CS182 Spring 2016: Homework 2

Due date: Friday, February 19, 2016 (before class).

1. (8 pts) Prove or disprove the following statements:

   (a) For all positive integers n, if n is a perfect square then $n+3$ is not a perfect square. (Recall the definition of a *perfect square*: an integer n is a perfect square if and only if there exists an integer a such that $a^2 = n$.)

   Solution: The claim is false.

   Proof: Let $n = 1$. Then, $n + 3 = 4$ and $4 = 2^2$. Therefore, 4 is a perfect square. Thus, the claim is false.

   (b) For every real number x, there is a nonzero real number y such that $x \cdot y = x + y$.

   Solution: The claim is false.

   Proof: Let $x = 0$. Then for any nonzero real number y, we have: $x \cdot y = 0$ and $x + y = y \neq 0$. Thus, the claim is false.

   (c) There is a real number x such that for every integer n we have $\frac{n}{x} > 0$.

   Solution: The claim is false.

   Proof: Let x be any real number. Then, let $n = 0$ and we have $\frac{n}{x} = 0 \leq 0$ for any x. Thus, the claim is false.

   (d) The following statements are equivalent for all nonnegative integers a and b:

   - $a < b$
   - $(a+b)^2 < 4b^2$
   - $4a^2 < (a+b)^2$

   Solution: The claim is true.

   Proof: First we show that $a < b \to (a+b)^2 < 4b^2$. Assume $a < b$. Then, consider $(a+b)^2$:

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2 \\ &< b^2 + 2bb + b^2 \text{ (since } a < b \text{ and are nonnegative)} \\ &= 4b^2\end{aligned}$$

   Thus we have shown the first result. Next, we show that $(a+b)^2 < 4b^2 \to a < b$ by contrapositive. Assume that $a \geq b$. Then consider $(a+b)^2$:

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2 \\ &\leq b^2 + 2bb + b^2 \text{ since } a \geq b \text{ and are nonnegative} \\ &= 4b^2\end{aligned}$$

Thus the contrapositive is true and thus we have shown the second result. Next, we show that $a < b \to 4a^2 < (a+b)^2$. Assume $a < b$. Then,

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2 \text{ since } a < b \text{ and are nonnegative} \\ &> a^2 + 2aa + a^2 \\ &= 4a^2\end{aligned}$$

Thus we have shown the third result. Finally, we show that $4a^2 < (a+b)^2 \to a < b$ by contrapositive. Assume that $a \geq b$. Then,

$$\begin{aligned}(a+b)^2 &= a^2 + 2ab + b^2 \\ &\geq a^2 + 2aa + a^2 \text{ since } a \geq b \text{ and are nonnegative} \\ &= 4a^2\end{aligned}$$

Thus we have shown the contrapositive and thus the final result is true. Thus, the three statements are equivalent.

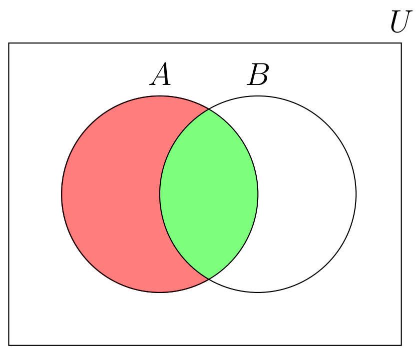
2. (6 pts) List all the elements of the following sets:
   (a) $S = \{i \mid i \in \mathbb{Z} \wedge i^2 \leq 4\} = \{-2, -1, 0, 1, 2\}$
   (b) $S = \{p \mid p \in \mathbb{Q},\ 0 < p < 1,\ p \text{ is even}\} = \emptyset$
   (c) $S = \{x \mid x \in \mathbb{C}, x \text{ is a root of } x^4 - 1\} = \{1, -1, i, -i\}$ (where $i = \sqrt{-1}$)
3. (10 pts) What are the cardinalities of the following sets?
   (a) $\emptyset$ Answer: 0
   (b) $\{\emptyset, 1\}$ Answer: 2
   (c) $\{1, 2, \{3, 4\}, \emptyset\}$ Answer: 4
   (d) $\{\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}\}$ Answer: 4
   (e) $\mathcal{P}(\emptyset)$ Answer: 1
4. (14 pts) Let $A = \{1, 2, 4, 6, 7\}$ and $B = \{3, 4, 5\}$, and let our universe be $U = \{n \mid n \in \mathbb{Z}, 1 \leq n \leq 10\}$.
   (a) List the elements of $A \cup B = \{1, 2, 3, 4, 5, 6, 7\}$
   (b) List the elements of $A \cap \overline{B} = \{1, 2, 6, 7\}$
   (c) List the elements of $\overline{A} - B = \{8, 9, 10\}$
   (d) List the elements of $\overline{A} - (A \cup \overline{B}) = \{3, 5\}$
   (e) List the elements of $\mathcal{P}(B) = \{\emptyset, \{3\}, \{4\}, \{5\}, \{3, 4\}, \{3, 5\}, \{4, 5\}, \{3, 4, 5\}\}$

(f) List the elements of $A \times B =$
$\{(1,3),(2,3),(4,3),(6,3),(7,3),$
$(1,4),(2,4),(4,4),(6,4),(7,4),$
$(1,5),(2,5),(4,5),(6,5),(7,5)\}$

(g) List the elements of $B \times A =$
$\{(3,1),(3,2),(3,4),(3,6),(3,7)$
$(4,1),(4,2),(4,4),(4,6),(4,7)$
$(5,1),(5,2),(5,4),(5,6),(5,7)\}$

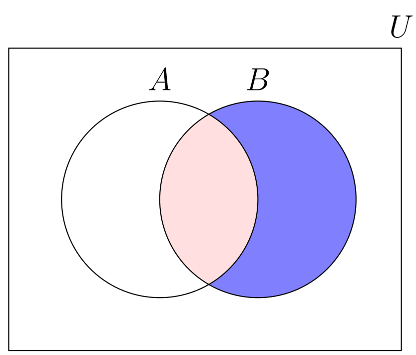
5. (8 pts) Let A and B be sets.

(a) Use a venn diagram to show that $A \cap B \subseteq A$.
Solution: Red = A, Green = $A \cap B$



Since the green section is within the bounds of A, the green section is a subset of A.

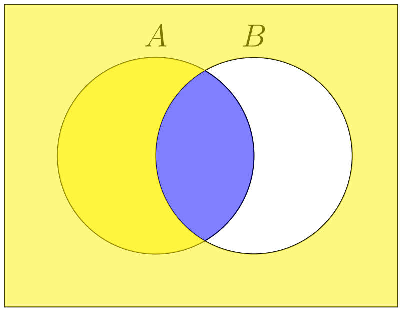
(b) Use a venn diagram to show that $A \cap B \subseteq B$.
Solution: Pink = B, Blue = $A \cap B$



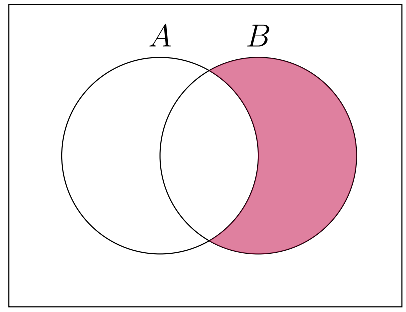
Since the blue section is within the bounds of B, the blue section is a subset of B.

(c) Use a venn diagram to show that $(A \cup \overline{B}) = \overline{(\overline{A} \cap B)}$
Solution:

$A \cup \overline{B}$



$\overline{A} \cap B$

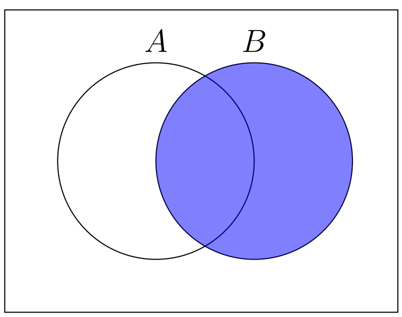


Note that in the diagram for $\overline{A} \cap B$, we can see that $\overline{\overline{A} \cap B}$ is represented by the white portion of the diagram, and this white portion is the same as the diagram for $A \cup \overline{B}$.

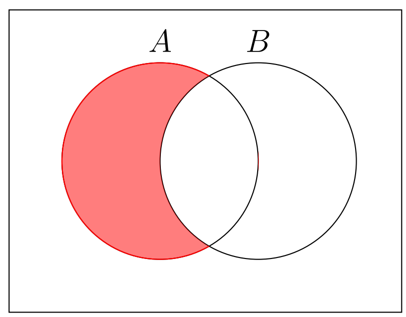
(d) Use a venn diagram to show that $B \subseteq \overline{(A - B)}$

Solution:

B



$A - B$



Note that in the diagram for $A - B$, the white area represents $\overline{A - B}$ and note that the area of B is within this white area. Thus, $B \subseteq \overline{A - B}$.

6. (4 pts) Let A and B be two sets. Define the *symmetric difference of A and B* as $A \oplus B = \{s \mid s \text{ is in } A \text{ or } B \text{ but not both}\}$ Prove that $A \oplus B = (A - B) \cup (B - A)$.

Proof: We use the definitions to prove.

$$\begin{aligned}
A \oplus B &= \{s \mid s \text{ is in } A \text{ or } B \text{ but not both}\} \\
&= \{s \mid (s \in A \vee s \in B) \wedge (s \notin A \cap B)\} \\
&= \{s \mid (s \in A \wedge s \notin B) \vee (s \notin A \wedge s \in B)\} \\
&= \{s \mid (s \in A \wedge s \notin B)\} \cup \{s \mid (s \notin A \wedge s \in B)\} \\
&= (A - B) \cup (B - A)
\end{aligned}$$

Thus, we have shown the result.

Note: Another way to show this result is to show that each side is a subset of the other.

7. (10 pts) Let $f : \mathbb{R} \to \mathbb{R}$. Given the following definitions of f, state whether or not it is a function. If it is a function, state the domain, codomain, and range. If it is not a function, state which domain (if any) will make it a function.

   (a) $f(x) = 1/x$.
   (b) $f(x) = x^2 + 1$.
   (c) $f(x) = 0$
   (d) $f(x) = \pm\sqrt{x}$
   (e) $f(x) = \log_e(x)$, where e is Euler's Number.

   Solutions:

   (a) Not a function. Domain to make it a function: $\mathbb{R} - \{0\}$
   (b) Is a function. Domain: $\mathbb{R}$, Codomain: $\mathbb{R}$; Range: $\{r \mid r \in \mathbb{R}, r \geq 1\}$
   (c) Is a function. Domain: $\mathbb{R}$, Codomain: $\mathbb{R}$; Range: $\{0\}$
   (d) Is not a function. A domain change to $\{0\}$ will make it a function.
   (e) Not a function. Domain to make it a function: $\{r \mid r \in \mathbb{R}, r > 0\}$

8. (6 pts) Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be functions.

   (a) If $f(x) = x^2 + 1$ and $g(x) = 2x + 3$, find $(f + g)(x)$ and $(fg)(x)$.
   (b) If $f(x) = x^3 + 2x$ and $g(x) = -x + 2$, find $(fg + f^2)(x)$.
   (c) If $f(x) = 2x$ and $g(x) = 3x^2$, find $(f + g)^2(x)$.

   Solutions:

   (a) $(f + g)(x) = f(x) + g(x) = x^2 + 2x + 4$; $(fg)(x) = f(x)g(x) = 2x^3 + 3x^2 + 2x + 3$
   (b) $(fg + f^2)(x) = f(x)g(x) + f(x)f(x) = x^6 + 3x^4 + 2x^3 + 2x^2 + 4x$
   (c) $(f + g)^2(x) = (f + g)(f + g)(x) = f^2(x) + 2f(x)g(x) + g^2(x) = 4x^2 + 12x^3 + 9x^4$

9. (8 pts) Let f be a function with domain and codomain defined below. For each f, show whether f is injective, surjective, both, or neither.

   (a) $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ and $f(x) = x + 1$. (Note: $\mathbb{Z}^+$ is the set of all positive integers.)
   (b) $f : \mathbb{R} \to \mathbb{Z}$ and $f(x) = \lfloor x \rfloor$.
   (c) $f : \mathbb{R} \to \mathbb{R}$ and $f(x) = x + 1$.
   (d) $f : \mathbb{Z}^+ \to \{0, 1, 2\}$ and $f(x) = (x \mod 2) + 1$. (Note: $x \mod n$ is the remainder of x when divided by n.)

   Solutions:

(a) This function is injective but not surjective. Proof: We show injective: Let $a, b \in \mathbb{Z}^+$ such that $f(a) = f(b)$. Then,

$$\begin{aligned} &f(a) = f(b) \\ &\rightarrow a + 1 = b + 1 \\ &\rightarrow a = b \end{aligned}$$

Thus the function is injective. We now show not surjective: Take $b = 1$ in the codomain $\mathbb{Z}^+$. Then, there is no such positive integer a such that $a + 1 = 1 = b$. Thus, the function is not surjective.

(b) The function is surjective but not injective. Proof: We show surjective. Let $b \in \mathbb{Z}$. Then, $b \in \mathbb{R}$ and $f(b) = \lfloor b \rfloor = b$. So the function is surjective. We now show not injective. Take $a = 0 \in \mathbb{R}$ and $b = 1/2 \in \mathbb{R}$. Then, $f(a) = 0$ and $f(b) = 0$ but $a \neq b$. Thus the function is not injective.

(c) The function is injective and surjective (it is a bijection). Proof: We show injective: Let $a, b \in \mathbb{R}$ such that $f(a) = f(b)$. Then,

$$\begin{aligned} &f(a) = f(b) \\ &\rightarrow a + 1 = b + 1 \\ &\rightarrow a = b \end{aligned}$$

Thus it is injective. We now show surjective: Take any $b \in \mathbb{R}$ and let $a = b - 1$. Then, $f(a) = a + 1 = (b - 1) + 1 = b$. Thus, the function is surjective.

(d) The function is neither injective nor surjective. Proof: We show not injective: Let $a = 1$ and $b = 3$. Then, $f(a) = 1+1 = 2$ and $f(b) = 1+1 = 2$. Thus, $f(a) = f(b)$ but $a \neq b$ and thus the function is not injective. We now show not surjective: let $b = 0 \in \{0, 1, 2\}$. Then, for any positive integer a, $a \mod 2$ is either 0 or 1. Thus, $f(a)$ is either $0 + 1$ or $1 + 1$. Thus, there is no such $a \in \mathbb{Z}^+$ such that $f(a) = b$. Thus, the function is not surjective.

10. (12 pts) For each of the following f and g, give f^{-1} (if it exists), g^{-1} (if it exists), $f \circ g$ (if it exists), and $g \circ f$ (if it exists), otherwise state that it does not exist (Note: f^{-1} denotes f inverse, not $1/f$):

(a) $f(x) = x^3$, $g(x) = x + 2$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$

(b) $f(x) = x/2$, $g(x) = x + 3$, where $f : \mathbb{Q} \rightarrow \mathbb{Q}$ and $g : \mathbb{Q} \rightarrow \mathbb{Q}$

(c) $f(x) = x^2$, $g(x) = \lceil x \rceil + 1$, where $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and $g : \mathbb{R}^+ \rightarrow \mathbb{Z}$

Solutions:

(a) $f^{-1}(x) = \sqrt[3]{x}$, $g^{-1}(x) = x - 2$, $(f \circ g)(x) = (x + 2)^3$, $(g \circ f) = x^3 + 2$

(b) $f^{-1}(x) = 2x$, $g^{-1} = x - 3$, $(f \circ g)(x), = (x + 3)/2$, $(g \circ f)(x) = (x/2) + 3$

(c) $f^{-1}(x) = \sqrt{x}$, g^{-1} does not exist, $(f \circ g)(x)$ does not exists, $(g \circ f)(x) = \lceil x^2 \rceil + 1$