Finding the k-grid points inside the reciprocal cell

Let L_K be the k-grid lattice. Let $\mathbb{K}$ be a matrix where the columns are the vectors of the generating grid. The matrix $\mathbb{S}$ that transforms the k-grid vectors into the reciprocal vectors must have all integer elements. Its determinant is equal to the number of reducible k-points. Then, the recriprocal lattice, L_R, is given by $\mathbb{R} = \mathbb{K}\mathbb{S}$ (where the columns of $\mathbb{R}$ are the reciprocal lattice vectors). With no loss of generality, we may choose $\mathbb{S}$ as Hermite Normal Form matrix.

We want to know which points of L_K are within the first unit cell of L_R. Let a point within the cell be denoted $\vec{x}$. The (lattice) components of $\vec{x}$ must be $0 \leq x_i < 1$. Since we are interested in points in the cell that are also points of L_K, we have $\mathbb{R}\vec{x} = \mathbb{K}\vec{z}$ where the components of z are integers. (If so, then $\vec{x}$ is obviously a lattice point of the k-grid [that is, $\vec{x} \in L_k$].)

But since $\mathbb{R} = \mathbb{K}\mathbb{S}$,

$$
\begin{aligned}
\mathbb{R}\vec{x} &= \mathbb{K}\vec{z} \\
\mathbb{K}\mathbb{S}\vec{x} &= \mathbb{K}\vec{z} \\
\mathbb{K}^{-1}\mathbb{K}\mathbb{S}\vec{x} &= \mathbb{K}^{-1}\mathbb{K}\vec{z} \\
\mathbb{S}\vec{x} &= \vec{z}
\end{aligned}
\quad \Rightarrow \quad
\begin{pmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{pmatrix}
\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} =
\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}
$$

So,

$$
\begin{aligned}
ax_1 &= z_1 \\
bx_1 + cx_2 &= z_2 \\
dx_1 + ex_2 + fx_3 &= z_3
\end{aligned}
$$

Since the components of $\vec{x}$ must be $[0, 1)$,

$$
\begin{aligned}
0 \leq x_1 = z_1/a < 1 \quad &\rightarrow \quad \boxed{0 \leq z_1 < a} \\
0 \leq x_2 = \frac{z_2}{c} - \frac{b}{ca}z_1 < 1 \quad &\rightarrow \quad \frac{b}{ca}z_1 \leq \frac{z_2}{c} < 1 + \frac{b}{ca}z_1 \\
&\rightarrow \quad \boxed{\frac{b}{a}z_1 \leq z_2 < c + \frac{b}{a}z_1}
\end{aligned}
$$

$$
\begin{aligned}
0 \leq x_3 = \frac{z_3}{f} - \frac{d}{f}\frac{z_1}{a} - e[\frac{z_2}{c} - \frac{b}{ca}z_1] < 1 \\
\frac{d}{f}\frac{z_1}{a} + \frac{e}{f}[\frac{z_2}{c} - \frac{b}{ca}z_1] \leq \frac{z_3}{f} < 1 + \frac{d}{f}\frac{z_1}{a} + \frac{e}{f}[\frac{z_2}{c} - \frac{b}{ca}z_1] \\
d\frac{z_1}{a} + \frac{e}{c}[z_2 - \frac{b}{a}z_1] \leq z_3 < f + z_1\frac{d}{a} + \frac{e}{c}[z_2 - \frac{b}{a}z_1] \\
\boxed{z_1[\frac{d}{a} - \frac{eb}{ca}] + \frac{e}{c}z_2 \leq z_3 < f + z_1[\frac{d}{a} - \frac{eb}{ca}] + \frac{e}{c}z_2}
\end{aligned}
$$