"Strong" mathematical induction

Let P be a property of integers.

Take any integer n_0.

To prove $P(n)$ for all integers $n \geq n_0$, it suffices to prove that

for all $n \geq n_0$,

if $P(m)$ for all m s.t. $n_0 \leq m < n$,

then $P(n)$.

Notice that here you get a "stronger" induction hypothesis:

instead of assuming $P(n-1)$ in order to prove $P(n)$,

you can assume $P(m)$ for all $m \in \{n_0, n_0 + 1, \ldots, n-1\}$.

That's why this is called "strong" mathematical induction.

But where's the base case?

Well, notice what happens when $n = n_0$: the IH is empty!

Example of strong mathematical induction

Typical first example involves Fibonacci numbers, defined recursively as follows.

$F_0 = 0$
$F_1 = 1$
$F_{n+2} = F_n + F_{n+1}$

"Weak" induction can be less convenient for proofs about F_n for all $n \in \mathcal{N}$, since F_{n+2} is defined in terms of both F_n and F_{n+1}.

Claim: For every $n \in \mathcal{N}$,

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1\,.$$

Proof by strong mathematical induction on n ($n \geq 0$).

IH: For all k s.t. $0 \leq k < n$,

$$\sum_{i=0}^{k} F_i = F_{k+2} - 1\,.$$

NTS:

$$\sum_{i=0}^{n} F_i = F_{n+2} - 1$$

We'll consider *three* cases...

Claim: For every $n \in \mathcal{N}$, $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

IH: For all $k < n$, $\sum_{i=0}^{k} F_i = F_{k+2} - 1$.

NTS: $\sum_{i=0}^{n} F_i = F_{n+2} - 1$.

Consider three cases.

Case 1: $n = 0$.

$$\begin{array}{rcll} \sum_{i=0}^{0} F_i & = & F_0 & \\ & = & (F_0 + 1) - 1 & \\ & = & (F_0 + F_1) - 1 & \text{(defn } F_1\text{)} \\ & = & F_2 - 1 & \text{(defn } F_2\text{)} \end{array}$$

Case 2: $n = 1$.

$$\begin{array}{rcll} \sum_{i=0}^{1} F_i & = & F_0 + F_1 & \\ & = & F_2 & \text{(defn } F_2\text{)} \\ & = & (1 + F_2) - 1 & \\ & = & (F_1 + F_2) - 1 & \text{(defn } F_1\text{)} \\ & = & F_3 - 1 & \text{(defn } F_3\text{)} \end{array}$$

Case 3: $n \geq 2$.

$$\begin{array}{rcll} \sum_{i=0}^{n} F_i & = & \sum_{i=0}^{n-2} F_i + (F_{n-1} + F_n) & (n \geq 2) \\ & = & (F_n - 1) + (F_{n-1} + F_n) & \text{(IH)} \\ & = & (F_n - 1) + F_{n+1} & \text{(defn } F_{n+1},\ n \neq 0\text{)} \\ & = & (F_n + F_{n+1}) - 1 & \\ & = & F_{n+2} - 1 & \text{(defn } F_{n+2}\text{)} \end{array}$$

Format for strong mathematical induction proof

In this class, please use the format of the previous example for proofs by strong mathematical induction.

Essential elements:

- Say what you are proving.
- Say that the proof is by strong mathematical induction, and make it clear what is playing the role of n_0.
- State the induction hypothesis (IH) and what you need to show (NTS).
- Divide the argument into cases, as needed.
- Indicate clearly where and how you use the IH.

Notice that in the proof format I am recommending, you are free to devise cases according to the needs of the argument. (Of course your cases must be exhaustive.) There is no "separate" base case or induction case.