**UNIVERSITY COLLEGE LONDON**
Department of Mathematics

**MATH0013- Complex Analysis**
December 2024 in-class test

Duration: 1 hour

CANDIDATE NUMBER ..............................................................

**INSTRUCTIONS:** (READ CAREFULLY!)

- This test booklet contains 12 pages including this one. It consists of **3** questions, for a total of **100** marks.
- **Do not write or draw anything on the QR code on the top corner of any page.**
- **All questions should be attempted.** Marks obtained in all solutions will count.
- **If you need extra space for a question, you may use pages 10 - 12 for this purpose.** If you do so, clearly indicate it on the corresponding problem page.
- **Organize your work.** Work that is scattered over the page, that has no clear order, that is messy and illegible, might receive little credit.
- **No aids are permitted on this examination.** Examples of illegal aids include, but are not limited to textbooks, notes, cheatsheets, calculators, mobile phones, or any other electronic device.
- **Do not turn this page over until the invigilators instruct you to do so.**

*Good luck!*

**Question 1.** **(35 marks)**

(a) (*15 marks*)

Let $f$ be an entire function and $M > 0$ be a constant such that $|f(z)| \leq Me^x$ for all $z = x + iy \in \mathbb{C}$. Show that there exists a constant $k \in \mathbb{C}$ such that $f(z) = ke^z$ for all $z \in \mathbb{C}$.

**Solution:**

The appearance of $e^z$ in the answer suggests that we consider the entire function

$$g(z) = \frac{f(z)}{e^z}.$$

It is holomorphic everywhere, as $e^z \neq 0$. We remark that $|e^z| = e^x$ with $z = x + iy$. We get

$$|g(z)| = \frac{|f(z)|}{|e^z|} = \frac{|f(z)|}{e^x} \leq M$$

by the given assumption. This means that $g$ is entire and bounded. By Liouville's theorem, $g$ is constant. Therefore, we can find a constant $k \in \mathbb{C}$ such that

$$g(z) = k \Leftrightarrow f(z) = ke^z, \quad \forall z \in \mathbb{C}.$$

(b) (*20 marks*) Prove that the Möbius transform

$$\phi(z) = \frac{z - \frac{1}{2}}{z - 2}$$

maps the unit disk $D(0,1) = \{z : |z| < 1\}$ onto the disk $D(0, 1/2) = \{z : |z| < 1/2\}$. Do not forget to check the surjectivity.

**Solution:** Check that the mapping is into, i.e. that $|z - \frac{1}{2}| < \frac{1}{2}|z - 2|$:

$$\begin{aligned}\left|z - \frac{1}{2}\right|^2 - \frac{1}{4}\left|z - 2\right|^2 &= |z|^2 - \mathrm{Re}\, z + \frac{1}{4} - \frac{1}{4}(|z|^2 - 4\mathrm{Re}\, z + 4) \\ &= \frac{3}{4}|z|^2 - \frac{3}{4} < 0,\end{aligned}$$

since $|z| < 1$.

Onto: pick a point $w \in D(0, 1/2)$, and solve the equation

$$\frac{z - \frac{1}{2}}{z - 2} = w,$$

which gives

$$z = \frac{1 - 4w}{2(1 - w)}.$$

Check that $|z| < 1$:

$$\begin{aligned}|1 - 4w|^2 - 4|1 - w|^2 &= 1 - 8\mathrm{Re}\, w + 16|w|^2 - 4(1 - 2\mathrm{Re}\, w + |w|^2) \\ &= \; -3|w|^2 + 12|w|^2 = -3(1 - 4|w|^2) < 0,\end{aligned}$$

since $|w| < 1/2$. This proves that $|z| < 1$, and hence the mapping $\phi$ is onto, as required.

**Question 2.** **(35 marks)**

(a) (*20 marks*) Let $f$ be holomorphic on a domain $\Omega \subset \mathbb{C}$, and let $a \in \Omega$. Prove that the function

$$g(z) = \frac{f(z) - f(a)}{z - a},$$

has a removable singularity at the point $z = a$.

Explain why $g$ can be considered as a function holomorphic on $\Omega$, and find $g(a), g'(a)$.

**Solution:** The function $g$ is clearly holomorphic everywhere on $\Omega$ apart from the point $a$. Expand $f$ in its Taylor series around $a$:

$$f(z) = \sum_{k=0}^{\infty} b_k (z-a)^k, \quad b_k = \frac{f^{(k)}(a)}{k!},$$

so that $b_0 = f(a)$, and substitute in the definition of $g$:

$$g(z) = \frac{\sum_{k=1}^{\infty} b_k (z-a)^k}{z - a} = \sum_{k=0}^{\infty} b_{k+1} (z-a)^k.$$

This means that the principal part of the Laurent expansion for $g$ is zero. Thus $g$ has a removable singularity. If we define $g(a) = b_1 = f'(a)$, then the function $g$ becomes holomorphic on $\Omega$, and $g'(a) = b_2 = f''(a)/2$.

(b) (*15 marks*) For each of the following functions determine the isolated singularities and their nature. For the poles, find the order of the pole, the principal part and residue at the pole.

$$\text{(i)}\ \frac{z^2}{(1+z)^2}, \quad \text{(ii)}\ \frac{\sin z}{z}.$$

**Solution:** (i) This function has a pole of order two at $z_0 = -1$. To find the principal part denote $w = z + 1$ and expand:

$$\frac{z^2}{(1+z)^2} = \frac{(w-1)^2}{w^2} = \frac{w^2 - 2w + 1}{w^2} = \frac{1}{w^2} - \frac{2}{w} + 1.$$

Thus the principal part is

$$\frac{1}{(z+1)^2} - \frac{2}{z+1}$$

and the residue equals $-2$.

(ii) Expand $\sin z$ in Taylor series:

$$\frac{\sin z}{z} = \frac{1}{z}\sum_{k=0}^{\infty}(-1)^k \frac{z^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty}(-1)^k \frac{z^{2k}}{(2k+1)!}.$$

Thus the principal part equals zero, and hence the singularity at $z = 0$ is removable. Residue $= 0$.

Solution to Question 2, continued

**Question 3.** **(30 marks)**

Let $f$ be holomorphic in a domain containing the closed unit disk $\overline{D}(0,1)$. Let $S = S(0,1)$ be the positively oriented circular contour of radius $1$ centred at zero. Show that

$$f(0) = \frac{1}{2\pi i} \int_S \frac{f(w)\, \sin w}{w^2}\, dw.$$

**Solution:**

We apply the Cauchy integral formula for first derivative of the function

$$h(z) = f(z) \sin z.$$

We have $h'(z) = f'(z)\sin z + f(z)\cos z$, by the product rule. This gives

$$h'(0) = f(0)\cos 0 = f(0).$$

Therefore,

$$f(0) = h'(0) = \frac{1}{2\pi i} \int_S \frac{h(w)}{w^2} dw = \frac{1}{2\pi i} \int_S \frac{f(w) \sin w}{w^2}\, dw.$$

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