CS182 Spring 2016: Homework 8

Due date: Wednesday, April 27th 2016 (before class).

1. (8pts) Pigeon Hole Principle and Permutations and Combinations

   (a) Show that if seven integers are selected from the first 10 positive integers, there must be at least two pairs of those integers with sum equal to 11.

   Solution: First, we think of how many pairs of the first 10 positive integers sum to 11. They are the pairs $\{1, 10\}, \{2, 9\}, \{3, 8\}, \{4, 7\}, \{5, 6\}$. Note that if the order is flipped, then the sums still remain the same. Note that we can pick at most 5 items so that no pair of them sums to 11. By the Pigeon Hole Principle, this means that if 6 items are picked, there must be at least one pair whose sum is 11. Thus, when 7 items are picked, there must be at least 2 pairs whose sum equals 11.

   (b) How many bit strings contain exactly five 0s and 14 1s if every 0 must be immediately followed by two 1s?

   Solution: In our bit string, since a 0 must be followed immediately by two 1s, we can think of the bit strings as having segments like "011". Let $x = 011$. Note that since we must have 5 0s, this means we have 5 xs in our bit string. Note this also means that we have 10 1s. So we need 4 more 1s. If we think of the string only in terms of x and 1s, then our strings will be of length 9. Thus, the number of bit strings with 5 xs and 4 1s is obtained by placing the 5 xs into the 9 positions (the remaining 1s will just fill the remaining slots). Thus, we have $\binom{9}{5}$ such strings.

2. (8 pts) Binomial Coefficients and Identities and Generalized Permutations and Combinations

   (a) Give a formula for the coefficient of x^k in the expansion of $(x + 1/x)^{100}$, where k is a nonnegative integer.

   Solution: Recall the binomial formula for $(a + b)^n$:

$$(a + b)^n = \sum_{i=0}^{n} \binom{n}{i} a^{n-i} b^i$$

   Substituting $a = x$ and $b = \frac{1}{x}$:

$$\left(x + \frac{1}{x}\right)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} \left(\frac{1}{x}\right)^i$$

So for any particular k, to figure out the coefficient of x^k, we must figure out when $x^{n-i}\left(\frac{1}{x}\right)^i = x^k$.

$$\begin{aligned} x^{n-i}\left(\frac{1}{x}\right)^i &= \frac{x^n}{x^i}\frac{1}{x^i} \\ &= x^{n-2i} \end{aligned}$$

So the coefficient of x^k occurs when $k = n - 2i$, so this means that $i = \dfrac{n-k}{2}$ is the term that gives us x^k. Thus, the coefficient of x^k will be $\binom{100}{i} = \binom{100}{\frac{100-k}{2}}$.

(b) How many ways are there to distribute five distinguishable objects into three indistinguishable boxes?

Solution: Let A, B, C, D, E represent the 5 distinguishable objects. To put them into 3 indistinguishable boxes, we can look at the number of ways we can place these 5 objects into disjoint sets.

One way to place these objects is to put them all in one box. So this would be $\{A, B, C, D, E\}$. There is exactly one way to do this.

Next, we can place 4 objects into one box and the final object in another box (since the boxes are indistinguishable, it does not matter which box the other one is placed in). The number of ways to do this is the number of ways to choose 4 objects from 5 (or equivalently, choose 1 object from 5). So there are $\binom{5}{4} = \binom{5}{1} = 5$ ways to do this.

Next, we can place 3 objects into one box and put the remaining two objects in another box (again, which other box does not matter). Thus, this is the number of ways to choose 3 objects from 5 (or equivalently, choose 2 objects from 5). So there are $\binom{5}{3} = \binom{5}{2} = 10$ ways to do this.

Next, we can place 3 objects into one box, put one of the remaining objects in another box, and put the final object in the final box. Because the other two boxes are indistinguishable, this is the same as the previous case. Thus, there are 10 ways to do this.

Finally, we can place 2 objects into one box, put two objects in another box, and one object in the final box. The number of ways to choose objects for the first box is $\binom{5}{2} = 10$. Then, the number of ways to choose objects for the second box after we've chosen for the first box is $\binom{3}{2} = 3$, and then we just place the last object in the remaining box. Note however that since the boxes are indistinguishable, when we place the first two then the next two, we could have done it in reverse order. So we've double counted. So the total number of ways will be $(10 \times 3)/2 = 15$.

Thus, the total number of ways is: $1 + 5 + 10 + 10 + 15 = 41$.

3. (8 pts) Discrete Probability

   (a) What is the probability that a player of a lottery wins the prize offered for correctly choosing five (but not six) numbers out of six integers chosen at random from the integers between 1 and 40, inclusive?

   Solution: For this problem, order does not matter and there is no replacement. The total number of ways to pick 6 integers from 1 to 40 is $\binom{40}{6}$. Next, we calculate the number of ways we can pick exactly 5 of the 6 winning numbers. We can pick 5 of the 6 winning numbers $\frac{6}{5}$ ways. Finally, we must pick the final non-winning number. There are precisely 34 non-winning numbers, so the number of ways to pick this final number is $\binom{34}{1}$. Thus, the probability is:

$$\frac{\binom{6}{5}\binom{34}{1}}{\binom{40}{6}}$$

   (b) What is the probability that a five-card poker hand contains a flush (five cards of the same suit)?

   Solution: The total number of 5-card hands is $\binom{52}{5}$. Next, we calculate the number of ways to choose a suit, which is $\binom{4}{1}$. Finally, of the cards from this suit, we calculate the number of 5 card hands we can get, which is $\binom{13}{5}$. Thus, the probability of getting a flush is

$$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}}$$

4. (8 pts) Probability Theory

   (a) What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?

   Solution: Let E be the event that exactly four heads appear when a fair coin is flipped five times. Let E be the event that the first of five coin flips is heads. Then, we wish to calculate $\Pr(E|F)$. This means we must calculate $\Pr(E \cap F)/\Pr(F)$. Note that the number of ways the first of five coin flips is heads is 2^4 since the first flip is heads and the remaining flips can be anything. Next, the number of ways exactly 4 heads appears when a fair coin is flipped five times is exactly 5 (the cases are $THHHH$, $HTHHH$, $HHTHH$, $HHHTH$, and $HHHHT$). This means that $E \cap F = \{HTHHH, HHTHH, HHHTH, HHHHT\}$. Finally, the total number of ways 5 coin flips can be is $2^5 = 32$. Thus,

$$\begin{aligned} \Pr(F) &= 2^4/2^5 = 1/2 \\ \Pr(E \cap F) &= 2^2/2^5 = 1/8 \\ \Pr(E \cap F)/\Pr(F) &= (1/8)/(1/2) = 1/4 \end{aligned}$$

   (b) Find the probability of each outcome when a loaded die is rolled if a 3 is twice as likely to appear as every other number on the die.

   Solution: The probability of 3 appearing is two times the probability of any other number appearing. Let x be the probability of a number not equal to 3 appearing. Then, $2x$ is the probability of 3 appearing. Note that $x + x + 2x + x + x + x = 1$, so $7x = 1$ and $x = 1/7$. So $\Pr(3) = 2/7$, and the probability of any other outcome is $1/7$.

5. (8 pts) Bayes' Theorem

   (a) When a test for steroids is given to soccer players, 98% of the players taking steroids test positive and 12% of the players not taking steroids test positive. Suppose that 5% of soccer players take steroids. What is the probability that a soccer player who tests positive takes steroids?

   Solution: Let F be the event that a player selected at random is taking steroids. Then $\Pr(F) = .05$. Let E be the event that a player selected at random tests positive for steroids. Then, we wish to calculate $\Pr(F|E)$. By Bayes' Theorem,

$$\Pr(F|E) = \frac{\Pr(E|F)\Pr(F)}{\Pr(E|F)\Pr(F) + \Pr(E|\overline{F})\Pr(\overline{F})}$$

where $\overline{F}$ is the event that F does not occur. Then,

$$\begin{aligned}
\Pr(F) &= .05 \\
\Pr(\overline{F}) &= 1 - \Pr(F) = .95 \\
\Pr(E|F) &= .98 \\
\Pr(E|\overline{F}) &= .12
\end{aligned}$$

Finally, we have that

$$\begin{aligned}
\Pr(F|E) &= \frac{\Pr(E|F)\Pr(F)}{\Pr(E|F)\Pr(F) + \Pr(E|\overline{F})\Pr(\overline{F})} \\
&= \frac{(.98)(.05)}{(.98)(.05) + (.12)(.95)} \\
&= \frac{.049}{0.163} \\
&= .300613497 \approx 30.1\%
\end{aligned}$$

(b) A space probe near Neptune communicates with Earth using bit strings. Suppose that in its transmissions it sends a 1 one-third of the time and a 0 two-thirds of the time. When a 0 is sent, the probability that it is received correctly is 0.9, and the probability that it is received incorrectly (as a 1) is 0.1. When a 1 is sent, the probability that it is received correctly is 0.8, and the probability that it is received incorrectly (as a 0) is 0.2. What's the probability a 0 was received? What's the probability that a 0 was transmitted given that a 0 was received? (Use Bayes' Theorem).

Solution: The probability that you received a 0 is the probability that a 0 was sent and that a 0 was correctly received plus the probability that a 1 was sent and that it was received incorrectly. In other words,

$$\begin{aligned}
\Pr(0\text{ received}) &= \Pr(0\text{ sent})\Pr(0\text{ received correctly}) + \Pr(1\text{ sent})\Pr(1\text{ received incorrectly}) \\
&= (2/3)(9/10) + (1/3)(2/10) \\
&= (20/30) = (2/3)
\end{aligned}$$

We use Bayes' Theorem to figure out the probability that a 0 was transmitted given that a 0 was received. Let E be the event that 0 was received, and let F be the event that 0 was transmitted. Note this means that $\overline{E}$ is the event that a 1 was received and $\overline{F}$ is the even that a 1 was transmitted. Then:

$$
\begin{aligned}
\Pr(F|E) &= \frac{\Pr(E|F)\Pr(F)}{\Pr(E|F)\Pr(F) + \Pr(E|\overline{F})\Pr(\overline{F})} \\
&= \frac{(9/10)(2/3)}{(9/10)(2/3) + (2/10)(1/3)} \\
&= \frac{(18/30)}{(20/30)} \\
&= (18/20) = (9/10).
\end{aligned}
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6. (8 pts) Expected Value and Variance

   (a) What is the expected sum of the numbers that appear when three fair dice are rolled?

   Solution: Note that the expected sum of the numbers that appear when one fair die is rolled is $\frac{1}{6}\sum_{i=1}^{6} i = 3.5$. Note that each of the three fair dice rolled are independent events. Then, by linearity of expectation, the expectation of the sum of rolling 3 fair dice is $3.5 + 3.5 + 3.5 = 10.5$.

   (b) What is the variance of the number of times a 6 appears when a fair die is rolled 10 times?

   Solution: Note that each roll of the die is independent of every other roll. Note also that there are exactly two outcomes for any event: a 6 was rolled or a 6 was not rolled. Then, we can think of random variables X_i as being "a 6 was rolled on trail i". Then, we are considering 10 Bernoulli trails. So the variance of these 10 trials is $10pq$, where p is the probability of success, and q is the probability of failure. Note that $p = 1/6$ and $q = 5/6$, so the variance is $(50/36) = (25/18)$.