

Sheet 4, solutions

Below $S(z, r)$ is a circular positively oriented contour of radius $r > 0$ centred at z .

1. Evaluate the integral

$$\frac{1}{2\pi i} \int_{\gamma} \frac{e^z}{z^2 + a^2} dz,$$

where $a > 0$ is a real number and γ is a contour such that $\overline{D}(0, a) \subset \text{Int } \gamma$.

Solution. Using partial fractions, we get

$$\frac{1}{z^2 + a^2} = \frac{i}{2a(z + ia)} - \frac{i}{2a(z - ia)}.$$

Thus the integral equals $I_1 + I_2$, where

$$I_1 = \frac{1}{2\pi i} \int_{\gamma} \frac{ie^z}{2a(z + ia)} dz, \quad I_2 = -\frac{1}{2\pi i} \int_{\gamma} \frac{ie^z}{2a(z - ia)} dz.$$

By the Cauchy formula,

$$I_1 = \frac{i}{2a} e^{-ia}, \quad I_2 = -\frac{i}{2a} e^{ia},$$

so that the integral equals

$$-\frac{i}{2a} e^{ia} + \frac{i}{2a} e^{-ia} = \frac{1}{a} \frac{e^{ia} - e^{-ia}}{2i} = \frac{\sin a}{a}$$

2. Let f be analytic on Ω , and let $\overline{D}(a, R) \subset \Omega$ with some $a \in \mathbb{C}$.

(a) Using the Cauchy Integral Formula prove that

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + Re^{it}) dt.$$

(b) Evaluate the integral

$$\int_{r < |z-a| < R} f(z) dx dy,$$

where $r < R$.

Solution.

(a) Use the Cauchy formula:

$$f(a) = \frac{1}{2\pi i} \int_{S(a,R)} \frac{f(z)}{z-a} dz.$$

Substitute the parametrisation $z = a + Re^{it}$, $t \in [0, 2\pi]$:

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + Re^{it})}{Re^{it}} iRe^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} f(a + Re^{it}) dt,$$

as required.

(b) Use the polar coordinates:

$$\begin{aligned} \int_{r < |z-a| < R} f(z) dx dy &= \int_{r < |z-a| < R} \int_0^{2\pi} f(a + \rho e^{i\phi}) d\phi \rho d\rho \\ &= 2\pi f(a) \int_{r < |z-a| < R} \rho d\rho = \pi f(a) (R^2 - r^2). \end{aligned}$$

3. Evaluating the integral

$$\frac{1}{2\pi i} \int_{S(0,1)} \frac{1}{(z-a)(z-a^{-1})} dz,$$

prove that for any $a \in (0, 1)$,

$$\int_0^{2\pi} \frac{dt}{1 + a^2 - 2a \cos t} = \frac{2\pi}{1 - a^2}.$$

Solution. By Cauchy formula, for $a \in (0, 1)$ we have

$$I := \frac{1}{2\pi i} \int_S \frac{1}{(z-a)(z-a^{-1})} dz = \frac{1}{a - a^{-1}}.$$

On the other hand, with the usual parametrisation $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$, we obtain that

$$I = i \frac{1}{2\pi i} \int_0^{2\pi} e^{it} \frac{1}{(e^{it} - a)(e^{it} - a^{-1})} dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{a}{2a \cos t - a^2 - 1} dt$$

This follows from this simple calculation:

$$w^{-1}(w-a)(w-a^{-1}) = \frac{1}{a}(1-aw^{-1})(aw-1) = \frac{1}{a}(aw-a^2-1+aw^{-1}).$$

Thus

$$-\frac{1}{2\pi} \int_0^{2\pi} \frac{a}{1+a^2-2a\cos t} dt = \frac{1}{a-a^{-1}},$$

which gives the required result.

4. Let f be continuous on $S(0, 1)$.

(a) Show that

$$\overline{\int_{S(0,1)} f(z) dz} = - \int_{S(0,1)} \frac{\overline{f(z)}}{z^2} dz.$$

Use the identity $\overline{\int_a^b g(t) dt} = \int_a^b \overline{g(t)} dt$ which holds for any continuous function $g : [a, b] \rightarrow \mathbb{C}$.

(b) Let f be an entire function. Prove that

$$\frac{1}{2\pi i} \int_{S(0,1)} \frac{\overline{f(z)}}{z-a} dz = \begin{cases} \overline{f(0)}, & \text{if } |a| < 1; \\ \overline{f(0) - f(\bar{a}^{-1})}, & \text{if } |a| > 1. \end{cases}$$

Solution.

(a) Use the parametrisation $\gamma(t) = e^{it}$, $t \in [0, 2\pi]$:

$$\begin{aligned} \overline{\int_S f(z) dz} &= -i \overline{\int_0^{2\pi} e^{it} f(e^{it}) dt} = -i \int_0^{2\pi} e^{-it} \overline{f(e^{it})} dt \\ &= -i \int_0^{2\pi} e^{-2it} \overline{f(e^{it})} e^{it} dt = - \int_S \frac{\overline{f(z)}}{z^2} dz, \end{aligned}$$

as claimed.

(b) Since $z\bar{z} = 1$, one can rewrite, using the first part of the exercise:

$$\int_S \frac{\overline{f(z)}}{z-a} dz = \int_S \overline{\left(\frac{f(z)}{z^2(\bar{z}-\bar{a})} \right)} \frac{1}{z^2} dz = - \int_S \frac{\overline{f(z)}}{z^2(\bar{z}-\bar{a})} dz = - \int_S \frac{\overline{f(z)}}{z(1-\bar{a}z)} dz.$$

Let us evaluate

$$I = \int_S \frac{f(z)}{z(1 - \bar{a}z)} dz.$$

By partial fractions,

$$I = I_1 + I_2, \quad I_1 = \int_S \frac{f(z)}{z} dz, \quad I_2 = \bar{a} \int_S \frac{f(z)}{1 - \bar{a}z} dz.$$

By Cauchy formula,

$$I_1 = 2\pi i f(0).$$

If $|a| > 1$, then by Cauchy formula

$$I_2 = - \int_S \frac{f(z)}{z - \bar{a}^{-1}} dz = -2\pi i f(\bar{a}^{-1}).$$

If $|a| < 1$, then $I_2 = 0$. Since

$$\int_S \frac{\overline{f(z)}}{z - a} dz = -\bar{I},$$

the required formula is now proved.

5. Let f be entire.

- (a) Show that, if e^f is bounded, then f is constant.
- (b) Assume that $\text{Im}(f)$ is bounded below. Show that f is a constant function.

Solution.

- (a) The function e^f is entire, as composition of e^w and f . Since it is given to be bounded, we can apply Liouville's theorem and deduce that e^f is constant, say $= k \neq 0$. Therefore

$$0 = \frac{d}{dz} e^{f(z)} = f'(z) e^{f(z)} = k f'(z).$$

As $k \neq 0$, we conclude that $f'(z) = 0$ for all $z \in \mathbb{C}$. Therefore $f = \text{const}$.

- (b) Assume that $\text{Im}(f(z)) \geq m$ for all $z \in \mathbb{C}$. Consider the entire function $g(z) = i f(z)$. We have

$$|e^{g(z)}| = |e^{i f(z)}| = e^{-\text{Im}(f(z))} \leq e^{-m}.$$

By (a) $g(z) = i f(z)$ is a constant, which implies that f is a constant.

6. Find the integral

$$h(z) = \int_{S(0,1)} \frac{dw}{w(w-z)},$$

for $|z| > 1$ and $0 < |z| < 1$.

Solution. If $|z| > 1$, then by the Cauchy Integral Formula,

$$h(z) = -\frac{2\pi i}{z}.$$

If $0 < |z| < 1$, then represent, using partial fractions:

$$\frac{1}{w(w-z)} = \frac{1}{z(w-z)} - \frac{1}{zw}.$$

Then by the Cauchy Integral Formula,

$$\int_{S(0,1)} \frac{dw}{z(w-z)} = \frac{2\pi i}{z}, \quad \int_{S(0,1)} \frac{dw}{zw} = \frac{2\pi i}{z}.$$

Thus $h(z) = 0, 0 < |z| < 1$.

7. Let the function g be holomorphic on a domain Ω which contains the disk $\overline{D}(0, 1)$. Show that

$$\frac{1}{2\pi i} \int_{S(0,1)} \frac{g(\zeta)}{z\zeta - 1} d\zeta = \begin{cases} 0, & |z| < 1; \\ z^{-1}g(z^{-1}), & |z| > 1. \end{cases}$$

Solution. Rewrite the integral:

$$I(z) = \frac{1}{2\pi i} \int_{S(0,1)} \frac{g(\zeta)}{z\zeta - 1} d\zeta = \frac{1}{2\pi i} \int_{S(0,1)} \frac{g(\zeta)}{z(\zeta - z^{-1})} d\zeta.$$

By the Cauchy-Goursat Theorem, if $|z| < 1$, then $I(z) = 0$. If $|z| > 1$, then the Cauchy Integral Formula gives:

$$I(z) = z^{-1}g(z^{-1}),$$

as claimed.

8. Let g be entire. Assume that $|g(z)| \leq |z| \ln(1 + |z|)$ for all $z \in \mathbb{C}$. Prove that $g(z) = 0$ for all $z \in \mathbb{C}$.

Solution. The Taylor series for g converges absolutely for all $z \in \mathbb{C}$:

$$g(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + \sum_{k=2}^{\infty} a_k z^k.$$

First we prove that $a_k = 0$ for all $k \geq 2$. Let $R > 0$. Then

$$M_R = \max_{|z|=R} |g(z)| \leq R \ln(1 + R).$$

Using Cauchy's inequalities we get

$$|a_k| = \frac{|g^{(k)}(0)|}{k!} \leq \frac{M_R}{R^k} \leq R^{1-k} \ln(1 + R), \quad (1)$$

for all $R > 0$. For $k \geq 2$ the right-hand side tends to zero as $R \rightarrow \infty$. Therefore $a_k = 0$ if $k \geq 2$, as claimed, and hence $g(z) = a_0 + a_1 z$.

It follows from the bound $|g(z)| \leq |z| \ln(1 + |z|)$ that $g(0) = 0$, so $a_0 = 0$. Furthermore, the same bound implies that $|g(z)| \leq |z| \ln(1 + |z|) \leq |z|^2$. Consequently, $a_1 = 0$.

To summarise, all Taylor coefficients equal zero and hence $g(z) = 0$ as required.

9. Let f be entire. Suppose that $\operatorname{Re}(f(z)) - \operatorname{Im}(f(z)) < 2$ for all $z \in \mathbb{C}$. Show that f is a constant function.

Solution. Consider a simpler example first: let $g = u + iv$ with real-valued u and v be an entire function, and assume that $v < 0$. Then $g = \text{const}$. Indeed, consider the entire function

$$e^{-ig} = e^{iu+v}.$$

Then

$$|e^{-ig}| = e^v \leq e^0 = 1.$$

Thus by Liouville's theorem, $e^{-ig} = \text{const}$, and hence $g = \text{const}$, as claimed.

To solve the question we introduce a new entire function:

$$g(z) = (a + ib)f(z) + ic.$$

Our objective is to find such real a, b and c such that $\operatorname{Im} g < 0$. Representing $f = u + iv$, calculate:

$$g(z) = (a + ib)(u + iv) + ic = au - bv + i(bu + av + c).$$

We now that $u - v - 2 < 0$. Thus we choose $b = 1, a = -1, c = -2$, so that $\operatorname{Im} g = u - v - 2 < 0$. Therefore, by the first part of the solution, $g = \text{const}$, and hence $f = \text{const}$ as well.

10. (a) Let f be an entire function and $M > 0$ be a constant such that $|f(z)| \geq M$ for all $z \in \mathbb{C}$. Show that f is a constant function.
- (b) Let f be an entire function and $M > 0$ be a constant such that $|f(z)| \leq Me^x$ for all $z = x + iy \in \mathbb{C}$. Show that there exists a constant k such that $f(z) = ke^z$ for all $z \in \mathbb{C}$.

Solution.

- (a) From the inequality $|f(z)| \geq M > 0$ we deduce that $f(z) \neq 0$ for all $z \in \mathbb{C}$. Therefore, the function

$$g(z) = \frac{1}{f(z)}$$

is holomorphic by the quotient rule on all of \mathbb{C} , i.e. it is entire. Moreover,

$$|g(z)| = \frac{1}{|f(z)|} \leq \frac{1}{M}.$$

Therefore, g is entire and bounded. This implies that g is a constant by Liouville's theorem. So we can find a constant k such that

$$g(z) = k \implies f(z) = 1/k.$$

Notice that the constant k cannot be zero, as g has no roots by construction.

- (b) The appearance of e^z in the answer suggests that we consider the entire function

$$g(z) = \frac{f(z)}{e^z}.$$

It is holomorphic everywhere, as $e^z \neq 0$. We remark that $|e^z| = e^x$ with $z = x + iy$. We get

$$|g(z)| = \frac{|f(z)|}{|e^z|} = \frac{|f(z)|}{e^x} \leq M$$

by the given assumption. This means that g is entire and bounded. Therefore, we can find a constant k such that

$$g(z) = k \Leftrightarrow f(z) = ke^z, \quad \forall z \in \mathbb{C}.$$

11. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$ be a series with a positive (possibly infinite) radius of convergence R . Prove that this series converges uniformly for all $|z| \leq R_1$ where $R_1 < R$.

Solution. The series converges absolutely for all $|z| < R$, so the series

$$\sum |a_k| R_1^k$$

converges. For all $|z| \leq R_1$ we have

$$|a_k z^k| \leq |a_k| |z|^k \leq |a_k| R_1^k.$$

Thus by the Weierstrass' M -test, the series f converges uniformly for all $|z| \leq R_1$, as claimed.

12. Let $f(z) = \sum_{k=0}^n a_k z^k$ be a polynomial. Using contour integration prove that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \sum_{k=0}^n |a_k|^2.$$

Solution. Rewrite the integral using the substitution

$$z = e^{i\theta}, d\theta = -iz^{-1} dz.$$

Since $\bar{z} = z^{-1}$ on the unit circle, the left-hand side equals

$$\frac{-i}{2\pi} \int_{|z|=1} |f(z)|^2 z^{-1} dz = \frac{1}{2\pi i} \sum_{k,l=0}^n a_k \bar{a}_l \int_{|z|=1} z^k z^{-l} z^{-1} dz.$$

The integral equals zero unless $k - l - 1 = -1$, that is $k = l$. In this case

$$\frac{1}{2\pi i} \int_{|z|=1} z^k z^{-l} z^{-1} dz = \frac{1}{2\pi i} \int_{|z|=1} z^{-1} dz = 1.$$

Consequently,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \sum_{k=0}^n |a_k|^2$$

holds, as claimed.

13. Find the value of the integrals

$$\int_S \frac{z^2 + 3}{z(z^2 + 9)} dz$$

taken around the contour

- (a) $S = S(0, 1)$,
 (b) $S = S(0, 4)$.

Solution. The denominator has zeros at 0 and $\pm 3i$.

For (a) we notice that only 0 is inside it. We define

$$f(z) = \frac{z^2 + 3}{z^2 + 9}.$$

which is holomorphic inside the disc $|z| \leq 1$. By Cauchy's integral formula

$$2\pi i f(0) = \int_S \frac{f(z)}{z} dz \implies \int_S \frac{z^2 + 3}{z(z^2 + 9)} dz = 2\pi i \frac{0^2 + 3}{0^2 + 9} = \frac{2\pi i}{3}.$$

For (b) we use partial fractions:

$$\frac{z^2 + 3}{z(z^2 + 9)} = \frac{A}{z} + \frac{B}{z + 3i} + \frac{C}{z - 3i},$$

which gives

$$z^2 + 3 = A(z^2 + 9) + B(z - 3i)z + C(z + 3i)z.$$

Plugging $z = 0$ we get $A = 1/3$. Plugging $z = 3i$ we get $-9 + 3 = C \cdot 6i \cdot 3i = -18C$, so that $C = 1/3$. Plugging $z = -3i$ we get $-9 + 3 = B \cdot (-6i) \cdot (-3i) = -18B$, so that $B = 1/3$. We use Cauchy's integral formula three times with $g(z) = 1/3$ to get

$$\begin{aligned} \int_S \frac{z^2 + 3}{z(z^2 + 9)} dz &= 2\pi i \left(\int_S \frac{1/3}{z} dz + \int_S \frac{1/3}{z + 3i} dz + \int_S \frac{1/3}{z - 3i} dz \right) \\ &= 2\pi i (1/3 + 1/3 + 1/3) = 2\pi i. \end{aligned}$$

14. Let f be holomorphic on a region containing the closed unit disk $\overline{D}(0, 1)$. Let $S = S(0, 1)$. Show that

$$2f'(0) = \frac{1}{2\pi i} \int_S \frac{f(w) - f(-w)}{w^2} dw.$$

Solution. We split the integral on the right-hand side:

$$\frac{1}{2\pi i} \int_S \frac{f(w) - f(-w)}{w^2} dw = \frac{1}{2\pi i} \int_S \frac{f(w)}{w^2} dw - \frac{1}{2\pi i} \int_S \frac{f(-w)}{w^2} dw.$$

Cauchy's formula for $f'(0)$ gives

$$f'(0) = \frac{1}{2\pi i} \int_S \frac{f(w)}{w^2} dw. \quad (2)$$

For the second integral, we set $g(z) = f(-z)$ motivated by the expression in the numerator. We have $g'(z) = -f'(-z)$ by the chain rule. Cauchy's integral formula for $g'(0)$ is

$$g'(0) = \frac{1}{2\pi i} \int_S \frac{g(w)}{w^2} dw = \frac{1}{2\pi i} \int_S \frac{f(-w)}{w^2} dw,$$

which gives

$$-f'(0) = \frac{1}{2\pi i} \int_S \frac{f(-w)}{w^2} dw.$$

We flip the sign in the above equation, add to Eq. (2) to get the result.

15. Let f be holomorphic in a domains containing the closed unit disk $\overline{D}(0, 1)$. Let $S = S(0, 1)$. Show that

$$f(0) + f'(0) = \frac{1}{2\pi i} \int_S \frac{f(w)e^w}{w^2} dw.$$

Solution. We apply the Cauchy integral formula for $n = 1$ and for the function

$$h(z) = f(z)e^z.$$

We have $h'(z) = f'(z)e^z + f(z)e^z$ by the product rule. This gives

$$h'(0) = f'(0)e^0 + f(0)e^0 = f(0) + f'(0).$$

We get

$$f(0) + f'(0) = h'(0) = \frac{1}{2\pi i} \int_S \frac{h(w)}{w^2} dw = \frac{1}{2\pi i} \int_C \frac{f(w)e^w}{w^2} dw.$$

16. Let $n \geq 2$ be a positive integer. Let f be holomorphic in a domain containing the closed unit disk $\overline{D}(0, 1)$. Let $S = S(0, 1)$. Show that

$$\frac{1}{2\pi i} \int_C \frac{f(w) + f(e^{2\pi i/n}w) + f(e^{4\pi i/n}w) + \dots + f(e^{2(n-1)\pi i/n}w)}{w^2} dw = 0.$$

Solution. Let

$$g(z) = f(z) + f(e^{2\pi i/n}z) + f(e^{4\pi i/n}z) + \dots + f(e^{2(n-1)\pi i/n}z).$$

Since f is holomorphic on a region containing the closed unit disk $\overline{D}(0, 1)$ and for $|z| \leq 1$, we have $|e^{2\pi i/n} z| \leq 1$, $|e^{4\pi i/n} z| \leq 1, \dots, |e^{2(n-1)\pi i/n} z| \leq 1$, the function g is holomorphic on a region containing the closed unit disk $\overline{D}(0, 1)$. We can apply Cauchy's integral formula for its first derivative to get

$$g'(0) = \frac{1}{2\pi i} \int_C \frac{f(w) + f(e^{2\pi i/n} w) + f(e^{4\pi i/n} w) + \dots + f(e^{2(n-1)\pi i/n} w)}{w^2} dw.$$

We compute the derivative of g using the chain rule:

$$g'(z) = f'(z) + e^{2\pi i/n} f'(e^{2\pi i/n} z) + e^{4\pi i/n} f'(e^{4\pi i/n} z) + \dots + e^{2(n-1)\pi i/n} f'(e^{2(n-1)\pi i/n} z),$$

so that

$$\begin{aligned} g'(0) &= f'(0) + e^{2\pi i/n} f'(e^{2\pi i/n} 0) + e^{4\pi i/n} f'(e^{4\pi i/n} 0) + \dots + e^{2(n-1)\pi i/n} f'(e^{2(n-1)\pi i/n} 0) \\ &= f'(0) + e^{2\pi i/n} f'(0) + e^{4\pi i/n} f'(0) + \dots + e^{2(n-1)\pi i/n} f'(0) \\ &= f'(0)(1 + e^{2\pi i/n} + e^{4\pi i/n} + \dots + e^{2(n-1)\pi i/n}) \\ &= f'(0) \frac{1 - (e^{2\pi i/n})^n}{1 - e^{2\pi i/n}} = f'(0) \frac{1 - 1}{1 - e^{2\pi i/n}} = 0, \end{aligned}$$

using the sum of the first n terms of a geometric series.

17. Let f be holomorphic on an open set Ω containing the contour γ and its interior. Suppose that $z_0 \in \text{Int } \gamma$. Show that

$$\int_{\gamma} \frac{f'(z)}{z - z_0} dz = \int_{\gamma} \frac{f(z)}{(z - z_0)^2} dz.$$

Solution. We use Cauchy's integral formula for the first derivative and the function f to see that

$$\int_{\gamma} \frac{f(z)}{(z - z_0)^2} dz = 2\pi i f'(z_0).$$

On the other hand we use Cauchy's integral formula for the function $g(z) = f'(z)$ to see that

$$f'(z_0) = g(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{z - z_0} dz.$$

The result becomes now obvious.