

**Sheet 4, solutions**

Below  $S(z, r)$  is a circular positively oriented contour of radius  $r > 0$  centred at  $z$ .

- Evaluate the integral

$$\frac{1}{2\pi i} \int_{\gamma} \frac{e^z}{z^2 + a^2} dz,$$

where  $a > 0$  is a real number and  $\gamma$  is a contour such that  $\overline{D}(0, a) \subset \text{Int } \gamma$ .

*Solution.* Using partial fractions, we get

$$\frac{1}{z^2 + a^2} = \frac{i}{2a(z + ia)} - \frac{i}{2a(z - ia)}.$$

Thus the integral equals  $I_1 + I_2$ , where

$$I_1 = \frac{1}{2\pi i} \int_{\gamma} \frac{ie^z}{2a(z + ia)} dz, \quad I_2 = -\frac{1}{2\pi i} \int_{\gamma} \frac{ie^z}{2a(z - ia)} dz.$$

By the Cauchy formula,

$$I_1 = \frac{i}{2a} e^{-ia}, \quad I_2 = -\frac{i}{2a} e^{ia},$$

so that the integral equals

$$-\frac{i}{2a} e^{ia} + \frac{i}{2a} e^{-ia} = \frac{1}{a} \frac{e^{ia} - e^{-ia}}{2i} = \frac{\sin a}{a}$$

- Let  $f$  be analytic on  $\Omega$ , and let  $\overline{D}(a, R) \subset \Omega$  with some  $a \in \mathbb{C}$ .

- Using the Cauchy Integral Formula prove that

$$f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + Re^{it}) dt.$$

(b) Evaluate the integral

$$\int_{r<|z-a|<R} f(z) dx dy,$$

where  $r < R$ .

*Solution.*

(a) Use the Cauchy formula:

$$f(a) = \frac{1}{2\pi i} \int_{S(a,R)} \frac{f(z)}{z-a} dz.$$

Substitute the parametrisation  $z = a + Re^{it}$ ,  $t \in [0, 2\pi]$ :

$$f(a) = \frac{1}{2\pi i} \int_0^{2\pi} \frac{f(a + Re^{it})}{Re^{it}} iRe^{it} dt = \frac{1}{2\pi} \int_0^{2\pi} f(a + Re^{it}) dt,$$

as required.

(b) Use the polar coordinates:

$$\begin{aligned} \int_{r<|z-a|<R} f(z) dx dy &= \int_{r<|z-a|<R} \int_0^{2\pi} f(a + \rho e^{i\phi}) d\phi \rho d\rho \\ &= 2\pi f(a) \int_{r<|z-a|<R} \rho d\rho = \pi f(a)(R^2 - r^2). \end{aligned}$$

3. Evaluating the integral

$$\frac{1}{2\pi i} \int_{S(0,1)} \frac{1}{(z-a)(z-a^{-1})} dz,$$

prove that for any  $a \in (0, 1)$ ,

$$\int_0^{2\pi} \frac{dt}{1 + a^2 - 2a \cos t} = \frac{2\pi}{1 - a^2}.$$

*Solution.* By Cauchy formula, for  $a \in (0, 1)$  we have

$$I := \frac{1}{2\pi i} \int_S \frac{1}{(z-a)(z-a^{-1})} dz = \frac{1}{a - a^{-1}}.$$

On the other hand, with the usual parametrisation  $\gamma(t) = e^{it}$ ,  $t \in [0, 2\pi]$ , we obtain that

$$I = i \frac{1}{2\pi i} \int_0^{2\pi} e^{it} \frac{1}{(e^{it}-a)(e^{it}-a^{-1})} dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{a}{2a \cos t - a^2 - 1} dt$$

This follows from this simple calculation:

$$w^{-1}(w-a)(w-a^{-1}) = \frac{1}{a}(1-aw^{-1})(aw-1) = \frac{1}{a}(aw-a^2-1+aw^{-1}).$$

Thus

$$-\frac{1}{2\pi} \int_0^{2\pi} \frac{a}{1+a^2-2a \cos t} dt = \frac{1}{a-a^{-1}},$$

which gives the required result.

4. Let  $f$  be continuous on  $S(0, 1)$ .

(a) Show that

$$\overline{\int_{S(0,1)} f(z) dz} = - \int_{S(0,1)} \frac{\overline{f(z)}}{z^2} dz.$$

Use the identity  $\overline{\int_a^b g(t) dt} = \int_a^b \overline{g(t)} dt$  which holds for any continuous function  $g : [a, b] \rightarrow \mathbb{C}$ .

(b) Let  $f$  be an entire function. Prove that

$$\frac{1}{2\pi i} \int_{S(0,1)} \frac{\overline{f(z)}}{z-a} dz = \begin{cases} \overline{f(0)}, & \text{if } |a| < 1; \\ \overline{f(0)} - \overline{f(\bar{a}^{-1})}, & \text{if } |a| > 1. \end{cases}$$

*Solution.*

(a) Use the parametrisation  $\gamma(t) = e^{it}$ ,  $t \in [0, 2\pi]$ :

$$\begin{aligned} \overline{\int_S f(z) dz} &= -i \overline{\int_0^{2\pi} e^{it} f(e^{it}) dt} = -i \int_0^{2\pi} e^{-it} \overline{f(e^{it})} dt \\ &= -i \int_0^{2\pi} e^{-2it} \overline{f(e^{it})} e^{it} dt = - \int_S \frac{\overline{f(z)}}{z^2} dz, \end{aligned}$$

as claimed.

(b) Since  $z\bar{z} = 1$ , one can rewrite, using the first part of the exercise:

$$\int_S \frac{\overline{f(z)}}{z-a} dz = \int_S \overline{\left( \frac{f(z)}{z^2(\bar{z}-\bar{a})} \right)} \frac{1}{z^2} dz = - \int_S \frac{\overline{f(z)}}{z^2(\bar{z}-\bar{a})} dz = - \int_S \frac{\overline{f(z)}}{z(1-\bar{a}z)} dz.$$

Let us evaluate

$$I = \int_S \frac{f(z)}{z(1 - \bar{a}z)} dz.$$

By partial fractions,

$$I = I_1 + I_2, \quad I_1 = \int_S \frac{f(z)}{z} dz, \quad I_2 = \bar{a} \int_S \frac{f(z)}{1 - \bar{a}z} dz.$$

By Cauchy formula,

$$I_1 = 2\pi i f(0).$$

If  $|a| > 1$ , then by Cauchy formula

$$I_2 = - \int_S \frac{f(z)}{z - \bar{a}^{-1}} dz = -2\pi i f(\bar{a}^{-1}).$$

If  $|a| < 1$ , then  $I_2 = 0$ . Since

$$\int_S \frac{\overline{f(z)}}{z - a} dz = -\bar{I},$$

the required formula is now proved.

5. Let  $f$  be entire.

- (a) Show that, if  $e^f$  is bounded, then  $f$  is constant.
- (b) Assume that  $\operatorname{Im}(f)$  is bounded below. Show that  $f$  is a constant function.

*Solution.*

- (a) The function  $e^f$  is entire, as composition of  $e^w$  and  $f$ . Since it is given to be bounded, we can apply Liouville's theorem and deduce that  $e^f$  is constant, say  $= k \neq 0$ . Therefore

$$0 = \frac{d}{dz} e^{f(z)} = f'(z) e^{f(z)} = k f'(z).$$

As  $k \neq 0$ , we conclude that  $f'(z) = 0$  for all  $z \in \mathbb{C}$ . Therefore  $f = \operatorname{const}$ .

- (b) Assume that  $\operatorname{Im}(f(z)) \geq m$  for all  $z \in \mathbb{C}$ . Consider the entire function  $g(z) = i f(z)$ . We have

$$|e^{g(z)}| = |e^{if(z)}| = e^{-\operatorname{Im}(f(z))} \leq e^{-m}.$$

By (a)  $g(z) = i f(z)$  is a constant, which implies that  $f$  is a constant.

6. Find the integral

$$h(z) = \int_{S(0,1)} \frac{dw}{w(w-z)},$$

for  $|z| > 1$  and  $0 < |z| < 1$ .

*Solution.* If  $|z| > 1$ , then by the Cauchy Integral Formula,

$$h(z) = -\frac{2\pi i}{z}.$$

If  $0 < |z| < 1$ , then represent, using partial fractions:

$$\frac{1}{w(w-z)} = \frac{1}{z(w-z)} - \frac{1}{zw}.$$

Then by the Cauchy Integral Formula,

$$\int_{S(0,1)} \frac{dw}{z(w-z)} = \frac{2\pi i}{z}, \quad \int_{S(0,1)} \frac{dw}{zw} = \frac{2\pi i}{z}.$$

Thus  $h(z) = 0, 0 < |z| < 1$ .

7. Let the function  $g$  be holomorphic on a domain  $\Omega$  which contains the disk  $\overline{D}(0, 1)$ . Show that

$$\frac{1}{2\pi i} \int_{S(0,1)} \frac{g(\zeta)}{z\zeta - 1} d\zeta = \begin{cases} 0, & |z| < 1; \\ z^{-1}g(z^{-1}), & |z| > 1. \end{cases}$$

*Solution.* Rewrite the integral:

$$I(z) = \frac{1}{2\pi i} \int_{S(0,1)} \frac{g(\zeta)}{z\zeta - 1} d\zeta = \frac{1}{2\pi i} \int_{S(0,1)} \frac{g(\zeta)}{z(\zeta - z^{-1})} d\zeta.$$

By the Cauchy-Goursat Theorem, if  $|z| < 1$ , then  $I(z) = 0$ . If  $|z| > 1$ , then the Cauchy Integral Formula gives:

$$I(z) = z^{-1}g(z^{-1}),$$

as claimed.

8. Let  $g$  be entire. Assume that  $|g(z)| \leq |z| \ln(1 + |z|)$  for all  $z \in \mathbb{C}$ . Prove that  $g(z) = 0$  for all  $z \in \mathbb{C}$ .

*Solution.* The Taylor series for  $g$  converges absolutely for all  $z \in \mathbb{C}$ :

$$g(z) = \sum_{k=0}^{\infty} a_k z^k = a_0 + a_1 z + \sum_{k=2}^{\infty} a_k z^k.$$

First we prove that  $a_k = 0$  for all  $k \geq 2$ . Let  $R > 0$ . Then

$$M_R = \max_{|z|=R} |g(z)| \leq R \ln(1+R).$$

Using Cauchy's inequalities we get

$$|a_k| = \frac{|g^{(k)}(0)|}{k!} \leq \frac{M_R}{R^k} \leq R^{1-k} \ln(1+R), \quad (1)$$

for all  $R > 0$ . For  $k \geq 2$  the right-hand side tends to zero as  $R \rightarrow \infty$ . Therefore  $a_k = 0$  if  $k \geq 2$ , as claimed, and hence  $g(z) = a_0 + a_1 z$ .

It follows from the bound  $|g(z)| \leq |z| \ln(1+|z|)$  that  $g(0) = 0$ , so  $a_0 = 0$ . Furthermore, the same bound implies that  $|g(z)| \leq |z| \ln(1+|z|) \leq |z|^2$ . Consequently,  $a_1 = 0$ .

To summarise, all Taylor coefficients equal zero and hence  $g(z) = 0$  as required.

9. Let  $f$  be entire. Suppose that  $\operatorname{Re}(f(z)) - \operatorname{Im}(f(z)) < 2$  for all  $z \in \mathbb{C}$ . Show that  $f$  is a constant function.

*Solution.* Consider a simpler example first: let  $g = u + iv$  with real-valued  $u$  and  $v$  be an entire function, and assume that  $v < 0$ . Then  $g = \operatorname{const}$ . Indeed, consider the entire function

$$e^{-ig} = e^{iu+v}.$$

Then

$$|e^{-ig}| = e^v \leq e^0 = 1.$$

Thus by Liouville's theorem,  $e^{-ig} = \operatorname{const}$ , and hence  $g = \operatorname{const}$ , as claimed.

To solve the question we introduce a new entire function:

$$g(z) = (a + ib)f(z) + ic.$$

Our objective is to find such real  $a, b$  and  $c$  such that  $\operatorname{Im} g < 0$ . Representing  $f = u + iv$ , calculate:

$$g(z) = (a + ib)(u + iv) + ic = au - bv + i(bu + av + c).$$

We now that  $u - v - 2 < 0$ . Thus we choose  $b = 1, a = -1, c = -2$ , so that  $\operatorname{Im} g = u - v - 2 < 0$ . Therefore, by the first part of the solution,  $g = \operatorname{const}$ , and hence  $f = \operatorname{const}$  as well.

10. (a) Let  $f$  be an entire function and  $M > 0$  be a constant such that  $|f(z)| \geq M$  for all  $z \in \mathbb{C}$ . Show that  $f$  is a constant function.
- (b) Let  $f$  be an entire function and  $M > 0$  be a constant such that  $|f(z)| \leq Me^x$  for all  $z = x + iy \in \mathbb{C}$ . Show that there exists a constant  $k$  such that  $f(z) = ke^z$  for all  $z \in \mathbb{C}$ .

*Solution.*

- (a) From the inequality  $|f(z)| \geq M > 0$  we deduce that  $f(z) \neq 0$  for all  $z \in \mathbb{C}$ . Therefore, the function

$$g(z) = \frac{1}{f(z)}$$

is holomorphic by the quotient rule on all of  $\mathbb{C}$ , i.e. it is entire. Moreover,

$$|g(z)| = \frac{1}{|f(z)|} \leq \frac{1}{M}.$$

Therefore,  $g$  is entire and bounded. This implies that  $g$  is a constant by Liouville's theorem. So we can find a constant  $k$  such that

$$g(z) = k \implies f(z) = 1/k.$$

Notice that the constant  $k$  cannot be zero, as  $g$  has no roots by construction.

- (b) The appearance of  $e^z$  in the answer suggests that we consider the entire function

$$g(z) = \frac{f(z)}{e^z}.$$

It is holomorphic everywhere, as  $e^z \neq 0$ . We remark that  $|e^z| = e^x$  with  $z = x + iy$ . We get

$$|g(z)| = \frac{|f(z)|}{|e^z|} = \frac{|f(z)|}{e^x} \leq M$$

by the given assumption. This means that  $g$  is entire and bounded. Therefore, we can find a constant  $k$  such that

$$g(z) = k \Leftrightarrow f(z) = ke^z, \quad \forall z \in \mathbb{C}.$$

11. Let  $f(z) = \sum_{k=0}^{\infty} a_k z^k$  be a series with a positive (possibly infinite) radius of convergence  $R$ . Prove that this series converges uniformly for all  $|z| \leq R_1$  where  $R_1 < R$ .

*Solution.* The series converges absolutely for all  $|z| < R$ , so the series

$$\sum |a_k|R_1^k.$$

converges. For all  $|z| \leq R_1$  we have

$$|a_k z^k| \leq |a_k| |z|^k \leq |a_k| R_1^k.$$

Thus by the Weierstrass'  $M$ -test, the series  $f$  converges uniformly for all  $|z| \leq R_1$ , as claimed.

12. Let  $f(z) = \sum_{k=0}^n a_k z^k$  be a polynomial. Using contour integration prove that

$$\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \sum_{k=0}^n |a_k|^2.$$

*Solution.* Rewrite the integral using the substitution

$$z = e^{i\theta}, d\theta = -iz^{-1} dz.$$

Since  $\bar{z} = z^{-1}$  on the unit circle, the left-hand side equals

$$\frac{-i}{2\pi} \int_{|z|=1} |f(z)|^2 z^{-1} dz = \frac{1}{2\pi i} \sum_{k,l=0}^n a_k \bar{a}_l \int_{|z|=1} z^k z^{-l} z^{-1} dz.$$

The integral equals zero unless  $k - l - 1 = -1$ , that is  $k = l$ . In this case

$$\frac{1}{2\pi i} \int_{|z|=1} z^k z^{-l} z^{-1} dz = \frac{1}{2\pi i} \int_{|z|=1} z^{-1} dz = 1.$$

Consequently,

$$\frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^2 d\theta = \sum_{k=0}^n |a_k|^2$$

holds, as claimed.

13. Find the value of the integrals

$$\int_S \frac{z^2 + 3}{z(z^2 + 9)} dz$$

taken around the contour

- (a)  $S = S(0, 1)$ ,
- (b)  $S = S(0, 4)$ .

*Solution.* The denominator has zeros at 0 and  $\pm 3i$ .

For (a) we notice that only 0 is inside it. We define

$$f(z) = \frac{z^2 + 3}{z^2 + 9}.$$

which is holomorphic inside the disc  $|z| \leq 1$ . By Cauchy's integral formula

$$2\pi i f(0) = \int_S \frac{f(z)}{z} dz \implies \int_S \frac{z^2 + 3}{z(z^2 + 9)} dz = 2\pi i \frac{0^2 + 3}{0^2 + 9} = \frac{2\pi i}{3}.$$

For (b) we use partial fractions:

$$\frac{z^2 + 3}{z(z^2 + 9)} = \frac{A}{z} + \frac{B}{z + 3i} + \frac{C}{z - 3i},$$

which gives

$$z^2 + 3 = A(z^2 + 9) + B(z - 3i)z + C(z + 3i)z.$$

Plugging  $z = 0$  we get  $A = 1/3$ . Plugging  $z = 3i$  we get  $-9 + 3 = C \cdot 6i \cdot 3i = -18C$ , so that  $C = 1/3$ . Plugging  $z = -3i$  we get  $-9 + 3 = B \cdot (-6i) \cdot (-3i) = -18B$ , so that  $B = 1/3$ . We use Cauchy's integral formula three times with  $g(z) = 1/3$  to get

$$\begin{aligned} \int_S \frac{z^2 + 3}{z(z^2 + 9)} dz &= 2\pi i \left( \int_S \frac{1/3}{z} dz + \int_S \frac{1/3}{z + 3i} dz + \int_S \frac{1/3}{z - 3i} dz \right) \\ &= 2\pi i(1/3 + 1/3 + 1/3) = 2\pi i. \end{aligned}$$

14. Let  $f$  be holomorphic on a region containing the closed unit disk  $\bar{D}(0, 1)$ . Let  $S = S(0, 1)$ . Show that

$$2f'(0) = \frac{1}{2\pi i} \int_S \frac{f(w) - f(-w)}{w^2} dw.$$

*Solution.* We split the integral on the right-hand side:

$$\frac{1}{2\pi i} \int_S \frac{f(w) - f(-w)}{w^2} dw = \frac{1}{2\pi i} \int_S \frac{f(w)}{w^2} dw - \frac{1}{2\pi i} \int_S \frac{f(-w)}{w^2} dw.$$

Cauchy's formula for  $f'(0)$  gives

$$f'(0) = \frac{1}{2\pi i} \int_S \frac{f(w)}{w^2} dw. \quad (2)$$

For the second integral, we set  $g(z) = f(-z)$  motivated by the expression in the numerator. We have  $g'(z) = -f'(-z)$  by the chain rule. Cauchy's integral formula for  $g'(0)$  is

$$g'(0) = \frac{1}{2\pi i} \int_S \frac{g(w)}{w^2} dw = \frac{1}{2\pi i} \int_S \frac{f(-w)}{w^2} dw,$$

which gives

$$-f'(0) = \frac{1}{2\pi i} \int_S \frac{f(-w)}{w^2} dw.$$

We flip the sign in the above equation, add to Eq. (2) to get the result.

15. Let  $f$  be holomorphic in a domains containing the closed unit disk  $\overline{D}(0, 1)$ . Let  $S = S(0, 1)$ . Show that

$$f(0) + f'(0) = \frac{1}{2\pi i} \int_S \frac{f(w)e^w}{w^2} dw.$$

*Solution.* We apply the Cauchy integral formula for  $n = 1$  and for the function

$$h(z) = f(z)e^z.$$

We have  $h'(z) = f'(z)e^z + f(z)e^z$  by the product rule. This gives

$$h'(0) = f'(0)e^0 + f(0)e^0 = f(0) + f'(0).$$

We get

$$f(0) + f'(0) = h'(0) = \frac{1}{2\pi i} \int_S \frac{h(w)}{w^2} dw = \frac{1}{2\pi i} \int_C \frac{f(w)e^w}{w^2} dw.$$

16. Let  $n \geq 2$  be a positive integer. Let  $f$  be holomorphic in a domain containing the closed unit disk  $\overline{D}(0, 1)$ . Let  $S = S(0, 1)$ . Show that

$$\frac{1}{2\pi i} \int_C \frac{f(w) + f(e^{2\pi i/n}w) + f(e^{4\pi i/n}w) + \cdots + f(e^{2(n-1)\pi i/n}w)}{w^2} dw = 0.$$

*Solution.* Let

$$g(z) = f(z) + f(e^{2\pi i/n}z) + f(e^{4\pi i/n}z) + \cdots + f(e^{2(n-1)\pi i/n}z).$$

Since  $f$  is holomorphic on a region containing the closed unit disk  $\overline{D}(0, 1)$  and for  $|z| \leq 1$ , we have  $|e^{2\pi i/n}z| \leq 1$ ,  $|e^{4\pi i/n}z| \leq 1, \dots, |e^{2(n-1)\pi i/n}z| \leq 1$ , the function  $g$  is holomorphic on a region containing the closed unit disk  $\overline{D}(0, 1)$ . We can apply Cauchy's integral formula for its first derivative to get

$$g'(0) = \frac{1}{2\pi i} \int_C \frac{f(w) + f(e^{2\pi i/n}w) + f(e^{4\pi i/n}w) + \dots + f(e^{2(n-1)\pi i/n}w)}{w^2} dw.$$

We compute the derivative of  $g$  using the chain rule:

$$g'(z) = f'(z) + e^{2\pi i/n}f'(e^{2\pi i/n}z) + e^{4\pi i/n}f'(e^{4\pi i/n}z) + \dots + e^{2(n-1)\pi i/n}f'(e^{2(n-1)\pi i/n}z),$$

so that

$$\begin{aligned} g'(0) &= f'(0) + e^{2\pi i/n}f'(e^{2\pi i/n}0) + e^{4\pi i/n}f'(e^{4\pi i/n}0) + \dots + e^{2(n-1)\pi i/n}f'(e^{2(n-1)\pi i/n}0) \\ &= f'(0) + e^{2\pi i/n}f'(0) + e^{4\pi i/n}f'(0) + \dots + e^{2(n-1)\pi i/n}f'(0) \\ &= f'(0)(1 + e^{2\pi i/n} + e^{4\pi i/n} + \dots + e^{2(n-1)\pi i/n}) \\ &= f'(0) \frac{1 - (e^{2\pi i/n})^n}{1 - e^{2\pi i/n}} = f'(0) \frac{1 - 1}{1 - e^{2\pi i/n}} = 0, \end{aligned}$$

using the sum of the first  $n$  terms of a geometric series.

17. Let  $f$  be holomorphic on an open set  $\Omega$  containing the contour  $\gamma$  and its interior. Suppose that  $z_0 \in \text{Int } \gamma$ . Show that

$$\int_{\gamma} \frac{f'(z)}{z - z_0} dz = \int_{\gamma} \frac{f(z)}{(z - z_0)^2} dz.$$

*Solution.* We use Cauchy's integral formula for the first derivative and the function  $f$  to see that

$$\int_{\gamma} \frac{f(z)}{(z - z_0)^2} dz = 2\pi i f'(z_0).$$

On the other hand we use Cauchy's integral formula for the function  $g(z) = f'(z)$  to see that

$$f'(z_0) = g(z_0) = \frac{1}{2\pi i} \int_{\gamma} \frac{g(z)}{z - z_0} dz = \frac{1}{2\pi i} \int_{\gamma} \frac{f'(z)}{z - z_0} dz.$$

The result becomes now obvious.