Quadcopter Platform Linearization
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The following is rough code (unpolished) to linearize the platform model. A few checks or symbolic/numeric substitutions are included.
Provided “as-is” to save time.

Define the rotation matrices about the axes.

In[44]:=
```
Rx[θ_] := {{1, 0, 0}, {0, Cos[θ], -Sin[θ]}, {0, Sin[θ], Cos[θ]}};
Ry[θ_] := {{Cos[θ], 0, Sin[θ]}, {0, 1, 0}, {-Sin[θ], 0, Cos[θ]}};
Rz[θ_] := {{Cos[θ], -Sin[θ], 0}, {Sin[θ], Cos[θ], 0}, {0, 0, 1}};
```

Define the full rotation matrix (ZYX parameterization).

In[47]:=
```
Rot[ϕ_, θ_, ψ_] := Rz[ψ].Ry[θ].Rx[ϕ];
```

Assume a diagonal inertia matrix.

In[48]:=
```
J = DiagonalMatrix[{J11, J22, J33}];
```

Vee operation:

In[49]:=
```
vee[M_] := {M[[3, 2]], M[[1, 3]], M[[2, 1]]};
```

Relationship between body frame angular velocities and Euler angle derivatives

In[50]:=
```
H = Normal[CoefficientArrays[FullSimplify[vee[Transpose[Rot[ϕ[t], θ[t], ψ[t]]].
        D[Rot[ϕ[t], θ[t], ψ[t]], t]]], {ϕ'[t], θ'[t], ψ'[t]}][[2]]];
```

In[51]:=
```
omega = FullSimplify[H.{ϕ'[t], θ'[t], ψ'[t]}];
```

Nonlinear equations of motion.

In[52]:=
```
nonlinEqns = FullSimplify[Join[1/m * (Rot[ϕ[t], θ[t], ψ[t]].{0, 0, f[t]} - {0, 0, m * g}),
     Inverse[J].({M1[t], M2[t], M3[t]} - Cross[omega, J.omega])]];
```

In[53]:=
```
nonlinEqns // MatrixForm
```

Out[53]//MatrixForm=

$$\begin{pmatrix} \frac{f[t]\,(\mathrm{Cos}[\phi[t]]\,\mathrm{Cos}[\psi[t]]\,\mathrm{Sin}[\theta[t]]+\mathrm{Sin}[\phi[t]]\,\mathrm{Sin}[\psi[t]])}{m} \\ \frac{f[t]\,(-\mathrm{Cos}[\psi[t]]\,\mathrm{Sin}[\phi[t]]+\mathrm{Cos}[\phi[t]]\,\mathrm{Sin}[\theta[t]]\,\mathrm{Sin}[\psi[t]])}{m} \\ -g+\frac{\mathrm{Cos}[\theta[t]]\,\mathrm{Cos}[\phi[t]]\,f[t]}{m} \\ \frac{M1[t]+(J22-J33)\left(-\mathrm{Cos}[\phi[t]]\,\mathrm{Sin}[\phi[t]]\,\theta'[t]^2+\mathrm{Cos}[\theta[t]]\,\mathrm{Cos}[2\,\phi[t]]\,\theta'[t]\,\psi'[t]+\mathrm{Cos}[\theta[t]]^2\,\mathrm{Cos}[\phi[t]]\,\mathrm{Sin}[\phi[t]]\,\psi'[t]^2\right)}{J11} \\ \frac{M2[t]+(J11-J33)\,(\mathrm{Sin}[\phi[t]]\,\theta'[t]-\mathrm{Cos}[\theta[t]]\,\mathrm{Cos}[\phi[t]]\,\psi'[t])\,(\phi'[t]-\mathrm{Sin}[\theta[t]]\,\psi'[t])}{J22} \\ \frac{M3[t]+(J11-J22)\,(\phi'[t]-\mathrm{Sin}[\theta[t]]\,\psi'[t])\,(\mathrm{Cos}[\phi[t]]\,\theta'[t]+\mathrm{Cos}[\theta[t]]\,\mathrm{Sin}[\phi[t]]\,\psi'[t])}{J33} \end{pmatrix}$$

Create the state space model, linearized about ϕi, θi, etc.

In[54]:=
```
ss = StateSpaceModel[
    {Join[{x'[t], y'[t], z'[t], ϕ'[t], θ'[t], ψ'[t]}, nonlinEqns],
     {x[t], y[t], z[t], ϕ[t],
      θ[t], ψ[t], x'[t], y'[t], z'[t], ϕ'[t], θ'[t], ψ'[t]}},
    {{x[t], 0},
     {y[t], 0},
     {z[t], 0},
     {ϕ[t], ϕi},
     {θ[t], θi},
     {ψ[t], ψi},
     {x'[t], 0},
     {y'[t], 0},
     {z'[t], 0},
     {ϕ'[t], ϕdi},
     {θ'[t], θdi},
     {ψ'[t], ψdi}},
    {{f[t], fm},
     {M1[t], m1},
     {M2[t], m2},
     {M3[t], m3}}
   ];
```

Extract A and B for visual inspection.

In[55]:=
```
A = ss[[1, 1]];
B = ss[[1, 2]];
```

In[57]:=
```
A // MatrixForm
B // MatrixForm
```

Out[57]//MatrixForm=

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\text{fm}\,(-\text{Cos}[\psi i]\,\text{Sin}[\theta i]\,\text{Sin}[\phi i]+\text{Cos}[\phi i]\,\text{Sin}[\psi i])}{\text{m}} \\
0 & 0 & 0 & \frac{\text{fm}\,(-\text{Cos}[\phi i]\,\text{Cos}[\psi i]-\text{Sin}[\theta i]\,\text{Sin}[\phi i]\,\text{Sin}[\psi i])}{\text{m}} \\
0 & 0 & 0 & -\frac{\text{fm}\,\text{Cos}[\theta i]\,\text{Sin}[\phi i]}{\text{m}} \\
0 & 0 & 0 & \frac{(\text{J22}-\text{J33})\left(-\theta \text{di}^2\,\text{Cos}[\phi i]^2+\psi \text{di}^2\,\text{Cos}[\theta i]^2\,\text{Cos}[\phi i]^2+\theta \text{di}^2\,\text{Sin}[\phi i]^2-\psi \text{di}^2\,\text{Cos}[\theta i]^2\,\text{Sin}[\phi i]^2-2\,\theta \text{di}\,\psi \text{di}\,\text{Cos}[\theta i]\,\text{Sin}[2\,\phi i]\right)}{\text{J11}} \\
0 & 0 & 0 & \frac{(\text{J11}-\text{J33})\,(\phi \text{di}-\psi \text{di}\,\text{Sin}[\theta i])\,(\theta \text{di}\,\text{Cos}[\phi i]+\psi \text{di}\,\text{Cos}[\theta i]\,\text{Sin}[\phi i])}{\text{J22}} \\
0 & 0 & 0 & \frac{(\text{J11}-\text{J22})\,(\phi \text{di}-\psi \text{di}\,\text{Sin}[\theta i])\,(\psi \text{di}\,\text{Cos}[\theta i]\,\text{Cos}[\phi i]-\theta \text{di}\,\text{Sin}[\phi i])}{\text{J33}}
\end{pmatrix}
$$

Out[58]//MatrixForm=

$$
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\frac{\text{Cos}[\phi i]\,\text{Cos}[\psi i]\,\text{Sin}[\theta i]+\text{Sin}[\phi i]\,\text{Sin}[\psi i]}{\text{m}} & 0 & 0 & 0 \\
\frac{-\text{Cos}[\psi i]\,\text{Sin}[\phi i]+\text{Cos}[\phi i]\,\text{Sin}[\theta i]\,\text{Sin}[\psi i]}{\text{m}} & 0 & 0 & 0 \\
\frac{\text{Cos}[\theta i]\,\text{Cos}[\phi i]}{\text{m}} & 0 & 0 & 0 \\
0 & \frac{1}{\text{J11}} & 0 & 0 \\
0 & 0 & \frac{1}{\text{J22}} & 0 \\
0 & 0 & 0 & \frac{1}{\text{J33}}
\end{pmatrix}
$$

Process model at hover.

In[59]:=
```
subs = {ϕi → 0, θi → 0, ϕdi → 0, θdi → 0, fm → m * g};
```

In[60]:=
```
A //. subs // MatrixForm
B //. subs // MatrixForm
```

Out[60]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & g\,\text{Sin}[\psi\text{i}] & g\,\text{Cos}[\psi\text{i}] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -g\,\text{Cos}[\psi\text{i}] & g\,\text{Sin}[\psi\text{i}] & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{(J22-J33)\,\psi\text{di}^2}{J11} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{(J22-J33)\,\psi\text{di}}{J11} & 0 \\ 0 & 0 & 0 & 0 & \frac{(J11-J33)\,\psi\text{di}^2}{J22} & 0 & 0 & 0 & 0 & -\frac{(J11-J33)\,\psi\text{di}}{J22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Out[61]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{m} & 0 & 0 & 0 \\ 0 & \frac{1}{J11} & 0 & 0 \\ 0 & 0 & \frac{1}{J22} & 0 \\ 0 & 0 & 0 & \frac{1}{J33} \end{pmatrix}$$

Attitude dynamics (lower block)

In[62]:=
```
FullSimplify[ss[[1, 1]][[7 ;; 12, 7 ;; 12]]] // MatrixForm
A22 = FullSimplify[ss[[1, 1]][[7 ;; 12, 7 ;; 12]]];
```

Out[62]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & 0 & \\ 0 & 0 & 0 & 0 & \frac{(J22-J33)\,(\psi\text{di}\,\text{Cos}[\theta\text{i}]\,\text{Cos}[2\,\phi\text{i}]-2\,\theta\text{di}\,\text{Cos}[\phi\text{i}]\,\text{Sin}[\phi\text{i}])}{J11} & \\ 0 & 0 & 0 & \frac{(J11-J33)\,(-\psi\text{di}\,\text{Cos}[\theta\text{i}]\,\text{Cos}[\phi\text{i}]+\theta\text{di}\,\text{Sin}[\phi\text{i}])}{J22} & \frac{(J11-J33)\,(\phi\text{di}-\psi\text{di}\,\text{Sin}[\theta\text{i}])\,\text{Sin}[\phi\text{i}]}{J22} & -\text{(J11-} \\ 0 & 0 & 0 & \frac{(J11-J22)\,(\theta\text{di}\,\text{Cos}[\phi\text{i}]+\psi\text{di}\,\text{Cos}[\theta\text{i}]\,\text{Sin}[\phi\text{i}])}{J33} & \frac{(J11-J22)\,\text{Cos}[\phi\text{i}]\,(\phi\text{di}-\psi\text{di}\,\text{Sin}[\theta\text{i}])}{J33} & -\text{(J11-} \end{pmatrix}$$

Should be controllable, verify:

In[64]:=
```
ControllableModelQ[ss]
```

Out[64]= True

Select full-state feedback gains via LQR:

In[65]:=
```
Q = 0.1 * IdentityMatrix[12];
R = 2 * IdentityMatrix[4];
```

In[67]:= `Chop[LQRegulatorGains[ss //. subs, {Q, R}], 10^-8] // MatrixForm`

RiccatiSolve: RiccatiSolve has received a matrix with non-numerical elements.

RiccatiSolve: RiccatiSolve has received a matrix with non-numerical elements.

Out[67]//MatrixForm=

$$\text{LQRegulatorGains}\left[\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & g\,\mathrm{Sin}[\psi i] & g\,\mathrm{Cos}[\psi i] & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -g\,\mathrm{Cos}[\psi i] & g\,\mathrm{Sin}[\psi i] & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{(J22-J33)\,\psi di^2}{J11} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{(J11-J33)\,\psi di^2}{J22} & 0 & 0 & 0 & 0 & -\frac{(J11-J33)\,\psi di}{J22} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}\right.$$

```
{{{0.1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0.1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0.1, 0, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0.1, 0, 0, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0.1, 0, 0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0.1, 0, 0, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0.1, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0.1, 0, 0, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0.1, 0, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0.1, 0, 0},
  {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0.1}},
 {{2, 0, 0, 0}, {0, 2, 0, 0}, {0, 0, 2, 0}, {0, 0, 0, 2}}}]
```

In[68]:= `A22 // MatrixForm`

Out[68]//MatrixForm=

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{(J22-J33)\,(\psi di\,\mathrm{Cos}[\theta i]\,\mathrm{Cos}[2\,\phi i]-2\,\theta di\,\mathrm{Cos}[\phi i]\,\mathrm{Sin}[\phi i])}{J11} \\
0 & 0 & 0 & \frac{(J11-J33)\,(-\psi di\,\mathrm{Cos}[\theta i]\,\mathrm{Cos}[\phi i]+\theta di\,\mathrm{Sin}[\phi i])}{J22} & \frac{(J11-J33)\,(\phi di-\psi di\,\mathrm{Sin}[\theta i])\,\mathrm{Sin}[\phi i]}{J22} \\
0 & 0 & 0 & \frac{(J11-J22)\,(\theta di\,\mathrm{Cos}[\phi i]+\psi di\,\mathrm{Cos}[\theta i]\,\mathrm{Sin}[\phi i])}{J33} & \frac{(J11-J22)\,\mathrm{Cos}[\phi i]\,(\phi di-\psi di\,\mathrm{Sin}[\theta i])}{J33}
\end{pmatrix}$$