CS182 Spring 2016: Homework 6

Due date: Friday, April 1, 2016 (before class).

1. (5 pts)Using induction, verify that for all $n \geq 1$, the sum of the squares of the first 2n positive integers is given by the formula:

$$1^2 + 2^2 + 3^2 + ... + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$$

Soln: For any integer n 1, let P_n be the statement that

$$1^2 + 2^2 + 3^2 + ... + (2n)^2 = \frac{n(2n+1)(4n+1)}{3}$$

Base Case. The statement P_1 says that

$$1^2 + 2^2 = \frac{(1)(2(1)+1)(4(1)+1)}{3} = 5$$

which is true.
Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is,

$$1^2 + 2^2 + 3^2 + ... + (2k)^2 = \frac{k(2k+1)(4k+1)}{3}$$

. It remains to show that P_{k+1} holds, that is,

$$1^2 + 2^2 + 3^2 + ... + (2(k+1))^2 = \frac{n(2(k+1)+1)(4(k+1)+1)}{3}$$

$$\begin{aligned}
LHS = 1^2 + 2^2 + 3^2 + ... + (2(k+1))^2 &= 1^2 + 2^2 + 3^2 + ... + (2k+2)^2 \\
&= 1^2 + 2^2 + 3^2 + ...(2k)^2 + (2k+1)^2 + (2k+2)^2 \\
&= \frac{k(2k+1)(4k+1)}{3} + (2k+1)^2 + (2k+2)^2 \\
&= \frac{k(2k+1)(4k+1)}{3} + \frac{3(2k+1)^2 + 3(2k+2)^2}{3} \\
&= \frac{8k^3 + 30k^2 + 37k^2 + 15}{3}
\end{aligned}$$

$$RHS = \frac{(k+1)(2k+2+1)(4k+4+1)}{3}$$

$$= \frac{(k+1)(2k+3)(4k+5)}{3}$$
$$= \frac{8k^3 + 30k^2 + 37k^2 + 15}{3}$$

Therefore P_{k+1} holds. Thus, by the principle of mathematical induction, for all $n \geq 1, P_n$ holds.

2. (5 pts)Consider the sequence of real numbers defined by the relations:

$$x_1 = 1, x_{n+1} = \sqrt{1 + 2x_n}$$

for $n \geq 1$.
Use the Principle of Mathematical Induction to show that $x_n < 4$ for all $n \geq 1$.

Soln: For any $n \geq 1$, let P_n be the statement that $x_n < 4$.
Base Case. The statement P_1 says that $x_1 = 1 < 4$, which is true.
Inductive Step. Fix $k \geq 1$, and suppose that P_k holds, that is, $x_k < 4$. It remains to show that P_{k+1} holds, that is, that $x_{k+1} < 4$.

$$x_{k+1} = \sqrt{1 + 2x_k} = \sqrt{1 + 2(4)} = \sqrt{9} = 3 < 4$$

Therefore P_{k+1} holds.
Thus by the principle of mathematical induction, for all $n \geq 1, P_n$ holds.

3. (5 pts)Show that $n! > 3^n$ for $n \geq 7$ via induction.

Soln: For any $n \geq 7$, let P_n be the statement that $n! > 3^n$.
Base Case. The statement P_7 says that $7! = 7 * 6 * 5 * 4 * 3 * 2 * 1 = 5040 > 3\ 7 = 2187$, which is true.
Inductive Step. Fix $k \geq 7$, and suppose that P_k holds, that is, $k! > 3^k$.
It remains to show that P_{k+1} holds, that is,
$(k+1)! > 3^{k+1}$.

$$(k+1)! = (k+1)k! > (k+1)3k \geq (7+1)3k = 8 * 3k > 3 * 3k = 3^{k+1}.$$

Therefore P_{k+1} holds.
Thus by the principle of mathematical induction, for all $n \geq 1, P_n$ holds.

4. (5 pts)Let $p_0 = 1, p_1 = cos\theta$ (for θ some fixed constant) and $p_{n+1} = 2p_1p_n - p_{n-1}$ for $n \geq 1$. Use Principle of Mathematical Induction to prove that $p_n = cos(n\theta)$ for $n \geq 0$.

   Soln:
   For any $n \geq 0$, let P_n be the statement that $p_n = cos(n\theta)$.
   Base Cases. The statement P_0 says that $p_0 = 1 = cos(0\theta) = 1$, which is true. The statement P_1 says that $p_1 = cos = cos(1\theta)$, which is true.
   Inductive Step. Fix $k \geq 0$, and suppose that both P_k and P_{k+1} hold, that is, $p_k = cos(k)$, and $p_{k+1} = cos((k+1))$. It remains to show that P_{k+2} holds, that is, $p_{k+2} = cos((k+2)\theta)$. We have the following identities:
   cos(a + b) = cos a cos b - sin a sin b
   cos(a b) = cos a cos b + sin a sin b
   Therefore, using the first identity when $a = \theta$ and $b = (k+1)\theta$, we have

   $$cos(\theta + (k+1)\theta) = cos\theta cos(k+1)\theta - sin\theta sin(k+1)\theta$$

   , and using the second identity when $a = (k+1)\theta$ and $b = \theta$, we have

   $$cos((k+1)\theta - \theta) = cos(k+1)\theta cos\theta + sin(k+1)\theta sin\theta$$

   Therefore,

   $$\begin{aligned} p_{k+2} &= 2p_1p_{k+1}p_k \\ &= 2(cos\theta)(cos((k+1)\theta)) - cos(k\theta) \\ &= (cos\theta)(cos((k+1)\theta)) + (cos\theta)(cos((k+1)\theta)) - cos(k\theta) \\ &= cos(\theta + (k+1)\theta) + sin\theta sin(k+1)\theta + (cos\theta)(cos((k+1)\theta)) - cos(k\theta) \\ &= cos((k+2)\theta) + sin\theta sin(k+1)\theta + (cos\theta)(cos((k+1)\theta)) - cos(k\theta) \\ &= cos((k+2)\theta) + sin\theta sin(k+1)\theta + cos((k+1)\theta - \theta) - sin(k+1)\theta sin\theta - cos(k\theta) \\ &= cos((k+2)\theta) + cos(k\theta) - cos(k\theta) \\ &= cos((k+2)\theta) \end{aligned}$$

   . Therefore P_{k+2} holds.
   Thus by the principle of mathematical induction, for all $n \geq 1, P_n$ holds

5. (5 pts)Using strong induction, prove that the Fibonacci sequence:
   $a_0 = 1, a_1 = 1$, $a_{k+1} = a_k + a_{k-1}$ for $k \geq 1$:

   $$a_k \geq (\frac{3}{2})^{k-2}.$$

Soln: P(1) is true since $a_1 = 1 = \frac{2}{3} = (\frac{3}{2})^{1-2}$
Now consider any $k \geq 1$. If we assume $a_{k-1} \geq (\frac{3}{2})^{k-3}$ and $a_k \geq (\frac{3}{2})^{k-2}$, then

$$\begin{aligned} a_{k+1} = a_k + a_{k-1} = (\frac{3}{2})^{k-3} + (\frac{3}{2})^{k-2} \\ \geq (\frac{3}{2} + 1)(\frac{3}{2})^{k-3} \\ \geq (\frac{5}{2})(\frac{3}{2})^{k-3} \\ \geq (\frac{9}{4})(\frac{3}{2})^{k-3} \\ \geq (\frac{3}{2})^2(\frac{3}{2})^{k-3} \\ \geq (\frac{3}{2})^{(k+1)-2} \end{aligned}$$

Hence proved.