CS_18200 Spring 2016
Midterm 2

First Name: ________________

Last Name: ________________

Purdue ID#: ________________

PSO Number: ____________

Wednesday, April 20, 2016, 8:00 PM - 9:15 PM

Important Note: This is a closed book, closed notes exam. No electronic devices, other than a non-programmable calculator, are allowed. Any evidence of academic dishonesty will be dealt with strictly in accordance with established rules at Purdue University.

Question 1. (a) i. -17 mod 7 = __4__ (1 point)

ii. 43 mod 11 = __10__ (1 point)

(b) i. Convert $(122)_3$ from its base 3 expansion to its binary expansion.

$$\begin{aligned}(122)_3 &= 1 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0 \\ &= 17 \\ &= (10001)_2 \quad (1\,point)\end{aligned}$$

ii. Convert $(\mathrm{BACD})_{16}$ from its hexadecimal expansion to its octal expansion.

$$(BACD)_{16} = (135315)_8 \quad (2\,points)$$

(c) How many zeros are there at the end of 150!?

The number of zeros would be equal to the exponent of 5 in the prime factorization of 150! (1 point)

$$\begin{aligned} &= \left\lfloor \frac{150}{5} \right\rfloor + \left\lfloor \frac{150}{25} \right\rfloor + \left\lfloor \frac{150}{125} \right\rfloor \\ &= 37 \quad (4\,points)\end{aligned}$$

Question 2. (a) Show that if a and b are both positive integers, then $(2^a - 1)\,mod\,(2^b - 1) = (2^{a\,mod\,b} - 1)$. (A proof by example will result in zero credit)

Consider two cases,

- Case 1: $a < b$
  L.H.S: $(2^a - 1)\,mod\,(2^b - 1) = (2^a - 1)$

  R.H.S: $(2^a - 1)$, as $a\,mod\,b = a$ ■ (1 point)

- Case 2: $a \geq b$
  Let $a = bq + r$, where q is the quotient and r is the remainder.

$$\begin{aligned}
(2^{bq+r} - 1) &= (2^b - 1)(2^{b(q-1)+r}) + 2^{b(q-1)+r} - 1 \\
(2^{b(q-1)+r} - 1) &= (2^b - 1)(2^{b(q-2)+r}) + 2^{b(q-2)+r} - 1 \\
(2^{b(q-2)+r} - 1) &= (2^b - 1)(2^{b(q-3)+r}) + 2^{b(q-3)+r} - 1 \\
\cdots so\,on \cdots & \\
(2^{b(q-(q-1))+r} - 1) &= (2^b - 1)(2^{b(q-q)+r}) + 2^{b(q-q)+r} - 1
\end{aligned}$$

Final remainder: $2^{b(q-q)+r} - 1 =$ R.H.S ■ (4 points)

(b) Express the greatest common divisor of 252 and 198 as a linear combination of 252 and 198.

$$\begin{aligned}
252 &= 198 \cdot 1 + 54 && (1) \\
198 &= 54 \cdot 3 + 36 && (2) \\
54 &= 36 \cdot 1 + 18 && (3) \\
36 &= 18 \cdot 2 \quad (2\,points) && (4)
\end{aligned}$$

Using eqn. 2 in 3,

$$\begin{aligned}
18 &= 54 - 36 \cdot 1 \\
&= 54 - (198 - 3 \cdot 54) \\
&= 4 \cdot 54 - 1 \cdot 198 \\
&= 4(252 - 198 \cdot 1) - 198 \quad Using\ eqn.1 \\
&= 4 \cdot 252 - 5 \cdot 198 \quad \blacksquare \quad (3\,points)
\end{aligned}$$

Question 3. (a) Solve the congruence $x^2 \equiv 29\ (mod\, 35)$

Due to Chinese Remainder Theorem it is equivalent to solving the system

$$
\begin{gathered}
x^2 \equiv 29\ (mod\, 5) \quad x^2 \equiv 29\ (mod\, 7) \\
x^2 \equiv 4\ (mod\, 5) \quad x^2 \equiv 1\ (mod\, 7) \\
(x-2)(x+2) = 5k_1 \quad (x-1)(x+1) = 7k_2 \quad (k_1, k_2 \in \mathbb{Z}) \\
x \equiv \pm 2\ (mod\, 5) \quad x \equiv \pm 1\ (mod\, 7) \quad (k_1, k_2 = 0)
\end{gathered}
$$

The solution of $x \equiv a_1\ (mod\, 5)$ and $x \equiv a_2\ (mod\, 7)$ by Chinese remainder theorem is given by $x \equiv a_1 \cdot 7 \cdot 3 + a_2 \cdot 5 \cdot 3\ (mod\, 35)$. Setting $a_1 = \pm 2$ and $a_2 = \pm 1$ we obtain x = 8, 13, 22, and 27 ■ (5 points)

(b) Find the remainder when 2^{520} is divided by 7.

By Fermat's little theorem $2^6 \equiv 1\ (mod\, 7)$

$$2^{520} = (2^6)^{86} \cdot 2^4 \equiv 16 \equiv 2\ (mod\, 7) \blacksquare \quad (2\, points)$$

(c) Find the number that leaves a remainder 2 when divided by 3, 3 when divided by 5 and 5 when divided by 7.
The number is a solution to the system of congruences

$$
\begin{gathered}
x \equiv 2\ (mod\, 3) \\
x \equiv 3\ (mod\, 5) \\
x \equiv 5\ (mod\, 7)
\end{gathered}
$$

The solution to this system are those x such that

$$x \equiv 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 5 \cdot 15 \cdot 1\ (mod\, 7) x \equiv 68\ (mod\, 105)$$

$$x = 68 + 105k_1 \quad k_1 \in \mathbb{Z} \blacksquare \quad (3\, points)$$

Question 4. (a) Use mathematical induction to prove that the sum of the squares of the first n non-negative integers is $\dfrac{n(n-1)(2n-1)}{6}$

Let $P(n)$ be the proposition : $0^2+1^2+\cdots+(n-1)^2 = \dfrac{n(n-1)(2n-1)}{6}$

Basis Step: $P(1)$ is true, because $0^2 = 0$ (1 point)

Inductive Step: For the inductive hypothesis we assume that P(k) holds, that is,

$0^2+1^2+\cdots+(k-1)^2 = \dfrac{k(k-1)(2k-1)}{6}$

$P(k+1) : 0^2+1^2+\cdots+(k-1)^2+k^2 = \dfrac{k(k-1)(2k-1)}{6} + k^2 = k\left[\dfrac{(k-1)(2k-1)+6k}{6}\right]$

$= k\left[\dfrac{2k^2+3k+1}{6}\right] = \dfrac{(k+1)(k+1-1)(2(k+1)-1)}{6}$

Hence $P(k+1)$ is true under the assumption that $P(k)$ is true. ■ (4 points)

(b) What amounts of money can be formed using only \$2 and \$5 bills. Prove your answer using strong induction.

\$2, \$4, \$5, \$6, \$7, ... (1 point)

Let $P(n)$ be the statement that amount of \$n can be formed using \$2 and \$5 bills.

Basis Step: We can form amounts of \$2, \$4, and \$5 using one \$2 bill, two \$2 bills, and one \$5 bill respectively. Hence $P(2)$, $P(4)$, and $P(5)$ are true. This completes the basis step.(1 point)

Inductive Step: The inductive hypothesis is the statement that $P(j)$ is true for $4 \leq j \leq k$ with $k \geq 5$. Under this assumption we have to show that $P(k+1)$ is true. Using inductive hypothesis we can assume that $P(k-1)$ is true because $k \geq 5$ and hence $k-1 \geq 4$ which is already true due to inductive hypothesis, we only need to add a \$2 bill to to form the amount \$(k+1). Hence $P(k+1)$ is also true. This completes the inductive step. ■ (3 points)

Question 5. (a) There are 13 stations between two points A and B. In how many different ways can a train stop at exactly four stations between A and B such that no two stops are adjacent.

The train has to stop at 4 stations so the remaining 9 have to be arranged around these 4 to satisfy the constraint. If the five spaces around these four stations are denoted by x_1, x_2, x_3, x_4, and x_5. We have to find the number of nonnegative integral solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 9$ with the x_2, x_3, and x_4 ≥ 1. This can be reduced to $x_1 + x_2 + x_3 + x_4 + x_5 = 6$ under the assumption that one station each is given to x_2, x_3, and x_4. Hence the total number of ways the train can stop is $\binom{6+4}{4} = \binom{10}{4}$ ■(5 points)

(b) In how many ways can you select exactly 5 fruits from a basket containing pears, peaches, oranges, and apples. The order of selection does not matter and there are at least 5 fruits of each kind in the basket.

If x_1, x_2, x_3, and x_4 are used to denote the number of chosen pears, peaches, oranges, and apples. We want the number of nonnegative integral solutions to $x_1 + x_2 + x_3 + x_4 = 5$, which is $\binom{5+3}{3} = \binom{8}{3}$ ■

Alternatively, if the string 01010010 is used to denote the choice 1 pear (1st group of zeros), 1 peach (2nd group of zeros), 2 oranges (3rd group of zeros) and 1 apple (4th group of zeros). Then the total number of selections of 5 fruits are $\frac{8!}{5!3!} = \binom{8}{3}$ ■ (5 points)

Question 6. (a) i. How many ways are there to distribute hands of 5 cards to 6 people from a standard deck of 52 cards?

$\binom{52}{5}\binom{47}{5}\binom{42}{5}\binom{37}{5}\binom{32}{5}\binom{27}{5}$ ■ (2 points)

ii. Consider 10 randomly selected bits. How many of the selections start with a 1 or have exactly two 1's in them?

$2^9 + \binom{10}{2}$ - $1 \cdot \binom{9}{1}$ ■ (2 points)

(b) In a Discrete Mathematics quiz consisting of 4 questions the maximum points for questions 1 through 4 are 50, 50, 50, and 100 respectively. Find the total number of ways to score a total of exactly 155 points on the quiz if only non-negative integral points can be scored in each question.

Scoring 155 points is the same as losing 95 points. Let x_1, x_2, x_3, x_4 denote the respective points lost in the four questions. This is equivalent to finding the number of nonnegative integral solutions of $x_1 + x_2 + x_3 + x_4 = 95$ with $x_1, x_2, x_3 \leq 50$. Let us first solve it for an unconstrained version, the total solutions being $\binom{95+3}{3} = \binom{98}{3}$. We have to subtract the cases in which either of x_1, x_2, x_3 get more than 50. The number of cases where x_1 receives more than 50 is given by the number of nonnegative integral solutions to the unconstrained equation $x_1 + x_2 + x_3 + x_4 = 44$ which is $\binom{44+3}{3} = \binom{47}{3}$. The same can be said for x_2 and x_3. Hence the total number of ways to score exactly 155 points is $\binom{98}{3} - 3 \cdot \binom{47}{3}$ ■ (6 points)

Question 7. (a) A bowl contains 8 red and 5 blue balls. What is the minimum number of balls one must pick to have

i. at least three balls of the same color?
5 ■(1 point)

ii. at least three blue balls?
11■(1 point)

(b) What should be the minimum number of students at Purdue to ensure that at least 100 come from any one of the 50 states?(3 points)

4951. (Due to pigeonhole principle, placing $99 \cdot 50 + 1 = 4951$ pigeons into 50 holes would cause one hole to have 100 pigeons) ■ (3 points)

(c) Prove that if five integers are selected from the first 8 positive integers, there must be a pair of selected integers that sum to 9.

Consider four pigeonholes corresponding to the sets $\{1,8\}$, $\{2,7\}$, $\{3,6\}$, $\{4,5\}$.
Due to pigeonhole principle placing 5 numbers into the four holes means one hole has to contain two numbers, due to construction of the holes those two numbers sum to 9.■

Alternative argument: If you were to avoid choosing a pair that sums to 9, you can only choose at most 4 numbers; all from different holes. Pigeonhole principle says choosing one more number would invariably involve picking two numbers from one hole and hence a pair that sums to 9. ■ (5 points)