CS 18200 Fall 2016
Midterm Solution Sketch

First Name: ________________

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Monday, October 24, 2016, 8:00 PM

Important Note: This is an open book, open notes exam. You are not allowed to use any electronic devices(including but not limited to smart watches, calculators, and phones). Any evidence of this or other forms of academic dishonesty will be dealt with strictly in accordance with established policies at Purdue University. Provide clear and concise answers. Answers without justification will not be awarded any points.Where necessary, make clearly stated assumptions.

Q 1 (a) Give a truth table for the following logical expression:

$$(\neg q \wedge (p \to q)) \to \neg p$$

p	q	$\neg p$	$\neg q$	$p \to q$	$\neg q \wedge (p \to q)$	$(\neg q \wedge (p \to q)) \to \neg p$
T	T	F	F	T	F	T
T	F	F	T	F	F	T
F	F	T	T	T	T	T
F	T	T	F	T	F	T

■

(b) Establish using identities the logical equivalence between $(p \to q) \vee (p \to r)$ and $p \to (q \vee r)$. At each step, indicate the name of the identity used.

$$\begin{aligned}
(p \to q) \vee (p \to r) &\equiv (\neg p \vee q) \vee (\neg p \vee r) && (\textit{logical equivalence}) \\
&\equiv (q \vee \neg p) \vee (\neg p \vee r) && (\textit{commutative law}) \\
&\equiv q \vee (\neg p \vee \neg p) \vee r && (\textit{associative law}) \\
&\equiv q \vee \neg p \vee r && (\textit{idempotent law}) \\
&\equiv \neg p \vee q \vee r && (\textit{commutative law}) \\
&\equiv \neg p \vee (q \vee r) && (\textit{associative law}) \\
&\equiv p \to (q \vee r) && (\textit{logical equivalence})
\end{aligned}$$

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Q 2 Translate the following sentences to logical expressions

(a) i. I like to eat ice cream when it rains. Use the following propositions:
   - e: I like to eat ice cream
   - r: it rains

$$r \rightarrow e \blacksquare$$

ii. You cannot go into water if you are under 5 years of age unless you know how to swim. Use the following propositions:
   - g: You can go into water
   - u: You are under five years of age
   - s: You know how to swim

Consider it as A unless B where A itself is ($\neg g$ if u).

$$(u \rightarrow \neg g) \vee s \blacksquare$$

(b) Assume that the universe contains of all human beings and use he following propositions:
   - C(x): x is in the class
   - T(x): x enjoys Thai food
   - H(x): x plays hockey

i. Everyone in your class enjoys Thai food.

$$\forall x(C(x) \rightarrow T(x)) \blacksquare$$

ii. Someone in your class does not play hockey.

$$\exists x(C(x) \wedge \neg H(x)) \blacksquare$$

Q 3 (a) Use rules of inference to show that if $\forall x(A(x) \rightarrow Q(x))$ and $\exists x(A(x) \wedge \neg B(x))$ are true, then $\exists x(Q(x) \wedge \neg B(x))$ is also true, where the domains of all quantifiers are the same. At each step clearly indicate the rule of inference used.

Look at Section 1.6, Page 77, Example 13 in the text■

(b) Prove that $\sqrt[3]{2}$ is irrational

On the lines of Section 1.7, Page 86, Example 10 in the text■

Q 4 (a) What is the Cartesian product of $A = \{a, b\}$, $B = \{9, 10\}$, and $C = \{\{1, \{2\}\}\}$?

{(a, 9, {1, {2}}), (a, 10, {1, {2}}), (b, 9, {1, {2}}), (b, 10, {1, {2}})}■

(b) Use set builder notation and logical equivalences to establish the second De Morgan law $\overline{A \cup B} = \overline{A} \cap \overline{B}$. Clearly name the logical equivalence used at each step.

On the lines of Section 2.2, Page 131, Example 11 in the text■

(c) What is the cardinality of the set $\{1, 2, 3, 3, 4, 5, \{5, \{5\}\}\}$?

6 ■

Q 5 (a) State if the function $f : \mathbb{Z} \to \mathbb{Z}$ where $f(n) = \left\lceil \frac{n}{2} \right\rceil + \left\lfloor \frac{n}{2} \right\rfloor$ is one-to-one, onto, both or neither. Support your answer with a proof or counterexample.

$\forall n \in \mathbb{Z}$, we have $f(n) = n$ because for any $k \in \mathbb{Z}$

$$\begin{aligned} f(2k+1) &= \left\lceil k + \frac{1}{2} \right\rceil + \left\lfloor k + \frac{1}{2} \right\rfloor \\ &= 2k+1 \\ f(2k) &= \left\lceil k \right\rceil + \left\lfloor k \right\rfloor \\ &= 2k \end{aligned}$$

As $(\forall n_1, n_2 \in \mathbb{Z}, f(n_1) = f(n_2) \to (n_1 = n_2)) \land (\forall y \in \mathbb{Z}, \exists x \in \mathbb{Z} : f(x) = y)$, f is both one-to-one and onto.

(b) Justify through a proof or counterexample whether the inverse of the function $f : \mathbb{R} \to \mathbb{R}^+$ where $f(x) = e^x$ exists? If yes, state the inverse.

$\forall x_1, x_2 \in \mathbb{R}, (f(x_1) = f(x_2)) \to x_1 = x_2) \land (\forall y \in \mathbb{R}^+ \exists x \in \mathbb{R} : f(x) = y)$

Hence, f is a bijection and inverse exists.
$f^{-1}(x) = ln(x)$ ■

(c) Find $\sum_{i=6}^{19} 2^i$

$$\begin{aligned}\sum_{i=6}^{19} 2^i &= \sum_{i=0}^{19} 2^i - \sum_{i=0}^{5} 2^i \\ &= \frac{2^{20}-1}{2-1} - \frac{2^6-1}{2-1} \\ &= 2^{20} - 2^6\end{aligned}$$

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(d) If f and $f \circ g$ are one-to-one does it follow that g is one-to-one? Justify.

Yes.

Assume g is not one-to-one and $\exists x_1, x_2 : (g(x_1) = g(x_2)) \wedge (x_1 \neq x_2)$, then $f(g(x_1)) = f(g(x_2))$ as f is one-to-one and $g(x_1) = g(x_2)$ thereby contradicting that $f \circ g$ is one-to-one for $x_1 \neq x_2$ ■

Q 6 (a) Give a tight big-O estimate for $f(x) = (x+6)^2$. Justify by providing witnesses.

$$\begin{aligned} f(x) &= x^2 + 12x + 36 \\ &\leq x^2 + 12x^2 + 36x^2 \\ &\leq 49x^2 \end{aligned}$$

Hence, $f(x)$ is $O(x^2)$ with $C = 49$ and $k = 1$ ■

(b) Give the tight big-O estimate for $f(n) = (n! + n^n + 2^n)(3^n + n^n + \log(n^7 + 17))$. Justify through sums and products of big-Os of different functions.

$f(n) \leq (4n^n)(6n^n) \leq 24n^{2n}$

Hence $f(n)$ is $O(n^{2n})$ ■

(c) What is the tight big-O running time of the following code segment? Provide a clear justification for the run time.

```
a = 1
i = 1
while i < n
    a = a * i
    i = i + 1
endwhile
j = i
while j > 1
    k = 1
    while k < m
        a = a + k - j
        k = k + 1
    endwhile
    j = j - 1
endwhile
```

The first loop runs n times and the second loop runs for mn times. Hence, the total run time is $O(mn)$ ■