

**Due date: Friday, February 5, 2016 (before class).**

1. (12 pts) Let the universe of discourse be the set of all real numbers. Let  $P(x)$  be the statement “ $x$  is an integer”,  $Q(x)$  be the statement “ $x$  is a rational number”, and  $R(x)$  be the statement “ $x$  is greater than zero”.
  - (a) Translate the following English sentence to a logical expression using the statements above:  
“Every real number is rational and greater than zero”  
Solution:  $\forall x(Q(x) \wedge R(x))$
  - (b) What is the negation of the expression  $\exists x(R(x) \rightarrow [P(x) \wedge Q(x)])$ ? (Choose 1)
    - i.  $\forall x(\neg R(x) \vee [(P(x) \wedge Q(x))])$
    - ii.  $\forall x(R(x) \wedge [\neg P(x) \vee \neg Q(x)])$  [Correct]
  - (c) Translate the following logical expression to English:  $\forall x(Q(x) \rightarrow P(x))$ .  
Solution: For all real numbers, if that real number is rational, then it is an integer.  
Another possibility: Every rational number is an integer.
  - (d) Translate the following English sentence to a logical expression: “All irrational numbers are not integers”  
Solution:  $\forall x(\neg Q(x) \rightarrow \neg P(x))$  [or equivalently:  $\forall x(Q(x) \vee \neg P(x))$ ]
  - (e) Translate the following logical expression to English:  $\exists x(\neg Q(x) \wedge R(x))$   
Solution: There is a real number such that that number is not rational and is greater than zero. Alternatively, There is a real number that is irrational and greater than zero. Alternatively, there is an irrational number greater than zero.
  - (f) What is the negation of the expression  $\forall x([P(x) \vee Q(x)] \rightarrow \neg R(x))$ ? (Choose 1)
    - i.  $\exists x([P(x) \vee Q(x)] \wedge R(x))$  [Correct]
    - ii.  $\exists x([\neg P(x) \wedge \neg Q(x)] \vee \neg R(x))$
2. (10 pts) Let the universe of discourse be the set of all positive integers. let  $P(x, y)$  be the statement “ $x + y < 2$ ”
  - (a) Translate the following logical expression to English:  $\forall x \exists y P(x, y)$   
Solution: For all positive integers  $x$ , there is a positive integer  $y$  such that  $x + y < 2$ .
  - (b) What is  $\neg P(x, y)$ ? (Choose 1)
    - i.  $x + y \geq 2$  [Correct]
    - ii.  $x + y > 2$
  - (c) Is the expression in part (a) true? Justify your answer.  
Solution: No, it is not true. Let  $x = 2$  Then, for every positive integer  $y$ ,  $x + y \geq 2$ .

- (d) Translate the following logical expression to English:  $\exists x \forall y \neg P(x, y)$   
 Solution: There is a positive integer  $x$  such that for all positive integers  $y$ ,  $x + y \geq 2$ .
- (e) Translate the following English sentence to a logical expression: “There are two distinct positive integers such that their sum is less than two”  
 Solution:  $\exists x \exists y (P(x, y) \wedge (x \neq y))$
3. (12 pts) Let the universe of discourse be all students at Purdue. Let  $P(x, y)$  be the statement “ $x$  is friends with  $y$ ”. Let  $C(x, y)$  be the statement “ $x$  shares a class with  $y$ ”. Let  $Q(x)$  be the statement “ $x$  is in CS 182”.
- (a) Translate the following English sentence to a logical expression: “If a student is in CS 182, then there is different student in CS 182 that is friends with them”  
 Solution:  $\forall x [Q(x) \rightarrow \exists y ((x \neq y) \wedge Q(y) \wedge P(x, y))]$
- (b) Translate the following logical expression to English:  $\forall x \forall y (P(x, y) \rightarrow C(x, y))$   
 Solution: For all pairs of students at Purdue, if those students are friends, then they share a class together. Alternatively, for all students  $x$  and  $y$  at Purdue, if  $x$  is friends with  $y$ , then  $x$  shares a class with  $y$ .
- (c) What is the negation of  $\forall x \forall y (P(x, y) \rightarrow C(x, y))$ ?  
 i.  $\exists x \exists y (P(x, y) \wedge \neg C(x, y))$  [Correct]  
 ii.  $\exists x \forall y (P(x, y) \rightarrow C(x, y))$
- (d) Translate the following logical expression to English:  $\exists x \forall y (C(x, y) \rightarrow P(x, y))$   
 Solution: There is a student  $x$  such that for every student  $y$  if  $x$  shares a class with  $y$ , then  $x$  is friends with  $y$ . Alternatively, there is a student that is friends with every other student in their class.
- (e) What is the negation of  $\exists x \forall y (C(x, y) \rightarrow P(x, y))$ ? (Choose 1)  
 i.  $\forall x \exists y (C(x, y) \wedge \neg P(x, y))$  [Correct]  
 ii.  $\forall x \exists y (\neg C(x, y) \vee P(x, y))$
- (f) Translate the following English sentence to a logical expression: “There is a student not in CS 182 such that for every student not friends with them, they share a class together”  
 Solution:  $\exists x [\neg Q(x) \wedge \forall y (\neg P(x, y) \rightarrow C(x, y))]$
4. (12 + 5 pts) Let  $P$  be the proposition “you baked some cookies”. Let  $Q$  be the proposition “you went to the store”. Let  $R$  be the proposition “it was snowing outside”. Let  $S$  be the proposition “your oven was broken”.

- (a) Given the premises “If you baked some cookies, then you went to the store” and “If it was snowing outside, then you baked some cookies”, which of the following can be concluded? (Choose 1)

- i. “If you went to the store, then it was snowing outside”
- ii. “You went to the store or it wasn’t snowing outside” [Correct]

- (b) Given the premises  $(\neg P \wedge \neg Q) \rightarrow S$ ,  $\neg P$ ,  $\neg Q$ , and  $S \rightarrow R$ , which of the following can be concluded? (Choose 1)

- i.  $\neg S$
- ii.  $R$  [Correct]

- (c) Translate the following argument to English:

$\neg P \vee \neg S$   
 $P$

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$\therefore \neg S$

Solution: “You didn’t bake some cookies or your oven wasn’t broken”, “You baked some cookies”, “Therefore, your oven wasn’t broken”

- (d) Translate the following argument to English:

$\neg R \rightarrow Q$   
 $Q \rightarrow P$

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$\therefore \neg R \rightarrow P$

Solution: “If it wasn’t snowing outside, you went to the store”, “If you went to the store, then you baked some cookies”, “Therefore, if it wasn’t snowing outside, you baked some cookies”.

- (e) In the following argument, label which rules of inference are used:

- i.  $Q$  (Premise)
- ii.  $R \rightarrow \neg Q$  (Premise)
- iii.  $\neg R$  (a. **modus tollens** of i and ii)
- iv.  $P \rightarrow R$  (Premise)
- v.  $R \rightarrow Q$  (Premise)
- vi.  $P \rightarrow Q$  (b. **hypothetical syllogism** of iv and v)
- vii.  $\neg R \vee S$  (c. **addition** of iii with  $S$ )
- viii.  $R \vee (P \rightarrow Q)$  (d. **addition** of vi with  $R$ )
- ix.  $\therefore S \vee (P \rightarrow Q)$  (e. **resolution** of vii and viii)

- (f) Write each premise and the conclusion from part (e) in English.

Solution: “You went to the store”, “If it was snowing outside, then you didn’t go to the store”, “If you baked some cookies, then it was snowing outside”, “If it was snowing outside, then you went to the store”, “Therefore, your oven was broken or if you baked some cookies then you went to the store”.

- (g) (5 pts) Let  $A(x)$ ,  $B(x)$ , and  $C(x)$  be propositional functions. Use rules of inference to show if  $\forall x(A(x) \vee B(x))$  and  $\forall x((\neg A(x) \wedge B(x)) \rightarrow C(x))$  are true, then  $\forall x(\neg C(x) \rightarrow A(x))$  is also true, where the domains of all the quantifiers are the same.

Solution:

1.  $\forall x(A(x) \vee B(x))$  [Premise]
2.  $A(c) \vee B(c)$  for arbitrary  $c$  [Universal Instantiation of 1]
3.  $\forall x((\neg A(x) \wedge B(x)) \rightarrow C(x))$  [Premise]
4.  $(\neg A(c) \wedge B(c)) \rightarrow C(c)$  for arbitrary  $c$  [Universal Instantiation of 3]
5.  $\neg(\neg A(c) \wedge B(c)) \vee C(c)$  [Implication Equivalence of 4]
6.  $A(c) \vee \neg B(c) \vee C(c)$  [DeMorgan’s Law of 5 and Association Law]
7.  $A(c) \vee C(c) \vee A(c)$  [Resolution of 2 and 6]
8.  $A(c) \vee C(c)$  [Idempotent Law of 7]
9.  $\neg(\neg C(c)) \vee A(c)$  [Double Negation Law of 8]
10.  $\neg C(c) \rightarrow A(c)$  [Implication Equivalence of 9]
11.  $\forall x(\neg C(x) \rightarrow A(x))$  [Universal Generalization of 10]

5. (15 pts) For each of the following questions, if the proof method is not specified, you are free to choose a proof method.

- (a) Use a direct proof to show that the difference of two rational numbers is rational.

Proof: Let  $r = \frac{a}{b}$  and  $s = \frac{c}{d}$  be rational numbers, where  $a$  and  $c$  are integers and  $b$  and  $d$  are positive integers. Then,  $r - s = \frac{a}{b} - \frac{c}{d} = \frac{ad-bc}{bd}$ . Since  $a$ ,  $b$ ,  $c$ , and  $d$  are integers, we know that  $bd$  is an integer and is positive since both  $b$  and  $d$  are positive,  $ad$  is an integer, and  $bc$  is an integer. This means that  $ad - bc$  is an integer. Thus,  $r - s$  is equal to an integer divided by a positive integer. Thus,  $r - s$  is a rational number.

- (b) Use a proof by contradiction to show that if  $n$  is an integer and  $3n + 2$  is even, then  $n$  is even.

Proof: Let  $n$  be an integer and  $3n + 2$  be even and assume by way of contradiction that  $n$  is odd. Since  $n$  is odd, there exists some integer  $m$  such that  $n = 2m + 1$ . Substituting for  $n$ , we have:

$$\begin{aligned}
3n + 2 &= 3(2m + 1) + 2 \\
&= 6m + 3 + 2 \\
&= 6m + 5 \\
&= 6m + 4 + 1 \\
&= 2(3m + 2) + 1
\end{aligned}$$

Let  $h = 3m + 2$ . Note that  $h$  is an integer. We have shown that  $3n + 2 = 2h + 1$ , for some integer  $h$ . Thus we have shown that  $3n + 2$  is odd. But we assumed that  $3n + 2$  was even. This is a contradiction, and thus we have shown that  $n$  is even.

- (c) Use a proof by contrapositive to show that if  $n$  is an integer and  $n^2$  is odd, then  $n$  is odd.

Proof: By contrapositive, we show that if  $n$  is even, then  $n^2$  is also even. Assume  $n$  is an even integer. As such, there exists an integer  $k$  such that  $n = 2k$ . Then,  $n^2 = n * n = (2k) * (2k) = 4k^2 = 2(2k^2)$ . Since  $k$  is an integer,  $k^2$  is also an integer, and so is  $2k^2$ . Let  $m = 2k^2$ . Then we have shown that  $n^2 = 2m$  for some integer  $m$ . Thus, by definition,  $n^2$  is even. Thus we have shown if  $n$  is even, then  $n^2$  is also even. By way of contrapositive, we have shown that if  $n^2$  is odd, then  $n$  is odd.

- (d) Prove that if  $n$  is a positive integer, then  $n$  is even if and only if  $7n + 4$  is even.

Proof: First, we show that if  $n$  is even, then  $7n + 4$  is also even. Let  $n$  be an even integer. As such, there exists some integer  $k$  such that  $n = 2k$ . Then, substituting for  $n$ , we have:

$$\begin{aligned}
7n + 4 &= 7(2k) + 4 \\
&= 14k + 4 \\
&= 2(7k + 2)
\end{aligned}$$

Let  $h = 7k + 2$ . Then, since  $k$  is an integer, we know that  $h$  is also an integer. Thus,  $7n + 4 = 2h$ , and by definition,  $7n + 4$  is even.

Now we show that if  $7n + 4$  is even, then  $n$  must be even. We prove the contrapositive. That is, we show that if  $n$  is odd, then  $7n + 4$  is odd. Let  $n$  be an odd integer. As such, there exists an integer  $m$  such that  $n = 2m + 1$ . Then, substituting for  $n$ , we have:

$$\begin{aligned}
 7n + 4 &= 7(2m + 1) + 4 \\
 &= 14m + 7 + 4 \\
 &= 14m + 7 + 3 + 1 \\
 &= 14m + 10 + 1 \\
 &= 2(7m + 5) + 1
 \end{aligned}$$

Since  $7m + 5$  is an integer, we have shown that  $7n + 4$  is equal to two times an integer plus one. Thus, by definition,  $7n + 4$  must be odd.

- (e) Prove that  $m^2 = n^2$  if and only if  $m = n$  or  $m = -n$ , where  $m$  and  $n$  are any real number.

Solution: First, we show that if  $m = n$  or  $m = -n$  then  $m^2 = n^2$ . Let  $m = n$ . Then,  $m^2 = m * m = n * n = n^2$ . Now let  $m = -n$ . Then,  $m^2 = m * m = (-n) * (-n) = (-1) * (-1) * n^2 = n^2$ . Thus we have shown the result.

Now, we show that if  $m^2 = n^2$  then  $m = n$  or  $m = -n$ . We show this by contrapositive. Assume by way of contrapositive that  $m \neq n$  and  $m \neq -n$ . Then,  $m^2 = m * m \neq n * n = n^2$  and  $m^2 = m * m \neq (-n) * (-n) = (-1) * (-1) * n^2 = n^2$ . Thus,  $m^2 \neq n^2$ .