MATH0013 Complex Analysis (Year 2021/22)
Questions and solutions

In this examination $D(z_0, R) = \{z : |z - z_0| < R\}$ denotes the open disk of radius $R > 0$ centred at $z_0 \in \mathbb{C}$, and $S(z_0, R)$ denotes a positively oriented circular contour of radius R centred at z_0. The notation $H(\Omega)$ is used for the set of all holomorphic functions on the domain Ω.

Section A

1. (a) Let $f = f(z)$ be entire. Assume that for two complex numbers a, b such that $|a| + |b| > 0$, we have $af(z) + b\overline{f(z)} = 0$ for all $z \in \mathbb{C}$. Show that f is a constant on $\mathbb{C}$.

   (b) Let g be entire. Assume that $|g(z)| \le |z| \ln(1 + |z|)$ for all $z \in \mathbb{C}$. Prove that $g(z) = 0$ for all $z \in \mathbb{C}$.

   (c) Explain why the function $\exp(z)$ is conformal in $\mathbb{C}$.
   Show that $\exp(z)$ maps the strip $\Lambda = \{z : 0 < \operatorname{Im} z < \pi\}$ **onto** the upper half-plane $\Pi = \{z : \operatorname{Im} z > 0\}$. Fully justify your answer.

Solution.

(a) If one of the constants a, b is zero then the other is non-zero, and hence $f(z) = 0$ for all $z \in \mathbb{C}$.
If $a \ne 0, b \ne 0$, then

$$\overline{f(z)} = -\frac{a}{b} f(z).$$

Thus $\overline{f} = u - iv$ is entire together with $f = u + iv$. Thus both satisfy Cauchy-Riemann equations:

$$u_x = v_y, u_y = -v_x \quad \text{and} \quad u_x = -v_y, u_y = v_x.$$

Consequently, $u_x = u_y = v_x = v_y = 0$, which implies that $f'(z) = 0$. Therefore f is constant in $\mathbb{C}$.

(b) The Taylor series for g converges absolutely for all $z \in \mathbb{C}$:

$$g(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + \sum_{k=2}^{\infty} a_k z^k.$$

First we prove that $a_k = 0$ for all $k \geq 2$. Let $R > 0$. Then

$$M_R = \max_{|z|=R} |g(z)| \leq R \ln(1+R).$$

Using Cauchy's inequalities we get

$$|a_k| = \frac{|g^{(k)}(0)|}{k!} \leq \frac{M_R}{R^k} \leq R^{1-k} \ln(1+R), \tag{1}$$

for all $R > 0$. For $k \geq 2$ the right-hand side tends to zero as $R \to \infty$. Therefore $a_k = 0$ if $k \geq 2$, as claimed, and hence $g(z) = a_0 + a_1 z$.

It follows from the bound $|g(z)| \leq |z| \ln(1 + |z|)$ that $g(0) = 0$, so $a_0 = 0$. Furthermore, the same bound implies that $|g(z)| \leq |z| \ln(1 + |z|) \leq |z|^2$. Consequently, $a_1 = 0$.

To summarise, all Taylor coefficients equal zero and hence $g(z) = 0$ as required.

(c) Since the derivative of $\exp(z)$ coincides with $\exp(z)$, it does not vanish on $\mathbb{C}$, and hence $\exp(z)$ is conformal.

Let $z \in \Lambda$, i.e. $z = x + iy$ with $0 < y < \pi$. Therefore $\exp(z) = \exp(x) \exp(iy)$. The argument of $\exp(z)$ is y and it is between 0 and π, i.e. z is in the upper half-plane.

Let us show that for every w with $\operatorname{Im} w > 0$, there exists a $z \in \Lambda$ such that $\exp(z) = w$. Define $z = \operatorname{Log} w$ (principal log), i.e. $z = \ln |w| + i \arg w$, where $\arg w \in (-\pi, \pi]$. In fact, since $\operatorname{Im} w > 0$, we have $\arg w \in (0, \pi)$, and hence $z \in \Pi$, as required.

2. (a) Let f be analytic in the disk $D(z_0, R)$. Let $r < R$. Show that for all integer $k \geq 1$ and $m \geq 0$ the formula holds:

$$(k-1)! \int\limits_{S(z_0,r)} \frac{f^{(m)}(z)}{(z-w)^k} dz = (m+k-1)! \int\limits_{S(z_0,r)} \frac{f(z)}{(z-w)^{m+k}} dz,$$

for all $w \in D(z_0, R)$, $w \notin S(z_0, r)$.

(b) Find and classify the singularities of the function in the disk $D(0,1)$:

$$\frac{(\sin z)^3}{\cos(z^2) - 1}$$

Find the residues at every singularity that you found.

(c) Let f be entire. Prove that for all $n = 1, 2, \ldots,$ we have

$$\frac{1}{2\pi i} \int\limits_{S(0,1)} \frac{f(e^z) + f(e^{2z}) + \cdots + f(e^{nz})}{z^3} dz = \frac{n(n+1)(2n+1)}{12} \big(f''(1) + f'(1)\big).$$

Solution.

(a) If w is outside $S(z_0, r)$, then both integrals equal zero by the Cauchy Theorem, since both integrands are analytic in $\overline{D(z_0, r)}$.
If $z \in D(z_0, R)$ but $z \notin \overline{D(z_0, r)}$, then both integrals equal zero as the contour $S(z_0, r)$ does not contain singularities.
Assume that $z \in \operatorname{Int} S(z_0, r)$. By the Cauchy formula for derivatives,

$$(k-1)! \int\limits_{S(z_0,r)} \frac{f^{(m)}(z)}{(z-w)^k} dz = 2\pi i f^{(m+k-1)}(w),$$

and

$$(m+k-1)! \int\limits_{S(z_0,r)} \frac{f(z)}{(z-w)^{m+k}} dz = 2\pi i f^{(m+k-1)}(w).$$

Thus the integrals coincide, as claimed.

(b) Since $|z|^2 < 1 < 2\pi$, the only singular point is $z = 0$. To determine the type of this singularity expand the numerator and denominator noticing that

$$(\sin z)^3 = (z - \frac{z^3}{6} + \dots)^3 = z^3 + \dots,$$

and

$$\cos(z^2) = 1 - \frac{z^4}{2} + \dots,$$

so that

$$\begin{aligned} f(z) = \frac{z^3 + \dots}{1 - \frac{z^4}{2} - 1 + \dots} &= \frac{2z^3 + \dots}{-z^4(1 + \dots)} \\ &= -\frac{2 + \dots}{z(1 + \dots)}. \end{aligned}$$

Since

$$\lim_{z \to 0} z f(z) = -2,$$

the singularity at $z = 0$ is a simple pole and the residue equals -2.

(c) The function $F(z) = f(e^z) + f(e^{2z}) + \dots + f(e^{nz})$ is also entire. By the Cauchy formula for the second derivative,

$$\frac{1}{2\pi i} \int_{S(0,1)} \frac{F(z)}{z^3}\, dz = \frac{F''(0)}{2}.$$

To calculate the derivative $F''(z)$ observe that

$$\begin{aligned} \frac{d^2}{dz^2}\big(f(e^{kz})\big) &= k^2 e^{2kz} f''(e^{kz}) + k^2 e^{kz} f'(e^{kz}) \\ &= k^2\big(e^{2kz} f''(e^{kz}) + e^{kz} f'(e^{kz})\big). \end{aligned}$$

Therefore,

$$F''(0) = \sum_{k=1}^{n} k^2\big(f''(1) + f'(1)\big) = \frac{n(n+1)(2n+1)}{6}\big(f''(1) + f'(1)\big),$$

which leads to the claimed result.

3. (a) Let $P = P(z)$ be a polynomial of degree at least two. Let $r > 0$ be a number such that $P(z) \neq 0$ for all $z : |z| \geq r$. Prove that

$$\int_{S(0,r)} \frac{1}{P(z)} dz = 0.$$

Would the same conclusion hold for the linear polynomial $P(z)$, i.e. for $P(z) = az + b$ with complex $a \neq 0, b$?

Fully justify your answer.

(b) As usual, we denote $z = x + iy$. Let $f(z) = u(x, y) + iv(x, y)$ be an entire function. Suppose that $u(x, y) \leq x$ for all $z \in \mathbb{C}$. Show that $f(z) = z + c$ with a constant c. Justify your answer.

Provide all details of your argument.

Solution.

(a) Since $f(z) = P(z)^{-1}$ is holomorphic on and outside the circle $S(0, r)$, by the Key-hole Lemma, we have

$$\int_{S(0,r)} f(z)dz = \int_{S(0,R)} f(z)dz,$$

for all $R \geq r$.

Estimate $f(z)$ on the circle $S(0, R)$. Let

$$P(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0, \quad a_n \neq 0,$$

where $n \geq 2$. Since

$$|P(z)| \geq |z|^n \big(|a_n| - |a_{n-1}||z|^{-1} - \cdots - |a_0||z|^{-n}\big),$$

then for sufficiently large R we have

$$|P(z)| \geq \frac{|a_n|}{2} R^n, \quad \text{for} \quad |z| = R,$$

so that $|f(z)| \leq 2|a_n|^{-1} R^{-n}$, $|z| = R$. Estimate the integral using the Standard Integral Bound from the lectures:

$$\left| \int_{S(0,R)} f(z)dz \right| \leq 2\pi R \cdot 2|a_n|^{-1} R^{-n} = 4|a_n|^{-1} \pi R^{1-n}.$$

As $n \geq 2$, the right-hand side tends to zero as $R \to \infty$. Consequently, the integral around $S(0,r)$ equals zero, as required.

For the function $P(z) = az + b$ such an integral would be equal to

$$2\pi i \mathrm{Res}\,(P, -b/a) = \frac{2\pi i}{a} \neq 0,$$

by the Cauchy integral formula, so the answer to the question is "no".

<u>Alternative solution.</u> **This method was not taught in class, but it may be used in the exam.**

Represent

$$\int_{S(0,r)} f(z)\,dz = -2\pi i\,\mathrm{Res}\,(f,\infty) = 2\pi i\,\mathrm{Res}\,\big(f(1/z)1/z^2, 0\big).$$

Write:

$$\begin{aligned} \frac{1}{P\left(\frac{1}{z}\right)z^2} &= \frac{1}{z^2\big(a_n z^{-n} + a_{n-1}z^{-n+1} + \cdots + a_0\big)} \\ &= \frac{z^{n-2}}{a_n + a_{n-1}z + \cdots + a_0 z^n}. \end{aligned}$$

For $n \geq 2$ this function is holomorphic at $z = 0$, and hence the residue equals zero.

The solution for the linear function is the same as in the main version of the solution.

(b) Let $g(z) = e^{f(z)-z}$. Then $|g(z)| = e^{u(x,y)-x}$. Since $u(x,y) \leq x$, we have $|g(z)| \leq 1$ for all z. And since $g(z)$ is entire, $g(z)$ must be constant by Louville's theorem. Therefore, $g'(z) = 0$. That is, $(f'(z) - 1)e^{f(z)-z} = 0$ and hence $f'(z) = 1$ for all z. So $f(z) = z + c$ for some constant c.

Section B

4. (a) Let $\Omega \subset \mathbb{C}$ be a domain.

   i. Let $w \in H(\Omega)$, and let $u(z) = e^{w(z)}$. Write a formula expressing the derivative of w via the function u and its derivative.

   ii. Let $g \in H(\Omega)$. Assume that Ω is convex and that $g(z) \neq 0$ for all $z \in \Omega$. Prove that there exists a function $f \in H(\Omega)$ such that $e^{f(z)} = g(z)$.
   Explain why such a function f is not unique by exhibiting two distinct functions with the same property.
   Fully justify your answer.
   Hint: use part (i) of the question and apply the antiderivative theorem.

   (b) Let a_n be a sequence of complex numbers such that $\sum_{n=0}^{\infty} |a_n| < \infty$. Suppose that

$$\sum_{n=0}^{\infty} a_n k^{-n} = 0 \quad \text{for} \quad k = 1, 2, \ldots.$$

   Prove that $a_n = 0$ for all n.

Solution.

(a) i. Differentiate w: $u' = w'e^w = w'u$, so that $w' = u'/u$.

ii. Pick a point $z_0 \in \Omega$. Since Ω is convex, by the antiderivative theorem, the function

$$h(z) = \int\limits_{[z_0, z]} \frac{g'(w)}{g(w)}\, dw, \quad z \in \Omega,$$

is an antiderivative of $g'g^{-1}$. The latter function is holomorphic on Ω as g does not vanish on Ω. Recall also that $[z_0, z]$ is a segment from z_0 to z. It lies entirely within Ω thanks to the convexity of Ω.
Calculate for an arbitrary complex constant a:

$$\frac{d}{dz}\big(g(z)e^{-h(z)-a}\big) = g'(z)e^{-h(z)-a} - h'(z)g(z)e^{-h(z)-a} = 0.$$

As Ω is connected, this means that ge^{-h-a} is constant on Ω, and hence

$$g(z)e^{-h(z)-a} = g(z_0)e^{-a}.$$

Choose $a = \log g(z_0)$, so that the right-hand side of the above equality is equal to 1. Choice of the log-branch is inessential, for instance one can take the principal log. Therefore

$$g(z)e^{-f(z)} = 1, \quad \text{where} \quad f(z) = h(z) + a,$$

and $g(z) = e^{f(z)}$, as required.
The function f is not unique – for instance, $f(z) + 2\pi i$ also satisfies the required property: $g(z) = e^{f(z)+2\pi i}$.

(b) Define the function

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

Since $\sum |a_n|$ converges, the radius of convergence is at least 1. We know that for the sequence $z_k = k^{-1}$ the values of f vanish, i.e. $f(z_k) = 0$, Since $z_k \to 0$ as $k \to \infty$, by the unique continuation theorem, $f(z) = 0$ for all $|z| < 1$. Consequently, $a_n = 0$ for all n, as claimed.

<u>Alternative solution.</u> Denote

$$b_j(k) = \sum_{n=j}^{\infty} a_n k^{-n}, \quad j = 0, 1, \dots.$$

We have $b_0(k) = 0$ for all $k = 1, 2, \dots$. Also, for each $j \geq 1$ we have

$$k^{j-1}|b_j(k)| = k^{-1}\left|\sum_{n=j}^{\infty} a_n k^{-n+j}\right| \leq k^{-1}\sum_{n=0}^{\infty}|a_n| \to 0, \tag{2}$$

as $k \to \infty$. According to (2) with $j = 1$,

$$0 = a_0 + b_1(k) \to a_0, \quad \text{as} \quad k \to \infty,$$

so that $a_0 = 0$. Assume that $a_0 = a_1 = a_l = 0$, and let us prove that $a_{l+1} = 0$. To this end write:

$$0 = k^{l+1}\sum_{n=l+1}^{\infty} a_n k^{-n} = a_{l+1} + k^{l+1}b_{l+2}(k) \to a_{l+1},$$

as $k \to \infty$, so that $a_{l=1} = 0$, as claimed. By induction, all coefficients a_n vanish.

5. (a) Consider the path $\gamma(t) = c + it$, $t \in (-\infty, \infty)$, where $c > 0$. For every $x > 0$ prove that

$$\frac{1}{2\pi i}\int\limits_{\gamma} \frac{x^s}{s(s+1)}ds = \begin{cases} 1 - x^{-1}, & x \geq 1, \\ 0, & x < 1. \end{cases}$$

The power x^s is defined as $x^s = e^{s \ln x}$.

(b) Let f be a function holomorphic on the domain $\Lambda = \{z : |z| > 1\}$. Let

$$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n, \quad z \in \Lambda,$$

be its Laurent expansion. Assuming that f is a bounded function, prove that $a_n = 0$ for all $n > 0$.

Solution.

(a) Rewrite the integral by substituting $s = c + it$:

$$\begin{aligned} &\frac{x^c}{2\pi}\int_{-\infty}^{\infty} \frac{x^{it}}{(c+it)(c+1+it)}\,dt \\ =&\ \frac{x^c}{2\pi}\int_{-\infty}^{\infty} \frac{x^{it}}{-(-ic+t)(-i(c+1)+t)}\,dt \\ =&\ -\frac{x^c}{2\pi}\int_{-\infty}^{\infty} \frac{e^{iat}}{(t-ic)(t-i(c+1))}\,dt = -\frac{x^c}{2\pi}I, \end{aligned}$$

where $a = \ln x$.
Write:

$$I = \lim_{R\to\infty} I_R, \quad I_R = \int_{-R}^{R} \frac{e^{iat}}{(t-ic)(t-i(c+1))}\,dt$$

Assume first that $x \geq 1$, and hence $a \geq 0$. We consider the integral

$$\tilde{I}_R = \int_{\sigma_R} f(z)\,dz, \quad f(z) = \frac{e^{iaz}}{(z-ic)(z-i(c+1))},$$

around the contour σ_R formed by the straight segment $[-R, R]$ and the semi-circular path $\gamma_R(\theta) = Re^{i\theta}, \theta \in [0, \pi]$. For sufficiently large R both zeroes of the denominator are in the upper half-plane, and hence by the Residue Theorem,

$$\tilde{I}_R = 2\pi i\big(\mathrm{Res}\,(f, ic) + \mathrm{Res}\,(f, i(c+1))\big).$$

The singularities are simple poles, and hence

$$\mathrm{Res}\,(f, ic) = \lim_{z\to ic} \frac{e^{iaz}}{z - i(c+1)} = ie^{-ac},$$
$$\mathrm{Res}\,(f, i(c+1)) = \lim_{z\to i(c+1)} \frac{e^{iaz}}{z - ic} = -ie^{-a(c+1)},$$

Consequently,

$$\tilde{I}_R = -2\pi e^{-ac}(1 - e^{-a}) = -2\pi x^{-c}(1 - x^{-1}).$$

Estimate the contribution from the semicicrle. To this end observe that the function f satisfies the bound

$$\max_{|z|=R,\ \mathrm{Im}\, z\ge 0} |f(z)| \le \frac{e^{-a\,\mathrm{Im}\, z}}{(|z|-c)(|z|-(c+1))} \le M\frac{1}{R^2},$$

with some constant $M > 0$. Therefore by the Standard Integral Bound,

$$\left|\int_{\gamma_R} f(z)dz\right| \le \pi \cdot \max_{|z|=R,\ \mathrm{Im}\, z\ge 0} |f(z)| \le \frac{\pi M}{R} \to 0,\ R\to\infty.$$

As a result,

$$I = \lim_{R\to\infty} \tilde{I}_R = -2\pi x^{-c}(1 - x^{-1}),$$

and

$$\frac{1}{2\pi i}\int\limits_{\gamma} \frac{x^s}{s(s+1)} ds = 1 - x^{-1},$$

as claimed.

Assume now that $0 < x < 1$, i.e. $a < 0$. We consider the integral

$$\tilde{I}_R = \int_{\omega_R} f(z)\, dz,$$

around the contour ω_R formed by the straight segment $[-R, R]$ and the semi-circular path $\gamma_R(\theta) = Re^{i\theta}, \theta \in [0, -\pi]$, in the lower half-plane. Since the integrand has no singularities in the lower half-plane, the integral equals zero. Furthermore, estimating the function on the semicircle in the same way as before, we conclude that the contribution from γ_R tends to zero as $R \to \infty$. This means that $I = 0$, as required.

(b) Assume that $|f(z)| \le M$, for all $|z| > 1$. Recall the formula for the Laurent coefficients:

$$a_n = \frac{1}{2\pi i} \int\limits_{S(0,R)} \frac{f(z)}{z^{n+1}}\, dz, \quad n \in \mathbb{Z},$$

where $R > 1$ is arbitrary. Estimate a_n using the Standard Integral Bound:

$$|a_n| \le \frac{1}{2\pi} 2\pi R \cdot \frac{\max_{|z|=R} |f(z)|}{R^{n+1}} \le MR^{-n}.$$

If $n > 0$, then the right-hand side tends to zero as $R \to \infty$. Therefore $a_n = 0$, as claimed.