Normal pdf

Also called the "Gaussian" pdf

$$p_X(x;\mu,\sigma) = N(x;\mu,\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$P(X \in [x, x+dx]) = p_X(x)dx$: probability of getting a value in $[x, x+dx]$

μ : mean of the pdf

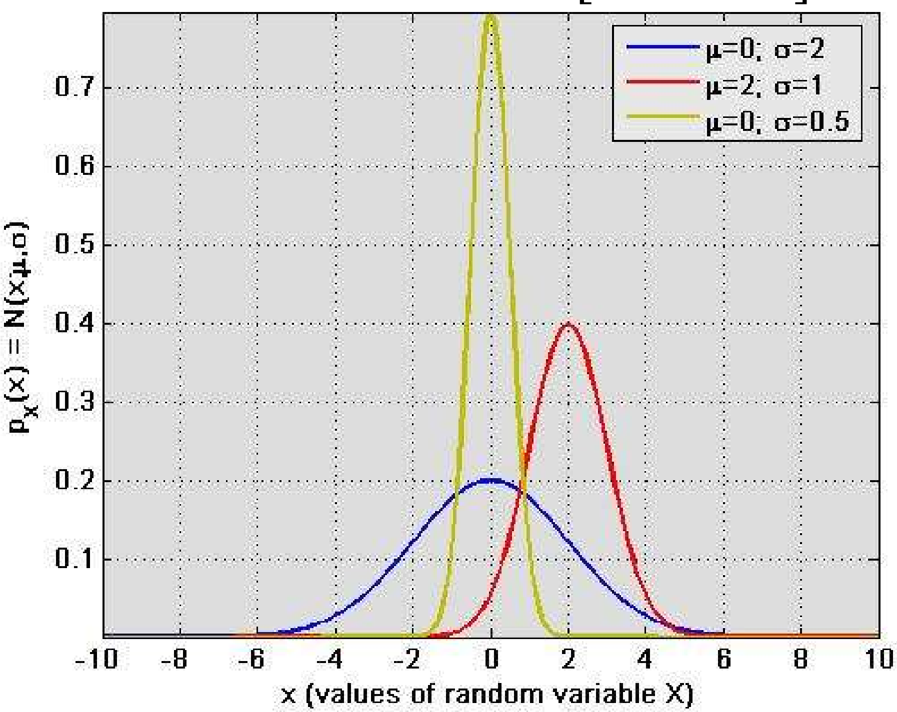
$\mu = E[X]$

σ^2 : variance of the pdf

$\sigma^2 = E[(X - E[X])^2]$

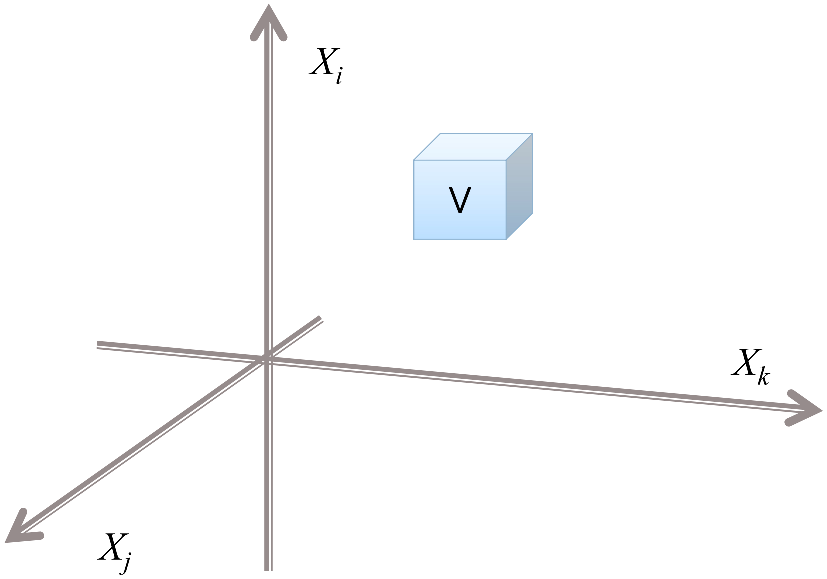
σ : standard deviation

Usually we say μ and σ^2 are the mean and variance of the random variable X



N random variables

- $\{ X_1, X_2, \dots, X_N \}$
- Outcome of a trial is a vector $\bar{x} = (x_1, x_2, \dots, x_N)$ in $\mathbb{R}^N$



Probability that a trial outcome will fall in a volume V

$$P(X \in V) = \int_V p_X(\bar{x}) \mathrm{d}^n x$$

$p_X(\bar{x})$: Joint pdf
$p_{Y \subset X}(x)$: marginal pdf of $Y \subset X$ obtained by integrating $p_X(\bar{x})$ over $X \backslash Y$
$X_i \sim p_{X_i}(x)$: X_i has the pdf $p_{X_i}(x)$

Expectation

$$E[f(X_1,\ldots,X_N)] = \int_{-\infty}^{\infty} f(x_1,\ldots,x_N)\, p_{X_1,\ldots,X_N}(x_1,\ldots,x_N)\, dx_1,\ldots,dx_N$$

$E[(X_1 - E[X_1])^{m_1}(X_2 - E[X_2])^{m_2}\ldots(X_N - E[X_N])^{m_N}]$ is called the $m_1 + m_2 + \ldots + m_N$ order (central) moment of the joint pdf

- Of course, there are many possible moments of a given order
- Of special importance is the central moment of second order

$$C_{ij} = E[(X_i - E[X_i])(X_j - E[X_j])]$$

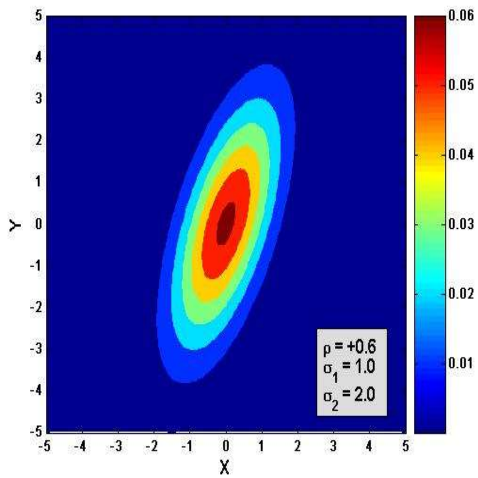
C_{ij} is called the Covariance of X_i and X_j

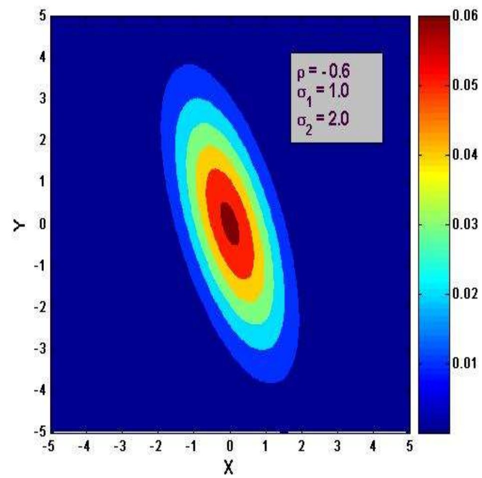
The matrix with elements C_{ij} is called the Covariance Matrix of the random variables $\{X_1,\ldots X_N\}$

Bivariate Normal pdf

$$p_{XY}(\bar{x};\bar{\mu},\mathbf{C}) = \frac{1}{2\pi\,|\,\mathbf{C}\,|^{1/2}}\exp\left(-\frac{1}{2}(\bar{x}-\bar{\mu})^T\mathbf{C}^{-1}(\bar{x}-\bar{\mu})\right)$$

$$\mathbf{C} = \begin{pmatrix} \sigma_x^2 & \rho\sigma_x\sigma_y \\ \rho\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix};\ \bar{x} = \begin{pmatrix} x \\ y \end{pmatrix};\ \bar{\mu} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}$$

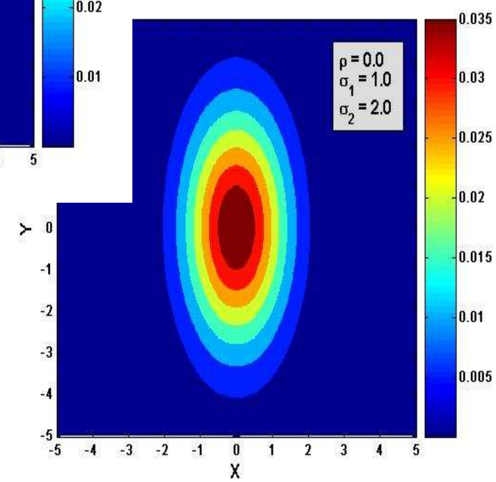




$$\mu_x = \mu_y = 0$$

$$\sigma_x = 1.0, \sigma_y = 2.0$$

$$\rho \in [-0.6, 0, 0.6]$$



One can prove that **C** is also the covariance matrix of the bivariate normal pdf

Stochastic process

A sequence of random variables

$$\{\ldots, X_{-2}, X_{-1}, X_0, X_1, X_2, \ldots\}$$

- The sequence need not be a vector in a finite dimensional space
- The indices can be taken to indicate "time" instants$\rightarrow$ "Time Series"
  - True continuous time theory for stochastic processes is more complicated

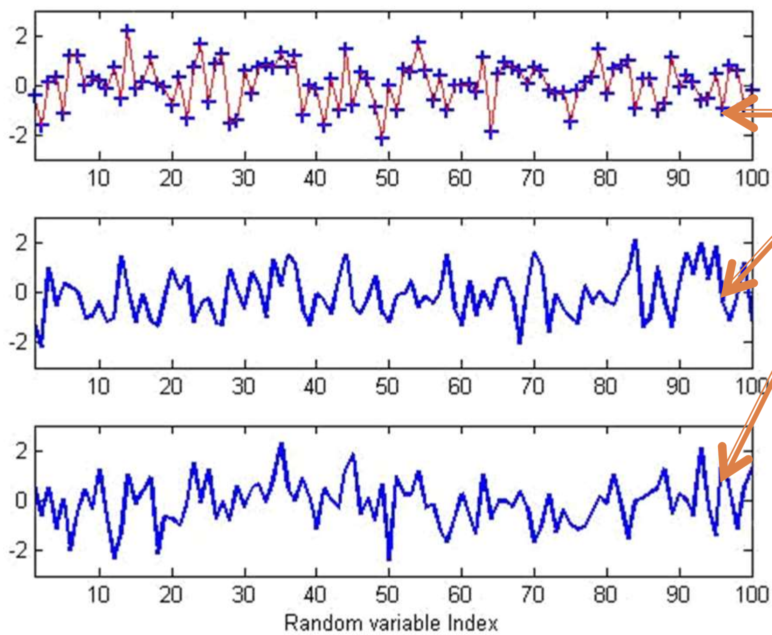
Each trial gives a set of values (a "realization")

$$\{\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots\}$$

- The stochastic processes we will mostly consider are time-series (i.e, only one index)
- The study of stochastic processes extends to spatial stochastic processes
- Notation: $X_{\mathrm{k}} \equiv X[\mathrm{k}]$

Textbook: *Stochastic Processes* by Ross; *Statistical Optics* by Goodman

Mathematical description of a stochastic process



- Different realizations of the same stochastic process
- Each time series is the outcome of **one** trial
- X-axis is the index number of the random variables
- The **ordering** of random variables distinguishes a stochastic process from a simple collection of variables: change the ordering and you change the stochastic process

Ensemble average

Ensemble average

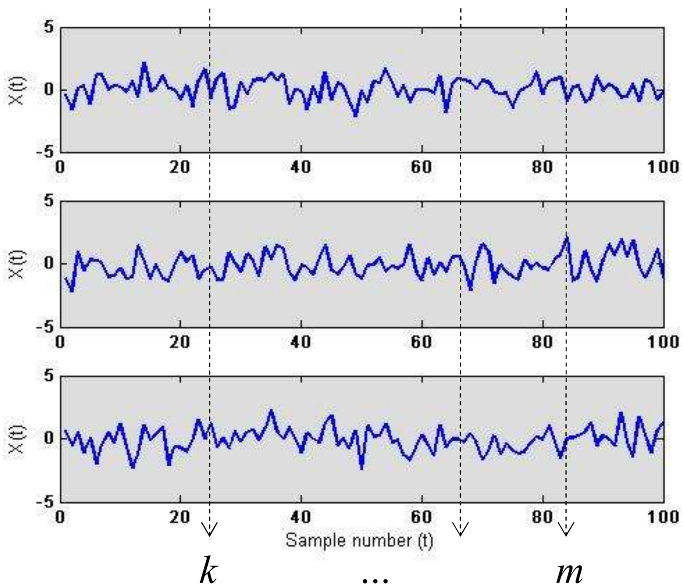
Same as the joint expectation defined earlier

$$E[f(X_k,\ldots,X_m)] = \int_{-\infty}^{\infty} f(x_k,\ldots,x_m)\, p_{X_k,\ldots,X_m}(x_k,\ldots,x_m)\, dx_k,\ldots,dx_m$$

Ensemble averages are constructed by averaging $f(X_k, \ldots X_m)$ over an infinite number of realizations of the stochastic process

- Compute $f_p = f(X_k, \ldots X_m)$ for the p^{th} realization ($1 \le p \le N$)

- Then compute $\frac{1}{N}\sum_{p=1}^{N} f_p$ with $N \to \infty$



Stationary stochastic process

- Wide-sense stationary process
  - $E[\,X_k\,]$ is constant, independent of k
  - $E[\,X_k X_{k+m}\,]$ is independent of k and dependent only on *m*
    - *First and second order moments are time-translation independent*
  - Makes sense to talk about the "mean and variance of the stochastic process"

- Strictly stationary stochastic process: ***All*** joint moments are time-translation independent

Gaussian noise: multivariate Normal pdf

- Multivariate Normal pdf:
- $p_{\bar{X}}(\bar{x}) = \frac{1}{(2\pi)^{N/2}|\boldsymbol{C}|^{1/2}} \exp(-\frac{1}{2}\|\bar{x} - \bar{\mu}\|^2)$
- $\bar{x} \in R^N$ (row vector)
- $E[X_i] = \mu_i$
- $C_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)]$: Covariance matrix
- $|\boldsymbol{C}|$: Determinant of $\boldsymbol{C}$
- $\|\bar{x}\|^2 = \langle \bar{x}, \bar{x} \rangle$ where $\langle \bar{x}, \ \bar{y} \rangle = \ \bar{x}\boldsymbol{C}^{-1}\bar{y}^T$
- Gaussian noise: The joint pdf of any subset of the random variable sequence is a multivariate normal pdf

Textbook: *Introduction to multivariate statistics*, T.W. Anderson

Wide-sense stationary Gaussian noise

- $E[X_i] = \mu$ (Independent of i)
- $E\big[(X_i - \mu)\big(X_{i\pm m} - \mu\big)\big] = \phi(m)$ (dependent only on the separation): Autocovariance sequence of the noise
- Covariance matrix is a Toeplitz matrix

$$A = \begin{bmatrix} a_0 & a_{-1} & a_{-2} & \cdots & \cdots & a_{-(n-1)} \\ a_1 & a_0 & a_{-1} & \ddots & & \vdots \\ a_2 & a_1 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & a_{-1} & a_{-2} \\ \vdots & & \ddots & a_1 & a_0 & a_{-1} \\ a_{n-1} & \cdots & \cdots & a_2 & a_1 & a_0 \end{bmatrix}$$

Power spectral density

- $\phi(0) = E[(X_i - \mu)^2] = \sigma^2$
- $\phi(k)$: even function of k (Because $\boldsymbol{C}$ is always symmetric)

We use $\tilde{s}$ to denote the discrete Fourier transform (DFT) of $\bar{s}$,

$$\tilde{s}^T := \mathbf{F}\bar{s}^T, \tag{1}$$

$$F_{\text{km}} = e^{-2\pi i k m/N}, \tag{2}$$

with $\tilde{s}_j$ being its jth element. The inverse DFT is given by,

$$\mathbf{F}^{-1} = \frac{1}{N}\mathbf{F}^{\dagger}. \tag{3}$$

The symbol "./" denotes element-by-element division

$$\begin{aligned} \|\bar{x}\|^2 &= \frac{1}{N}\tilde{x}^*(\mathbf{FCF}^{-1})^{-1}\tilde{x}^T \\ &\approx \frac{1}{Nf_s}\tilde{x}(\tilde{x}^{\dagger}./\bar{S}^T), \end{aligned} \tag{6}$$

where $\bar{S}$ is the two-sided power spectral density of the noise

$$\bar{S} = \frac{1}{f_s}\boldsymbol{F}\bar{\phi}$$

(Where $\bar{\phi}$ is a circular sequence)

$$S_i = \frac{1}{Nf_s}E[|\tilde{n}_i|^2]$$

Convenient definition

$$\delta_f \sum_{m=0}^{N-1} S_m = \sigma^2,$$

Power Spectral Density

The definitions are more convenient if one switches to continuous time

$$S_n(f) = \int_{-\infty}^{\infty} d\tau\, \phi(\tau) e^{-2\pi i f \tau}$$

$S_n(f)$ is real and symmetric as $\phi(\tau)$ is an even function

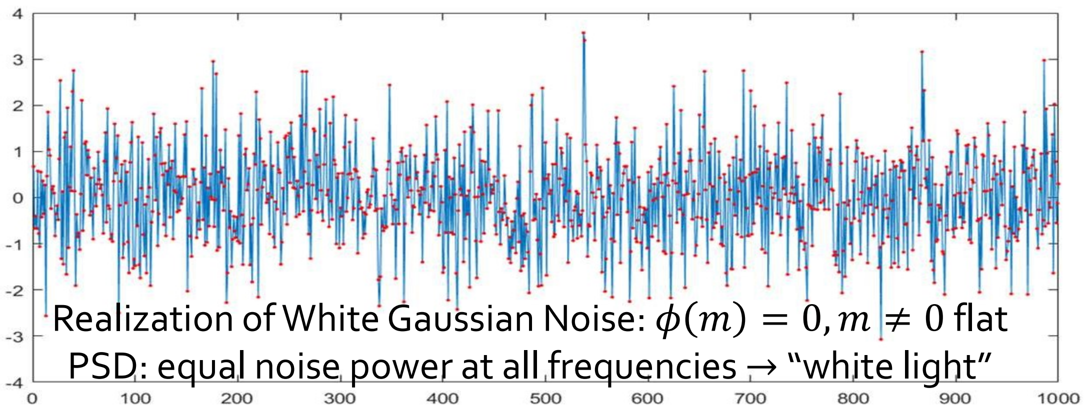
(Sign of phase in the Fourier transform does not matter)

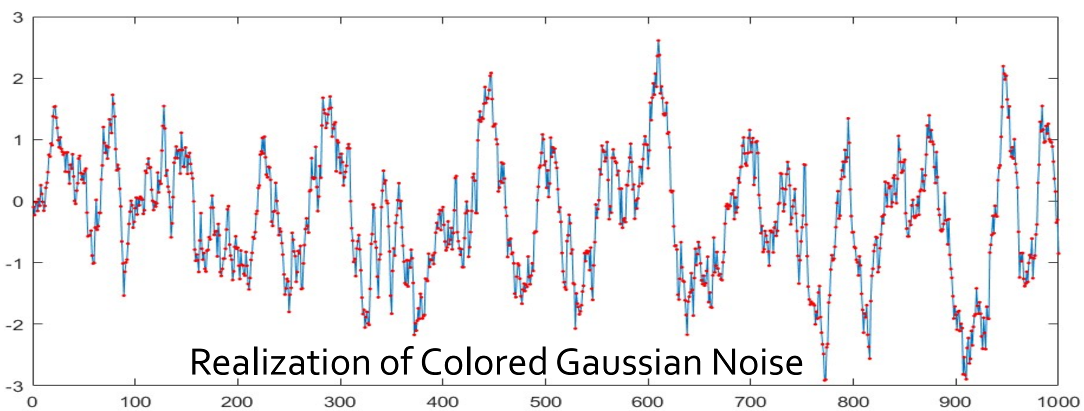
Physical interpretation

The variance of the noise process $\sigma^2 = \phi(0) = \int_{-\infty}^{\infty} df\; S_n(f)$

Hence, $S_n(f)df$ can be interpreted as the noise variance contributed by the band $[f, f + df]$

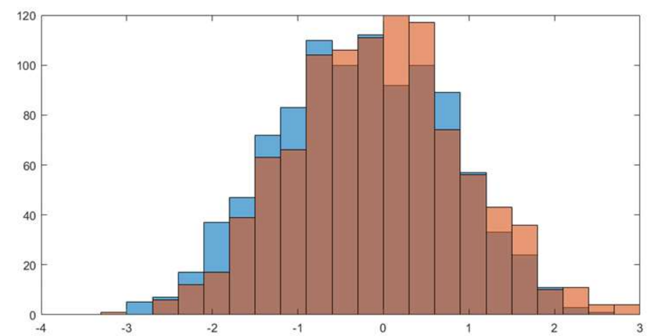
Gaussian noise nomenclature



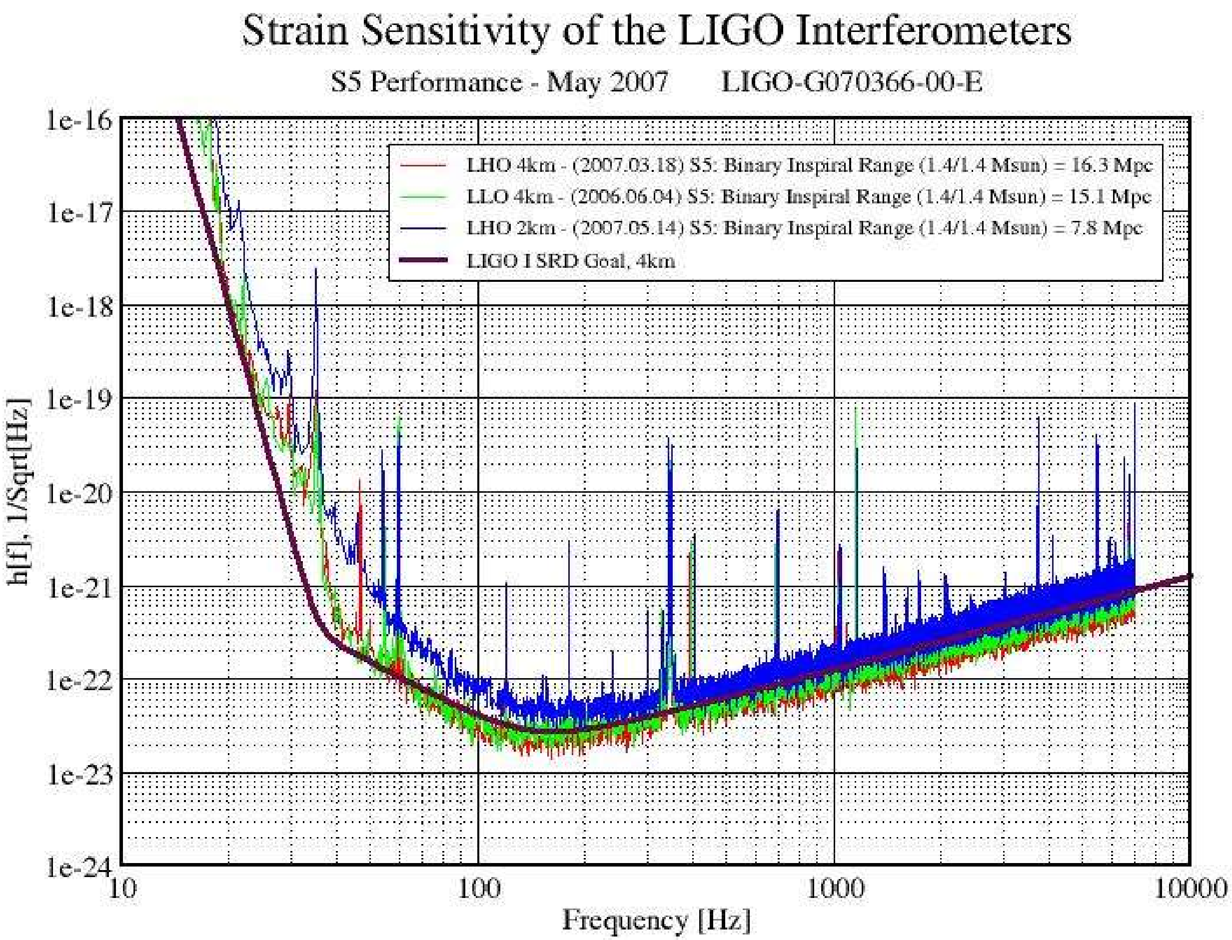


The terms ‘white’ and ‘colored’ refer to the shape of the PSD

Here, the two noise processes have the same marginal pdf $\mathrm{X_i} \sim N(0,1) \Rightarrow$ marginal pdf is not enough to describe noise



$\sqrt{S(f)}$



Strain Sensitivity of the LIGO Interferometers
S5 Performance - May 2007
LIGO-G070366-00-E
LHO 4km - (2007.03.18) S5: Binary Inspiral Range (1.4/1.4 Msun) = 16.3 Mpc
LLO 4km - (2006.06.04) S5: Binary Inspiral Range (1.4/1.4 Msun) = 15.1 Mpc
LHO 2km - (2007.05.14) S5: Binary Inspiral Range (1.4/1.4 Msun) = 7.8 Mpc
LIGO I SRD Goal, 4km
h[f], 1/Sqrt[Hz]
1e-16
1e-17
1e-18
1e-19
1e-20
1e-21
1e-22
1e-23
1e-24
10
100
1000
10000
Frequency [Hz]

Estimation problem in GW

$\bar{x}$: given data from GW detector

If a signal is present

$\bar{x} = \bar{h}(\Theta) + \bar{n}$

$\bar{h}$: Signal time series

Θ : Parameters defining the signal

$\bar{n}$: noise realization

Let $p_{\bar{X}}(\bar{x})$ be the pdf of noise.

Then, $p_{\bar{X}}(\bar{x};\Theta) = p_{\bar{X}}(\bar{x} - \bar{h}(\Theta))$

Examples

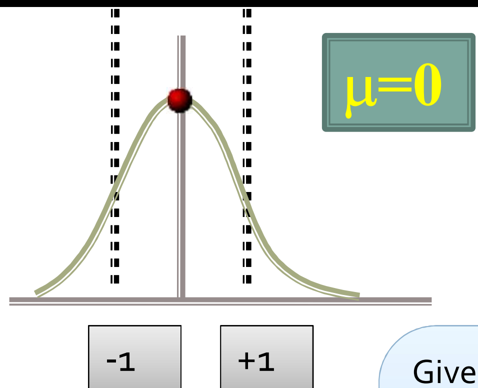
$$\bar{h} = h_0 \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \rightarrow \text{"dc signal"}$$

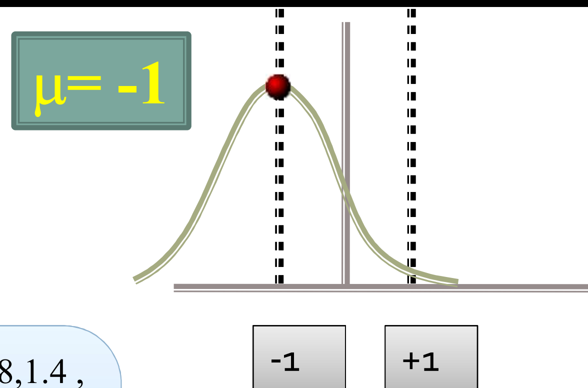
Θ Is h_0

$$h_k = A_0 \sin(\omega_0 k \Delta)$$

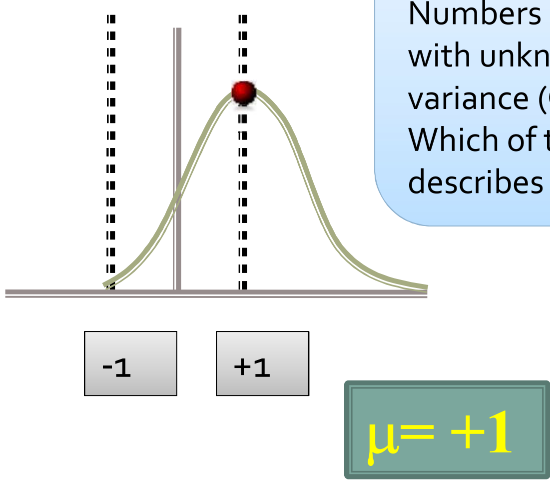
Θ Is (A_0, ω_0)

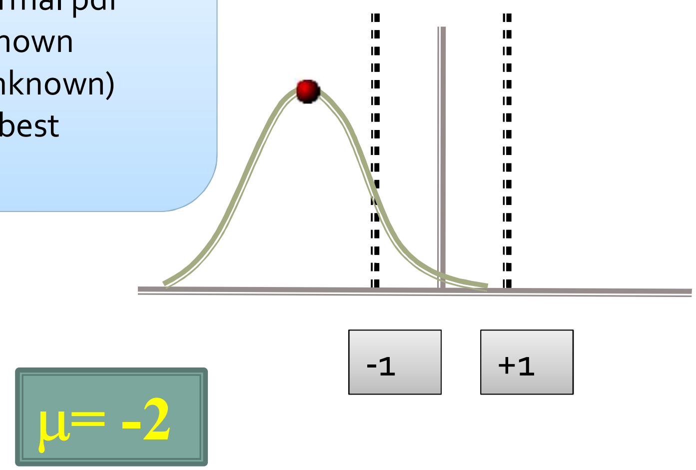
A simple estimation problem





Given data: 0.5, -0.08, 1.2, 1.5, 0.8,1.4 , -0.1,0.6
Numbers drawn from a normal pdf with unknown mean but known variance ($\Theta = (\mu,\sigma)$, μ is unknown)
Which of the mean values best describes the data?





Likelihood

- In the first example, our judgment was based on which pdf appeared to be more "likely" as the correct one
- One way to make this idea mathematically precise is the to use the **likelihood function**
- Joint pdf of the data: $p_{\bar{X}}(\bar{x};\Theta)$
- Likelihood function: consider the joint pdf as a function of Θ for fixed x (data)
  - Alternative notation: $\Lambda(\Theta;\bar{x})$
- A high likelihood value means the corresponding pdf gives higher probability of occurrence for the data

In example 1: estimating the mean of the normal pdf,
The joint pdf of the data is

$$p_{\bar{X}}(\{x_1, x_2, \ldots, x_8\};\mu) = \prod_{i=1}^{8} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_i-\mu)^2\right)$$

$$\Lambda(\mu;\{x_1, x_2, \ldots, x_8\}) = \prod_{i=1}^{8} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x_i-\mu)^2\right)$$

Which μ gives highest value of Λ?

Example of likelihood function: dc signal in WGN

Data (realization)

$\overline{x} = \{x_1, x_2, \ldots, x_N\}$

Parameter μ

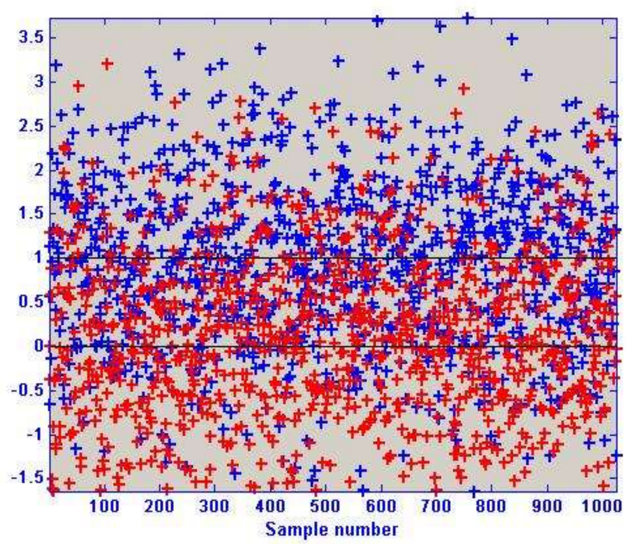
$\overline{\mu} = \mu\,(1,1,\ldots,1)^T$

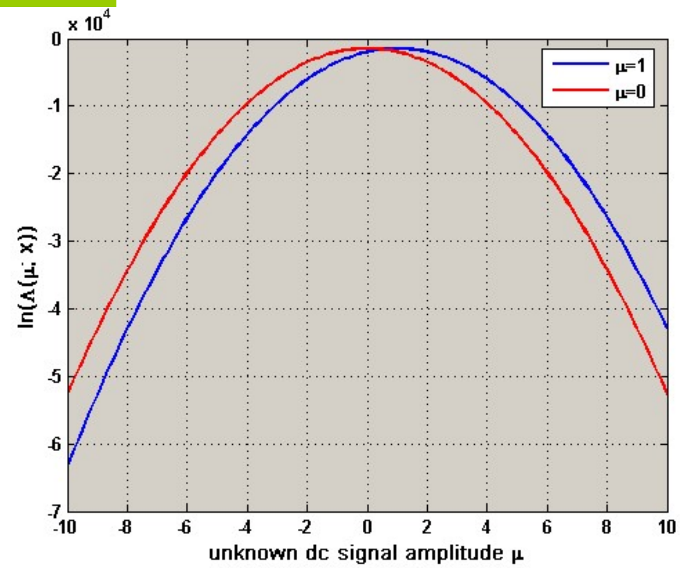
$1 \times N$ vector

"dc signal"

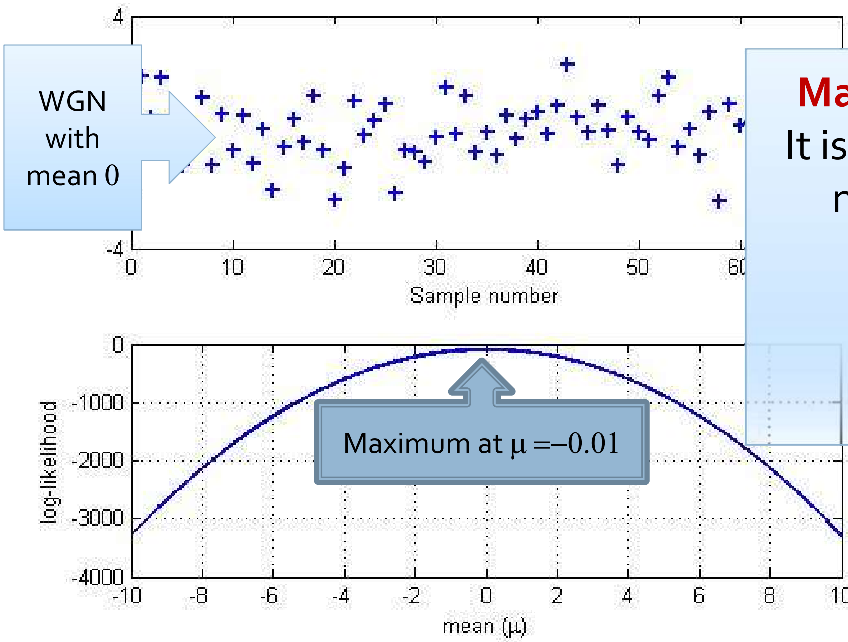
Likelihood

$$\frac{1}{\left(\sqrt{2\pi}\right)^N} \exp\left(-\frac{1}{2}(\overline{x}-\overline{\mu})^T(\overline{x}-\overline{\mu})\right)$$





Maximum Likelihood Estimator



Maximum Likelihood Estimator (MLE)

It is the point at which the Likelihood has maximum value

$\hat{\Theta}$ is such that

$$\max_{\Theta}\left(\Lambda(\Theta;\bar{x})\right) = \Lambda(\hat{\Theta};\bar{x})$$

Instead of maximizing the likelihood, we can use any monotonic function of the likelihood → e.g., $\ln(\Lambda(x;\Theta))$ (“log-likelihood”)

MLE for GW signals in Gaussian noise

GW data : $\bar{x} = \bar{h}(\Theta) + \bar{n}$

$\bar{n}$: realization of Gaussian, stationary noise

$$p_{\bar{X}}(\bar{x}) = \frac{1}{(2\pi)^{N/2}\,|\mathbf{C}|^{1/2}} \exp\left(-\frac{1}{2}\bar{x}^T\mathbf{C}^{-1}\bar{x}\right)$$

Let $\langle \bar{z}, \bar{y} \rangle = \bar{z}^T \mathbf{C}^{-1} \bar{y}$, where $\mathbf{C}$ is the covariance matrix of the noise,

and $\|\bar{z}\|^2 = \langle \bar{z}, \bar{z} \rangle$. Then, the likelihood is

$$\Lambda(\Theta;\bar{x}) = p_{\bar{X}}(\bar{x} - \bar{h}(\Theta)) = \mathrm{N}\,\exp\left(-\frac{1}{2}\|\bar{x} - \bar{h}(\Theta)\|^2\right)$$

$$\ln\Lambda(\Theta;\bar{x}) = \ln\mathrm{N} - \frac{1}{2}\|\bar{x} - \bar{h}(\Theta)\|^2$$

$$\mathrm{N} = \frac{1}{(2\pi)^{N/2}\,|\mathbf{C}|^{1/2}}$$

$$= \left(\ln\mathrm{N} - \frac{1}{2}\|\bar{x}\|^2\right) + \langle \bar{x}, \bar{h}(\Theta) \rangle - \frac{1}{2}\|\bar{h}(\Theta)\|^2$$

MLE for Θ : find $\hat{\Theta}$ such that $\max_{\Theta}\left(\ln\Lambda(\Theta;\bar{x})\right) = \ln\Lambda(\hat{\Theta};\bar{x})$

Or, get $\hat{\Theta}$ by maximizing $\lambda(\Theta;\bar{x}) = \langle \bar{x}, \bar{h}(\Theta) \rangle - \frac{1}{2}\|\bar{h}(\Theta)\|^2$

Least Squares

- Least squares estimation is simply MLE for the case of Gaussian noise
- Maximizing Log-likelihood

$$\ln \Lambda(\bar{x}\,|\Theta) = \ln N \; - \frac{1}{2}\left\|\bar{x} - \bar{h}(\Theta)\right\|^2$$

- Is the same as minimizing Least squares

$$\left\|\bar{x} - \bar{h}(\Theta)\right\|^2$$

- For White Gaussian Noise: $\phi(m) = 0, m \neq 0$ (Random variables in the sequence are statistically independent) ⇒ $C_{ij} = \delta_{ij} \Rightarrow$ $\left\|\bar{x} - \bar{h}(\Theta)\right\|^2 = \sum_{k=0}^{N-1}\left(x_k - h_k(\Theta)\right)^2$

Matched filtering for Colored noise

- Maximizing log-likelihood (= minimizing least squares) is equivalent to maximizing
$$\left\langle \bar{x}, \bar{h}(\Theta) \right\rangle - \frac{1}{2} \left\| \bar{h}(\Theta) \right\|^2 = \left\langle \bar{x}, \bar{h}(\Theta) \right\rangle - \frac{1}{2} \langle \bar{h}(\Theta), \bar{h}(\Theta) \rangle$$
- (See Lecture 2)
- Everything stays the same except
$$\langle \bar{x}, \bar{y} \rangle = \frac{1}{N f_s} \tilde{x} (\tilde{y}^{\dagger} \, ./ \, \bar{S}^T)$$

Estimating Power Spectral Density

FFT

$$\tilde{x}[k] = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x[j] e^{-2\pi ijk/N}$$

$$x[j] = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \tilde{x}[k] e^{2\pi ijk/N}$$

$$\sum_{k=0}^{N-1} x[j+k] e^{-2\pi i\frac{(k+j)l}{N}} e^{2\pi i\frac{jl}{N}}$$

$$= e^{2\pi i\frac{jl}{N}} \tilde{x}[l]$$

$$\hat{\phi}[k] = \frac{1}{N} \sum_{j=0}^{N-1} x[j]x[j+k] \text{ ; Circular time-shift}$$

Consider the estimator:

$$\hat{S}_n(l) = \sum_{k=0}^{N-1} \hat{\phi}[k]\, e^{-2\pi ikl/N} = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} x[j]\, x[j+k] e^{-2\pi ikl/N}$$

$$= \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} x[j] \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x[j+k] e^{-2\pi ikl/N}$$

$$= \frac{1}{\sqrt{N}} \tilde{x}[l] \sum_{j=0}^{N-1} x[j]\, e^{2\pi ijl/N} = \tilde{x}^*[l]\, \tilde{x}[l] = |\, \tilde{x}[l]\,|^2$$

Periodogram

- The variance of this estimator does not go down with increase in N
- So, simply taking DFT of the auto-covariance function does not work
- Need to average the Periodograms over several data segments

Welch's method

- Welch's method of overlapping windows
  - Already a built-in function in Matlab: `psd` (old) `pwelch` (new)

Compute FFTs of K short segments

Calculate periodogram for each: mod square the FFT values

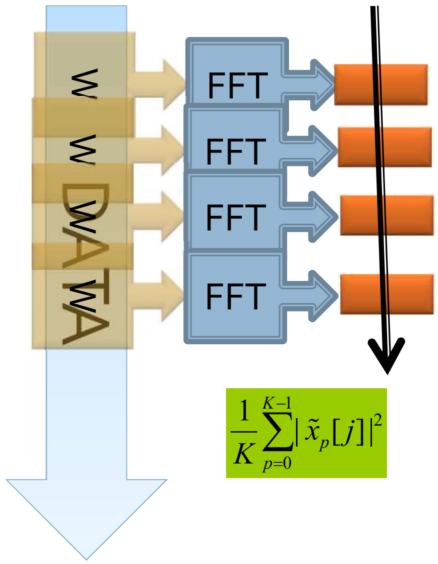
PSD estimate: $S_n[j] = \frac{1}{K}\sum_{p=0}^{K-1} |\tilde{x}_p[j]|^2$

Matlab: psd(x,nfft,fs)

x : data vector

nfft: number of samples in each short segment

fs : sampling frequency of the data



Wiener-Khinchin theorem

Input noise process $x(t)$ with PSD $S_{in}(f)$

Linear time invariant system

$$y(t) = \int_{-\infty}^{\infty} x(t')\, h(t-t')dt'$$

$h(t)$: system impulse response

$T(f) = \int_{-\infty}^{\infty} dt\; h(t) e^{-2\pi i\, ft}$: System transfer function

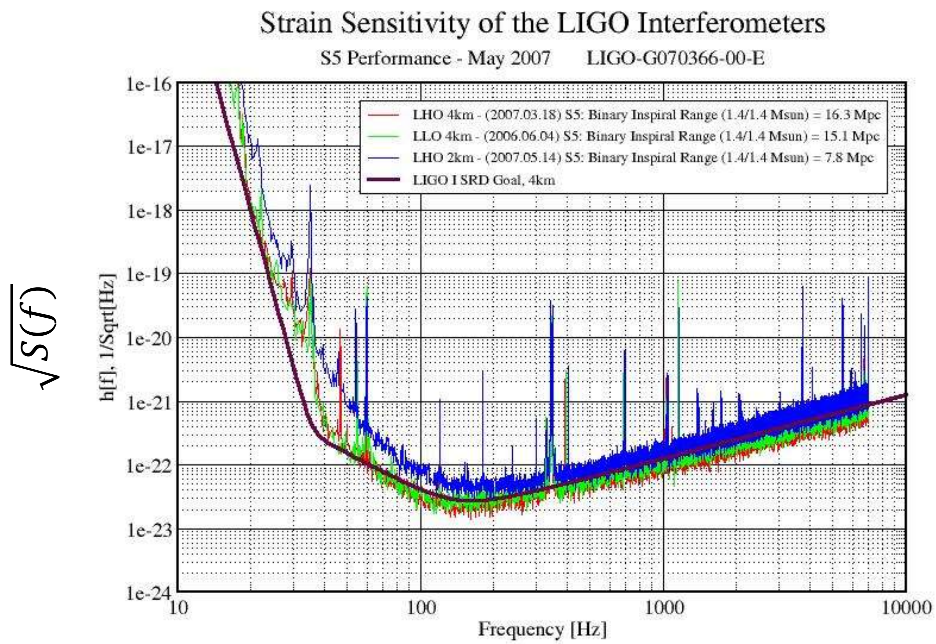
Output noise process $y(t)$ with PSD $S_{out}(f)$

$$S_{out}(f) = S_{in}(f)|T(f)|^2$$

Whitening

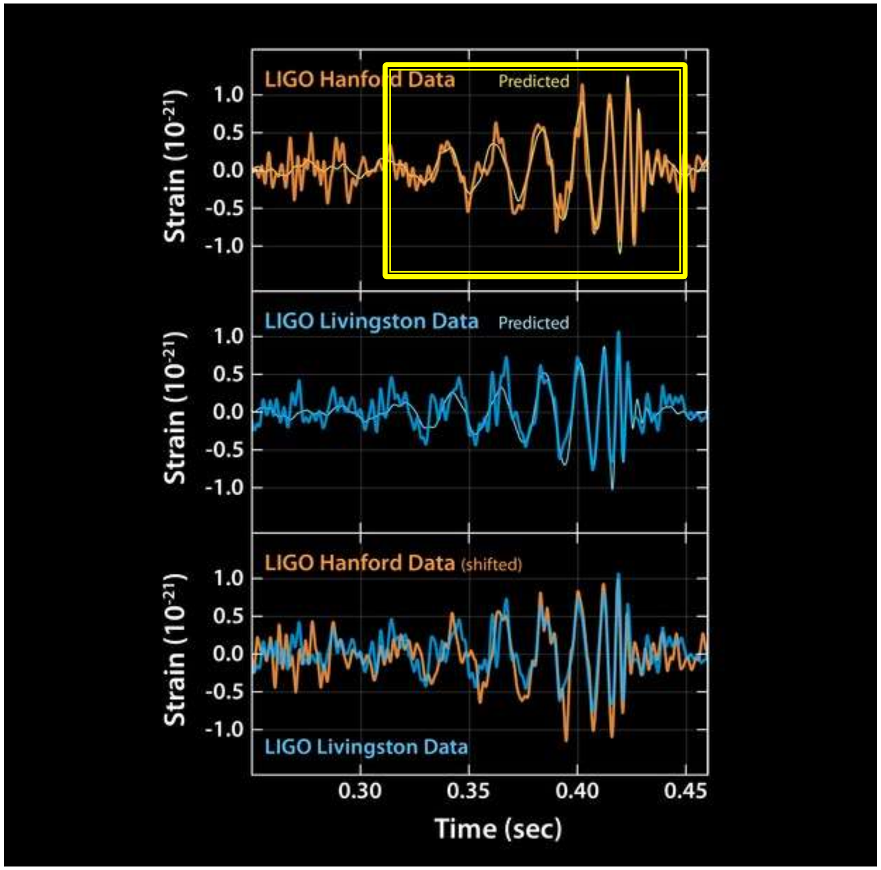
- From the Wiener-Khinchin theorem
  - Input noise PSD: $S_{in}(f)$
  - Transfer function: $T(f) = \frac{1}{\sqrt{S_{in}(f)}}$
  - Output PSD: $S_{out}(f) = S_{in}(f)|T(f)|^2 = const$ $\Rightarrow$White noise
- Matched filter in colored noise
  - $\langle q_a(\Theta), \bar{y} \rangle = \frac{1}{Nf_s} \tilde{y}(\tilde{q}_a{}^{\dagger} ./ \bar{S}^T) = \frac{1}{Nf_s} \left(\tilde{y}./\bar{S}^{1/2}\right)\left(\tilde{q}_a^{\dagger}./\bar{S}^{1/2}\right)$ (where the square root is taken over each element of $\bar{S}$.
  - Since $\tilde{y} = \tilde{s}(\Theta_{true}) + \tilde{n}$ where the PSD of the noise is $\bar{S}$, we see that the mathed filter can also be seen as the estimation of a "whitened" signal $\left(\tilde{s}(\Theta)./\bar{S}^{1/2}\right)$ in white noise.

Seismic wall in ground-based IFOs



- Whitened signal: $\tilde{s}(\Theta)./\bar{S}^{1/2}$
- The signal power is reduced where the noise PSD is higher
  - Note the log-scale on the plot
  - Essentially no power left in the whitened signal below some low frequency ("Seismic Wall")
- Current low frequency cutoff in LIGO is at about $\approx$ 40 Hz
- At final design sensitivity, it will be reduced to $\approx$ 10 Hz

GW150914



- Whitened data shown in the plots
- Whitened signal cuts off below the seismic wall frequency
- Actual signal stretches back into the past from the moment the Black Hole Binary was formed!
- (The noise is whitened but also band-passed)

Generation of colored noise

- Use Wiener-Khinchin theorem
- Pass white noise through an LTI system
- Discrete time: Pass white noise sequence through a digital filter
- Example: $y_i = b_0 x_i + b_1 x_{i-1} + \cdots + b_m x_{i-m}$ for some coefficients $\bar{b}$ and white noise sequence $\bar{x}$
- Fourier domain: Multiply discrete fourier transform of white noise sequence with $\sqrt{S(f)}$ where $S(f)$ is the desired output PSD (Caution: this will generate a circular auto-covariance noise sequence!)

Digital filtering

- Filtering operation: $y(t) = \int_{-\infty}^{\infty} x(t')\, h(t-t')dt'$
- $h(t)$: Impulse response
  - Output produced when the input is an impulse: $x(t') = \delta(t')$
- Discrete time: $y_i = b_0 x_i + b_1 x_{i-1} + \cdots + b_m x_{i-m}$
- Finite Impulse Response (FIR) filter: m is finite
- Infinite Impulse Response (IIR) filter:
  - $y_i = a_1 y_{i-1} + a_2 y_{i-2} + \cdots + a_p y_{i-p} + b_0 x_i + \cdots + b_m x_{i-m}$
  - In statistics (time series analysis): FIR filtered output=Moving Average (MA) process; pure IIR ($b_i = 0$) output = Auto-regressive (AR) process; General IIR process=ARMA process

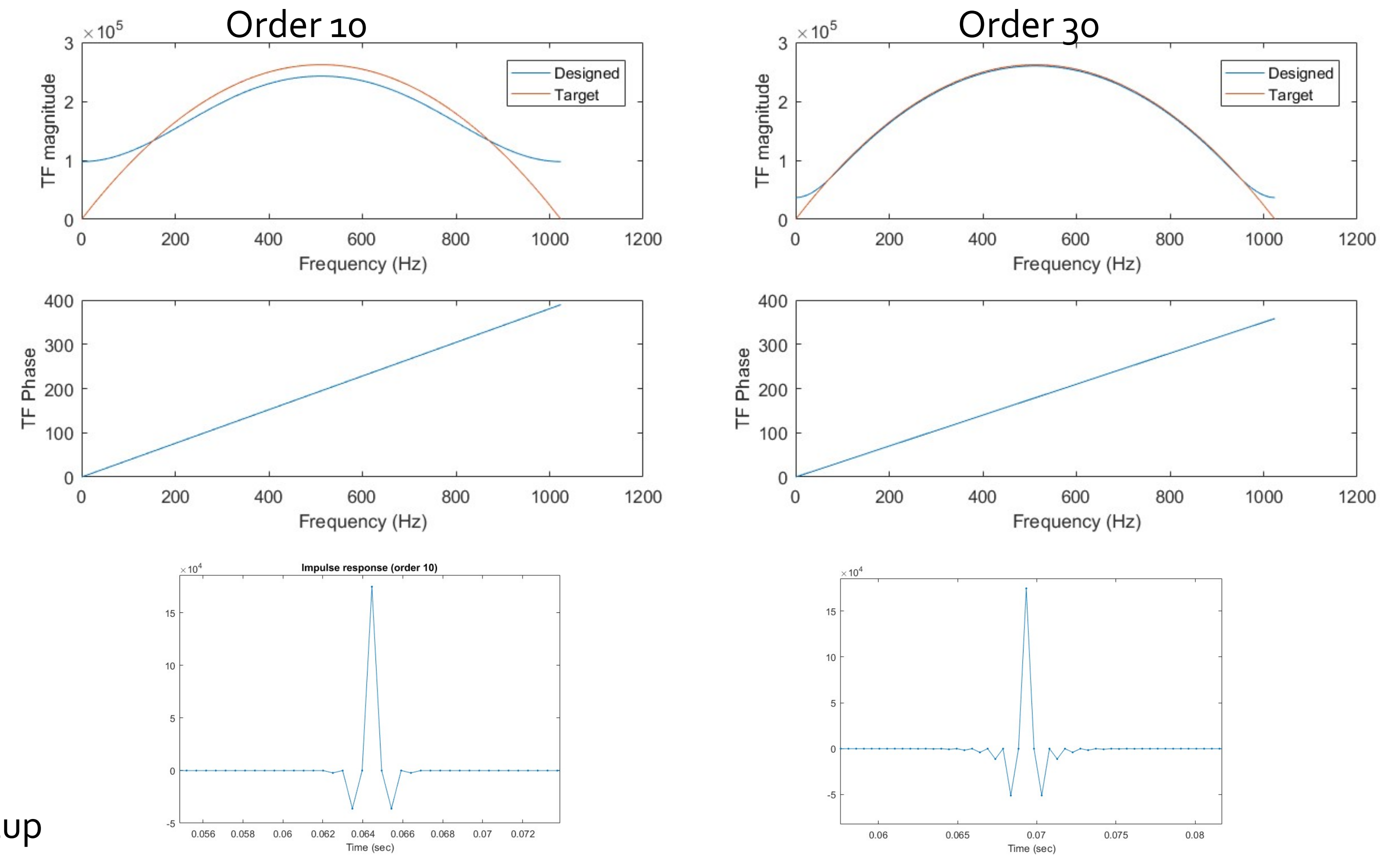
Digital filter design

- One of the main topics in Digital Signal Processing (DSP) is the design of digital filters
  - Given a desired transfer function $T(f)$, what should be the value of the filter coefficients $\bar{a}$, $\bar{b}$
- Various issues to consider:
  - FIR filters produced a linear phase shift between the input and output signals while IIR produced non-linear phase shift; However the magnitude of the transfer function can be better matched by a lower order IIR filter
  - FIR filters can be implemented using FFT (very fast); IIR filters are slower
  - Ref. Manolakis, Proakis, *Digital Signal Processing*

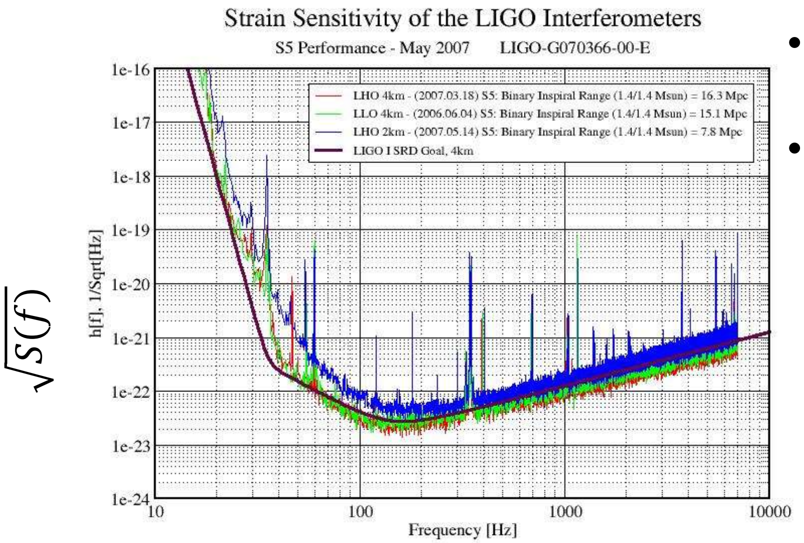
Filter design example

```
%Sampling frequency
fs = 2048 ;%Hz;

%Filter order
fN = 10;

%Frequency values at which to specify
%the target transfer function
f = 0:2:1024;

%Target transfer function
targetTf = f.*(1024-f);

%Design the digital filter
b = fir2(fN,f/(fs/2),targetTf);

%Get the impulse response
impVec = zeros(1,256);
impVec(128)=1;
impResp = fftfilt(b,impVec);

%Get the transfer function
designTf = fft(impResp);
```

Start of data acts like impuse ⇒ startup transient in filtered output



GW detector noise simulation



- Task: generate colored noise with the LIGO design PSD (smooth curve) $S(f)$
- Approach:
  - Fourier domain: circular autocovariance (but straightforward to implement numerically)⇒only fixed length segments and they cannot be joined
  - Time domain: Pass white noise sequence through a digital filter (FIR or IIR) → requires filter design tool ⇒data can be generated in arbitrary lengths and joined together after removing startup-transient
    - Target transfer function is $\sqrt{S(f)}$
- Caution: Neither approach can handle the steeply rising seismic part (also rising high frequency part → aliasing issues)
  - But the matched filter will cut these parts off anyway ⇒ need to generate noise with PSD matching only the middle part and set $S(f) = 0$ outside this range

Steps to follow for LIGO noise simulation

- Download design sensitivity ($\sqrt{S(f)}$) data:
  - https://dcc.ligo.org/LIGO-T0900288/public (advanced LIGO)
  - https://dcc.ligo.org/LIGO-E950018/public (initial LIGO)
- Convert to $\sqrt{S(f)}$
- Select high and low frequency cutoffs (e.g., 40Hz and 1000 Hz for initial LIGO)
- Use Matlab filter design tool `fir1` to generate FIR digital filter with $\sqrt{S(f)}$ as the target transfer function
- Use Matlab function `fftfilt` or `filtfilt` to pass white noise sequence through the filter
- See code `ligonoisesim` in LDACSchool repository

LISA

- Everything remains more or less the same for LISA as far as data analysis goes
- LISA data is obtained using time-delay interferometry
  - Response of the detector to a high frequency GW source requires more work to obtain
  - Low-frequency response obtained in essentially the same way as LIGO
- LISA noise can be assumed to be Gaussian and stationary when studying the performance of data analysis algorithms