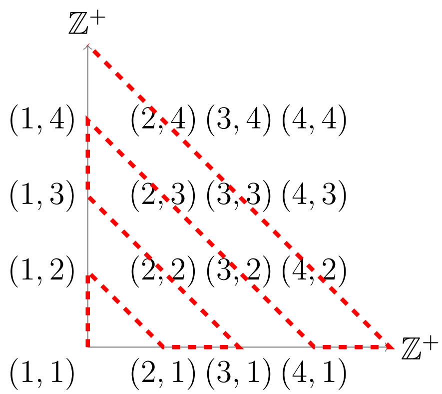
CS182 Spring 2016: Homework 4

Due date: Friday, March 4, 2016 (before class).

1. (4 pts) Prove that $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countably infinite. Proof:

   Consider the grid of $\mathbb{Z}^+ \times \mathbb{Z}^+$:

   We follow the dotted red line to find our bijection with $\mathbb{Z}^+$:

$$\begin{aligned}
1 &\to (1,1)\\
2 &\to (1,2)\\
3 &\to (2,1)\\
4 &\to (3,1)\\
5 &\to (2,2)\\
6 &\to (1,3)\\
7 &\to (1,4)\\
8 &\to (2,3)\\
9 &\to (3,2)
\end{aligned}$$

   and so on. This mapping takes every positive integer to a unique ordered pair of $\mathbb{Z}^+ \times \mathbb{Z}^+$, and will map to every element of $\mathbb{Z}^+ \times \mathbb{Z}^+$. Thus this map is a bijection and $\mathbb{Z}^+ \times \mathbb{Z}^+$ is countable.

2. (6 pts) Consider the set $\mathbb{R}$ consisting of all real numbers. Give an example of a subset S of $\mathbb{R}$ with $S \neq \mathbb{R}$ such that:

   (a) S is countably infinite

   (b) S is finite

   (c) S is uncountably infinite

   Solutions:

(a) Possible answers:

$$S = \begin{cases} \mathbb{Q} \\ \mathbb{Q}^+ \\ \mathbb{Z} \\ \mathbb{Z}^+ \\ \text{the set of all even integers} \end{cases}$$

(b) $S = \{\pi\}$, $S = \{0, 1, \ldots, 100\}$, $S = \emptyset$, and so on.

(c) $S = \mathbb{R} - \mathbb{Q}$, $S = \{r : r \in \mathbb{R},\ 0 < r < 1\}$ are just some examples.

3. (4 pts) Let B be the set of all possible bit strings. That is, $B = \{b_0 b_1 \ldots b_n \mid n \in \mathbb{Z}^+ \cup \{0\}$ and $b_i \in \{0, 1\}\}$. Show that B is countably infinite.

Proof: Note that we can order B as follows: $B = \{0, 1, 00, 01, 10, 11, 000, 001, 010, 011, 100, 101, 110, 11$
So we can define a mapping from $\mathbb{Z}^+$ to B as

$$\begin{aligned} 1 &\to 0 \\ 2 &\to 1 \\ 3 &\to 00 \\ 4 &\to 01 \\ 5 &\to 10 \\ 6 &\to 11 \\ 7 &\to 000 \\ &\vdots \end{aligned}$$

continuing to infinity. This mapping is a bijection and thus B is countably infinite. **Alternative:** If leading zeros were ignored (i.e., $01 = 1$, $000 = 0$, etc.), we can define the mapping from $\mathbb{Z}^+ \cup \{0\}$ to B as B is the binary representation of elements of $\mathbb{Z}^+ \cup \{0\}$. That is

$$\begin{aligned} 0 &\to 0 \\ 1 &\to 1 \\ 2 &\to 10 \\ 3 &\to 11 \\ 4 &\to 100 \\ &\vdots \end{aligned}$$

which is a bijection.

4. (8 pts) For each of the following sets, state the cardinality (finite, countably infinite, or uncountably infinite) (you do not need to justify your answer):

   (a) The set $\mathbb{R} - \mathbb{Q}$ **Uncountably Infinite**

   (b) The set $\mathbb{Z} \cap \mathbb{Q}$ **Countably Infinite**

   (c) The set $\mathcal{P}(\mathbb{Z})$ **Uncountably Infinite**

   (d) The set $\mathbb{Z} \cap \{r \mid r \in \mathbb{R}, 0 \leq r \leq 1\}$ **Finite**

5. (8 pts) List the next 5 terms in each of the following sequences:

   (a) $a_0 = 7$, $a_n = 2 * a_{n-1} - 5$

   (b) $b_0 = 1$, $b_1 = 2$, $b_n = 2 * b_{n-1} + b_{n-2}$

   (c) $c_0 = 2$, $c_n = (c_{n-1})^n$

   (d) $a_0 = 2$, $a_1 = 3$, $a_n = \dfrac{a_{n-1}}{a_{n-2}}$

   Solutions:

   (a) $a_1 = 9$, $a_2 = 13$, $a_3 = 21$, $a_4 = 37$ $a_5 = 69$

   (b) $b_2 = 5$, $b_3 = 12$, $b_4 = 29$, $b_5 = 70$, $b_6 = 169$

   (c) $c_1 = 2$, $c_2 = 4$, $c_3 = 64$, $c_4 = 16777216$, $c_5 = (16777216)^5$

   (d) $a_2 = \dfrac{3}{2}$, $a_3 = \dfrac{1}{2}$, $a_4 = \dfrac{1}{3}$, $a_5 = \dfrac{2}{3}$, $a_6 = 2$

6. (12 pts) Evaluate the following sums (note: page 166 in the book can be helpful for these):

   (a) $\sum_{i=1}^{m} i^2 + i$

   (b) $\sum_{j=17}^{n} 6j^2$

   (c) $\sum_{k=20}^{\infty} (\frac{1}{2})^k$

   (d) $\sum_{h=10}^{20} h^3 - 2h^2$

   Solutions:

   (a)

$$\begin{aligned}\sum_{i=1}^{m} i^2 + i &= \sum_{i=1}^{m} i^2 + \sum_{i=1}^{m} i \\ &= \frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2}\end{aligned}$$

(b)

$$\begin{aligned}
\sum_{j=17}^{n} 6j^2 &= 6\sum_{j=17}^{n} j^2 \\
&= 6\left(\sum_{j=1}^{n} j^2 - \sum_{j=1}^{16} j^2\right) \\
&= 6\left(\frac{n(n+1)(2n+1)}{6} - \frac{16(17)(33)}{6}\right) \\
&= n(n+1)(2n+1) - 8976
\end{aligned}$$

(c)

$$\begin{aligned}
\sum_{k=20}^{\infty}\left(\frac{1}{2}\right)^k &= \sum_{k=0}^{\infty}\left(\frac{1}{2}\right)^k - \sum_{k=0}^{19}\left(\frac{1}{2}\right)^k \\
&= 2 - \frac{(1/2)^{20} - 1}{1/2 - 1} \\
&= 2 + (1/2)^{19} - 2 = (1/2)^{19}
\end{aligned}$$

(d)

$$\begin{aligned}
\sum_{h=10}^{20} h^3 - 2h^2 &= \sum_{h=10}^{20} h^3 = 2\sum_{h=10}^{20} h^2 \\
&= \left(\sum_{h=1}^{20} h^3 - \sum_{h=1}^{9} h^3\right) - 2\left(\sum_{h=1}^{20} h^2 - \sum_{h=1}^{9} h^2\right) \\
&= \frac{20^2(21)^2}{4} - \frac{9^2(10)^2}{4} - \frac{20(21)(41)}{3} + \frac{9(10)(19)}{3} \\
&= 44100 - 2025 - 5740 + 570 \\
&= 36905
\end{aligned}$$

7. (6 pts) Find a solution to each of these recurrence relations with their given initial conditions:

   (a) $a_n = -a_{n-1}$, $a_0 = 6$

   (b) $a_n = (n+1)a_{n-1}$, $a_0 = 2$

   (c) $a_n = a_{n-1} - n$, $a_0 = 3$

   Solutions:

   (a) $a_n = (-1)^n 6$

(b) $a_n = 2(n+1)!$

(c) $a_n = 3 - \dfrac{n(n+1)}{2}$

8. (10 pts) Let $\mathbf{A} = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 4 & -1 \end{bmatrix}$ and let $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ -2 & 5 \\ -1 & 7 \end{bmatrix}$

   (a) What is the size of $\mathbf{A}$?

   (b) What is the size of $\mathbf{B}$?

   (c) What is $\mathbf{A}^T + \mathbf{B}$?

   (d) What is $\mathbf{AB}$?

   (e) What is $\mathbf{BA}$?

   Solutions:

   (a) 2×3

   (b) 3×2

   (c) $\begin{bmatrix} 3 & 6 \\ 1 & 9 \\ -1 & 6 \end{bmatrix}$

   (d) $\begin{bmatrix} -4 & 18 \\ -1 & 22 \end{bmatrix}$

   (e) $\begin{bmatrix} 11 & 18 & -3 \\ 13 & 14 & -5 \\ 20 & 25 & -7 \end{bmatrix}$