CS 18200 Spring 2016
Midterm

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PSO Number: _______________

Wednesday, March 9, 2016, 9:30 AM

Question 1. Propositional Logic

(a) Give a truth table for the following logical expression:

$$[p \iff (q \wedge r)] \implies (\neg r \vee p)$$

p	q	r	$\neg r$	$(\neg r \vee p)$	$(q \wedge r)$	$p \iff (q \wedge r)$	$[p \iff (q \wedge r)] \implies (\neg r \vee p)$
T	T	T	F	T	T	T	T
T	T	F	T	T	F	F	T
T	F	T	F	T	F	F	T
T	F	F	T	T	F	F	T
F	T	T	F	F	T	F	T
F	T	F	T	T	F	T	T
F	F	T	F	F	F	T	F
F	F	F	T	T	F	T	T

(b) Translate the following English sentence to a logical expression, defining appropriate propositional variables:

A sufficient condition for you driving to the store or ordering takeout is that it was not raining outside
Let $p =$"You drove to the store", $q =$"You ordered takeout", and $r =$"It was raining outside". Then, the statement in propositional logic is:

$$\neg r \rightarrow (p \vee q)$$

Question 2. Quantified Logic

(a) Show that the following two logical expressions are equivalent:

$$(1)\ \forall x\,[Q(x) \implies \exists y\,(P(x,y) \lor \neg Q(y))]$$
$$(2)\ \neg\,[\exists x \forall y\,(\neg P(x,y) \land Q(x) \land Q(y))]$$

$$\begin{aligned}
\forall x[Q(x) \implies \exists y(P(x,y) \lor \neg Q(y)] &\equiv \forall x[\neg Q(x) \lor \exists y(P(x,y) \lor \neg Q(y)] \\
&\equiv \forall x \exists y(P(x,y) \lor \neg Q(x) \lor \neg Q(y)) \\
&\equiv \forall x \exists y(P(x,y) \lor \neg(Q(x) \land Q(y))) \\
&\equiv \neg[\exists x \forall y \neg(P(x,y) \lor \neg(Q(x) \land Q(y)))] \\
&\equiv \neg[\exists x \forall y(\neg P(x,y) \land Q(x) \land Q(y))]
\end{aligned}$$

(b) Translate the following English sentence to quantified logic where the universe of discourse is "all students at Purdue", defining appropriate propositional functions:

There is a student who sells cookies and another student with straight A's whenever all students are home for Winter Break.

Let $C(x)$ = "x sells cookies", $A(x)$ = "x has straight A's", and $W(x)$ = " x is home for Winter Break". Then, the statement in propositional logic is:

$$\forall x(W(x) \to \exists y \exists z(C(y) \land A(z)))$$

Question 3. Sets

(a) Let $A = \{1, 3, 6\}$, $B = \{6, 7\}$, and let the universe be $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

i. List the elements of $B - \overline{A}$

Answer: $B - \overline{A} = \{6\}$

ii. List the elements of $B \times A$

Answer: $B \times A = \{(6, 1), (6, 3), (6, 6), (7, 1), (7, 3), (7, 6)\}$

iii. List the elements of $\mathcal{P}(B)$ (the power set of B)

Answer: $\mathcal{P}(B) = \{\emptyset, \{6\}, \{7\}, \{6, 7\}\}$

(b) Using the definition of set difference, if A, B, and C are sets, show that $(A - B) - C = (A - C) - (B - C)$.

Proof:

$$\begin{aligned}
(A - C) - (B - C) &= \{s \colon s \in (A - C) \wedge s \notin (B - C)\} \\
&= \{s \colon s \in (A \cap \overline{C}) \wedge s \notin (B \cap \overline{C})\} \\
&= \{s \colon s \in (A \cap \overline{C}) \wedge s \in \overline{(B \cap \overline{C})}\} \\
&= \{s \colon s \in (A \cap \overline{C}) \wedge s \in (\overline{B} \cup C)\} \\
&= \{s \colon s \in [(A \cap \overline{C}) \cap (\overline{B} \cup C)]\} \\
&= \{s \colon s \in [A \cap \overline{C} \cap (\overline{B} \cup C)]\} \\
&= \{s \colon s \in [A \cap ((\overline{C} \cap \overline{B}) \cup (\overline{C} \cap C))]\} \\
&= \{s \colon s \in A \cap [(\overline{C} \cap \overline{B}) \cup \emptyset]\} \\
&= \{s \colon s \in A \cap \overline{C} \cap \overline{B}\} \\
&= \{s \colon s \in (A \cap \overline{B}) \cap \overline{C}\} \\
&= \{s \colon s \in (A - B) \cap \overline{C}\} \\
&= \{s \colon s \in (A - B) - C\} \\
&= (A - B) - C
\end{aligned}$$

Question 4. Functions

(a) Let $f : \mathbb{R}^+ \to \mathbb{Z}^+ \cup \{0\}$ defined by $f(x) = \lceil x \rceil - 1$. Using the definition of injective and surjective, show that f is surjective but not injective.

Not Injective: Let $x_1 = \frac{1}{2}$ and let $x_2 = 1$. Then, $f(x_1) = \lceil \frac{1}{2} \rceil - 1 = 1 - 1 = 0$ and $f(x_2) = \lceil 1 \rceil - 1 = 1 - 1 = 0$. So $f(x_1) = f(x_2)$ but $x_1 \neq x_2$. Thus, f is not injective.

Surjective: Let $b \in \mathbb{Z}^+ \cup \{0\}$. Take $a = b+1$. Then, $a \in \mathbb{R}^+$ and $f(a) = \lceil a \rceil - 1 = \lceil b+1 \rceil - 1 = b+1-1 = b$. Thus, f is surjective.

(b) Let $g : A \to B$ and $h : B \to C$ be functions. Under what conditions is $h \circ g$ invertible? What is $(h \circ g)^{-1}$ if it exists?

Conditions: h and g must both be bijections and $A = C$ must be the case. If $(h \circ g)^{-1}$ exists, then $(h \circ g)^{-1} = (g^{-1} \circ h^{-1})$

(c) Evaluate the following summation: $\sum_{i=7}^{19} i + 2\left(\frac{1}{2}\right)^i$

$$
\begin{aligned}
\sum_{i=7}^{19} i + 2\left(\frac{1}{2}\right)^i &= \sum_{i=7}^{19} i + \sum_{i=7}^{19} 2\left(\frac{1}{2}\right)^i \\
&= \left(\sum_{i=0}^{19} i - \sum_{i=0}^{6} i\right) + \left(\sum_{i=0}^{19} 2\left(\frac{1}{2}\right)^i - \sum_{i=0}^{6}\left(\frac{1}{2}\right)^i\right) \\
&= \left(\sum_{i=1}^{19} i - \sum_{i=1}^{6} i\right) + \left(\frac{2\left(\frac{1}{2}\right)^{20} - 2}{\frac{1}{2} - 1} - \frac{2\left(\frac{1}{2}\right)^{7} - 2}{\frac{1}{2} - 1}\right) \\
&= \left(\frac{(19)(20)}{2} - \frac{(6)(7)}{2}\right) + \left(\frac{2\left(\frac{1}{2}\right)^{20} - 2 - 2\left(\frac{1}{2}\right)^{7} + 2}{\frac{-1}{2}}\right) \\
&= 169 + 4\left(\frac{1}{2}\right)^{7} - 4\left(\frac{1}{2}\right)^{20} \\
&= 169 + \left(\frac{1}{2}\right)^{5} - \left(\frac{1}{2}\right)^{18}
\end{aligned}
$$

Question 5. Growth of Functions

(a) Let $f(n) = 2n^2 + n$. Give witnesses that show $f(n)$ is $\mathcal{O}(n^2)$.

Solution: Let $c = 3$, $k = 1$, and $g(n) = n^2$. Then, for all $n > k$, we have that $f(n) = 2n^2 + n < 2n^2 + n^2 = 3n^2 = cg(n)$. Thus, $f(n)$ is $\mathcal{O}(g(n)) = \mathcal{O}(n^2)$.

(b) Give a good big-Oh bound on the function $f(n) = 2^{\log_2 n} n^2 + 3n^2 \log_2 n + n - 17$

Answer: $\mathcal{O}(n^3)$

(c) State the big-Oh running time of the following code segment:

```
a = 1
for i = 1 to n
    a = a + 1
endfor
for i = 1 to n
    for j = 1 to m
        a = a + i*j
    endfor
    a = a / 2
endfor
```

Runtime: $\mathcal{O}(mn)$

Question 6. Proofs

(a) Use a direct proof to show that for every integer n, if n is even, then $3n^2 + 2n + 7$ is odd.

Proof: Let $n \in \mathbb{Z}$ be even. As such, there exists $k \in \mathbb{Z}$ such that $n = 2k$. Then, evaluating the expression:

$$\begin{aligned} 3n^2 + 2n + 7 &= 3(2k)^2 + 2(2k) + 7 \\ &= 12k^2 + 4k + 7 \\ &= 12k^2 + 4k + 6 + 1 \\ &= 2(6k^2 + 2k + 3) + 1 \end{aligned}$$

Thus, since k is an integer, k^2 is also an integer, and any other integer multiplied by k is an integer, and the sum of integers is an integer, we have that $6k^2 + 2k + 3$ is an integer. Let $m = 6k^2 + 2k + 3$. Then, we have shown that $3n^2 + 2n + 7 = 2m + 1$. Thus, by definition, $3n^2 + 2n + 7$ is odd, as desired.

(b) Use a proof by contradiction to show that the product of a nonzero rational number and an irrational number is irrational.

Proof: Let $p \in \mathbb{Q}$ and let $m \in \mathbb{R} - \mathbb{Q}$. Assume by way of contradiction that $m * p \in \mathbb{Q}$. By definition of rational, there exist $a, b, c, d \in \mathbb{Z} - \{0\}$ such that $p = \frac{a}{c}$ and $mp = \frac{b}{d}$. Without loss of generality, we assume that $\frac{a}{c}$ and $\frac{b}{d}$ are written in lowest terms. Thus, we have an equation:

$$\frac{b}{d} = mp = m\left(\frac{a}{c}\right)$$

This implies that

$$m = \frac{bc}{ad}$$

Note that the product of two integers is again an integer. So we have expressed m as a fraction of two integers. Thus m is rational. But we assumed m is irrational. This is a contradiction. Thus, mp is irrational.