LMDC: Model for SMBH (comparable masses).

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(Dated: May 22, 2006)

I. MODELS OVERVIEW

Currently the following PPn models are available:

- Circular orbit, non-spinning bodies

  Here we know the phase up to 3.5PN order and amplitude up to 2.5PN in usual Taylor expansion in velocity or equivalently in $x = (M\omega)^{2/3}$, where $M = m1 + m2$ is total mass and ω is orbital angular frequency (we work in geometrical units: $G = c = 1$). Besides this different resummation techniques could be applied: Pade, EOB (effective one body), see [1] for overview of various models.

- Circular orbit, spinning bodies.

  Here we know the carrier phase with (almost) the same precision as non-spinning (3.5PN): namely we have spin contribution only formally on 1.5PN level (spin-orbit) and on 2PN (spin-spin). PN corrections to leading 1.5PN (spin-orbit) term will be available soon. Amplitude corrections are known up to 1.5PN order [2].

- Eccentric orbit, non-spinning bodies

  Here we know phase up to 3.5 PN (very recent result) and amplitude up to 2PN beyond leading term. Expressions are very complicated. Spin effects will be included in the near future.

I want to propose to use only one model for BBH injections in the first/second parts of LMDC, this is the model for two spinning bodies in quasi-circular orbit.

Pros:

We expect that SMBH will be spinning (realism). We can study effect of spins on the parameter determination. We can study effect of higher order harmonics on the parameter determination. Relatively easy to generate (software).

Cons:

I am not aware of the existing search algorithm for these signals with reliable parameter determination. Less well known as compared to signals from non-spinning binaries. Despite the mentioned 'cons', it could have positive effect as it could boost research in those directions.

II. DETAILS OF THE PROPOSED MODEL

In first two subsections we define orbital evolution and waveform in the source and in radiation frames as defined in [2–4].

We start with defining the notations. The masses of bodies are defined by m_1, m_2. The total mass is $M = m_1 + m_2$, symmetric mass ration $\eta = m_1 m_2/M^2$, the reduced mass $\mu = m_1 m_2/M$ and $\delta m = m_1 - m_2$ is mass difference. Inspiral is described by the the orbital angular frequency ω, the orbital phase Φ, the direction $\hat{\mathbf{L}}_{\mathbf{N}} \sim \mathbf{r} \times \mathbf{v}$ of the orbital angular momentum, and the two spins $\mathbf{S_1} = \chi_1 m_1^2 \hat{\mathbf{S}}_{\mathbf{1}}$ and $\mathbf{S_2} = \chi_2 m_2^2 \hat{\mathbf{S}}_{\mathbf{2}}$, where $\hat{\mathbf{S}}_{\mathbf{1}}, \hat{\mathbf{S}}_{\mathbf{2}}$ are unit vectors and $0 \le \chi_{1,2} < 1$. Bold fonts denote 3-d vectors and hats denote the unit vectors. We use geometrical units $G = c = 1$.

A. Orbital evolution

For orbital angular frequency evolution we can use simple Taylor expression with spin-orbital (1.5PN) and spin-spin (formal 2PN) terms as given in [5] eqns.(1-7) with $\hat{\theta} = \theta - 3/7\lambda = \frac{1039}{4620}$. Or explicitly:

$$\frac{d\omega}{dt} = \frac{96}{5}\frac{\eta}{M^2}(M\omega)^{11/3}\left\{1 + 1PN + 1.5PN + SO + 2PN + SS + 2.5PN + 3PN + 3.5PN\right\} \tag{2.1}$$

and

$$
\begin{aligned}
1PN &= -\frac{743+924\eta}{336}(M\omega)^{2/3} && (2.2)\\
1.5PN &= 4\pi(M\omega) && (2.3)\\
SO &= -\frac{1}{12}\sum_{i=1,2}\left[\chi_i\left(\mathbf{\hat{L}_N}.\mathbf{\hat{S}_i}\right)\left(113\frac{m_i^2}{M^2}+75\eta\right)\right](M\omega) && (2.4)\\
2PN &= \left(\frac{34103}{18144}+\frac{13661}{2016}\eta+\frac{59}{16}\eta^2\right)(M\omega)^{4/3} && (2.5)\\
SS &= -\frac{1}{48}\eta\chi_1\chi_2\left[247(\mathbf{\hat{S}_1}.\mathbf{\hat{S}_2})-721(\mathbf{\hat{L}_N}.\mathbf{\hat{S}_1})(\mathbf{\hat{L}_N}.\mathbf{\hat{S}_2})\right](M\omega)^{4/3} && (2.6)\\
2.5PN &= -\frac{1}{672}(4159+14532\eta)\pi(M\omega)^{5/3} && (2.7)\\
3PN &= \left[\left(\frac{16447322263}{139708800}-\frac{1712}{105}\gamma_E+\frac{16}{3}\pi^2\right)+\left(-\frac{273811877}{1088640}+\frac{451}{48}\pi^2-\frac{88}{3}\hat{\theta}\right)\eta\right.\\
&\quad\left.+\frac{541}{896}\eta^2-\frac{5605}{2592}\eta^3-\frac{856}{105}\ln\left(16(M\omega)^{2/3}\right)\right](M\omega)^2 && (2.8)\\
3.5PN &= \left(-\frac{4415}{4032}+\frac{661775}{12096}\eta+\frac{149789}{3024}\eta^2\right)\pi(M\omega)^{7/3}, && (2.9)
\end{aligned}
$$

where γ_E is Euler's constant. Note that transformation $t \to t/M, \omega \to \omega M$, leads to eliminating total mass. This implies that the waveform for different total masses can be obtained by simple re-scaling.

For precession of spins and newtonian angular momentum $\mathbf{L}_N = \eta M^2(M\omega)^{-1/3}\mathbf{\hat{L}_N}$ we can use eqns.(4.17a-c) in [2] or equivalently eqns.(8-10) in [5]:

$$
\begin{aligned}
\frac{d\mathbf{\hat{S}_1}}{dt} &= \frac{(M\omega)^2}{2M}\left\{\eta(M\omega)^{-1/3}\left(4+3\frac{m_2}{m_1}\right)\mathbf{\hat{L}_N}+\chi_2\frac{m_2^2}{M^2}\left[\mathbf{\hat{S}_2}-3(\mathbf{\hat{S}_2}.\mathbf{\hat{L}_N})\mathbf{\hat{L}_N}\right]\right\}\times\mathbf{\hat{S}_1} && (2.10)\\
\frac{d\mathbf{\hat{S}_2}}{dt} &= \frac{(M\omega)^2}{2M}\left\{\eta(M\omega)^{-1/3}\left(4+3\frac{m_1}{m_2}\right)\mathbf{\hat{L}_N}+\chi_1\frac{m_1^2}{M^2}\left[\mathbf{\hat{S}_1}-3(\mathbf{\hat{S}_1}.\mathbf{\hat{L}_N})\mathbf{\hat{L}_N}\right]\right\}\times\mathbf{\hat{S}_2} && (2.11)\\
\frac{d\mathbf{\hat{L}_N}}{dt} &= \mathbf{V_{L_N}}\times\mathbf{\hat{L}_N}\equiv\frac{(M\omega)^2}{2M}\left\{\left(4+3\frac{m_2}{m_1}\right)\chi_1\frac{m_1^2}{M^2}\mathbf{\hat{S}_1}+\left(4+3\frac{m_1}{m_2}\right)\chi_2\frac{m_2^2}{M^2}\mathbf{\hat{S}_2}-\right.\\
&\quad\left.3(M\omega)^{1/3}\eta\chi_1\chi_2\left[(\mathbf{\hat{S}_2}.\mathbf{\hat{L}_N})\mathbf{\hat{S}_1}+(\mathbf{\hat{S}_1}.\mathbf{\hat{L}_N})\mathbf{\hat{S}_2}\right]\right\}\times\mathbf{\hat{L}_N}. && (2.12)
\end{aligned}
$$

Alternatively, instead of the last equation we can use

$$\mathbf{\hat{L}_N} = \{\sin(i)\cos(\alpha), \sin(i)\sin(\alpha), \cos(i)\}$$

which leads to evolution of α, i:

$$
\begin{aligned}
\frac{di}{dt} &= V^y_{L_N}\cos(\alpha)-V^x_{L_N}\sin(\alpha) && (2.13)\\
\frac{d\alpha}{dt} &= V^z_{L_N}-\frac{\cos(i)}{\sin(i)}\left[V^x_{L_N}\cos(\alpha)+V^y_{L_N}\sin(\alpha)\right] && (2.14)
\end{aligned}
$$

Here we have used condition that $\sin(i) \neq 0$, and $\mathbf{V_{L_N}} = \{V^x_{L_N}, V^y_{L_N}, V^z_{L_N}\}$. In the implementation of this scheme (described below), I integrate equations for (i, α) and eqn.(2.12) and check consistency (error of integration).

Finally, for the phase evolution we have:

$$\frac{d\Phi}{dt} = \omega - \frac{d\alpha}{dt}\cos(i). \qquad (2.15)$$

Besides masses m_1, m_2 and amplitude of spins χ_1, χ_2, one has to specify the initial conditions. Initial data for the differential equations specified at $t = t_0$, those are initial directions of spins: $\mathbf{\hat{S}_1}(t_0), \mathbf{\hat{S}_2}(t_0)$, initial direction of the orbital angular momentum $i(t_0), \alpha(t_0)$, initial frequency $\omega_0 = \omega(t_0)$ and initial phase $\Phi_0 = \Phi(t_0)$. These values are defined in source coordinate frame (see nice figure in [4]).

Termination point (follow [4]) is defined as

$$min_\omega \left\{ \dot\omega = 0; \frac{dE_{3PN}}{d\omega} = 0 \right\}$$

and referred to as MECO. One can write $\frac{dE_{3PN}}{d\omega} = 0$ explicitly using equations above and

$$\frac{d}{d\omega} = \frac{1}{\dot\omega}\frac{d}{dt}$$

when we need to differentiate vectors. I use $\dot\omega$ in the above equation only up to 1PN order, neglecting others. Following this prescription and neglecting term cubic in spins: $O(S^3)$ one can arrive at the following:

$$\begin{aligned}\frac{1}{M}\frac{dE_{3PN}}{d\omega} = -\frac{\mu}{3}(M\omega)^{-1/3}\Bigg\{1 - \frac{9+\eta}{6}(M\omega)^{2/3} + \frac{20}{3M^2}(\mathbf{\hat{L}_N}.\mathbf{S_{eff}})(M\omega)+ \\ \frac{5}{64}\left(1-\frac{3}{8}\eta\right)(M\omega)^{1/3}\frac{\delta m}{M}\chi_1\chi_2\left[1+\frac{743+924\eta}{336}(M\omega)^{2/3}\right](\mathbf{\hat{S}_1}\times\mathbf{\hat{S}_2})\mathbf{\hat{L}_N} \\ \frac{1}{8}(-81+57\eta-\eta^2)(M\omega)^{4/3} + \frac{3\eta}{4}\chi_1\chi_2\left[(\mathbf{\hat{S}_1}.\mathbf{\hat{S}_2}) - 3(\mathbf{\hat{L}_N}.\mathbf{\hat{S}_1})(\mathbf{\hat{L}_N}.\mathbf{\hat{S}_2})\right](M\omega)^{4/3} \\ 4\left[-\frac{675}{64} + \left(\frac{34445}{576} - \frac{205}{96}\pi^2\right)\eta - \frac{155}{96}\eta^2 - \frac{35}{5184}\eta^3\right](M\omega)^2\Bigg\},\end{aligned} \tag{2.16}$$

where

$$\mathbf{S_{eff}} = \left(1+\frac{3}{4}\frac{m_2}{m_1}\right)\mathbf{S_1} + \left(1+\frac{3}{4}\frac{m_1}{m_2}\right)\mathbf{S_2}$$

For non-spinning binaries $\omega_{MECO} = 0.129M^{-1}$, compared to the LSO for test mass case ($\eta \to 0$), $\omega_{LSO} = 0.068M^{-1}$. I find that MECO gives quite high ending frequency (almost twice LSO).

B. Gravitational Waveform

The waveform is defined in [2], eqn.(4.8) up to 1.5PN. and it is given explicitly in the AppendixB up to 1PN. In the radiation coordinate frame (see [3] eqns.(3.1-3.3), [2] eqns.(4.22a-c), [4] eqns.(21-23)) waveform is given as

$$h^{ij}_{TT} = h_+ T^{ij}_+ + h_\times T^{ij}_\times \tag{2.17}$$

$$\mathbf{T}_\times = \mathbf{e}^R_x \otimes \mathbf{e}^R_x - \mathbf{e}^R_y \otimes \mathbf{e}^R_y \tag{2.18}$$

$$\mathbf{T}_\times = \mathbf{e}^R_x \otimes \mathbf{e}^R_y + \mathbf{e}^R_y \otimes \mathbf{e}^R_x \tag{2.19}$$

$$h_+ = \frac{1}{2}h^{ij}(T_+)_{ij}, \;\; h_\times = \frac{1}{2}h^{ij}(T_\times)_{ij} \tag{2.20}$$

Radiative system introduces one more angle θ which is defined as inclination of direction to the observer $\mathbf{\hat{N}}$ to the total angular momentum at t_0[1].

Finally waveform (plus and cross polarizations in the radiation coordinate frame) can be written as

$$h_{+,\times} = \frac{2\mu}{D}\frac{M}{r}\left[Q_{+,\times} + \left(\frac{M}{r}\right)^{1/2} Q^1_{+,\times} + \left(\frac{M}{r}\right) Q^2_{+,\times} + \mathcal{O}\left(\frac{M}{r}\right)^{3/2}\right] \tag{2.21}$$

where expressions for $Q^1_{+,\times} = P^{0.5}_{+,\times}$, $Q^2_{+,\times} = PQ_{+,\times}$ are given in AppendixB of [2]. In order to show the conventions used here we will give the formulae explicitly (should be useful if one wants to use different coordinate frame or wants to check our result).

$$Q_{+,\times} = \frac{1}{2}Q^{ij}(T_{+,\times})_{ij}; \quad Q^1_{+,\times} = \frac{1}{2}(Q^1)^{ij}(T_{+,\times})_{ij}; \quad Q^2_{+,\times} = \frac{1}{2}(Q^2)^{ij}(T_{+,\times})_{ij}. \tag{2.22}$$

---

[1] Another angle, ϕ, defining direction to the observer, can always be put to zero by rotating coordinate frame around z-axis

where

$$
\begin{aligned}
Q^{ij} &= 2(\lambda^i\lambda^j - n^i n^j), && (2.23)\\
(Q^1)^{ij} &= \frac{\delta m}{m}\Big\{6(\hat{\mathbf{N}}.\mathbf{n})n^{(i}\lambda^{j)} + (\hat{\mathbf{N}}.\lambda)(n^i n^j - 2\lambda^i\lambda^j)\Big\} && (2.24)\\
(Q^2)^{ij} &= \frac{2}{3}(1-3\eta)\Big\{(\hat{\mathbf{N}}.\mathbf{n})^2(5n^i n^j - 7\lambda^i\lambda^j) - 16(\hat{\mathbf{N}}.\mathbf{n})(\hat{\mathbf{N}}.\lambda)n^{(i}\lambda^{j)}(\hat{\mathbf{N}}.\lambda)^2(3\lambda^i\lambda^j - n^i n^j)\Big\}\\
&\quad \frac{1}{3}(19-3\eta)(n^i n^j - \lambda^i\lambda^j) + \frac{2}{M^2}n^{(i}(\mathbf{\Delta}\times\hat{\mathbf{N}})^{j)}, && (2.25)
\end{aligned}
$$

where λ is unit vector along relative velocity $\mathbf{v}$, $\mathbf{n}$ is unit vector along the separation vecotr of binary $\mathbf{r}$, $\hat{\mathbf{N}}$ is the unit vector in the direction of the observer (SSB) and

$$\frac{\mathbf{\Delta}}{M^2} = \chi_2\frac{m_2}{M}\hat{\mathbf{S}}_\mathbf{2} - \chi_1\frac{m_1}{M}\hat{\mathbf{S}}_\mathbf{1}.$$

In the source coordinate frame:

$$
\begin{aligned}
\mathbf{n} &= \{-\sin(\alpha)\sin(\Phi) - \cos(\alpha)\cos(i)\cos(\Phi), -\cos(\alpha)\sin(\Phi) - \sin(\alpha)\cos(i)\cos(\Phi), \sin(i)\cos(\Phi)\} && (2.26)\\
\lambda &= \{\sin(\alpha)\sin(\Phi) - \cos(\alpha)\cos(i)\cos(\Phi), -\cos(\alpha)\sin(\Phi) - \sin(\alpha)\cos(i)\cos(\Phi), \sin(i)\cos(\Phi)\}, && (2.27)\\
\hat{\mathbf{N}} &= \{\sin(\theta), 0, \cos(\theta)\}. && (2.28)
\end{aligned}
$$

The radiation frame related to source frame in the following way

$$
\begin{aligned}
\hat{\mathbf{e}}^\mathbf{R}_\mathbf{x} &= \hat{\mathbf{e}}^\mathbf{S}_\mathbf{x}\cos(\theta) - \hat{\mathbf{e}}^\mathbf{S}_\mathbf{z}\sin(\theta) && (2.29)\\
\hat{\mathbf{e}}^\mathbf{R}_\mathbf{y} &= \hat{\mathbf{e}}^\mathbf{S}_\mathbf{y} && (2.30)\\
\hat{\mathbf{e}}^\mathbf{R}_\mathbf{z} &= \hat{\mathbf{e}}^\mathbf{S}_\mathbf{x}\sin(\theta) + \hat{\mathbf{e}}^\mathbf{S}_\mathbf{z}\cos(\theta) = \hat{\mathbf{N}}. && (2.31)
\end{aligned}
$$

So that

$$
\begin{aligned}
\mathbf{T}_+ &= \hat{\mathbf{e}}^\mathbf{S}_\mathbf{x}\otimes\hat{\mathbf{e}}^\mathbf{S}_\mathbf{x}\cos^2(\theta) - \hat{\mathbf{e}}^\mathbf{S}_\mathbf{x}\otimes\hat{\mathbf{e}}^\mathbf{S}_\mathbf{z}\sin(\theta)\cos(\theta) - \hat{\mathbf{e}}^\mathbf{S}_\mathbf{z}\otimes\hat{\mathbf{e}}^\mathbf{S}_\mathbf{x}\sin(\theta)\cos(\theta) - \hat{\mathbf{e}}^\mathbf{S}_\mathbf{y}\otimes\hat{\mathbf{e}}^\mathbf{S}_\mathbf{y} + \hat{\mathbf{e}}^\mathbf{S}_\mathbf{y}\otimes\hat{\mathbf{e}}^\mathbf{S}_\mathbf{y}\sin^2(\theta) && (2.32)\\
\mathbf{T}_\mathbf{x} &= (\hat{\mathbf{e}}^\mathbf{S}_\mathbf{x}\otimes\hat{\mathbf{e}}^\mathbf{S}_\mathbf{y} + \hat{\mathbf{e}}^\mathbf{S}_\mathbf{y}\otimes\hat{\mathbf{e}}^\mathbf{S}_\mathbf{x})\cos(\theta) - (\hat{\mathbf{e}}^\mathbf{S}_\mathbf{y}\otimes\hat{\mathbf{e}}^\mathbf{S}_\mathbf{z} + \hat{\mathbf{e}}^\mathbf{S}_\mathbf{z}\otimes\hat{\mathbf{e}}^\mathbf{S}_\mathbf{y})\sin(\theta) && (2.33)
\end{aligned}
$$

Using these relationships we have for Qs (we give expression for "+" polarization only, "x" polarization can be obtained by replacing "+" with "x"):

$$
\begin{aligned}
Q_+ &= -2(C_+\cos(2\Phi) + S_+\sin(2\Phi)) && (2.34)\\
Q^1_+ &= \frac{1}{4}\frac{\delta m}{M}\{9(aS_+ + bC_+)\cos(3\Phi) + 9(bS_+ - aC_+)\sin(3\Phi) +\\
&\quad (3aS_+ - 3bC_+ - 2bK+)\cos(\Phi) - (3bS_+ + 3aC_+ - 2aK_+)\sin(\Phi)\} && (2.35)\\
Q^2_+ &= \frac{8}{3}(1-3\eta)\left\{[(a^2-b^2)C_+ - 2abS_+]\cos(4\Phi) + [(a^2-b^2)S_+ + 2abC_+]\sin(4\Phi)\right\}\\
&\quad +\frac{1}{3}\left\{2(1-3\eta)[(b^2-a^2)K_+ + 2(a^2+b^2)C_+] + (19-3\eta)C_+\right\}\cos(2\Phi) +\\
&\quad +\frac{1}{3}\left\{4(1-3\eta)[(a^2+b^2)S_+ - abK_+] + (19-3\eta)S_+\right\}\sin(2\Phi) + DC_+\cos(\Phi) + DS_+\sin(\Phi). && (2.36)
\end{aligned}
$$

where

$$
\begin{aligned}
C_+ &= \frac{1}{2}\cos^2(\theta)\left(\sin^2(\alpha) - \cos^2(i)\cos^2(\alpha)\right) + \frac{1}{2}\left(\cos^2(i)\sin^2(\alpha) - \cos^2(\alpha)\right) - \\
&\quad \frac{1}{2}\sin^2(\theta)\sin^2(i) - \frac{1}{4}\sin(2\theta)\sin(2i)\cos(\alpha), && (2.37)\\
C_\times &= -\frac{1}{2}\cos(\theta)\sin(2\alpha)\left(1+\cos^2(i)\right) - \frac{1}{2}\sin(\theta)\sin(2i)\sin(\alpha), && (2.38)\\
S_+ &= \frac{1}{2}\left(1+\cos^2(\theta)\right)\cos(i)\sin(2\alpha) + \frac{1}{2}\sin(2\theta)\sin(i)\sin(\alpha), && (2.39)\\
S_\times &= -\cos(\theta)\cos(i)\cos(2\alpha) - \sin(\theta)\sin(i)\cos(\alpha) && (2.40)\\
K_+ &= \frac{1}{2}\cos^2(\theta)\left(\sin^2(\alpha) + \cos^2(i)\cos^2(\alpha)\right) - \frac{1}{2}\left(\cos^2(i)\sin^2(\alpha) + \cos^2(\alpha)\right) + \\
&\quad \frac{1}{2}\sin^2(\theta)\sin^2(i) + \frac{1}{4}\sin(2\theta)\sin(2i)\cos(\alpha), && (2.41)\\
K_\times &= -\frac{1}{2}\cos(\theta)\sin(2\alpha)\sin^2(i) + \frac{1}{2}\sin(\theta)\sin(2i)\sin(\alpha), && (2.42)\\
DC_+ &= -\frac{1}{M^2}\left[\Delta^y\sin(\alpha)\cos(\theta) + d\cos(\alpha)\right], && (2.43)\\
DS_+ &= -\frac{1}{M^2}\left[c\Delta^y - d\cos(i)\sin(\alpha)\right], && (2.44)\\
DC_\times &= \frac{1}{M^2}\left[\Delta^y\cos(\alpha) - d\cos(\theta)\sin(\alpha)\right], && (2.45)\\
DS_\times &= \frac{1}{M^2}\left[-\Delta^y\cos(i)\sin(\alpha) + cd\right], && (2.46)\\
a &= -\sin(\theta)\sin(\alpha), && (2.47)\\
b &= \cos(\theta)\sin(i) - \sin(\theta)\cos(i)\cos(\alpha), && (2.48)\\
c &= \cos(\theta)\cos(i)\cos(\alpha) + \sin(i)\sin(\theta), && (2.49)\\
d &= \Delta^z\sin(\theta) - \Delta^x\cos(\theta). && (2.50)
\end{aligned}
$$

Since we integrate orbital angular frequency we need to relate M/r to $(M\omega)$ (with sufficient for our purpose accuracy):

$$\frac{M}{r} \approx (M\omega)^{2/3}\left[1 + \frac{1}{3}(3-\eta)(M\omega)^{2/3}\right]$$

Taking the above into account we can rewrite $h_{+,\times}$ as follows:

$$h_{+,\times} = \frac{2\mu}{D}(M\omega)^{2/3}\left[Q_{+,\times} + (M\omega)^{1/3}Q^1_{+,\times} + (M\omega)^{2/3}\left(Q^2_{+,\times} + \frac{1}{3}(3-\eta)Q_{+,\times}\right) + \mathcal{O}(M\omega)\right] \tag{2.51}$$

The last term in (2.51) yields the following change in (2.36), instead of $(19-3\eta)C_+$, $(19-3\eta)S_+$ we have $(13-\eta)C_+$, $(13-\eta)S_+$ correspondingly.

In order to generate waveform using the inspiralling trajectory described in the previous subsection one need to specify distance between observer and the source, and the direction to the observer in the source frame $\theta, \phi = 0$.

III. REDUCTION TO NON-SPINNING CASE

Reducing from spinning to non-spinning case can be done pretty easily We start with orbital evolution

A. Non-spinning case: orbital evolution

The orbital evolution is described by only one differential equation which can be integrated analytically. This is the equation for the orbital frequency (2.1) with $SO = SS = 0$. The analytic expressions for frequency and phase are given in [6] we also give them here. First introduce

$$\tau = \frac{\eta}{5M}(t_c - t) \tag{3.1}$$

$$x = v^2 = (M\omega)^{2/3} \tag{3.2}$$

Then

$$
\begin{aligned}
(M\omega)^{2/3} &= \frac{1}{4}\tau^{-1/4}\Big\{1+\left(\frac{743}{4032}+\frac{11}{48}\eta\right)\tau^{-1/4}-\frac{1}{5}\pi\tau^{-3/8}+ \\
&\left(\frac{19583}{254016}+\frac{24401}{193536}\eta+\frac{31}{288}\eta^2\right)\tau^{-1/2}+\left(-\frac{11891}{53760}+\frac{109}{1920}\eta\right)\pi\tau^{-5/8}- \\
&\left[-\frac{10052469856691}{6008596070400}+\frac{1}{6}\pi^2+\frac{107}{420}\gamma_E-\frac{107}{3360}\ln\left(\frac{\tau}{256}\right)+\left(\frac{15335597827}{3901685760}-\frac{451}{3072}\pi^2-\frac{77}{72}\lambda+\frac{11}{24}\theta\right)\eta-\right. \\
&\left.\frac{15211}{442368}\eta^2+\frac{25565}{331776}\eta^3\right]\tau^{-3/4}+\left(-\frac{113868647}{433520640}-\frac{31821}{143360}\eta+\frac{294941}{3870720}\eta^2\right)\pi\tau^{-7/8}\Big\}
\end{aligned} \tag{3.3}
$$

where $\lambda=-\frac{1987}{3080},\theta=-\frac{11831}{9240}$. Or taking the power 3/2:

$$
\begin{aligned}
M\omega &= \frac{1}{8}\tau^{-3/8}\Big\{1+\left(\frac{11}{32}\eta+\frac{743}{2688}\right)\tau^{-1/4}-\frac{3}{10}\pi\tau^{-3/8}+\left(\frac{1855099}{14450688}+\frac{371}{2048}\eta^2+\frac{56975}{258048}\eta\right)\tau^{-1/2}+ \\
&\left(\frac{13}{256}\eta-\frac{7729}{21504}\right)\pi\tau^{-5/8}+\left[\frac{235925}{176472}\eta^3-\frac{30913}{1835008}\eta^2+\left(-\frac{451}{2048}\pi^2+\frac{25302017977}{4161798144}\right)eta-\right. \\
&\left.\frac{720817631400877}{288412611379200}+\frac{53}{200}\pi^2+\frac{107}{280}\gamma_E-\frac{107}{2240}\ln\left(\frac{\tau}{256}\right)\right]\tau^{-3/4}+ \\
&\left(\frac{141769}{1290240}\eta^2-\frac{188516689}{433520640}-\frac{97765}{258048}\eta\right)\pi\tau^{-7/8}\Big\}
\end{aligned} \tag{3.4}
$$

The termination frequency is defined by MECO [4]: $\omega_{MECO}=0.129M^{-1}$ or by condition $\dot\omega=0$.
The orbital phase could be expressed in terms of orbital frequency

$$
\begin{aligned}
\Phi &= -\frac{1}{32\eta}(M\omega)^{-5/3}\Big\{1+\left(\frac{3715}{1008}+\frac{55}{12}\eta\right)(M\omega)^{2/3}-10\pi(M\omega)+ \\
&\left(\frac{15293365}{1016064}+\frac{27145}{1008}\eta+\frac{3085}{144}\eta^2\right)(M\omega)^{4/3}+\left(\frac{38645}{1344}-\frac{65}{16}\eta\right)\pi\ln\left(\frac{\omega}{\omega_0}\right)(M\omega)^{5/3} \\
&\left[\frac{12348611926451}{18776862720}-\frac{160}{3}\pi^2-\frac{1712}{21}\gamma_E-\frac{856}{21}\ln\left[16(m\omega)^{2/3}\right]+\left(-\frac{15335597827}{12192768}+\frac{2255}{48}\pi^2+\frac{3080}{9}\lambda-\right.\right. \\
&\left.\left.\frac{440}{3}\theta\right)\eta-\frac{76055}{6912}\eta^2-\frac{127825}{5184}\eta^3\right](M\omega)^2+\left(\frac{77096675}{2032128}+\frac{378515}{12096}\eta-\frac{74045}{6048}\eta^2\right)\pi(M\omega)^{7/3}\Big\}
\end{aligned} \tag{3.5}
$$

This is waveform "TaylorT3" in classification described in [7] if one substitutes orbital angular frequency (3.4) in the expression for phase above.

B. Gravitational waveform: non-spinning case

The reduction to non-spinning case can be easily done by using (2.51) and setting

$$\alpha=0,\quad i=0.$$

Together with $\chi_1=\chi_2=0$. This reduces the long expression to relatively short[2]:

$$C_+=-\frac{1}{2}(1+\cos^2(\theta)),\quad C_\times=0 \tag{3.6}$$

$$S_+=0,\quad S_\times=-\cos(\theta) \tag{3.7}$$

$$K_+=-\frac{1}{2}\sin^2(\theta),\quad K_\times=0 \tag{3.8}$$

$$DC_+=DC_\times=DS_+=DS_\times=0 \tag{3.9}$$

$$a=0,\quad d=0,\quad b=-\sin(\theta),\quad c=\cos(\theta). \tag{3.10}$$

[2] Note that here θ angle correspond to i used in [3]

Substituting these expressions into (2.36) we get:

$$\begin{aligned}
Q_+ &= (1+\cos^2(\theta))\cos(2\Phi) && (3.11)\\
Q_\times &= 2\cos(\theta)\sin(2\Phi) && (3.12)\\
Q_+^{(1)} &= \frac{1}{8}\frac{\delta m}{M}\sin(\theta)\left[9(1+\cos^2(\theta))\cos(3\Phi) - (5+\cos^2(\theta))\cos(\Phi)\right] && (3.13)\\
Q_\times^{(1)} &= \frac{3}{4}\frac{\delta m}{M}\sin(\theta)\cos(\theta)\left[3\sin(3\Phi) - \sin(\Phi)\right] && (3.14)\\
\frac{1}{3}(3-\eta)Q_+ + Q_+^{(2)} &= \frac{4}{3}(1-3\eta)(1-\cos^4(\theta))\cos(4\Phi) + \\
&\quad \frac{1}{6}\left[-19+19\eta-(9+11\eta)\cos^2(\theta)+(1-3\eta)\cos^4(\theta)\right]\cos(2\Phi) && (3.15)\\
\frac{1}{3}(3-\eta)Q_\times + Q_\times^{(2)} &= \frac{8}{3}(1-3\eta)(1-\cos^2(\theta))\cos(\theta)\sin(4\Phi) - \\
&\quad \frac{1}{3}\left[17-13\eta-4(1-3\eta)\cos^2(\theta)\right]\cos(\theta)\sin(2\Phi) && (3.16)
\end{aligned}$$

which is in agreement with result derived by number of authors.

IV. TRANSFORMATION TO SSB

Transformation to Solar-System-Baricentric ecliptic coordinates is given by O_1 rotation matrix given in [10] eqn.7-9. Transformation to the LISA frame is given by eqn.6,9 [10]. For completeness we will write it here explicitly. At the origin of the SSB frame, the transverse-traceless metric perturbation due to a source located at ecliptic latitude β and longitude λ can be written as

$$H(t) = O_1 H^S(t) O_1^{-1} \qquad (4.1)$$

where the metric perturbation in the source frame is taken to be

$$H^S(t) = \begin{pmatrix} h_+(t) & h_\times(t) & 0 \\ h_\times(t) & -h_+(t) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad (4.2)$$

The two polarizations are given by eqn.(2.51) and the rotation matrix O_1:

$$O_1 = \begin{pmatrix} \sin(\lambda)\cos(\psi) - \cos(\lambda)\sin(\beta)\sin(\psi) & -\sin(\lambda)\sin(\psi) - \cos(\lambda)\sin(\beta)\cos(\psi) & -\cos(\lambda)\cos(\beta) \\ -\cos(\lambda)\cos(\psi) - \sin(\lambda)\sin(\beta)\sin(\psi) & \cos(\lambda)\sin(\psi) - \sin(\lambda)\sin(\beta)\cos(\psi) & -\sin(\lambda)\cos(\beta) \\ \cos(\beta)\sin(\psi) & \cos(\beta)\cos(\psi) & -\sin(\beta) \end{pmatrix}. \qquad (4.3)$$

These equations describe the plane waves, which are propagating from a source located in the direction

$$\hat{\mathbf{k}} = \{\cos(\lambda)\cos(\beta), \sin(\lambda)\cos(\beta), \sin(\beta)\}. \qquad (4.4)$$

The polarization angle ψ encodes a rotation around the direction of wave propagation, $-\hat{\mathbf{k}}$, setting the convention used to define the two polarizations, "$+$" and "$\times$".

So in order to describe the gravitational wave signal in SBB one requires 3 more parameters determining the position of the source and the polarization angle

$$\lambda, \beta, \psi \qquad (4.5)$$

V. PARAMETER DISTRIBUTION

Here I will try to define the distribution of the parameter for injections.

- *Masses.* Individual masses are uniformly distributed in the range $[10^5 - 10^7]M_\odot$. For first challenges we can reduce upper bound to $10^6 M_\odot$ as this will increase duration of the waveform in the LISA band.

- *Spins.* Initial conditions for spins. For spinning binaries χ_1, χ_2 are uniformly distributed on $[0,1]$, $\hat{S}_i^z$ are uniformly distributed on $[-1,1]$, and angles ϕ_{S_i} are uniformly distributed on $[0, 2\pi]$

- *Initial orbit* Initial orbit is defined by initial direction of the orbital angular momentum $\hat{\mathbf{L}}_{\mathbf{N}}$, initial phase and initial orbital frequency. $\hat{\mathbf{L}}_{\mathbf{N}}^z = \cos(i)$ is uniformly distributed on $[-1,1]$, α and Φ_0 are uniformly distributed on $[0, 2\pi]$. The initial orbital angular frequency ω_0 will be defined by the length of the duration of the signal t_{cut}. User should define $\Delta t = t_c - t_{cut}$ (say 1 week before t_c).

- *Direction to the observer.* As mentioned in the text $\phi = 0$ and $\cos(\theta)$ is distributed uniformly on $[-1,1]$

- *Distance to the observer.* Distance to the source should be defined by SNR for a given direction. Here there is an option: 1. try to estimate actual SNR for a given parameters 2. estimate SNR for non-spinning binary for the same location on the sky, the same masses 3. Restricted SPA waveform for moving or static LISA

- *Location on the sky in SSB.* In SSB we need to set the direction to the source and polarization angle: $\sin(\beta)$ is distributed uniformly on $[-1,1]$, and λ, ψ are uniformly distributed on $[0, 2\pi]$

VI. IMPLEMENTATION

The computation of GW from the spinning binary (in the source frame) is implemented as C++ class. It has the following methods:

- `SpinBBHWaveform(float mass1, float mass2)`, Constructor, the input are two masses in units of solar mass.

- `SetInitialSpins(float chi1, float S1z0, float phiS10, float chi2, float S2z0, float phiS20)`. Here user specifies the initial value for spins. The spins are specified by its amplitude `chi1, chi2`, and the orientation is defined $\hat{\mathbf{S}}_{\mathbf{1}} = \{\sqrt{1-(S1z0)^2}\cos(phiS10), \sqrt{1-(S1z0)^2}\sin(phiS10), S1z0\}$ (similar for $\hat{\mathbf{S}}_{\mathbf{2}}$). This parametrization was chosen as the most convenient for generating the random signal, since `S1z0, S2z0` are distributed uniformly on $[-1,1]$ and `phiS10, phiS20` are distributed uniformly on $[0, 2\pi)$.

- `void SetInitialOrbit(float omega0, float phi0, float iota0, float alpha0)` Here user specifies initial orbit: initial frequency `omega0`, initial phase `phi0`, and initial direction of the orbital angular momentum `iota0, alpha0` (corresponding to i, α).

- `void ComputeInspiral(float timeStep, std::string order, std::string route)`. This method integrates equations of motion and as a result produces the inspiralling trajectory. User must specify sampling step `timeStep` in seconds. The parameter `order` specifies the PPN order of phase and orbital frequency, and parameter `route` specifies for non-spinniung binaries method to compute orbit (analytic or numerical integration).

- `void SetObserver(float thetaD, double D)`. Here user specifies direction `thetaD` and distance `D` (in pc) to the observer.

- `EstimateTc(double omega0, std::string order)`. This method returns estimated value of coalescence time using orbital evolution of non-spinning binary up to PPN order specified by `order` and with initial frequency `omega0`.

- `void ComputeWaveform(int order, float truncateTime, Matrix<double>& hPlus, Matrix<double>& hCross)`. This method computes the waveform $h_+, h_\times$ for observer described in `SetObserver` and trajectory computed in `ComputeInspiral`. This can be called multiple number of times for different observers without re-computing inspiral. The parameter `order` allows user to get only leading order amplitude (0), only 0.5PN amplitude (1), only 1PN amplitude or all of them together (default). User can also truncate the waveform at the time $\Delta T =$ `truncateTime` (in sec) before coalescence.

In the figure below one can see the example of the waveform.

As I have mentioned above the MECO condition gives quite high frequency. I have used the condition $\frac{1}{M}|dE/d\omega| < 3.5 \times 10^{-6}$ to terminate inspiral.

FIG. 1: Gravitational waveform in the source frame for the following parameters: $m_1 = 2.0M_\odot, m_2 = 4.0M_\odot, \chi_1 = 0.7, \chi_2 = 0.9, \hat{S}_1^z(t_0) = 0.4, \hat{S}_2^z(t_0) = 0.7, \phi_{S1}(t_0) = \pi/7, \phi_{S2}(t_0) = \pi/3.5, \omega(t_0) = 40\pi, \Phi(t_0) = 0, i(t_0) = \pi/7, \alpha(t_0) = \pi/3, \theta = \pi/2.2, D = 10^6(pc)$

VII. MODEL OF THE SIGNAL FOR THE FIRST MOCK DATA CHALLENGE

Here we summarize the model which will be used for the first LISA mock data challenge. It is a restricted non-spinning model with the phase up to 2PN.

The orbital evolution is presented by orbital angular frequency and orbital phase up to 2PN order:

$$M\omega = \frac{1}{8}\tau^{-3/8}\left\{1 + \left(\frac{11}{32}\eta + \frac{743}{2688}\right)\tau^{-1/4} - \frac{3}{10}\pi\tau^{-3/8} + \left(\frac{1855099}{14450688} + \frac{371}{2048}\eta^2 + \frac{56975}{258048}\eta\right)\tau^{-1/2}\right\} \tag{7.1}$$

And the phase is described by:

$$\Phi = -\frac{1}{32\eta}(M\omega)^{-5/3}\left\{1 + \left(\frac{3715}{1008} + \frac{55}{12}\eta\right)(M\omega)^{2/3} - 10\pi(M\omega) + \left(\frac{15293365}{1016064} + \frac{27145}{1008}\eta + \frac{3085}{144}\eta^2\right)(M\omega)^{4/3}\right\} \tag{7.2}$$

The evolution is terminated by condition $r = 6M$ which corresponds to

$$\omega_{LSO} = 0.068(M_\odot/M)Hz$$

unless user wants to terminate it Δt sec before the coalescence time t_c.

Waveform is described by two polarizations in the radiation frame:

$$h_+ = \frac{2\mu}{D}(M\omega)^{2/3}(1 + \cos^2(\theta))\cos(2\Phi) \tag{7.3}$$

$$h_\times = \frac{4\mu}{D}(M\omega)^{2/3}\cos(\theta)\sin(2\Phi) \tag{7.4}$$

The transformation to the SSB system is given in the Section IV. This is finalizes the time domain description of the signal which will be an input to the simulator.

A. Stationary phase approximation

Here we derive the fourier image of the time-domain signal described above using stationary phase approximation (SPA).

The signal in the radiation frame is given by eqns.(7.3, 7.4), this signal has to be passed through a particular transfer function to get a TDI observable, this can be presented in a rather symbolic form as follows

$$h^I = \sum_{j=1}^{3}\sum_k A^I_{k,j}(t)\left[F_j^+(t)h_+(t - t_k) + F_j^\times(t)h_\times(t - t_k)\right] \tag{7.5}$$

where index I corresponds to a different TDI combination α_i, $\bar{A}, \bar{E}, \bar{T}, ...$; t_k are delays and $A_{k,j}(t)$ scales an amplitude of each delayed waveform. The antenna pattern if given as

$$F_j^+ = u_j(t)\cos(2\psi) + v_j(t)\sin(2\psi), \tag{7.6}$$

$$F_j^\times = v_j(t)\cos(2\psi) - u_j(t)\sin(2\psi) \tag{7.7}$$

where ψ is polarization angle introduced in Section IV. The functions $u_j(t), v_j(t)$ depend on the rotation matrix O_1 and on LISA-to-SSB transformation, thus, they depend on time through motion of the guiding center and cartwheelng motion of the spacecrafts, and on the position of the source in the sky, given by the ecliptic coordinates β, λ. The explicit expressions for those functions are given in [10], eqns.(27-39). Following [8] introduce polarization phase $\Phi_p^j(t)$:

$$F_j^+(t)h_+(t-t_k) + F_j^\times(t)h_\times(t-t_k) = \mathcal{A}^j(t)(M\omega)^{2/3}\cos(2\Phi(t-t_k) - \Phi_p^j(t)) \tag{7.8}$$

where

$$\mathcal{A}^j(t) = \frac{2\mu}{D}\sqrt{(F_j^+(t)h_0^+)^2 + (F_j^\times(t)h_0^\times)^2}, \tag{7.9}$$

$$\tan(\Phi_p^j) = \frac{F_j^\times(t)h_0^\times}{F_j^+(t)h_0^+} \tag{7.10}$$

$$h_0^\times = 2\cos(\theta), \quad h_0^+ = \frac{1}{2}(1+\cos^2(\theta)) \tag{7.11}$$

As a next step we expand the phase around t:

$$\Phi(t-t_k) \approx \Phi(t) - \omega t_k + ... \tag{7.12}$$

here we neglected second order Doppler corrections (see [8] for justification). It also convenient to represent cos in exponential form:

$$\cos(2\Phi(t-t_k) - \Phi_p^j(t)) = \frac{1}{2}e^{-i\left[2\Phi(t) - 2\omega t_k - \Phi_p^j(t)\right]} + c.c. \tag{7.13}$$

where $c.c.$ stands for complex conjugate, which we will omit in further expressions (I assume that we mainly interested in positive frequencies). Here we can identify the Doppler phase $\Phi_D^k(t) = 2\omega t_k$. The delays can be further expanded according to:

$$t_j = \omega R\cos(\beta)\cos(\Omega t + \eta_0 - \lambda) + \hat{\mathbf{k}}(O_2(t)\mathbf{p}_j^L) - m_j L \tag{7.14}$$

where $R = 1au$, η_0 is initial phase of the LISA's guiding center, O_2 is the LISA-to-SSB rotation matrix (see [10]), $\mathbf{p}_j^L$ is the vector connecting the guiding center and j-th spacecraft and m_j is integer coming from a particular TDI combination. Note that $\omega\hat{\mathbf{k}}(O_2(t)\mathbf{p}_j^L) \sim \omega L$.

Put all pieces together:

$$h^I = \Lambda^I(t)(M\omega)^{2/3}e^{-i(2\Phi(t))}, \tag{7.15}$$

$$\Lambda^I(t) = \sum_{j=1}^{3}\sum_k \frac{1}{2}A_{k,j}^I(t)\mathcal{A}^j(t)e^{i(\Phi_D^k(t) + \Phi_p^j(t))} \tag{7.16}$$

Since $\Lambda(t)$ varies on the timescale $1yr >> 2\pi/\omega$ we can apply SPA to evaluate the waveform in the frequency domain:

$$\tilde{h}_{spa}^I = \Lambda(t_f)(M\omega(t_f))^{2/3}\left[\frac{\pi}{\dot{\omega}(t_f)}\right]^{1/2} e^{i(2\pi f t_f - 2\Phi(t_f) - \pi/4)} \tag{7.17}$$

where t_f is defined by equation

$$\omega(t_f) = \pi f \tag{7.18}$$

For restricted waveform it is sufficient to use only leading order term in amplitude:

$$(\dot{\omega}(t_f))^{-1/2} = M\sqrt{\frac{5}{96\eta}}(\pi M f)^{-11/6} \tag{7.19}$$

The phase (up to 2PN order) is given in [7]:

$$\begin{aligned} 2\pi f t_f - 2\Phi(t_f) &= 2\pi f t_c - \phi_c + \frac{3}{128\eta}(\pi M f)^{-5/3}\left\{1 + \frac{5}{9}\left(\frac{743}{84} + 11\eta\right) - 16\pi(\pi M f) + \right. \\ &\left. \left(\frac{15293365}{1016064} + \frac{27145}{1008}\eta + \frac{3085}{144}\eta^2\right)(\pi M f)^{4/3}\right\} \end{aligned} \tag{7.20}$$

and inverting (7.1) we obtain

$$t_f = t_c - \frac{5M}{256\eta}(\pi M f)^{-8/3}\left\{1 + \left(\frac{743}{252} + \frac{11}{3}\eta\right)(\pi M f)^{2/3} - \frac{32}{5}\pi(\pi M f) + \left(\frac{8579163}{508032} + \frac{29555}{1008}\eta + \frac{203}{8}\eta^2\right)(\pi M f)^{4/3}\right\} \tag{7.21}$$

Confirm 2PN term in t_f.

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