, |s - y| < \varepsilon \leq |y|$. This implies that

$$\frac{|t - x|}{|x|} < 1, \quad \frac{|s - y|}{|y|} < 1.$$

Consequently,

$$\frac{t}{x} = 1 + \frac{t - x}{x} \geq 1 - \frac{|t - x|}{|x|} > 0$$

and

$$\frac{s}{y} = 1 + \frac{s - y}{y} \geq 1 - \frac{|s - y|}{|y|} > 0.$$

As a consequence,

$$ts = xy\frac{ts}{xy} > 0,$$

and hence $w + t + is \in \mathcal{D}$, as claimed.