MATH0013 Complex Analysis (Year 2019/20)
Questions and solutions

1. (a) Assume that v is a harmonic conjugate of the function u in $\mathbb{C}$. Find a harmonic conjugate of the function v.

   (b) Find the radii of convergence for the following power series:

$$\sum_{n=0}^{\infty} n^{59}(z+i)^n, \quad \sum_{n=0}^{\infty} n! e^{-n^2} z^n, \quad \sum_{n=1}^{\infty} (-1)^n \frac{1}{n^2}(2z+3)^n.$$

   (c) Let f be an entire function such that $|f(z)| \geq 10$ for all $z \in \mathbb{C}$. Prove that f is a constant function.

   (d) Establish the following integration formula with the aid of residues:

$$\int_0^{\infty} \frac{\cos(2x)}{(x^2+1)(x^2+4)} dx = \frac{\pi(2e^{-2} - e^{-4})}{12}.$$

   Complete explanations are required.

Solution.

(a) It is given that u and v are harmonic in $\mathbb{C}$ and the pair u, v satisfies the Cauchy-Riemann equaltions, i.e. $u_x = v_y$, $u_y = -v_x$. Therefore so does the pair $v, -y$:

$$v_x = -u_x = (-u)_x, \quad v_y = u_x = -(-u)_x.$$

Therefore $-u$ is a harmonic conjugate to v.

(b) For the first series use the ratio test:

$$\frac{(n+1)^{59}}{n^{59}} \frac{|z+i|^{n+1}}{|z+i|^n} = \left(1 + \frac{1}{n}\right)^{59} |z+i| \to |z+i|, n \to \infty.$$

If $|z+i| < 1$, then the series converges absolutely, and if $|z+i| > 1$, then it diverges. Thus the radius of convergence equals 1.

For the second series use the ratio test:

$$\frac{(n+1)!e^{-(n+1)^2}|z|^{n+1}}{n!e^{-n^2}|z|^n} = (n+1)e^{-2n-1}|z| \to 0, n \to \infty,$$

so that the radius of convergence equals infinity.
For the third series use the ratio test again:

$$\frac{|2z+3|^{n+1}n^2}{(n+1)^2|2z+3|^n} = |2z+3|\left(1+\frac{1}{n}\right)^{-2} \to |2z+3|, \ n \to \infty.$$

If $|z+3/2| < 1/2$, then the series converges absolutely, and if $|z+3/2| > 1/2$, then it diverges. Thus the radius of convergence equals $1/2$.

(c) Since $|f(z)| \geq 10$, the function $g(z) = \frac{1}{f(z)}$ is entire together with f, and $|g(z)| \leq 10^{-1}$. By Liouville's Theorem, g is constant on $\mathbb{C}$, and hence so is f, as required.

(d) Since the integrand is even, it coincides with the half of the integral

$$\mathrm{Re} \int_{-\infty}^{\infty} f(x)dx,$$

where

$$f(z) = \frac{e^{2iz}}{(z^2+1)(z^2+4)}.$$

We consider the contour σ_R formed by the straight segment $[-R, R]$ and the semi-circular path $\gamma_R(t) = Re^{it}, t \in [0, \pi]$. Factor the denominator as

$$(z+i)(z-i)(z+2i)(z-2i).$$

The function f is holomorphic in $\mathbb{C}$ apart from four simple poles at the roots of the denominator $\pm i$, $\pm 2i$. Only the poles i and $2i$ lie inside the contour and for $R > 2$ only. We calculate the residues at these points.

$$\begin{aligned}\mathrm{Res}\,(f, i) &= \lim_{z\to i}(z-i)f(z) = \lim_{z\to i}\frac{e^{2iz}}{(z+i)(z^2+4)} = \frac{e^{2i^2}}{(i+i)((i)^2+4)} \\ &= \frac{e^{-2}}{2i(-1+4)} = \frac{e^{-2}}{6i}.\end{aligned}$$

$$\begin{aligned}\mathrm{Res}\,(f, 2i) &= \lim_{z\to 2i}(z-2i)f(z) = \lim_{z\to 2i}\frac{e^{2iz}}{(z+2i)(z^2+1)} = \frac{e^{4i^2}}{(2i+2i)((2i)^2+1)} \\ &= \frac{e^{-4}}{4i(-4+1)} = \frac{e^{-4}}{-12i}.\end{aligned}$$

(Calculation of the residues by any other method is acceptable). Therefore, by the Cauchy Residue Theorem,

$$\int_{\sigma_R} f(z)dz = 2\pi i\left(\frac{e^{-2}}{6i} - \frac{e^{-4}}{12i}\right) = \pi\frac{2e^{-2} - e^{-4}}{6},$$

for all $R > 2$. The contribution from the semicircle γ_R can be bounded as

$$\left|\int_{\gamma_R} f(z)\,dz\right| \le \pi R \max_{z\in\gamma_R} |f(z)| \le \pi R\frac{1}{(R^2-1)(R^2-4)} \to 0, \quad R \to \infty.$$

Here we have used that on γ_R we have

$$|z^2+1| \ge |z^2| - 1 = R^2 - 1, \quad |z^2+4| \ge |z^2| - 4 = R^2 - 4,$$
$$|e^{2iz}| = e^{\mathrm{Re}(2iz)} = e^{-2\,\mathrm{Im}(z)} \le e^0 = 1.$$

Consequently,

$$\int_{-\infty}^{\infty} f(x)dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)dx = \lim_{R\to\infty}\int_{\sigma_R} f(z)dz = \pi\frac{2e^{-2} - e^{-4}}{6}.$$

This integral is real-valued, and hence the sought integral equals 1/2 of the right-hand side, as required.

2. (a) Consider the linear fractional transformation

$$f(z) = \frac{az+b}{cz+d}, \qquad z \in \mathbb{C}, cz+d \neq 0,$$

with some complex coefficients a, b, c, d. We say that this transformation is non-degenerate if the coefficients satisfy the condition

$$ad - bc \neq 0. \tag{1}$$

i. Show that a non-degenerate f is conformal for all z such that $cz + d \neq 0$.

ii. Prove that the composition of two non-degenerate linear fractional transformations is again a non-degenerate linear fractional transformation.

(b) In each of the following two cases is it possible to find a holomorphic function on $D(0, 1) = \{z \in \mathbb{C} : |z| < 1\}$ satisfying the condition

i. $f(1/n) = (-1)^n$, $n = 2, 3, \dots$?

ii. $f(1/n) = \frac{n}{n+1}$, $n = 2, 3, \dots$?

If it is possible, in each case indicate how many functions satisfy the requirement. Explain your answer.

(c) Find the Laurent expansions of the function

$$g(z) = \frac{1}{(z+1)(z-3)},$$

valid for

i. $1 < |z| < 3$;

ii. $1 < |z - 2| < 3$.

Solution.

(a) i. Since f is a rational function it is holomorphic away from the roots of the denominator, i.e. for $cz + d \neq 0$. Furthermore, a direct calculation shows that

$$f'(z) = \frac{a(cz+d) - c(az+b)}{(cz+d)^2} = \frac{ad - bc}{(cz+d)^2},$$

and hence, in view of (1), the right hand side is non-zero for all $cz+d \neq 0$. This implies the required conformality.

ii. Let

$$f_1(z) = \frac{a_1 z + b_1}{c_1 z + d_1}, \quad f_2(z) = \frac{a_2 z + b_2}{c_2 z + d_2}$$

with $a_1 d_1 - b_1 c_1 \neq 0, a_2 d_2 - b_2 c_2 \neq 0$. Calculate the composition

$$f_1(f_2(z)) = \frac{a_1 \frac{a_2 z + b_2}{c_2 z + d_2} + b_1}{c_1 \frac{a_2 z + b_2}{c_2 z + d_2} + d_1} = \frac{(a_1 a_2 + c_2 b_1) z + a_1 b_2 + d_2 b_1}{(c_1 a_2 + c_2 d_1) z + c_1 b_2 + d_1 d_2}.$$

Observe that the coefficients can be obtained by multiplying two matrices:

$$A_1 A_2 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} = \begin{pmatrix} a_1 a_2 + b_1 c_2 & a_1 b_2 + b_1 d_2 \\ c_1 a_2 + d_1 c_2 & c_1 b_2 + d_1 d_2 \end{pmatrix}.$$

The condition (1) is also satisfied for this new transformation, since the determinant of the product of two matrices equals the product of determinants:

$$\det(A_1 A_2) = \det A_1 \det A_2 \neq 0,$$

as required.

(b) i. No, since the function would have no limit as $z \to 0$, and hence can't be holomorphic in $D(0,1)$.

ii. Yes,

$$f(z) = \frac{1}{1+z}.$$

This is the unique function satisfying the requirement, by the Unique Continuation Theorem.

(c) Expand, using partial fractions:

$$g(z) = -\frac{1}{4(z+1)} + \frac{1}{4(z-3)}.$$

i. Re-write the above expansion:

$$\begin{aligned} g(z) = & -\frac{1}{4z(1+\frac{1}{z})} - \frac{1}{12(1-\frac{z}{3})} = -\frac{1}{4z} \sum_{k=0}^{\infty} \frac{(-1)^k}{z^k} - \frac{1}{12} \sum_{k=0}^{\infty} \frac{z^k}{3^k} \\ = & -\frac{1}{4} \sum_{k=0}^{\infty} \frac{(-1)^k}{z^{k+1}} - \frac{1}{12} \sum_{k=0}^{\infty} \frac{z^k}{3^k}. \end{aligned}$$

ii. Denote $w = z - 2$ and re-write:

$$\begin{aligned} g(z) &= -\frac{1}{4(w+3)} + \frac{1}{4(w-1)} = -\frac{1}{12(1+\frac{w}{3})} + \frac{1}{4w(1-\frac{1}{w})} \\ &= -\frac{1}{12}\sum_{k=0}^{\infty}(-1)^k \frac{w^k}{3^k} + \frac{1}{4w}\sum_{k=0}^{\infty}\frac{1}{w^k} \\ &= -\frac{1}{12}\sum_{k=0}^{\infty}(-1)^k \frac{(z-2)^k}{3^k} + \frac{1}{4}\sum_{k=0}^{\infty}\frac{1}{(z-2)^{k+1}}. \end{aligned}$$

3. (a) Let f be holomorphic in the punctured neghbourhood $D'(z_0, r) = \{z \in \mathbb{C} : 0 < |z - z_0| < r\}$. Classify possible singularity types of the function f at the point z_0.

(b) Locate and classify the singularities of the following functions:

$$\frac{1}{e^z - 1}, \qquad \frac{1 - \cos z}{z^2}, \qquad z^{2020} \sin \frac{1}{z}.$$

Find the residues at every isolated singularity. Explain your answer.

(c) i. Show that the map

$$w = \frac{z - 4i}{z + 4i}$$

maps the upper half-plane $\{z : \mathrm{Im}(z) > 0\}$ conformally onto the unit disc $\{w : |w| < 1\}$.

ii. Show that the map $w = e^z$ maps the horizontal strip

$$\{z : 0 < \mathrm{Im}(z) < \pi\}$$

conformally onto the upper half-plane

$$\{w : \mathrm{Im}(w) > 0\}.$$

iii. Find a conformal map from the vertical strip

$$\{z : 0 < \mathrm{Re}(z) < \pi/2\}$$

onto the unit disk $\{w : |w| < 1\}$. Please give full explanations.

Please remember that a map is *onto* if and only if it is *surjective.*

Solution.

(a) The type of singularity depends on the Laurent expansion in the neighbourhood $D'(z_0, r)$:

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n.$$

The part of this series containing only negative powers of $(z - z_0)$ is called the principle part:

$$\sum_{n=-\infty}^{-1} a_n (z - z_0)^n.$$

If the principle part equals zero, i.e. $a_n = 0$ for all $n < 0$, then the singularity is said to be removable.

If the principle part contains finitely many terms, then the function is said to have a pole at z_0. The number k, such that $a_{-k} \neq 0$ and $a_{-n} = 0$, $n > k$, is said to be the order of the pole.

If the principal part contains infinitely many terms then the singularity is said to be essential.

(b) The isolated singularities coincide with the roots of the denominator, and hence they are located at the points z where $e^z = 1$, i.e. at $z_k = 2\pi i k, k \in \mathbb{Z}$. These are all isolated singularities of this function. Each of them is a simple pole. Indeed,

$$\lim_{z\to z_k} \frac{z - z_l}{e^z - 1} = \lim_{z\to z_k} \frac{z - z_l}{e^{z-z_k} - 1} = \lim_{z\to 0} \frac{z}{e^z - 1} = 1.$$

Thus, according to a Theorem, proved in the lectures, the singularities are simple poles. Moreover, the residue at each pole equals 1.

The second function is holomorphic except at $z = 0$. Expand around zero:

$$\begin{aligned} \frac{1 - \cos z}{z^2} &= \frac{1}{z^2}\left(1 - 1 + \frac{z^2}{2} - \frac{z^4}{4!} + \dots\right) \\ &= \frac{1}{2} - \frac{z^2}{4!} + \dots . \end{aligned}$$

Therefore the principal part of the expansion vanishes, and hence the singularity is removable. As a result, the residue equals zero.

The third function is holomorphic except at $z = 0$. Expand around zero:

$$\begin{aligned} z^{2020} \sin\frac{1}{z} &= z^{2020} \sum_{k=0}^{\infty} (-1)^k \frac{z^{-(2k+1)}}{(2k+1)!} \\ &= \sum_{k=0}^{1009} (-1)^k \frac{z^{-(2k+1)+2020}}{(2k+1)!} + \frac{1}{2021!}\frac{1}{z} + \sum_{k=1011}^{\infty} (-1)^k \frac{z^{-(2k+1)+2020}}{(2k+1)!}. \end{aligned}$$

The last sum contains infinitely many terms, and hence the singularity at zero is essential. By definition, the residue equals $\frac{1}{2021!}$.

(c) i. The map

$$f(z) = \frac{z - 4i}{z + 4i}$$

is conformal on the upper half-plane, since

$$f'(z) = \frac{8i}{(z+4i)^2} \neq 0,$$

for all z with $\operatorname{Im} z > 0$. Calculate:

$$\begin{aligned}\left|\frac{z-4i}{z+4i}\right|^2 &= \frac{(z-4i)(\overline{z}+4i)}{(z+4i)(\overline{z}-4i)} \\ &= \frac{|z|^2 - 8\operatorname{Im} z + 16}{|z|^2 + 8\operatorname{Im} z + 16}.\end{aligned}$$

The right-hand side is strictly less than 1 since $\operatorname{Im} z > 0$. Thus f maps the upper-half-plane into the unit disk $|w| < 1$. To prove that it is surjective, solve for z:

$$w = \frac{z-4i}{z+4i}, \quad |w| < 1,$$

which gives

$$\begin{aligned} z = & -4i\frac{w+1}{w-1} = -4i\frac{(w+1)(\overline{w}-1)}{|w-1|^2} \\ = & -4i\frac{|w|^2 + \overline{w} - w - 1}{|w-1|^2} = -4i\frac{|w|^2 - 1 - 2i\operatorname{Im} w}{|w-1|^2} \\ = & -\frac{8\operatorname{Im} w}{|w-1|^2} + 4i\frac{1-|w|^2}{|w-1|^2}.\end{aligned}$$

Since $|w| < 1$, the imaginary part is strictly positive. This proves the required surjectivity.

ii. First observe that the map e^z is conformal on the entire complex plane, since its derivative coincides with e^z and hence it is non-zero. Thus it is conformal on the strip as well.
Write $w = e^z = e^x e^{iy}$, so that $\arg w = y \in (0, \pi)$, so that $\operatorname{Im} w > 0$. Let us show that it is onto. For every w with $\operatorname{Im} w > 0$ we use the polar representation $w = re^{i\varphi}$ with $r > 0$ and $\varphi \in (0, \pi)$. Therefore $w = e^{x+iy}$ with $x = \ln r$ with the standard natural logarithm ln, and $y = \varphi \in (0, \pi)$, as required.

iii. The plan is to

A. map the vertical strip onto a horizontal strip,
B. map the horizontal strip onto the upper half-plane, as in the previous question,
C. map the upper half-plane onto the unit disk using a suitable linear fractional transformation.

The vertical strip is mapped onto the horizontal strip $\{z : 0 < \operatorname{Im} z < \pi\}$ by $w = 2iz$.
According to the second part of the question, the map e^z maps this strip onto the upper half-plane.
The upper half-plane is mapped onto the unit disk by the linear fractional transformation in the first part of the question.
To summarise: by combining the mappings

$$z \to 2iz \to e^{2iz} \to \frac{e^{2iz} - 4i}{e^{2iz} + 4i},$$

we obtain that the function

$$\phi(z) = \frac{e^{2iz} - 4i}{e^{2iz} + 4i}$$

satisfies the required properties. It is conformal as a composition of three conformal maps, and it has the required mapping properties.

4. (a) Let f be holomorphic in the annulus $D_{R_1,R_2}(z_0) = \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$, and let $a_n, n \in \mathbb{Z}$, be the coefficients in its Laurent expansion. Prove that

$$|a_n| \le \max_{z:|z-z_0|=r} |f(z)|\, r^{-n},\ n \in \mathbb{Z},$$

for all $r \in (R_1, R_2)$.

(b) Let u be holomorphic in the punctured neighbourhood $D'(0,1) = \{z \in \mathbb{C} : 0 < |z| < 1\}$. Suppose that

$$|u(z)| \le M|z|^{-\alpha}, \qquad z \in D'(0,1),$$

with some positive constant M and some $\alpha \in [0,1)$. Using the estimate in Part (a) prove that u has a removable singularity at 0.

(c) Suppose that f and g are entire functions, and that $|f(z)| \le |g(z)|$ for all $z \in \mathbb{C}$. Prove that there exists a number $\beta \in \mathbb{C}$ such that $f(z) = \beta g(z)$ for all $z \in \mathbb{C}$.

(d) Evaluate the integral using residues: $\displaystyle\int_0^{2\pi} \frac{\cos\theta}{5 + 3\cos\theta} d\theta$. Explain your solution.

Solution.

(a) Let $S = S(z_0, r)$ be the positively oriented circular contour centred at z_0 of radius $r \in (R_1, R_2)$. The coefficients a_n are given by the formula

$$a_n = \frac{1}{2\pi i} \int_S \frac{f(z)}{(z - z_0)^{n+1}} dz, n \in \mathbb{Z}.$$

Denote

$$C = \max_{z:|z-z_0|=r} |f(z)|,$$

and estimate $|a_n|$, using the standard estimate for integrals:

$$|a_n| \le (2\pi)^{-1} C r^{-n-1} \text{Length}(S) = C r^{-n}.$$

This is the required result.

(b) In order to prove that the singularity at 0 is removable, we need to show that $a_n = 0$ for all $n < 0$, where a_n are the Laurent coefficients in the expansion of u in the annulus $D'(0,1)$. For any $r \in (0,1)$ we have

$$\max_{|z|=r} |u(z)| \le Mr^{-\alpha}.$$

Using the first part of the question we can estimate:

$$|a_n| \le Mr^{-\alpha-n}, \ n \in \mathbb{Z}.$$

In particular, since $\alpha < 1$ for $n \le -1$ we have $-\alpha - n > 0$, and therefore the right-hand side of the above bound tends to zero as $r \to 0$. Since a_n does not depend on r, this implies that $a_n = 0$ for all $n < 0$, as required.

(c) Assume that g is not identically zero, otherwise $f = 0$ and the conclusion is obvious. Let $h(z) = f(z)/g(z)$. This function is holomorphic outside the roots of g, and $|h(z)| \le 1$. All roots of g are isolated, and hence the singularities of h are also isolated. By Part (b), the roots of g are removable singularities of h, and after removing the singularities, the function h becomes entire, and it satisfies the bound $|h(z)| \le 1$ for all $z \in \mathbb{C}$. By Liouville's Theorem, $h(z) \equiv \beta$ with some constant $\beta \in \mathbb{C}$, so that $f(z) = \beta g(z)$ as required.

(d) Let

$$I = \int_0^{2\pi} \frac{\cos\theta}{5 + 3\cos\theta}\, d\theta.$$

Substitute $z = e^{i\theta}$ to transform the integral to a complex integral around the unit circle S. Then $dz = ie^{i\theta} d\theta$, so that

$$d\theta = \frac{dz}{iz}, \quad \cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

So we get

$$I = \int_S \frac{\frac{1}{2}\left(z + \frac{1}{z}\right)}{5 + \frac{3}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz} = -i \int_S \frac{z^2+1}{z(3z^2 + 10z + 3)}\, dz.$$

Let

$$f(z) = \frac{z^2+1}{z(3z^2+10z+3)} = \frac{z^2+1}{z(3z+1)(z+3)},$$

and observe that $f(z)$ has simple poles at $z = 0$, $z = -1/3$ and $z = -3$. Only $z = 0$ and $z = -1/3$ are inside S. Thus, by the Residue Theorem

$$I = -i(2\pi i)\left(\operatorname{Res}(f, 0) + \operatorname{Res}(f, -1/3)\right).$$

Calculate:

$$\operatorname{Res}(f, 0) = \lim_{z\to 0} \frac{z(z^2+1)}{z(3z+1)(z+3)} = \frac{1}{3},$$

and

$$\operatorname{Res}(f, -1/3) = \lim_{z\to -1/3} \frac{(z+1/3)(z^2+1)}{3z(z+1/3)(z+3)} = \frac{(1/9+1)}{(-1)(-1/3+3)} = -\frac{5}{12}.$$

Therefore,

$$\int_0^{2\pi} \frac{\cos\theta}{5+3\cos\theta}\, d\theta = 2\pi\left(\frac{1}{3} - \frac{5}{12}\right) = -\frac{\pi}{6}.$$

5. (a) Suppose that f is a function holomorphic on Ω except for finitely many poles. Let $\gamma \subset \Omega$ be a positively oriented contour such that $\operatorname{Int}\gamma \subset \Omega$, and there are no zeroes or poles on γ. Let $z_1, z_2, \dots, z_N$ and $w_1, w_2, \dots, w_P$ be the zeros and poles inside γ, counting multiplicity. Prove that

$$\frac{1}{2\pi i}\int_\gamma z\frac{f'(z)}{f(z)}dz = \sum_{k=1}^{N} z_k - \sum_{k=1}^{P} w_k.$$

(b) Integrating along a suitable contour in the complex plane and using Jordan's Lemma, evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+1)^2}dx.$$

Explain how Jordan's Lemma is used.

Solution.

(a) Due to the Cauchy Residue Theorem,

$$\frac{1}{2\pi i}\int_\gamma z\frac{f'(z)}{f(z)}dz = \sum_{j=1}^{N} \operatorname{Res}(zf'f^{-1}, z_j) + \sum_{j=1}^{P} \operatorname{Res}(zf'f^{-1}, w_j).$$

Pick a zero $z_0 = z_k$, and assume that its order is m, i.e.

$$f(z) = a_m(z-z_0)^m + a_{m+1}(z-a)^{m+1} + \cdots = (z-z_0)^m g(z),\ \ a_m \neq 0,$$

where g is a holomorphic function in $D(z_0, r)$ having no zeroes in $D(z_0, r)$, if r is small enough. Thus

$$\begin{aligned} z\frac{f'(z)}{f(z)} &= z\frac{m(z-z_0)^{m-1}g(z) + (z-z_0)^m g'(z)}{(z-z_0)^m g(z)} \\ &= \frac{mz}{z-z_0} + z\frac{g'(z)}{g(z)} = \frac{mz_0}{z-z_0} + m + z\frac{g'(z)}{g(z)}. \end{aligned}$$

This means that the function zf'/f has a simple pole at z_0 with residue mz_0, and hence

$$\operatorname{Res}(zf'f^{-1}, z_0) = mz_0.$$

Now pick a pole $w_0 = w_j$ and assume that its order is p, that is

$$f(z) = (z - w_0)^{-p} h(z),$$

where h is a holomorphic function in $D(w_0, r)$ having no zeroes in $D(w_0, r)$, if r is small enough. Thus

$$\begin{aligned} z\frac{f'(z)}{f(z)} &= z\frac{-p(z-w_0)^{-p-1}h(z) + (z-w_0)^{-p}h'(z)}{(z-w_0)^{-p}h(z)} \\ &= \frac{-pz}{z-w_0} + z\frac{h'(z)}{h(z)} = \frac{-pz_0}{z-w_0} - p + z\frac{h'(z)}{h(z)}. \end{aligned}$$

This means that the function zf'/f has a simple pole at w_0 with residue $-pz_0$, and hence

$$\operatorname{Res}(zf'f^{-1}, w_0) = -pz_0.$$

Collecting contributions from all roots and poles we arrive at the claimed result.

(b) First we find the integral

$$I = \int_{-\infty}^{\infty} \frac{x^3 e^{ix}}{(x^2+1)^2} dx$$

as the limit

$$I = \lim_{R\to\infty} \int_{-R}^{R} \frac{x^3 e^{ix}}{(x^2+1)^2} dx.$$

Attach to the interval of integration a semi-circular path in the upper half-plane:

$$\gamma_R(t) = Re^{it}, \quad t \in [0, \pi].$$

Consider the integral

$$I_R = \int\limits_{[-R,R]\cup\gamma_R} \frac{z^3 e^{iz}}{(z^2+1)^2} dz,$$

where the contour $[-R, R] \cup \gamma_R$ is positively oriented. The integrand has two poles: at i and $-i$. Only the pole i is inside the contour. Thus by the Residue Theorem,

$$I_R = 2\pi i \mathrm{Res}\,(f, i), \quad f(z) = \frac{z^3 e^{iz}}{(z^2+1)^2}.$$

The pole is of order 2, so that

$$\mathrm{Res}\,(f, i) = g'(i), \quad g(z) = (z-i)^2 f(z) = \frac{z^3 e^{iz}}{(z+i)^2}.$$

Differentiate:

$$g'(z) = \frac{3z^2 e^{iz} + iz^3 e^{iz}}{(z+i)^2} - 2\frac{z^3 e^{iz}}{(z+i)^3}.$$

Consequently,

$$\begin{aligned}\mathrm{Res}\,(f, i) = g'(i) &= \frac{3i^2 e^{i^2} + ii^3 e^{i^2}}{(2i)^2} - 2\frac{i^3 e^{i^2}}{(2i)^3}\\ &= -\frac{-3e^{-1} + e^{-1}}{4} - \frac{e^{-1}}{4} = \frac{e^{-1}}{4}.\end{aligned}$$

Since

$$\max_{z \in \gamma_R} \frac{|z|^3}{|z^2+1|^2} \le \frac{R^3}{(R^2-1)^2} \to 0, \quad R \to \infty,$$

we can use Jordan's Lemma, which implies that

$$\int\limits_{\gamma_R} \frac{z^3 e^{iz}}{(z^2+1)^2} dz \to 0, \quad R \to \infty.$$

Therefore

$$I = \lim_{R \to \infty} I_R = 2\pi i \,\mathrm{Res}\,(f, i) = \frac{\pi i e^{-1}}{2}.$$

Since $\sin x = \mathrm{Im}\, e^{ix}$, the original integral coincides with $\mathrm{Im}\, I$, i.e.

$$\int_{-\infty}^{\infty} \frac{x^3 \sin x}{(x^2+1)^2} dx = \frac{\pi e^{-1}}{2}.$$