CS182 Spring 2016: Homework 3

Due date: Friday, February 19, 2016 (before class).

1. Prove or disprove the following statements:

   (a) For all positive integers n, if n is a perfect square then $n+3$ is not a perfect square. (Recall the definition of a *perfect square*: an integer n is a perfect square if and only if there exists an integer a such that $a^2 = n$.)

   (b) For every real number x, there is a nonzero real number y such that $x \cdot y = x + y$.

   (c) There is a real number x such that for every integer n we have $\frac{n}{x} > 0$.

   (d) The following statements are equivalent for all nonnegative integers a and b:

   - $a < b$
   - $(a+b)^2 < 4b^2$
   - $4a^2 < (a+b)^2$

2. List all the elements of the following sets:

   (a) $S = \{i \mid i \in \mathbb{Z} \wedge i^2 \leq 4\}$

   (b) $S = \{p \mid p \in \mathbb{Q},\ 0 < p < 1,\ p \text{ is even}\}$

   (c) $S = \{x \mid x \in \mathbb{C}, x \text{ is a root of } x^4 - 1\}$

3. What are the cardinalities of the following sets?

   (a) $\emptyset$

   (b) $\{\emptyset, 1\}$

   (c) $\{1, 2, \{3, 4\}, \emptyset\}$

   (d) $\{\mathbb{R}, \mathbb{Q}, \mathbb{Z}, \mathbb{N}\}$

   (e) $\mathcal{P}(\emptyset)$

4. Let $A = \{1, 2, 4, 6, 7\}$ and $B = \{3, 4, 5\}$, and let our universe be $U = \{n \mid n \in \mathbb{Z}, 1 \leq n \leq 10\}$.

   (a) List the elements of $A \cup B$.

   (b) List the elements of $A \cap \overline{B}$.

   (c) List the elements of $\overline{A} - B$.

   (d) List the elements of $\overline{A} - (A \cup \overline{B})$.

   (e) List the elements of $\mathcal{P}(B)$.

   (f) List the elements of $A \times B$.

   (g) List the elements of $B \times A$.

5. Let A and B be sets.
   (a) Use a venn diagram to show that $A \cap B \subseteq A$.
   (b) Use a venn diagram to show that $A \cap B \subseteq B$.
   (c) Use a venn diagram to show that $(A \cup \overline{B}) = \overline{(\overline{A} \cap B)}$
   (d) Use a venn diagram to show that $B \subseteq \overline{(A - B)}$
6. Let A and B be two sets. Define the *symmetric difference of A and B* as $A \oplus B = \{s \mid s \text{ is in } A \text{ or } B \text{ but not both}\}$ Prove that $A \oplus B = (A - B) \cup (B - A)$. (Hint: show that each side is a subset of the other side).
7. Let $f : \mathbb{R} \rightarrow \mathbb{R}$. Given the following definitions of f, state whether or not it is a function. If it is a function, state the domain, codomain, and range. If it is not a function, state which domain (if any) will make it a function.
   (a) $f(x) = 1/x$.
   (b) $f(x) = x^2 + 1$.
   (c) $f(x) = 0$
   (d) $f(x) = \pm\sqrt{x}$
   (e) $f(x) = \log_e(x)$, where e is Euler's Number.
8. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$ be functions.
   (a) If $f(x) = x^2 + 1$ and $g(x) = 2x + 3$, find $(f + g)(x)$ and $(fg)(x)$.
   (b) If $f(x) = x^3 + 2x$ and $g(x) = -x + 2$, find $(fg + f^2)(x)$.
   (c) If $f(x) = 2x$ and $g(x) = 3x^2$, find $(f + g)^2(x)$.
9. Let f be a function with domain and codomain defined below. For each f, show whether f is injective, surjective, both, or neither.
   (a) $f : \mathbb{Z}^+ \rightarrow \mathbb{Z}^+$ and $f(x) = x + 1$. (Note: $\mathbb{Z}^+$ is the set of all positive integers.)
   (b) $f : \mathbb{R} \rightarrow \mathbb{Z}$ and $f(x) = \lfloor x \rfloor$.
   (c) $f : \mathbb{R} \rightarrow \mathbb{R}$ and $f(x) = x + 1$.
   (d) $f : \mathbb{Z}^+ \rightarrow \{0, 1, 2\}$ and $f(x) = (x \mod 2) + 1$. (Note: $x \mod n$ is the remainder of x when divided by n.)
10. For each of the following f and g, give f^{-1} (if it exists), g^{-1} (if it exists), $f \circ g$ (if it exists), and $g \circ f$ (if it exists), otherwise state that it does not exist (Note: f^{-1} denotes f inverse, not $1/f$):
    (a) $f(x) = x^3$, $g(x) = x + 2$, where $f : \mathbb{R} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}$

(b) $f(x) = x/2$, $g(x) = x + 3$, where $f : \mathbb{Q} \to \mathbb{Q}$ and $g : \mathbb{Q} \to \mathbb{Q}$

(c) $f(x) = x^2$, $g(x) = \lceil x \rceil + 1$, where $f : \mathbb{R}^+ \to \mathbb{R}^+$ and $g : \mathbb{R}^+ \to \mathbb{Z}$