Let $A = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{pmatrix}$, so our primary lattice $L$ consists of all integer points with each coordinate divisible by 5. This has the usual set of rotational symmetries (all permutations of the coordinates with every combination of sign changes, thus $6 \cdot 8 = 48$ orthogonal transformations.

Let

$$D = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix} \right\}.$$

Thus our multilattice $M$ consists of four translated copies of $L$. There are, however, only 16 of the original 48 transformations which still correspond to symmetries of $M$. Let's focus on one of those.

We'll consider the orthogonal transformation

$$N = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

which interchanges the top two coordinates, then changes the sign of the middle and bottom coordinates. (It's a rotation of $90^0$ around the $z$-axis, followed by an inversion.) Thus

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \to \begin{pmatrix} y \\ -x \\ -z \end{pmatrix}.$$

We apply this to the set $D$, and get

$$D' = \left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} \right\}.$$

Note that, except for the first, none of these are $L$-equivalent to elements in the set $D$, but if we now add the vector $\begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$ (an element of $D$) to all of the vectors in the latter set, we get

$$D'' = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} \right\},$$

all of which vectors are $L$-congruent to $D$ vectors. In fact

$$D'' \equiv \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 5 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 5 \\ -5 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ -5 \\ 0 \end{pmatrix} \right\} =$$

$$\left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 3 \\ 0 \end{pmatrix} \right\},$$

which is a permutation of the original set $D$.

In other words, the transformation $x \to Nx + t$ (where $t = \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}$) is a symmetry of the multilattice $M$, and its action on the vectors of $D$ can be described by $Nd_i + t - v_i = d_{\pi(i)}$ (or $Nd_i + t = d_{\pi(i)} + v_i$), where $\pi(i)$ is the permutation described by $\pi(1) = 4, \pi(2) = 1, \pi(3) = 2, \pi(4) = 3$, and the vectors $v_i$ are (members of $L$, thus lvecs in the program) $\begin{pmatrix}0\\0\\0\end{pmatrix}$, $\begin{pmatrix}5\\0\\0\end{pmatrix}$, $\begin{pmatrix}5\\-5\\0\end{pmatrix}$, and $\begin{pmatrix}0\\-5\\0\end{pmatrix}$ for $i = 1, 2, 3, 4$ respectively. Note that $t$ and $\pi$ and the $v_i$ all depend on $N$, though I didn't bother to denote that by subscripting them in this example. All of this information, *for this particular* $N$, is encoded in the seventh output line in the worksheet after the lines which note that the set is good and there are 16 symmetries.

Now what happens if we add the superlattice defined by the HNF matrix

$$H = \begin{pmatrix}1 & 0 & 0\\1 & 2 & 0\\0 & 0 & 2\end{pmatrix}?$$

The Smith Normal form is

$$\begin{pmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 2\end{pmatrix},$$

and a left transition matrix is

$$T = \begin{pmatrix}0 & 1 & 0\\0 & 0 & 1\\1 & -1 & 1\end{pmatrix}.$$

[Thus our superlatice is all integer points of the form $\begin{pmatrix}5i\\5j\\5k\end{pmatrix}$ where $k$ must be even and $i - j + k$ must be even.] It turns out that our transformation $N$ above is still a symmetry of this superlattice, thus we may continue our analysis further.

Let $\phi(x) = TA^{-1}x$ be the projection from elements of $L$ into the quotient group (so $L_A$ is the kernel – the elements of $L$ which get projected to the zero in $G$). Our quotient group is $G = \mathbf{Z}_1 \oplus \mathbf{Z}_2 \oplus \mathbf{Z}_2$, and each element $g \in G$ represents a distinct translate (namely $\phi^{-1}(g)$, the set of all pre-images of $g$) of the superlattice $L_H$ within $L$, all of whose points must get the same label. Each translated lattice $d_i + L$ of $L$ within the superlattice also has corresponding translates of the superlattice (of the form $d_i + \phi^{-1}(g)$) which must all get the same label. Thus for every $g \in G$ and for every $d_i \in D$, there is a unique translated lattice consisting of points which are $L$-equivalent and must all receive the same label.

In other words, we may label the set of pairs $(d, g)$ (with $d \in D$ and $g \in G$) as we wish, and satisfy the requirement that our labeling be $L$-periodic. But which of these are equivalent under the various symmetries of $M$. To answer that question, we must examine what translations, and transformations of the form $x \to Nx + t$ (as above) do to the set of pairs (which we've been calling the "table").

Note that an entry $(d_i, g)$ corresponds to the $L$ translate $d_i + \phi^{-1}(g)$. We may discover what our symmetry does to that set, by computing what the symmetry does to any element of $d_i + \phi^{-1}(g)$. A convenient representative is $x = d_i + AT^{-1}g$ (remember that $g$ is a column vector). Now

$$x \to Nx + t = N\big(d_i + AT^{-1}g\big) + t = Nd_i + t + NAT^{-1}g = d_{\pi(i)} + v_i + NAT^{-1}g.$$

Since $v_i$ and $NAT^{-1}g$ are elements of $L$, it follows that $d_{\pi(i)}$ is the left hand entry of $(d_i, g)$. In other words, this transformation $x \to Nx + t$ maps $(d_i, g)$ to $(d_{\pi(i)}, g')$. We may then apply the projection $\phi$ to get $g' = TA^{-1}(v_i + NAT^{-1}g$. This explains the permutation of our table which is displayed in the seventh line under "orthogonal permutations" in the worksheet. Thus the rows are permuted according to $\pi$ for this particular symmetry, and action within the row is described by the $g'$ computation shown.

Phew. I'm exhausted. Going to bed.