Analytic kludge waveforms for extreme-mass-ratio inspirals

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I. INTRODUCTION

This note summarizes the analytic kludge waveforms of Barack & Cutler [1], hereafter referred to as BC. The orbit is approximated as a Newtonian ellipse at any instant, but one whose perihelion direction, orbital plane, semi-major axis, and eccenticity evolve according to post-Newtonian evolution equations. At any instant, the emitted waveform is taken to be the Peters-Matthews waveform corresponding to the instantaneous Newtonian orbit. While these waveforms are not particularly accurate in the highly relativistic regime of interest for EMRI searches, they do exhibit the main qualitative features of true waveforms, and are considerably simpler to generate. It is expected that any search strategy that works for "analytic kludge" waveforms could be converted over fairly easily to true, GR waveforms, once these become available. For more details we refer the reader to BC (from which pieces were chopped up and re-pasted to make this note).

Our index notation is the following. Indices for vectors and tensors on parameter space are chosen from the beginning of the Latin alphabet $(a, b, c, \ldots)$. Vectors and tensors on three-dimensional space have indices chosen from the middle of the Latin alphabet $(i, j, k, \ldots)$, and run over $1, 2, 3$; their indices are raised and lowered with the flat 3-metric, η_{ij}. Actually, we adopt a mixed notation for spatial vectors, sometimes labelling them with spatial indices $(i, j, k, \ldots)$, but sometimes suppressing the indices and instead using the standard $3-d$ vector notation: an over-arrow (as in $\vec{A}$) to represent a vector, $\vec{A}\cdot\vec{B}$ to represent a scalar ("dot") product, and $\vec{A}\times\vec{B}$ to represent the vector ("cross") product. An over-hat (as in $\hat{n}$) will indicate that a vector is normalized, i.e., has unit length. We trust our meaning will always be clear, despite this mixed notation. Throughout this note we use units in which $G = c = 1$.

II. PARAMETER SPACE

The two-body system is described by 17 parameters. The spin of the CO can be marginally relevant (see Appendix C of BC), but in this paper we shall ignore it, leaving us with 14 parameters. We shall denote a vector in the 14-d parameter space by λ^a $(a = 0, \ldots, 13)$. We choose our parameters as follows:

$$\begin{aligned}\lambda^a \;\equiv\;& (\lambda^0, \ldots, \lambda^{13}) = \\ & \left[t_0\, \ln\mu,\, \ln M,\, S/M^2,\, e_0,\, \tilde{\gamma}_0,\, \Phi_0,\, \mu_S \equiv \cos\theta_S,\, \phi_S,\, \cos\lambda,\, \alpha_0, \mu_K \equiv \cos\theta_K,\, \phi_K,\, \ln(\mu/D)\right]. \end{aligned} \tag{1}$$

Here, t_0 is a time parameter that allows us to specify "when" the inspiral occurs—we shall generally choose t_0 to be the instant of time when the (radial) orbital frequency sweeps through some fiducial value ν_0 (typically, we shall choose ν_0 of order 1 mHz), μ and M are the masses of the CO and MBH, respectively, and S is the magnitude of the MBH's spin angular momentum (so $0 \le S/M^2 \le 1$). The parameters e_0, $\tilde{\gamma}_0$, and Φ_0 describe, respectively, the eccentricity, the direction of the pericenter within the orbital plane, and the mean anomaly—all at time t_0. More specifically, we take $\tilde{\gamma}_0$ to be the angle (in the plane of the orbit) from $\hat{L}\times\hat{S}$ to pericenter, and, as usual, Φ_0 to be the mean anomaly with respect to pericenter passage. The parameter $\alpha_0 \equiv \alpha(t = t_0)$ [where $\alpha(t)$ is defined in Eq. (14)] describes the direction of $\hat{L}$ around $\hat{S}$ at t_0. The angles (θ_S, ϕ_S) are the direction to the source, in ecliptic-based coordinates; $(\theta_K,\, \phi_K)$ represent the direction $\hat{S}$ of the MBH's spin (approximated as constant) in ecliptic-based coordinates; and λ is the angle between $\hat{L}$ and $\hat{S}$ (also approximated as constant[17]). Finally, D is the distance to the source.

The various parameters and their meaning are summarized in Table I. Fig. 1 illustrates the various angles involved in our parameterization.

Note for simplicity we are treating the background spacetime as Minkowski space, not Robertson-Walker. To correct this, for a source at redshift z, requires only the simple translation: $M \to M(1+z)$, $\mu \to \mu(1+z)$, $S \to S(1+z)^2$, $D \to D_L$, where D_L is the "luminosity distance" [8].

The parameters can be divided into "intrinsic" and "extrinsic" parameters, following Buonanno, Chen, and Vallisneri [6] (hereafter, BCV). Extrinsic parameters refer to the observer's position or orientation, or to the zero-of-time on the observer's watch. There are seven extrinsic parameters: the four parameters t_0, μ_S, ϕ_S, and D correspond to the freedom to translate the same binary in space and time, and the three parameters μ_K, ϕ_K, and α_0 are basically

FIG. 1: The MBH-CO system: setup and notation. M and μ are the masses of the MBH and the CO, respectively. The axes labeled x–y–z represent a Cartesian system *based on ecliptic coordinates* (the Earth's motion around the Sun is in the x–y plane). The spin $\vec{S}$ of the MBH is parametrized by its magnitude S and the two angular coordinates θ_K, ϕ_K, defined (in the standard manner) based on the system x–y–z. $\vec{L}(t)$ represents the (time-varying) orbital angular momentum; its direction is parametrized by the (constant) angle λ between $\vec{L}$ and $\vec{S}$, and by an azimuthal angle $\alpha(t)$ (not shown in the figure). The angle $\tilde{\gamma}(t)$ is the (intrinsic) direction of pericenter, as measured with respect to $\vec{L} \times \vec{S}$. Finally, $\Phi(t)$ denotes the mean anomaly of the orbit, i.e., the average orbital phase with respect to the direction of pericenter.

λ^0	t_0	t_0 is time where orbital frequency sweeps through fiducial value (e.g., 1mHz)
λ^1	$\ln \mu$	(ln of) CO's mass
λ^2	$\ln M$	(ln of) MBH's mass
λ^3	S/M^2	magnitude of (specific) spin angular momentum of MBH
λ^4	e_0	$e(t_0)$, where $e(t)$ is the orbital eccentricity
λ^5	$\tilde{\gamma}_0$	$\tilde{\gamma}(t_0)$, where $\tilde{\gamma}(t)$ is the angle (in orbital plane) between $\hat{L} \times \hat{S}$ and pericenter
λ^6	Φ_0	$\Phi(t_0)$, where $\Phi(t)$ is the mean anomaly
λ^7	$\mu_S \equiv \cos\theta_S$	(cosine of) the source direction's polar angle
λ^8	ϕ_S	azimuthal direction to source
λ^9	$\cos\lambda$	$\hat{L} \cdot \hat{S} (= \text{const})$
λ^{10}	α_0	$\alpha(t_0)$, where $\alpha(t)$ is the azimuthal direction of $\hat{L}$ (in the orbital plane)
λ^{11}	$\mu_K \equiv \cos\theta_K$	(cosine of) the polar angle of MBH's spin
λ^{12}	ϕ_K	azimuthal direction of MBH's spin
λ^{13}	$\ln(\mu/D)$	(ln of) CO's mass divided by distance to source

TABLE I: Summary of physical parameters and their meaning. The angles (θ_S,ϕ_S) and (θ_K,ϕ_K) are associated with a spherical coordinate system attached to the ecliptic. $\hat{L}$ and $\hat{S}$ are unit vectors in the directions of the orbital angular momentum and the MBH's spin, respectively. For further details see figure 1 and the description in the text.

Euler angles that specify the orientation of the orbit with respect to the observer (at t_0). The intrinsic parameters are the ones that control the detailed dynamical evolution of the system, without reference to the observer's location or orientation. In our parametrization, the seven intrinsic parameters are $\ln\mu$, $\ln M$, S/M^2, $\cos\lambda$, e_0, $\tilde{\gamma}_0$, and Φ_0. BCV observed (in the context of circular-orbit binaries with spin) that extrinsic parameters are generally much "cheaper" to search over than intrinsic parameters, which can be important for constructing efficient search strategies.

III. PRINCIPAL AXES

Let $\hat{n}$ be the unit vector pointing from the detector to the source, and let $\hat{L}(t)$ be the unit vector along the CO's orbital angular momentum. We find it convenient to work in a (time-varying) wave frame defined with respect to $\hat{n}$ and $\hat{L}(t)$. We define unit vectors $\hat{p}$ and $\hat{q}$ by

$$\begin{aligned} \hat{p} &\equiv (\hat{n} \times \hat{L})/|\hat{n} \times \hat{L}|, \\ \hat{q} &\equiv \hat{p} \times \hat{n}, \end{aligned} \tag{2}$$

based on which we then define the two polarization basis tensors

$$\begin{aligned} H^{+}_{ij}(t) &\equiv \hat{p}_i\hat{p}_j - \hat{q}_i\hat{q}_j, \\ H^{\times}_{ij}(t) &\equiv \hat{p}_i\hat{q}_j + \hat{q}_i\hat{p}_j. \end{aligned} \tag{3}$$

The general GW strain field at the detector can then be written as

$$h_{ij}(t) = A^{+}(t)H^{+}_{ij}(t) + A^{\times}(t)H^{\times}_{ij}(t), \tag{4}$$

where $A^{+}(t)$ and $A^{\times}(t)$ are the amplitudes of the two polarizations. In the next section we derive expressions for $A^{+}(t)$ and $A^{\times}(t)$.

IV. PETERS-MATHEWS WAVEFORMS

In the quadrupole approximation, the metric perturbation far from the source is given (in the "transverse/traceless" gauge) by [7]

$$h_{ij} = (2/D)\big(P_{ik}P_{jl} - \frac{1}{2}P_{ij}P_{kl}\big)\ddot{I}^{kl} \tag{5}$$

where D is the distance to the source, the projection operator P_{ij} is given by $P_{ij} \equiv \eta_{ij} - \hat{n}_i\hat{n}_j$, and $\ddot{I}^{ij}$ is the second time derivative of the inertia tensor. In this paper we work in the limit of small mass ratio, $\mu/M \ll 1$, where μ and M are the masses of the CO and MBH, respectively. In this limit, the inertia tensor is just $I^{ij}(t) = \mu\, r^i(t)r^j(t)$, where $\vec{r}$ is the position vector of the CO with respect to the MBH.

Consider now a CO-MBH system described as a Newtonian binary, with semi-major axis a, eccentricity e, and orbital frequency $\nu = (2\pi M)^{-1}(M/a)^{3/2}$. Let $\hat{e}_1$ and $\hat{e}_2$ be orthonormal vectors pointing along the major and minor axes of the orbital ellipse, respectively. Since the orbit is planar, I^{ij} has only 3 independent components: I^{11}, I^{21}, and I^{22}, and as the motion is periodic, we can express I^{ij} as a sum of harmonics of the orbital frequency ν: $I^{ij} = \sum_n I^{ij}_n$.

We next denote

$$\begin{aligned} a_n &\equiv \frac{1}{2}(\ddot{I}^{11}_n - \ddot{I}^{22}_n), \\ b_n &\equiv \ddot{I}^{12}_n, \\ c_n &\equiv \frac{1}{2}(\ddot{I}^{11}_n + \ddot{I}^{22}_n). \end{aligned} \tag{6}$$

Peters and Matthews showed [3] that

$$\begin{aligned} a_n &= -n\mathcal{A}\big[J_{n-2}(ne) - 2eJ_{n-1}(ne) + (2/n)J_n(ne) + 2eJ_{n+1}(ne) - J_{n+2}(ne)\big]\cos[n\Phi(t)], \\ b_n &= -n\mathcal{A}(1-e^2)^{1/2}\big[J_{n-2}(ne) - 2J_n(ne) + J_{n+2}(ne)\big]\sin[n\Phi(t)], \\ c_n &= 2\mathcal{A}J_n(ne)\cos[n\Phi(t)], \end{aligned} \tag{7}$$

where

$$\mathcal{A} \equiv (2\pi\nu M)^{2/3}\frac{\mu}{D}, \tag{8}$$

J_n are Bessel functions of the first kind, and $\Phi(t)$ is the mean anomaly (measured from pericenter). For a strictly Newtonian binary we have

$$\Phi(t) = 2\pi\nu(t - t_0) + \Phi_0, \tag{9}$$

where Φ_0 is the mean anomaly at t_0. Decomposing Eq. (4) into n-harmonic contributions and using Eq. (5), one then easily obtains explicit expressions for the n-harmonic components of the two polarization coefficients,

$$A^{+} \equiv \sum_n A^{+}_n \tag{10}$$

$$A^{\times} \equiv \sum_n A^{\times}_n\,. \tag{11}$$

$$\tag{12}$$

where the $A_n^{+,\times}$ are

$$
\begin{aligned}
A_n^{+} &= -[1+(\hat{L}\cdot\hat{n})^2]\,[a_n\cos(2\gamma)-b_n\sin(2\gamma)]+[1-(\hat{L}\cdot\hat{n})^2]c_n,\\
A_n^{\times} &= 2(\hat{L}\cdot\hat{n})\,[b_n\cos(2\gamma)+a_n\sin(2\gamma)]\,,
\end{aligned} \tag{13}
$$

where γ is an azimuthal angle measuring the direction of pericenter with respect to $\hat{x}\equiv[-\hat{n}+\hat{L}(\hat{L}\cdot\hat{n})]/[1-(\hat{L}\cdot\hat{n})^2]^{1/2}$. In practice, we truncate the sums in Eq. (10) at $n=20$.

The angular momenum direction vector $\hat{L}$ is not constant, since $\hat{L}$ precesses about the MBH's spin direction $\hat{S}$. Let $\theta_L(t),\phi_L(t)$ be the angles specifying the instantaneous direction of $\hat{L}$. These can be expressed in terms of the other angles in the problem as follows. Recall that θ_K,ϕ_K give the direction of $\vec{S}$ in the ecliptic-based system ('K' standing for 'Kerr'). Let λ be the angle *between* $\hat{L}$ and $\hat{S}$, and let $\alpha(t)$ be an azimuthal angle (in the orbital plane) that measures the precession of $\hat{L}$ *around* $\hat{S}$: Specifically, let

$$
\hat{L}=\hat{S}\,\cos\lambda+\frac{\hat{z}-\hat{S}\cos\theta_K}{\sin\theta_K}\sin\lambda\cos\alpha+\frac{\hat{S}\times\hat{z}}{\sin\theta_K}\,\sin\lambda\sin\alpha, \tag{14}
$$

where $\hat{z}$ is a unit vector normal to the ecliptic. Then the angles $\theta_L(t),\phi_L(t)$ are given in terms of θ_K, ϕ_K, λ, $\alpha(t)$ by

$$
\begin{aligned}
\cos\theta_L(t) &= \cos\theta_K\cos\lambda+\sin\theta_K\sin\lambda\cos\alpha(t),\\
\sin\theta_L(t)\cos\phi_L(t) &= \sin\theta_K\cos\phi_K\cos\lambda-\cos\phi_K\cos\theta_K\sin\lambda\cos\alpha(t)+\sin\phi_K\sin\lambda\sin\alpha(t),\\
\sin\theta_L(t)\sin\phi_L(t) &= \sin\theta_K\sin\phi_K\cos\lambda-\sin\phi_K\cos\theta_K\sin\lambda\cos\alpha(t)-\cos\phi_K\sin\lambda\sin\alpha(t).
\end{aligned} \tag{15}
$$

The evolution equation for $\alpha(t)$ is given in Sec. V below.

A. The pericenter angle $\tilde{\gamma}$

As mentioned above, the angle γ that appears in Eqs. (13) measures the direction of pericenter with respect to $\hat{x}\equiv[-\hat{n}+\hat{L}(\hat{L}\cdot\hat{n})]/[1-(\hat{L}\cdot\hat{n})^2]^{1/2}$. With this definition, γ is neither purely extrinsic nor purely intrinsic. (In the terminology of BCV, "intrinsic" parameters describe the system without reference to the location or orientation of the observer.) We will find it convenient to introduce a somewhat different convention for the zero-point of this angle: We shall define $\tilde{\gamma}$ to be the direction of pericenter with respect to $\hat{L}\times\hat{S}$. Then $\tilde{\gamma}$ is a purely intrinsic quantity.

Clearly, γ and $\tilde{\gamma}$ are related by

$$
\gamma=\tilde{\gamma}+\beta, \tag{16}
$$

where β is the angle from $\hat{x}\propto[\hat{L}(\hat{L}\cdot\hat{n})-\hat{n}]$ to $(\hat{L}\times\hat{S})$. It is straightforward to show that β is given by

$$
\begin{aligned}
\sin\beta &= \frac{\cos\lambda\,\hat{L}\cdot\hat{n}-\hat{S}\cdot\hat{n}}{\sin\lambda\big[1-(\hat{L}\cdot\hat{n})^2\big]^{1/2}},\\
\cos\beta &= \frac{\hat{n}\cdot(\hat{S}\times\hat{L})}{\sin\lambda\big[1-(\hat{L}\cdot\hat{n})^2\big]^{1/2}}.
\end{aligned} \tag{17}
$$

To evaluate $\beta(t)$ in practice, it is useful to know the following relations:

$$
\hat{S}\cdot\hat{n}=\cos\theta_S\cos\theta_K+\sin\theta_S\sin\theta_K\cos(\phi_S-\phi_K), \tag{18}
$$

$$
\hat{n}\cdot(\hat{S}\times\hat{L})=\sin\theta_S\sin(\phi_K-\phi_S)\sin\lambda\cos\alpha+\frac{\hat{S}\cdot\hat{n}\cos\theta_K-\cos\theta_S}{\sin\theta_K}\,\sin\lambda\sin\alpha, \tag{19}
$$

and

$$
\hat{L}\cdot\hat{n}=\hat{S}\cdot\hat{n}\cos\lambda+\frac{\cos\theta_S-\hat{S}\cdot\hat{n}\cos\theta_K}{\sin\theta_K}\,\sin\lambda\cos\alpha+\frac{(\hat{S}\times\bar{z})\cdot\hat{n}}{\sin\theta_K}\,\sin\lambda\sin\alpha, \tag{20}
$$

or, equivalently,

$$
\hat{L}\cdot\hat{n}=\cos\theta_S\cos\theta_L+\sin\theta_S\sin\theta_L\cos(\phi_S-\phi_L). \tag{21}
$$

Note that the time-variation of $\hat{S}\cdot\hat{n}$ is very small in the extreme mass-ratio case considered here. In our kludged model we approximate $\hat{S}$—and hence $\hat{S}\cdot\hat{n}$—as strictly constant.

V. ORBITAL EVOLUTION EQUATIONS

We evolve $\Phi(t)$, $\nu(t)$, $\tilde{\gamma}(t)$, $e(t)$, and $\alpha(t)$ using the following PN formulae:

$$\frac{d\Phi}{dt} = 2\pi\nu, \tag{22}$$

$$\begin{aligned}\frac{d\nu}{dt} &= \frac{96}{10\pi}(\mu/M^3)(2\pi M\nu)^{11/3}(1-e^2)^{-9/2}\big\{\big[1+(73/24)e^2+(37/96)e^4\big](1-e^2)\\ &+(2\pi M\nu)^{2/3}\big[(1273/336)-(2561/224)e^2-(3885/128)e^4-(13147/5376)e^6\big]\\ &-(2\pi M\nu)(S/M^2)\cos\lambda(1-e^2)^{-1/2}\big[(73/12)+(1211/24)e^2\\ &+(3143/96)e^4+(65/64)e^6\big]\big\},\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{d\tilde{\gamma}}{dt} &= 6\pi\nu(2\pi\nu M)^{2/3}(1-e^2)^{-1}\left[1+\frac{1}{4}(2\pi\nu M)^{2/3}(1-e^2)^{-1}(26-15e^2)\right]\\ &-12\pi\nu\cos\lambda(S/M^2)(2\pi M\nu)(1-e^2)^{-3/2},\end{aligned} \tag{24}$$

$$\begin{aligned}\frac{de}{dt} &= -\frac{e}{15}(\mu/M^2)(1-e^2)^{-7/2}(2\pi M\nu)^{8/3}\big[(304+121e^2)(1-e^2)\big(1+12(2\pi M\nu)^{2/3}\big)\\ &-\frac{1}{56}(2\pi M\nu)^{2/3}\big((8)(16705)+(12)(9082)e^2-25211e^4\big)\big]\\ &+e(\mu/M^2)(S/M^2)\cos\lambda\,(2\pi M\nu)^{11/3}(1-e^2)^{-4}\,\big[(1364/5)+(5032/15)e^2+(263/10)e^4\big],\end{aligned} \tag{25}$$

$$\frac{d\alpha}{dt} = 4\pi\nu(S/M^2)(2\pi M\nu)(1-e^2)^{-3/2}. \tag{26}$$

Equations (23), (24), and (25) are from Junker and Schäfer [9], except (i) the second line of Eq. (24) is from Brumberg [10] (cf. our Appendix A), and the last term in Eq. (23)—the term $\propto S/M^2$—is from Ryan [11]. Eq. (26) is from Barker and O'Connell [12]. The dissipative terms $d\nu/dt$ and de/dt are given accurately through 3.5PN order (i.e., one order higher than 2.5PN order, where radiation reaction first becomes manifest). The non-dissipative equations, for $d\tilde{\gamma}/dt$ and $d\alpha/dt$, are accurate through 2PN order.[18]

In solving the above time-evolution equations, the initial values (at time t_0) of Φ, ν, $\tilde{\gamma}$, e, and α are just the parameters Φ_0, ν_0, $\tilde{\gamma}_0$, e_0, and α_0.

We use our PN Eqs. (25) and (23) to evolve $e(t)$ and $\nu(t)$ forward in time, up to the point when the CO plunges over the top of the effective potential barrier. For a point particle in Schwarzschild, the plunge occurs at $a_{\rm min} = M(6+2e)(1-e^2)^{-1}$ [14], so we set

$$\nu_{\rm max} = (2\pi M)^{-1}[(1-e^2)/(6+2e)]^{3/2}\,, \tag{27}$$

so we cut off the integration when ν reaches this $\nu_{\rm max}$.

A. Doppler phase modulation

Doppler phase modulation due to LISA's orbital motion becomes important for integration times longer than a few weeks. We incorporate this effect by shifting the phase $\Phi(t)$, according to

$$\Phi(t) \to \Phi(t) + \Phi^D(t), \tag{28}$$

where

$$\Phi^D(t) = 2\pi\nu(t)R\sin\theta_S\cos[2\pi(t/T)+\bar{\phi}_0-\phi_S]. \tag{29}$$

Here $R \equiv 1\,\text{AU} = 499.00478$ sec, and $\bar{\phi}_0$ specificies the location of the LISA detector (around the Sun) at time $t=0$.

VI. THE POLARIZATION ANGLE

Eq. (4) express the waveform in terms of polarization tensors $H^{\pm}_{ij}(t)$ that are time-varying. However to generate the LISA responses with Synthetic LISA or LISA Simulator, it is useful to re-express Eq. (4) in terms of fixed polarization tensors. How do we do this? First note that at any instant, within the Synthetic LISA conventions, the polarization angle is given by

$$\psi_{SL} = -arctan(\frac{\cos\theta_S \sin\theta_L \cos(\phi_S - \phi_L) - \cos\theta_L \sin\theta_S}{\sin\theta_L \sin(\phi_S - \phi_L)}) . \tag{30}$$

I.e., for this polarization angle, $h_+ = A^+$ and $h_\times = A^\times$. To use Synthetic LISA, though, we want to choose some fixed ψ– call it ψ_0. (Again, this just corresponds to fixing the basis of polarization tensors. One could just set ψ_0 equal to 0, but we'll allow it to be arbitrary here.) With respect to this new basis, $h_{+,\times}(t)$ are given by

$$\begin{aligned} h_+(t) &= A^+(t)cos(2\psi_0 - 2\psi_{SL}(t)) + A^\times(t)sin(2\psi_0 - 2\psi_{SL}(t)) \\ h_\times(t) &= A^\times(t)cos(2\psi_0 - 2\psi_{SL}(t)) - A^+(t)sin(2\psi_0 - 2\psi_{SL}(t)) \end{aligned} \tag{31}$$

[[Note: I'm actually not sure I understand Michele's sign convention wrt polarization angle, so there could be some sign errors in this last bit. I'll sort it out with him. CC]]

VII. PUTTING THE PIECES TOGETHER

The algorithm for constructing our approximate waveform is then: Fix some fiducial frequency ν_0 and choose waveform parameters (t_0, $\ln\mu$, $\ln M$, S/M^2, e_0, $\tilde\gamma_0$, Φ_0, $\cos\theta_S$, ϕ_S, $\cos\lambda$, $\alpha_0, \cos\theta_K$, ϕ_K, D). Solve the ODEs (22)–(26) for $\Phi(t)$, $\nu(t)$, $\tilde\gamma(t)$, $e(t)$, $\alpha(t)$. Use $e(t)$ and $\nu(t)$ to calculate $a_n(t), b_n(t), c_n(t)$ in Eqs. (7), remembering to include the Doppler modulation via $n\Phi(t) \to n[\Phi(t) + \Phi^D(t)]$, a la Eqs. (28) and (29). Calculate $\theta_L(t), \phi_L(t)$ using Eqs. (15), and then calculate $\gamma(t)$ from $\tilde\gamma(t)$ using Eqs. (16)–(21). Calculate the amplitude coefficients $A_n^{+,\times}$ using Eqs. (13) and (21). Calculate ψ_{SL} using Eq. (30). Then finally calculate $h_\pm(t)$ (for the ψ_0 of your choice) using Eqs. (31).

Note that, in our kludge treatment, pericenter precession and Lense-Thirring precession have no effect on the a_n, b_n, c_n. The effect of these motions is simply to rotate the binary system, which modifies the amplitude harmonics (via Eqs. 13) and the also polarization angle $\psi_{SL}(t)$. The latter enters the time-dependence of $h_{+,\times}(t)$ via Eqs. (31).

VIII. CHOICE OF PARAMETERS FOR THE FIRST DATA CHALLENGE

IX. IMPLEMENTATION

First of all we should mention that some parameters in the code were fixed to the following values: (i) the initial LISA's phase $\bar\phi_0$ was taken to be zero and (ii) $1yr$ is sidedreal year = 31556925.2(sec).

The waveform described here was implemented as C++ class with the following methods:

- `AKWaveform(float spin, float mu, float MBHmass, float tfin, float timestep)`. This is constructor with parameters: `spin` is dimensional (reduced) spin of MBH (between 0 and 1), `mu` is the mass of CO in $M_\odot$, `MBHmass` is the mass of MBH in $M_\odot$, `tfin` is the duration time in sec (inital time is assumed to be zero), and `timestep` defines sampling interval (in sec.).

- `SetSourceLocation(float thS, float phS, float thK, float phK, float D)`. In this function user defines location of the source through the following parameters: `thS` this is θ_S, `phS` this is ϕ_S, `thK` corresponds to θ_K and `phK` to ϕ_K, the distance to the source is `D` and it is given in pc.

- `EstimateInitialParams(float elso, float nulso, float ein, float nuin)` This function estimates inital frequency `nuin` and initial eccentricity `ein` for values given at plunge `elso` by integrating simplified evolution equations backwards.

- `EvolveOrbit(float nu0, float eccen, float gamma0, float Phi0, float al0, float lam)`. This method is used to evolve the orbit with initial conditions specified by `nu0, eccen, gamma0, Phi0, al0` and for a given `lam` = λ

- `GetOrbitalEvolution(Matrix<float>& time, Matrix<float>& Phit, Matrix<float>& nut, Matrix<float>& gammat, Matrix<float>& et, Matrix<float>& alt)`. Using this function user can have a look at the result of integration of eqns. (22 - 26)

- `GetWaveform(float ps0, Matrix<float>& time, Matrix<float>& hplus, Matrix<float>& hcross)`. Finally this function will fill up 1-d matrices (vectors) with waveforms with initial polarization angle specified by `ps0`$= \psi_0$.

Note that orbital evolution is decoupled from the source location, so one can compute waveform for many direction on the sky, for the same intrinsic parameters (multiple call to `SetSourceLocation` and `GetWaveform`, with only one run of `EvolveOrbit`).

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[17] In reality, radiation reaction will impose a small time variation in λ; however, this variation is known to be very small (See Ref. [**?**]) and we shall ignore it here. When a model of the time-variation of λ is eventually at hand, it would be trivial to generalize our treatment to incorporate it: one would just need an equation for $d\lambda/dt$, and in the parameter list λ would be replaced by λ_0—the value of λ at time t_0.
[18] In fact, the equations for $d\tilde{\gamma}/dt$ and $d\alpha/dt$ are missing terms proportional to $(S/M^2)^2$, which, according to usual "order counting" are classified as 2PN. However, this usual counting is misleading when the central object is a spinning BH: Because BHs are ultracompact, their spins are smaller than suggested by the usual counting, and the missing terms $\propto (S/M^2)^2$ have, in fact, the same magnitude as 3PN terms. Similarly, the terms $\propto (S/M^2)$ in Eqs. (24) and (26) can be viewed as effectively 1.5PN terms.