Sheet 3, solutions

1. Let R be the radius of convergence of the series $f(z) = \sum_{k=0}^{\infty} a_k z^k$. What is the radius of convergence of the series $\sum_{k=0}^{\infty} a_k(1 + (k+1)^{-1})z^k$? Explain.

   Solution. The radius of convergence is the same. Indeed, let R_1 be the radius of convergence of the new series. Since

   $$|a_k z^k| \le |a_k(1 + (k+1)^{-1})z^k|,$$

   by the comparison principle, the original series converges absolutely for all $|z| < R_1$, and hence $R_1 \le R$. On the other hand,

   $$|a_k(1 + (k+1)^{-1})z^k| \le 2\,|a_k z^k|, \quad \text{for all} \quad k \ge 0,$$

   so by the comparison principle again, the new series converges absolutely for all $|z| < R$, which implies that $R_1 \ge R$. Consequently, $R = R_1$, as claimed.

   Alternative solution: use the formula

   $$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}},$$

   i.e. the Cauchy-Hadamard formula. Then

   $$\begin{aligned} \frac{1}{R_1} &= \limsup_{n\to\infty} \left[|a_n|^{\frac{1}{n}} \left(1 + \frac{1}{n+1}\right)^{\frac{1}{n}} \right] \\ &= \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} \lim_{n\to\infty} \left(1 + \frac{1}{n+1}\right)^{\frac{1}{n}} = \frac{1}{R}. \end{aligned}$$

2. Show that the map

   $$w = \frac{z-3}{z+3}$$

   maps the right half-plane $\{z : \mathrm{Re}(z) > 0\}$ conformally onto the unit disc $D(0,1)$.

Solution. The map is holomorphic in the right half-plane, since the denominator does not vanish there. Also, the derivative equals

$$\frac{6}{(z+3)^2} \neq 0, \quad \operatorname{Re} z > 0.$$

Thus the map is conformal.

Now we need to show that it maps onto the disk. First we prove that it maps into the unit disk. Indeed,

$$\left|\frac{z-3}{z+3}\right| = \frac{|z-3|^2}{|z+3|^2} = \frac{|z|^2 - 6\operatorname{Re} z + 9}{|z|^2 + 6\operatorname{Re} z + 9} < 1,$$

if $\operatorname{Re} z > 0$. Alternatively, geometrically the distance to 3 is less than the distance to -3 iff $\operatorname{Re} z > 0$.

To prove surjectivity, one can find the inverse map:

$$\begin{aligned} w = \frac{z-3}{z+3} &\Leftrightarrow wz + 3w = z - 3 \\ &\Leftrightarrow z(w-1) = -3 - 3w = -3(w+1) \Leftrightarrow z = -3\frac{w+1}{w-1}. \end{aligned}$$

This is holomorphic on $D(0,1)$ as it is a rational function and $1 \notin D(0,1)$. Let us show that for every $|w| < 1$ we have $\operatorname{Re} z > 0$:

$$z = -3\frac{(w+1)(\overline{w}-1)}{|w-1|^2} = -3\frac{|w|^2 - 1 + \overline{w} - w}{|w-1|^2} = 3\frac{1-|w|^2}{|1-w|^2} + 6i\frac{\operatorname{Im} w}{|1-|w|^2}.$$

Consequently,

$$\operatorname{Re} z = 3\frac{1-|w|^2}{|w-1|^2} > 0,$$

since $|w| < 1$.

3. Consider the Möbius map

$$f(z) = \frac{(1+4i)z - 21 - 9i}{z - 1 + 5i}.$$

Show that it is non-degenerate, and find the image of the upper half-plane $\{z : \operatorname{Im} z > 0\}$ under this map. Explain your answer.

Hint: rewrite $f(z)$ is the form

$$f(z) = \frac{z - 1 - 5i}{z - 1 + 5i} + g(z). \tag{1}$$

The expression for the function g should give you a clear idea what the image is.

Solution. Non-degeneracy is easily checked:

$$(1 + 4i)(-1 + 5i) - (-21 - 9i) = 10i \neq 0.$$

Rewriting the function as in (1) we get

$$f(z) = \frac{z - 1 - 5i}{z - 1 + 5i} + 4i.$$

The first term on the right-hand side maps the upper half-plane onto the unit disk centred at $z = 0$. Then the second term shifts the whole disk by $4i$ along the imaginary axis. Thus the answer is: the unit disk centred at $4i$.

4. (a) Show that $e^{\bar{z}}$ is not holomorphic at any point in $\mathbb{C}$.
   (b) Show that e^z is real iff $\mathrm{Im}(z) = n\pi$, $n \in \mathbb{Z}$.
   (c) Show that $|e^{iz}| < 1$ iff $\mathrm{Im}(z) > 0$.
   (d) Show that $|e^{-2z}| < 1$ iff $\mathrm{Re}(z) > 0$.

Solution.

(a) Let $z = x + iy$. We have

$$\overline{e^z} = e^x(\cos y - i \sin y).$$

We have that $u = \mathrm{Re}(e^{\bar{z}}) = e^x \cos y$ and $v = \mathrm{Im}(e^{\bar{z}}) = -e^x \sin y$. The first Cauchy–Riemann equation means

$$e^x \cos y = u_x = v_y = -e^x \cos y \Longrightarrow \cos y = 0.$$

The second Cauchy–Riemann equation means

$$-e^x \sin y = u_y = -v_x = e^x \sin y \Longrightarrow \sin y = 0.$$

There is no $y \in \mathbb{R}$ with $\sin y = \cos y = 0$, by the basic trigonometric identity $\cos^2 t + \sin^2 t = 1$.

(b) Since $e^z = e^x(\cos y + i \sin y)$, this is real iff $\sin y = 0 \Leftrightarrow y = n\pi$, $n \in \mathbb{Z}$.

(c) We have $|e^{iz}| = e^{\mathrm{Re}(i(x+iy))} = e^{-y}$. We have $e^{-y} < 1$ iff $-y \Leftrightarrow y > 0$, as the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is increasing and $e^0 = 1$.

(d) We have $|e^{-2z}| = e^{\mathrm{Re}(-2z)} = e^{-2x}$. We have $e^{-2x} < 1$ iff $-2x < 0 \Leftrightarrow x > 0$, as the exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is increasing and $e^0 = 1$.

5. Expand the function $\sin z$ in a power series about the point z_0.

Solution. Write

$$\sin z = \sin(z_0 + z - z_0) = \sin z_0 \cos(z - z_0) + \cos z_0 \sin(z - z_0).$$

Now use the known series for sin and cos:

$$\sin z = \sin z_0 \sum_{k=0}^{\infty} (-1)^k \frac{(z - z_0)^{2k}}{(2k)!} + \cos z_0 \sum_{k=0}^{\infty} (-1)^k \frac{(z - z_0)^{2k+1}}{(2k+1)!}.$$

6. (a) Solve the equation $\cosh(z) = \sqrt{2}/2$.
   (b) Solve the equation $\sin(z) = \cosh 4$.

Solution.

(a) We have $\cosh z = (e^z + e^{-z})/2$ so that

$$\cosh(z) = \sqrt{2}/2 \Leftrightarrow \frac{e^z + e^{-z}}{2} = \sqrt{2}/2 \Leftrightarrow e^z + e^{-z} = \sqrt{2} \Leftrightarrow e^{2z} + 1 = \sqrt{2} e^z,$$

as $e^z \neq 0$. We rewrite the equation to see that it is a quadratic equation with unknown e^z, so that we can set $w = e^z$:

$$\begin{aligned} (e^z)^2 - \sqrt{2} e^z + 1 = 0 &\Leftrightarrow w^2 - \sqrt{2} w + 1 = 0 \\ &\Leftrightarrow w = \frac{\sqrt{2} \pm \sqrt{2 - 4}}{2} = \frac{\sqrt{2} \pm i\sqrt{2}}{2} = \frac{1}{\sqrt{2}}(1 \pm i) = e^{\pm i\pi/4}. \end{aligned}$$

We get

$$e^z = e^{\pm i\pi/4} \Leftrightarrow z = \pm i\pi/4 + 2\pi i k, \quad k \in \mathbb{Z}.$$

(b) We have $\sin z = (e^{iz} + e^{-iz})/(2i)$ so that

$$\begin{aligned}\sin(z) = \cosh 4 &\Leftrightarrow \frac{e^{iz} - e^{-iz}}{2i} = \cosh 4\\ &\Leftrightarrow e^{iz} - e^{-iz} = 2i\cosh 4 \Leftrightarrow e^{2iz} - 1 = 2i\cosh 4 e^{iz},\end{aligned}$$

as $e^{iz} \neq 0$. We rewrite the equation to see that it is a quadratic equation with unknown e^{iz}, so that we can set $w = e^{iz}$:

$$\begin{aligned}(e^{iz})^2 - 2i\cosh 4 e^{iz} - 1 = 0 &\Leftrightarrow w^2 - 2i\cosh 4 w - 1 = 0\\ &\Leftrightarrow w = i\cosh 4 \pm \sqrt{-\cosh^2 4 + 1} = i\cosh 4 \pm i\sinh 4 = ie^4 \text{ or } ie^{-4},\end{aligned}$$

since $\cosh^2 u - \sinh^2 u = 1$. Since $i = e^{\pi i/2}$ we get

$$\begin{aligned}e^{iz} = e^{i\pi/2 \pm 4} &\Leftrightarrow iz = i\pi/2 \pm 4 + 2\pi ik, \quad k \in \mathbb{Z}\\ &\Leftrightarrow z = \pi/2 + 2\pi k \pm i4, \quad k \in \mathbb{Z}.\end{aligned}$$

7. Denote by Log (z) the principal logarithm. Show that if $\mathrm{Re}(z_1) > 0$ and $\mathrm{Re}(z_2) > 0$, then

$$\mathrm{Log}\,(z_1 z_2) = \mathrm{Log}\,(z_1) + \mathrm{Log}\,(z_2).$$

Show that this formula is not true in general by finding a counterexample.

Solution. Recall that the principal logarithm of z is defined as

$$\mathrm{Log}\, z = \log |z| + i\mathrm{Arg}\,(z), \quad \mathrm{Arg}\,(z) \in (-\pi, \pi].$$

If z_k satisfy $\mathrm{Re}(z_k) > 0$, the principal arguments Arg (z_k) are in the interval $(-\pi/2, \pi/2)$. We know that

$$\log |z_1 z_2| = \log(|z_1||z_2|) = \log |z_1| + \log |z_2|$$

by the familiar properties of the standard logarithms, as here $|z_1|$ and $|z_2|$ are real. As far as the principal argument of $z_1 z_2$, we choose the argument of $z_1 z_2$ in the range $(-\pi, \pi]$. As $\mathrm{Arg}\, z_1 + \mathrm{Arg}\, z_2 \in (-\pi, \pi)$ and the argument of the product can be taken to be the sum of the arguments $(+2\pi ik,\ k \in \mathbb{Z})$, we get

$$\mathrm{Arg}\,(z_1 z_2) = \mathrm{Arg}\, z_1 + \mathrm{Arg}\,(z_2).$$

By summing the real and imaginary parts, the result is the identity:

$$\mathrm{Log}\,(z_1 z_2) = \mathrm{Log}\,(z_1) + \mathrm{Log}\,(z_2).$$

In general the formula is not correct and it suffices to choose two complex numbers with principal arguments summing up to an angle $> \pi$. E.g. we can take $z_1 = i$, $z_2 = -1$. Then

$$\begin{aligned} \mathrm{Log}\,(z_1) = \mathrm{Log}\,(i) = &\ \log|i| + i\pi/2 = \log 1 + i\pi/2 = i\pi/2, \\ \mathrm{Log}\,(z_2) = &\ \log|-1| + i\pi = i\pi. \end{aligned}$$

These give

$$\mathrm{Log}\,(z_1) + \mathrm{Log}\,(z_2) = 3\pi i/2.$$

On the other hand $z_1 z_2 = -i$, and hence

$$\mathrm{Log}\,(-i) = \log|-i| + i(-\pi/2) = -i\pi/2.$$

8. Find the image of the semi-infinite strip $x > 0$ and $0 < y < \pi$ under the transformation $w = e^z$. Justify your answer.

Solution. Denote

$$S = \{z = x + iy : x > 0, 0 < y < \pi\}.$$

Set $w = u + iv$. Since $w = e^{x+iy} = e^x e^{iy}$ has argument y, the image w belongs to the upper half-plane. Furthermore, $|e^x| > 1$, so that $w \in \Omega$ where

$$\Omega = \{w : |w| > 1,\ \mathrm{Im}\, w > 0\}.$$

Let us show that e^z is surjective. Pick a $w \in \Omega$. On the domain Ω the function $\mathrm{Log}\, w$ is well-defined. Let

$$z = \mathrm{Log}\, w = \ln|w| + i\,\mathrm{Arg}\, w.$$

Since $|w| > 1$, we have $\mathrm{Re}\, z = \ln|w| > 0$. Furthermore, since $\mathrm{Im}\, w > 0$, we have $\mathrm{Arg}\, w \in (0, \pi)$, and hence $\mathrm{Im}\, z = \mathrm{Arg}\, w \in (0, \pi)$. Thus $z \in S$, as claimed.

9. For $z = x + iy$ evaluate

$$I_{m,n} = \int\int_{|z|<1} z^m \bar{z}^n \, dxdy,$$

where $m, n = 0, 1, 2, ...$

Solution. Using the polar coordinates, write $z = \rho e^{i\phi}$, $\rho = |z|$, $\phi = \arg z$, so

$$I_{m,n} = \int_0^1 \int_0^{2\pi} \rho^{m+n} e^{i(m-n)\phi} d\phi \rho d\rho.$$

It is straightforward to see that

$$\int_0^{2\pi} e^{i(m-n)\phi} d\phi = \begin{cases} 0, & m \neq n; \\ 2\pi, & m = n. \end{cases}$$

Thus

$$I_{m,n} = 0, \quad \text{if} \quad m \neq n.$$

If $m = n$, then

$$I_{m,n} = I_{m,m} = 2\pi \int_0^1 \rho^{2m} \rho d\rho = \frac{\pi}{m+1}.$$

10. Evaluate $\displaystyle\int_\gamma \frac{z+i}{z-2i}\, dz$ where γ is:

 (a) the semi-circle from 0 to $4i$ that lies to the right of the imaginary axis, with initial point $z = 0$;

 (b) the boundary of the circle $|z - 2i| = 2$ taken in the positive (anti-clockwise) sense.

Solution.

(a) Here $\gamma(t) = 2i + 2e^{it}$ where $t \in [-\pi/2, \pi/2]$ so $\gamma'(t) = i2e^{it}$. Then we have

$$\begin{aligned}
\int_\gamma \frac{z+i}{z-2i}\, dz &= \int_{-\pi/2}^{\pi/2} \frac{2i + 2e^{it} + i}{2e^{it}} i2e^{it}\, dt = \int_{-\pi/2}^{\pi/2} i(3i + 2e^{it})\, dt \\
&= \int_{-\pi/2}^{\pi/2} (-3 + 2ie^{it})\, dt \\
&= \left[-3t + 2e^{it}\right]_{-\pi/2}^{\pi/2} = -\frac{3\pi}{2} + 2e^{i\pi/2} - \frac{3\pi}{2} - 2e^{-i\pi/2} \\
&= -3\pi + 2i - 2(-i) = -3\pi + 4i.
\end{aligned}$$

Alternatively, note that $f(z) = 1 + \frac{3i}{z-2i}$ and that f has antiderivative $F(z) = z + 3i \log(z - 2i)$, which is holomorphic on Γ (this should be stated explicitly to get full marks). Thus we may apply the antiderivative theorem to get

$$\int_\gamma \frac{z+i}{z-2i}\, dz = F(4i) - F(0) = -3\pi + 4i.$$

(b) Here $\gamma(t) = 2i + 2e^{it}$ where $t \in [-\pi, \pi]$ so $\gamma'(t) = i2e^{it}$. Then we have (similarly as above)

$$\int_\gamma \frac{z+i}{z-2i}\,dz = \left[-3t + 2e^{it}\right]_{-\pi}^{\pi} = -3\pi + 2e^{i\pi} - 3\pi - 2e^{-i\pi}$$
$$= -6\pi + 2(-1) - 2(-1) = -6\pi.$$

Alternatively, apply Cauchy's integral formula for this part to get

$$\int_\gamma \frac{z+i}{z-2i}\,dz = 2\pi i \cdot (2i + i) = -6\pi.$$

11. Evaluate the integral

$$\int_\gamma z^2 dz,$$

where γ is the circular contour of the form $\gamma(t) = 2r + re^{it}$, $t \in [0, 2\pi]$, with some $r > 0$.

Solution. By definition of the integral,

$$I = \int_\gamma z^2 dz = \int_0^{2\pi} (2r + re^{it})^2 ire^{it} dt = ir^3 \int_0^{2\pi} (2 + e^{it})^2 e^{it} dt$$
$$= ir^3 \int_0^{2\pi} (4 + 4e^{it} + e^{2it}) e^{it} dt = 0.$$

12. Evaluate $I = \displaystyle\int_\gamma |z|\bar{z}dz$ along each of the following paths joining the point $z = 0$ to the point $z = 1 + i$:

(a) $y = x$;

(b) the polygonal line through the points $z = 0$, $z = i$ and $z = 1 + i$ with initial point $z = 0$.

Solution.

(a) Parametrisation: $\gamma(t) = t(1 + i), t \in [0, 1]$. Thus $|z|\bar{z} = |1 + i|(1 - i)t^2$, and

$$I = (1 + i)|1 + i|(1 - i) \int_0^1 t^2 dt = \frac{2\sqrt{2}}{3}.$$

(b) Represent $\gamma = \gamma_1 \cup \gamma_2$ where $\gamma_1(t) = it, t \in [0,1]$ and $\gamma_2(t) = i + t, t \in [0,1]$. Thus

$$I_1 := \int_{\gamma_1} |z|\bar{z}dz = (-i)i \int_0^1 t^2 dt = \frac{1}{3},$$

and

$$\begin{aligned} I_2 := \int_{\gamma_2} |z|\bar{z}dz &= \int_0^1 |i+t|(t-i)dt = \int_0^1 \sqrt{1+t^2}(t-i)dt \\ &= \int_0^1 \sqrt{1+t^2}t dt - i \int_0^1 \sqrt{1+t^2}dt =: I_{21} - iI_{22}. \end{aligned}$$

The first one is easy:

$$I_{21} = \frac{1}{2}\int_0^1 \sqrt{1+s}ds = \frac{1}{3}(2\sqrt{2}-1).$$

For the second one integrate by parts:

$$\begin{aligned} I_{22} &= t\sqrt{1+t^2}\Big|_0^1 - \int_0^1 \frac{t^2}{\sqrt{t^2+1}}dt = \sqrt{2} - I_{22} + \int_0^1 \frac{1}{\sqrt{1+t^2}}dt \\ &= \sqrt{2} - I_{22} + \ln(t + \sqrt{1+t^2})\Big|_0^1 \\ &= \sqrt{2} - I_{22} + \ln(1+\sqrt{2}). \end{aligned}$$

Thus

$$I_{22} = \frac{1}{2}(\sqrt{2} + \ln(1+\sqrt{2})).$$

Therefore

$$\begin{aligned} I = I_1 + I_{21} - iI_{22} &= \frac{1}{3} + \frac{1}{3}(2\sqrt{2}-1) - \frac{i}{2}(\sqrt{2} + \ln(1+\sqrt{2})) \\ &= \frac{2\sqrt{2}}{3} - \frac{i}{2}(\sqrt{2} + \ln(1+\sqrt{2})). \end{aligned}$$

13. Find the length of the path $\gamma(t) = \sin t$, $t \in [0,\pi]$.

Solution. Since $\gamma'(t) = \cos t$, by definition we have

$$L(\gamma) = \int_0^\pi |\cos t|\, dt = 2\int_0^{\frac{\pi}{2}} \cos t\, dt = 2\, \sin t\Big|_0^{\frac{\pi}{2}} = 2.$$

14. By considering the contour integral

$$\int_{|z|=2} \left(\frac{z}{2} - \frac{2}{z}\right)^{2n} \frac{dz}{z}$$

prove that

$$\int_0^{2\pi} \sin^{2n} t\, dt = 2\pi \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

Solution. We use the binomial theorem for $(z/2 - 2/z)^n$ to get

$$\begin{aligned}
&\int_{|z|=2} \sum_{j=0}^{2n} \binom{2n}{j} (z/2)^j (-2z^{-1})^{2n-j} z^{-1} dz \\
&\qquad = \sum_{j=0}^{2n} \binom{2n}{j} (-1)^{2n-j} 2^{2(n-j)} \int_{|z|=2} z^{2j-2n-1} dz \\
&\qquad = (-1)^n 2\pi i \binom{2n}{n},
\end{aligned}$$

since $\int_{|z|=2} z^k dz = 2\pi i$ if $k = -1$ and 0 otherwise. Now we substitute $z = 2e^{it}$ to get (use $2i \sin t = e^{it} - e^{-it}$)

$$\int_{|z|=2} \left(\frac{z}{2} - \frac{2}{z}\right)^{2n} \frac{dz}{z} = \int_0^{2\pi} (2i \sin t)^{2n} \frac{2ie^{it}}{2e^{it}} dt = 2^{2n} (-1)^n i \int_0^{2\pi} \sin^{2n} t\, dt.$$

Comparing the two results we get

$$\int_0^{2\pi} \sin^{2n} t\, dt = \frac{2\pi}{2^{2n}} \binom{2n}{n} = \frac{2\pi}{2^{2n}} \frac{(2n)!}{n!n!} = \frac{2\pi}{2^{2n}} \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdots (2n-1) \cdot (2n)}{1 \cdot 2 \cdots n \cdot 1 \cdot 2 \cdots n}.$$

We cancel the even numbers from the numerator with a factor of 2 and an integer from 1 to n. This gives

$$\int_0^{2\pi} \sin^{2n} t\, dt = \frac{2\pi \cdot 1 \cdot 3 \cdots (2n-1)}{2^n \cdot 1 \cdot 2 \cdots n}$$

and this gives the result by doubling every integer from 1 to n to get the even integers from 2 to $2n$ in the denominator.

15. Using the formula for the sum of the geometric series prove the following formulae:

$$\frac{1}{z-b} = -\sum_{k=0}^{\infty} b^{-k-1} z^k, \quad |z| < |b|; \quad \frac{1}{z-b} = \sum_{k=-\infty}^{-1} b^{-k-1} z^k, \quad |z| > |b|.$$

$$\frac{1}{z-b} = \sum_{n=-\infty}^{-1} (b-a)^{-n-1}(z-a)^n, \quad a \neq b, \quad |z-a| > |b-a|.$$

Solution. For $|z| < |b|$ write:

$$\frac{1}{z-b} = -\frac{1}{b(1-zb^{-1})} = -b^{-1}\sum_{k=0}^{\infty} z^k b^{-k} = -\sum_{k=0}^{\infty} z^k b^{-k-1}.$$

For $|z| > |b|$ use the above formula with b and z interchanged:

$$\frac{1}{z-b} = \sum_{k=0}^{\infty} b^k z^{-k-1}.$$

Let $n = -k-1$, so that the series transforms into

$$\frac{1}{z-b} = \sum_{n=-\infty}^{-1} b^{-n-1} z^n.$$

If $|z-a| > |b-a|$ and $a \neq b$, then $z - b = z - a - (b-a)$, and one can use the second formula again replacing z, b by $z-a$, $b-a$ respectively, so that

$$\frac{1}{z-b} = \sum_{n=-\infty}^{-1} (b-a)^{-n-1}(z-a)^n.$$

16. Let f be holomorphic on a region Ω and satisfy $|f(z) - 1| < 1$ for $z \in \Omega$. Let γ be a contour in Ω. Show that

$$\int_\gamma \frac{f'(z)}{f(z)}\, dz = 0.$$

Hint: Find a primitive of the integrand.

Solution. By integration techniques we know that the primitive should be $\log f(z)$, as the differentiation rules give

$$(\log(f(z)))' = \frac{f'(z)}{f(z)}.$$

Then, since γ is a closed contour, the corollary to the fundamental theorem of calculus gives the answer to be zero. The question is whether $\log f(z)$ is holomorphic. The problem is that $\log(w)$ is not holomorphic on the whole complex plane. We need to choose an argument e.g. make a cut somewhere in the complex plane starting at 0. The principal logarithm uses the principal argument defined with $\text{Arg}(z) \in (-\pi, \pi]$. In our case we see from the inequality

$$|f(z) - 1| < 1 \implies |\text{Re}(f(z) - 1)| < 1 \implies 1 - \text{Re}(f(z)) < 1 \implies \text{Re}(f(z)) > 0.$$

Geometrically: if the complex number $f(z)$ is inside the disk $D(1, 1)$, which is in the right-half plane, then $f(z)$ being in the right-half plane has positive real part. Therefore, the holomorphic function $f(z)$ avoids the negative real axis (including 0). Therefore, we can define $\log f(z)$ using the principal logarithm. This logarithm is holomorphic on $\mathbb{C} \setminus \{z : z \leq 0\}$. Then the composition of the two holomorphic functions $\log w$ and $f(z)$ is holomorphic.

17. Let C be the boundary of the square with vertices at the points $z = 1 + i$, $z = 1 - i$, $z = -1 + i$ and $z = -1 - i$, traced anticlockwise. Without evaluating the integral show that

$$\left| \int_C z^5 \, dz \right| \leq 32\sqrt{2}.$$

Solution. We use the inequality

$$\left| \int_\gamma f(z) \, dz \right| \leq \max_{z \in \gamma} |f(z)| \cdot L(\gamma),$$

where γ is any path with length $L(\gamma)$. Since $f(z) = z^5$, with $|f(z)| = |z|^5$, we need upper bounds for $|z|$ on the contour. The largest $|z|$ will become is the largest distance from 0 to the sides of the square. Since the diagonal has length $\sqrt{2^2 + 2^2} = 2\sqrt{2}$, the maximum distance is $\sqrt{2}$. This gives

$$\max_{z \in \gamma} |f(z)| = \sqrt{2}^5 = 2^{5/2} = 4\sqrt{2}.$$

The length of each side is 2, so $L(\gamma) = 8$. The result now follows.